Critical points of the one dimensional Ambrosio-Tortorelli functional with an obstacle condition

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Abstract : We consider a family of critical points of the Ambrosio-Tortorelli energy with an obstacle condition on the phase field variable. This problem can be interpreted as a time discretization of a quasistatic evolution problem where the obstacle at step n is defined as the solution at step $n-1$. The obstacle condition now reads as an irreversibility condition (the crack can only increase in time). The questions tackled here are the regularity of the critical points, the properties inherited from the obstacle sequence, the position of the limit points and the equipartition of the phase field energy. The limits of such critical points turn out to be critical points of the Mumford-Shah energy that inherit the possible discontinuities induced by the obstacle sequence.

1 Introduction

The Mumford-Shah functional, introduced by D.Mumford and J.Shah in 1989 in [MS89] in the context of image segmentation, is also used in fracture mechanics to describe brittle fractures [FM98]. If Ω is a bounded open subset of $\mathbb{R}^N$, it is defined, for any $u \in SBV^2(\Omega)$, as

$$MS(u) = \int_\Omega |\nabla u|^2 + 2\mathcal{H}^{N-1}(J_u).$$

In the spirit of the Allen-Cahn model for phase transition ([Mod87], [Ste88]), Ambrosio and Tortorelli proposed in [AT92] the following variational phase field regularization, defined for $(u,v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ by

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 dx + \int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{(1-v)^2}{\varepsilon}\right) dx$$

where $0 < \eta_\varepsilon << \varepsilon$ is a small parameter that ensures ellipticity. This regularization has also a mechanical interpretation. The phase field variable v can be seen as a damage variable taking values in the interval $[0,1]$, between the state $\{v=1\}$ where the material is sane and the state $\{v=0\}$ where the material is broken. This regularized formulation is also at the basis of numerical simulations (see [BFM08]).

In [AT92], a $\Gamma-$convergence result of AT_ε to MS is proved by suitably extending MS as a two variable functional :

$$MS(u,v) = \begin{cases} MS(u) & \text{if } v \equiv 1\ , \\ +\infty & \text{otherwise.} \end{cases}$$

Under suitable boundary conditions, the existence of minimizers for AT_ε for a fixed value of ε is obtained via the direct method of the Calculus of Variation. The Γ-convergence then ensures that a converging sequence $(u_\varepsilon, v_\varepsilon)$ of minimizers of AT_ε converges toward a minimizer of MS.

Concerning the numerical implementation, the term $v^2|\nabla u|^2$ implies a lack of convexity for the functional AT_ε. Consequently, numerical methods might fail to converge toward a minimizer of AT_ε. Using the fact that the functional remains separately strictly convex, [BFM08] proposes to perform an alternate minimization algorithm. It is proven in [Bou07, Theorem 1] that the sequence of iterates converge toward a critical point of AT_ε, but not always to a minimizer. This motivates to investigate the convergence of critical points of the Ambrosio-Tortorelli functional as $\varepsilon \to 0$ and compare this convergence with the original goal to minimize the Mumford-Shah functional.

A critical point for the Ambrosio-Tortorelli functional is a pair $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ such that

$$\frac{d}{dt}|_{t=0}\ AT_\varepsilon(u_\varepsilon + t\psi, v_\varepsilon + t\varphi) = 0 \quad \forall\, (\psi,\varphi) \in\ H_0^1(\Omega)^2.$$

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with suitable conditions at the boundary $\partial\Omega$ (for example Dirichlet conditions, as in [BMR24]).

The Γ-convergence provides little help to study the convergence of critical points. The extension of the fundamental theorem of $\Gamma-$convergence to the convergence of critical points has been studied in various settings, see [HT00], [Ton02], [Ton05] for the Allen-Cahn functional and [BBH94], [BBO01], [LR01], [SS07] for the Ginzburg-Landau functional. For the Ambrosio-Tortorelli functional, the convergence has been established in all dimensions in [BMR24], with the additional hypothesis of the convergence of the energy $AT_\varepsilon(u_\varepsilon, v_\varepsilon)$ to $MS(u, 1)$. In dimension one, this hypothesis is no longer required, and it is proved in [FLS09] and [BMR23] that a sequence of critical points for the one-dimensional Ambrosio-Tortorelli converges to critical points of MS. A similar result is proved in [Le10] for the one-dimensional Mumford-Shah functional adapted to image segmentation. (See also [BI23] in cohesive fracture).

In [FL03], a model for the quasi-static evolution of a brittle fracture using the Mumford-Shah functional is proposed. The model starts with a discretized version of the evolution of the crack, where the irreversibility through time is translated into the condition that the crack $K(t_i)$ is included in the crack $K(t_{i+1})$, and the Mumford-Shah functional is then minimized at time t_{i+1} over all the cracks K containing $K(t_i)$, and with a variable Dirichlet condition g_i at the boundary $\partial\Omega$. In [Gia05], it is proved that a quasi-static evolution of AT_ε approximates a quasi-static evolution for MS using minimizers of AT_ε. The irreversibility condition translates into the minimization of AT_ε at time t_{i+1} among all couples (u, v) such that $v \leqslant v_{t_i,\varepsilon}$, where $(u_{t_i,\varepsilon}, v_{t_i,\varepsilon})$ is the minimum of AT_ε at time t_i.

A natural question is whether the result of [Gia05] can be extended to critical points. Since in high dimension, results on the convergence of critical points of AT_ε require stronger hypothesis, such an extension can be investigated only in dimension one for the moment. This paper is the first step in that direction, namely, we study the convergence of the critical points of the functional AT_ε defined on the pairs (u, v) in $H^1([0, L])$ satisfying $v \leqslant v_{0,\varepsilon}$, where $(u_{0,\varepsilon}, v_{0,\varepsilon})$ is a critical point of AT_ε. We use the same setting as [BMR23] (Dirichlet conditions on the variables (u, v)).

In Section 2, we define the one-dimensional functional $\overline{AT}_\varepsilon$ at study, which corresponds to the functional AT_ε restricted to the pairs (u, v) satisfying $v \leqslant v_{0,\varepsilon}$, we recall the results established by [BMR23] and state the main result in Theorem 2.6 : the critical points of $\overline{AT}_\varepsilon$ with the irreversibility condition $v \leqslant v_{0,\varepsilon}$ converge to critical point of MS. Theorem 2.7, proved in the appendix, deals with the $\Gamma-$convergence of the sequence $\overline{AT}_\varepsilon$. Theorem 2.6 can then be seen as an extension of Theorem 2.7 for critical points.

When we study the convergence of a functional with a phase field term, a natural question is the validity of an equipartition of the energy principle, i.e whether the terms of the phase field energy share the same limit. This principle, as well as the convergence of the Dirichlet approximation term is also stated in Theorem 2.6.

In Section 3, we establish the properties concerning the critical points of $\overline{AT}_\varepsilon$, namely the regularity and the variations of v_ε. In Section 4, we study the asymptotic behaviour of the sequence $(u_\varepsilon, v_\varepsilon)$ and prove Theorem 2.6.

Similarly to [FLS09], [BMR23] and [Le10], the key point to the main estimates is the study of the quantity called *discrepancy* (Equation 4.5) associated to $\overline{AT}_\varepsilon$ (which can be interpreted as the Hamiltonian of the functional). In [FLS09], [BMR23], this discrepancy is constant, which is no longer the case in our situation. However, we are able to control the variations of the discrepancy in Lemma 4.7 to obtain analogous estimates and results as in [BMR23].

2 Notations and statement of the results

2.1 Notations and preliminary results

Throughout the paper, L is a positive number. Letters like C, m and M will denote constants that do not depend on any other parameter and that may change from line to line. We also denote by $I_r(x)$ the interval $(x - r, x + r)$.

For any open subset $\Omega \subset (0, L)$, we denote by

$$||f||_{\infty,\Omega} := \sup_{x\in\Omega} |f(x)| \,,$$

$$[\,f\,]_{1/2,\Omega} := \sup_{x\neq y\in\Omega} \frac{|f(x) - f(y)|}{|x - y|^{1/2}} \,,$$

$$||f||_{C^{0,1/2},\Omega} := ||f||_{\infty,\Omega} + [\,f\,]_{1/2,\Omega}.$$

The one-dimensional Ambrosio-Tortorelli functional, is defined as

$$AT_\varepsilon(u, v) := \int_0^L \left((\eta_\varepsilon + v^2)|u'|^2 + \frac{(1 - v)^2}{\varepsilon} + \varepsilon|v'|^2 \right) dx, \qquad \forall (u, v) \in H^1(0, L)^2,$$

where η_ε is a positive parameter such that $\eta_\varepsilon/\varepsilon \to 0$. Before defining the one-dimensional Mumford-Shah functional, we recall the definition of the space $SBV^2(0,L)$, which is the space of the functions $u \in L^1$ such that their distributional derivative is a Radon measure which writes as

$$Du = u' d\mathcal{L} + (u^+ - u^-)\nu_u \mathcal{H}^0_{|J_u} \quad \text{with} \quad \int_0^L |u'|^2 < \infty,\ \mathcal{H}^0(J_u) < \infty.$$

The one-dimensional Mumford-Shah functional MS defined on $SBV^2(0,L) \times L^2(0,L)$ reads as

$$MS(u,v) := \begin{cases} \displaystyle\int_0^L |u'|^2 + 2\mathcal{H}^0(J_u) & \text{if } v \equiv 1\ , \\ +\infty & \text{otherwise.} \end{cases} .$$

We define a critical point of AT_ε as follows

Definition 2.1. *Let $a_0 \in \mathbb{R}$. A critical point of AT_ε is a pair $(u,v) \in H^1(0,L)^2$ that satisfies for all $(\psi,\varphi) \in \mathcal{C}_c^\infty(0,L)$,*

$$\begin{cases} dAT_\varepsilon(u,v)[(\psi,\varphi)] := \lim\limits_{t\to 0} \dfrac{AT_\varepsilon(u+t\psi, v+t\varphi) - AT_\varepsilon(u,v)}{t} = 0\ , \\ u_{0,\varepsilon}(0) = 0,\ u_{0,\varepsilon}(L) = a_0\ , \\ v_{0,\varepsilon}(0) = v_{0,\varepsilon}(L) = 1. \end{cases}$$

A critical point $(u_{0,\varepsilon}, v_{0,\varepsilon})$ of AT_ε satisfies the following equations on $(0,L)$

$$\begin{cases} -\varepsilon v''_{0,\varepsilon} + \frac{v_{0,\varepsilon}-1}{\varepsilon} + v_{0,\varepsilon}|u'_{0,\varepsilon}|^2 = 0 & \text{in } (0,L)\ , \\ \left[u'_{0,\varepsilon}(\eta_\varepsilon + v^2_{0,\varepsilon})\right]' = 0 & \text{in } (0,L)\ , \\ u_{0,\varepsilon}(0) = 0,\ u_{0,\varepsilon}(L) = a_0\ , \\ v_{0,\varepsilon}(0) = v_{0,\varepsilon}(L) = 1. \end{cases} \tag{2.1}$$

In the sequel, we consider a sequence $(u_{0,\varepsilon}, v_{0,\varepsilon})$ of critical points of AT_ε satisfying the following energy bound

$$\limsup_{\varepsilon>0} AT_\varepsilon(u_{0,\varepsilon}, v_{0,\varepsilon}) < C. \tag{2.2}$$

In [BMR23, Theorem 1.1], the following has been established :

Theorem 2.2. *Up to a subsequence, $(u_{0,\varepsilon}, v_{0,\varepsilon}) \to (u_0, 1)$ in $L^2(0,L)^2$, where either $u_0 = u_{jump} \equiv a_0 1_{[L/2,L]}$ or $u_0(x) = u_{aff}(x) = a_0 x/L$.*

When integrating the second equation of (2.1), we obtain a constant $c_{0,\varepsilon}$ such that

$$u'_{0,\varepsilon}(\eta_\varepsilon + v^2_{0,\varepsilon}) = c_{0,\varepsilon}.$$

[BMR23, Lemma 2.4] states that

Lemma 2.3. *Up to a subsequence, the sequence*

$$c_{0,\varepsilon} \to c_0; \tag{2.3}$$

with c_0 a real number satisfying $c_0 \in \{0, a_0/L\}$.

Theorem 2.2 enlightens a selection phenomenon : the accumulation points of the sequence $(u_{0,\varepsilon}, v_{0,\varepsilon})$ are specific critical points of MS. Indeed, critical points of MS are either the affine function u_{aff} or the piecewise constant functions. It is also proved in [BMR23, Theorem 1.2] that u_{jump} can be reached by a sequence of critical points of AT_ε. The crack of the material at time 0 then corresponds to J_{u_0}, which is, according to [BMR23, Theorem 1.1]

$$\Gamma_0 = J_{u_0} = \emptyset \qquad \text{or} \qquad \Gamma_0 = J_{u_0} = \left\{\frac{L}{2}\right\}. \tag{2.4}$$

The Ambrosio-Tortorelli functional with obstacle is defined as

$$\overline{AT_\varepsilon}(u,v) := \begin{cases} AT_\varepsilon(u,v) & \text{if } v \leqslant v_{0,\varepsilon}\ , \\ +\infty & \text{else,} \end{cases} \qquad \forall\,(u,v) \in H^1(0,L). \tag{2.5}$$

In order to account for the constraint on the phase field variable, we introduce the following definition for constrained critical points.

Definition 2.4. *A critical point of $\overline{AT}_\varepsilon$ is a pair $(u,v) \in H^1(0,L)^2$ that satisfies $v \leqslant v_{0,\varepsilon}$ and, for all $(\psi,\varphi) \in \mathcal{C}_c^\infty(0,L)$ with $\varphi \geqslant 0$,*

$$\begin{cases} d\overline{AT}_\varepsilon(u,v)[(\psi,\varphi)] := \lim\limits_{t\to 0^+} \dfrac{\overline{AT}_\varepsilon(u,v) - \overline{AT}_\varepsilon(u-t\psi, v-t\varphi)}{t} = 0\ , \\ u(0) = 0,\ u(L) = a_1\ , \\ v(0) = v(L) = 1. \end{cases}$$

Note that the Dirichlet condition at L is different and carry the evolution in time of the material.

In the sequel, we denote

$$O_\varepsilon = \{v_\varepsilon < v_{0,\varepsilon}\} \quad \text{and} \quad F_\varepsilon = \{v_\varepsilon = v_{0,\varepsilon}\}. \tag{2.6}$$

Since $v_\varepsilon, v_{0,\varepsilon} \in H^1(0,L)$, there difference is continuous by Sobolev embedding so O_ε and F_ε are respectively open and closed in $[0,L]$.

2.2 Euler-Lagrange equations

In this subsection, we establish the Euler-Lagrange equations for $\overline{AT}_\varepsilon$. Let $(u_\varepsilon, v_\varepsilon)$ be a critical point of $\overline{AT}_\varepsilon$. Take $\varphi = 0$ in Definition 2.4. Similarly to [FLS09, Introduction], the equation satisfied by u_ε reads as

$$\left[u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)\right]' = 0. \tag{2.7}$$

This means that there exists a constant c_ε such that

$$u_\varepsilon' = \frac{c_\varepsilon}{(\eta_\varepsilon + v_\varepsilon^2)} \qquad \text{a.e in } (0,L). \tag{2.8}$$

As v_ε is continuous by Sobolev embedding, the left hand side is also continuous. Similarly to [FLS09], we obtain that u_ε' has a constant sign, corresponding to the sign of c_ε, meaning that u_ε is monotone on $[0,L]$.

We next move on to the equation for v_ε. Taking $\psi = 0$ in Definition 2.4, we obtain the following inequality, where v_ε'' denotes the distributional derivative of v_ε', which exists as $v_\varepsilon \in H^1(0,L)$,

$$\left\langle \varphi\ ,\ v_\varepsilon|u_\varepsilon'|^2 - \varepsilon v_\varepsilon'' + \frac{v_\varepsilon - 1}{\varepsilon} \right\rangle \leqslant 0 \tag{2.9}$$

which we rewrite in the following way : there exists a non positive Radon measure distribution μ_ε such that

$$-\varepsilon v_\varepsilon'' + \frac{v_\varepsilon - 1}{\varepsilon} + v_\varepsilon|u_\varepsilon'|^2 = \mu_\varepsilon \leqslant 0. \tag{2.10}$$

Taking φ compactly supported on $\{v_\varepsilon < v_{0,\varepsilon}\}$ gives the reversed inequality of (2.9). This means that the measure μ_ε is supported on the closed set F_ε.

Using (2.10), we see that v_ε'' is also a Radon measure, so that $v_\varepsilon' \in BV(0,L)$, which is embedded in $L^\infty(0,L)$, and that $v_\varepsilon \in C^{0,1}(0,L)$. In section 3.1, we provide a better regularity result on v_ε. This allows us to establish an explicit formula for μ_ε at the end of section 3.1 (Corollary 3.3).

In summary, the equations satisfied by $(u_\varepsilon, v_\varepsilon)$ are

$$\begin{cases} -\varepsilon v_\varepsilon'' + \frac{v_\varepsilon - 1}{\varepsilon} + v_\varepsilon|u_\varepsilon'|^2 = \mu_\varepsilon \leqslant 0 & \text{in } (0,L) \\ \left[u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)\right]' = 0 & \text{in } (0,L) \\ v_\varepsilon \leqslant v_{0,\varepsilon} & \text{in } (0,L)\ , \\ u_\varepsilon(0) = 0,\ u_\varepsilon(L) = a_0\ , \\ v_\varepsilon(0) = v_\varepsilon(L) = 1. \end{cases} \tag{2.11}$$

2.3 Statement of the theorems

We consider a sequence $(u_\varepsilon, v_\varepsilon)$ of critical points of $\overline{AT}_\varepsilon$. We suppose that the sequence $(u_\varepsilon, v_\varepsilon)$ satisfies the following energy bound

$$\limsup_{\varepsilon\to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \mathcal{C}. \tag{2.12}$$

Our first result concerns the regularity and is proved in section 3.

Theorem 2.5. *Let $(u_\varepsilon, v_\varepsilon)$ be a critical point of $\overline{AT}_\varepsilon$ in $H^1(0,L)\times H^1(0,L)$. Then $v_\varepsilon \in C^{1,1}([0,L])$.*

We now state the main result of this work concerning the limit points of $(u_\varepsilon, v_\varepsilon)$.

Theorem 2.6 (Convergence of the critical points)**.** *Let $(u_\varepsilon, v_\varepsilon)$ be a family of critical points of $\overline{AT}_\varepsilon$ satisfying (2.12). Then there exists a real number $x_{min} \in (0,L)$ and a subsequence such that*

(i) $(u_\varepsilon, v_\varepsilon) \to (u_1, 1)$ *in* $L^2(0,L)$, *with either* $u_1(x) = u_{jump}(x) = a_1 1_{[x_{min},L]}(x)$ *or* $u_1(x) = u_{aff}(x) = \dfrac{a_1 x}{L}$. *Moreover, if* $|F_\varepsilon| \neq 0$ *and* $|a_1| \geqslant a_0$, *then* $u_1 = u_{jump}$.

(ii) $\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon(v_\varepsilon')^2\right) dx \overset{\star}{\rightharpoonup} \alpha\delta_{x_{min}}$ *in* $\mathcal{M}([0,L])$ *with* $\alpha = 0$ *or* $\alpha = 2$. *Moreover, if* $u_1 = u_{jump}$ *then* $\alpha = 2$.

(iii) $(\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 dx \overset{\star}{\rightharpoonup} |u_1'|^2 dx$ *in* $\mathcal{M}([0,L])$.

(iv) *(Equipartition of the energy principle)* $\left|\dfrac{(1-v_\varepsilon)^2}{\varepsilon} - \varepsilon|v_\varepsilon'|^2\right| \to 0$ *in* $L^1(0,L)$.

(v) $\varepsilon|v_\varepsilon'|^2 dx \overset{\star}{\rightharpoonup} \dfrac{\alpha}{2}\delta_{x_{min}}$, *and* $\dfrac{(1-v_\varepsilon)^2}{\varepsilon} dx \overset{\star}{\rightharpoonup} \dfrac{\alpha}{2}\delta_{x_{min}}$ *in* $\mathcal{M}([0,L])$.

Remark 2.1. Theorem 2.6 extends the following $\Gamma-$ convergence result, which we prove in the appendix :

Theorem 2.7 (Γ convergence of $\overline{AT}_\varepsilon$)**.** *The functional $\overline{AT}_\varepsilon$ Γ-converges to the functional $\overline{MS}$, defined for every $(u,v) \in L^2(0,L)^2$ as*

$$\overline{MS}(u,v) := \begin{cases} \displaystyle\int_0^L |u'|^2 + 2\mathcal{H}^0(J_u \cup \Gamma_0) & \text{if } v = 1\,, \\ +\infty & \text{else}\,. \end{cases}$$

where Γ_0 is defined by (2.4).

While the Γ-convergence implies that a converging sequence of minimizers of $\overline{AT}_\varepsilon$ converges to a minimizer of $\overline{MS}(u)$, Theorem 2.6 acts as a selection criteria : the limit of critical points of $\overline{AT}_\varepsilon$ are critical points of $\overline{MS}$ that have at most one discontinuity point. Observe that we recover a similar selection phenomenon as in [BMR23, Theorem 1.2].

Considering (2.7), it is natural to define the crack at time 1 by

$$\Gamma_1 = \Gamma_0 \cup J_{u_1} \tag{2.13}$$

The different possible values for Γ_1 according to Theorem 2.6 are represented in Figure 1.

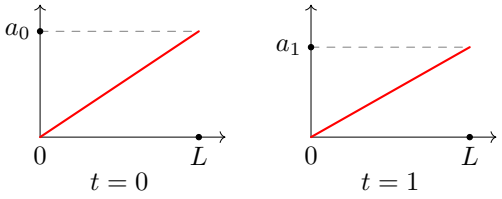
Remark 2.2. The item *(iii)* of Theorem 2.6 is called the *equipartition of energy principle* and is to be compared with the equality case of the arithmetico-geometric inequality

$$\frac{(1-v)^2}{\varepsilon} + \varepsilon|v'|^2 \geqslant 2|1-v||v'|$$

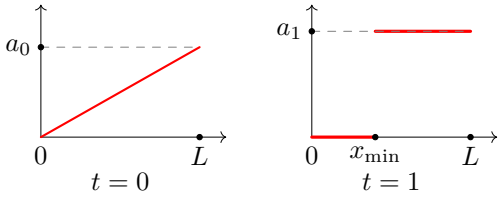
used in the proof of Gamma-convergence of AT_ε to MS.

Remark 2.3. Let us stress that the implication in *(ii)* is not an equivalence, and that the converse is still an open question. However, the fact that $\alpha > 0$ is equivalent to the existence of a so-called $v-$jump for the sequence (v_ε) defined as follows :

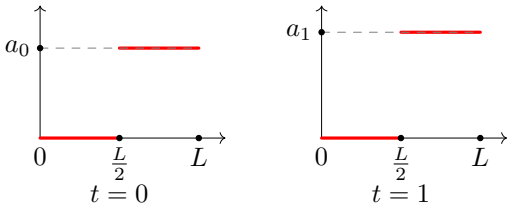
Definition 2.8. *A point $x \in [0,L]$ is a $v-$jump if there exists a sequence (x_ε) such that $x_\varepsilon \to x$ and $v_\varepsilon(x_\varepsilon) \to 0$.*



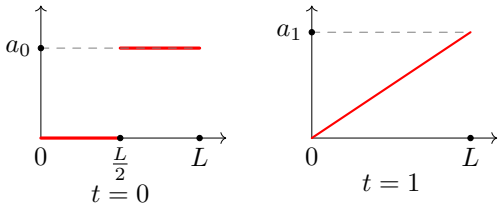
(a) $\Gamma_0 = \emptyset$, $J_{u_1} = \emptyset$ and $\Gamma_1 = \emptyset$



(b) $\Gamma_0 = \emptyset$, $J_{u_1} = \{x_{\min}\}$ and $\Gamma_1 = \{x_{\min}\}$



(c) $\Gamma_0 = \{L/2\}$, $J_{u_1} = \{L/2\}$ and $\Gamma_1 = \{L/2\}$



(d) $\Gamma_0 = \{L/2\}$, $J_{u_1} = \emptyset$ and $\Gamma_1 = \{L/2\}$

Figure 1: Possible values of Γ_0, J_{u_1} and Γ_1.

It turns out that a $v-$jump does not necessarily lead to a discontinuity point in the limit. The variations and the symmetry of the function $v_{0,\varepsilon}$ around $L/2$ imply that the only possible $v-$jump for $(v_{0,\varepsilon})$ is $L/2$. This symmetry disappears at time 1, as x_{min} is not necessarily equal to $L/2$.

Remark 2.4. A similar result holds for a sequence $(u_\varepsilon, v_\varepsilon)$ of critical points of $\overline{AT}_\varepsilon$ with Neumann conditions on v_ε, which is the framework of [FLS09, Theorem 2.2]. More precisely, if Γ_0 and Γ_1 are defined as in (2.4) and (2.13), [FLS09, Theorem 2.2] states that $\Gamma_0 = \emptyset$ or Γ_0 is a regular subdivision of $(-L, L)$. It can be proved that

- in the former case, $\Gamma_1 = \{x_0, \dots, x_{n+1}\}$, where $x_0 = 0, x_{n+1} = L$ and $x_i \in \left[\frac{(2i-1)L}{2n}, \frac{(2i+1)L}{2n}\right]$ for $1 \leqslant i \leqslant n$ or $\{x_1, \dots, x_n\}$ where $x_i \in \left[\frac{(i-1)L}{n}, \frac{iL}{n}\right]$ for $1 \leqslant i \leqslant n$. Observe that, as for Theorem 2.6, the symmetry of the "perfect staircase" observed in [FLS09, Remark 2.3] is lost at time 1.
- in the latter case, $\Gamma_1 = \Gamma_0$.

We would like to close this section by explaining what happens if we intend to iterate Theorem 2.6 for $t = 2$. Denote by $(u_{1,\varepsilon}, v_{1,\varepsilon})$ a sequence of critical points at time 1, i.e satisfying $v_{1,\varepsilon} \leqslant v_{0,\varepsilon}$, $\mu_{1,\varepsilon}$ the right-hand side of the first equation of (2.11) and $(u_{2,\varepsilon}, v_{2,\varepsilon})$ a sequence of critical points at time 2, i.e satisfying $v_{2,\varepsilon} \leqslant v_{1,\varepsilon}$. The right hand-side in the first equation of (2.11) corresponding to $v_{2,\varepsilon}$ is a non positive measure $\mu_{2,\varepsilon}$. The expression of $\mu_{2,\varepsilon}$ now depends both on $\{v_{0,\varepsilon} = v_{1,\varepsilon}\}$ and $\{v_{1,\varepsilon} = v_{2,\varepsilon}\}$, so that we do not always have a similar result as Corollary 3.3 in the case $|\{v_{1,\varepsilon} = v_{2,\varepsilon}\}| \neq 0$. As a consequence, the behavior of $(u_{2,\varepsilon}, v_{2,\varepsilon})$ is less predictable, in the sense that we do not have an analogous of item (i) of Theorem 2.6. A possible reason to that phenomenon is that, as we consider critical points and not only minimizers, we do not have an energy balance at our disposal, which is crucial in the iteration argument proposed in [Gia05, Theorem 2.2].

3 Critical points of $\overline{AT}_\varepsilon$

3.1 $C^{1,1}$ regularity of the critical points

In this subsection, we investigate the regularity of v_ε. More precisely, we establish the following proposition :

Proposition 3.1. *Let $(u_\varepsilon, v_\varepsilon)$ be a critical point of $\overline{AT}_\varepsilon$ in $H^1(0, L) \times H^1(0, L)$. Then $v_\varepsilon \in C^{1,1}([0, L])$.*

The methods of the proof are similar to those of the classical obstacle problem, see e.g. [FRRO22, Theorem 5.5]. Set $w_\varepsilon = v_{0,\varepsilon} - v_\varepsilon$, which is a non-negative function on $[0, L]$. Note that by Sobolev embedding, $v_\varepsilon \in C^{0,1/2}([0, L])$, so that, using the equation (2.8), $u_\varepsilon \in C^{1,1/2}([0, L])$. In particular, u'_ε is bounded on $[0, L]$. On $(0, L)$, w_ε satisfies the following equation

$$\begin{aligned}\varepsilon w_\varepsilon'' &= \varepsilon(v_{0,\varepsilon}''-v_\varepsilon'') = \frac{v_{0,\varepsilon}-1-(v_\varepsilon-1)}{\varepsilon} + v_{0,\varepsilon}|u_{0,\varepsilon}'|^2 - v_\varepsilon|u_\varepsilon'|^2+\mu_\varepsilon\\ &= \frac{w_\varepsilon}{\varepsilon} + (v_{0,\varepsilon}-v_\varepsilon)|u_{0,\varepsilon}'|^2 + v_\varepsilon(|u_{0,\varepsilon}'|^2-|u_\varepsilon'|^2)+\mu_\varepsilon\\ &= \frac{w_\varepsilon}{\varepsilon} + w_\varepsilon|u_{0,\varepsilon}'|^2 + v_\varepsilon\left(\frac{c_{0,\varepsilon}^2}{(\eta_\varepsilon+v_{0,\varepsilon}^2)^2}-\frac{c_\varepsilon^2}{(\eta_\varepsilon+v_\varepsilon^2)^2}\right)+\mu_\varepsilon\end{aligned}$$

Since $\mu_\varepsilon \leqslant 0$, we obtain the following expressions

$$\begin{cases} \varepsilon w_\varepsilon'' \leqslant w_\varepsilon\left(\frac{1}{\varepsilon}+|u_{0,\varepsilon}'|^2\right)+v_\varepsilon\left(\frac{c_{0,\varepsilon}^2}{(\eta_\varepsilon+v_{0,\varepsilon}^2)^2}-\frac{c_\varepsilon^2}{(\eta_\varepsilon+v_\varepsilon^2)^2}\right) & \text{on } [0,L],\\ \varepsilon w_\varepsilon'' = w_\varepsilon\left(\frac{1}{\varepsilon}+|u_{0,\varepsilon}'|^2\right)+v_\varepsilon\left(\frac{c_{0,\varepsilon}^2}{(\eta_\varepsilon+v_{0,\varepsilon}^2)^2}-\frac{c_\varepsilon^2}{(\eta_\varepsilon+v_\varepsilon^2)^2}\right) & \text{on } O_\varepsilon. \end{cases} \tag{3.1}$$

Since $v_{0,\varepsilon}$ is smooth, it is sufficient to prove that w_ε is in $C^{1,1}([0,L])$. We first need to prove that w_ε grows at most quadratically around ∂O_ε (Lemma 3.2). Then, we can use Schauder estimates on (3.1) to ensure that w_ε' is continuous at ∂O_ε.

Lemma 3.2. *There exists a constant $C>0$ only depending on $\|u_{0,\varepsilon}'\|_{\infty,[0,L]}, \|u_\varepsilon'\|_{\infty,[0,L]}$ and ε such that for any x such that $d(x,\partial O_\varepsilon) \leqslant \frac{1}{3}d(x,\{0,L\})$, one has*

$$0 \leqslant w_\varepsilon(x) \leqslant C d(x,\partial O_\varepsilon)^2.$$

Proof. Since $0\leqslant v_\varepsilon \leqslant v_{0,\varepsilon}\leqslant 1$, w_ε is bounded. As u_ε' and $u_{0,\varepsilon}'$ are also bounded on $[0,L]$, the first inequality of the system (3.1) gives

$$\varepsilon w_\varepsilon'' \leqslant \frac{1}{\varepsilon}w_\varepsilon + |u_{0,\varepsilon}'|^2 w_\varepsilon + v_\varepsilon(|u_{0,\varepsilon}'|^2-|u_\varepsilon'|^2) \leqslant C, \tag{3.2}$$

where this inequality is written in the distributional sense.

Fix any $z\in(0,L)$. The mean value formula applied to the superharmonic function $\varepsilon w_\varepsilon - C\frac{|z-\cdot|^2}{2}$ yields, for any r such that $I_r(z)\subset[0,L]$,

$$\varepsilon w_\varepsilon(z) \geqslant \frac{1}{2r}\int_{I_r(z)}\left(\varepsilon w_\varepsilon(y) - C\frac{|z-y|^2}{2}\right)dy \geqslant \frac{\varepsilon}{2r}\int_{I_r(z)} w_\varepsilon(y)dy - Cr^2 \tag{3.3}$$

Secondly, for any $y\in O_\varepsilon$, the second equation of (3.1) yields

$$\varepsilon w_\varepsilon''(y) \geqslant -v_\varepsilon(y)\frac{c_\varepsilon^2}{(\eta_\varepsilon+|v_\varepsilon|^2)^2} \geqslant -\frac{c_\varepsilon^2}{\eta_\varepsilon^2} = -C'.$$

Fix then $z\in O_\varepsilon$. The mean value formula applied to the subharmonic function $\varepsilon w_\varepsilon + C'\frac{|z-\cdot|^2}{2}$ yields for any r such $I_r(z)\subset O_\varepsilon$,

$$\varepsilon w_\varepsilon(z) \leqslant \frac{1}{2r}\int_{I_r(z)}\left(\varepsilon w_\varepsilon(y) + C'\frac{|z-y|^2}{2}\right)dy \leqslant \frac{\varepsilon}{2r}\int_{I_r(z)} w_\varepsilon(y)dy + C'r^2. \tag{3.4}$$

Now fix $x\in O_\varepsilon$ such that $r := d(x,\partial O_\varepsilon)$ satisfies $r\leqslant \frac{1}{3}d(x,\{0,L\})$ and take any $x'\in \overline{I_r(x)}\cap\partial O_\varepsilon$. Then $I_{2r}(x')\subset I_{3r}(x)\subset(0,L)$ and $x'\in F_\varepsilon$ so, using (3.3),

$$0=\varepsilon w_\varepsilon(x') \geqslant \frac{1}{4r}\int_{I_{2r}(x')}\varepsilon w_\varepsilon(y)dy - Cr^2. \tag{3.5}$$

Then, since $w_\varepsilon\geqslant 0$ and $I_r(x)\subset I_{2r}(x')$, we get

$$\varepsilon w_\varepsilon(x) - C'r^2 \overset{(3.4)}{\leqslant} \frac{\varepsilon}{2r}\int_{I_r(x)} w_\varepsilon(y)dy \leqslant \frac{\varepsilon}{2r}\int_{I_{2r}(x')} w_\varepsilon(y)dy \overset{(3.5)}{\leqslant} 2Cr^2.$$

which is the desired inequality. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. First, we focus on the regularity on $(0,L)$. Let $x_1 \in O_\varepsilon$ and take $x_0 \in \partial O_\varepsilon$ such that $r := |x_0 - x_1| = d(x_1, \partial O_\varepsilon)$ and suppose that $r < \frac{1}{3} d(x_1, \{0, L\})$. Notice that the right-hand side of the second equation of (3.1) belongs to $C^{0,1/2}([0,L])$. Applying Schauder estimates ([GT01, Theorem 4.6]) on $I_r(x_1) \subset O_\varepsilon$, there exists a constant C independent of x_1, r and w_ε such that

$$r\|w_\varepsilon'\|_{\infty, I_{r/2}(x_1)} + r^2 \|w_\varepsilon''\|_{\infty, I_{r/2}(x_1)} \leqslant C \left(\underbrace{\|w_\varepsilon\|_{\infty, I_r(x_1)}}_{:=I} + r^2 \underbrace{\left\| w_\varepsilon \left(\frac{1}{\varepsilon} + |u_\varepsilon'|^2 \right) \right\|_{C^{0,1/2}, I_r(x_1)}}_{:=II} + r^2 \underbrace{\|v_\varepsilon(|u_{0,\varepsilon}'|^2 - |u_\varepsilon'|^2)\|_{C^{0,1/2}, I_r(x_1)}}_{:=III} \right). \quad (3.6)$$

We now estimate the terms in the right hand side of (3.6).

I : Using Lemma 3.2, one has $\|w_\varepsilon\|_{\infty, I_r(x_1)} \leqslant C d(x_1, \partial O_\varepsilon)^2 = C r^2$.

II : We split the norm $\|\cdot\|_{C^{0,1/2}}$ into the norm $\|\cdot\|_\infty$ and the semi-norm $[\cdot]_{1/2}$. Concerning the norm $\|\cdot\|_\infty$, one has

$$\left\| w_\varepsilon \left(\frac{1}{\varepsilon} + |u_\varepsilon'|^2 \right) \right\|_{\infty, I_r(x_1)} \leqslant 2 \left(\frac{1}{\varepsilon} + \|u_\varepsilon'\|_{\infty,[0,L]}^2 \right) = C,$$

where $C > 0$ is a constant which depends on $\|u_\varepsilon'\|_{\infty,[0,L]}$. Concerning the semi-norm $[\cdot]$, we use the inequality $[|u_\varepsilon'|^2]_{1/2} \leqslant 2\|u_\varepsilon'\|_\infty [u_\varepsilon']_{1/2}$. This yields

$$\begin{aligned} \left[w_\varepsilon \left(\frac{1}{\varepsilon} + |u_\varepsilon'|^2 \right) \right]_{1/2, I_r(x_1)} &\leqslant \|w_\varepsilon\|_{\infty, I_r(x_1)} \left[\frac{1}{\varepsilon} + |u_\varepsilon'|^2 \right]_{1/2, I_r(x_1)} + [w_\varepsilon]_{1/2, I_r(x_1)} \left\| \frac{1}{\varepsilon} + |u_\varepsilon'|^2 \right\|_{\infty, I_r(x_1)} \\ &\leqslant 2\|u_\varepsilon'\|_{\infty,[0,L]} [u_\varepsilon']_{1/2,[0,L]} + [w_\varepsilon]_{1/2,[0,L]} \left\| \frac{1}{\varepsilon} + |u_\varepsilon'|^2 \right\|_{\infty,[0,L]} \\ &\leqslant C \end{aligned}$$

where $C > 0$ is a constant depending on $\|u_\varepsilon'\|_{C^{0,1/2},[0,L]}$ and $\|w_\varepsilon\|_{C^{0,1/2},[0,L]}$.

III : Similarly, one has

$$\|v_\varepsilon(|u_{0,\varepsilon}'|^2 - |u_\varepsilon'|^2)\|_{\infty, I_r(x_1)} \leqslant \|u_{0,\varepsilon}'\|_{\infty,[0,L]}^2 + \|u_\varepsilon'\|_{\infty,[0,L]}^2,$$

and

$$\begin{aligned} & [v_\varepsilon(|u_{0,\varepsilon}'|^2 - |u_\varepsilon'|^2)]_{1/2, I_r(x_1)} \\ \leqslant\ & \|v_\varepsilon\|_{\infty, I_r(x_1)} ([|u_{0,\varepsilon}'|^2]_{1/2, I_r(x_1)} + |u_\varepsilon'|_{1/2, I_r(x_1)}^2] + [v_\varepsilon]_{1/2, I_r(x_1)} (\|u_{0,\varepsilon}'\|_{\infty, I_r(x_1)}^2 + \|u_\varepsilon'\|_{\infty, I_r(x_1)}^2) \\ \leqslant\ & 2\|u_{0,\varepsilon}'\|_{\infty,[0,L]} [u_{0,\varepsilon}']_{1/2,[0,L]} + 2\|u_\varepsilon'\|_{\infty,[0,L]} [u_\varepsilon']_{1/2,[0,L]} + [v_\varepsilon]_{1/2,[0,L]} (\|u_{0,\varepsilon}'\|_{\infty,[0,L]}^2 + \|u_\varepsilon'\|_{\infty,[0,L]}^2) \\ \leqslant\ & C \end{aligned},$$

where $C > 0$ is a constant depending only on $\|u_{0,\varepsilon}'\|_{C^{0,1/2},[0,L]}, \|u_\varepsilon'\|_{C^{0,1/2},[0,L]}$ and $\|v_\varepsilon'\|_{C^{0,1/2},[,L]}$. It follows from (3.6) that

$$r\|w_\varepsilon'\|_{\infty, I_{r/2}(x_1)} + r^2 \|w_\varepsilon''\|_{\infty, I_{r/2}(x_1)} \leqslant C r^2.$$

In particular, $|w_\varepsilon'(x_1)| \leqslant \|w_\varepsilon'\|_{\infty, I_{r/2}(x_1)} \leqslant C d(x_1, \partial O_\varepsilon)$. This means that $w_\varepsilon'(x_1)$ tends to 0 as $d(x_1, \partial O_\varepsilon) \to 0$. Moreover, as w_ε vanishes on F_ε, $w_\varepsilon' = 0$ a.e on F_ε (see [GT01, Lemma 7.7]). So w_ε' admits a representative on $(0,L)$ that is zero on F_ε and is continuous on $(0,L)$, so that $w_\varepsilon \in C^1(0,L)$.

From (3.2), we also have

$$\|w_\varepsilon''\|_{\infty,(0,L)} \leqslant C_2,$$

where $C_2 > 0$ is a constant depending on $\|u'_{0,\varepsilon}\|_{\infty,[0,L]}$, from which we deduce that w'_ε is Lipschitz on $(0, L)$.

To recover the regularity at the boundary points 0 and L, we see that w'_ε is uniformly continuous on $(0, L)$(because it is Lipschitz) so it admits a continuous extension on $[0, L]$. This implies that w_ε is differentiable at 0 and L with $w'_\varepsilon(0) = \lim_{x\to 0^+} w'_\varepsilon(x)$ and $w'_\varepsilon(L) = \lim_{x\to L^-} w'_\varepsilon(x)$. We deduce that $w_\varepsilon \in C^1([0, L])$. Then, as w'_ε is Lipschitz on $(0, L)$, its extension to $[0, L]$ is Lipschitz as well so that $w_\varepsilon \in C^{1,1}([0, L])$. Hence $v_\varepsilon \in C^{1,1}([0, L])$. □

The following result gives an explicit pointwise formula for μ_ε.

Corollary 3.3. *Let μ_ε be the right-hand side in (2.10). We have the following expression for μ_ε,*

$$\mu_\varepsilon = \frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2 + \eta_\varepsilon]^2}((c_\varepsilon)^2 - (c_{0,\varepsilon})^2)1_{F_\varepsilon} = v_{0,\varepsilon}(|u'_\varepsilon|^2 - |u'_{0,\varepsilon}|^2)1_{F_\varepsilon} \tag{3.7}$$

In particular, if $|F_\varepsilon| \neq 0$, then $|c_\varepsilon| \leqslant c_{0,\varepsilon}$.

Proof. As $v_\varepsilon \in C^{1,1}([0, 1])$, μ_ε is absolutely continuous with respect to the Lebesgue measure. Moreover, as $v_\varepsilon = v_{0,\varepsilon}$ on F_ε, we have $v'_\varepsilon = (v_{0,\varepsilon})'$ and $v''_\varepsilon = (v_{0,\varepsilon})''$ a.e on F_ε (see [GT01, Lemma 7.7]). Hence

$$\begin{aligned}
\mu_\varepsilon &= \left(-\varepsilon v''_\varepsilon + \frac{v_\varepsilon - 1}{\varepsilon} + v_\varepsilon|u'_\varepsilon|^2\right)1_{F_\varepsilon} &&= \left(-\varepsilon v''_{0,\varepsilon} + \frac{v_{0,\varepsilon} - 1}{\varepsilon} + v_{0,\varepsilon}|u'_\varepsilon|^2\right)1_{F_\varepsilon} \\
&= \left(-v_{0,\varepsilon}(u'_{0,\varepsilon})^2 + v_{0,\varepsilon}|u'_\varepsilon|^2\right)1_{F_\varepsilon} &&= v_{0,\varepsilon}(|u'_\varepsilon|^2 - |u'_{0,\varepsilon}|^2)1_{F_\varepsilon} \\
&= v_{0,\varepsilon}\left(\frac{c_\varepsilon^2}{(v_\varepsilon)^2 + \eta_\varepsilon} - \frac{c_{0,\varepsilon}^2}{(v_{0,\varepsilon})^2 + \eta_\varepsilon}\right)1_{F_\varepsilon} &&= \frac{v_{0,\varepsilon}}{(v_{0,\varepsilon}^2 + \eta_\varepsilon)^2}(c_\varepsilon^2 - c_{0,\varepsilon}^2)1_{F_\varepsilon}
\end{aligned}$$

As $\mu_\varepsilon \leqslant 0$, provided that $|F_\varepsilon| \neq 0$, one has $c_\varepsilon^2 \leqslant c_{0,\varepsilon}^2$, and $|c_\varepsilon| \leqslant c_{0,\varepsilon}$. □

3.2 Shape of v_ε

The aim of this section is to take advantage of the ODE (2.10) satisfied by v_ε to determine its variations. We prove that v_ε admits a unique global minimum, denoted by $x_\varepsilon \in [L/4, 3L/4]$ and that v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$. We start with the following lemma on the possible positions of the critical points :

Lemma 3.4. *Let x_ε be a critical point of v_ε. If $x_\varepsilon \in F_\varepsilon$, then $x_\varepsilon = L/2$.*

Proof. Note that statement is non empty, by Rolle's Theorem, as $v_\varepsilon(0) = v_\varepsilon(L)$. Suppose $x_\varepsilon \in F_\varepsilon \setminus \{0, L\}$. Then

$$0 = v'_\varepsilon(x_\varepsilon) = \lim_{h\to 0^+} \frac{v_\varepsilon(x_\varepsilon + h) - v_\varepsilon(x_\varepsilon)}{h} = \lim_{h\to 0^+} \frac{v_\varepsilon(x_\varepsilon + h) - v_{0,\varepsilon}(x_\varepsilon)}{h} \leqslant \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(x_\varepsilon + h) - v_{0,\varepsilon}(x_\varepsilon)}{h} = v'_{0,\varepsilon}(x)$$

and

$$0 = v'_\varepsilon(x_\varepsilon) = \lim_{h\to 0^+} \frac{v_\varepsilon(x_\varepsilon) - v_\varepsilon(x_\varepsilon - h)}{h} = \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(x_\varepsilon) - v_\varepsilon(x_\varepsilon - h)}{h} \geqslant \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(x_\varepsilon) - v_{0,\varepsilon}(x_\varepsilon - h)}{h} = v'_{0,\varepsilon}(x)$$

so that $v'_{0,\varepsilon}(x_\varepsilon) = 0$, which implies $x_\varepsilon = L/2$ by [BMR23, Proposition 2.1]. If $x_\varepsilon = 0$, then, using again [BMR23, Proposition 2.1],

$$0 > v'_{0\varepsilon}(0) = \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(h) - v_{0,\varepsilon}(0)}{h} \geqslant \lim_{h\to 0^+} \frac{v_\varepsilon(h) - 1}{h} = v'_\varepsilon(0)$$

and if $x_\varepsilon = L$,

$$0 < v'_{0,\varepsilon}(L) = \lim_{h\to 0^+} \frac{v_{0,\varepsilon}(L) - v_{0,\varepsilon}(L - h)}{h} \leqslant \lim_{h\to 0^+} \frac{1 - v_\varepsilon(L - h)}{h} = v'_\varepsilon(L)$$

so that 0 and L are not critical points of v_ε. □

The following proposition establishes the variations of v_ε on $[0, L]$.

Proposition 3.5. *The function v_ε admits a unique local minimum at x_ε on $[0, L]$. This local minimum is also a global minimum and is either located at $L/2$ or it belongs to the connected component of O_ε containing $L/2$.*

Observe that while $v_{0,\varepsilon}$ is symmetric with respect to $L/2$, the previous proposition states that it is not necessarily the case for v_ε.

Proof. According to Lemma 3.4, a local minimum of v_ε is either $L/2$ or in O_ε. The proof is divided in two steps. In the first step, we prove that any connected component of O_ε contains at most one critical point y. Using this first step, we prove in the second step that the only connected component of O_ε that contains a local minimum is (if it exists) the connected component containing $L/2$.

Step 1 : Any connected component of O_ε contains at most one critical point.

Take (β,γ) a non-empty connected component of O_ε and suppose that it contains a critical point of v_ε. Assume that $v_\varepsilon(\beta) \leqslant v_\varepsilon(\gamma)$ (the case $v_\varepsilon(\beta) \geqslant v_\varepsilon(\gamma)$ can be treated similarly). As $\beta,\gamma \in F_\varepsilon$, we also have $v_{0,\varepsilon}(\beta) \leqslant v_{0,\varepsilon}(\gamma)$. As $v_{0,\varepsilon}$ is decreasing on $[0,L/2]$ and increasing on $[L/2,L]$ ([BMR23, Proposition 2.1]), we have in particular that $\gamma > L/2$ and $v'_{0,\varepsilon}(\gamma) > 0$.

Assume that $\{v'_\varepsilon = 0\} \cap (\beta,\gamma)$ is non-empty. First, γ cannot be critical points of v_ε, as

$$v'_\varepsilon(\gamma) = \lim_{h\to 0} \frac{v_\varepsilon(\gamma) - v_\varepsilon(\gamma-h)}{h} \geqslant \lim_{h\to 0} \frac{v_{0,\varepsilon}(\gamma) - v_{0,\varepsilon}(\gamma-h)}{h} = v'_{0,\varepsilon}(\gamma) > 0.$$

Hence, the supremum y of $\{v'_\varepsilon = 0\}\cap(\beta,\gamma)$ belongs to (β,γ). First, suppose that $y > \frac{\beta+\gamma}{2}$. The Cauchy problem

$$\begin{cases} -\varepsilon v''_\varepsilon + \dfrac{v_\varepsilon - 1}{\varepsilon} + v_\varepsilon |u'_\varepsilon|^2 &= 0 \qquad \text{on } (\beta,\gamma), \\ v_\varepsilon(y) &= v_\varepsilon(y), \\ v'_\varepsilon(y) &= 0, \end{cases}$$

admits a unique solution on the maximal interval (β,γ) (by Cauchy-Lipschitz Theorem). But the function $v_\varepsilon(2y-\cdot)$, defined on $(2y-\gamma,\gamma)$, is also a solution of the Cauchy problem on $(2y-\gamma,\gamma) \subset (\beta,\gamma)$, so that by uniqueness, v_ε is symmetric around y on $(2y-\gamma,\gamma)$ (see Figure 2a). By continuity, v_ε is symmetric around y on $[2y-\gamma,\gamma]$ and

$$v_{0,\varepsilon}(2y-\gamma) \geqslant v_\varepsilon(2y-\gamma) = v_\varepsilon(\gamma) = v_{0,\varepsilon}(\gamma).$$

Since $2y-\gamma \in (\beta,\gamma)$, it follows from the variations of $v_{0,\varepsilon}$ (see [BMR23, Proposition 2.1]) that $v_{0,\varepsilon}(2y-\gamma) < \max(v_{0,\varepsilon}(\beta), v_{0,\varepsilon}(\gamma)) = v_{0,\varepsilon}(\gamma)$. This contradiction shows that

$$y \leqslant \frac{\beta+\gamma}{2}. \tag{3.8}$$

Now, suppose that there exists another critical point z of v_ε on (β,γ). As y is supposed to be the largest critical point, $z < y$. Similarly to the previous argument of symmetry, $v_\varepsilon(2y-\cdot)$ is a solution of the Cauchy problem on $(\beta,2y-\beta) \subset (\beta,\gamma)$ so v_ε is symmetric around y on $[\beta,2y-\beta]$. It follows that $2y-z$ is also critical point of v_ε on (β,γ), which implies $2y-z=y$, hence $y=z$.

Finally, y is the unique critical point of v_ε on (β,γ).

Step 2 : (β,γ) contains a local minimum if and only if $L/2 \in (\beta,\gamma)$.

We distinguish two cases.

Case 1 : Suppose that $\dfrac{L}{2} \notin (\beta,\gamma)$ and that y is a local minimum of v_ε. As v'_ε does not vanish on $(\beta,\gamma)\setminus\{y\}$, v_ε is decreasing on (β,y) and increasing on (y,γ).

On one hand, the symmetry of v_ε around y yields :

$$v'_\varepsilon(\beta) = \lim_{x\to\beta^+} v'_\varepsilon(x) = -\lim_{x\to(2y-\beta)^-} v'_\varepsilon(x) = -v'_\varepsilon(2y-\beta) < 0.$$

But on the other hand :

$$v'_\varepsilon(\beta) = \lim_{h\to 0} \frac{v_\varepsilon(\beta) - v_\varepsilon(\beta-h)}{h} \geqslant \lim_{h\to 0} \frac{v_{0,\varepsilon}(\beta) - v_{0,\varepsilon}(\beta-h)}{h} = v'_{0,\varepsilon}(\beta) > 0.$$

Thus, it follows that (β,γ) does not contain any local minimum.

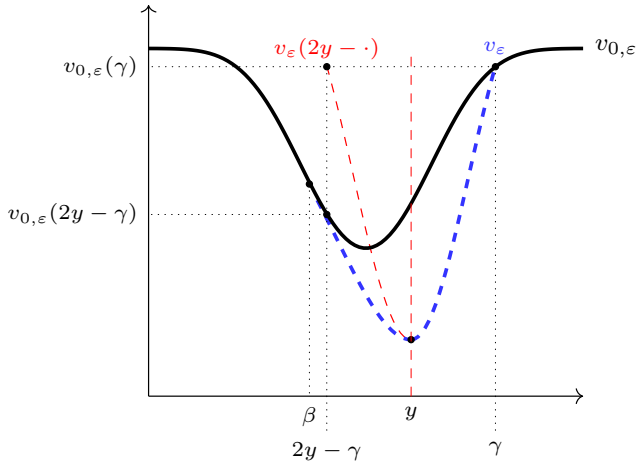
Case 2 : Suppose that $L/2 \in (\beta,\gamma)$. Since

$$v_\varepsilon(L/2) \leqslant v_{0,\varepsilon}(L/2) < \min(v_{0,\varepsilon}(\beta), v_{0,\varepsilon}(\gamma)) = \min(v_\varepsilon(\beta), v_\varepsilon(\gamma)),$$

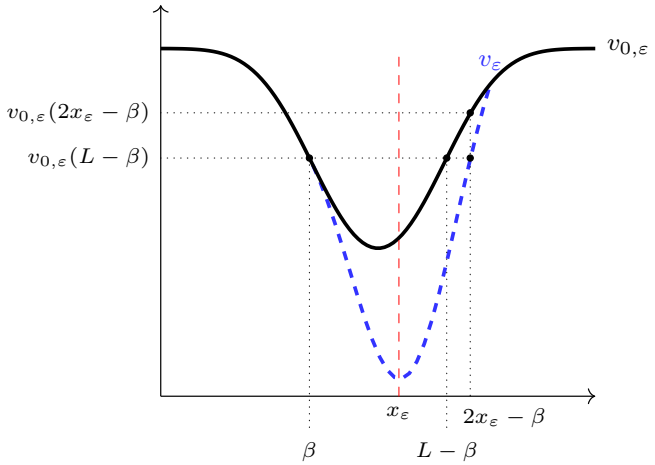
v_ε admits at least one local minimum on (β,γ). By uniqueness, this local minimum is y.

This concludes the proof.

□



(a) $y \leqslant \dfrac{\gamma+\beta}{2}$



(b) $x_\varepsilon \leqslant \dfrac{3}{4}L$

Figure 2: Critical points of v_ε

Corollary 3.6. *The function v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$.*

Proof. Let us prove that v_ε is non-increasing on $[0, x_\varepsilon]$. Let $a, b \in [0, x_\varepsilon]$ such that $a < b$. If $v_\varepsilon(a) < v_\varepsilon(b)$, then $v_\varepsilon(a) < \min(v_\varepsilon(0), v_\varepsilon(b))$, so v_ε admits a local minimum on $(0, b)$, which contradicts Proposition 3.5. So $v_\varepsilon(a) \geqslant v_\varepsilon(b)$ and v_ε is non increasing on $[0, x_\varepsilon]$. Similarly, v_ε is non decreasing on $[x_\varepsilon, 1]$. □

Notice that the sequence (x_ε) of global minimum of v_ε is bounded so, up to an extraction, we can suppose that

$$x_\varepsilon \longrightarrow x_{min} \in [0, L]. \tag{3.9}$$

The previous proof implies the following on x_{min} :

Corollary 3.7. *Let x_ε denote the global minimum of v_ε and let x_{min} be the limit of a converging subsequence of (x_ε). Then $x_{min} \in [L/4, 3L/4]$.*

Proof. Suppose that $x_\varepsilon \in (\beta, \gamma)$ the connected component of O_ε containing $L/2$. Suppose also that $v_\varepsilon(\beta) \leqslant v_\varepsilon(\gamma)$. According to (3.8), $x_\varepsilon \leqslant \dfrac{\gamma+\beta}{2} \leqslant \dfrac{3}{4}L < L$. On the other hand, the symmetries of $v_{0,\varepsilon}$ around $L/2$ imply

$$v_{0,\varepsilon}(2x_\varepsilon - \beta) \geqslant v_\varepsilon(2x_\varepsilon - \beta) = v_\varepsilon(\beta) = v_{0,\varepsilon}(\beta) = v_{0,\varepsilon}(L-\beta).$$

From the variations of $v_{0,\varepsilon}$ (see [BMR23, Proposition 2.1]) and the inequality $v_{0,\varepsilon}(2x_\varepsilon - \beta) \geqslant v_{0,\varepsilon}(\beta)$, we deduce that $2x_\varepsilon \geqslant L/2$. Then, from the the variations of $v_{0,\varepsilon}$ on $[L/2, L]$ and the inequality $v_{0,\varepsilon}(2x_\varepsilon - \beta) \geqslant v_{0,\varepsilon}(L-\beta)$, we deduce that $2x_\varepsilon - \beta \geqslant L - \beta$, so $x_\varepsilon \geqslant L/2$ (see Figure 2b). The case where $v_\varepsilon(\beta) \geqslant v_\varepsilon(\gamma)$ can be treated similarly. □

In particular, x_{min} is not localized at the boundaries of $[0, L]$.

4 Asymptotic analysis

In this section, we prove Theorem 2.6. The proofs of the several items of Theorem 2.6 are disseminated within the subsections : (i) is proved in subsection 4.3, (ii) in subsection 4.5 and subsection 4.7, (iii) in subsection 4.4 and (iv) and (v) in subsection 4.6.

4.1 Preliminary estimates

The following estimates are classical and can be found in [FLS09], [BMR23] and [Le10].

First, notice that the energy bound (2.12) provides the following inequality on c_ε :

$$\mathcal{C} \geqslant \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \geqslant \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 = \int_0^L |c_\varepsilon u_\varepsilon'| = |c_\varepsilon||u_\varepsilon(L) - u_\varepsilon(0)| = |c_\varepsilon a_1|.$$

where we used that u'_ε is of constant sign (see (2.8)). It follows that $|c_\varepsilon| \leqslant C/|a_1|$. The sequence (c_ε) is bounded, so, up to an extraction, we can suppose that it converges to a constant c_1. The following proposition establishes the connection between c_1 and the limit function of (u_ε).

Proposition 4.1. *One has $v_\varepsilon \to 1$ in $L^2(0,L)$ and, up to an extraction, there exists $u_1 \in BV(0,L)$ such that $u_\varepsilon \to u_1$ in $L^1(0,L)$. Moreover, $u'_\varepsilon \to c_1$ a.e in $[0,L]$ with $|Du_1|(0,L) \leqslant |a_1|$. We also have that $c_1 \in [0, a_1/L]$ if $a_1 \geqslant 0$ and $c_1 \in [a_1/L, 0]$ if $a_1 \leqslant 0$.*

Proof. The convergence of v_ε comes from the energy bound (2.12) :

$$\int_0^L (1-v_\varepsilon)^2 \leqslant \varepsilon \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \mathcal{C}\varepsilon \to 0.$$

As u_ε is monotone, $u_\varepsilon(0) = 0$ and $u_\varepsilon(L) = a_1$, it is of constant sign and $\int_0^L |u_\varepsilon| = |\int_0^L u_\varepsilon| \leqslant |a_1|L$. So (u_ε) is bounded in $L^1(0,L)$. Secondly, as u'_ε is of constant sign,

$$|Du_\varepsilon|(0,L) = \int_0^L |u'_\varepsilon| = |\int_0^L u'_\varepsilon| = |a_1|.$$

So the total variation of u_ε is also bounded. Then, the sequence (u_ε) is bounded in $BV(0,L)$ and we can extract a converging subsequence in $L^1(0,L)$ to a function $u_1 \in BV(0,L)$. The lower semi-continuity of the total variation implies

$$|Du_1|(0,L) \leqslant \liminf_{\varepsilon\to 0} |Du_\varepsilon|(0,L) \leqslant |a_1|.$$

Using equation (2.8), as one can extract a converging subsequence of (v_ε) that converges to 1 almost everywhere, one gets $u'_\varepsilon \to c_1$ almost everywhere. If $u_\varepsilon \geqslant 0$, then by Fatou's lemma :

$$c_1 L = \int_0^L \lim_{\varepsilon\to 0} u'_\varepsilon \leqslant \liminf_{\varepsilon\to 0} \int_0^L u'_\varepsilon = a_1.$$

If $u_\varepsilon \leqslant 0$, then similarly

$$-c_1 L = \int_0^L \lim_{\varepsilon\to 0} (-u'_\varepsilon) \leqslant \liminf_{\varepsilon\to 0} \int_0^L (-u'_\varepsilon) = -a_1.$$

We obtain that if $a_1 \geqslant 0$, then $c_1 \in [0, a_1/L]$ and if $a_1 \leqslant 0$, then $c_1 \in [a_1/L, 0]$. □

4.2 Estimates on v_ε

In this section we establish the two main estimates on v_ε, namely :

Proposition 4.2. *Let $x \in [0,L]$. Suppose that $v_\varepsilon \geqslant \alpha > 0$ over $I_r(x)$ for all ε small enough. Then there exists $\varepsilon_0 > 0$ and $b, M > 0$ depending on α such that on $I_{r/2}(x)$*

$$|\mu_\varepsilon| \leqslant M, \qquad |u'_\varepsilon| \leqslant M, \qquad v_\varepsilon \geqslant 1 - b\varepsilon, \quad \forall \varepsilon < \varepsilon_0. \tag{4.1}$$

Corollary 4.3. *For any compact set $K \in [0,L] \setminus \{x_{min}\}$, there exists $\varepsilon_0 > 0$ and $b, M > 0$ depending on K such that (4.1) holds.*

Proposition 4.4. *If $c_0 > 0$, then,*

$$\|v'_\varepsilon\|_\infty \leqslant \frac{C}{\varepsilon}.$$

If $|c_1| > 0$, then,

$$\|u'_\varepsilon\|_\infty \leqslant \frac{C}{\varepsilon}, \qquad \inf_{(0,L)} v_\varepsilon \geqslant m\sqrt{\varepsilon}.$$

Note that the first estimate relies on a condition concerning c_0 and not c_1. The proof of Proposition 4.2 and its corollary is inspired by the proof of the Proposition 3.12 of [Le10] and the proof of Proposition 3.2 of [HT00].

Proof of Proposition 4.2. Suppose that the assertion is false for every b. We will reach a contradiction for b large enough. To this end, fix $b > 0$ and suppose on the contrary that, for infinitely many ε, $\inf_{I_{r/2}(x)} v_\varepsilon < 1 - b\varepsilon$.

First we can already bound u'_ε and μ_ε uniformly in ε on $I_r(x)$, namely :

$$|u'_\varepsilon| \leqslant \frac{|c_\varepsilon|}{\alpha^2 + \eta_\varepsilon} \leqslant M \tag{4.2}$$

Concerning μ_ε, since $v_{0,\varepsilon} = v_\varepsilon$ on F_ε, one has by Corollary 3.3 on $I_r(x)$:

$$-\mu_\varepsilon = |\mu_\varepsilon| = \frac{v_{0,\varepsilon}}{(v_{0,\varepsilon}^2 + \eta_\varepsilon)^2} |c_\varepsilon^2 - c_{0,\varepsilon}^2| 1_{F_\varepsilon} \leqslant \frac{1}{(\alpha^2 + \eta_\varepsilon)^2} M 1_{F_\varepsilon} \leqslant M. \tag{4.3}$$

Let ζ_ε be a smooth function such that $\zeta_\varepsilon(x - r) = \zeta_\varepsilon(x + r) = \alpha$, $\zeta_\varepsilon = 1 - b/2\varepsilon$ on $I_{r/2}(x)$, $\zeta_\varepsilon \leqslant 1$ on $I_r(x)$ and $\|\zeta''_\varepsilon\| = O(1)$.

Set $g_\varepsilon = v_\varepsilon - \zeta_\varepsilon$. Then $g_\varepsilon(x + r), g_\varepsilon(x - r) \geqslant 0$, while $\inf\limits_{I_{r/2}(x)} g_\varepsilon \leqslant (1 - b\varepsilon) - (1 - (b\varepsilon)/2) = -(b\varepsilon)/2 < 0$ since $\inf\limits_{I_{r/2}(x)} v_\varepsilon \leqslant 1 - b\varepsilon$. In particular, g_ε reaches its minimum at a point x_1 of $(x - r, x + r)$ and this minimum satisfies $g_\varepsilon(x_1) \leqslant -(b\varepsilon)/2$. Then :

$$\begin{aligned} 0 \quad &\leqslant \quad \varepsilon g''_\varepsilon(x_1) = \varepsilon v''_\varepsilon(x_1) - \varepsilon \zeta''_\varepsilon(x_1) = \frac{v_\varepsilon(x_1) - 1}{\varepsilon} + v_\varepsilon(x_1)|u'_\varepsilon(x_1)|^2 - \mu_\varepsilon(x_1) - \varepsilon\zeta''_\varepsilon(x_1) \\ &= \quad g_\varepsilon(x_1)\left(\frac{1}{\varepsilon} + |u'_\varepsilon(x_1)|^2\right) + \zeta_\varepsilon(x_1)\left(\frac{1}{\varepsilon} + |u'_\varepsilon(x_1)|^2\right) - \frac{1}{\varepsilon} - \varepsilon\zeta''_\varepsilon(x_1) - \mu_\varepsilon(x_1) \end{aligned}$$

We deduce, using (4.2) and (4.3),

$$-b/2 \geqslant -\frac{b\varepsilon}{2}\left(\frac{1}{\varepsilon} + |u'_\varepsilon(x_1)|^2\right) \geqslant \frac{1 - \zeta_\varepsilon(x_1)}{\varepsilon} + \varepsilon\zeta''_\varepsilon(x_1) - \zeta_\varepsilon(x_1)|u'_\varepsilon(x_1)|^2 + \mu_\varepsilon \geqslant -\varepsilon\|\zeta''_\varepsilon\|_\infty - M^2 - M \geqslant -2M^2$$

which is absurd for b large enough. □

Proof of Corollary 4.3. Let K be a compact set of $[0, L] \setminus \{x_{min}\}$ and take $\delta > 0$ such that $(x_{min} - \delta, x_{min} + \delta) \subset [0, L] \setminus K$. Then, for $\varepsilon > 0$ small enough, $x_\varepsilon \in (x_{min} - \delta/2, x_{min} + \delta/2)$ so using Corollary 3.6, v_ε is non-increasing on $(x_{min} - \delta, x_{min} - \delta/2)$. The L^2 convergence of v_ε then implies

$$\frac{\delta}{2}(1 - v_\varepsilon(x_{min} - \delta))^2 \leqslant \int_{x_{min}-\delta}^{x_{min}-\delta/2} (1 - v_\varepsilon)^2 \to 0. \tag{4.4}$$

It follows that for ε small enough, $v_\varepsilon(x_{min} - \delta) \geqslant 1/2$ and then $v_\varepsilon \geqslant 1/2$ on $[0, x_{min} - \delta]$. Similarly, $v_\varepsilon \geqslant 1/2$ on $[x_{min} + \delta, L]$. Hence $v_\varepsilon \geqslant 1/2$ on K.

Let $x \in K$ and r_x such that $I_{r_x}(x) \subset (0, x_{min} - \delta) \cup (x_{min} + \delta, L)$. Then $v_\varepsilon \geqslant 1/2$ on $I_{r_x}(x)$. According to Proposition 4.2, there exists $\varepsilon_x > 0$ and constants b_x, M_x such that

$$v_\varepsilon \geqslant 1 - b_x\varepsilon \quad , \quad |u'_\varepsilon| \leqslant M_x \quad , \quad |\mu_\varepsilon| \leqslant M_x \quad \forall \varepsilon < \varepsilon_x.$$

As K is compact, there exists a finite covering of K of the form $K \subset \bigcup_{i=1}^n I_{r_{x_i}/2}(x_i)$. Taking $\varepsilon_0 = \min\limits_{1 \leqslant i \leqslant n} \varepsilon_{x_i}, b = \max\limits_{1 \leqslant i \leqslant n} b_{x_i}$ and $M = \min\limits_{1 \leqslant i \leqslant n} M_{x_i}$ yields Corollary 4.3. □

Before proving Proposition 4.4, we introduce the so-called discrepancy of $\overline{AT}_\varepsilon$.

Definition 4.5. *We denote by d_ε, also named as the discrepancy of $\overline{AT}_\varepsilon$ the following quantity :*

$$d_\varepsilon := \frac{(1 - v_\varepsilon)^2}{\varepsilon} - \varepsilon|v'_\varepsilon|^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2. \tag{4.5}$$

Remark that

$$\int_0^L |d_\varepsilon| = \int_0^L \left|\frac{(1 - v_\varepsilon)^2}{\varepsilon} - \varepsilon(v'_\varepsilon)^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2\right| \leqslant AT_\varepsilon(v_\varepsilon, u_\varepsilon) < C.$$

As in [FLS09], [BMR23] and [Le10], we use the variations of the discrepancy to infer estimates on u_ε and v_ε. The following proposition gives the expression of the derivative of the discrepancy :

Proposition 4.6. *There holds*

$$d'_\varepsilon = 2v'_{0,\varepsilon}\mu_\varepsilon. \tag{4.6}$$

Consequently, d_ε is non-decreasing on $[0, L/2]$ and non-increasing on $[L/2, L]$.

Proof. Note that $d_\varepsilon \in C^{0,1}([0,L])$, so that we can define its derivative almost everywhere but here in the sense of $\mathcal{D}'$, which reads as :

$$\begin{aligned}\left\{\frac{(1-v_\varepsilon)^2}{\varepsilon} - \varepsilon(v'_\varepsilon)^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2\right\}' &= 2v'_\varepsilon[\frac{v_\varepsilon - 1}{\varepsilon} - \varepsilon v''_\varepsilon] - 2(v_\varepsilon^2+\eta_\varepsilon)u''_\varepsilon u'_\varepsilon - 2v_\varepsilon v'_\varepsilon|u'_\varepsilon|^2 \\ &= 2v'_\varepsilon[\mu_\varepsilon - v_\varepsilon|u'_\varepsilon|^2] - 2(v_\varepsilon^2+\eta_\varepsilon)u''_\varepsilon u'_\varepsilon - 2v_\varepsilon v'_\varepsilon|u'_\varepsilon|^2 \\ &= 2v'_\varepsilon\mu_\varepsilon - 2u'_\varepsilon \underbrace{[(\eta_\varepsilon + v_\varepsilon^2)u'_\varepsilon]'}_{=0} = 2v'_\varepsilon\mu_\varepsilon = 2v'_{0,\varepsilon}\mu_\varepsilon\end{aligned}$$

where the last equality uses the fact that $v_\varepsilon = v_{0,\varepsilon}$ on the support of μ_ε. Since $\mu_\varepsilon \leqslant 0$ and $v'_{0,\varepsilon} \leqslant 0$ on $[0, L/2]$ while $v'_{0,\varepsilon} \geqslant 0$ on $[L/2, L]$, we deduce that d_ε is non-decreasing on $[0, L/2]$ and non-increasing on $[L/2, L]$. □

We use this expression to prove the following estimate on the discrepancy.

Lemma 4.7. *Let d_ε be the discrepancy associated to $(u_\varepsilon, v_\varepsilon)$, defined by (4.5). If $c_0 > 0$, then :*

$$\|d_\varepsilon\|_\infty \leqslant \frac{C}{\varepsilon}$$

The hypothesis on c_0 comes from the necessity, in the proof, to bound from below $v_{0,\varepsilon}(L/2)$. As stated in Lemma 2.2 of [BMR23], if $c_0 > 0$, then $v_{0,\varepsilon}(L/2) \geqslant C\sqrt{\varepsilon}$.

Proof. The proof of Lemma 4.7 consists in estimating d_ε at its local extremum $L/2$ and controlling the integral of the derivative d'_ε to prove that the difference $|d_\varepsilon(x) - d_\varepsilon(L/2)|$ is bounded by C/ε.

We start by estimating $d_\varepsilon(L/2)$. We distinguish two cases. Suppose first that $d_\varepsilon(L/2) \geqslant 0$. Then we have

$$0 \leqslant d_\varepsilon(L/2) \leqslant \frac{(1 - v_\varepsilon(L/2))^2}{\varepsilon} \leqslant \frac{1}{\varepsilon}.$$

Suppose now that $d_\varepsilon(L/2) \leqslant 0$. Recall that $\|d_\varepsilon\|_1$ is bounded by a constant C independent from ε. Then, because $L/2$ is a global maximum of d_ε, d_ε is non-positive and

$$C \geqslant \int_0^L |d_\varepsilon| = -\int_0^L d_\varepsilon \geqslant L(-d_\varepsilon(L/2))$$

so that $|d_\varepsilon(L/2)| \leqslant C$. Both cases imply that $|d_\varepsilon(L/2)| \leqslant C/\varepsilon$.

Now we prove that the total variation of d_ε is bounded by C/ε. By Corollary 3.3 :

$$\begin{aligned}\int_0^L |d'_\varepsilon| &= 2\int_0^L |v'_{0,\varepsilon}\mu_\varepsilon| = 2\int_0^L \left|v'_{0,\varepsilon}\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2}((c_\varepsilon)^2 - (c_{0,\varepsilon})^2)1_{F_\varepsilon}\right| \\ &= 2((c_{0,\varepsilon})^2 - (c_\varepsilon)^2)\int_0^L |v'_{0,\varepsilon}|\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2} \leqslant 2C\int_0^L |v'_{0,\varepsilon}|\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2}.\end{aligned}$$

Using that $v'_{0,\varepsilon} \leqslant 0$ on $[0, L/2]$ and $v'_{0,\varepsilon} \geqslant 0$ on $[L/2, L]$, along with the estimate $v_{0,\varepsilon}(L/2) \geqslant C\sqrt{\varepsilon}$, we have :

$$\int_0^{L/2} |v'_{0,\varepsilon}|\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2} = -\int_0^{L/2} v'_{0,\varepsilon}\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2} = \frac{1}{2}\left[\frac{1}{(v_{0,\varepsilon})^2+\eta_\varepsilon}\right]_0^{L/2} = \frac{1}{2(v_{0,\varepsilon}^2(L/2)+\eta_\varepsilon)} - \frac{1}{2(1+\eta_\varepsilon)} \leqslant \frac{C}{\varepsilon}.$$

Similarly, we obtain that $\displaystyle\int_{L/2}^L |v'_{0,\varepsilon}|\frac{v_{0,\varepsilon}}{[(v_{0,\varepsilon})^2+\eta_\varepsilon]^2} \leqslant \frac{C}{\varepsilon}$, so that $\displaystyle\int_0^L |d'_\varepsilon| \leqslant C/\varepsilon$.

We deduce that for a.e x in $[0, L]$, one has

$$|d_\varepsilon(x)| \leqslant |d_\varepsilon(L/2)| + \left|\int_{L/2}^x d'_\varepsilon\right| \leqslant |d_\varepsilon(L/2)| + \int_0^L |d'_\varepsilon| \leqslant C/\varepsilon,$$

which gives the desired result. □

We are now in position to complete the proof of Proposition 4.4 :

Proof of Proposition 4.4. First we focus on the estimate on v_ε. For a.e x in $[0, L]$, one has

$$0 \leqslant \varepsilon(v_\varepsilon')^2 = \frac{(1-v_\varepsilon)^2}{\varepsilon} - d_\varepsilon - (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \leqslant \frac{1}{\varepsilon} - d_\varepsilon \leqslant \frac{C}{\varepsilon},$$

hence the result. Now we focus on the estimate on u_ε'. One has

$$|c_\varepsilon u_\varepsilon'| = (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 = \frac{(1-v_\varepsilon)^2}{\varepsilon} - \varepsilon|v_\varepsilon'|^2 - d_\varepsilon \leqslant \frac{C}{\varepsilon}.$$

Dividing by $|c_\varepsilon|$, we obtain the desired result. Concerning v_ε, we have

$$\eta_\varepsilon + v_\varepsilon^2 = \frac{c_\varepsilon}{u_\varepsilon'} \geqslant C\varepsilon.$$

Since $\eta_\varepsilon << \varepsilon$, we conclude that $v_\varepsilon \geqslant m\varepsilon$. □

In [FLS09] and [BMR23], the first estimate of Proposition 4.4 is true without any assumption on the slope c_0. This comes from the fact that the discrepancy d_ε is constant in their case, so an estimate as in Lemma 4.7 follows. However, the hypothesis $c_0 > 0$ is not that restrictive in our case, since $c_0 = 0$ implies that $c_1 = 0$ as well, as long as $|F_\varepsilon| \neq 0$ (see Corollary 4.9).

4.3 Possible values for the limit slope

The goal of this subsection is to establish the following property :

Proposition 4.8. $c_1 \in \{0, \frac{a_1}{L}\}$.

Corollary 4.9. *If $|a_1| > a_0$ and $|F_\varepsilon| \neq 0$ for infinitely many values of ε, then $c_1 = 0$.*

Proof of Corollary 4.9. If $|F_\varepsilon| \neq 0$, then by Corollary 3.3 $|c_\varepsilon| \leqslant |c_{0,\varepsilon}|$. Hence, by [BMR23, Lemma 2.1] $|c_1| \leqslant c_0 \leqslant a_0/L < |a_1|/L$, so Proposition 4.8 yields $c_1 = 0$. □

The proof of Proposition 4.8 is similar to that of [FLS09, Lemma 4.4].

Proof of Proposition 4.8. We already know by Proposition 4.1 that c_1 is within the interval $[0, a_1/L]$ (or $[a_1/L, 0]$).

Suppose now that $|c_1| > 0$. The proof is divided into two steps. In the first step, we integrate (2.10) over sets of the form

$$K_{\varepsilon,A} := \{v_\varepsilon \leqslant A\sqrt{\varepsilon}\}$$

to obtain an estimate on $|K_{\varepsilon,A}|$. In the second step, we split the integral $\int_0^L u_\varepsilon'$ into three terms which we study using the previous estimate and Proposition 4.4.

Step 1 : We estimate $|K_{\varepsilon,A}|$. Fix $A > 0$. We integrate the inequality

$$-\varepsilon v_\varepsilon'' + \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{\varepsilon} \leqslant 0$$

over the set $\{v_\varepsilon \leqslant A\sqrt{\varepsilon}\}$. We obtain

$$\int_{K_{\varepsilon,A}} \varepsilon v_\varepsilon'' \geqslant \int_{K_{\varepsilon,A}} \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \int_{K_{\varepsilon,A}} \frac{v_\varepsilon - 1}{\varepsilon}. \tag{4.7}$$

According to Proposition 3.5, the set $K_{A,\varepsilon}$ is an interval that we denote by $[a_\varepsilon, b_\varepsilon]$. Then we have, using Proposition 4.4,

$$\left|\int_{a_\varepsilon}^{b_\varepsilon} \varepsilon v_\varepsilon''\right| = \varepsilon|v_\varepsilon'(b_\varepsilon) - v_\varepsilon'(a_\varepsilon)| \leqslant C.$$

Next, recall by Proposition 4.4 that $v_\varepsilon^2 \geqslant m\varepsilon >> \eta_\varepsilon$. We can then bound from below the right-hand side of (4.7) by

$$\int_{K_{\varepsilon,A}} \frac{v_\varepsilon c_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \int_{K_{\varepsilon,A}} \frac{v_\varepsilon - 1}{\varepsilon} \geqslant \int_{K_{\varepsilon,A}} \frac{v_\varepsilon c_1^2/2}{2v_\varepsilon^4} - \frac{|K_{\varepsilon,A}|}{\varepsilon} \geqslant C\frac{|K_{\varepsilon,A}|}{A^3\varepsilon^{3/2}} - \frac{|K_{\varepsilon,A}|}{\varepsilon} \geqslant C\frac{|K_{\varepsilon,A}|}{A^3\varepsilon^{3/2}}.$$

Combining the two bounds, we obtain $|K_{\varepsilon,A}| \leqslant CA^3\varepsilon^{3/2}$. This ends Step 1.

Step 2 : In this step, we investigate the weight of u'_ε in each of the three regimes $\{v_\varepsilon \leqslant A\sqrt{\varepsilon}\}, \{A\sqrt{\varepsilon} \leqslant v_\varepsilon \leqslant 1/2\}$ and $\{v_\varepsilon \geqslant 1/2\}$. Indeed, one has :

$$a_1 = \int_0^L u'_\varepsilon = \underbrace{\int_{K_{\varepsilon,A}} u'_\varepsilon}_{:=I_1} + \underbrace{\int_{\{A\sqrt{\varepsilon}\leqslant v_\varepsilon \leqslant 1/2\}} u'_\varepsilon}_{:=I_2} + \underbrace{\int_{\{v_\varepsilon \geqslant 1/2\}} u'_\varepsilon}_{:=I_3}.$$

First, thanks to the previous estimate on $K_{\varepsilon,A}$ and Proposition 4.4,

$$I_1 \leqslant \int_{K_{\varepsilon,A}} |u'_\varepsilon| \leqslant \frac{C}{\varepsilon}|K_{\varepsilon,A}| \leqslant CA^3\varepsilon^{1/2}.$$

As $u'_\varepsilon \to c_1$ and $v_\varepsilon \to 1$ a.e, we have $u'_\varepsilon 1_{v_\varepsilon \geqslant 1/2} \to c_1$ a.e. As

$$\left|\frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} 1_{\{v_\varepsilon \geqslant 1/2\}}\right| \leqslant 2|c_1| \times 4,$$

we can apply the dominated convergence theorem to obtain

$$I_3 \to \int_0^L c_1 = c_1 L.$$

For I_2, the energy bound (2.2) gives :

$$C \geqslant \int_0^L \frac{(1-v_\varepsilon)^2}{\varepsilon} \geqslant \int_{\{v_\varepsilon \leqslant 1/2\}} \frac{(1-v_\varepsilon)^2}{\varepsilon} \geqslant \frac{1}{4\varepsilon}|\{v_\varepsilon \leqslant 1/2\}|$$

On $\{A\sqrt{\varepsilon} \leqslant v_\varepsilon \leqslant 1/2\}$, we recover

$$|u'_\varepsilon| = \frac{|c_\varepsilon|}{\eta_\varepsilon + v_\varepsilon^2} \leqslant \frac{|c_\varepsilon|}{v_\varepsilon^2} \leqslant \frac{2|c_1|}{A^2\varepsilon}.$$

Thus

$$|I_2| \leqslant |\{A\sqrt{\varepsilon} \leqslant v_\varepsilon \leqslant 1/2\}| \times \frac{2|c_1|}{A^2\varepsilon} \leqslant \frac{C}{A^2}$$

Combining the estimates on I_1, I_2 and I_3, we obtain

$$|a_1| \leqslant \frac{C}{A^2} + |c_1|L$$

By letting A tending to ∞, we obtain $|c_1|L \geqslant |a_1|$. Combining with $|c_1| \leqslant |a_1|/L$, we get the desired result. □

We can now conclude the proof of (i) in Theorem 2.6.

Proof of (i) *in Theorem 2.6.* Let $\delta > 0$. We have already seen with Corollary 4.3 that $v_\varepsilon \to 1$ uniformly on $[0, L] \setminus I_\delta(x_{min})$ for every $\delta > 0$. This combined with (2.8) implies that $u'_\varepsilon \to c_1$ uniformly on $[0, L] \setminus I_\delta(x_{min})$ for every $\delta > 0$. Then we get that :

$$u_\varepsilon(x) = \int_0^x u'_\varepsilon(t)dt \to c_1 x,$$

uniformly with respect to $x \in [0, x_{min} - \delta]$. Similarly, $u_\varepsilon(x) \to a_1 - c_1(L - x)$ uniformly with respect to $x \in [x_{min} + \delta, L]$. Using that $c_1 = 0$ or $c_1 = a_1/L$, we see that the limit function u_1 of (u_ε) (in the point-wise convergence sense) is either $u_1 = u_{aff}$ or $u_1 = u_{jump}$. Moreover, as u_ε is monotone and u_ε is uniformly bounded by a_1 on $[0, L]$, by dominated convergence, we get that $u_\varepsilon \to u_1$ in $L^2(0, L)$. The additionnal conclusion corresponds to Corollary 4.9. □

4.4 Convergence of the Dirichlet energy

In this subsection, we prove the following, which proves part (iii) of Theorem 2.6. The proof is similar to [FLS09, section 6] and [BMR23, section 3].

Lemma 4.10. $(\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 dx \overset{\star}{\rightharpoonup} |u_1'|^2 dx$ *weak** *in* $\mathcal{M}([0,L])$.

Proof. Let φ be a test-function in $\mathcal{C}^\infty([0,L])$. According to Proposition 4.1 :

$$\int_0^L (v_\varepsilon^2 + \eta_\varepsilon)|u_\varepsilon'|^2\varphi = c_\varepsilon \int_0^L u_\varepsilon'\varphi = -c_\varepsilon \int_0^L u_\varepsilon\varphi' + c_\varepsilon a_1\varphi(L) \to -c_1 \int_0^L u_1\varphi' + c_1 a_1\varphi(L).$$

If $c_1 = 0$, then $u_1 = u_{jump}$ and $u_1' = 0$ so

$$-c_1 \int_0^L u_1\varphi' + c_1 a_1\varphi(L) = 0 = \int_0^L |u_{jump}'|^2\varphi.$$

If $c_1 = a_1/L$, then $u_1 = u_{aff}$ and $u_1' = a_1/L$, so by integration by part,

$$-c_1 \int_0^L u_1\varphi' + c_1 a_1\varphi(L) = c_1 \int_0^L u_{aff}'\varphi = \int_0^L |u_{aff}'|^2\varphi.$$

□

4.5 Convergence of the phase-field term

In this subsection, we prove the first part of (ii) in Theorem 2.6, namely

Lemma 4.11. $\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|v_\varepsilon'|^2\right) dx \overset{\star}{\rightharpoonup} \alpha\delta_{x_{min}}$ *weak** *in* $\mathcal{M}([0,L])$ *for some* $\alpha \in \mathbb{R}$.

The computation of the weight α is made in section subsection 4.7.

We start by noticing that the energy bound (2.2) implies that the measure $\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|v_\varepsilon'|^2\right) dx$ is bounded, thus we can extract a converging subsequence such that

$$\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|v_\varepsilon'|^2\right) dx \overset{\star}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}([0,L])$$

for some non negative $\nu \in \mathcal{M}([0,L])$. In [BMR23] and in [FLS09], the proof that ν is atomic relies on the following estimate (see [BMR23, Lemma 3.1] and [FLS09, Lemma 6.1]): if K is a compact subset of $[0,L] \setminus \{x_{min}\}$, then

$$\int_K \left(\frac{(1-v_{0,\varepsilon})^2}{\varepsilon} + \varepsilon|v_{0,\varepsilon}'|^2\right) dx \leqslant C_K \varepsilon^{1/4}.$$

The proof of this estimate relies on an integration by part of the PDE satisfied by $v_{0,\varepsilon}$ on K, and strongly uses the estimate $\|v_{0,\varepsilon}'\|_\infty \leqslant C/\varepsilon$. In our case, such an estimate $\|v_\varepsilon'\|_\infty \leqslant C/\varepsilon$ is proved only in the case where $c_0 > 0$.

We are going to show the following result, which provides a sharper estimate (an order 1 on ε instead of an order $1/4$) :

Proposition 4.12. *Let K be a compact subset of $[0,L] \setminus \{x_{min}\}$. There exists a constant C_K such that :*

$$\int_K \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|v_\varepsilon'|^2\right) \leqslant C_K \varepsilon.$$

The proof is divided in three steps. In the first one, we prove the result for a compact set K of $(0,L) \setminus \{x_{min}\}$:

Lemma 4.13. *Let K be a compact subset of $(0,L) \setminus \{x_{min}\}$. For any positive test-function φ supported in K, there exists a constant $C_{K,\varphi}$ such that :*

$$\int_K \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|v_\varepsilon'|^2\right) \varphi \leqslant C_{K,\varphi}\varepsilon.$$

In the second one, we use Lemma 4.13 to obtain a sharper estimate on $\|v_\varepsilon'\|_\infty$ on $(0,L) \setminus \{x_{min}\}$.

Lemma 4.14. *Let K be a connected compact subset of $(0,L)\setminus\{x_{min}\}$. Then there exists a constant C_K such that,*

$$\sup_K \varepsilon|v_\varepsilon'|^2 \leqslant C_K\varepsilon^{1/3}.$$

Finally, we use Lemma 4.14 to prove Proposition 4.12, in a similar way than [BMR23, Lemma 3.1] and [FLS09, Lemma 6.1].

Proof of Lemma 4.13. The argument is inspired from [HT00, Proposition 4.1] to prove a monotonicity formula for the Allen-Cahn functional.

Take φ a smooth and positive test function supported on K. First, using Corollary 4.3, there exists $b, M > 0$ such that

$$u_\varepsilon' \leqslant M, \quad |\mu_\varepsilon| \leqslant M, \quad v_\varepsilon \geqslant 1 - b\varepsilon \quad \text{on } K.$$

Then

$$\int_K \frac{(1-v_\varepsilon)^2}{\varepsilon}\varphi \leqslant \int_K \frac{b^2\varepsilon^2}{\varepsilon}\varphi \leqslant C_K\varepsilon. \tag{4.8}$$

It remains to estimate $\int_K \varepsilon|v_\varepsilon'|^2\varphi$. To do so, we compare $\varepsilon|v_\varepsilon'|^2$ with its second derivative (in the distributional sense). One has

$$[\varepsilon|v_\varepsilon'|^2]' = 2\varepsilon v_\varepsilon'' v_\varepsilon' = 2v_\varepsilon'\left(\frac{v_\varepsilon - 1}{\varepsilon} + v_\varepsilon|u_\varepsilon'|^2 - \mu_\varepsilon\right).$$

Note that $v_\varepsilon'\mu_\varepsilon \in L^1(0,L)$, so it admits a derivative in the distributional sense and then

$$[\varepsilon|v_\varepsilon'|^2]'' = \underbrace{2v_\varepsilon''\left(\frac{v_\varepsilon-1}{\varepsilon} + v_\varepsilon|u_\varepsilon'|^2\right)}_{=A_\varepsilon}\underbrace{-2(v_\varepsilon'\mu_\varepsilon)'}_{=B_\varepsilon} + 2\frac{v_\varepsilon'^2}{\varepsilon} + \underbrace{2|v_\varepsilon'|^2|u_\varepsilon'|^2}_{=C_\varepsilon} + \underbrace{4v_\varepsilon' v_\varepsilon u_\varepsilon'' u_\varepsilon'}_{=D_\varepsilon}.$$

It follows that

$$2\int_K \varepsilon|v_\varepsilon'|^2\varphi = \varepsilon^2\int_K (\varepsilon(v_\varepsilon')^2)''\varphi - \varepsilon^2\left(\int_K A_\varepsilon\varphi + \langle\, B_\varepsilon\,,\,\varphi\,\rangle + \int_K C_\varepsilon\varphi + \int_0^L D_\varepsilon\varphi\right). \tag{4.9}$$

We now study each terms of the right hand side.

$\underline{\int_0^L (\varepsilon(v_\varepsilon')^2)''\varphi}$: As $\varphi = 0$ outside of K, then φ and φ' vanish at 0 and L. Applying (4.8) to φ'' yields

$$\lim_{\varepsilon\to 0}\int_0^L (\varepsilon|v_\varepsilon'|^2)''\varphi = \lim_{\varepsilon\to 0}\int_0^L \varepsilon|v_\varepsilon'|^2\varphi'' = \lim_{\varepsilon\to 0}\int_0^L \left(\varepsilon|v_\varepsilon'|^2 + \frac{(1-v_\varepsilon)^2}{\varepsilon}\right)\varphi'' \to \int_0^L \varphi''d\nu.$$

Hence,

$$\varepsilon^2\int_K (\varepsilon(v_\varepsilon')^2)''\varphi = \varepsilon^2\int_0^L (\varepsilon(v_\varepsilon')^2)''\varphi \leqslant C_\varphi\varepsilon^2. \tag{4.10}$$

$\underline{\int_K A_\varepsilon\varphi}$:

$$A_\varepsilon = \frac{1}{\varepsilon}\underbrace{\left(\frac{v_\varepsilon-1}{\varepsilon} + v_\varepsilon|u_\varepsilon'|^2\right)^2}_{\geqslant 0} + \frac{1}{\varepsilon}\underbrace{(-\mu_\varepsilon)}_{\leqslant M}\left(\underbrace{\frac{v_\varepsilon-1}{\varepsilon}}_{\geqslant -b} + \underbrace{v_\varepsilon|u_\varepsilon'|^2}_{\geqslant 0}\right) \geqslant \frac{-bM}{\varepsilon}.$$

so that

$$-\varepsilon^2\int_K A_\varepsilon\varphi \leqslant C_{K,\varphi}\varepsilon. \tag{4.11}$$

$\underline{\langle\, B_\varepsilon\,,\,\varphi\,\rangle}$: We use Cauchy-Schwarz inequality and the fact that $|\mu_\varepsilon| \leqslant M$ to get that

$$\left|2\varepsilon^2\langle\,(v_\varepsilon'\mu_\varepsilon)'\,,\,\varphi\,\rangle\right| = 2\varepsilon^2\left|\int_0^L v_\varepsilon'\mu_\varepsilon\varphi'\right| \leqslant 2\varepsilon^{3/2}\underbrace{\|\mu_\varepsilon\|_{L^2(0,L)}}_{\leqslant C}\left\|\varepsilon^{1/2}|v_\varepsilon'|\varphi'\right\|_{L^2(0,L)}. \tag{4.12}$$

On the other hand,

$$\left|\left|\varepsilon^{1/2}|v_\varepsilon'|\varphi'\right|\right|^2_{L^2(0,L)} = \int_0^L \varepsilon|v_\varepsilon'|^2|\varphi'|^2 \leqslant \int_0^L \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|v_\varepsilon'|^2\right)|\varphi'|^2 \to \int_0^L |\varphi'|^2 d\nu,$$

From which we deduce that $\|\varepsilon^{1/2}(v_\varepsilon')\varphi'\|^2_{L^2(0,L)} \leqslant C_\varphi$. (4.12) then yields

$$\left|\varepsilon^2 \langle B_\varepsilon , \varphi\rangle\right| \leqslant C_\varphi \varepsilon^{3/2}. \tag{4.13}$$

$\underline{\int_K C_\varepsilon\varphi : C_\varepsilon \geqslant 0}$, so

$$-\varepsilon^2 \int_K C_\varepsilon\varphi \leqslant 0. \tag{4.14}$$

$\underline{\int_K D_\varepsilon\varphi}$: We have $u_\varepsilon'' = \left(\frac{c_\varepsilon}{v_\varepsilon^2+\eta_\varepsilon}\right)' = -2c_\varepsilon \frac{v_\varepsilon' v_\varepsilon}{(v_\varepsilon^2+\eta_\varepsilon)^2}$, so that

$$D_\varepsilon = -8v_\varepsilon' v_\varepsilon c_\varepsilon \frac{v_\varepsilon' v_\varepsilon}{(v_\varepsilon^2+\eta_\varepsilon)^2}\cdot\frac{c_\varepsilon}{v_\varepsilon^2+\eta_\varepsilon} = -8c_\varepsilon^2|v_\varepsilon'|^2 \underbrace{\frac{v_\varepsilon^2}{(v_\varepsilon^2+\eta_\varepsilon)^3}}_{\leqslant C}.$$

where we used that $1-b\varepsilon \leqslant v_\varepsilon \leqslant 1$ on K. As (c_ε) is bounded, we can write :

$$|D_\varepsilon| \leqslant M|v_\varepsilon'|^2.$$

Thus

$$|\varepsilon^2 \int_K D_\varepsilon\varphi| \leqslant M\varepsilon \int_K \varepsilon|v_\varepsilon'|^2\varphi. \tag{4.15}$$

Injecting (4.14) and (4.15) in (4.9) yields the following inequality, which is true for every ε small enough and every positive smooth test function φ :

$$(1-M\varepsilon)\int_K \varepsilon|v_\varepsilon'|^2\varphi \leqslant \varepsilon^2 \int_K (\varepsilon|v_\varepsilon'|^2)''\varphi - \varepsilon^2\left(\int_K A_\varepsilon\varphi + \langle B_\varepsilon,\varphi\rangle\right) \tag{4.16}$$

Now combining with (4.10),(4.11) and (4.13), we obtain

$$(1-M\varepsilon)\int_K \varepsilon|v_\varepsilon'|^2\varphi \leqslant C\varepsilon^2 + C\varepsilon + C\varepsilon^{3/2} \leqslant C_{K,\varphi}\varepsilon \tag{4.17}$$

Which concludes the proof. □

Note that a corollary of this property is that

$$\int_K \varepsilon|v_\varepsilon'|^2 \leqslant C_K\varepsilon \qquad \text{for any compact set } K \subset (0,L)\setminus\{x_{min}\}. \tag{4.18}$$

We now move on to the proof of Lemma 4.14.

Proof of Lemma 4.14. Let K be a connected compact subset of $(0,L)\setminus\{x_{min}\}$ and let $y \in K$. The strategy is the following : we consider a mollifier ρ_ε (i.e ρ is positive smooth function, compactly supported on $[-1,1]$ with integral 1 and $\rho_\varepsilon = \frac{1}{\varepsilon^\alpha}\rho\left(\frac{\cdot}{\varepsilon^\alpha}\right)$, whith $\alpha > 0$ to be determined). Then we take $\rho_\varepsilon(\cdot - y)$ as a test function in the inequality (4.16) and we prove the two estimates

$$\int_K \rho_\varepsilon(x-y)\varepsilon|v_\varepsilon'(x)|^2dx \leqslant C_K\varepsilon^{1-\alpha} \tag{4.19}$$

and

$$\left|\int_K \rho_\varepsilon(x-y)\varepsilon|v_\varepsilon'(x)|^2dx - \varepsilon|v_\varepsilon'(y)|^2\right| \leqslant C_K\varepsilon^{\alpha/2}. \tag{4.20}$$

It follows that for $\alpha = 2/3$, one has $\varepsilon|v_\varepsilon'(y)|^2 \leqslant C_K\varepsilon^{1/3}$.

We start by proving (4.19). One already has from inequalities (4.16), (4.11) and (4.12) that

$$(1-M\varepsilon)\int_K \rho_\varepsilon(x-y)\varepsilon|v_\varepsilon'|^2(x)dx \leqslant \underbrace{\varepsilon^2\int_K(\varepsilon|v_\varepsilon'|^2)''\rho_\varepsilon}_{:=I_1} + \underbrace{\varepsilon\int_K bM\rho_\varepsilon}_{:=I_2} - \underbrace{\varepsilon^2\int_K(v_\varepsilon'\mu_\varepsilon)'\rho_\varepsilon}_{:=I_3}.$$

$\underline{I_1}$: Using Lemma 4.13,

$$\left|\int_K \varepsilon|v_\varepsilon'|^2\rho_\varepsilon''\right| = \varepsilon^{-2\alpha}\left|\int_K \varepsilon|v_\varepsilon'|^2\frac{1}{\varepsilon^\alpha}\rho''\left(\frac{x}{\varepsilon^\alpha}\right)\right| \leqslant \varepsilon^{-3\alpha}\|\rho''\|_\infty\int_K\varepsilon|v_\varepsilon'|^2 \leqslant C_K\varepsilon^{1-3\alpha}. \tag{4.21}$$

Hence,

$$I_1 \leqslant C_K\varepsilon^{3-3\alpha} \tag{4.22}$$

$\underline{I_2}$:

$$|I_2| = \left|\varepsilon bM\int_K\rho_\varepsilon\right| \leqslant bM|K|\varepsilon^{1-\alpha}\|\rho\|_\infty \leqslant C_K\varepsilon^{1-\alpha}. \tag{4.23}$$

$\underline{I_3}$: Using the Cauchy-Schwarz inequality, (4.3) and Lemma 4.13,

$$|I_3| \leqslant \left|\varepsilon^2\int_K v_\varepsilon'\mu_\varepsilon\rho_\varepsilon'\right| \leqslant \varepsilon^2\underbrace{\left(\int_K\mu_\varepsilon^2\right)^{1/2}}_{\leqslant C_K}\varepsilon^{-1/2}\left(\int_K\varepsilon|v_\varepsilon'|^2|\rho_\varepsilon'|^2\right)^{1/2} \leqslant C_K\varepsilon^2\varepsilon^{-2\alpha-1/2}\|\rho'\|_\infty\underbrace{\left(\int_K\varepsilon|v_\varepsilon'|^2\right)^{1/2}}_{\leqslant C_K\varepsilon^{1/2}}. \tag{4.24}$$

And thus,

$$|I_3| \leqslant C_K\varepsilon^{2-2\alpha}. \tag{4.25}$$

Combining (4.22), (4.23) and (4.25) yields (4.19).

We now move on to the proof of (4.20). First, using that $\int_K\rho_\varepsilon(x-y)=1$, we rewrite (4.20) as

$$\int_K\rho_\varepsilon(x-y)(\varepsilon|v_\varepsilon'|^2(x)dx - \varepsilon|v_\varepsilon'|^2(y))dx.$$

Then, for all x in K, considering the fact that $|v_\varepsilon'|^2$ is $C^{0,1}([0,L])$ and using the Cauchy-Schwarz inequality,

$$|\varepsilon|v_\varepsilon'(x)|^2 - \varepsilon|v_\varepsilon'(y)|^2| = 2\varepsilon\left|\int_y^x v_\varepsilon''(t)v_\varepsilon'(t)dt\right| \leqslant 2\varepsilon\varepsilon^{-1}\left|\int_y^x\varepsilon^2|v_\varepsilon''(t)|^2dt\right|^{1/2}\varepsilon^{-1/2}\underbrace{\left|\int_y^x\varepsilon|v_\varepsilon'(t)|^2dt\right|^{1/2}}_{\leqslant C_K\varepsilon^{1/2}}.$$

By Corollary 4.3,

$$\|\varepsilon v_\varepsilon''\|_{L^\infty(K)} = \left\|\frac{v_\varepsilon-1}{\varepsilon} - \mu_\varepsilon + v_\varepsilon|u_\varepsilon'|^2\right\|_{L^\infty(K)} \leqslant b+M+M^2 \leqslant C$$

so that

$$\left|\int_y^x\varepsilon^2v_\varepsilon''(t)^2\right|^{1/2} \leqslant C|x-y|^{1/2}.$$

We deduce

$$\int_K\rho_\varepsilon(x-y)\left|\varepsilon|v_\varepsilon'(x)|^2 - \varepsilon|v_\varepsilon'(y)|^2\right|dx \leqslant C_K\int_K|x-y|^{1/2}\frac{1}{\varepsilon^\alpha}\rho_\varepsilon\left(\frac{x-y}{\varepsilon^\alpha}\right)dx.$$

Changing variable $s=\dfrac{x-y}{\varepsilon^\alpha}$, we rewrite the integral as

$$\int_{(K-y)/\varepsilon^\alpha}\varepsilon^{\alpha/2}|s|^{1/2}\rho(s)ds \leqslant C\varepsilon^{\alpha/2}.$$

Finally,

$$\int_K\rho_\varepsilon(x-y)\left|\varepsilon|v_\varepsilon'(x)|^2 - \varepsilon|v_\varepsilon'(y)|^2\right|dx \leqslant C_K\varepsilon^{\alpha/2}.$$

Combining (4.19) and (4.20) finishes the proof. □

We are now in position to prove Proposition 4.12.

Proof of Proposition 4.12. It is sufficient to prove the inequality for $K = [\,0, x_{min} - \delta] \cup [x_{min} + \delta, L]$ for any $\delta > 0$. In the sequel, we fix $\delta > 0$. We adapt the argument used by [BMR23, Lemma 3.1] and [FLS09, Lemma 6.3]. Multiplying both sides of the equation (2.10) by $v_\varepsilon - 1$ and integrating by parts over $[0, x_{min} - \delta]$ yields :

$$\int_0^{x_{min}-\delta} \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon |v_\varepsilon'|^2 \right) = \varepsilon v_\varepsilon'(x_{min}-\delta)(v_\varepsilon(x_{min}-\delta)-1) + \int_0^{x_{min}-\delta} \mu_\varepsilon (v_\varepsilon - 1) + \int_0^{x_{min}-\delta} c_\varepsilon^2 \frac{v_\varepsilon(1-v_\varepsilon)}{(\eta_\varepsilon + v_\varepsilon^2)^2}.$$

We now bound each term on the right-hand side. Let $K' = K \cap [L/8, 7L/8]$ which is a compact subset of $[0, L] \setminus \{x_{min}\}$. According to Lemma 4.14, $|v_\varepsilon'| \leqslant C_{K'} \varepsilon^{-1/3}$ on K'. Using Corollary 4.3, there exists a constant $b > 0$ such that $v_\varepsilon \geqslant 1 - b\varepsilon$ on K. This implies that μ_ε is uniformly bounded on K (see the inequality (4.3)). Consequently :

$$\varepsilon \underbrace{|v_\varepsilon'(x_{min}-\delta)|}_{\leqslant C_{K'}\varepsilon^{-1/3}} \underbrace{|v_\varepsilon(x_{min}-\delta)-1|}_{\leqslant b\varepsilon} \leqslant C_\delta \varepsilon^{5/3}.$$

We also have

$$\left| \int_0^{x_{min}-\delta} \underbrace{\mu_\varepsilon}_{\leqslant M} \underbrace{(v_\varepsilon - 1)}_{\geqslant -b\varepsilon} \right| \leqslant C_\delta \varepsilon,$$

as well as

$$\left| \int_0^{x_{min}-\delta} c_\varepsilon^2 (1-v_\varepsilon) \frac{v_\varepsilon}{(\eta_\varepsilon + v_\varepsilon^2)^2} \right| \leqslant c_\varepsilon^2 \int_0^{x_{min}-\delta} \underbrace{|1-v_\varepsilon|}_{\leqslant b\varepsilon} \underbrace{\frac{1}{v_\varepsilon^3}}_{\leqslant C} \leqslant C_\delta \varepsilon.$$

We deduce that

$$\int_0^{x_{min}-\delta} \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon |v_\varepsilon'|^2 \right) \leqslant C_\delta \varepsilon.$$

We proceed similarly on $[x_{min} + \delta, L]$ to conclude. □

Once we get Proposition 4.12, we can adapt the proof of Lemma 4.14 to obtain the following :

Corollary 4.15. *Let K be a connected compact set of $[\,0, L\,] \setminus \{x_{min}\}$. Then there exists a constant C_K such that for all $y \in K$:*

$$\sup_K \varepsilon |v_\varepsilon'|^2 \leqslant C_K \varepsilon^{1/3}.$$

Proof. Fix $y \in K$. Take ρ_ε a mollifier, with parameter ε^α as in the proof of Lemma 4.14 and $\alpha > 0$ to be determined. The same argument used to establish (4.20) yields

$$\left| \int_K \rho_\varepsilon(x-y) \varepsilon |v_\varepsilon'(x)|^2 dx - \varepsilon |v_\varepsilon'(y)|^2 \right| \leqslant C_K \varepsilon^{\alpha/2}.$$

On the other hand, using Proposition 4.12,

$$\begin{aligned} \int_K \rho_\varepsilon(x-y) \varepsilon |v_\varepsilon'(x)|^2 dx &\leqslant \int_K \rho_\varepsilon(x-y) \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon |v_\varepsilon'(x)|^2 \right) dx \\ &\leqslant \frac{\|\rho\|_\infty}{\varepsilon^\alpha} \int_K \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon |v_\varepsilon'(x)|^2 \right) dx \leqslant C_K \varepsilon^{1-\alpha}. \end{aligned}$$

Combining both estimates for $\alpha = 2/3$ yields the desired result. □

The Proposition 4.12 implies that ν is supported only on $\{x_{min}\}$ and thus $\nu = \alpha \delta_{x_{min}}$ for some α.

4.6 Equipartition of the energy

In this subsection, our goal is to prove the following result, which corresponds to the item (iv) of Theorem 2.6, and which implies (v).

Proposition 4.16. *We have*

$$\lim_{\varepsilon\to 0}\int_0^L \left|\frac{(1-v_\varepsilon)^2}{\varepsilon}-\varepsilon(v_\varepsilon')^2\right| = 0. \tag{4.26}$$

We recover here a classical result in the case of the minimizers of AT_ε, as it corresponds to the equality case of the arithmetico-geometric inequality used in the proof of the Γ-convergence of AT_ε to MS. It is used in [FLS09] and [BMR23] to compute the value of the weight α associated to the limit Dirac-measure of the phase field term of AT_ε. An equipartition of the energy result also exists for the Ambrosio-Tortorelli functional with an additional fidelity term, see [Le10].

In [FLS09] and [Le10], the Neumann conditions on (2.10) imply that the difference $\frac{(1-v_\varepsilon)^2}{\varepsilon}-\varepsilon|v_\varepsilon'|^2$ is positive. In [BMR23] and [FLS09], the proof of the equipartition of the energy hinges on the fact that d_ε is constant and tends to $-c_0^2$.

In our case, recalling the equality :

$$\frac{(1-v_\varepsilon)^2}{\varepsilon}-\varepsilon|v_\varepsilon'|^2 = d_\varepsilon + c_\varepsilon u_\varepsilon',$$

the first ingredient is to prove that $d_\varepsilon + c_\varepsilon u_\varepsilon' \to 0$ at the boundary point 0 and L of $(0,L)$. It rests on Corollary 4.15. The second ingredient is to bound d_ε' to control the variation of v_ε on the full interval. To deal with the signs of each terms and get rid of the absolute values, we need to separate our integrals along $L/2$ and x_ε.

Proof of Proposition 4.16. Corollary 4.15 implies in particular that $\varepsilon|v_\varepsilon'|^2(0), \varepsilon|v_\varepsilon'|^2(L) \to 0$ as $\varepsilon\to 0$, by recalling that $x_{min}\in[L/4,3L/4]$ (Corollary 3.7). Then we have

$$d_\varepsilon(0)+c_\varepsilon u_\varepsilon'(0) = -\varepsilon|v_\varepsilon'|^2(0)\to 0 \tag{4.27}$$

and similarly, $d_\varepsilon(L)+c_\varepsilon u_\varepsilon'(L)\to 0$.

By Proposition 4.6, $d_\varepsilon\in C^{0,1}([0,L])$, so we can write $d_\varepsilon(x)=d_\varepsilon(0)+\int_0^x d_\varepsilon'$ on $[0,L/2]$ and $d_\varepsilon(x)=d_\varepsilon(L)+\int_L^x d_\varepsilon'$ on $[L/2,L]$, with $d_\varepsilon'\geqslant 0$ on $[0,L/2]$ and $d_\varepsilon'\leqslant 0$ on $[L/2,L]$. However, the sign of $u_\varepsilon'(x)-u_\varepsilon'(0)$ depends on the position of x with respect to x_ε. So we split the integral in the left-hand side of (4.26) along x_ε and later along $L/2$.

$$\begin{aligned}
\int_0^L\left|\frac{(1-v_\varepsilon)^2}{\varepsilon}-\varepsilon|v_\varepsilon'|^2\right| &= \int_0^{x_\varepsilon}\left|d_\varepsilon(0)+c_\varepsilon u_\varepsilon'(0)+\int_0^x d_\varepsilon'(t)dt+c_\varepsilon(u_\varepsilon'(x)-u_\varepsilon'(0))\right|dx\\
&+\int_{x_\varepsilon}^L\left|d_\varepsilon(L)+c_\varepsilon u_\varepsilon'(L)+\int_L^x d_\varepsilon'(t)dt+c_\varepsilon(u_\varepsilon'(x)-u_\varepsilon'(L))\right|dx\\
&\leqslant \underbrace{x_\varepsilon|d_\varepsilon(0)+c_\varepsilon u_\varepsilon'(0)|}_{:=I}+\underbrace{\int_0^{x_\varepsilon}\left|\int_0^x d_\varepsilon'\right|}_{:=II}+\underbrace{\int_0^{x_\varepsilon}|c_\varepsilon(u_\varepsilon'(x)-u_\varepsilon'(0))|dx}_{:=III}\\
&+\underbrace{(L-x_\varepsilon)|d_\varepsilon(L)+c_\varepsilon u_\varepsilon'(L)|}_{:=IV}+\underbrace{\int_{x_\varepsilon}^L\left|\int_x^L d_\varepsilon'\right|}_{:=V}+\underbrace{\int_{x_\varepsilon}^L|c_\varepsilon(u_\varepsilon'(x)-u_\varepsilon'(L))|dx}_{:=VI}
\end{aligned}$$

We deal separately with $I+IV$, $II+V$ and $III+VI$.

$\underline{I+IV}$: This tends to 0 according to (4.27).

$\underline{II+V}$: We prove that

$$II + V \leqslant \int_0^{L/2} \left(\int_0^x d'_\varepsilon \right) dx - \int_{L/2}^L \left(\int_x^L d'_\varepsilon \right) dx + \left| \frac{L}{2} - x_\varepsilon \right| \left| \int_0^L d'_\varepsilon \right|$$

Suppose that $x_\varepsilon \geqslant L/2$ (the case $x_\varepsilon \leqslant L/2$ can be treated in a similar way). Recall that $d'_\varepsilon \geqslant 0$ on $[0, L/2]$ and $d'_\varepsilon \leqslant 0$ on $[L/2, L]$. Then $II + V$ can be rewritten as

$$\begin{aligned}
\int_0^{x_\varepsilon} \left| \int_0^x d'_\varepsilon \right| dx + \int_{x_\varepsilon}^L \left| \int_x^L d'_\varepsilon \right| dx &= \int_0^{L/2} \left| \int_0^x \underbrace{d'_\varepsilon}_{\geqslant 0} \right| dx + \int_{L/2}^{x_\varepsilon} \left| \int_0^L d'_\varepsilon - \int_x^L \underbrace{d'_\varepsilon}_{\leqslant 0} \right| dx + \int_{x_\varepsilon}^L \left| \int_x^L \underbrace{d'_\varepsilon}_{\leqslant 0} \right| dx \\
&\leqslant \int_0^{L/2} \left(\int_0^x d'_\varepsilon \right) dx - \int_{L/2}^{x_\varepsilon} \left(\int_x^L d'_\varepsilon \right) dx + \int_{L/2}^{x_\varepsilon} \left| \int_0^L d'_\varepsilon \right| - \int_{x_\varepsilon}^L \left(\int_x^L d'_\varepsilon \right) dx \\
&= \int_0^{L/2} \left(\int_0^x d'_\varepsilon \right) dx - \int_{L/2}^L \left(\int_x^L d'_\varepsilon \right) dx + \left| \frac{L}{2} - x_\varepsilon \right| \left| \int_0^L d'_\varepsilon \right|.
\end{aligned}$$

Now, as $u'_\varepsilon(0) = u'_\varepsilon(L) = \dfrac{c_\varepsilon}{\eta_\varepsilon + 1}$, by (4.27) $d_\varepsilon(0)$ and $d_\varepsilon(L)$ both converge to $-c_1^2$. Then

$$\left| \frac{L}{2} - x_\varepsilon \right| \left| \int_0^L d'_\varepsilon \right| = \left| \frac{L}{2} - x_\varepsilon \right| |d_\varepsilon(L) - d_\varepsilon(0)| \to 0.$$

We study the remaining terms separately, recalling the expressions (4.6) and (3.7).

$$\begin{aligned}
\int_0^{L/2} \int_0^x d'_\varepsilon dx &\leqslant 2(c_{0,\varepsilon}^2 - c_\varepsilon^2) \int_0^{L/2} \int_0^x - \frac{v'_{0,\varepsilon} v_{0,\varepsilon}}{(v_{0,\varepsilon}^2 + \eta_\varepsilon)^2} dx &&= (c_{0,\varepsilon}^2 - c_\varepsilon^2) \int_0^{L/2} \left[\frac{1}{v_{0,\varepsilon}^2 + \eta_\varepsilon} \right]_0^x dx \\
&\leqslant c_{0,\varepsilon}^2 \int_0^{L/2} \left(\frac{1}{v_{0,\varepsilon}^2(x) + \eta_\varepsilon} - \frac{1}{v_{0,\varepsilon}^2(0) + \eta_\varepsilon} \right) dx &&= \int_0^{L/2} \frac{c_{0,\varepsilon}^2}{v_{0,\varepsilon}^2(x) + \eta_\varepsilon} dx - \frac{c_{0,\varepsilon}^2}{v_{0,\varepsilon}^2(0) + \eta_\varepsilon} \cdot \frac{L}{2} \\
&= \int_0^{L/2} c_{0,\varepsilon} u'_{0,\varepsilon}(x) dx - \frac{c_{0,\varepsilon}^2}{1 + \eta_\varepsilon} \cdot \frac{L}{2}
\end{aligned}$$

Similarly, one has

$$- \int_{L/2}^L \left(\int_x^L d'_\varepsilon \right) dx \leqslant \int_{L/2}^L c_{0,\varepsilon} u'_{0,\varepsilon}(x) dx - c_{0,\varepsilon}^2 \frac{1}{1 + \eta_\varepsilon} \cdot \frac{L}{2}.$$

Summing both inequalities yields

$$\int_0^{L/2} \left(\int_0^x d'_\varepsilon \right) dx - \int_{L/2}^L \left(\int_x^L d'_\varepsilon \right) dx \leqslant c_{0,\varepsilon} \underbrace{\int_0^L u'_{0,\varepsilon}(x) dx}_{u_{0,\varepsilon}(L) - u_{0,\varepsilon}(0)} - L c_{0,\varepsilon}^2 \frac{1}{1 + \eta_\varepsilon} = c_{0,\varepsilon} \left(a_0 - L \frac{c_{0,\varepsilon}}{1 + \eta_\varepsilon} \right) \to c_0(a_0 - L c_0) = 0,$$

where we used that $c_0 \in \{0, a_0/L\}$ (see [BMR23, Lemma 2.4]). We conclude that $II + V \to 0$.

$\underline{III + VI}$: Recall that v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$. Then, $c_\varepsilon u'_\varepsilon$ is itself non-decreasing on $[0, x_\varepsilon]$ and non-increasing on $[x_\varepsilon, L]$. This allows us to get rid of the absolute values in the expressions, so that $III + VI$ rewrites as :

$$\begin{aligned}
III + VI &= \int_0^{x_\varepsilon} c_\varepsilon (u'_\varepsilon(x) - u'_\varepsilon(0)) dx + \int_{x_\varepsilon}^L c_\varepsilon (u'_\varepsilon(x) - u'_\varepsilon(L)) dx \\
&= \int_0^L c_\varepsilon u'_\varepsilon(x) dx - c_\varepsilon (x_\varepsilon u'_\varepsilon(0) + (L - x_\varepsilon) u'_\varepsilon(L)) \;=\; c_\varepsilon (u_\varepsilon(L) - u_\varepsilon(0)) - L c_\varepsilon \frac{c_\varepsilon}{1 + \eta_\varepsilon} \\
&\to c_1(a_1 - c_1 L) = 0
\end{aligned}$$

where we used Proposition 4.8. In conclusion, we get $III + VI \to 0$.

This completes the proof of the equipartition of the energy. □

4.7 Computation of the weight of the limit measure

In this section, we compute the weight α of the Dirac mass $\delta_{x_{min}}$, which completes the proof of the item (ii) in Theorem 2.6.

To do so, we use the equipartition principle, as it is done in [FLS09, section 6] and [BMR23, section 3].

Proof of (ii). The equipartition principle ensures that :

$$\left| \int_0^L \left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon |v_\varepsilon'|^2 \right) - 2|v_\varepsilon'|(1-v_\varepsilon) \right| = \int_0^L \left| \frac{1-v_\varepsilon}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} v_\varepsilon' \right|^2 \leqslant \int_0^L \left| \frac{(1-v_\varepsilon)^2}{\varepsilon} - \varepsilon |v_\varepsilon'|^2 \right| \to 0,$$

so that it is enough to evaluate $\int_0^L 2|v_\varepsilon'|(1-v_\varepsilon)$. Recall that v_ε is non-increasing on $[0, x_\varepsilon]$ and non-decreasing on $[x_\varepsilon, L]$, where x_ε denotes the global minimum of v_ε. We have :

$$\int_0^L 2|v_\varepsilon'|(1-v_\varepsilon) = \int_0^{x_\varepsilon} -2v_\varepsilon'(1-v_\varepsilon) + \int_{x_\varepsilon}^L 2v_\varepsilon'(1-v_\varepsilon) = \left[(1-v_\varepsilon)^2\right]_0^{x_\varepsilon} + \left[-(1-v_\varepsilon)^2\right]_{x_\varepsilon}^L = 2(1-v_\varepsilon(x_\varepsilon))^2.$$

So the value of α depends on the limit of $v_\varepsilon(x_\varepsilon)$.

First, note that $\alpha \geqslant 0$ as the measures $\left(\frac{(1-v_\varepsilon)^2}{\varepsilon} + \varepsilon|v_\varepsilon'|^2\right) dx$ are non negative. Suppose that $\alpha > 0$. We prove that $v_\varepsilon(x_\varepsilon) \to 0$. Suppose that it is not the case and that there exists a subsequence such that $v_\varepsilon(x_\varepsilon) \geqslant \beta > 0$. Then $v_\varepsilon \geqslant \beta$ on $[0, L]$. According to Proposition 4.2, it implies that $v_\varepsilon(x_\varepsilon) \geqslant 1 - b\varepsilon$. But then

$$\left[(1-v_\varepsilon)^2\right]_0^{x_\varepsilon} + \left[-(1-v_\varepsilon)^2\right]_{x_\varepsilon}^L \to 0$$

which contradicts that $\alpha > 0$. Hence, $v_\varepsilon(x_\varepsilon) \to 0$ and

$$\left[(1-v_\varepsilon)^2\right]_0^{x_\varepsilon} + \left[-(1-v_\varepsilon)^2\right]_{x_\varepsilon}^L \to 2$$

We conclude that if $\alpha \neq 0$, then $\alpha = 2$. □

As a last remark, we explicit the link between the convergence of $\varepsilon|v_{0,\varepsilon}'|^2$ and the value of x_{min}. According to [BMR23, Theorem 1.1], there are to cases :

- If $\varepsilon|v_{0,\varepsilon}'|^2 \overset{\star}{\rightharpoonup} \delta_{L/2}$, then $L/2$ is a $v-$jump for $v_{0,\varepsilon}$. The obstacle condition implies that $L/2$ is also a $v-$jump for v_ε, which means that $x_{min} = L/2$. Note that $\varepsilon|v_{0,\varepsilon}'|^2 \overset{\star}{\rightharpoonup} \delta_{L/2}$ can happen both when $u_0 = u_{aff}$ and $u_0 = u_{jump}$.
- If $\varepsilon|v_{0,\varepsilon}'|^2 \overset{\star}{\rightharpoonup} 0$, then $v_{0,\varepsilon}$ admits no $v-$jump and x_{min} is not always equal to $L/2$ (but still belongs to $[L/4, 3L/4]$, see Corollary 3.7). Note that in this case, we always have $u_0 = u_{aff}$.

5 Appendix : Proof of Theorem 2.7

The proof of the $\Gamma-$convergence of $\overline{AT}_\varepsilon$ is an adaptation the proof of the classical Γ-convergence result of AT_ε to MS (see [AT92]). The main difficulty is to take into account the obstacle Γ_0 in the proof of the Γ-limsup. The argument is inspired by [Gia05, Lemma 6.2].

5.1 Proof of the Γ-liminf

Lemma 5.1. *Let $(u_\varepsilon, v_\varepsilon)$ be a sequence in $L^2(0,L)^2$ converging to (u,v) in $L^2(0,L)^2$. Then*

$$\overline{MS}(u,v) \leqslant \liminf_{\varepsilon \to +\infty} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon).$$

Proof. We suppose that $\liminf_{\varepsilon\to\infty} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$. We extract a subsequence for which the energy converges to the liminf. From now on, we keep the notations $(u_\varepsilon, v_\varepsilon)$ for simplicity but we only work with this subsequence.

Since $u_\varepsilon \in H^1(0,L) \subset L^\infty(0,L)$, we can write $\|u_\varepsilon\|_\infty \leqslant M$. We also have by a maximum principle (i.e replacing v_ε by a better competitor) that $0 \leqslant v_\varepsilon \leqslant 1$. Now, applying the arithmetico-geometric inequality to the second member of the energy yields

$$\mathcal{C} \geqslant \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \geqslant \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 + 2\int_0^L |v_\varepsilon'|(1-v_\varepsilon). \tag{5.1}$$

We first focus on the second term of the right-hand-side. Fix $0 < a \leqslant b < 1$. The coarea formula allows us to compute

$$\int_0^L |v_\varepsilon'|(1-v_\varepsilon) = \int_0^1 (1-t)\mathcal{H}^0(\partial^\star\{v_\varepsilon < t\})dt \geqslant \int_a^b (1-t)\mathcal{H}^0(\partial^\star\{v_\varepsilon < t\})dt. \tag{5.2}$$

Set $f(t) = t - t^2/2$. Then there exists $t_\varepsilon \in (a,b)$ such that

$$\int_a^b f'(t)\mathcal{H}^0(\partial^\star\{v_\varepsilon < t\})dt \geqslant (f(b)-f(a))\mathcal{H}^0(\partial^\star\{v_\varepsilon < t_\varepsilon\}). \tag{5.3}$$

Set $X_\varepsilon = \{v_\varepsilon < t_\varepsilon\}$ and $Y_\varepsilon = \{v_{0,\varepsilon} < t_\varepsilon\}$. Note that since $v_\varepsilon \leqslant v_{0,\varepsilon}$, one has $Y_\varepsilon \subset X_\varepsilon$. Combining (5.1), (5.2) and (5.3) yields

$$\mathcal{H}^0(\partial^\star X_\varepsilon) \leqslant C. \tag{5.4}$$

Using the variations of $v_{0,\varepsilon}$ (see [BMR23, Proposition 2.1], $\mathcal{H}^0(\partial^\star Y_\varepsilon) \leqslant 2$ and Y_ε is an open interval $(c^\varepsilon, d^\varepsilon)$. Moreover, since $t_\varepsilon < b$, one has

$$|Y_\varepsilon| \subset |X_\varepsilon| \leqslant |\{1-v_\varepsilon > 1-b\}| \leqslant \frac{1}{(1-b)^2}\int_0^L (1-v_\varepsilon)^2 \leqslant \frac{C}{(1-b)^2}\varepsilon \to 0. \tag{5.5}$$

Because $v_\varepsilon \in H^1(0,L)$, v_ε is continuous and X_ε is open set and can be written as the union of a countable number of open intervals and $\partial^\star X_\varepsilon = \partial X_\varepsilon$. By (5.4), up to an extraction, there exists an integer m such that for all ε,

$$X_\varepsilon = \bigcup_{i=1}^m (a_i^\varepsilon, b_i^\varepsilon).$$

There exists an index i_0 such that $(c^\varepsilon, d^\varepsilon) \subset (a_{i_0}^\varepsilon, b_{i_0}^\varepsilon)$. Since $|X_\varepsilon|, |Y_\varepsilon| \to 0$, one has $b_i^\varepsilon - a_i^\varepsilon \to 0$ and since $[0,L]$ is compact, up to another extraction, the sequences $(a_i^\varepsilon), (b_i^\varepsilon)$ converge to the same limit z_i. In particular, $c^\varepsilon, d^\varepsilon \to z_{i_0}$.

We will prove that $\Gamma_0 \cup J_u \subset \cup_i\{z_i\}$, with Γ_0 being empty or $\{L/2\}$. We first prove that $J_u \subset \cup_i\{z_i\}$, which is given if we prove that $u \in H^1(z_i, z_{i+1})$. Take an index $i \in \{1,\dots,m\}$ and take $z_i < \alpha < \beta < z_{i+1}$. Let us show that $u_\varepsilon \rightharpoonup u$ in $H^1(\alpha,\beta)$.

First, $\|u_\varepsilon\|_2$ is bounded. Second, thanks to the energy bound (2.2), and since for ε large enough, one has $[\alpha,\beta] \subset [0,L] \setminus X_\varepsilon \subset [0,L] \setminus \{v_\varepsilon < b\}$:

$$C \geqslant \int_\alpha^\beta (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \geqslant b\int_\alpha^\beta |u_\varepsilon'|^2 = b\|u_\varepsilon'\|_2^2.$$

So (u_ε) is bounded in $H^1(0,L)$. Thus, up to an extraction, $u_\varepsilon \rightharpoonup \hat{u}$ in $H^1(\alpha,\beta)$. But according to Rellich embedding, up to a subsequence, $u_\varepsilon \to \hat{u}$ in $L^2(\alpha,\beta)$. Thus $\hat{u} = u$ and

$$\int_\alpha^\beta |u'|^2 \leqslant \liminf_{\varepsilon\to 0}\int_\alpha^\beta |u_\varepsilon'|^2 \leqslant \liminf_{\varepsilon\to 0}\int_{b_i^\varepsilon}^{a_{i+1}^\varepsilon} |u_\varepsilon'|^2. \tag{5.6}$$

Letting α and β tend to z_i and z_{i+1} respectively shows that $u \in H^1(z_i, z_{i+1})$ with

$$\int_{z_i}^{z_{i+1}} |u'|^2 \leqslant \liminf_{\varepsilon\to 0}\int_{b_i^\varepsilon}^{a_{i+1}^\varepsilon} |u_\varepsilon'|^2. \tag{5.7}$$

Similarly, we obtain

$$\int_0^{z_1} |u'|^2 \leqslant \liminf_{\varepsilon\to 0}\int_0^{a_1^\varepsilon} |u_\varepsilon'|^2 \qquad \text{and} \qquad \int_{z_m}^L |u'|^2 \leqslant \liminf_{\varepsilon\to 0}\int_{b_m^\varepsilon}^L |u_\varepsilon'|^2. \tag{5.8}$$

Combining (5.8) with (5.7) for $i = 1, \dots, m$, we deduce that $u \in SBV^2(0, L)$ with $J_u \subset \cup_{i=1}^m \{z_i\}$ and with

$$\int_0^L |u'|^2 \leqslant \liminf_{\varepsilon \to 0} \int_{[0,L]\setminus X_\varepsilon} |u'_\varepsilon|^2. \tag{5.9}$$

The same argument applied to the sequences $(u_{0,\varepsilon}, v_{0,\varepsilon})$ proves that $J_{u_0} \subset \{z_{i_0}\}$. Recalling (2.4), this proves that $\Gamma_0 \subset \cup_i \{z_i\}$. We then have

$$2\mathcal{H}^0(J_u \cup \Gamma_0) \leqslant 2m = \mathcal{H}^0(\partial^\star\{v_\varepsilon < t_\varepsilon\})$$

Combining this inequality with (5.2) and (5.3) yields

$$\liminf_{\varepsilon \to 0} 2\int_0^L |v_\varepsilon|(1 - v_\varepsilon) \geqslant 2(f(b) - f(a)) \lim_{\varepsilon \to 0} \mathcal{H}^0(\partial^\star X_\varepsilon) = 4(f(b) - f(a))\mathcal{H}^0(J_u \cup \Gamma_0). \tag{5.10}$$

Combining (5.10) with (5.1) and letting a and b tending to 0 and 1 respectivelygives

$$\liminf_{\varepsilon \to 0} \int_0^L \left(\frac{(1 - v_\varepsilon)^2}{\varepsilon} + \varepsilon |v'_\varepsilon|^2 \right) \geqslant 2\mathcal{H}^0(J_u \cup \Gamma_0). \tag{5.11}$$

Now focus on the first term of $\overline{AT}_\varepsilon$. Indeed,

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \geqslant \int_{[0,L]\setminus X_\varepsilon} v_\varepsilon^2 |u'_\varepsilon|^2 \geqslant a^2 \int_{[0,L]\setminus X_\varepsilon} |u'_\varepsilon|^2.$$

and using (5.9),

$$\liminf_{\varepsilon \to 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \geqslant a^2 \liminf_{\varepsilon \to 0} \int_{[0,L]\setminus X_\varepsilon} |u'_\varepsilon|^2 \geqslant a^2 \int_0^L |u'|^2.$$

Since this inequality is true for every $a < 1$, letting $a \to 1$ leads to

$$\liminf_{\varepsilon \to 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \geqslant \int_0^L |u'|^2. \tag{5.12}$$

Summing (5.11) and (5.12), gives $\liminf_{\varepsilon \to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \geqslant \overline{MS}(u)$. □

5.2 Proof of the Γ-limsup

Lemma 5.2. *Let $u \in L^2((0, L))$. Then there exists a sequence $(u_\varepsilon, v_\varepsilon)$ converging in $L^2 \times L^2$ to $(u, 1)$ such that*

$$\limsup_{\varepsilon \to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \overline{MS}(u).$$

In the sequel, we denote by $MM_\varepsilon^V(v) := \displaystyle\int_V \frac{(1 - v)^2}{\varepsilon} + \varepsilon |v'|^2$ and $MM_\varepsilon(v) := MM_\varepsilon^{[0,L]}(v)$.

We suppose that u is in $SBV^2(0, L)$, so that $\overline{MS}(u) < \infty$. The proof goes the following way : we approach $J_u \setminus \Gamma_0$ with the classical recovery sequence $(w_\varepsilon, z_\varepsilon)$ used the proof of the $\Gamma-$limsup of AT_ε to MS (Proposition 5.3). This sequence might not satisfy the obstacle condition $w_\varepsilon \leqslant v_{0,\varepsilon}$ so we apply the modification $w_{1,\varepsilon} = \min(w_\varepsilon, v_{0,\varepsilon})$ (Lemma 5.4). In a neighbourhood W of Γ_0, we might not have $\limsup_{\varepsilon \to 0} MM_\varepsilon^V(w_{1,\varepsilon}) \leqslant 2\mathcal{H}^0(\Gamma_0)$. For this reason we construct a sequence $w_{2,\varepsilon}$ and a cutoff function φ_ε that both vanish around Γ_0(Lemma 5.5). Finally, setting $v_\varepsilon = \min(w_{1,\varepsilon}, w_{2,\varepsilon})$ and $u_\varepsilon = \varphi_\varepsilon z_\varepsilon$, we prove that $(u_\varepsilon, v_\varepsilon)$ satisfies the requirements of Lemma 5.2.

Proof. Let us start by recalling the construction of a recovery sequence $(w_\varepsilon, z_\varepsilon)$ for $J_u \setminus \Gamma_0$. This construction can be found in [AT92].

Proposition 5.3. *Let $d(x)$ denote the distance from x to the set $J_u \setminus \Gamma_0$ and $\alpha_\varepsilon = \sqrt{\eta_\varepsilon \varepsilon}$. We set*

$$h_\varepsilon(t) = 1 - e^{-\frac{t - \alpha_\varepsilon}{\varepsilon}},$$

Then the pair $(w_\varepsilon, z_\varepsilon)$ given by

$$w_\varepsilon = \begin{cases} h_\varepsilon(d(x)) & \text{if } d(x) > \alpha_\varepsilon \\ 0 & \text{if } 0 \leqslant d(x) \leqslant \alpha_\varepsilon \end{cases} \tag{5.13}$$

and

$$z_\varepsilon = \begin{cases} u(x) & \text{if } d(x) > \alpha_\varepsilon \\ \left(\frac{2d(x)}{\alpha_\varepsilon} - 1\right) u(x) & \text{if } \frac{\alpha_\varepsilon}{2} < d(x) \leqslant \alpha_\varepsilon \\ 0 & \text{if } d(x) \leqslant \frac{\alpha_\varepsilon}{2} \end{cases} \tag{5.14}$$

satisfies

$$(w_\varepsilon, z_\varepsilon) \overset{L^2(0,L)^2}{\longrightarrow} (1, u) \tag{5.15}$$

and, for all neighbourhood W of $J_u \setminus \Gamma_0$

$$\limsup_{\varepsilon \to 0} MM_\varepsilon^W(w_\varepsilon) \leqslant 2\mathcal{H}^0(J_u \setminus \Gamma_0) \tag{5.16}$$

and,

$$\limsup_{\varepsilon \to 0} \int_W (\eta_\varepsilon + w_\varepsilon^2)|z_\varepsilon'|^2 \leqslant \int_W |u'|^2. \tag{5.17}$$

Now we adapt this recovery sequence to satisfy the obstacle condition :

Lemma 5.4. *Let*

$$w_{1,\varepsilon} = \min(w_\varepsilon, v_{0,\varepsilon}) \tag{5.18}$$

Then $w_{1,\varepsilon} \leqslant v_{0,\varepsilon}$,

$$w_{1,\varepsilon} \overset{L^2(0,L)}{\longrightarrow} 1, \tag{5.19}$$

and for any neighbourhood W_1 of $J_u \setminus \Gamma_0$ not containing Γ_0,

$$\limsup_{\varepsilon \to 0} MM_\varepsilon^{W_1}(w_{1,\varepsilon}) \leqslant 2\mathcal{H}^0(J_u \setminus \Gamma_0) \tag{5.20}$$

and,

$$\limsup_{\varepsilon \to 0} \int_{W_1} (\eta_\varepsilon + w_{1,\varepsilon}^2)|z_\varepsilon'|^2 \leqslant \int_{W_1} |u'|^2. \tag{5.21}$$

Proof of Lemma 5.4. First, by definition of (5.18), $w_{1,\varepsilon} \leqslant v_{0,\varepsilon}$.

Second, according to (5.15), $w_\varepsilon \to 1$ in $L^2(0,L)$ and according to [BMR23, Theorem 1.1 (i)], $v_{0,\varepsilon} \to 1$ in $L^2(0,L)$, so that $w_{1,\varepsilon} \to 1$ in $L^2(0,L)$ as well.

Third, take W_1 a neighbourhood of $J_u \setminus \Gamma_0$ not containing Γ_0. Using [BMR23, Theorem 1.1 (iv)], $\limsup_{\varepsilon \to 0} MM_\varepsilon^{W_1}(v_{0,\varepsilon}) \to 0$, so using (5.16) and (5.17) :

$$\limsup_{\varepsilon \to 0} MM_\varepsilon^{W_1}(w_{1,\varepsilon}) \leqslant \limsup_{\varepsilon \to 0} \left(\underbrace{MM_\varepsilon^{W_1}(v_{0,\varepsilon})}_{\to 0} + MM_\varepsilon^{W_1}(w_\varepsilon) \right) \leqslant 2\mathcal{H}^0(J_u \setminus \Gamma_0)$$

and

$$\limsup_{\varepsilon \to 0} \int_{W_1} (\eta_\varepsilon + w_{1,\varepsilon}^2)|z_\varepsilon'|^2 \leqslant \limsup_{\varepsilon \to 0} \int_{W_1} (\eta_\varepsilon + w_\varepsilon^2)|z_\varepsilon'|^2 \leqslant \int_{W_1} |u'|^2.$$

□

Now we deal with Γ_0. Let $\mathcal{C} > 0$ be such that

$$\limsup MM_\varepsilon(v_{0,\varepsilon}) \leqslant \mathcal{C}. \tag{5.22}$$

We construct a function $w_{2,\varepsilon}$ and a cut-off function φ_ε that vanish around Γ_0. The construction is given in the following lemma (which is the analogue of [Gia05, Lemma 6.2]) :

Lemma 5.5. *Set $r_\varepsilon = \sqrt[4]{\eta_\varepsilon/\varepsilon}$, $s_\varepsilon = 2r_\varepsilon$, $t_\varepsilon = \sqrt{s_\varepsilon}$ and $k_\varepsilon = t_\varepsilon/s_\varepsilon$.*

There exists a cut-off function φ_ε such that

$$\varphi_\varepsilon = \begin{cases} 1 & \text{if } \quad s_\varepsilon \leqslant v_{0,\varepsilon} \\ 0 & \text{if } \quad v_{0,\varepsilon} \leqslant r_\varepsilon \end{cases} \tag{5.23}$$

and

$$\int_{\{r_\varepsilon \leqslant v_{0,\varepsilon} \leqslant s_\varepsilon\}} \eta_\varepsilon |\varphi_\varepsilon'|^2 \to 0 \tag{5.24}$$

Second, set

$$w_{2,\varepsilon} = \begin{cases} v_{0,\varepsilon} & \text{if} \quad t_\varepsilon \leqslant v_{0,\varepsilon} \\ \frac{k_\varepsilon}{k_\varepsilon - 1}(v_{0,\varepsilon} - t_\varepsilon) + t_\varepsilon & \text{if} \quad s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon \\ 0 & \text{if} \quad v_{0,\varepsilon} \leqslant s_\varepsilon \end{cases} \tag{5.25}$$

Then $w_{2,\varepsilon} \leqslant v_{0,\varepsilon}$

$$w_{2,\varepsilon} \overset{L^2(0,L)}{\to} 1, \tag{5.26}$$

and, for some constant $\mathcal{C} > 0$,

$$\limsup_{\varepsilon \to 0} MM_\varepsilon(w_{2,\varepsilon}) - MM_\varepsilon(v_{0,\varepsilon}) \leqslant \mathcal{C}\left(\frac{2k_\varepsilon}{(k_\varepsilon - 1)^2} + \frac{2t_\varepsilon}{(k_\varepsilon - 1)(1 - t_\varepsilon)^2} + \frac{2t_\varepsilon}{(1 - t_\varepsilon)^2}\right). \tag{5.27}$$

Remark 5.1. Observe that the right hand side of (5.27) tends to 0 with ε as $t_\varepsilon \to 0$ and $k_\varepsilon \to +\infty$.

Remark 5.2. The transition from the two states of φ_ε is done on a domain on which $w_{2,\varepsilon}$ is equal to zero.

Remark 5.3. The set $\{v_{0,\varepsilon} < t_\varepsilon\}$ might be empty. In this case, $w_{2,\varepsilon} = v_{0,\varepsilon}$ and $\varphi_\varepsilon = 1$.

We postpone the proof of Lemma 5.5 to the end of the proof of Lemma 5.2 and now we construct the sequence $(u_\varepsilon, v_\varepsilon)$ satisfying the requirements of Lemma 5.2, using Lemma 5.4 and Lemma 5.5.

Set

$$v_\varepsilon = \min(w_{1,\varepsilon}, w_{2,\varepsilon}), \tag{5.28}$$

and

$$u_\varepsilon = \varphi_\epsilon z_\varepsilon. \tag{5.29}$$

$v_\varepsilon \leqslant v_{0,\varepsilon}$: This is a consequence of Lemma 5.4 and Lemma 5.5.

$(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^2(0,L)^2$: The fact that $v_\varepsilon \to 1$ in $L^2(0,L)$ is a consequence of (5.19) and (5.26). Concerning the convergence of u_ε, one has

$$\|u_\varepsilon - u\|_{L^2} \leqslant \|u_\varepsilon - z_\varepsilon\|_{L^2} + \|z_\varepsilon - u\|_{L^2}.$$

One already has from (5.15) that $\|z_\varepsilon - u\|_{L^2} \to 0$. On the other hand, as u belongs to $SBV^2((0,L))$, it is a bounded function and using the definition (5.14) of z_ε and the definition (5.23) of φ_ε,

$$\int_0^L |\varphi_\varepsilon z_\varepsilon - z_\varepsilon|^2 = \int_0^L |z_\varepsilon|^2 |\varphi_\varepsilon - 1|^2 \leqslant \int_0^L |u|^2 |\varphi_\varepsilon - 1|^2 \leqslant \int_{\{v_{0,\varepsilon} \leqslant s_\varepsilon\}} |u|^2 \leqslant \|u\|_\infty^2 |\{v_{0,\varepsilon} \leqslant s_\varepsilon\}|$$

But the energy bound (5.22) yields

$$\mathcal{C} \geqslant \int_{\{v_{0,\varepsilon} \leqslant s_\varepsilon\}} \frac{(1 - v_{0,\varepsilon})^2}{\varepsilon} \geqslant \frac{(1 - t_\varepsilon)^2}{\varepsilon} |\{v_{0,\varepsilon} \leqslant s_\varepsilon\}| \tag{5.30}$$

Thus $|\{v_{0,\varepsilon} \leqslant s_\varepsilon\}| \to 0$ and $\|u_\varepsilon - z_\varepsilon\|_{L^2} \to 0$.

$\limsup_{\varepsilon \to 0} \overline{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant \overline{MS}(u)$:

Fix W_1 a neighbourhood of $J_u \setminus \Gamma_0$ and W_2 a neighbourhood of Γ_0 so that $W_1 \cap W_2 = \emptyset$. Then one has

$$v_\varepsilon = w_{1,\varepsilon} \text{ on } W_1 \quad ; \quad u_\varepsilon = \begin{cases} z_\varepsilon & \text{on } W_1, \\ \varphi_\epsilon u & \text{on } W_2, \\ u & \text{outside of } W_1 \sqcup W_2. \end{cases} \tag{5.31}$$

Indeed, outside of W_2, $v_{0,\varepsilon} \geqslant 1 - M\varepsilon > t_\varepsilon$ (see Corollary 4.3 applied to $\mu_\varepsilon = 0$ and $x_{min} = L/2$) so, outside of W_2, $w_{2,\varepsilon} = v_{0,\varepsilon}$ and $v_\varepsilon = \min(w_{1,\varepsilon}, v_{0,\varepsilon}) = w_{1,\varepsilon}$ and φ_ε is equal to 1 outside of W_2. Similarly, for ε small enough, $\{d \leqslant \alpha_\varepsilon\} \subset W_1$, so $z_\varepsilon = u$ outside of W_1.

Concerning the convergence of $MM_\varepsilon(v_\varepsilon)$, by (5.20) and (5.27).

$$\begin{aligned} \limsup_{\varepsilon \to 0} MM_\varepsilon(v_\varepsilon) &\leqslant \limsup_{\varepsilon \to 0} MM_\varepsilon^{W_1}(w_{1,\varepsilon}) + \limsup_{\varepsilon \to 0} MM_\varepsilon(w_{2,\varepsilon}) \\ &\leqslant 2\mathcal{H}^0(J_u \setminus \Gamma_0) + \limsup_{\varepsilon \to 0} \left(MM_\varepsilon(v_{0,\varepsilon}) + \mathcal{C}\left(\frac{2k_\varepsilon}{(k_\varepsilon - 1)^2} + \frac{2t_\varepsilon}{(k_\varepsilon - 1)(1 - t_\varepsilon)^2} + \frac{2t_\varepsilon}{(1 - t_\varepsilon)^2}\right)\right) \end{aligned}$$

Then, according to [BMR23, Theorem 1.1 (iii),(iv)], $MM_\varepsilon(v_{0,\varepsilon}) \to 2\mathcal{H}^0(\Gamma_0)$. Thus, using Remark 5.1,

$$\limsup_{\varepsilon\to 0} MM_\varepsilon(v_\varepsilon) \leqslant 2\mathcal{H}^0(J_u \setminus \Gamma_0) + 2\mathcal{H}^0(\Gamma_0) = 2\mathcal{H}^0(J_u \cup \Gamma_0).$$

Now we deal with the convergence of the elastic energy $\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2$. According to (5.21),

$$\limsup_{\varepsilon\to 0} \int_{W_1} (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \leqslant \limsup_{\varepsilon\to 0} \int_{W_1} (\eta_\varepsilon + w_{1,\varepsilon}^2)|z_\varepsilon'|^2 \leqslant \int_{W_1} |u'|^2,$$

and using the expression (5.31) of u_ε,

$$\limsup_{\varepsilon\to 0} \int_{[0,L]\setminus(W_1\sqcup W_2)} (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2 \leqslant \limsup_{\varepsilon\to 0} \int_{[0,L]\setminus(W_1\sqcup W_2)} (\eta_\varepsilon + 1)|u'|^2 = \int_{[0,L]\setminus(W_1\sqcup W_2)} |u'|^2.$$

It remains to deal with

$$\limsup_{\varepsilon\to 0} \int_{W_2} (\eta_\varepsilon + v_\varepsilon^2)|u_\varepsilon'|^2.$$

To compute $\int_{W_2} (\eta_\varepsilon+v_\varepsilon^2)|u_\varepsilon'|^2$, we recall the definition (5.23) of φ_ε and we apply the inequality $|p+q|^2 \leqslant 2(|p|^2+|q|^2)$ to $p = \varphi_\varepsilon' z$ and $q = \varphi_\varepsilon z'$ on the set $\{r_\varepsilon \leqslant v_{0,\varepsilon} \leqslant s_\varepsilon\}$. Namely,

$$\begin{aligned}
\int_{W_2} (\eta_\varepsilon + v_\varepsilon^2)|(\varphi_\varepsilon u)'|^2 &= \int_{\{s_\varepsilon\leqslant v_{0,\varepsilon}\}\cap W_2} (\eta_\varepsilon + v_\varepsilon)^2|\varphi_\varepsilon' u + \varphi_\varepsilon u'|^2 + \int_{\{r_\varepsilon\leqslant v_{0,\varepsilon}\leqslant s_\varepsilon\}} (\eta_\varepsilon + v_\varepsilon)^2|\varphi_\varepsilon' u + \varphi_\varepsilon u'|^2 \\
&\leqslant \int_{\{s_\varepsilon\leqslant v_\varepsilon\}\cap W_2} (\eta_\varepsilon + 1)|u'|^2 + 2\int_{\{r_\varepsilon\leqslant v_\varepsilon\leqslant s_\varepsilon\}} \eta_\varepsilon \underbrace{|\varphi_\varepsilon|^2}_{\leqslant 1}|u'|^2 + 2\int_{\{r_\varepsilon\leqslant v_\varepsilon\leqslant s_\varepsilon\}} \eta_\varepsilon|\varphi_\varepsilon'|^2|u|^2 \\
&\leqslant \int_{\{s_\varepsilon\leqslant v_\varepsilon\}\cap W_2} (\eta_\varepsilon + 1)|u'|^2 + 2\eta_\varepsilon\|u'\|_{L^2} + 2\|u\|_\infty^2 \int_{\{r_\varepsilon\leqslant v_\varepsilon\leqslant s_\varepsilon\}} \eta_\varepsilon|\varphi_\varepsilon'|^2.
\end{aligned}$$

The limsup of this last term is smaller than $\int_{W_2}|u'|^2$, the second term goes to 0 and the third term goes to 0 according (5.24), which concludes the proof of the $\Gamma-$limsup. □

We now turn to the proof of the lemma Lemma 5.5 :

Proof of Lemma 5.5. We start with the construction of a cut-off function φ_ε satisfying (5.23) and (5.24). Divide the interval $[r_\varepsilon, s_\varepsilon]$ into h_ε intervals $I_1, \dots, I_{h_\varepsilon}$ of the same size, such that $\frac{\eta_\varepsilon}{\varepsilon}\cdot\frac{h_\varepsilon}{(s_\varepsilon - r_\varepsilon)^2} \to 0$ (which is possible as $\frac{\eta_\varepsilon}{\varepsilon}\cdot\frac{1}{(s_\varepsilon - r_\varepsilon)^2} \to 0$, take for example $h_\varepsilon = \left\lfloor \left(\frac{\eta_\varepsilon}{\varepsilon}\cdot\frac{1}{(s_\varepsilon - r_\varepsilon)^2}\right)^{-1/2} \right\rfloor$). Note that by the energy bound (5.22) :

$$\mathcal{C} \geqslant \sum_{j=1}^{h_\varepsilon} \int_{I_j} \varepsilon|v_{0,\varepsilon}'|^2.$$

So there exists an interval $I_\varepsilon = [\alpha_\varepsilon, \beta_\varepsilon]$ of length $(s_\varepsilon - r_\varepsilon)/h_\varepsilon$ such that

$$\int_{I_\varepsilon} \varepsilon|v_{0,\varepsilon}'|^2 \leqslant \frac{\mathcal{C}}{h_\varepsilon}.$$

Define $\varphi_\varepsilon = \min\left(\frac{(v_{0,\varepsilon}-\alpha_\varepsilon)^+}{\beta_\varepsilon-\alpha_\varepsilon}, 1\right)$. Then φ_ε takes the desired values on $\{v_{0,\varepsilon} \leqslant r_\varepsilon\}$ and $\{v_{0,\varepsilon} \geqslant s_\varepsilon\}$ and

$$\eta_\varepsilon \int_{\{r_\varepsilon\leqslant v_{0,\varepsilon}\leqslant s_\varepsilon\}} |\varphi_\varepsilon'|^2 = \eta_\varepsilon \int_{\{\alpha_\varepsilon\leqslant v_{0,\varepsilon}\leqslant \beta_\varepsilon\}} \frac{|v_{0,\varepsilon}'|^2}{(\beta_\varepsilon - \alpha_\varepsilon)^2} \leqslant \frac{\eta_\varepsilon}{\varepsilon}\frac{\mathcal{C}}{h_\varepsilon}\frac{h_\varepsilon^2}{(s_\varepsilon - r_\varepsilon)^2} \to 0$$

which proves (5.24).

Now we turn to the proof of (5.26) and (5.27). Observe that $w_{2,\varepsilon} \leqslant v_{0,\varepsilon}$ as $v_{0,\varepsilon}$ is positive and on $\{s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon\}$, according to (5.25) :

$$w_{2,\varepsilon} = v_{0,\varepsilon} + \frac{1}{k_\varepsilon - 1}(v_{0,\varepsilon} - t_\varepsilon) \leqslant v_{0,\varepsilon}.$$

To prove (5.26), observe that $w_{2,\varepsilon}$ and $v_{0,\varepsilon}$ are taking values in $[0,1]$, so that

$$\int_0^L |w_{2,\varepsilon} - v_{0,\varepsilon}|^2 = \int_{\{v_{0,\varepsilon} \leqslant t_\varepsilon\}} |w_{2,\varepsilon} - v_{0,\varepsilon}|^2 \leqslant 4|\{v_{0,\varepsilon} \leqslant t_\varepsilon\}|.$$

Then, using the energy bound (5.22), we see that

$$\mathcal{C} \geqslant \int_{\{v_{0,\varepsilon} \leqslant t_\varepsilon\}} \frac{(1-v_{0,\varepsilon})^2}{\varepsilon} \geqslant \frac{(1-t_\varepsilon)^2}{\varepsilon} |\{v_{0,\varepsilon} \leqslant t_\varepsilon\}|. \tag{5.32}$$

So $|\{v_{0,\varepsilon} \leqslant t_\varepsilon\}| \to 0$ and $\|w_{2,\varepsilon} - v_{0,\varepsilon}\|_2 \to 0$, which proves (5.26).

We turn to the proof of (5.27). We compute the difference in energy, starting with the difference between the Dirichlet energies. The expression (5.25) of $w_{2,\varepsilon}$ and the energy bound (5.22) yields

$$\begin{aligned}
\int_0^L \varepsilon|w'_{2,\varepsilon}|^2 - \varepsilon|v'_{0,\varepsilon}|^2 &= \int_{\{s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon\}} \varepsilon\left(\frac{k_\varepsilon^2}{(k_\varepsilon-1)^2} - 1\right)|v'_{0,\varepsilon}|^2 - \int_{\{v_{0,\varepsilon} \leqslant s_\varepsilon\}} \varepsilon|v'_{0,\varepsilon}|^2 \\
&\leqslant \left(\frac{k_\varepsilon^2}{(k_\varepsilon-1)^2} - 1\right)\int_{\{s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon\}} \varepsilon|v'_{0,\varepsilon}|^2 \leqslant \mathcal{C}\left(\frac{k_\varepsilon^2}{(k_\varepsilon-1)^2} - 1\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_0^L \frac{(1-w_{2,\varepsilon})^2}{\varepsilon} - \frac{(1-v_{0,\varepsilon})^2}{\varepsilon} &= \frac{1}{\varepsilon}\int_{\{v_{0,\varepsilon} \leqslant t_\varepsilon\}} (v_{0,\varepsilon} - w_{2,\varepsilon})(2 - v_{0,\varepsilon} - w_{2,\varepsilon}) \\
&= \frac{1}{\varepsilon}\int_{\{s_\varepsilon \leqslant v_{0,\varepsilon} \leqslant t_\varepsilon\}} \frac{1}{k_\varepsilon - 1} \underbrace{(t_\varepsilon - v_{0,\varepsilon})}_{\leqslant t_\varepsilon} \underbrace{(2 - v_{0,\varepsilon} - w_{2,\varepsilon})}_{\leqslant 2} + \frac{1}{\varepsilon}\int_{\{v_{0,\varepsilon} \leqslant s_\varepsilon\}} \underbrace{v_{0,\varepsilon}}_{\leqslant s_\varepsilon} \underbrace{(2 - v_{0,\varepsilon})}_{\leqslant 2} \\
&\leqslant \frac{2t_\varepsilon}{k_\varepsilon - 1}\frac{|\{v_{0,\varepsilon} \leqslant t_\varepsilon\}|}{\varepsilon} + \frac{2s_\varepsilon}{k_\varepsilon - 1}\frac{|\{v_{0,\varepsilon} \leqslant s_\varepsilon\}|}{\varepsilon}
\end{aligned}$$

But the energy bound (5.22) yields, in a similar fashion than (5.32), for any $\alpha > 0$,

$$\mathcal{C} \geqslant \int_{\{v_{0,\varepsilon} \leqslant \alpha\}} \frac{(1-v_{0,\varepsilon})^2}{\varepsilon} \geqslant \frac{(1-\alpha)^2}{\varepsilon} |\{v_{0,\varepsilon} \leqslant \alpha\}|.$$

Hence, the last term is smaller than for

$$\mathcal{C}\frac{2t_\varepsilon}{(k_\varepsilon - 1)(1-t_\varepsilon)^2} + \mathcal{C}\frac{2s_\varepsilon}{(1-s_\varepsilon)^2} \leqslant \mathcal{C}\left(\frac{2t_\varepsilon}{(k_\varepsilon - 1)(1-t_\varepsilon)^2} + \frac{2t_\varepsilon}{(1-t_\varepsilon)^2}\right).$$

Summing both estimates yields the desired result. This achieves the proof of Lemma 5.5. □

5.3 Critical Points of $\overline{MS}$

Theorem 2.7 ensures that the minimizers of $\overline{AT}_\varepsilon$ converge to a pair $(1,u)$ that minimizes $\overline{MS}$ if we have the additional assumption of an energy bound on (AT_ε).

A natural extension is to ask whether the critical points of $\overline{AT}_\varepsilon$ also converge to critical points of $\overline{MS}$. Here, we prove that the critical points of $\overline{MS}$ are either affine or piecewise constant.

Take u a minimizer of $\overline{MS}$. First, u has a finite number of discontinuity points, that we denote by $\{x_1, \dots, x_d\}$. Let φ in $SBV^2(0,L)$ with $J_\varphi \subset J_u$ and take $u + t\varphi$ as a competitor for t a real number. Then, $J_{u+t\varphi} = J_u$ so one has

$$\int_0^L |u'|^2 \leqslant \int_0^L |u' + t\varphi'|^2,$$

which rewrites, after dividing by t, letting t tend successively to 0^+ and 0^-, as :

$$\int_0^L u'\varphi' = 0.$$

So u is affine on each interval (x_i, x_{i+1}). In the case where u admits at least one discontinuity point, say x_i, and take φ equal to 0 outside (x_i, x_{i+1}) and equal to $\dfrac{x - x_{i+1}}{x_i - x_{i+1}}$ on (x_i, x_{i+1}) (where x_{i+1} can be replaced by L). Then $\varphi \in SBV^2(0, L)$ with $\varphi' = 1_{(x_i, x_{i+1})}$, $J_\varphi = \{x_i\}$ and

$$u(x_{i+1}) - u(x_i) = \int_{x_i}^{x_{i+1}} u' = \int_0^L u'\varphi' = 0,$$

so that $u(x_i) = u(x_{i+1})$, i.e u is piecewise constant.

Hence, critical points of $\overline{AT}_\varepsilon$ converge to critical points of $\overline{MS}$. However, there is a selection criterion as limit functions of critical points of AT_ε must be non decreasing and admit at most one discontinuity point.

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