State reconstruction for stochastic nonlinear systems with unknown local nonlinearities via output injection

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Abstract: This paper addresses state estimation for dynamical systems involving localized unknown nonlinearities. Direct application of linear state estimation techniques, e.g., the Kalman filter, would yield erroneous state estimates. Existing approaches in the literature either assume or estimate the nonlinearities. Alternatively, the present paper proposes to reject the unknown nonlinearities as if they were unknown disturbances. By applying an existing disturbance rejection technique, the need to know or to estimate the nonlinearities is avoided. The efficiency of the proposed method is demonstrated through numerical simulations on a nonlinear mechanical system.

Keywords: Estimation and filtering, Disturbance rejection, Mechanical and aerospace estimation

1. INTRODUCTION

Linear systems can be effectively modeled and estimated using well-established techniques (Ljung, 2004). In contrast, nonlinear (NL) systems often exhibit intricate behaviours, making state estimation challenging due to modeling difficulties and the lack of efficient state estimation theory for general NL systems (Ljung, 2010). This complexity arises from various nonlinear interactions among system variables, external inputs (forces), and real-life uncertainties, making the development of accurate models and estimation algorithms particularly difficult (Paduart et al., 2010).

Several methodologies have been developed to model NL systems and estimate their state dynamics. In this regard, discrete-time models have been preferred since the observed measurement outputs (sensor data) are commonly sampled at discrete time instants and their computational efficiency compared to continuous-time models is known to be superior (Paduart et al., 2010). Various estimation methods - such as maximum likelihood estimation (MLE) (Schon et al., 2005), expectation maximization (EM) (Schön et al., 2011), etc. - estimate the states of the NL systems by approximating the nonlinearity present in the system. Since these methods faced irregular estimation due to the nonlinearity of the system, smoothing in combination with particle-based approaches emerged, wherein the particle weights are filtered to achieve better approximation for the nonlinearity (Schön et al., 2011). However this method proves computationally expensive.

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With both input and system nonlinearity considered as unknown, a subspace identification technique has been developed to estimate the nonlinearities of the system (Schrangl et al., 2020). The method iteratively estimates the coefficients of the initially assumed polynomial description of the nonlinearities on each of the locations of the multi-degree of freedom (*mdof*) system. The iterations at each time step is continued until convergence of the nonlinearity coefficients, making the algorithm computationally costly. In (Wei et al., 2023), a two step method for estimating the nonlinearity was developed. It first identifies a subset range where nonlinearity seems to be acting, then using candidate nonlinear basis functions to estimate the localised nonlinearity. Though the nonlinearity is being estimated, an initial assumption has been made about the form of the nonlinearity (Wei et al., 2023). By utilizing observer/Kalman filter identification with nonlinear feedback interpretation, (Liu and Cai, 2024) estimated a nonlinear *mdof* system. However, the input was assumed to be known. (Huang, 2024) introduced a center-manifold frequency domain subspace method for NL system identification. It analytically solves the NL system identification problem by decomposing it into series of linear-least square problems, bypassing the need of iterative estimation of the nonlinearities. Analytically solving equations makes the computation relatively more expensive. Further it was reported that the CM method was noise sensitive which is inevitable for real-life applications. Similar, problem has been observed in (Pan et al., 2015).

This study considers NL systems where the nonlinearities are unknown but reside in some subspace of the state-space. Typically, such system models result from space discretization of spatially distributed systems with localized nonlinearities. This class of systems is of high practical

importance in mechanical engineering, where structures are frequently modelled using the theory of linear elasticity with the assumption of small strains or displacements, which yield linear properties in most parts of the system (mass, damping and stiffness). However, mechanical systems are usually composed of assembled components linked by connecting elements, such as bolts, joints or gears, whose behaviour is often highly nonlinear. Characterizing and modelling such nonlinearities often prove challenging because NL systems can exhibit a wide variety of behaviours and are extremely sensitive to parameter variations (Ahlquist et al., 2011).

To address this issue, this paper proposes rejecting nonlinearities through output injection, treating them as unknown disturbances (Zhang and Zhang, 2018), with the fundamental assumption that nonlinearities are localized within a known subspace of the state space whose dimension must be smaller than that of the observable subspace. This approach is related to unknown input observers (Kitanidis, 1987; Darouach and Zasadzinski, 1997; Hsieh, 2000; Gillijns and De Moor, 2007; Gao et al., 2016; Bezzaoucha et al., 2017). Usually, in unknown input observer design problems, state estimation and unknown input decoupling are made jointly in a single step. A particularity of the approach, initially proposed in (Zhang and Zhang, 2018), is its two-step nature: unknown input decoupling is first realized by output injection (in this paper unknown nonlinearities are treated as unknown inputs). By rejecting these unknown nonlinearities, robust state estimation can be achieved using a classical Kalman filter on the transformed state-space model, which becomes linear. We emphasize here that this approach does not require knowledge of the type or amplitude of the nonlinearities, only their locations.

In the following, Section 2 presents the considered nonlinear state-space model. Section 3 recalls the output injection technique that is utilized. In Section 4, the developed methodology has been numerically tested on a nonlinear 8 degree-of-freedom (*dof*) chain of oscillators, where the output measurement has been simulated wile considering localized nonlinearity and reconstructed using a linear counterpart of the simulated system infused with the input rejection technique.

2. NONLINEAR STATE-SPACE FORMULATION

A general NL system state-space model in discrete time would be in the form of

$$\begin{aligned} \mathbf{x}_k &= g(\mathbf{x}_{k-1}, \mathbf{w}_k) \\ \mathbf{y}_k &= h(\mathbf{x}_k, \mathbf{v}_k). \end{aligned} \tag{1}$$

In this paper, it is assumed that a state-space model of an *mdof* NL system is available in the following particular form, typically resulting from the discretization (both in space and in time) of a spatially distributed system,

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{E}f(\mathbf{x}_{k-1}) + \mathbf{w}_k \tag{2}$$

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k \tag{3}$$

with $\mathbf{x}_k \in \mathbb{R}^{n_x}$ and $\mathbf{y}_k \in \mathbb{R}^{n_y}$ being the state and measurement vectors housing n_x and n_y state variables and measurement outputs. $\mathbf{w}_k \in \mathbb{R}^{n_x}$ and $\mathbf{v}_k \in \mathbb{R}^{n_y}$ are both unknown unmeasured Gaussian random input vectors acting (respectively) on the NL system and the measurement noise (with covariances $\mathbf{Q}_k$ and $\mathbf{R}_k$). $\mathbf{A}(\in \mathbb{R}^{n_x \times n_x})$ is the system matrix, respectively. $\mathbf{H}(\in \mathbb{R}^{n_y \times n_x})$ is the output matrix which relates the state vector $(\mathbf{x}_k)$ to the observed measurements $(\mathbf{y}_k)$. The nonlinearities of the system are some unknown function $f(\mathbf{x}_{k-1})$ of the state vector $\mathbf{x}_{k-1}$, whereas the linear part is described by the linear term, $\mathbf{A}\mathbf{x}_{k-1}$. The matrix $\mathbf{E} \in n_x \times n_\xi$ is a known coefficient matrix specifying the subspace in which the nonlinear term resides. In this paper, $\mathbf{A}$, and $\mathbf{H}$ are assumed to be constant and known. Time-variations (LTV systems), parameter estimation and change detection will be the subjecst of further studies.

When the finite dimensional state vector $\mathbf{x}_k$ results from the space discretization of the spatially distributed state, the columns of the matrix $\mathbf{E}$ correspond to the positions of the nonlinearities in the state-space. The localized nature of the nonlinearities implies that $\mathbf{E}$ has much less columns than rows, i.e., $n_\xi << n_x$. It is proposed in this paper to deal with the unknown nonlinearities by rejecting them, as if they were arbitrary unknown disturbances. To simplify notations, $f(\mathbf{x}_{k-1})$ will be replaced by ξ_k, i.e., in what follows,

$$\xi_k = f(\mathbf{x}_{k-1}), \tag{4}$$

which will be treated as an unknown arbitrary vector sequence.

3. METHODOLOGY

To estimate the state trajectory of system (cf. Equations (2) and (3)) from available sensor data, despite the unknown nonlinearities $f(\mathbf{x}_{k-1})$, this paper is based on the elimination of the nonlinear term by means of output injection.

Following (Zhang and Zhang, 2018), let $\mathbf{G}(\in \mathbb{R}^{n_x \times n_y})$ be a non-zero matrix to be specified later, it follows from the measurement Equation (3) that

$$0 = \mathbf{G}\left(\mathbf{y}_k - \mathbf{H}\mathbf{x}_k - \mathbf{v}_k\right). \tag{5}$$

Adding Equation (5) to the process equation (2) yields

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}\mathbf{x}_{k-1} + \mathbf{E}\xi_k + \mathbf{w}_k + \mathbf{G}\left(\mathbf{y}_k - \mathbf{H}\mathbf{x}_k - \mathbf{v}_k\right) \\ &= \mathbf{A}\mathbf{x}_{k-1} + \mathbf{E}\xi_k + \mathbf{w}_k + \mathbf{G}\mathbf{y}_k \\ &\quad - \mathbf{G}\mathbf{H}(\mathbf{A}\mathbf{x}_{k-1} + \mathbf{E}\xi_k + \mathbf{w}_k) - \mathbf{G}\mathbf{v}_k \\ &= \tilde{A}\mathbf{x}_{k-1} + \tilde{E}\xi_k + \mathbf{G}\mathbf{y}_k + \tilde{w}_k \end{aligned} \tag{6}$$

where, $\tilde{A} = \mathcal{L}\mathbf{A}$, $\tilde{E} = \mathcal{L}\mathbf{E}$, $\tilde{w}_k = \mathcal{L}\mathbf{w}_k - \mathbf{G}\mathbf{v}_k$, with

$$\mathcal{L} = \mathbf{I}_{n_x \times n_x} - \mathbf{G}\mathbf{H}. \tag{7}$$

Choose

$$\mathbf{G} = \mathbf{E}(\mathbf{H}\mathbf{E})^\dagger \tag{8}$$

with † denoting matrix Moore-Penrose Pseudo-inverse operation, then

$$\tilde{E} = \mathcal{L}\mathbf{E} = (\mathbf{I}_{n_x \times n_x} - \mathbf{E}(\mathbf{H}\mathbf{E})^\dagger\mathbf{H})\mathbf{E} \tag{9}$$

$$= \mathbf{E} - \mathbf{E} \tag{10}$$

$$= 0, \tag{11}$$

hence $\tilde{E}$ is equal to zero. Thus the corresponding process equation becomes

$$\mathbf{x}_k = \tilde{A}\mathbf{x}_{k-1} + \mathbf{G}\mathbf{y}_k + \tilde{w}_k. \tag{12}$$

3.1 robust Kalman filter (rKF)

Since the transformed state-space model (cf. Equations (12) and (3)) is linear, the Kalman filter (KF) is readily applicable for state estimation. To distinguish from the KF applied to truely linear systems, the KF applied to the considered NL system after nonlinearity rejection is referred to as robust Kalman Filter (rKF) in the present study.

Prediction: The state ($\mathbf{x}_{k|k-1}$) and its covariance ($\mathbf{P}_{k|k-1}$) are estimated based on the updated model, followed by computation of the corresponding measurement,

$$\mathbf{x}_{k|k-1} = \tilde{A}\mathbf{x}_{k-1|k-1} + \mathbf{G}\mathbf{y}_k \tag{13}$$

$$\mathbf{P}_{k|k-1} = \tilde{A}\mathbf{P}_{k-1|k-1}\tilde{A}^T + \mathcal{L}\mathbf{Q}_k\mathcal{L}^T + \mathbf{G}\mathbf{R}_k\mathbf{G}^T \tag{14}$$

Innovation:

$$\epsilon_k = \mathbf{y}_k - \mathbf{H}\mathbf{x}_{k|k-1} \tag{15}$$

Kalman gain:

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1} \tag{16}$$

Update:

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k\epsilon_k \tag{17}$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_{k|k-1} \tag{18}$$

4. NUMERICAL STUDY

The efficiency of the proposed methodology has been presented by undertaking different numerical studies on an 8 *dof* chain of oscillators (Figure 1). The corresponding equation of motion reads as follows,

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathcal{K}(\mathbf{q}(t)) = \mathbf{u}(t) \tag{19}$$

where, $\mathbf{M}$ is the time invariant mass matrix, $\mathbf{C}$ is the time invariant linear damping, and $\mathcal{K}(\mathbf{q}(t))$ is the elastic restoring force which is a nonlinear function of the displacements $\mathbf{q}(t)$. The overhead dot $\dot{\bullet}$ denotes derivation with respect to time such that $\dot{\mathbf{q}}(t)$ and $\ddot{\mathbf{q}}(t)$ correspond to the velocities and accelerations, respectively. It is assumed here that the nonlinearity is localized, i.e. the nonlinear restoring force only acts on a reduced subset of *dof*s. This type of nonlinearity is ubiquitous in mechanical systems due to the presence of joints and contacts. For instance, localized nonlinearities were experimentally observed in the elastomeric engine mounts of the Airbus A400M aircraft Ahlquist et al. (2011). Without loss of generality, the nonlinear force can be expressed as:

$$\mathbf{F}_{nl}(t) = \mathbf{E}\mathbf{f}_{nl}\left(\mathbf{q}(t)\right) \tag{20}$$

Further, the NL system can be defined by replacing the nonlinear term, $\mathcal{K}(\mathbf{q}(t))$, by linear and nonlinear mono/polynomials:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) + \mathbf{F}_{nl}(\mathbf{q}(t)) = \mathbf{f}_{ex}(t) \tag{21}$$

where, $\mathbf{F}_{nl}(\mathbf{q}(t))$ is assummed to be the unknown nonlinearity. $\mathbf{E}$ is a binary matrix which localizes the presence of nonlinearity on the corresponding *dof*, i.e., 1 if some nonlinearity is attached to the particular *dof*, otherwise 0. Meanwhile, $\mathbf{K}$ is the linear stiffness matrix, consisting of the linear part of stiffness, wrt. to each *dof*.

Subsequently, the state-space formulation of the above NL system can be expressed by Equations (12) and (3), with states $\mathbf{x} = \{\mathbf{q}\ \dot{\mathbf{q}}\}^T$, and measurement $\mathbf{y} = \dot{\mathbf{q}}^T$.

It should be noted that the proposed methodology estimates and reconstructs the states by assuming the external forces and nonlinearities as unknown. Therefore the support predictor model in the filter comprises only the linear parts of the system without the term $\mathbf{F}_{nl}(t)$.

In the subsequent sections, first the simulated data has been described, followed by a numerical study wherein state reconstruction using different existing filters has been compared. Further, a sensor density and noise study has been undertaken which showcases the performance of the proposed methodology with reduced number of measurement outputs and varying levels of sensor noise. The proposed methodology is also tested for its efficiency under different types, locations and amplitudes of nonlinearities.

4.1 Simulation data

A localized polynomial nonlinearity is modelled as a cubic polynomial $\mathbf{f}_{nl}(t) = k_{nl}\mathbf{q}(t)^3$ acting on one of the masses. A band-limited white Gaussian noise with root mean square value of 1000 N in frequency range [5, 150] Hz is considered to act as external force $\mathbf{f}_{ex}(t)$, on the third degree of freedom. Linear damping is introduced in the system by considering a stiffness proportional damping term $\mathbf{C} = \beta\mathbf{K}$ with coefficient $\beta = 4.7\text{e}^{-3}$ s^{-1} chosen to ensure the damping ratio equals 1% at the frequency of the first linear mode. Multiple datasets are generated by numerically integrating the equations of motion using Matlab's ODE45 solver with the properties given in Table 1.

Table 1. Material properties of the NL system.

$m_{1:8}$ (kg)	k (N/m)	k_{nl} (N/m^3)
1400	3e6	1e9

The measurement outputs, here velocities at each *dof*, are contaminated with a 1% signal-to-noise ratio (SNR) stationary white Gaussian noise (SWGN). The material properties and external force of the simulation data are kept the same for all the following cases, unless specified otherwise for different cases.

4.2 State estimation via proposed methodology

The relevance of the proposed methodology has been highlighted by comparing the proposed methodology that utilizes robust Kalman filter (rKF) with the different established filters, such as the Kalman (KF), Extended Kalman (EKF), and Ensemble Kalman (EnKF) Filters. The same measurement output data has been supplied to each of the filters while considering that the nonlinearity and the external force are unknown.

First, KF has been used to estimate the states from the known measurement outputs. The support predictor numerical model for all the filters, is an 8 *dof* linear chain of oscillators with the same m, c and k as present in the simulated data without any provision of nonlinearity (cf. Figure 1). Figure 2a compares the measurement sensor output (velocity) with the velocity of the first *dof* estimated by the KF only. One can see that although the KF provides a relatively good qualitative agreement with the simulated states, there is a poor quantitative agreement. This is clearly visible in Figure 3 which shows symmetric

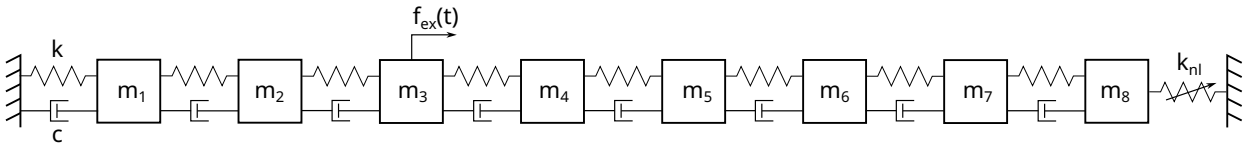
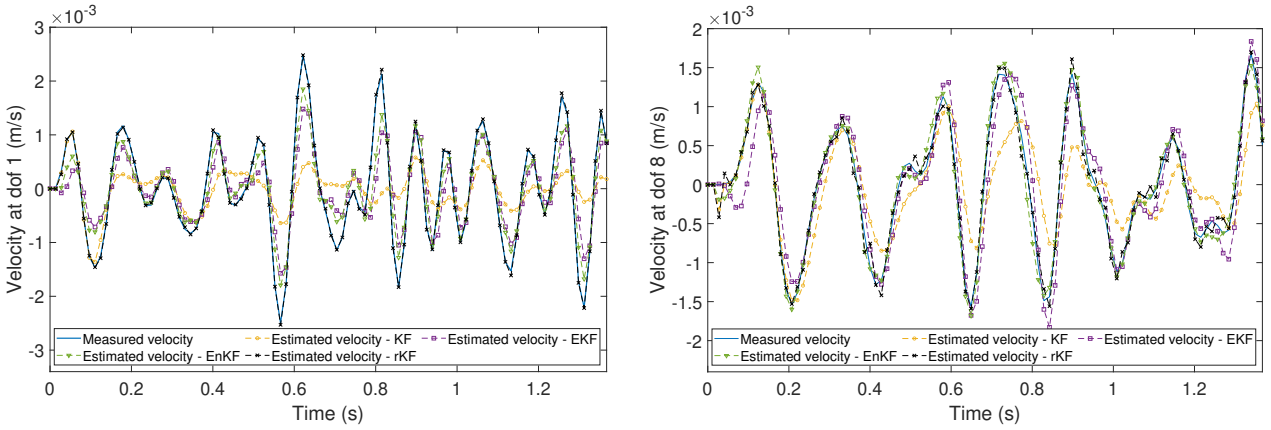


Fig. 1. Schematic representation of the systems under consideration - nonlinear system with localized nonlinearity.



(a) (b)

Fig. 2. State estimation - velocities at *dof* 1 (a) and 8 (b) - using for KF, EKF, EnKF and rKF.

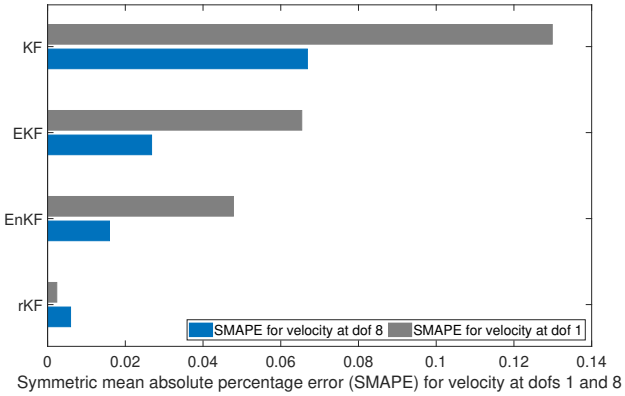


Fig. 3. SMAPE on the state estimation - velocities at *dof* 1 and 8) - using KF, EKF, EnKF and rKF.

mean absolute percentage error (SMAPE) as 12.99% for the compared velocities. This result is coherent with the fact that KF is not suitable to deal with NL systems.

In order to further highlight the performances and advantages of the proposed approach, a comparison with other Kalman filter-based techniques, namely the extended Kalman filter (EKF) and ensemble Kalman filter (EnKF) is performed. The state estimation by EKF shown in Figure 2a shows a significant improvement compared to the results obtained with KF. Nevertheless, the time series of the velocities, as well as the SMAPE value shown in Figure 3, show that its accuracy is still limited with a SMAPE value of 6.54%. A similar analysis on the states reconstructed with EnKF (utilizing 200 ensembles, chosen after a convergence study) and depicted in Figure 2a shows greater accuracy with a SMAPE value of 4.78%.

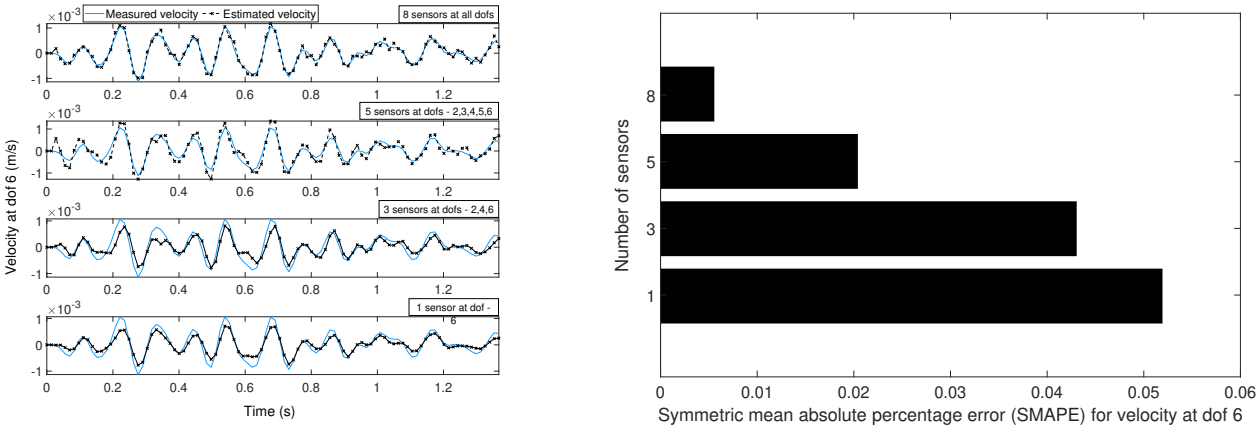
However, the rKF methodology still outperforms it with an even smaller SMAPE value of 0.24%. Figure 2a shows the states reconstructed with the proposed methodology. A similar analysis has been carried out for the state reconstruction of *dof* 8 where the nonlinearity is attached. Figures 2b and 3 show that the methodology outperforms the other three tested methodologies in spite of an overall decrease in accuracy due to the closeness of the nonlinearity that is being rejected. It is also worth noting that the computational effort of the methodology is similar to that of classical KF since, in this case, matrix $\mathbf{G}$ has to be evaluated only once.

4.3 Sensor density and noise study

Often, while estimating a real-life NL system, access to measurement data associated with each *dof* chosen for its counterpart support numerical model in the filters, is not feasible. Thus, the present study explores the efficiency of the proposed methodology in such cases. The nonlinear force is here set to act on the sixth *dof* of the numerical model. Hence, the velocity (measurement output) at *dof* 6 has been chosen for observation, while the other assumptions remain the same as before (cf. Section 4.2).

Figure 4a shows the time series of the velocity at sixth *dof* for several number of sensor outputs. The accuracy of the state estimation decreases as the number of sensor outputs is reduced from 8 (all *dof*s) to 1 (*dof* 6 only). The SMAPE values, shown in Figure 4, increase from less than 1% to more than 5%. However, a satisfactory qualitative agreement is observed for all the cases.

The efficiency of the proposed method is further evaluated for different cases where the simulated data is contaminated with $1, 2, 5$ and 10% SNR. Figure 5 shows the SMAPE values of the velocities of the first and eighth *dof*. It appears that the SMAPE values of both *dof* follow the same increasing trend with the SNR, going from 0.24% to



(a) (b)

Fig. 4. Time series (a) and SMAPE values (b) of the states reconstructed with rKF for varying sensor density.

1.57% on the first *dof* and from approximately 0.6% with 1% SNR to 5.5% with 10% SNR on the eighth *dof*.

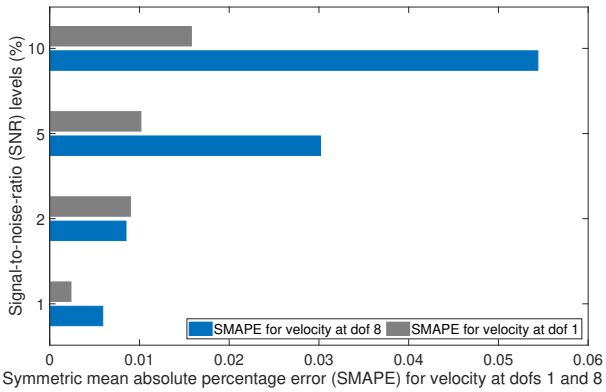


Fig. 5. SMAPE values of the state estimation with rKF for different SNR. Velocities at *dof* 1 and *dof* 8.

4.4 Effectiveness against different nonlinearities

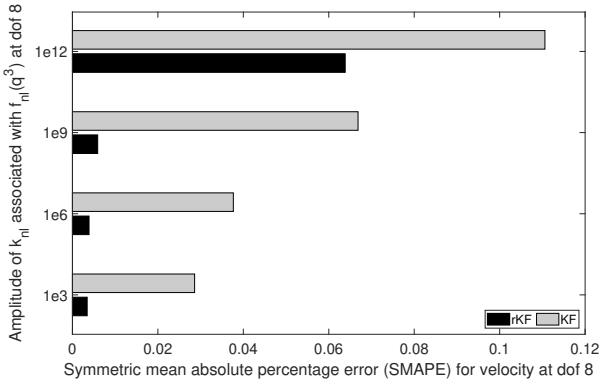


Fig. 6. SMAPE values of the state estimation with KF and rKF for different amplitudes of nonlinear coefficient.

The aim of this section is to investigate the robustness of the methodology to different amplitudes, locations and forms of nonlinearity. Hence, several simulations were carried out for different values of the nonlinear coefficient k_{nl}. For each value, the proposed methodology was used to estimate the states and the accuracy to the reference solution was evaluated and, for each case, the performances of rKF have been compared with those of KF. Note that, for these cases, the nonlinearity was assumed to act on *dof* 8.

Figure 6 depicts the SMAPE values of both rKF and KF estimations for amplitudes of the nonlinear coefficient k_{nl} ranging from $1e3$ N/m^3 to $1e12$ N/m^3. It is observed that the SMAPE values increase with the amplitude of the nonlinear coefficient for both KF and rKF estimations. However, the proposed methodology provides much more accurate estimations of the states. For instance, for a weakly NL system with a nonlinear coefficient equal to $1e3$ N/m^3 where one would expect relatively small deviations from nonlinear behaviour, the SMAPE value of the rKF estimation is 0.34% while that of the KF estimation reaches approximately 3% for the estimated state of the eighth *dof* where the nonlinearity is located. The SMAPE values increase monotonically with the amplitude of the nonlinear coefficient, reaching about 11% and 6.5% for the states estimated with KF and rKF, respectively. This indicates that the methodology is capable of providing much more accurate state estimations, even with relatively strong local nonlinearities.

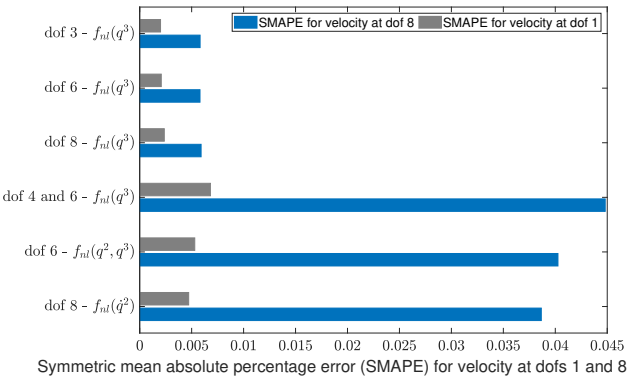


Fig. 7. SMAPE values of the state estimation with rKF for different functional forms and location of nonlinearities. Velocities at *dof* 1 and *dof* 8.

The effectiveness of the proposed methodology is further demonstrated by changing the location of the attached nonlinearity. For the first 3 cases, cf. Figure 7, the nonlinearity is attached to *dof* 3, 6, and 8 while keeping the nonlinear coefficient same for all cases (k_{nl} = 1e9). The proposed methodology accurately estimates the states (cf. Figure 7), with SMAPE values smaller than 0.3% for the velocity at *dof* 1 and smaller than 1% for *dof* 8.

A similar analysis was carried out by introducing, on the one hand, a nonlinearity on two *dof*s, here the fourth and sixth, both having the same nonlinear coefficient (k_{nl} = 1e9) and, on the other hand, a nonlinearity with both quadratic and cubic terms of the form ($f_{nl} = k_{nl,2}q^2 + k_{nl,3}q^3$) where the values of $k_{nl,2}$ and $k_{nl,3}$ are set to $1e6$ N/m^3 and $1e9$ N/m^3, respectively, acting only on the sixth *dof*. For these two cases, Figure 7 illustrates the accuracy of the proposed methodology with SMAPE values smaller than 5%.

To show the generality of the methodology, i.e., no assumption on the model of nonlinearity is required; and only the location of the nonlinearity must be known, a nonlinear damping term is introduced in the form $f_{nl} = c_{nl}\dot{q}^2$, with c_{nl} = 1e6 Ns^2/m^2 on the eighth *dof* of the simulation model. Figure 7 shows that the proposed methodology is also capable of accurately estimating the states with a SMAPE value smaller that 4%.

5. CONCLUSION

In this paper, it has been shown that the state estimation related to a nonlinear system can be handled by the classical Kalman filter through an output injection approach provided the nonlinearity is local. Even if the gain seems rather low compared to applying directly Kalman filtering, it has to be evaluated how a small error in state estimates can lead to non trivial errors in subsequent works such as model estimation in related applications in such nonlinear setting. Further works will include this evaluation together with the joint parameter/state estimation when the parameters are varying and the nonlinearity position is unknown, for both numerical and real applications.

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