



HAL
open science

Stability of Singular Periodic Motions in a Vibro-Impact Oscillator

Olivier Janin, Claude-Henri Lamarque

► **To cite this version:**

Olivier Janin, Claude-Henri Lamarque. Stability of Singular Periodic Motions in a Vibro-Impact Oscillator. *Nonlinear Dynamics*, 2002, 28 (3-4), pp.231-241. 10.1023/A:1015632510298 . hal-04765236

HAL Id: hal-04765236

<https://hal.science/hal-04765236v1>

Submitted on 4 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Stability of Singular Periodic Motions in a Vibro-Impact Oscillator

O. JANIN and C. H. LAMARQUE

*Ecole Nationale des Travaux Publics de l'Etat, Laboratoire GéoMatériaux, URA-CNRS 1652, Rue Maurice
Audin, 69518 Vaulx-En-Velin Cedex, France*

Abstract. A single-degree-of-freedom vibro-impact oscillator is considered. For some values of parameters, a non-differentiable fixed point of the Poincaré map exists: a local expansion of the Poincaré map around such a point is given, including a square root term on the impact side. From this approximate map, the stability of the fixed point can be investigated, and it is shown that the periodic solution is stable when the Floquet multipliers are real.

Keywords: Impact, Poincaré map, stability.

1. Introduction

Impacts in mechanical systems have been the subject of a great number of publications, on the one hand because they often occur in the models of mechanical systems for applications, and on the other hand because they give rise to complex and unusual behaviours from the theoretical point of view. General considerations about mechanical systems with impacts have been developed in [1–3], and many papers have been devoted to the study of global behaviours in vibro-impact systems, including analytical studies, for example, in [4–9]. Some authors have also considered the issue of controlling systems where impacts take place (see [10–12]). The main feature of impacts, in the way that they are commonly taken into account, is that they lead to discontinuities, unless a nonlinear change of variables is performed as defined in [13]. Therefore, new behaviours can be exhibited for examples from the point of view of bifurcations, as in [13–15]. Moreover, singularities arise because of the occurrence of discontinuities, as described in [16, 17]. Some authors have also been interested in mathematical questions, including the development of numerical schemes adapted to systems with impacts, for instance, in [18, 19].

In this paper, we focus on the singularities in the Poincaré map of a single-degree-of-freedom vibro-impact oscillator: our aim is to complete the general work of Nordmark in [17], for which the calculations have not been given in details, as it have been underlined by Molenaar et al. [20]. In Section 2, we analytically write the Poincaré map corresponding to the vibro-impact oscillator considered, and from this expression we derive a local expansion of the map around a non-differentiable fixed point, for which we must distinguish between two cases. This approximate map enables to investigate the stability of the fixed point in Section 3, and this study is completed by numerical experiments in Section 4 that confirm the theoretical results of the preceding sections. Finally we conclude about the behaviour of solutions in the neighborhood of a non-differentiable fixed point of the Poincaré map in Section 5.

2. Local Mapping of the Vibro-Impact Oscillator

A single-degree-of-freedom vibro-impact oscillator is considered in this paper. The equation of motion is given by

$$\begin{aligned} \ddot{x}(t) + 2\varepsilon\omega_1\dot{x}(t) + \omega_1^2x(t) &= f \cos(\omega t), \\ x(t) &\leq x_{\max}, \\ x(t) = x_{\max} &\Rightarrow \dot{x}(t^+) = -e\dot{x}(t^-), \end{aligned} \quad (1)$$

where $e \in [0, 1]$ and $\varepsilon \in]0, 1]$.

In order to study the dynamics of this system, a Poincaré map can be built in $\mathbb{R}^2 \times \mathbb{S}^1$ by mapping a point (x_0, \dot{x}_0, t_0) to $(x(t_0 + T), \dot{x}(t_0 + T), t_0 + T)$ where x is the solution starting from (x_0, \dot{x}_0) at t_0 and T is the period of the external forcing. Such a Poincaré map exhibits particular fixed points for which the solution is tangent to the obstacle, and consequently the map is not differentiable at these points. We are going to focus on non-differentiable fixed points (x_0^0, \dot{x}_0^0) with one impact with zero velocity at t^0 in $[0, T]$.

We first perform a linear transformation of the co-ordinates defined by

$$\begin{aligned} u &= e^{-\varepsilon\omega_1 t^0} \left((x_0 - x_0^0)[\cos(\tilde{\omega}_1 t^0) + \eta \sin(\tilde{\omega}_1 t^0)] + (\dot{x}_0 - \dot{x}_0^0) \frac{\sin(\tilde{\omega}_1 t^0)}{\tilde{\omega}_1} \right) \\ v &= -e^{-\varepsilon\omega_1 t^0} \left(\tilde{\omega}_1(1 + \eta^2) \sin(\tilde{\omega}_1 t^0)(x_0 - x_0^0) \right. \\ &\quad \left. + (\dot{x}_0 - \dot{x}_0^0)[- \cos(\tilde{\omega}_1 t^0) + \eta \sin(\tilde{\omega}_1 t^0)] \right), \end{aligned} \quad (2)$$

where $\tilde{\omega}_1 = \omega_1 \sqrt{1 - \varepsilon^2}$ and $\eta = \varepsilon(\omega_1 / \tilde{\omega}_1)$. In these new variables, the Poincaré map reads as

$$\mathbf{\Pi}(u, v) = e^{-\varepsilon\omega_1 T} \left[\mathbf{M} \begin{pmatrix} u \\ v \end{pmatrix} + (1 + e)\varphi(u, v)\mathbf{\Lambda}(u, v) \right], \quad (3)$$

where

$$\mathbf{M} = \begin{pmatrix} \cos(\tilde{\omega}_1 T) + \eta \sin(\tilde{\omega}_1 T) & \frac{\sin(\tilde{\omega}_1 T)}{\tilde{\omega}_1} \\ -\tilde{\omega}_1(1 + \eta^2) \sin(\tilde{\omega}_1 T) & \cos(\tilde{\omega}_1 T) - \eta \sin(\tilde{\omega}_1 T) \end{pmatrix}$$

and φ is zero if no impact occurs. If there is one impact at $t^0 + \tau \in [0, T]$, then we have

$$\begin{aligned} \varphi(u, v) &= \tilde{\omega}_1(1 + \eta^2) \sin(\tilde{\omega}_1 \tau)u + [- \cos(\tilde{\omega}_1 \tau) + \eta \sin(\tilde{\omega}_1 \tau)]v \\ &\quad + \omega x_{\max} e^{\varepsilon\omega_1 \tau} \sin(\omega \tau), \end{aligned} \quad (4)$$

$$\mathbf{\Lambda} = \begin{pmatrix} \sin[\tilde{\omega}_1(T - \tau)]/\tilde{\omega}_1 \\ \cos[\tilde{\omega}_1(T - \tau)] - \eta \sin[\tilde{\omega}_1(T - \tau)] \end{pmatrix}. \quad (5)$$

2.1. APPROXIMATE MAP IN THE GENERAL CASE

When restricting to the second order, the impact condition becomes

$$P(u, v) = v^2 + 2\omega_1^2 u^2 + 4\varepsilon\omega_1 uv + 2\omega^2 x_{\max} u \geq 0, \quad (6)$$

and in that case

$$\tau(u, v) = \frac{v - \sqrt{P(u, v)}}{\omega_1^2 u + 2\varepsilon\omega_1 v + \omega^2 x_{\max}}. \quad (7)$$

Following the procedure explained in [17], we can derive a local expansion of Π by means of a nonlinear transformation of the co-ordinates. By setting

$$\chi = -P(u, v) \quad (8)$$

and

$$\begin{aligned} \psi = & \omega_1^2 [2 - (1 + \eta^2) \sin^2(\tilde{\omega}_1 T)] u^2 + (2 - [\cos(\tilde{\omega}_1 T) + \eta \sin(\tilde{\omega}_1 T)]^2) v^2 \\ & + 2\tilde{\omega}_1 (2\eta - (1 + \eta^2) \sin(\tilde{\omega}_1 T) [\cos(\tilde{\omega}_1 T) + \eta \sin(\tilde{\omega}_1 T)]) uv \\ & + 2e^{-\varepsilon\omega_1 T} \omega^2 x_{\max} \left([\cos(\tilde{\omega}_1 T) - \eta \sin(\tilde{\omega}_1 T)] u - \frac{\sin(\tilde{\omega}_1 T)}{\tilde{\omega}_1} v \right), \end{aligned} \quad (9)$$

we obtain a change of variables valid if and only if $\sin(\tilde{\omega}_1 T) \neq 0$.

When this condition is verified by the parameters of the system, and if we only keep the terms of order at most one, the local Poincaré map expressed in the new co-ordinates is given by

$$\left. \begin{aligned} \Pi_1(\chi, \psi) &= (\lambda_1 + \lambda_2)\chi + \psi \\ \Pi_2(\chi, \psi) &= -\lambda_1\lambda_2\chi \end{aligned} \right\} \quad \text{if } \chi \geq 0, \quad (10)$$

and

$$\left. \begin{aligned} \Pi_1(\chi, \psi) &= B\sqrt{-\chi} + \alpha\chi + \psi \\ \Pi_2(\chi, \psi) &= -e^2\lambda_1\lambda_2\chi \end{aligned} \right\} \quad \text{if } \chi < 0, \quad (11)$$

where $\lambda_1 = e^{-T(\varepsilon\omega_1 + i\tilde{\omega}_1)}$ and $\lambda_2 = e^{-T(\varepsilon\omega_1 - i\tilde{\omega}_1)}$ are the eigenvalues of M , and B and α are given by:

$$B = 2(1 + e)\omega^2 x_{\max} e^{-\varepsilon\omega_1 T} \frac{\sin(\tilde{\omega}_1 T)}{\tilde{\omega}_1}, \quad (12)$$

$$\begin{aligned} \alpha = & 2e^{-\varepsilon\omega_1 T} [-e \cos(\tilde{\omega}_1 T) + (1 + e)\eta \sin(\tilde{\omega}_1 T)] \\ & + (1 + e)^2 e^{-2\varepsilon\omega_1 T} (2 - [\cos(\tilde{\omega}_1 T) - \eta \sin(\tilde{\omega}_1 T)]^2). \end{aligned} \quad (13)$$

Comparing with what is given in [17], one can notice a difference in the map on the impact side, first because of the presence of a term $\alpha\chi$, and also because of the sign of the coefficient B of the square root, which can vary as it has already been pointed out in [20].

2.2. STUDY OF A PARTICULAR CASE

In the case when $\sin(\tilde{\omega}_1 T) = 0$, it is impossible to obtain the same kind of term in $\sqrt{-\chi}$ by means of a valid nonlinear transformation of the co-ordinates. The Poincaré map can then be approximated from the expressions given in (3) to obtain:

$$\left\{ \begin{aligned} \Pi_1(u, v) &= \varsigma e^{-\varepsilon\omega_1 T} \left(-(1 + 2e)u + (1 + e)v \frac{\sqrt{v^2 + 2vu - v}}{v} \right) \\ \Pi_2(u, v) &= \varsigma e^{-\varepsilon\omega_1 T} (v - (1 + e)\sqrt{v^2 + 2vu}) \end{aligned} \right. \quad \text{if } v^2 + 2vu > 0, \quad (14)$$

and

$$\begin{cases} \Pi_1(u, v) = \zeta e^{-\varepsilon\omega_1 T} u \\ \Pi_2(u, v) = \zeta e^{-\varepsilon\omega_1 T} v \end{cases} \quad \text{if } v^2 + 2\nu u \leq 0, \quad (15)$$

where $\nu = \omega^2 x_{\max}$ and $\zeta = \cos(\tilde{\omega}_1 T) \in \{-1, 1\}$. This local mapping is continuous.

3. Stability of the Non-Differentiable Fixed Point

In [17], it is argued that the fixed point is stable if and only if the eigenvalues λ_1 and λ_2 are both real and positive. Nevertheless, this is not true for the case that we are studying.

3.1. GENERAL CASE

In the case of Section 2.1, namely when $\sin(\tilde{\omega}_1 T) \neq 0$, the eigenvalues are complex and a linear change of variables is convenient to study the stability. If we set

$$x = \chi \quad \text{and} \quad y = \frac{\cos(\tilde{\omega}_1 T)\chi + e^{\varepsilon\omega_1 T}\psi}{\sin(\tilde{\omega}_1 T)},$$

the impact condition remains unchanged ($x < 0$) and on the non-impact side the mapping becomes

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = e^{-\varepsilon\omega_1 T} \mathcal{R} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad (16)$$

where \mathcal{R} is the rotation of angle $\tilde{\omega}_1 T$. Therefore, any point entering the non-impact area will eventually leave it after a finite number of iterations of the local mapping. The local Poincaré map consists then of a rotation combined with a strict linear contraction on the non-impact side, and a mapping with a square root term that yields infinite stretching on the impact side. These local properties show that the fixed point is unstable when $\sin(\tilde{\omega}_1 T) \neq 0$, i.e. when the eigenvalues λ_1 and λ_2 are complex.

3.2. PARTICULAR CASE

In the case of Section 2.2, namely when $\sin(\tilde{\omega}_1 T) = 0$, the eigenvalues are real. The local Poincaré map is then given by (14) and (15): the behaviour of this map depends on the value of ζ . As it can be seen from (14), for (u, v) close to $(0, 0)$ on the impact side, we have $\Pi_1(u, v) \sim \alpha_1 u$ and $\Pi_2(u, v) \sim \alpha_2 \sqrt{u}$ with $\alpha_2 > 0$. Hence it is sufficient to study the local mapping on the impact side for initial conditions such that $u = \lambda v^2$ with $1 + 2\nu\lambda > 0$.

3.2.1. Negative Eigenvalues

If we consider any initial condition (u, v) close to $(0, 0)$ and such that $v^2 + 2\nu u \leq 0$, the first iterate leads to $\Pi_2^2 - 2\nu\Pi_1 = e^{-2\varepsilon\omega_1 T} v^2 - 2\nu u e^{-\varepsilon\omega_1 T}$ which is positive since by assumption $-2\nu u \geq v^2$. The point (u, v) is then mapped into the impact side.

Let us now deal with an initial condition $(\lambda v^2, v)$ verifying $1 + 2\nu\lambda > 0$ and $v > 0$. At the first step, we obtain $\Pi_1(u, v) = f_{-1}(\lambda)\Pi_2(u, v)^2$ with

$$f_{-1}(\lambda) = e^{\varepsilon\omega_1 T} \frac{\lambda(1 + 2e) + (1 + e) \frac{1 - \sqrt{1 + 2\nu\lambda}}{v}}{[1 - (1 + e)\sqrt{1 + 2\nu\lambda}]^2}. \quad (17)$$

The behaviour of the iterates of Π is thus governed by the sequence $x_{n+1} = f_{-1}(x_n)$ with $x_0 = \lambda$. The function f_{-1} is defined by (17) on

$$\left[\frac{-1}{2\nu}, \frac{-1}{2\nu} + \frac{1}{2\nu(1+e)^2} \right] \cup \left[\frac{-1}{2\nu} + \frac{1}{2\nu(1+e)^2}, +\infty \right],$$

and when $\lambda < -1/2\nu$ it is given by

$$f_{-1}(\lambda) = -e^{\varepsilon\omega_1 T} \lambda. \quad (18)$$

This function admits three fixed points

$$\mu_0 \in \left] \frac{-1}{2\nu}, \frac{-1}{2\nu} + \frac{1}{2\nu(1+e)^2} \right[, \quad 0 \quad \text{and} \quad \lambda_0 \in]0, +\infty[,$$

and it can be analytically shown that $f'_{-1}(\mu_0) < -1$, $f'_{-1}(0) > 1$ and $|f'_{-1}(\lambda_0)| < 1$ (see the Appendix): the only stable fixed point of f_{-1} is λ_0 . An analytical study of the iterative sequence is possible to determine its asymptotic behaviour, depending on the initial condition λ . Four cases have to be distinguished:

- if $\lambda > 0$, $(x_n)_{n \geq 0}$ converges to λ_0 .
- if $\lambda = 0$, $(x_n)_{n \geq 0}$ is constant and equal to 0.
- if

$$\lambda \leq \frac{-1}{2\nu} + \frac{1}{2\nu(1+2e)^2},$$

- $x_1 > 0$ so the behaviour of $(x_n)_{n \geq 0}$ is settled by the previous cases.
- if

$$\lambda = \frac{-1}{2\nu} + \frac{1}{2\nu(1+e)^2},$$

- f_{-1} is not defined at λ but by considering directly Π_1 and Π_2 in that case, it can be seen that after two iterations the point $(\lambda\nu^2, \nu)$ is mapped into the area where $\lambda < -1/2\nu$, and thus the behaviour of the sequence $(x_n)_{n \geq 0}$ is the same as the preceding case.
- if

$$\lambda \in \left] \frac{-1}{2\nu} + \frac{1}{2\nu(1+e)^2}, 0 \right[- \left\{ \frac{-1}{2\nu} + \frac{1}{2\nu(1+e)^2} \right\},$$

$(x_n)_{n \geq 0}$ converges to λ_0 or is constantly equal to 0 for n sufficiently large, unless λ lies in a set K . This set is defined by $K = \bigcap_{n=0}^{+\infty} I_n$, where

$$I_0 = \left] \frac{-1}{2\nu} + \frac{1}{2\nu(1+2e)^2}, 0 \right[- \left\{ \frac{-1}{2\nu} + \frac{1}{2\nu(1+e)^2} \right\}$$

and $I_{n+1} = f_{-1}^{-1}(I_n)$.

Taking into account the properties of the sequence $(x_n)_{n \geq 0}$, the stability of the fixed point can be investigated. First of all, the n th iterate of the local mapping is given by

$$\Pi^{on}(u, v) = \begin{pmatrix} \alpha_n v^2 \\ \beta_n v \end{pmatrix}, \quad (19)$$

where

$$\alpha_n = \beta_n^2 f^{on}(\lambda), \quad \beta_n = \prod_{i=0}^n g(x_i) \quad \text{and} \quad g(x) = e^{-\varepsilon \omega_1 T} \left((1+e)\sqrt{1+2vx} - 1 \right).$$

For every λ such that $\lim_{n \rightarrow +\infty} x_n = 0$, since $g(0) = e e^{-\varepsilon \omega_1 T} \in]0, 1[$, it is clear that $\lim_{n \rightarrow +\infty} \beta_n = 0$. Moreover, it can be shown that $g(\lambda_0) \in]0, 1[$ (see the Appendix for further details). Hence for every λ such that $\lim_{n \rightarrow +\infty} x_n = \lambda_0$, we infer that $\lim_{n \rightarrow +\infty} \beta_n = 0$, and then that $\lim_{n \rightarrow +\infty} \alpha_n = 0$ as well. Thus we have proved that $\lim_{n \rightarrow +\infty} \Pi^{on}(u, v) = (0, 0)$ if $\lambda \notin K$.

If $\lambda \in K$, the asymptotic behaviour of the sequence $(x_n)_{n \geq 0}$ cannot be exactly determined. Nevertheless, we know that in that case

$$x_n \in \left] \frac{-1}{2v} + \frac{1}{2v(1+2e)^2}, 0 \right[, \quad \forall n,$$

which implies

$$g(x_n) \in \left] -\frac{e}{1+2e} e^{-\varepsilon \omega_1 T}, e e^{-\varepsilon \omega_1 T} \right[,$$

and then $|g(x_n)| < k, \forall n \geq 0$ with $k = e e^{-\varepsilon \omega_1 T} < 1$. Therefore, the sequence $(\beta_n)_{n \geq 0}$ verifies $|\beta_n| \leq k^{n+1}$ which converges to 0. Thus in that case, the iterates of Π still converge to $(0, 0)$.

The preceding results show that whatever the initial condition (u, v) considered, the local map asymptotically tends to $(0, 0)$, and consequently that the non-differentiable fixed point is stable.

3.2.2. Positive Eigenvalues

If we consider any initial condition (u, v) close to $(0, 0)$ and such that $v^2 + 2vu \leq 0$, then the iterates of Π will obviously converge to $(0, 0)$, for on the non-impact side the mapping is a strict contraction with positive ratio.

Let us now deal with an initial condition $(\lambda v^2, v)$ verifying $1 + 2v\lambda > 0$ and $v < 0$. At the first iteration, we obtain $\Pi_2^2(\lambda v^2, v) < 0$ and $\Pi_1(\lambda v^2, v) = f_1(\lambda) \Pi_2^2(\lambda v^2, v)$, where

$$f_1(\lambda) = -e^{\varepsilon \omega_1 T} \frac{\lambda(1+2e) + (1+e) \frac{1+\sqrt{1+2v\lambda}}{v}}{[1 + (1+e)\sqrt{1+2v\lambda}]^2}. \quad (20)$$

The behaviour of the iterates of Π is thus governed by the sequence $x_{n+1} = f_1(x_n)$ with $x_0 = \lambda$. The function f_1 is defined by (20) on $[-1/2v, +\infty[$, and f_1 is increasing on this interval. To determine the fixed points of f_1 , the function $h_1(\lambda) = f_1(\lambda) - \lambda$ can be studied: we can show that this function admits a maximum at $\lambda_0 \in]-1/2v, +\infty[$ if $e^2 e^{\varepsilon \omega_1 T} \geq 1$ and at $-1/2v$ otherwise, and that the maximum is always negative. Therefore, f_1 admits no fixed points, and the sequence $(x_n)_{n \geq 0}$ is decreasing as long as it stays inside $[-1/2v, +\infty[$. It is then clear that for n sufficiently large we have $x_n < -1/2v$, which implies that the iterate of

Π has entered the non-impact side. According to what has been shown before, the iterates of Π converge to $(0, 0)$. The non-differentiable fixed is hence stable in that case.

4. Numerical Aspects

In this section, we are going to provide numerical evidence for the existence of a stable non-differentiable fixed point. First of all, such a fixed point only exists in a special range of parameters. The conditions of existence are given by

$$\begin{cases} \varepsilon \in \left[0, \frac{1}{\sqrt{2}}\right], \\ 2\varepsilon\sqrt{1-\varepsilon^2}|x_{\max}|\omega_1^2 \leq f \leq |x_{\max}|\omega_1^2. \end{cases} \quad (21)$$

In this range of parameters, the eigenvalues λ_1 and λ_2 are real for a countable infinite number of values of f given by

$$f_k = |x_{\max}|\omega_1^2 \sqrt{4\varepsilon^2(1-\varepsilon^2) + \left[\left(1 - \frac{4}{k^2}\right)(1-\varepsilon^2) - \varepsilon^2\right]^2}, \quad \forall k \in \mathbb{N}^*. \quad (22)$$

For these values, we have $\sin(\tilde{\omega}_1 T) = 0$ and $\cos(\tilde{\omega}_1 T) = (-1)^k$, and ω is given by $\omega = (2\omega_1\sqrt{1-\varepsilon^2})/k$. The study carried out in Section 3.2 shows that the fixed point is stable in that case.

For any other value of f , the eigenvalues λ_1 and λ_2 are complex, hence the non-differentiable fixed point is unstable according to Section 3.1.

In order to verify and to complete the theoretical study of stability of Section 3.2, we plotted in Figures 2 and 1 the basins of attraction for $\omega_1 = 2.5$, $e = 0.9$, $x_{\max} = 14$, $\varepsilon = 1\%$ and $k \in \{3, 5\}$. These basins were computed using the exact analytical writing of the solution, and the brute force method to determine the nature of the solution. Figure 2 shows that two stable solutions exist for this set of parameters and $k = 5$: the T -periodic solution with one impact per cycle (in white), corresponding to the non-differentiable fixed point, and a T -periodic solution with two impacts per cycle (in black). When $k = 3$, it can be seen in Figure 1 that the T -periodic solution with one impact per cycle (in white) is stable as the theory predicted, but its basin of attraction is tiny: most of the initial conditions lead to chaos (in black).

5. Conclusion

We have investigated the behaviour of the Poincaré map in the neighborhood of a non-differentiable fixed point, for the classical single-degree-of freedom harmonic oscillator with impacts. We have shown that when the eigenvalues of the Poincaré map without impacts are complex, the local mapping on the impact side includes a square root term in addition to the usual linear terms: in that case, the non-differentiable fixed point occurs to be unstable. Furthermore, when the eigenvalues are real, we have seen that the local mapping could not be expressed in a same kind of form, but that square root terms were still present. The study of stability proved that the non-differentiable fixed point was stable in that case, with two different behaviours according to the sign of these eigenvalues: if they are positive, the solution is stable by remaining in the non-impact side after some iterations whereas when they are negative, the stability comes from the impact side (in spite of the presence of a square root term).

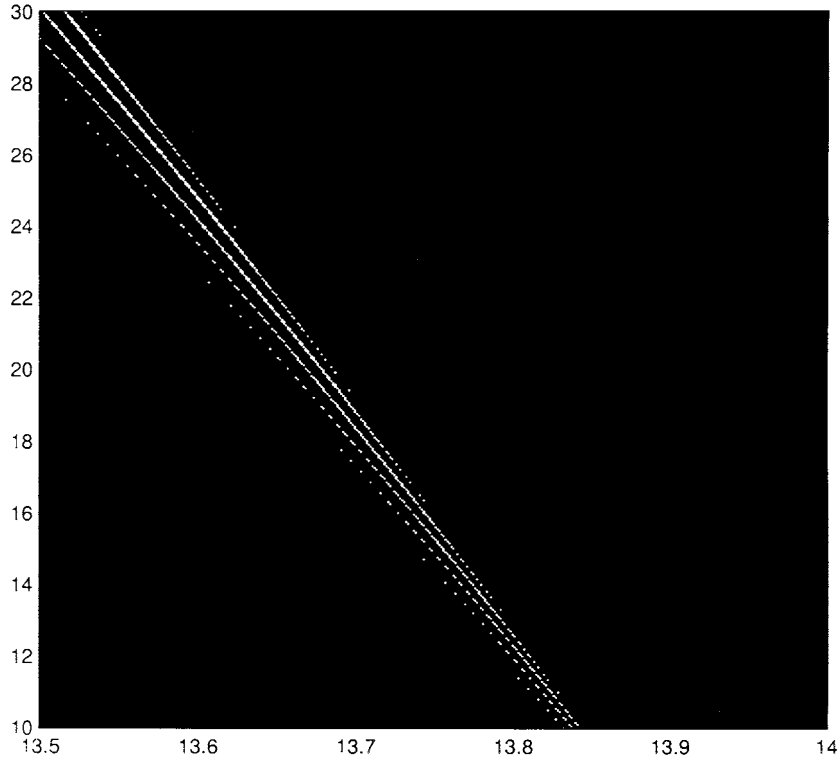


Figure 1. Basin of attraction for $k = 3$.

An extension to this work would be the local study of a general single-degree-of-freedom mechanical system with impacts, to determine whether the cases of stability are peculiar to the harmonic linear oscillator or not.

Appendix

The fixed points of f_{-1} (defined in (17)) can be analytically determined by setting $\lambda = (X^2 - 1)/2\nu$ with $X > 0$: they correspond to the positive roots of the polynomial:

$$P(X) = (1 + e)^2 X^3 + (e^2 - 1)X^2 - (1 + 2e)(1 + e^{\varepsilon\omega_1 T})X + 1 + e^{\varepsilon\omega_1 T}. \quad (23)$$

Moreover, with the same notations, f'_{-1} is given by

$$f'_{-1}(\lambda) = -\frac{e^2 e^{\varepsilon\omega_1 T}}{[1 - (1 + e)X]^3}. \quad (24)$$

For $\lambda = 0$, we have $X = 1$ and therefore $f'_{-1}(0) = (e^{\varepsilon\omega_1 T})/e > 1$.

Because μ_0 corresponds to $X_\mu \in]0, 1/(1 + e)[$, $f'_{-1}(\mu_0) + 1$ has the same sign as $X_0 - X_\mu$ where

$$X_0 = \frac{1 - e^{\varepsilon\omega_1 T/3} e^{2/3}}{1 + e}.$$

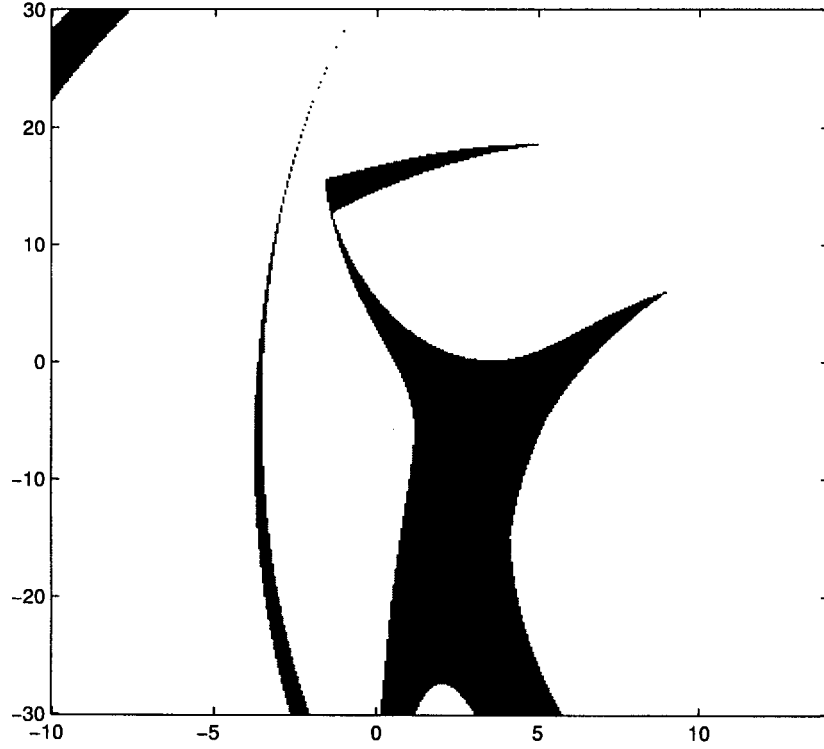


Figure 2. Basin of attraction for $k = 5$.

If X_0 is positive, we have $X_0 \in [0, 1/(1 + e)]$, and since P is decreasing in that interval, $X_\mu - X_0$ has the same sign as $P(X_0)$. After some calculations, we obtain

$$P(X_0) = \frac{e^{2/3} e^{2\varepsilon\omega_1 T/3}}{1 + e} [e^{5/3} + 2e^{2/3} + e e^{2\varepsilon\omega_1 T/3} + (1 + e) e^{\varepsilon\omega_1 T/3} (e^{\varepsilon\omega_1 T/3} - e^{1/3})] > 0. \quad (25)$$

Hence we infer that $f'_{-1}(\mu_0) < -1$.

Furthermore, λ_0 corresponds to $X_\lambda > 1$. Therefore, we have $f'_{-1}(\lambda_0) > 0$ and $f'_{-1}(\lambda_0) - 1$ has the same sign as $X_1 - X_\lambda$ where

$$X_1 = \frac{1 + e^{\varepsilon\omega_1 T/3} e^{2/3}}{1 + e}.$$

If X_1 is greater than 1, then $X_1 - X_\lambda$ has the same sign as $P(X_1)$ because P is increasing on $[1, +\infty[$. Analytical calculations yield

$$P(X_1) = \frac{e^{2/3} e^{2\varepsilon\omega_1 T/3}}{1 + e} [(e^{\varepsilon\omega_1 T/3} + e^{1/3})(e^{4/3} - e^{\varepsilon\omega_1 T/3}) + 2e^{2/3}(1 - e^{1/3} e^{2\varepsilon\omega_1 T/3})]. \quad (26)$$

By using the assumption $X_1 \geq 1$, we obtain

$$P(X_1) \leq \frac{e e^{2\varepsilon\omega_1 T/3}}{1 + e} (e - 1)(e^{\varepsilon\omega_1 T/3} - e^{1/3}) < 0. \quad (27)$$

Hence we have shown that $f'_{-1}(\lambda_0) \in]0, 1[$.

240 *O. Janin and C.H. Lamarque*

Finally, we have $g(\lambda_0) = e^{-\varepsilon\omega_1 T}[(1+e)X_\lambda - 1] > 0$, and $1 - g(\lambda_0)$ has the same sign as $X_2 - X_\lambda$ where

$$X_2 = \frac{1 + e^{\varepsilon\omega_1 T}}{1 + e}.$$

Since $X_2 > 1$, $X_2 > X_\lambda$ if and only if $P(X_2) > 0$. Yet we have:

$$P(X_2) = \frac{e^{\varepsilon\omega_1 T}(1 + e^{\varepsilon\omega_1 T})(e^{\varepsilon\omega_1 T} - e)}{1 + e} > 0. \quad (28)$$

Therefore, the fixed point λ_0 verifies $g(\lambda_0) \in]0, 1[$.

References

1. Brogliato, B., *Nonsmooth Impact Mechanics*, Lecture Notes in Control and Information Sciences, Vol. 220, Springer-Verlag, London, 1996.
2. Pfeiffer, F. and Glocker, C., *Multibody Dynamics with Unilateral Contacts*, Wiley, New York, 1996.
3. Babitsky, V., *Theory of Vibro-Impact Systems and Applications*, Springer-Verlag, Berlin, 1998. (Revised translation from Russian, Nauka, Moscow, 1978.)
4. Peterka, F., 'Laws of impact motion of mechanical systems with one degree of freedom. Part II – Results of analogue computer modelling of the motion', *Acta Technica CSAV* **5** 1974, 569–580.
5. Shaw, S. and Rand, R., 'The transition to chaos in a simple mechanical system', *International Journal of Non-Linear Mechanics* **24**(1), 1989, 41–56.
6. Peterka, F. and Vacik, J., 'Transition to chaotic motion in mechanical systems with impacts', *Journal of Sound and Vibration* **154**(1), 1992, 95–115.
7. Whiston, G., 'Global dynamics of a vibro-impacting linear oscillator', *Journal of Sound and Vibration* **18**(3), 1987, 395–429.
8. Lamarque, C. and Janin, C., 'Modal analysis of mechanical systems with impact non-linearities: Obstructions to a modal superposition', *Journal of Sound and Vibration* **235**(4), 2000, 567–609.
9. de Vorst, E. V., van Campen, D., and de Kraker, A., 'Periodic solutions of a multi-DOF beam system with impact', *Journal of Sound and Vibration* **192**, 1996, 913–925.
10. Lenci, S. and Rega, G., 'Controlling nonlinear dynamics in a two-well impact system. I. Attractors and bifurcation scenario under symmetric excitations', *International Journal of Bifurcation and Chaos* **8**(12), 1998, 2387–2407.
11. Lenci, S. and Rega, G., 'Controlling nonlinear dynamics in a two-well impact system. II. Attractors and bifurcation scenario under unsymmetric optimal excitation', *International Journal of Bifurcation and Chaos* **8**(12), 1998, 2409–2424.
12. Awrejcewicz, J., Tomczak, K., and Lamarque, C.-H., 'Controlling systems with impacts', *International Journal of Bifurcation and Chaos* **3**, 1999, 547–553.
13. Ivanov, A., 'Analytical methods in the theory of vibro-impact systems', *Journal of Applied Mathematics and Mechanics* **57**(2), 1993, 221–236.
14. Nordmark, A., 'Effects due to low velocity impact in mechanical oscillators', *International Journal of Bifurcation and Chaos* **2**(3), 1992, 597–605.
15. Lamba, H. and Budd, C., 'Scaling of Lyapunov exponents at nonsmooth bifurcations', *Physical Review E* **50**(1), 1994, 84–90.
16. Whiston, G., 'Singularities in vibro-impact dynamics', *Journal of Sound and Vibration* **152**(3), 1992, 427–460.
17. Nordmark, A., 'Non-periodic motion caused by grazing incidence in an impact oscillator', *Journal of Sound and Vibration* **145**(2), 1991, 279–297.
18. Panet, M., Paoli, L., and Schatzman, M., 'Theoretical and numerical study for a model of vibrations with unilateral constraints', in *Proceedings of Contact Mechanics International Symposium* Vol. 2, 19–23 September, M. Raous, M. Jean, and J. J. Moreau (eds.), Plenum Press, New York, 1995, pp. 457–464.

19. Paoli, L. and Schatzman, M., 'Mouvement à nombre fini de degrés de liberté avec contraintes unilatérales: Cas avec perte d'énergie', *Modèl. Math. Anal. Numér.* **27**, 1993, 673–717.
20. Molenaar, J., de Weger, J., and v.d. Water, W., 'Near-grazing behaviour of the harmonic oscillator', in *Proceedings of the 1999 ASME Design Technical Conferences*, Las Vegas, NV, S. C. Sinha and G. T. Flowers (eds.), ASME, New York, 1999, DETC 99/VIB-8036 (CD Rom).