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Regis Monneau

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Structure of Riemann solvers on networks (preliminary version 2)

R. Monneau*

January 11, 2025

Simplicity is the ultimate improvement.

Abstract

In this paper we consider scalar Riemann solvers on networks, associated to scalar conservation laws. A junction is a particular network which is a finite set of half lines glued together at the origin. Riemann solvers solve uniquely the Riemann problem on the junction. We also assume that Riemann solutions are stable by passage to the limit.

In part I of the paper, we only address fundamental questions concerning Riemann problems on junctions. We show a characterization of Riemann solvers either by their set of stationary solutions (the germ), or equivalently by their Godunov flux at the junction. Moreover, we show that the gluing of two junctions with Riemann solvers is well defined and leads to a new junction with a new Riemann solver. Our theory encompasses in particular Kruřkov germs, Hamilton-Jacobi germs, monotone germs, conservative and non-conservative germs.

In part II of the work, we give an existence and uniqueness theory for conservation laws on networks in the special case where Riemann solvers are associated to Kruřkov germs.

MSC2020: 35R02, 35F30.

Keywords: network, junction, Riemann problem, Riemann solver, conservation laws, discontinuous flux.

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1 Introduction

In part I of this paper, we consider scalar conservation laws on a junction. A junction of type $n : m$ consists in a set of $N := n + m$ half lines (branches) glued together at the origin, with n ingoing branches and m outgoing branches. We consider fluxes $f = (f^1, \dots, f^N)$, with one flux on each branch. We assume that the solution at the origin (the junction point) is given by a Riemann solver. Recall that the Riemann problem consists to solve the problem with initial data which is constant on each branch. By definition, a Riemann solver allows to solve uniquely the Riemann problem on the junction. Moreover, by definition, the germ is the set $\mathcal{G} \subset \mathbb{R}^N$ of stationary solutions for this Riemann problem on the junction. We show that Riemann solvers are characterized by their germ \mathcal{G} . We assume moreover that the Riemann solver is stable, i.e. that the set of solutions to the \mathcal{G} -Riemann problem is closed by passage to the limit.

We show that there is a 1-1 correspondence between any germ \mathcal{G} and its associated Godunov flux $\hat{f}_{\mathcal{G}}$ at the junction, such that $\mathcal{G} = \{\hat{f}_{\mathcal{G}} = f\}$. We also show that the Godunov flux $\hat{f}_{\mathcal{G}}$ enjoys a certain Riemann monotonicity property which implies for instance that $\hat{f}_{\mathcal{G}} + \varepsilon Id$ is injective for all positive ε , where $Id: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the identity. This monotonicity property allows to define the gluing $\mathcal{G}_1 \# \mathcal{G}_2$ of two Riemann germs \mathcal{G}_γ for junctions of type $n_\gamma : m_\gamma$ for $\gamma = 1, 2$. We glue together two branches with the same flux, one outgoing branch from germ \mathcal{G}_1 with one ingoing branch of germ \mathcal{G}_2 . Then the glued set $\mathcal{G}_1 \# \mathcal{G}_2$ is again a Riemann germ and is of type $(n_1 + n_2 - 1) : (m_1 + m_2 - 1)$.

This paper is the result of a project that we started more than ten years ago, strongly inspired by the work of ANDREIANOV, KARLSEN, RISEBRO in [3]. There, the convenient notion of (what we call Kruřkov) germ was introduced in order to describe the transmission condition between two domains. We were also motivated by problems coming from traffic on networks (see for instance the book of GARAVELLO, PICCOLI [12]). Notice that in the present paper we do not address at all the further difficulties that arise when drivers' turning preferences are transported by the traffic itself (see BRESSAN, FANG YU [6]). The notion of (Kruřkov) germs has been recently generalized by MUSCH, FJORDHOLM, RISEBRO in [18] to the case of networks (see also the recent work CARDALIAGUET, FORCADEL, MONNEAU [7] for an application of [18] to Kruřkov germs for traffic). Our reflection reached a certain maturity that allows us to deliver a quite general theory of germs/Riemann solvers, that we develop in the present paper.

Let us call \mathcal{G} -entropy solution any entropy solution with traces in \mathcal{G} at the origin. Then our work opens the door to the following natural question.

Open question: *for which Riemann germ \mathcal{G} , is there existence and/or uniqueness of \mathcal{G} -entropy solutions?*

There are potentially as many open problems as Riemann germs. This question is largely open, and we will try to provide partial answers in future works. For instance, a satisfactory theory can be developed for Hamilton-Jacobi germs (see FORCADEL, MONNEAU [11] and FORCADEL, IMBERT, MONNEAU [29]).

In part II of this paper, we provide an existence and uniqueness theory for PDE solutions associated to germs in the special class of Kruřkov germs. A subclass of Kruřkov germs is the one of monotone Kruřkov germs. For this subclass, we show that a theory of subsolutions/supersolutions is available, with a natural L^1 -comparison principle.

2 Framework and main results

2.1 Preliminaries

2.1.1 Riemann germ on a junction

Let $N \geq 1$. A branch is an oriented half line. We describe a junction of type $m : n$, i.e. with m ingoing branches $(-\infty, 0)$ and n outgoing branches $(0, +\infty)$, where $N = m + n$. Precisely, consider branches $J^k \simeq (0, +\infty)$ or $(-\infty, 0)$ for $k = 1, \dots, N$, and the following junction set

$$J := \{0\} \bigcup_{k=1, \dots, N} J^k$$

with the topology of N branches glued together at the junction point 0. For later use, we also define the orientation of the branch

$$(2.1) \quad -\sigma^j = \begin{cases} -1 & \text{if } J^j \simeq (-\infty, 0) \\ +1 & \text{if } J^j \simeq (0, +\infty) \end{cases}$$

For $a = (a^1, \dots, a^N), b = (b^1, \dots, b^N) \in \bar{\mathbb{R}}^N$ with $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we write $a < b$ if $a^k < b^k$ for all k , and $a \leq b$ if $a^k \leq b^k$ for all k . Then we consider a box $[a, b]$ defined as follows and a vectorial flux function

f satisfying

$$(2.2) \quad \left\{ \begin{array}{l} [a, b] := \prod_{k=1, \dots, N} [a^k, b^k] \subset \mathbb{R}^N, \quad \text{with } a < b \\ \text{the function } f = (f^k)_{k=1, \dots, N}, \text{ s.t.} \end{array} \right\} \quad \left\{ \begin{array}{l} f^k : [a^k, b^k] \rightarrow \mathbb{R} \text{ is locally Lipschitz continuous} \\ \text{and} \\ \text{there exists } \theta^k \in \{\pm 1\} \text{ s.t.} \\ \theta^k f^k(p^k) \rightarrow +\infty \text{ if } |p^k| \rightarrow +\infty \text{ and } p^k \in [a^k, b^k] \quad \text{(coercivity)} \end{array} \right.$$

where we take the convention in the whole paper that for all $c, d \in \bar{\mathbb{R}}$ with $c \leq d$, we set

$$(2.3) \quad \left\{ \begin{array}{l} [c, d] := \mathbb{R} \cap \{x \in \bar{\mathbb{R}}, \quad c \leq x \leq d\} \\ (c, d] := \mathbb{R} \cap \{x \in \bar{\mathbb{R}}, \quad c < x \leq d\} \\ [c, d) := \mathbb{R} \cap \{x \in \bar{\mathbb{R}}, \quad c \leq x < d\} \end{array} \right. \quad \text{(box convention)}$$

We mainly have in mind this convention for c, d finite, but allow more generally convention $c, d \in \{\pm\infty\}$. In particular, coercivity assumption in (2.2) is useful only when $a^k = -\infty$ and/or $b^k = +\infty$.

Even if the local Lipschitz continuity of f seems technical in assumption (2.2), it is very useful, because this insures that the velocity of propagation is always finite. For $p = (p^1, \dots, p^N)$, we will also denote

$$f^k(p) := f^k(p^k)$$

by abuse of notation.

We then consider functions $u^k : [0, +\infty) \times J^k \rightarrow [a^k, b^k]$, with $u^k(t, x)$ solution of scalar conservation laws on the branch J^k

$$(2.4) \quad \left\{ \begin{array}{ll} \partial_t u^k + \partial_x (f^k(u^k)) = 0 & \text{on } (0, +\infty) \times J^k, \quad k = 1, \dots, N \\ u(t, 0) \in \mathcal{G} & \text{for a.e. time } t \in (0, +\infty) \end{array} \right.$$

and where the junction condition satisfied by the trace of $u = (u^1, \dots, u^N)$ on $\{x = 0\}$, is encoded by a given set $\mathcal{G} \subset [a, b]$. We will see later why the trace is well defined in the case of a Riemann problem.

The box $[a, b]$ where the solutions take their values presents the advantage to be preserved by the PDE, under suitable conditions. Moreover, even if the box is equal to \mathbb{R}^N , we will see that under our assumptions, if the initial data is bounded, then there is a general procedure that allows to replace \mathbb{R}^N by a bounded box $[a, b]$ (see Proposition 6.1).

We now want to recall the definition of Kruřkov entropy solutions (see Kruřkov [16]). For $x \in \mathbb{R}$, we set $\text{sign}(x) := 1_{\{x>0\}} - 1_{\{x<0\}}$. We recall that the Kruřkov pairs (entropy/ flux of entropy) for $u = (u^1, \dots, u^N), v = (v^1, \dots, v^N) \in \mathbb{R}^N$ are given by

$$(2.5) \quad \eta^k(u, v) := \eta^k(u^k, v^k) := |u^k - v^k| \quad \text{and} \quad \psi^k(u, v) := \psi^k(u^k, v^k) := \text{sign}(u^k - v^k) \cdot \{f^k(u^k) - f^k(v^k)\}$$

Then we recall the following standard notion.

Definition 2.1 (Kruřkov entropy solution, [16])

We say that $u^k \in L^\infty([0, +\infty) \times J^k; [a^k, b^k])$ is a Kruřkov entropy solution of the first line of (2.4), with initial data $u_0^k \in L^\infty(J^k; [a^k, b^k])$, if for any constant $c = (c^1, \dots, c^N) \in \mathbb{R}^N$, and for any (test) function $0 \leq \varphi^k \in C_c^1([0, +\infty) \times J^k)$, we have

$$\int_{(0, +\infty) \times J^k} \{\eta^k(u, c)\varphi_t^k + \psi^k(u, c)\varphi_x^k\} dt dx + \int_{\{0\} \times J^k} \eta^k(u_0, c)\varphi^k dx \geq 0$$

Definition 2.2 (\mathcal{G} -entropy solution)

Under assumption (2.2), we say that $u = (u^k)_{k=1, \dots, N}$ is a \mathcal{G} -entropy solution, if each component u^k is a Kruřkov entropy solution of the first line of (2.4), and the trace $u(\cdot, 0)$ (when it is defined) satisfies the second line of (2.4).

We introduce the following definition.

Definition 2.3 (\mathcal{G} -Riemann problem)

Assume (2.2). Given any $p = (p^1, \dots, p^N) \in [a, b]$, we say that $u = (u^k)_{k=1, \dots, N}$ is a \mathcal{G} -entropy solution to the \mathcal{G} -Riemann problem with initial data p , if $u^k : [0, +\infty) \times J^k \rightarrow [a^k, b^k]$ is a Kružkov entropy solution of (2.4) such that for all k , we have

$$(2.6) \quad \begin{cases} u^k(t, x) = u^k\left(1, \frac{x}{t}\right) & \text{for all } (t, x) \in (0, +\infty) \times J^k & \text{(0-homogeneity)} \\ u^k = p^k & \text{on } \{0\} \times J^k, & \text{(initial data)} \\ u^k(t, 0) = \hat{p}^k & \text{for a.e. time } t \in (0, +\infty) & \text{(trace at the origin along } J^k) \end{cases}$$

for some $\hat{p} = (\hat{p}^1, \dots, \hat{p}^N) \in \mathcal{G}$. The solution is then denoted by $u = u_{p, \hat{p}}^{\mathcal{G}}$.

Notice that the meaning of the trace is the following

$$\operatorname{ess\,lim}_{J^j \ni x \rightarrow 0} \int_{(0, T)} |u^j(t, x) - \hat{p}^j| dt = 0 \quad \text{for all index } j \text{ and all } T > 0$$

Still, in part I of this paper, we will never have to use such a delicate notion of trace; here the notion will be much more elementar. Indeed, for classical Riemann problem (and then also for our \mathcal{G} -Riemann problem), it is known that each component map $x \mapsto u^k(t, x)$ is monotone (see Lemma 9.1), and then the trace $u^k(t, 0)$ is well-defined without requiring further assumptions on f which are usually required to get strong traces.

We have the following result.

Proposition 2.4 (L^1 estimate for Riemann problem)

Assume (2.2) and let $p \in [a, b]$ and some set $\mathcal{G} \subset [a, b]$ such that there exists $\hat{p} \in \mathcal{G}$ and a \mathcal{G} -entropy solution $u := u_{p, \hat{p}}^{\mathcal{G}}$. Then for all $t \geq 0$ and $j = 1, \dots, N$, we have

$$(2.7) \quad \int_{\{t\} \times J^j} (u^j - p^j) = t \{f^j(\hat{p}^j) - f^j(p^j)\} \quad \text{and} \quad \int_{\{t\} \times J^j} |u^j - p^j| = t |f^j(\hat{p}^j) - f^j(p^j)|$$

Definition 2.5 (Generalized Riemann germ, Godunov flux, Riemann germ)

Assume (2.2).

i) (Generalized Riemann germ and the projection $\pi_{\mathcal{G}}$)

A set $\mathcal{G} \subset [a, b]$ is called a generalized Riemann germ (with respect to (J, f)) if for any initial data $p \in [a, b]$, there exists a unique \mathcal{G} -entropy solution $u_{p, \hat{p}}^{\mathcal{G}}$ to \mathcal{G} -Riemann problem (2.6) with some trace $\hat{p} \in \mathcal{G}$ at $x = 0$. Then it defines a map $\pi := \pi_{\mathcal{G}}$

$$\begin{aligned} \pi_{\mathcal{G}} : [a, b] &\rightarrow \mathcal{G} \\ p &\mapsto \pi(p) := \hat{p} \end{aligned}$$

which is a nonlinear projection, i.e. satisfies $\pi \circ \pi = \pi$.

ii) (Godunov flux)

Given a generalized Riemann germ \mathcal{G} , we define the Godunov flux at the junction $\hat{f} := \hat{f}_{\mathcal{G}}$ as

$$\begin{aligned} \hat{f}_{\mathcal{G}} : [a, b] &\rightarrow \mathbb{R}^N \\ p &\mapsto \hat{f}_{\mathcal{G}}(p) := (f \circ \pi_{\mathcal{G}})(p) \end{aligned}$$

iii) (Stability and Riemann germ)

Given a generalized Riemann germ \mathcal{G} , we say that \mathcal{G} -Riemann problem (2.6) is stable if for any sequence $(p_n)_{n \in \mathbb{N}}$ with $[a, b] \ni p_n \rightarrow p_{\infty} \in [a, b]$, we have

$$u_{p_n, \pi(p_n)}^{\mathcal{G}} \rightarrow u_{p_{\infty}, \pi(p_{\infty})}^{\mathcal{G}} \quad \text{in } L_{loc}^1([0, +\infty) \times J) \quad \text{as } n \rightarrow +\infty$$

By abuse of terminology, we will say that \mathcal{G} is stable. In such a case, \mathcal{G} is called a Riemann germ (with respect to (J, f)).

Remark 2.6 The (generalized) Riemann solver is the map $p \mapsto u_{p, \pi(p)}^{\mathcal{G}}$. By abuse of terminology (in the literature), the map π itself is sometimes also called a (generalized) Riemann solver.

The Godunov flux at the junction $\hat{f}_{\mathcal{G}}$ has to be distinguished from the standard Godunov flux G^{f^j} .

Definition 2.7 (Standard Godunov flux)

Assume (2.2). The standard Godunov flux associated to the flux f^j is given by

$$G^j(p^j, q^j) := G^{f^j}(p^j, q^j) := \begin{cases} \inf_{[p^j, q^j]} f^j & \text{if } p^j \leq q^j \\ \sup_{[q^j, p^j]} f^j & \text{if } p^j \geq q^j \end{cases}$$

with monotonicities (not in the strict sense) indicated by the arrows $G^j(\uparrow, \downarrow)$.

We introduce the following.

Definition 2.8 (Subclasses of germs)

Assume (2.2). Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ.

i) (Kruřkov germs = D-germs)

We say that \mathcal{G} is a Kruřkov germ (also called a D-germ), if it satisfies

$$D^f(p, q) \geq 0 \quad \text{for all } p, q \in \mathcal{G}$$

where the dissipation is defined by

$$(2.8) \quad D^f(p, q) := \sum_{k=1, \dots, N} D^{f^k}(p, q) = IN - OUT \quad \text{with} \quad D^{f^k}(p, q) := \sigma^k \cdot \text{sign}(p^k - q^k) \cdot \{f^k(p^k) - f^k(q^k)\}$$

(which is a Kruřkov entropy production at the junction point).

i') (Monotone Kruřkov germs = D_+ -germs)

We say that \mathcal{G} is a D_+ -germ, if it satisfies

$$D_+^f(p, q) \geq 0 \quad \text{for all } p, q \in \mathcal{G}$$

where the semi-dissipation is defined by

$$(2.9) \quad D_+^f(p, q) := \sum_{k=1, \dots, N} D_+^{f^k}(p, q) = IN - OUT \quad \text{with} \quad D_+^{f^k}(p, q) := \sigma^k \cdot \text{sign}^+(p^k - q^k) \cdot \{f^k(p^k) - f^k(q^k)\}$$

(which is a Kruřkov semi-entropy production at the junction point).

ii) (HJ germs)

We say that \mathcal{G} is a Hamilton-Jacobi germ (HJ germ for short) if there exists a scalar function $h : \mathcal{G} \rightarrow \mathbb{R}$ such that

$$f = (h, \dots, h) \quad \text{on } \mathcal{G}$$

(with our convention $f^j(p) := f^j(p^j)$).

iii) (Monotone germs)

We say that \mathcal{G} is a monotone germ if for all $p, q \in [a, b]$ we have

$$(2.10) \quad p \geq q \quad \text{implies} \quad \pi_{\mathcal{G}}(p) \geq \pi_{\mathcal{G}}(q)$$

iv) (Conservative germs)

We say that \mathcal{G} is a conservative germ if $IN = OUT$, i.e. if

$$\sum_{J^j \simeq (-\infty, 0)} f^j(p) = \sum_{J^j \simeq (0, +\infty)} f^j(p) \quad \text{for all } p \in \mathcal{G}$$

For later use, we also define the Rankine-Hugoniot function

$$RH^f(p) := \sum_{j=1, \dots, N} \sigma^j f^j(p) = IN - OUT$$

which vanishes on conservative germs.

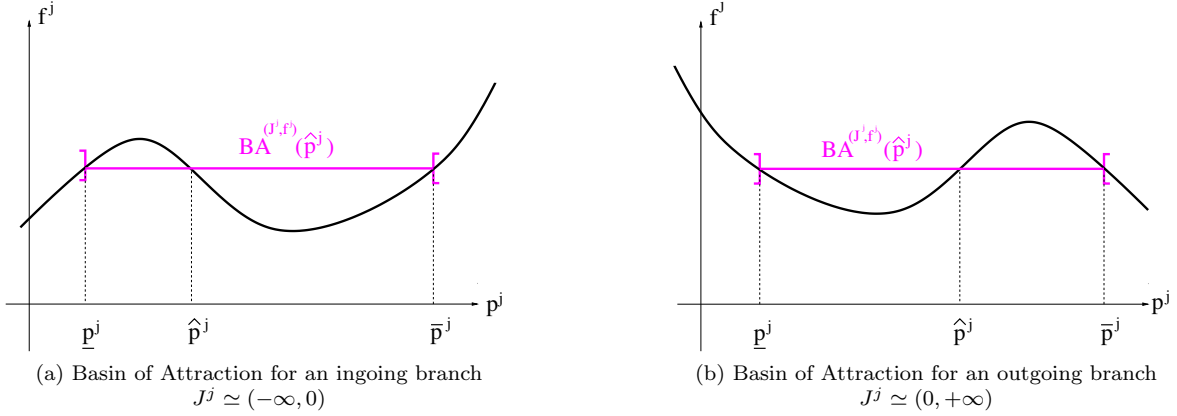
Notice that Kruřkov, monotone Kruřkov, HJ, monotone and conservative germs \mathcal{G} can equivalently (and conveniently) be defined by the properties of their Godunov flux $\hat{f}_{\mathcal{G}}$ (see Lemma 5.5). We will see that the terminology "monotone Kruřkov" germs is justified for D_+ -germs, because we have exactly

$$\{\text{Kruřkov germs}\} \cap \{\text{monotone germs}\} = \{D_+\text{-germs}\}$$

as shows Theorem 2.23.

2.1.2 Basin of Attraction

We will need the following key notion of basin of attraction of a point $\hat{p} \in [a, b]$. As we will see later, the basin of attraction of \hat{p} is the set of initial data $p \in [a, b]$ such that Riemann problem (2.6) can reach the value \hat{p} at the junction point $x = 0$. It turns out that this notion is independent on the germ \mathcal{G} , and indeed reduces to the question for each component j . Hence the basin of attraction of \hat{p} , is simply given by the product of the basins of attraction of each component \hat{p}^j . The basin of attraction of \hat{p}^j depends on the orientation of the branch J^j . For a single branch J^j with flux f^j , the "Basin of Attraction" around \hat{p}^j is called $BA^{(J^j, f^j)}(\hat{p}^j)$ and is pictured in the generic case on the associated figure.



The limit cases correspond to $\underline{p}^j = \hat{p}^j, a^j$ or $\bar{p}^j = \hat{p}^j, b^j$. For $J^j \simeq (-\infty, 0)$, the Basin of Attraction of \hat{p}^j is itself given by the largest interval $I^j := BA^{(J^j, f^j)}(\hat{p}^j)$ containing \hat{p}^j , such that for $\lambda^j := f^j(\hat{p}^j)$, we have $I^j \cap \{f^j = \lambda^j\} = \{\hat{p}^j\}$ with f^j strictly bigger than λ^j on the left of \hat{p}^j and f^j strictly less than λ^j on the right of \hat{p}^j . In the case where f^j is increasing, then I^j is reduced to the singleton $\{\hat{p}^j\}$.

The basin of attraction of \hat{p} is simply given by the product of the basins of attraction for each component \hat{p}^j . We now give the precise definition.

Definition 2.9 (Basin of attraction ¹)

Assume (2.2) and let $\hat{p} \in [a, b]$.

Then the Basin of Attraction of the point \hat{p} is the set

$$BA(\hat{p}) = \prod_{j=1, \dots, N} BA^{(J^j, f^j)}(\hat{p}^j)$$

i) (Case A: $J^j \simeq (-\infty, 0)$)

Then

$$BA^{(J^j, f^j)}(\hat{p}^j) := BA_-^j(\hat{p}^j) \cup \{\hat{p}^j\} \cup BA_+^j(\hat{p}^j)$$

with for $\lambda^j := f^j(\hat{p}^j)$

$$\begin{cases} BA_-^j(\hat{p}^j) := \{q^j \in [a^j, p^j], & f^j > \lambda^j \text{ on } [q^j, p^j]\} \\ BA_+^j(\hat{p}^j) := \{q^j \in (p^j, b^j], & f^j < \lambda^j \text{ on } (p^j, q^j]\} \end{cases}$$

where the intervals $BA_+^j(\hat{p}^j) = BA^j(\hat{p}^j) \cap (\hat{p}^j, +\infty)$ and $BA_-^j(\hat{p}^j) = BA^j(\hat{p}^j) \cap (-\infty, \hat{p}^j)$ may be empty.

ii) (Case B: $J^j \simeq (0, +\infty)$)

Then

$$BA^{(J^j, f^j)}(\hat{p}^j) := BA^{(-J^j, -f^j)}(\hat{p}^j) \quad \text{with} \quad -J^j \simeq (-\infty, 0)$$

¹In case $J^j \simeq (-\infty, 0)$, we have more precisely

$$\underline{p}^j := \begin{cases} \inf BA_-^j(\hat{p}^j) & \text{if } BA_-^j(\hat{p}^j) \neq \emptyset \\ \hat{p}^j & \text{otherwise} \end{cases} \quad \text{and} \quad BA_-^j(\hat{p}^j) = \begin{cases} [a^j, \hat{p}^j] & \text{if } \underline{p}^j = a^j \text{ and } f^j(a^j) > \lambda^j \\ (\underline{p}^j, \hat{p}^j] & \text{otherwise} \end{cases}$$

and

$$\bar{p}^j := \begin{cases} \sup BA_+^j(\hat{p}^j) & \text{if } BA_+^j(\hat{p}^j) \neq \emptyset \\ \hat{p}^j & \text{otherwise} \end{cases} \quad \text{and} \quad BA_+^j(\hat{p}^j) = \begin{cases} (\hat{p}^j, b^j] & \text{if } \bar{p}^j = b^j \text{ and } f^j(b^j) < \lambda^j \\ (\hat{p}^j, \bar{p}^j] & \text{otherwise} \end{cases}$$

In both Cases A or B, and when there will be no ambiguity, we will denote the basin of attraction simply by $BA^j(\hat{p}^j)$, or $BA^j(\hat{p})$.

2.1.3 Riemann monotonicity and local constancy

We now introduce certain properties of functions which will be satisfied by the the projection $\pi_{\mathcal{G}}$ and the Godunov flux $\hat{f}_{\mathcal{G}}$, under natural conditions. To this end, we first need to recall the following definition.

Definition 2.10 (Hadamard product of two vectors)

Assume $N \geq 1$. For $p, q \in \mathbb{R}^N$, we define $p \diamond q \in \mathbb{R}^N$ as the vector of components given by the product of components

$$(p \diamond q)^k = p^k q^k$$

Then we introduce the following.

Definition 2.11 (Riemann monotonicity)

Assume (2.2) and consider a function $h : [a, b] \rightarrow \mathbb{R}^N$ for $N \geq 1$. We set $[h]_q^p := h(p) - h(q)$. We say that h is Riemann monotone if it satisfies for all $p, q \in [a, b]$

$$(2.11) \quad (p - q) \diamond [h]_q^p \leq 0 \quad \implies \quad [h]_q^p = 0$$

Notice that condition (2.11) means $\max_{j=1, \dots, N} (p^j - q^j) \cdot \{h^j(p) - h^j(q)\} \leq 0$ implies $h(p) = h(q)$.

Notice in particular that if h is Riemann monotone, then the map $p^j \mapsto h^j(p)$ is nondecreasing. Moreover, for every $\varepsilon > 0$, the map $h + \varepsilon Id : [a, b] \rightarrow \mathbb{R}^N$ is injective (see Lemma 5.3).

Definition 2.12 (Local constancy)

Assume (2.2) and consider a map $h : [a, b] \rightarrow \mathbb{R}^N$. Let (e_1, \dots, e_N) be the canonical basis of \mathbb{R}^N . Then we say that h is locally constant on $\{h \neq f\}$, if

$$(2.12) \quad \left\{ \begin{array}{l} \text{for all } p \in [a, b] \text{ and } K_p := \{j \in \{1, \dots, N\}, h^j(p) \neq f^j(p)\}, \quad \text{there exists } \varepsilon > 0 \\ \text{such that for } Q_\varepsilon(p) := p + \sum_{j \in K_p} (-\varepsilon, \varepsilon) e_j, \text{ we have} \\ h = \text{const} \quad \text{on} \quad [a, b] \cap Q_\varepsilon(p) \end{array} \right. \quad \text{(Constant on a local box)}$$

2.2 Main results of Part I

2.2.1 Fundamental results: properties and gluing of Riemann germs

Proposition 2.13 (Properties of generalized Riemann germs)

Assume (2.2). Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ in the sense of Definition 2.5. Let $\pi := \pi_{\mathcal{G}}$ be the projection map and $\hat{f} := \hat{f}_{\mathcal{G}}$ be the Godunov flux at the junction, as introduced in Definition 2.5). Then the following holds true.

i) (Inverse of π)

We have

$$\pi^{-1}(\hat{p}) = BA(\hat{p}) \quad \text{for all } \hat{p} \in \mathcal{G}$$

where $BA(\hat{p})$ is the basin of attraction of \hat{p} given in Definition 2.9. In particular π is locally constant on $\{\pi \neq Id\}$.

ii) (Dissipation)

We have

$$D^f(q, \hat{p}) < 0 \quad \text{for all } q \in BA(\hat{p}) \setminus \{\hat{p}\}$$

where the dissipation D^f is defined in (2.8).

iii) (Riemann monotonicity of π)

The map π is Riemann monotone.

iv) (Local constancy of \hat{f})

The function \hat{f} is locally constant on $\{\hat{f} \neq f\}$.

v) (Germ as a level set)

The generalized Riemann germ \mathcal{G} can be recovered as follows

$$(2.13) \quad \mathcal{G} = \mathcal{G}_{\hat{f}} \quad \text{where} \quad \mathcal{G}_{\hat{f}} := \left\{ p \in [a, b], \quad \hat{f}(p) = f(p) \right\} \quad (\text{Germ as a level set})$$

vi) (Partial relaxation formula)

Moreover fix some $p \in [a, b]$ and some index j , and define the map

$$\hat{f}_p^j(q^j) := \hat{f}^j(\iota_p^j(q^j)) \quad \text{for all} \quad q^j \in [a^j, b^j], \quad \text{with} \quad \iota_p^j(q^j) := (p^1, \dots, p^{j-1}, q^j, p^{j+1}, \dots, p^N)$$

Then this map satisfies the following partial relaxation formula

$$\left\{ \hat{f}_p^j(p^j) \right\} = \begin{cases} \bigcup_{q^j \in [a^j, b^j]} \left\{ \hat{f}_p^j(q^j) \right\} \cap \left\{ G^j(q^j, p^j) \right\} & \text{if } J^j \simeq (0, +\infty) \\ \bigcup_{q^j \in [a^j, b^j]} \left\{ G^j(p^j, q^j) \right\} \cap \left\{ \hat{f}_p^j(q^j) \right\} & \text{if } J^j \simeq (-\infty, 0) \end{cases} \quad (\text{partial relaxation formula})$$

where G^j is the standard Godunov flux associated to the function f^j .

vii) (Partial Lipschitz estimate and basic monotonicity)

Moreover for any $p \in [a, b]$ and any index j , the map $\hat{f}_p^j : [a^j, b^j] \rightarrow \mathbb{R}$ is locally Lipschitz continuous and satisfies with $\sigma^j \in \{\pm 1\}$

$$\sigma^j(\partial_j \hat{f}^j) \circ \iota_p^j = \sigma^j(\hat{f}_p^j)' \in \{0, \max\{0, \sigma^j(f^j)'\}\} \quad \text{a.e. on } [a^j, b^j] \quad \text{if } J^j \simeq \sigma^j(-\infty, 0)$$

Notice that π is not a continuous map in general. We have the following structure result.

Theorem 2.14 (Structure of generalized Riemann germs)

Assume (2.2) and consider a set $\mathcal{G} \subset [a, b]$.

i) (First characterization)

The set \mathcal{G} is a generalized Riemann germ if and only if $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}}$ is a partition of $[a, b]$.

ii) (Second characterization)

The set \mathcal{G} is a generalized Riemann germ if and only if $\mathcal{G} = \mathcal{G}_{\hat{f}}$ with $\mathcal{G}_{\hat{f}}$ given in (2.13) for some function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ which is locally constant on $\{\hat{f} \neq f\}$ and satisfying for all j the following additional conditions:

$$(2.14) \quad \begin{cases} p^j \mapsto \sigma^j \hat{f}^j(p) \quad \text{is nondecreasing on } [a, b] & (\text{Basic monotonicities}) \\ f_- \leq \hat{f} \leq f_+ & (\text{Monotone bounds}) \end{cases}$$

where σ^j is defined in (2.1) and

$$(2.15) \quad \begin{cases} f_-^j(p^j) := \inf_{[a^j, p^j]} f^j = G^j(a^j, p^j) \quad \text{and} \quad f_+^j(p^j) := \sup_{[p^j, b^j]} f^j = G^j(b^j, p^j) & \text{if } J^j \simeq (0, +\infty), \quad \sigma^j = -1 \\ f_-^j(p^j) := \inf_{[p^j, b^j]} f^j = G^j(p^j, b^j) \quad \text{and} \quad f_+^j(p^j) := \sup_{[a^j, p^j]} f^j = G^j(p^j, a^j) & \text{if } J^j \simeq (-\infty, 0), \quad \sigma^j = +1 \end{cases}$$

When it is the case, then we necessarily have $\hat{f} = \hat{f}_{\mathcal{G}}$ where $\hat{f}_{\mathcal{G}}$ is defined in Proposition 2.13.

Remark 2.15 Notice that the second line of (2.14) means simply

$$\inf_{[p^j, b^j]} \sigma^j f^j \leq \sigma^j \hat{f}^j(p) \leq \sup_{[a^j, p^j]} \sigma^j f^j \quad \text{for all } p \in [a, b]$$

For $N = 1$, it is a fact that \mathcal{G} must be a closed set and the function \hat{f} must be continuous (see Theorem 2.25 for $N = 1$). For $N \geq 2$, this is no longer the case (see counter-examples in Lemma 8.1).

We now introduce the following

Theorem 2.16 (Riemann germs)

Assume (2.2) and let \mathcal{G} be a generalized Riemann germ.

i) (Characterization of Riemann germs)

Then \mathcal{G} is a Riemann germ if and only if the Godunov flux $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous.

ii) (Characterization of \hat{f} for Riemann germs)

If \mathcal{G} is a Riemann germ, then the Godunov flux $\hat{f}_{\mathcal{G}} : [a, b] \rightarrow \mathbb{R}^N$ is then fully characterized as a continuous function \hat{f} , locally constant on $\{\hat{f} \neq f\}$ and satisfying $\mathcal{G} = \{\hat{f} = f\}$.

Notice that the continuity of $\hat{f}_{\mathcal{G}}$ implies in particular the closedness of Riemann germs \mathcal{G} in $[a, b]$.

Theorem 2.17 (Riemann monotonicity of $\sigma \diamond \hat{f}$)

Assume (2.2) with $N \geq 1$ and recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the branches. Assume moreover that f satisfies

(2.16)

$f^k : [a^k, b^k] \rightarrow \mathbb{R}$ is not constant on any nondegenerate interval, for all $k = 1, \dots, N$ (**Nondegeneracy**)

i) (General result)

Then for any Riemann germ \mathcal{G} , the Godunov flux $\hat{f}_{\mathcal{G}} : [a, b] \rightarrow \mathbb{R}^N$ is such that $\sigma \diamond \hat{f}_{\mathcal{G}}$ is Riemann monotone in the sense of Definition 2.11.

ii) (Case of Kruřkov germs)

If \mathcal{G} is a Kruřkov germ, then the conclusion of point i) still holds true, without assuming nondegeneracy condition (2.16).

Remark 2.18 Notice that Riemann monotonicity of $\sigma \diamond \hat{f}_{\mathcal{G}}$ means that for two distinct points p, q of the germ, the entropy flux can not be negative for each components, at the junction point. In other words, at least one branch must dissipate a positive entropy flux at the junction point.

Notice that without condition (2.16), there exist Riemann germs \mathcal{G} such that $\sigma \diamond \hat{f}_{\mathcal{G}}$ is not Riemann monotone (see counter-example Lemma 8.3). Moreover Riemann monotonicity is required for gluing of Riemann germs, as shows Lemma 8.4.

Theorem 2.19 (Gluing of Riemann germs along $J_{\alpha}^{j_{\alpha}}$ and $J_{\beta}^{j_{\beta}}$)

For $\gamma = \alpha, \beta$, consider Riemann germs $\mathcal{G}_{\gamma} \subset [a_{\gamma}, b_{\gamma}]$ for junctions of type $n_{\gamma} : m_{\gamma}$ with $N_{\gamma} := n_{\gamma} + m_{\gamma}$, and assume (2.2) with $J_{\gamma}^j = \sigma_{\gamma}^j \cdot (-\infty, 0)$.

For each $\gamma = \alpha$ or β , assume either 1) nondegeneracy condition (2.16) for the flux function $f_{\gamma} : [a_{\gamma}, b_{\gamma}] \rightarrow \mathbb{R}^{N_{\gamma}}$, or 2) that \mathcal{G}_{γ} is a Kruřkov germ. We allow mixing cases for α and β .

Define $[a, b]_{\gamma}^j := [a_{\gamma}^j, b_{\gamma}^j]$. Fix $j_{\gamma} \in \{1, \dots, N_{\gamma}\}$ such that

$$\begin{cases} f_{\alpha}^{j_{\alpha}} = f_{\beta}^{j_{\beta}} =: f^0 & \text{and} & \sigma_{\alpha}^{j_{\alpha}} = -\sigma_{\beta}^{j_{\beta}} \\ \text{with } f^0 : [a^0, b^0] \rightarrow \mathbb{R} & \text{where} & [a^0, b^0] := [a, b]_{\alpha}^{j_{\alpha}} = [a, b]_{\beta}^{j_{\beta}} \end{cases}$$

Define the gluing $\mathcal{G} := \mathcal{G}_{\alpha} \#_{j_{\alpha}:j_{\beta}} \mathcal{G}_{\beta}$ of the germs \mathcal{G}_{α} and \mathcal{G}_{β} along $J_{\alpha}^{j_{\alpha}}$ and $J_{\beta}^{j_{\beta}}$ as

$$\mathcal{G} := \mathcal{G}_{\alpha} \#_{j_{\alpha}:j_{\beta}} \mathcal{G}_{\beta} := \left\{ (p'_{\alpha}, p'_{\beta}) \in [a, b]_{\alpha}^{j_{\alpha}} \times [a, b]_{\beta}^{j_{\beta}}, \quad \text{there exists } r_{\alpha}, r_{\beta} \in [a^0, b^0] \text{ s.t.} \quad \begin{cases} (r_{\alpha}, p'_{\alpha}) \in \mathcal{G}_{\alpha}, \\ (r_{\beta}, p'_{\beta}) \in \mathcal{G}_{\beta}, \\ f^0(r_{\alpha}) = G^{f^0}(r_{\alpha}, r_{\beta}) = f^0(r_{\beta}) \end{cases} \right\}$$

where G^{f^0} is the standard Godunov flux associated to the function f^0 , and where we use the abuse of notation $(r_{\gamma}, p'_{\gamma}) := (p_{\gamma}^1, \dots, p_{\gamma}^{j_{\gamma}-1}, r_{\gamma}, p_{\gamma}^{j_{\gamma}+1}, \dots, p_{\gamma}^{N_{\gamma}})$, and

$$[a, b]_{\gamma}^{j_{\gamma}} := \prod_{j \in \{1, \dots, N_{\gamma}\} \setminus \{j_{\gamma}\}} [a, b]_{\gamma}^j$$

i) (Glued Riemann germ)

Then \mathcal{G} is a Riemann germ for a junction $(n_{\alpha} + n_{\beta} - 1) : (m_{\alpha} + m_{\beta} - 1)$.

ii) (Associativity of gluing)

The gluing of germs is associative.

iii) (Identity element of gluing)

For general gluing, the identity element is the standard Godunov germ for 1 : 1 junction

$$\mathcal{G}^{f^0} := \left\{ (r, s) \in [a^0, b^0]^2, \quad f^0(r) = G^{f^0}(r, s) = f^0(s) \right\}$$

i.e.

$$\mathcal{G}_\alpha \# \mathcal{G}^{f^0} = \mathcal{G}_\alpha \quad \text{and} \quad \mathcal{G}^{f^0} \# \mathcal{G}_\beta = \mathcal{G}_\beta$$

iv) (Nature of the glued germs)

Moreover if \mathcal{G}_γ are Kruřkov germs (resp. HJ germs, resp. monotone germs, resp. conservative germs), then \mathcal{G} is a Kruřkov germ (resp. HJ germ, resp. monotone germ, resp. conservative germ).

Notice that the identity element \mathcal{G}^{f^0} is a conservative germ for 1 : 1 junctions and then by Theorem 2.29, it is also a Kruřkov, HJ and monotone germ. Moreover, from i) of Definition 2.5, there exists the Godunov projection map $\pi := \pi_{\mathcal{G}^{f^0}} : [a^0, b^0]^2 \rightarrow \mathcal{G}^{f^0}$ with $\pi = (\pi^\alpha, \pi^\beta) =: (\pi^L, \pi^R)$, such that the standard Godunov flux satisfies

$$G^{f^0}(p^L, p^R) = (f^0 \circ \pi^L)(p^L, p^R) = (f^0 \circ \pi^R)(p^L, p^R) \quad \text{for all} \quad (p^L, p^R) \in [a^0, b^0]^2$$

and π is monotone in the sense of (2.10). Already for the standard Godunov flux G^{f^0} , this result seems new.

Remark 2.20 In Theorem 2.19, with abuse of notation, we have formally the gluing along the axis of flux f^0 as follows

$$\mathcal{G}_\alpha \# \mathcal{G}_\beta \ni (p'_\alpha, p'_\beta) = \underbrace{(p'_\alpha, r_\alpha)}_{\in \mathcal{G}_\alpha} \# \underbrace{(r_\beta, p'_\beta)}_{\in \mathcal{G}_\beta} \in \mathcal{G}^{f^0}$$

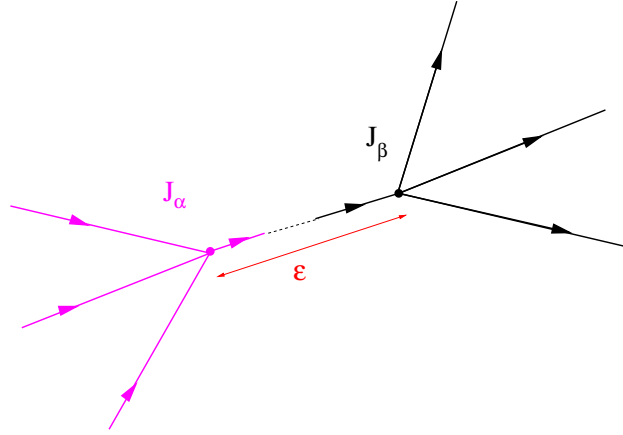


Figure 1: Illustration of a gluing of two particular junctions, formally as $\varepsilon \rightarrow 0$

2.2.2 Applications to Kruřkov, Hamilton-Jacobi, monotone or conservative germs

We have the following results.

Theorem 2.21 (Properties of Kruřkov germs)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ with respect to (J, f) .

i) (Characterization of Kruřkov germs)

Then \mathcal{G} is a Kruřkov germ if and only if $\mathcal{G} = \left\{ p \in [a, b], \hat{f}(p) = f(p) \right\}$ for some locally Lipschitz continuous function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ whose Jacobian matrix satisfies the following column diagonally dominant inequality

$$\sigma^i \partial_i \hat{f}^i \geq \sum_{j \in \{1, \dots, N\} \setminus \{i\}} |\partial_i \hat{f}^j| \quad \text{a.e. on } [a, b], \quad \text{for all } i = 1, \dots, N$$

When it is the case, then we have $\hat{f} = \hat{f}_{\mathcal{G}}$.

ii) (D-maximality)

If \mathcal{G} is a Kruřkov germ, then it satisfies the following D-maximality property: for all $p \in [a, b]$

$$(D^f(p, q) \geq 0 \quad \text{for all } q \in \mathcal{G}) \implies p \in \mathcal{G}$$

Theorem 2.22 (Characterization of monotone germs)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ with respect to (J, f) , and let $\hat{f} := \hat{f}_{\mathcal{G}}$ be its associated Godunov flux. Then \mathcal{G} is monotone if and only if

$$(2.17) \quad p \mapsto \sigma^j \hat{f}^j(p) \quad \text{is nonincreasing in } p^k \text{ for all } k \neq j$$

Theorem 2.23 (Characterization of D_+ -germs)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ with respect to (J, f) . Then we have as a germ

$$\mathcal{G} \text{ is a } D_+ \text{-germ} \iff (\mathcal{G} \text{ monotone and } \mathcal{G} \text{ Kruřkov})$$

Theorem 2.24 (Properties of conservative Riemann germs)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a Riemann germ with respect to (J, f) . Assume that \mathcal{G} is conservative. Then we have

$$\mathcal{G} \text{ monotone} \iff \mathcal{G} \text{ Kruřkov} \iff \mathcal{G} \text{ } D_+ \text{-germ}$$

Notice that even for $N = 2$, there exist monotone and nonmonotone nonconservative Kruřkov germs (see ii) of Lemma 8.6).

Notice also that for conservative germs \mathcal{G} , the fact that the monotonicity of \mathcal{G} is equivalent to the fact that \mathcal{G} is Kruřkov (which can itself be seen as equivalent to the fact that \mathcal{G} generates a L^1 -contraction semi-group), can be interpreted as a version of Crandall-Tartar's Lemma (see Proposition 1 in [8]).

Notice also that there exist monotone Kruřkov germs which are not necessarily conservative, as it is already the case for all non-conservative germs for $N = 1$ branch.

Theorem 2.25 (Properties of HJ germs)

Assume (2.2) with $N \geq 1$.

i) (Regularity of \hat{f})

Every HJ germ \mathcal{G} is a Riemann germ, i.e. $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous. Moreover there exists a function \hat{h} such that

$$(2.18) \quad \begin{cases} \hat{h} : [a, b] \rightarrow \mathbb{R} \quad \text{is continuous} \\ p \mapsto \sigma^j \hat{h}(p) \quad \text{is nondecreasing in } p^j \text{ for all } j = 1, \dots, N, \\ \hat{f} := (\hat{h}, \dots, \hat{h}) \text{ satisfies the monotone bounds given in the second line of (2.14).} \end{cases}$$

ii) (HJ-relaxation formula)

Then for each $p \in [a, b]$, we have

(2.19)

$$\{\hat{h}(p)\} = \bigcup_{q \in [a, b]} \left(\{\hat{h}(q)\} \cap \bigcap_{J^j \simeq (-\infty, 0)} \{G^j(p^j, q^j)\} \cap \bigcap_{J^j \simeq (0, +\infty)} \{G^j(q^j, p^j)\} \right) \quad \text{(HJ-relaxation formula)}$$

Remark 2.26 Notice that HJ-relaxation formula (2.19) can be used to define some HJ-relaxation operator (see Lemma 6.7) which computes the analogue for $N \geq 1$ of the effective boundary condition that was obtained for $N = 1$ by ANDREIANOV, SBIHI in [4].

Definition 2.27 (Characteristic subset of a HJ germ)

Assume (2.2) with $N \geq 1$. Let \mathcal{G} be a generalized Riemann germ which is a HJ germ. We introduce the characteristic subset of the HJ germ \mathcal{G} as the set $\chi\mathcal{G}$ defined by

$$\chi\mathcal{G} := \underline{\chi}\mathcal{G} \cup \overline{\chi}\mathcal{G} \quad \text{with} \quad \begin{cases} \underline{\chi}\mathcal{G} := \left\{ \hat{p} \in \mathcal{G}, \quad BA_{\sigma^j}^j \neq \emptyset \quad \text{for all } j = 1, \dots, N \right\} \\ \overline{\chi}\mathcal{G} := \left\{ \hat{p} \in \mathcal{G}, \quad BA_{-\sigma^j}^j(\hat{p}^j) \neq \emptyset \quad \text{for all } j = 1, \dots, N \right\} \end{cases}$$

with $BA_+^j(\hat{p}^j) := BA^j(\hat{p}^j) \cap (\hat{p}^j, +\infty)$ and $BA_-^j(\hat{p}^j) := BA^j(\hat{p}^j) \cap (-\infty, \hat{p}^j)$.

Theorem 2.28 (HJ germ determined by its characteristic subset)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G}, \mathcal{G}_0$ be two generalized Riemann germs which are both HJ germs. Then we have

$$\chi\mathcal{G} \subset \mathcal{G}_0 \quad \text{implies} \quad \mathcal{G}_0 = \mathcal{G}$$

Theorem 2.28 is very important, because in many practical situations, the characteristic subset $\chi\mathcal{G}$ is usually a finite set (see for instance example given in Lemma 8.10, where the characteristic subset is a set of four points for a HJ germ on a 1 : 1 junction).

Notice that monotone germs have less good properties than Kruřkov or HJ germs. For instance there exist monotone germs which are not Riemann germs (see Lemma 8.1 for $N = 2$). Notice also that for any Kruřkov or HJ germs, the Godunov flux at the junction is indeed always locally Lipschitz (see Theorem 2.21 for Kruřkov germs, and see Proposition 6.9 for HJ germs). On the contrary, there exists monotone germs \mathcal{G} such that $\hat{f}_{\mathcal{G}}$ is continuous but not locally Lipschitz (see Lemma 8.2).

Notice also that for $N = 1$, all generalized Riemann germs are indeed Kruřkov germs, HJ germs and monotone germs (but not conservative germs in general). This result extends to conservative germs for $N = 2$ in the case of 1 : 1 junctions.

Theorem 2.29 (Properties of conservative germs on 1 : 1 junctions)

Assume (2.2) for $N = 2$ and 1 : 1 junctions with $J^1 \simeq (-\infty, 0)$ and $J^2 \simeq (0, +\infty)$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ.

i) (Nature of the germ)

Then \mathcal{G} is conservative if and only if it is a HJ germ. Moreover in that case, \mathcal{G} is determined by its characteristic subset $\chi\mathcal{G}$ given in Definition 2.27, and \mathcal{G} is a D_+ -germ. Moreover we have

$$(2.20) \quad \mathcal{G} = \left\{ p \in [a, b], \quad RH^f(p) = 0, \quad D^f(p, q) \geq 0 \quad \text{for all } q \in \chi\mathcal{G} \right\}$$

ii) (HJ-relaxation formula)

When \mathcal{G} is conservative, then the Godunov flux satisfies $(\hat{f}^1, \hat{f}^2) = (\hat{h}, \hat{h})$ with $\hat{h} : [a, b] \rightarrow \mathbb{R}$ with monotonicities $\hat{h}(\uparrow, \downarrow)$. And for any $p \in [a, b]$, there exists some $q \in [a, b]$ (possibly non unique) such that

$$(2.21) \quad \hat{h}(p) = G^{f^1}(p^1, q^1) = \hat{h}(q) = G^{f^2}(q^2, p^2)$$

Moreover we have

$$(2.22) \quad \max \left\{ \inf_{[p^1, b^1]} f^1, \quad \inf_{[a^2, p^2]} f^2 \right\} \leq \hat{h}(p) \leq \min \left\{ \sup_{[a^1, p^1]} f^1, \quad \sup_{[p^2, b^2]} f^2 \right\}$$

Notice that for 1 : 1 junctions, there are examples of Kruřkov germs which are not conservative (see for instance ii) of Lemma 8.6 for 2 : 0 junctions). Notice also that HJ-relaxation formula (2.21) allows to construct all conservative germs for 1 : 1 junctions (see for instance the Relaxation operator in Lemma 6.7).

Proposition 2.30 (Germ product property for several conservative lines with concave fluxes)

Assume (2.2) for $N = 2n$ with $n : n$ junctions, and call f^{jL} and f^{jR} respectively the j -th ingoing and j -th

outgoing fluxes for $j = 1, \dots, n$. Assume that each $f^{j\alpha}$ is strictly concave with maximum at $c^{j\alpha} \in (a^{j\alpha}, b^{j\alpha})$. Let $\mathcal{G} \subset [a, b]$ be a closed generalized Riemann germ satisfying for $p = (p^{1L}, p^{1R}, \dots, p^{nL}, p^{nR})$

$$\mathcal{G} \subset \bigcap_{j=1, \dots, n} \Sigma^j, \quad \text{with } \Sigma^j := \{p \in [a, b], \quad f^{jL}(p^{jL}) = f^{jR}(p^{jR})\}$$

which means that each j -th line $J^{jL} \cup \{0\} \cup J^{jR} \simeq \mathbb{R}$ is conservative. Then

$$\mathcal{G} = \prod_{j=1, \dots, n} \mathcal{G}^j \quad \text{with } \mathcal{G}^j \subset \{(p^{jL}, p^{jR}) \in [a^{jL}, b^{jL}] \times [a^{jR}, b^{jR}], \quad f^{jL}(p^{jL}) = f^{jR}(p^{jR})\} \subset \mathbb{R}^2$$

where each \mathcal{G}^j is a conservative Riemann germ with respect to (f^{jL}, f^{jR}) for a 1 : 1 junction with $J^{jL} \simeq (-\infty, 0)$ and $J^{jR} \simeq (0, +\infty)$.

This result has implications in the theory of traffic on networks. In particular, if two conservative lines L_i for $i = 1, 2$ cross each other, then the flux limiter on line L_1 only depends on the structure of the junction (between the two lines), but not on the state of the traffic on line L_2 .

Notice that more generally, for bell-shaped fluxes, Godunov fluxes have a polar decomposition (see Proposition 7.23).

As a straightforward corollary of Theorem 2.19, we get the following result.

Corollary 2.31 (Semigroup of germs for the gluing of 1 : 1 junctions with the same flux)

Assume (2.2) and nondegeneracy assumption (2.16) for $N = 2$ and 1 : 1 junctions $J = \{0\} \cup \bigcup_{j=1,2} J^j$ with

$J^1 \simeq (-\infty, 0)$ and $J^2 \simeq (0, +\infty)$ and the same flux function

$$f^1 = f^2 =: f^0, \quad [a^1, b^1] = [a^2, b^2] =: [a^0, b^0], \quad [a, b] := [a^0, b^0]^2$$

Let $\mathfrak{G}^{(J,f)}$ be the set of Riemann germs $\mathcal{G} \subset [a, b]$ with respect to (J, f) . Let

$$\mathcal{G}^{f^0} := \left\{ (r, s) \in [a^0, b^0], \quad f^0(r) = G^{f^0}(r, s) = f^0(s) \right\}$$

where G^{f^0} is the standard Godunov flux associated to the flux $f^0 : [a^0, b^0] \rightarrow \mathbb{R}$.

Then $(\mathfrak{G}^{(J,f)}, \sharp)$ is a semigroup, with identity element equal to \mathcal{G}^{f^0} . We have

$$\begin{aligned} \sharp : \mathfrak{G}^{(J,f)} \times \mathfrak{G}^{(J,f)} &\rightarrow \mathfrak{G}^{(J,f)} \\ (\mathcal{G}_2, \mathcal{G}_1) &\mapsto \mathcal{G}_2 \sharp \mathcal{G}_1 \end{aligned}$$

where we recall that $\mathcal{G}_2 \sharp \mathcal{G}_1$ is the germ obtained by gluing of the outgoing branch $J^2 \simeq (0, +\infty)$ of \mathcal{G}_2 with the ingoing branch $J^1 \simeq (-\infty, 0)$ of \mathcal{G}_1 .

Notice that this semigroup can be not commutative (see Lemma 8.11 for an example).

2.3 Comments on Part II

2.3.1 Preliminaries: \mathcal{G} -entropy solutions, semi-solutions, strong traces

In part I of the paper we were focusing on self-similar solutions of Riemann problems on a junction. On the contrary, in Part II, we consider general Kruřkov entropy solutions/subsolutions/supersolutions, for which we will need to be able to define the trace at the junction point. This will be done using the work of Panov [35].

For functions $u^k : [0, +\infty) \times J^k \rightarrow [a^k, b^k]$, with $u^k(t, x)$, we consider the equation of the first line of (2.4), namely the scalar conservation laws on the branch J^k

$$(2.23) \quad \partial_t u^k + \partial_x (f^k(u^k)) = 0 \quad \text{on } (0, +\infty) \times J^k, \quad k = 1, \dots, N.$$

We want to recall the definition of Kruřkov entropy solutions, subsolutions and supersolutions (see Kruřkov [16]). For $x \in \mathbb{R}$, we set $\text{sign}(x) = 1_{\{x>0\}} - 1_{\{x<0\}}$. We recall that the Kruřkov pairs (entropy/ flux of entropy) for $u = (u^1, \dots, u^N), v = (v^1, \dots, v^N) \in \mathbb{R}^N$ are given by (2.5), i.e.

$$(2.24) \quad \eta^k(u^k, v^k) := |u^k - v^k| \quad \text{and} \quad \psi^{f^k}(u^k, v^k) = \psi^k(u^k, v^k) := \text{sign}(u^k - v^k) \cdot \{f^k(u^k) - f^k(v^k)\}$$

Similarly, for $x \in \mathbb{R}$, we set $|x|_{\pm} := \max(0, \pm x)$ and $\text{sign}^{\pm}(x) = 1_{\{\pm x > 0\}}$. We recall that the Kruřkov pairs (semi-entropy/ flux of semi-entropy) given by

$$(2.25) \quad \eta_{\pm}^k(u^k, v^k) := |u^k - v^k|_{\pm} \quad \text{and} \quad \psi_{\pm}^{f^k}(u^k, v^k) = \psi_{\pm}^k(u^k, v^k) := \text{sign}^{\pm}(u^k - v^k) \cdot \{f^k(u^k) - f^k(v^k)\}$$

We will also use shorthands notations $\eta^k(u, v) := \eta^k(u^k, v^k)$ and $\psi^{f^k}(u, v) = \psi^k(u, v) := \psi^k(u^k, v^k)$ and similarly for η_{\pm}^k and $\psi_{\pm}^k, \psi_{\pm}^{f^k}$. Then we recall the following standard notion (which recalls and contains Definition 2.32 for Kruřkov entropy solutions).

Definition 2.32 (Kruřkov entropy solution, subsolution and supersolution)

We say that u^k is a Kruřkov entropy solution (resp. subsolution, resp. supersolution) of (2.23), with initial data $u_0^k \in L^{\infty}(J^k; [a^k, b^k])$, if $u^k \in L^{\infty}([0, +\infty) \times J^k; [a^k, b^k])$ and for any constant $c = (c^1, \dots, c^N) \in \mathbb{R}^N$, and for any (test) function $0 \leq \varphi^k \in C_c^1([0, +\infty) \times J^k)$, we have

$$\int_{(0, +\infty) \times J^k} \{\eta^k(u, c)\varphi_t^k + \psi^k(u, c)\varphi_x^k\} dt dx + \int_{\{0\} \times J^k} \eta^k(u_0, c)\varphi^k dx \geq 0$$

$$\left(\text{resp.} \int_{(0, +\infty) \times J^k} \{\eta_{\pm}^k(u, c)\varphi_t^k + \psi_{\pm}^k(u, c)\varphi_x^k\} dt dx + \int_{\{0\} \times J^k} \eta_{\pm}^k(u_0, c)\varphi^k dx \geq 0 \right)$$

with $+$ for subsolutions and $-$ for supersolutions.

For subsolutions u^k , we write

$$\partial_t u^k + \partial_x(f^k(u^k)) \underset{\text{Kruřkov}}{\leq} 0 \quad \text{on} \quad (0, +\infty) \times J^k$$

and for supersolutions u^k , we write

$$\partial_t u^k + \partial_x(f^k(u^k)) \underset{\text{Kruřkov}}{\geq} 0 \quad \text{on} \quad (0, +\infty) \times J^k$$

Notice that the box $[a, b]$ is where all the values of the function u stay confined.

The standard Kruřkov theory on the real line (hence without junctions) shows that BV norm of the initial data is preserved by the evolution. As it has been shown in an important counter-example by ADIMURTHI, GHOSHAL, DUTTA, VEERAPPA GOWDA [21], already for $1 : 1$ junctions with convex fluxes on each branch the space BV norm of the solution may blow-up in finite time. For this reason, the notion of trace of the solution at the junction point can not be based on BV bounds which do not exist in general.

Fortunately, under suitable conditions on the fluxes, the notion of strong trace of the solution has first been shown to exist by Vasseur [37]. Then it has been generalized by Panov [35], in a way which is convenient for our work. We now recall this result, which plays a fundamental role in our analysis in Part II.

Theorem 2.33 (Existence of a strong trace; Theorem 1.1 in Panov [35])

Assume (2.2) for $N \geq 1$ and that f satisfies the nondegeneracy condition (2.16). Let u be a Kruřkov entropy solution (resp. subsolution, supersolution) of (2.23) in the sense of Definition 2.32. Then for each index $j = 1, \dots, N$, there exists a function $u^j(\cdot, 0) \in L_{loc}^1(0, +\infty)$ satisfying

$$(2.26) \quad \text{ess} \lim_{J^j \ni x \rightarrow 0} \int_{(0, T)} |u^j(t, 0) - u^j(t, x)| dt = 0 \quad \text{for all index } j \text{ and all } T > 0$$

Such function is called the strong trace of u^j on $(0, +\infty)_t \times \{0\}_x$.

Recall that without the nondegeneracy condition (2.16), the strong trace does not exist in general, but only a notion of quasi-trace is defined.

Definition 2.34 (Notion of \mathcal{G}^{SUB} and \mathcal{G}^{SUP})

Assume (2.2) for $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ, and let $\hat{f} := \hat{f}_{\mathcal{G}}$ be its associated Godunov flux. Then we define

$$\mathcal{G}^{SUB} := \left\{ p \in [a, b], \quad \sigma \diamond (\hat{f} - f)(p) \leq 0 \right\} \quad \text{and} \quad \mathcal{G}^{SUP} := \left\{ p \in [a, b], \quad \sigma \diamond (\hat{f} - f)(p) \geq 0 \right\}$$

Then we give the following definition (which recalls and contains Definition 2.2 for \mathcal{G} -entropy solutions).

Definition 2.35 (\mathcal{G} -entropy solution/subsolution/supersolution)

Assume (2.2) for $N \geq 1$ and that f satisfies the nondegeneracy condition (2.16). Let us consider some initial data $u_0 = (u_0^1, \dots, u_0^N)$ with $u_0^k \in L^\infty(J^k; [a^k, b^k])$. We say that $u = (u^1, \dots, u^N)$ is a \mathcal{G} -entropy solution (resp. subsolution, resp. supersolution) of (2.23) with initial data u_0 , if each u^k is a Kruřkov entropy solution (resp. subsolution, resp. supersolution) of (2.23) with initial data u_0^k in the sense of Definition 2.32, and if the strong trace $u(t, 0) = (u^1(t, 0), \dots, u^N(t, 0))$ of u given by Theorem 2.33 satisfies

$$\begin{aligned} u(t, 0) &\in \mathcal{G} \quad \text{for a.e. time } t \in (0, +\infty) \\ (\text{resp. } u(t, 0) &\in \mathcal{G}^{SUB} \quad \text{for a.e. time } t \in (0, +\infty)) \\ (\text{resp. } u(t, 0) &\in \mathcal{G}^{SUP} \quad \text{for a.e. time } t \in (0, +\infty)) \end{aligned}$$

If \mathcal{G} is a generalized Riemann germ, then a function u is naturally a \mathcal{G} -entropy solution if and only if it is both a \mathcal{G} -entropy subsolution and supersolution (see Lemma 10.1). The point with \mathcal{G} -entropy subsolutions/supersolutions is that we do not expect them to be interesting, except for certain subclasses of germs, like the subclass of monotone Kruřkov germs, as we will see below.

2.3.2 Main results of Part II: existence, uniqueness, contraction, comparison

Notice that we have a natural isomorphism $L^1(J) := L^1(J; \mathbb{R}) \simeq \prod_{j=1, \dots, N} L^1(J^j; \mathbb{R})$ with the norm

$$\int_J |u_0| dx := \sum_{k=1, \dots, N} \int_{J^k} |u_0^k| dx \quad \text{for } u_0 = (u_0^1, \dots, u_0^N)$$

that we use constantly.

Theorem 2.36 (Theory for Kruřkov germs)

Assume (2.2) for $N \geq 1$, nondegeneracy condition (2.16), and that $\mathcal{G} \subset [a, b] \ni 0_{\mathbb{R}^N}$ is a Kruřkov germ in the sense of i) of Definition 2.8. Let u_0 be an initial data satisfying

$$(2.27) \quad u_0^k \in (BV \cap L^1)(J^k; [a^k, b^k]) \quad \text{for all index } k = 1, \dots, N$$

where here BV denotes the space of functions with bounded variations.

i) (Existence and uniqueness)

Then there exists a unique \mathcal{G} -entropy solution u of (2.23) with initial data u_0 . Moreover we have

$$(2.28) \quad u \in C^0([0, +\infty); L^1(J^*)) \cap Lip([0, +\infty); \mathcal{M}(J^*)) \quad \text{with } J^* := J \setminus \{0\}$$

where $\mathcal{M}(J^*)$ is the set of measures on J^* . For a measure $w = (w^1, \dots, w^N)$, we set $|w|_{\mathcal{M}(J^*)} := \sum_{j=1, \dots, N} |w^j|_{\mathcal{M}(J^j)}$, where $\mathcal{M}(J^j)$ is the set of (real valued) measures on J^j and $|\cdot|_{\mathcal{M}(J^j)}$ is the total variation of the measure.

ii) (A priori bounds)

Either the box $[a, b]$ is bounded and we set $[\bar{a}, \bar{b}] := [a, b]$, or the box $[a, b]$ is not bounded and there exists a bounded box $[\bar{a}, \bar{b}] \subset [a, b]$ such that

$$u_0(J^*) \subset [\bar{a}, \bar{b}] \quad \text{and} \quad \mathcal{G} \cap [\bar{a}, \bar{b}] \quad \text{is a Riemann germ on the bounded box } [\bar{a}, \bar{b}]$$

Then we have the following bounds for a.e. $t > 0$

$$(2.29) \quad |u_t(t, \cdot)|_{\mathcal{M}(J^*)} \leq L|(u_0)_x|_{\mathcal{M}(J^*)} + K_0 \quad \text{with} \quad \begin{cases} L := \sup_{j=1, \dots, N} \text{Lip}(f^j; [\bar{a}^j, \bar{b}^j]) \\ K_0 := \sum_{j=1, \dots, N} |(f^j - \hat{f}^j)(u_0(0))| \end{cases}$$

where $\hat{f} := \hat{f}_{\mathcal{G}}$ is Godunov flux associated to the germ \mathcal{G} , and

$$(2.30) \quad |u_x(t, \cdot)|_{\mathcal{M}(J \setminus \bar{B}_{2\delta})} \leq (1 + \delta^{-1}L)|(u_0)_x|_{\mathcal{M}(J^*)} + \delta^{-1}K_0 \quad \text{for all } \delta \in (0, 1)$$

iii) (L^1 -contraction)

Moreover if u_0, v_0 are two initial data satisfying (2.27), and if u, v are respectively their associated \mathcal{G} -entropy solutions, then we have the following L^1 -contraction property

$$\int_{\{t\} \times J} |u - v|(t, \cdot) dx \leq \int_J |u_0 - v_0| dx \quad \text{for all time } t > 0$$

Remark 2.37 Notice that our BV assumption (2.27) on the initial data is technical, and indeed simplifies our proof of existence, and allows regularity (2.28) of the solutions, and gives nice a priori bounds. Moreover the condition $0_{\mathbb{R}^N} \in [a, b]$ is only here to allow the initial data to belong to $L^1(J)$. Obviously any shift from $0_{\mathbb{R}^N}$ can also be considered.

Notice that prior to Theorem 2.36, only a few existence and uniqueness results were available in several important and pionnering works. Existence results were available for complete and conservative D -maximal L^1 -dissipative sets \mathcal{G} (see [3], [22], [18] and [27]). Nevertheless completeness was not fully understood, and it was not understood that D -maximality is an automatic consequence of completeness and L^1 -dissipative properties.

Most of the time, existence was proved only for some particular Riemann solvers. Some nice uniqueness results were also obtained for Riemann solver \mathcal{RS}_2 in [30]. We indeed show (see Corollary 14.12) that this is due to the fact that Riemann solver \mathcal{RS}_2 is associated to a Kruřkov germ. As a consequence of Theorem 2.36, we also get existence of a solution in a systematic way. From this perspective point of view, it seems that Theorem 2.36 provides a new progress in the understanding of scalar conservation laws on junctions.

We also have the following result.

Theorem 2.38 (Properties of semisolutions for monotone Kruřkov germs)

Assume (2.2) for $N \geq 1$, nondegeneracy condition (2.16), and that $\mathcal{G} \subset [a, b] \ni 0_{\mathbb{R}^N}$ is a monotone Kruřkov germ in the sense of i') of Definition 2.8. We consider \mathcal{G} -entropy subsolutions/supersolutions of (2.23).

i) (Stability of sub/supersolutions)

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of \mathcal{G} -entropy subsolutions (resp. supersolutions) such that

$$u_n \rightarrow u \quad \text{in } L^1_{loc}([0, +\infty) \times J)$$

Then u is a \mathcal{G} -entropy subsolution (resp. supersolution).

ii) (Max/Min for sub/supersolutions)

Let u, w be two \mathcal{G} -entropy subsolutions (resp. supersolutions). Then $\max\{u, w\}$ (resp. $\min\{u, w\}$) is a \mathcal{G} -entropy subsolution (resp. supersolution).

iii) (L^1 -comparison principle)

Let u (resp. v) be a \mathcal{G} -entropy subsolution (resp. supersolution). Then for all $0 < s < t$, we have

$$\sum_{j=1, \dots, N} \int_{J^k} |u^k - v^k|_+(t, x) dx \leq \sum_{j=1, \dots, N} \int_{J^k} |u^k - v^k|_+(s, x) dx$$

We end this presentation with a known, but key result. This is classically the following result which garantees the stability of \mathcal{G} -entropy solutions for Kruřkov germs (see for instance [18]).

Proposition 2.39 (Equivalent characterization of \mathcal{G} -entropy solutions for Kruřkov germs)

Assume (2.2) with $N \geq 1$ and let $\mathcal{G} \subset [a, b] \ni 0_{\mathbb{R}^N}$ be a Kruřkov germ. Then $u = (u^1, \dots, u^N)$ is \mathcal{G} -entropy solution of (2.4) with initial data u_0 with $u_0 \in BV(J)$, if and only if $u^j : [0, +\infty) \times J^j \rightarrow [a^j, b^j]$ is a Kruřkov entropy solution of the first line of (2.4), and the trace condition in the second line of (2.4) is replaced by the following condition. For all test functions $0 \leq \varphi^j \in C_c^1([0, +\infty) \times \bar{J}^j)$ with $\bar{J}^j = \{0\} \cup J^j \simeq [0, +\infty)$ or $(-\infty, 0]$ with

$$\varphi^j(t, 0) = \varphi^k(t, 0) =: \varphi(t, 0) \quad \text{for all } t \in [0, +\infty) \text{ and all indices } j, k$$

and for all elements $c = (c^1, \dots, c^N) \in \mathcal{G}$, we have

$$(2.31) \quad \sum_k \left\{ \int_{(0, +\infty) \times J^k} \{ \eta^k(u, c) \varphi_t^k + \psi^k(u, c) \varphi_x^k \} dt dx + \int_{\{0\} \times J^k} \eta^k(u_0, c) \varphi dx \right\} \geq 0$$

for (η^k, ψ^k) given in (2.5).

2.4 Organization of the paper

The paper is organized in fifteen sections. The main results of Parts I and II presented in Section 2 are proved as indicated in the following table.

Main results	topics	proof for $N = 1$	proof for $N \geq 1$
Proposition 2.4	L^1 estimate for Riemann problem		Subsection 4.2
Proposition 2.13	properties of gen. Riemann germs i) inverse of π ii) dissipation iii) Riemann monotonicity of π iv) local constancy of \hat{f} v) germ as a level set vi) partial relaxation formula vii) $\left\{ \begin{array}{l} \text{partial Lipschitz estimate} \\ \text{and basic monotonicity} \end{array} \right.$	ii) of Proposition 3.6 Lemma 3.4 vi) of Proposition 3.6 i) of Proposition 3.6 iii) of Proposition 3.6 Proposition 3.10 Proposition 3.8	ii) of Proposition 4.10 Lemma 3.4 Proposition 5.1 i) of Proposition 4.10 iii) of Proposition 4.10 Proposition 4.15 Proposition 4.13
Theorem 2.14	character. of gen. Riemann germs i) 1st characterization ii) 2nd characterization	Subsection 3.4 ii) of Lemma 3.5 Proposition 3.9	Subsection 4.5 ii) of Lemma 4.9 Proposition 4.14
Theorem 2.16	Riemann germs		Subsection 4.7
Theorem 2.17	Riemann monotonicity of $\sigma \diamond \hat{f}$ i) general result ii) Kruřkov case	Proposition 3.8 Proposition 3.8	Subsection 5.4 Proposition 5.8 Corollary 5.10
Theorem 2.19	gluing of Riemann germs i) gluing and iv) nature of glued germs ii) associativity iii) identity element		Subsection 5.5 $\left\{ \begin{array}{l} \text{Proposition 5.11} \\ \text{i) of Corollary 5.12} \end{array} \right.$ Lemma 5.13 ii) of Corollary 5.12
Theorem 2.21	properties of Kruřkov germs i) characterization ii) D -maximality		Subsection 6.2 Proposition 6.4 Lemma 6.2
Theorem 2.22	characterization of monotone germs		iii) of Lemma 5.5
Theorem 2.23	characterization of D_+ -germs		$\left\{ \begin{array}{l} \text{Proposition 6.6} \\ \text{i') and iii) of Lemma 5.5} \end{array} \right.$
Theorem 2.24	conservative Riemann germs		Subsection 6.4
Theorem 2.25	properties of HJ germs i) regularity of \hat{f} ii) relaxation formula		Subsection 6.5 $\left\{ \begin{array}{l} \text{i) of Proposition 6.9} \\ \text{ii) of Theorem 2.14} \end{array} \right.$ ii) of Proposition 6.9
Theorem 2.28	HJ germ \mathcal{G} determined by $\chi\mathcal{G}$		Subsection 6.6
Theorem 2.29	properties of conservative 1 : 1 germs i) nature of the germs ii) relaxation formula i) relation (2.20)		Subsection 6.7 $\left\{ \begin{array}{l} \text{Lemma 6.11} \\ \text{Theorem 2.28} \end{array} \right.$ ii) of Proposition 6.9 Lemma 7.17
Proposition 2.30	germ product property		Subsection 6.8
Theorem 2.38	properties of semisolutions i) Stability ii) Max/Min iii) L^1 -comparison		Subsection 10 Lemma 10.2 Lemma 10.3 Lemma 10.4
Theorem 2.36	theory for Kruřkov germs i) existence and uniqueness ii) L^1 -contraction		Proposition 11.8 Lemma 10.4

For the precise material of the subsections, we refer the reader to the content given at the beginning of the paper. We insist below on the general structure/spirit of the paper.

In Section 3, we develop the theory for a single branch ($N = 1$) in the case of 1 : 0 junctions. The fundamental concepts (like basins of attraction $BA(\hat{p})$, the nonlinear projection $\pi_{\mathcal{G}}$, the Godunov flux $\hat{f}_{\mathcal{G}}$,

generalized Riemann germs, level set formulation of the germ \mathcal{G}_g , relaxation formula, Riemann monotonicity, local constancy) introduced in this paper can already easily be understood in this section. Only the gluing requires the understanding of the case $N > 1$. For $N = 1$, the flux is denoted by g , while for $N \geq 1$, the flux vector is denoted by $f = (f^1, \dots, f^N)$, in order to avoid any confusion between these cases.

In Section 4, we extend the theory to the case of $N \geq 1$ branches, and focus on the case of $N : 0$ junctions. We start the section, showing that the general case of $n : m$ junctions can always be reduced to the case of $N : 0$ junctions with $N := n + m$, and furthermore with coercive fluxes satisfying $f^j(p^j) \rightarrow +\infty$ as $|p^j| \rightarrow +\infty$. This reduction can be done using certain commutative transforms which are very convenient.

It turns out that there are two types of transforms that are interesting: the reversions (changing a branch $(-\infty, 0)$ into $(0, +\infty)$ and vice versa, and changing the sign of the flux) and the inversions (keeping the orientation of the branches unchanged, but changing the sign of the flux and of its arguments).

In the case $N \geq 1$, the key tool appears to be the slicing Lemma 4.12, which allows to reduce a germ from N branches to a germ with $N - k$ branches. Obviously any germ \mathcal{G} can not be sliced in a naive way. But using its Godunov flux $p \mapsto \hat{f}_{\mathcal{G}}(p)$, we can freeze the last k components of p , and then define a frozen flux which is associated to a new germ for $N - k$ branches. The slicing lemma is used in many ways to analyse generalized Riemann germs, and then to deduce their fundamental properties, similar to the case $N = 1$.

In Section 5, we excavate a fundamental property of germs: their Riemann monotonicity. Except for pathological fluxes (for instance fluxes which are constant on some intervals), this property is satisfied by any Riemann germ. Riemann monotonicity may be seen as a property coming from nowhere, but this is not the case. This property is very natural and necessary, once we are interested in the gluing of germs. This is the careful study of the gluing of germs that made this property to appear as a natural property. We show that the natural projection $\pi_{\mathcal{G}}$ is already Riemann monotone, and in some sense, this implies that $\sigma \diamond \hat{f}_{\mathcal{G}}$ is also Riemann monotone, where σ encodes the orientations of the branches. With this key property in hands, it becomes then easy to glue Riemann germs together (when their fluxes and branch orientations agree in a suitable way). Again, we first perform the gluing on the Godunov fluxes, and then deduce from it and justify the natural gluing of the germs.

Still in Section 5, we show that gluing preserves certain classes of germs (Kruřkov, HJ, monotone, conservative). Again, we first show it at the level of the associated Godunov fluxes, and then deduce these properties at the level of germs.

In Section 6, we give some applications of the theory mainly to the cases of Kruřkov germs and of HJ germs. For both, we show that all generalized Riemann germs have necessarily continuous Godunov fluxes, and then are Riemann germs, just by definition. A key result for HJ germs is that they are completely characterized by a characteristic subset $\chi\mathcal{G}$, which is a finite set in many applications. A culminating application is the case of $1 : 1$ junctions with conservative (generalized) Riemann germs. Such germs have all the best properties that we can expect: they are at the same time Kruřkov, HJ, monotone and conservative germs. They are characterized by their characteristic subset $\chi\mathcal{G}$ which is finite in many applications. And their Godunov flux $\hat{f}_{\mathcal{G}} =: (\hat{h}, \hat{h})$ is such that the function $\hat{h} : [a, b] \rightarrow \mathbb{R}$ satisfies the best relaxation formula that we can expect.

Section 7 provides complementary results. The main contributions concern duality for monotone Kruřkov germs, and polar decomposition for bell-shaped fluxes.

Section 8 gives various exemples and counter-exemples to illustrate the general theory.

Section 9 is an appendix. In the first subsection, we mainly recall the theory for solving the standard scalar Riemann problem on the real line, with initial data p involving two constant values (p^L, p^R) , one on the left, and one on the right. The explicit expression of the solution involving concave/convex envelopes of the flux, is here a key ingredient. This ingredient allows us, say for a solution defined on $(0, +\infty)_t \times (0, +\infty)_x$, to decide which trace \hat{p} the solution may reach on the axis $\{x = 0\}$, given its constant initial data $p := p^R$ on the set $\{0\}_t \times (0, +\infty)_x$. This allows us to study the set $\hat{\mathcal{P}}_p$ of such values \hat{p} . Indeed we can then see that the basin of attraction $BA(\hat{p})$ for $J^R = (0, +\infty)$, is nothing else than the set of p such that $\hat{p} \in \hat{\mathcal{P}}_p$.

In Section 9, the second subsection gives some important (independent) results about reduction of test functions for Hamilton-Jacobi equations. This is a key result which allows us to show that any HJ germ \mathcal{G}

is determined completely by its characteristic subset $\chi\mathcal{G}$.

Part II of the paper is composed of six sections.

Finally the bibliography is quite reduced for part I. This is in particular due to the novelty of the notion of germ, and to the new point of view that we develop here. The bibliography of Part II provides additional materials which are either works of fundamental interest, either useful illustrations for some examples, or useful results that we need for technical proofs.

2.5 Main notations

(e_1, \dots, e_N)	= canonical basis of \mathbb{R}^N
$J^j \simeq (0, +\infty)$ or $(-\infty, 0)$	= j -th (oriented) branch
$\sigma^j = \pm 1$	= "orientation" of j -th branch $J^j \simeq \sigma^j \cdot (-\infty, 0)$
$f^j : [a^j, b^j] \rightarrow \mathbb{R}$	= j -th flux
$[a, b] := \prod_{j=1, \dots, N} [a^j, b^j]$	= the box (with box convention (2.3))
$f = (f^j)_{j=1, \dots, N} : [a, b] \rightarrow \mathbb{R}^N$	= the flux function
$f^j(p) := f^j(p^j)$	= abuse of notation, for $p = (p^1, \dots, p^N)$
$p = (p^1, \dots, p^N) \leq 0$	= means $p^j \leq 0$ for all j
$p = (p^1, \dots, p^N) < 0$	= means $p^j < 0$ for all j
$\mathcal{G} \subset [a, b]$	= set or germ
$\pi = \pi_{\mathcal{G}} : [a, b] \rightarrow \mathcal{G}$	= natural projection on \mathcal{G}
$\hat{f} = \hat{f}_{\mathcal{G}} = f \circ \pi_{\mathcal{G}}$	= Godunov flux at the junction
$\mathcal{G}_{\hat{f}} := \{ \hat{f} = f \}$	= level set formulation of the germ
$f_{\pm} = (f_{\pm}^1, \dots, f_{\pm}^N)$	= monotone bounds
$BA^j(\hat{p}^j) = BA^{(J^j, f^j)}(\hat{p}^j)$	= Basin of Attraction of the point $\hat{p}^j \in [a^j, b^j]$
$BA_-^j(\hat{p}^j), BA_+^j(\hat{p}^j)$	= lower, upper Basins of Attraction
$BA(\hat{p}) = \prod_{j=1, \dots, N} BA^j(\hat{p}^j)$	= Basin of Attraction of the point $\hat{p} \in \mathcal{G}$
$G^j = G^{f^j} : [a^j, b^j]^2 \rightarrow \mathbb{R}$	= standard Godunov flux associated to f^j
$D^j = D^{f^j} = \sigma^j \psi^{f^j} : [a^j, b^j]^2 \rightarrow \mathbb{R}$	= j -th dissipation (see (2.8))
$D^f = \sum_{j=1, \dots, N} D^{f^j}$	= dissipation
$D_+^j = D_+^{f^j} = \sigma^j \psi_+^{f^j} : [a^j, b^j]^2 \rightarrow \mathbb{R}$	= j -th semi-dissipation (see (2.9))
$D_+^f = \sum_{j=1, \dots, N} D_+^{f^j}$	= semi-dissipation
$RH^{\hat{f}}(p) = \sum_{j=1, \dots, N} \sigma^j \hat{f}^j(p)$	= Rankine-Hugoniot function
$u_{p, \hat{p}} = (u_{p^1, \hat{p}^1}^1, \dots, u_{p^N, \hat{p}^N}^N)$	= solution with initial data p and trace \hat{p}
$u_{p, \hat{p}}^{\mathcal{G}}$	= \mathcal{G} -entropy solution $u_{p, \hat{p}}$, i.e. with $\hat{p} \in \mathcal{G}$
$\hat{\mathcal{P}}_{p^j}^j$	= set of $\{ \hat{p}^j = \hat{p}_{\lambda, p^j}^j \}$ for solutions u_{p^j, \hat{p}^j} (see (4.7))
$\hat{\mathcal{P}}_p = \prod_{j=1, \dots, N} \hat{\mathcal{P}}_{p^j}^j$	= set of \hat{p} for solutions $u_{p, \hat{p}}$
$p \diamond q$	= Hadamard product of two vectors
$[\hat{f}]_q^p := \hat{f}(p) - \hat{f}(q)$	= bracket of \hat{f}
$\mathcal{G}_{p_0''}$	= slicing of germ \mathcal{G} w.r.t. p_0'' (see (4.12))
$i_p^j(q^j)$	= injection of j -th coordinate
$\mathcal{G}_1 \# \mathcal{G}_2$	= gluing of germs
$\underline{\chi} \mathcal{G}, \bar{\chi} \mathcal{G}, \chi \mathcal{G}$	= sub/super/characteristic subset of HJ germ \mathcal{G}

Recall that Section 3 focuses on the case $N = 1$. Then the index j is dropped everywhere, and f is replaced by g in order to avoid any confusion.

Part I

Structure of germs

3 Riemann problem on a single branch $N = 1$

3.1 Characterization of trace values on a junction $1 : 0$

In this section, we work with a single branch with $N = 1$ which is a junction $1 : 0$. Then we drop the index $k = 1$ everywhere and use the notation $g := f^1$, $a := a^1$, $b := b^1$ and $J^1 := (-\infty, 0)$ and work on the junction $J := (-\infty, 0]$. We assume

$$(3.1) \quad \left\{ \begin{array}{l} g : \mathbb{R} \supset [a, b] \rightarrow \mathbb{R} \text{ is Lipschitz continuous with } -\infty \leq a < b \leq +\infty, \quad \text{with box convention (2.3)} \\ \text{and} \\ g(p) \rightarrow +\infty \text{ if } |p| \rightarrow +\infty \text{ and } p \in [a, b] \end{array} \right. \quad (\text{coercivity})$$

which at this stage are conditions slightly less general than in (2.2), because we impose the orientation of the junction, and also the direction of the coercivity.

We want to understand the \mathcal{G} -Riemann problem for $\mathcal{G} := \{\hat{p}\}$ with $\hat{p} \in [a, b]$ and initial data $p \in [a, b]$, namely we look for Kruřkov entropy solutions $v(t, x) = v$ of the following Riemann problem

$$(3.2) \quad \left\{ \begin{array}{ll} v(t, x) = v\left(1, \frac{x}{t}\right) & v : [0, +\infty) \times (-\infty, 0] \rightarrow [a, b] \\ & \text{on } (0, +\infty) \times (-\infty, 0) \\ v_t + (g(v))_x = 0 & \text{on } (0, +\infty) \times (-\infty, 0) \\ v(0, \cdot) = p & \text{on } \{0\} \times (-\infty, 0) \\ v(\cdot, 0^-) = \hat{p} & \text{a.e. on } (0, +\infty) \times \{0\} \end{array} \right.$$

where the last condition arises in the sense of traces. We want to determine the set of p such that such a solution does exist.

To this end, we recall that the Godunov flux $G(\uparrow, \downarrow) : [a, b]^2 \rightarrow \mathbb{R}$ associated to g is given by

$$(3.3) \quad G(q, r) := \begin{cases} \inf_{[q, r]} g & \text{if } q \leq r \\ \sup_{[r, q]} g & \text{if } q \geq r \end{cases}$$

We set the following nondecreasing functions of p

$$(3.4) \quad g_-(p) := \inf_{[p, b]} g = G(p, b) \leq g_+(p) := \sup_{[a, p]} g = G(p, a)$$

and for $\lambda \in [g_-(p), g_+(p)]$, we define the following element of $[a, b]$

$$(3.5) \quad \hat{p}_{\lambda, p} := \begin{cases} p & \text{if } g(p) = \lambda \\ \sup \{q \in (p, b], \quad g > \lambda \text{ on } (p, q)\} & \text{if } g(p) > \lambda \\ \inf \{q \in [a, p), \quad g < \lambda \text{ on } (q, p)\} & \text{if } g(p) < \lambda \end{cases}$$

which is nonincreasing in λ . We then define the following set

$$(3.6) \quad \hat{\mathcal{P}}_p := \{\hat{p}_{\lambda, p} \in [a, b], \quad \lambda \in [g_-(p), g_+(p)]\}$$

Precisely we have the following result

Lemma 3.1 (Set of \hat{p} 's for which $\{\hat{p}\}$ -Riemann solutions exist with initial data p)

Assume (3.1). For any $p, \hat{p} \in [a, b]$, there exists an entropy solution $v = v_{p, \hat{p}}$ of (3.2) if and only if $\hat{p} \in \hat{\mathcal{P}}_p$, with $\hat{\mathcal{P}}_p$ given in (3.6).

Proof of Lemma 3.1
Step 1: Equivalence

Let v be an entropy solution to (3.2). We set $\bar{p} := \hat{p}$ and extend v as $\tilde{v}(t, x) := \begin{cases} v(t, x) & \text{for } x < 0 \\ \bar{p} & \text{for } x \geq 0 \end{cases}$. Notice that for $J = (-\infty, 0]$, $J^1 = (-\infty, 0)$ and $\varphi \in C_c^1([0, +\infty) \times J^1)$ with $\varphi \geq 0$, we have for all $k \in \mathbb{R}$

$$\int_{(0, +\infty) \times J} |\tilde{v} - k| \varphi_t + \text{sign}(\tilde{v} - k) \cdot \{g(\tilde{v}) - g(k)\} \varphi_x + \int_{\{0\} \times J} |\tilde{v}_0 - k| \varphi \geq 0.$$

In particular, we have for all $k \in \mathbb{R}$

$$|\tilde{v} - k|_t + \partial_x (\text{sign}(\tilde{v} - k) \cdot \{g(\tilde{v}) - g(k)\}) \leq 0 \quad \text{in } \mathcal{D}'((0, +\infty) \times (\mathbb{R} \setminus \{0\}))$$

Using the fact that $\tilde{v}(t, x)$ is bounded in L^∞ and has trace \bar{p} at $x = 0$, it is then easy to check that \tilde{v} is an entropy solution to the following standard Riemann problem

$$(3.7) \quad \begin{cases} \tilde{v}_t + (g(\tilde{v}))_x = 0 & \text{on } (0, +\infty) \times \mathbb{R} \\ \tilde{v}(0, x) = \tilde{v}_0(x) := \begin{cases} \bar{p} & \text{if } x > 0 \\ p & \text{if } x < 0 \end{cases} \end{cases}$$

i.e. for all $\varphi \in C_c^1([0, +\infty) \times \mathbb{R})$ with $\varphi \geq 0$, we have for all $k \in [a, b]$

$$\int_{(0, +\infty) \times \mathbb{R}} |\tilde{v} - k| \varphi_t + \text{sign}(\tilde{v} - k) \cdot \{g(\tilde{v}) - g(k)\} \varphi_x + \int_{\{0\} \times \mathbb{R}} |\tilde{v}_0 - k| \varphi \geq 0.$$

Then we see that v solves (3.2), if and only if \tilde{v} solves (3.7) and satisfies $\tilde{v}(t, 0^-) = \bar{p}$ for a.e. $t > 0$.

Step 2: Characterization

From Lemma 9.1, we know that the solution \tilde{v} to Riemann problem (3.7) is unique and has to satisfy $\tilde{v}(t, x) = U(x/t)$ with U given in (9.2) and $(p_L, p_R) := (p, \bar{p})$. In particular, when it makes sense, we have $\bar{p} = \tilde{v}(t, 0^-) = ((\tilde{g}_I)')^{-1}(0^-)$, where $I := [\min(p_L, p_R), \max(p_L, p_R)]$ and \tilde{g} is the convex (resp. the concave) envelope of g on the interval I if $p_L - p_R < 0$ (resp. $p_L - p_R > 0$).

Case 1: $p < \bar{p}$

This means $p_L < p_R$. Then $(\xi_L, \xi_R) = (\tilde{g}'(p_L^+), \tilde{g}'(p_R^-))$ with $\xi_L \leq \xi_R$, and we can not have $\xi_R > 0$. Hence $\tilde{g}'(p_R^-) = \xi_R \leq 0$. Either $\tilde{v}(t, 0^-) = p_R$ if $\xi_R < 0$. Or $\xi_R = 0$ and (using the fact that \tilde{g} is the convex envelop of g on some interval) we get

$$g(q) \geq g(p_R^-) + \xi_R(q - p_R) = g(p_R^-) \quad \text{for all } q \in [p_L, p_R]$$

Then either there exists some $\varepsilon > 0$ such that $\tilde{g}' = \xi_R = 0$ on $[p_R - \varepsilon, p_R]$, and then $p_R = \bar{p}$ is not the trace of \tilde{v} from the left side $\{x < 0\}$, because there is a jump just at the place $x = 0$. Or $\tilde{g}' < 0 = \xi_R = \tilde{g}'(p_R^-)$ on $[p_L, p_R]$, and the fact that \tilde{g}' is nondecreasing implies that $\tilde{v}(t, 0^-) = ((\tilde{g}_I)')^{-1}(0^-) = p_R$. In this case, we conclude that $\bar{p} = \tilde{v}(t, 0^-)$ if and only if $\tilde{g}'(\bar{p}^-) < 0$ or $\tilde{g}'(\bar{p}^-) = 0 > \tilde{g}'$ on $[p, \bar{p}]$. This means that

$$\bar{p} = \tilde{v}(t, 0^-) \quad \text{if and only if } \tilde{g}' < 0 \quad \text{on } [p, \bar{p}]$$

Case 2: $p > \bar{p}$

This case is very similar to Case 1 with \tilde{g} concave instead of convex. Again, we conclude that

$$\bar{p} = \tilde{v}(t, 0^-) \quad \text{if and only if } \tilde{g}' < 0 \quad \text{on } (\bar{p}, p]$$

Case 3: $\bar{p} = p$

Then the unique solution is $\tilde{v} \equiv p$, and then the trace condition $\bar{p} = \tilde{v}(t, 0^+)$ is obviously satisfied.

Conclusion

We conclude that $\bar{p} = \tilde{v}(t, 0^-)$ iff $\begin{cases} \bar{p} = p \\ \bar{p} < p & \text{and } \tilde{g}' < 0 \quad \text{on } (\bar{p}, p] \\ \bar{p} > p & \text{and } \tilde{g}' < 0 \quad \text{on } [p, \bar{p}] \end{cases}$ (with \tilde{g} concave), i.e.

$$(3.8) \quad \bar{p} = \tilde{v}(t, 0^-) \quad \text{iff} \quad \begin{cases} \bar{p} = p \\ \bar{p} < p & \text{with } \tilde{g} \text{ concave decreasing on } [\bar{p}, p] \\ \bar{p} > p & \text{with } \tilde{g} \text{ convex decreasing on } [p, \bar{p}] \end{cases}$$

Let us call $\bar{\mathcal{P}}_p$ the set of $\bar{p} \in [a, b]$ characterized by the right hand side of (3.8). Still because of Lemma 9.1, we see that such a solution \tilde{v} does exist in each case covered in (3.8).

Step 3: Equivalent characterization

For $\bar{p} \in [a, b]$ and $\lambda := g(\bar{p})$, it is easy to see from (3.8) that $\bar{p} = \hat{p}_{\lambda, p}$ and $\lambda \in [g_-(p), g_+(p)]$. This shows that $\bar{p} \in \hat{\mathcal{P}}_p$ and then $\bar{\mathcal{P}}_p \subset \hat{\mathcal{P}}_p$. The reverse inclusion is also straightforward to check. This shows $\bar{\mathcal{P}}_p = \hat{\mathcal{P}}_p$ and ends the proof of the lemma.

3.2 Basin of attraction: the map $\tilde{p} \mapsto BA(\tilde{p})$

The main result of this subsection is the following inverse characterization of the map $p \mapsto \hat{\mathcal{P}}_p$.

Lemma 3.2 (Inverse characterization of the map $p \mapsto \hat{\mathcal{P}}_p$)

Assume (3.1) with $J = (-\infty, 0]$. Let $p \in [a, b]$ and some $\tilde{p} \in \hat{\mathcal{P}}_p$. Then for all $p' \in [a, b]$, we have

$$\tilde{p} \in \hat{\mathcal{P}}_{p'} \quad \text{if and only if} \quad p' \in BA(\tilde{p})$$

where

$$(3.9) \quad BA(\tilde{p}) := BA_-(\tilde{p}) \cup \{\tilde{p}\} \cup BA_+(\tilde{p})$$

and $BA_{\pm} = BA_{\pm}(\tilde{p})$ are relative open sets of $[a, b]$ given for $\lambda := g(\tilde{p})$ by $\left\{ \begin{array}{ll} BA_+ := \{q \in (\tilde{p}, b], & g < \lambda \quad \text{on} \quad (\tilde{p}, q] \\ BA_- := \{q \in [a, \tilde{p}), & g > \lambda \quad \text{on} \quad [q, \tilde{p}) \end{array} \right.$.

Proof of Lemma 3.2

Let $p \in [a, b]$ and $\tilde{p} \in \hat{\mathcal{P}}_p$. Notice that it is much more simple to make the reasoning on a picture (see Figure just before Subsubsection 2.1.2).

Step 1: proof that $p' \in BA(\tilde{p})$ implies $\tilde{p} \in \hat{\mathcal{P}}_{p'}$

Consider some $p' \in BA(\tilde{p})$. Then whatever is the position of p' with respect to \tilde{p} , we get

$$g_-(p') = \inf_{[p', b]} g \leq g(\tilde{p}) = \lambda \leq \sup_{[a, p']} g = g_+(p')$$

Therefore, we can consider the quantity $\hat{p}_{\lambda, p'}$, and whatever is the position of $g(p')$ with respect to $\lambda = g(\tilde{p})$, we easily get from its definition (3.5) that $\hat{p}_{\lambda, p'} = \tilde{p}$. Therefore $\tilde{p} = \hat{p}_{\lambda, p'} \in \hat{\mathcal{P}}_{p'}$.

Step 2: proof that $p' \in BA(\tilde{p})$ is implied by $\tilde{p} \in \hat{\mathcal{P}}_{p'}$

Now consider some $p' \in [a, b]$ such that $\tilde{p} \in \hat{\mathcal{P}}_{p'}$. Hence $\tilde{p} = \hat{p}_{\lambda', p'}$ for some $\lambda' \in [g_-(p'), g_+(p')]$. Now whatever is the position of $\lambda' = g(\tilde{p})$ with respect to $g(p')$, we easily get from (3.9) that $p' \in BA(\tilde{p})$.

This ends the proof of the lemma.

Then we get immediately

Corollary 3.3 (Set of p 's for which $\{\hat{p}\}$ -Riemann solutions exist with initial data p)

Assume (3.1). For any $p, \hat{p} \in [a, b]$, there exists an entropy solution $v = v_{p, \hat{p}}$ of (3.2) if and only if $p \in BA(\hat{p})$, with $BA(\hat{p})$ given in (3.9).

3.3 Dissipation property of basins of attraction

We have the following result.

Lemma 3.4 (Dissipation property of basins of attraction, $N = 1$, junction 1 : 0)

Assume (3.1). Let $\hat{p}, \hat{q} \in [a, b]$ be such that $BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset$. Then we have

$$(3.10) \quad D^g(\hat{q}, \hat{p}) \leq 0 \quad \text{with} \quad D^g(\hat{q}, \hat{p}) := \text{sign}(\hat{q} - \hat{p}) \cdot \{g(\hat{q}) - g(\hat{p})\}$$

Moreover

$$(3.11) \quad \text{either} \quad D^g(\hat{q}, \hat{p}) < 0, \quad \text{or} \quad \hat{p} = \hat{q}$$

Proof of Lemma 3.4

Notice that $D^g(\hat{p}, \hat{q}) = D^g(\hat{q}, \hat{p})$. Assume by symmetry that $\hat{p} \leq \hat{q}$, and set $\underline{q} := \inf BA(\hat{q})$, $\bar{p} := \sup BA(\hat{p})$, and let us show that

$$(3.12) \quad g(\hat{q}) < g(\hat{p}) \quad \text{or} \quad \hat{p} = \hat{q}$$

We first notice that for all $p \in BA(\hat{p})$, we have $D^g(p, \hat{p}) \leq 0$, with moreover $D^g(p, \hat{p}) = 0$ if and only if $p = \hat{p}$. Therefore if $\hat{p} \in BA(\hat{q})$ or $\hat{q} \in BA(\hat{p})$, this implies (14.11).

If $\underline{q} = \hat{q}$, then $\hat{q} \in BA(\hat{p})$. Similarly, if $\bar{p} = \hat{p}$, then $\hat{p} \in BA(\hat{q})$.

Assume now that $\hat{p} < \bar{p}$ and $\underline{q} < \hat{q}$. If $\bar{p} = b$, and $g(b) < g(\hat{p})$, then $BA(\hat{p}) \cap [\hat{p}, b] = [\hat{p}, b]$ and then $\hat{q} \in BA(\hat{p})$. Otherwise, we have $BA(\hat{p}) \cap [\hat{p}, b] = [\hat{p}, \bar{p}]$ and $g(\bar{p}) = g(\hat{p})$.

Similarly, if $\underline{q} = a$ and $g(a) > g(\hat{q})$, then $BA(\hat{q}) \cap [a, \hat{q}] = [a, \hat{q}]$ and then $\hat{p} \in BA(\hat{q})$. Otherwise, we have $BA(\hat{q}) \cap [a, \hat{q}] = (\underline{q}, \hat{q})$ and $g(\underline{q}) = g(\hat{q})$.

Now $BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset$ implies $\underline{q} < \bar{p}$. Now if $\underline{q} < \hat{p}$, then $\hat{p} \in BA(\hat{q})$. Similarly, if $\hat{q} < \bar{p}$, then $\hat{q} \in BA(\hat{p})$. Hence we can now assume that $\hat{p} \leq \underline{q} < \bar{p} \leq \hat{q}$ with $g(\hat{p}) = g(\bar{p})$ and $g(\underline{q}) = g(\hat{q})$. Now recall that we have

$$g < g(\hat{p}) = g(\bar{p}) \quad \text{on} \quad (\hat{p}, \bar{p}) \supset (\underline{q}, \bar{p}) \quad \text{which implies} \quad g(\underline{q}) < g(\bar{p})$$

Hence $g(\hat{q}) < g(\hat{p})$ which implies (14.11).

Finally, we conclude that in all cases, we have (14.11) which implies both (3.10) and (3.11). This ends the proof of the lemma.

3.4 Characterizations of generalized Riemann germs

The following result follows immediately from the definitions and Lemma 3.2.

Lemma 3.5 (First characterizations of generalized Riemann germs, $N = 1$)

Assume (3.1) and let $\mathcal{G} \subset [a, b]$ be a set.

i) (First characterization)

The set \mathcal{G} is a generalized Riemann germ if and only if for all $p \in [a, b]$, we have the singleton property $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$ where $\hat{\mathcal{P}}_p$ is defined in (3.6).

ii) (Equivalent characterization)

The set \mathcal{G} is a generalized Riemann germ if and only if $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}}$ is a partition of $[a, b]$.

Proposition 3.6 (First properties of generalized Riemann germs, $N = 1$)

Assume (3.1) and let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ for a junction $1 : 0$ with $J = (-\infty, 0]$ and $f := g$. Given $p \in [a, b]$, the unique \mathcal{G} -entropy solution of (2.6) writes $u_{p, \hat{p}}^{\mathcal{G}}$ with $\pi(p) := \pi_{\mathcal{G}}(p) := \hat{p} \in \mathcal{G}$ and $\pi : [a, b] \rightarrow \mathcal{G}$ which satisfies $\pi \circ \pi = \pi$. We set $\hat{g} := \hat{g}_{\mathcal{G}} := g \circ \pi$.

i) (Local constancy): The map \hat{g} is locally constant on $\{\hat{g} \neq g\}$.

ii) (Inverse of π): For all $\hat{p} \in \mathcal{G}$, we have $\pi^{-1}(\hat{p}) = BA(\hat{p})$.

iii) (Level set formulation of the germ): We have $\mathcal{G} = \mathcal{G}_{\hat{g}} := \{p \in [a, b], \hat{g}(p) = g(p)\}$.

iv) (Characterization of \hat{g}): The function $\hat{g} : [a, b] \rightarrow \mathbb{R}$ is fully characterized as a continuous function which is locally constant on $\{\hat{g} \neq g\}$ such that $\mathcal{G} = \mathcal{G}_{\hat{g}}$.

v) (Monotone bounds): We have $g_- \leq \hat{g} \leq g_+$ with g_{\pm} defined in (3.4).

vi) (Monotonicity of π): The map π is nondecreasing on $[a, b]$.

vii) (Continuity): The map \hat{g} is continuous.

Remark 3.7 Notice that the monotonicity of \hat{g} is not proved in Proposition 3.6. This seems to be a delicate property. It will be proved later, using the locally Lipschitz properties of \hat{g} in order to clean the possible accumulation of basins of attraction.

Proof of Proposition 3.6

Step 1: proof of vi)

Let $p \leq q$. Assume by contradiction that $\pi(p) = \hat{p} > \hat{q} = \pi(q)$. If $BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$, then

$$BA(\hat{q}) < BA(\hat{p})$$

where for sets $A, B \subset \mathbb{R}$, for $A < B$, we mean $a < b$ for all $(a, b) \in A \times B$. This implies that $q < p$. Contradiction. Hence $BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset$, and the partition property in Lemma 3.5 shows that $\hat{p} = \hat{q}$. Again contradiction. Therefore, we conclude that $\hat{p} \leq \hat{q}$, and the map π is nondecreasing.

Step 2: proof of ii)

From i) of Lemma 3.5 and Lemma 3.2, we deduce that $\pi^{-1}(\hat{p}) = BA(\hat{p})$ for any $\hat{p} \in \mathcal{G}$.

Step 3: proof of iii)

Recall that $\hat{g} = g \circ \pi$. Because $\pi^{-1}(\hat{p}) = BA(\hat{p})$ for any $\hat{p} \in \mathcal{G}$, we see that \hat{g} is constant on each $BA(\hat{p})$ with value $\hat{g}(p) = g(\hat{p})$ for every $p \in BA(\hat{p})$. This implies $\mathcal{G} \subset \mathcal{G}_{\hat{g}}$. Conversely, let $q \in \mathcal{G}_{\hat{g}}$. Then $g(q) = \hat{g}(q) = g(\hat{q})$ for the unique $\hat{q} \in \mathcal{G}$ such that $q \in BA(\hat{q})$. Because $g \neq g(\hat{q})$ on $BA(\hat{q}) \setminus \{\hat{q}\}$, we deduce that $q \in \{\hat{g} = g\} \cap BA(\hat{q}) = \{\hat{q}\} \subset \mathcal{G}$, which shows that $\mathcal{G}_{\hat{g}} \subset \mathcal{G}$. Therefore we have equality $\mathcal{G}_{\hat{g}} = \mathcal{G}$.

Step 4: proof of i)

Let $p \in [a, b] \cap \{\hat{g} \neq g\}$ and let $\hat{p} \in \mathcal{G}$ such that $p \in BA(\hat{p})$. Then p belongs to $BA(\hat{p}) \setminus \{\hat{p}\}$ which is a relative open set of $[a, b]$. We deduce the existence of some $\varepsilon > 0$ such that

$$\omega := Q_p \cap [a, b] \subset BA(\hat{p}) \quad \text{with} \quad Q_p := (p - \varepsilon, p + \varepsilon)$$

Because $\hat{g} = \text{const} = g(\hat{p})$ on ω , this shows the local constancy of the map \hat{g} .

Step 5: proof of v)

Consider $p \in [a, b]$ and let $\hat{p} \in \mathcal{G}$ such that $p \in BA(\hat{p})$. From Lemma 3.2, we deduce that $\hat{p} \in \hat{\mathcal{P}}_p$, and then there exists $\lambda = g(\hat{p}) \in [g_-(p), g_+(p)]$ such that $\hat{p} = \hat{p}_{\lambda, p}$. Because $\hat{g}(\hat{p}) = g(\hat{p})$, this shows point v).

Step 6: proof that \mathcal{G} is closed

Consider a sequence $(\hat{p}_n)_{n \in \mathbb{N}}$ with $\hat{p}_n \in \mathcal{G}$ such that $\hat{p}_n \rightarrow \hat{p}_\infty \in [a, b]$. We set $\tilde{p}_\infty := \pi(\hat{p}_\infty) \in \mathcal{G}$. Either $\hat{p}_\infty = \tilde{p}_\infty$ and the proof is done, or

$$(3.13) \quad \hat{p}_\infty \in BA(\tilde{p}_\infty) \setminus \{\tilde{p}_\infty\}$$

Then Step 3 shows that

$$(3.14) \quad (\hat{p}_\infty - \varepsilon, \hat{p}_\infty + \varepsilon) \cap [a, b] \subset BA(\tilde{p}_\infty)$$

Therefore $\hat{p}_n \in BA(\tilde{p}_\infty)$ for n large enough, i.e. $\hat{p}_n = \pi(\hat{p}_n) = \tilde{p}_\infty$. This implies that $\hat{p}_\infty = \tilde{p}_\infty \in \mathcal{G}$. Contradiction. Hence we always have $\hat{p}_\infty = \tilde{p}_\infty \in \mathcal{G}$, and then \mathcal{G} is closed.

Step 7: proof of vii)

The continuity of \hat{g} is not straightforward because we may have accumulation of basins of attraction.

Consider a sequence $(p_n)_{n \in \mathbb{N}}$ with $p_n \in [a, b]$ such that we assume by contradiction that

$$(3.15) \quad p_n \rightarrow p_\infty, \quad \hat{g}(p_n) \not\rightarrow \hat{g}(p_\infty)$$

We set $\hat{p}_n = \pi(p_n)$ and $\tilde{p}_\infty = \pi(p_\infty)$, and then have $g(\hat{p}_n) = \hat{g}(p_n) \not\rightarrow \hat{g}(p_\infty) = g(\tilde{p}_\infty)$. Using the continuity of g , and up to extract a subsequence, we can assume furthermore (from the closedness of \mathcal{G}) that there exists $\hat{p}_\infty \in \mathcal{G}$ such that

$$(3.16) \quad p_n \rightarrow p_\infty, \quad \hat{p}_n \rightarrow \hat{p}_\infty \neq \tilde{p}_\infty$$

We exhaust the different possible cases.

Case A: $p_\infty \neq \tilde{p}_\infty$

Then $p_\infty \in BA(\tilde{p}_\infty) \setminus \{\tilde{p}_\infty\}$, and from the definition of the basin of attraction, we deduce that $p_n \in BA(\tilde{p}_\infty)$, and then $\pi(p_n) = \hat{p}_n = \tilde{p}_\infty$. Contradiction with (3.16).

Case B: $p_\infty = \tilde{p}_\infty$

From (3.16), we can assume that $\hat{p}_\infty < \tilde{p}_\infty$ (the case $\hat{p}_\infty > \tilde{p}_\infty$ can be dealt in a similar way). Hence we have $\hat{p}_n \rightarrow \hat{p}_\infty < \tilde{p}_\infty = \lim_{n \rightarrow +\infty} p_n$. Therefore $[\hat{p}_n, p_n) \subset BA(\hat{p}_n)$ with $(\hat{p}_n, p_n) \rightarrow (\hat{p}_\infty, \tilde{p}_\infty)$. The fact that the family $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}}$ forms a partition of $[a, b]$ implies that the sequence $(\hat{p}_n)_n$ is stationary for large n , i.e. that $\hat{p}_n = \hat{p}_\infty$. Then

$$(3.17) \quad [\hat{p}_\infty, \tilde{p}_\infty) \subset BA(\hat{p}_\infty) \not\ni \tilde{p}_\infty$$

By definition of the basin of attraction, we deduce that

$$g(\tilde{p}_\infty) = g(\hat{p}_\infty) \quad \text{or} \quad (\tilde{p}_\infty = b \quad \text{and} \quad g(\tilde{p}_\infty) < g(\hat{p}_\infty))$$

Case B.1: $g(\tilde{p}_\infty) = g(\hat{p}_\infty)$

Then $\hat{g}(p_n) = g(\hat{p}_n) = g(\hat{p}_\infty) = g(\tilde{p}_\infty) = \hat{g}(p_\infty)$. Contradiction with (3.15).

Case B.2: $\tilde{p}_\infty = b$ and $g(\tilde{p}_\infty) < g(\hat{p}_\infty)$

The strict inequality and the construction of $BA(\hat{p}_\infty)$, imply that $\tilde{p}_\infty \in BA(\hat{p}_\infty)$, which gives a contradiction with (3.17). We conclude that (3.15) is impossible, and then \hat{g} is continuous.

Step 8: proof of iv)

Assume that $\tilde{g} : [a, b] \rightarrow \mathbb{R}$ is continuous which is locally constant on $\{\tilde{g} \neq g\}$ such that $\mathcal{G} = \mathcal{G}_{\tilde{g}}$. Then we have $\tilde{g} = g = \hat{g}$ on \mathcal{G} , where $\mathcal{G} = \mathcal{G}_{\tilde{g}}$ is a closed set, because \hat{g} is continuous. Moreover, we have $\tilde{g}' = 0$ on $[a, b] \setminus \mathcal{G}$. By continuity of g , \tilde{g} and the structure of each basin on attraction, we deduce that $\tilde{g} = g(\hat{p}) = \hat{g}$ on $BA(\hat{p})$ for all $\hat{p} \in \mathcal{G}$. Because the family of basins of attraction forms a partition of $[a, b]$, we deduce that $\tilde{g} = \hat{g}$. This ends the proof of the proposition.

Proposition 3.8 (Monotonicity of \hat{g} , $N = 1$)

We work under assumptions of Proposition 3.6. Then \hat{g} is nondecreasing and locally Lipschitz continuous, satisfying

$$(3.18) \quad \max(0, g') \geq \hat{g}' \geq 0 \quad \text{a.e. on } [a, b].$$

and

$$(3.19) \quad (\hat{g}' \in \{0, \max(0, g')\}) \quad \text{and } \hat{g}' = g' > 0 \text{ implies } \hat{g} = g \text{ holds a.e. on } [a, b]$$

Proof of Proposition 3.8

Step 1: Lipschitz continuity of \hat{g}

Consider two points $p, q \in (a, b)$ such that $p < q$, and set $\hat{p} = \pi(p), \hat{q} = \pi(q)$. From the monotonicity of π , we deduce that $\hat{p} \leq \hat{q}$. Assume moreover that

$$\hat{p} < \hat{q}$$

With notation of Lemma 3.2, we can then write $BA_+(\hat{p}) = (\hat{p}, \bar{p})$, $BA_-(\hat{q}) = (\underline{q}, \hat{q})$, where we recall that

$$\left\{ \begin{array}{l} BA_+(\hat{p}) := \{r \in (\hat{p}, b], \quad g < g(\hat{p}) \quad \text{on } (\hat{p}, r]\} \\ BA_-(\hat{q}) := \{r \in [a, \hat{q}), \quad g > g(\hat{q}) \quad \text{on } [r, \hat{q})\} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} \bar{p} := \begin{cases} \hat{p} & \text{if } BA_+(\hat{p}) = \emptyset \\ \sup BA_+(\hat{p}) & \text{if } BA_+(\hat{p}) \neq \emptyset \end{cases} \\ \underline{q} := \begin{cases} \hat{q} & \text{if } BA_-(\hat{q}) = \emptyset \\ \inf BA_-(\hat{q}) & \text{if } BA_-(\hat{q}) \neq \emptyset \end{cases} \end{array} \right.$$

Because $BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$, we deduce that $\hat{p} \leq \bar{p} < \underline{q} \leq \hat{q}$. Moreover $p \in BA(\hat{p}), q \in BA(\hat{q})$ imply that

$$a < p \leq \bar{p} < \underline{q} \leq q < b \quad \text{with} \quad g(\bar{p}) = g(\hat{p}), \quad g(\underline{q}) = g(\hat{q})$$

We deduce that $\hat{g}(p) - \hat{g}(q) = g(\hat{p}) - g(\hat{q}) = g(\bar{p}) - g(\underline{q})$. Hence

$$\frac{|\hat{g}(p) - \hat{g}(q)|}{|p - q|} = \frac{|g(\bar{p}) - g(\underline{q})|}{|p - q|} \leq \frac{|g(\bar{p}) - g(\underline{q})|}{|\bar{p} - \underline{q}|} \leq \text{Lip}(g; K)$$

for any compact interval $K \subset [a, b]$ containing p and q . This implies that $\text{Lip}(\hat{g}; K) \leq \text{Lip}(g; K)$ and then \hat{g} is locally Lipschitz continuous on $[a, b]$. Moreover, consider a point $p_0 \in (a, b)$ where both \hat{g} and g have a derivative. Then choosing $p := p_0 < q$ or $p < q := p_0$, we see in the limit $|p - q| \rightarrow 0$ that $|\hat{g}'(p_0)| \leq |g'(p_0)|$. In particular, we get

$$(3.20) \quad |\hat{g}'| \leq |g'| \quad \text{a.e. on } [a, b]$$

Step 2: Monotonicity of \hat{g}

Assume by contradiction that \hat{g} is not nondecreasing on $[a, b]$. Then, because \hat{g} and g are Lipschitz continuous, by Rademacher's theorem, there exists at least a point $p_0 \in (a, b)$ where both \hat{g} and g are differentiable, such that

$$(3.21) \quad \hat{g}'(p_0) < 0$$

Because $\hat{g}' = 0$ on the open set $[a, b] \setminus \{\hat{g} = g\}$, we deduce that $\hat{g}(p_0) = g(p_0)$, i.e. $p_0 \in \mathcal{G}$, and then $p_0 = \hat{p}_0 := \pi(p_0)$.

Case A: p_0 is an accumulation point of \mathcal{G}

Then along the accumulation sequence, we get $g = \hat{g}$ and then $g'(p_0) = \hat{g}'(p_0) < 0$. Then the structure of each basin of attraction implies that $p_0 = \hat{p}_0$ satisfies for some small $\varepsilon > 0$

$$(3.22) \quad Q := (\hat{p}_0 - \varepsilon, \hat{p}_0 + \varepsilon) \subset BA(\hat{p}_0)$$

Case B: p_0 is an isolated point of \mathcal{G}

Again, the structure of each basin of attraction implies that $p_0 = \hat{p}_0$ satisfies (3.22).

Conclusion in both cases

Then (3.22) implies that $\hat{g} = \text{const}$ on Q . Therefore $\hat{g}'(p_0) = \hat{g}'(\hat{p}_0) = 0$. Contradiction with (3.21). This finally shows that \hat{g} is nondecreasing on $[a, b]$. This implies with (3.20) that we have $|g'| \geq \hat{g}' \geq 0$ a.e. on $[a, b]$. Now recall that \mathcal{G} is a closed set, and $\hat{g}' = 0$ on $(a, b) \setminus \mathcal{G}$, with $\hat{g} = g$ on \mathcal{G} . Therefore $\hat{g}' = g'$ a.e. on \mathcal{G} . We deduce that $\max(0, g') \geq \hat{g}' \geq 0$ a.e. on $[a, b]$, which shows (3.18). Moreover we also get (3.19).

This ends the proof of the proposition.

Proposition 3.9 (Second characterization of (generalized) Riemann germs, $N = 1$)

Assume (3.1). Let $\mathcal{G} \subset [a, b]$ be a set. Then \mathcal{G} is a generalized Riemann germ with respect to $(J, f) = ((-\infty, 0], g)$ if and only we have

$$(3.23) \quad \mathcal{G} = \mathcal{G}_{\tilde{g}} \quad \text{with} \quad \mathcal{G}_{\tilde{g}} := \{p \in [a, b], \quad \tilde{g}(p) = g(p)\}$$

for some function \tilde{g} satisfying

$$(3.24) \quad \tilde{g} : [a, b] \rightarrow \mathbb{R} \quad \text{nondecreasing, locally constant on } \{\tilde{g} \neq g\}, \text{ and satisfying } g_- \leq \tilde{g} \leq g_+$$

Moreover, when this is the case, the function \tilde{g} is continuous and \mathcal{G} is a Riemann germ.

Proof of Proposition 3.9

Part I: proof that $\mathcal{G} = \mathcal{G}_{\tilde{g}}$ with $\tilde{g} := \hat{g}$

Assume that \mathcal{G} is a generalized Riemann germ. Then from iii), iv) and v) of Proposition 3.6 and from monotonicity property of \hat{g} given in Proposition 3.8, we deduce that \hat{g} satisfies (3.23) with (3.24). Moreover vii) of Proposition shows that \hat{g} is continuous.

Part II: proof that (3.23)-(3.24) implies that \mathcal{G} is a generalized Riemann germ

We want to check that $\mathcal{G} := \mathcal{G}_{\tilde{g}}$ is a generalized Riemann germ, i.e. satisfies the following singleton property

$$\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\} \quad \text{for all } p \in [a, b]$$

Step 1: nonemptiness of $\mathcal{G} \cap \hat{\mathcal{P}}_p$

Recall that $g_- \leq \tilde{g} \leq g_+$. We first define $\hat{p} = \pi(p)$ for $p \in [a, b]$. Setting $\lambda := \tilde{g}(p)$, we define

$$\hat{p} := \pi(p) := \begin{cases} p & \text{if } g(p) = \lambda \\ \sup \{q \in [p, b], \quad g > \lambda \text{ on } (p, q]\} & \text{if } g(p) > \lambda \\ \inf \{q \in [a, p], \quad g < \lambda \text{ on } [q, p)\} & \text{if } g(p) < \lambda \end{cases}$$

Because \tilde{g} is locally constant where it differs from g , we deduce the following.

If $g(p) > \lambda$, then either $g(\hat{p}) = \lambda = \tilde{g}(\hat{p})$ or $g(\hat{p}) > \lambda = \tilde{g}(\hat{p})$ and $\hat{p} = a$. In the second case, we conclude that $\tilde{g}(\hat{p}) = \tilde{g}(a) \leq g_+(a) = g(a) = g(\hat{p})$. Contradiction. Therefore only the first case arises, i.e. $g(\hat{p}) = \lambda = \tilde{g}(\hat{p})$ and $\hat{p} \in \mathcal{G}$. Moreover with $\hat{p}_{\lambda, p}$ defined in (3.5), we have

$$\hat{p} = \hat{p}_{\lambda, p} \quad \text{with} \quad g_-(p) \leq g(p) < \lambda = \tilde{g}(\hat{p}) \leq \tilde{g}(p) \leq g_+(p)$$

Similarly, if $g(p) < \lambda$, then either $g(\hat{p}) = \lambda = \tilde{g}(\hat{p})$ or $g(\hat{p}) < \lambda$ and $\hat{p} = b$, and we exclude the second case. Hence in all cases, this defines $\mathcal{G} \ni \hat{p} = \hat{p}_{\lambda, p}$ with $\lambda \in [g_-(p), g_+(p)]$. Hence $\mathcal{G} \cap \hat{\mathcal{P}}_p \supset \{\hat{p}\}$.

Step 2: $\mathcal{G} \cap \hat{\mathcal{P}}_p$ is reduced to a singleton

Consider any $\hat{p}_\alpha \in \mathcal{G} \cap \hat{\mathcal{P}}_p$ for $\alpha = 1, 2$. Then there exists $\lambda_\alpha \in [g_-(p), g_+(p)]$ such that $\lambda_\alpha = g(\hat{p}_\alpha)$ and

moreover $\hat{p}_\alpha = \hat{p}_{\lambda_\alpha, p}$. Hence we have $\tilde{g}(\hat{p}_\alpha) = g(\hat{p}_\alpha) = \lambda_\alpha$. Notice that the definition of $\mathcal{G} := \mathcal{G}_{\tilde{g}}$ and the fact that \tilde{g} is locally constant on $\{\tilde{g} \neq g\}$ imply that

$$\tilde{g} = \text{const} = \tilde{g}(\hat{p}_\alpha) = g(\hat{p}_\alpha) = \lambda_\alpha \quad \text{on} \quad BA(\hat{p}_\alpha)$$

Because $p \in BA(\hat{p}_1) \cap BA(\hat{p}_2)$, this implies that $\lambda_1 = \tilde{g}(p) = \lambda_2$ and then $\hat{p}_1 = \hat{p}_2$, which shows that $\mathcal{G} \cap \hat{\mathcal{P}}_p$ is reduced to a singleton. We conclude that $\mathcal{G}_{\tilde{f}}$ is a generalized Riemann germ.

Part III: getting a Riemann germ

Moreover, from Proposition 3.6, we have $\hat{g}_{\mathcal{G}_{\tilde{g}}} = \tilde{g}$ which is continuous. Hence \mathcal{G} is a Riemann germ. This ends the proof of the proposition.

3.5 Basic relaxation formula

Proposition 3.10 (Basic relaxation formula, $N = 1$, junction $1 : 0$)

Assume (3.1). Let g_0 be a function satisfying

$$(3.25) \quad \begin{cases} g_0 : [a, b] \rightarrow \mathbb{R} \text{ is continuous nondecreasing} & \text{(Monotonicity)} \\ g_- \leq g_0 \leq g_+ \quad \text{with } g_\pm \text{ defined in (3.4)} & \text{(Bounds)} \end{cases}$$

i) (Basic relaxation formula)

Then the following formula defines uniquely a map $g_1 : [a, b] \rightarrow \mathbb{R}$

$$(3.26) \quad \{g_1(p)\} = \bigcup_{q \in [a, b]} \{G(p, q)\} \cap \{g_0(q)\} \quad \text{for all } p \in [a, b]$$

where G is the standard Godunov flux associated to the function g , defined in (3.3).

ii) (Application to generalized Riemann germs)

Moreover the map g_1 satisfies condition (3.25), and is such that $\mathcal{G}_{g_1} = \{g_1 = g\}$ is a generalized Riemann germ, and g_1 is the Godunov flux associated to \mathcal{G}_{g_1} on a junction $1 : 0$, i.e. $g_1 = \hat{g}_{\mathcal{G}_{g_1}}$.

Conversely, if \mathcal{G} is a generalized Riemann germ, then the associated function $\hat{g} := \hat{g}_{\mathcal{G}}$ satisfies (3.25) and (3.26) with $g_1 = g_0 = \hat{g}$.

Proof of Proposition 3.10

Step 1: g_1 is well-defined, nondecreasing, continuous and satisfies (3.25)

Let $g_0 : [a, b] \rightarrow \mathbb{R}$ satisfying (3.25). Given $p, q \in [a, b]$, we define $\Phi_p(q) := G(p, q) - g_0(q)$ where we recall the monotonicities $g_0(\uparrow)$ and $G(\uparrow, \downarrow)$. Recall that we have $g_-(p) = G(p, b) \leq g_0(p) \leq g_+(p) = G(p, a)$.

Step 1.1: discussion for a, b finite or not

Case A: finite a and b

Hence $\Phi_p(a) \geq 0 \geq \Phi_p(b)$ for finite a, b .

Case B: infinite a or b

If $a = -\infty$, then

$$\lim_{q \rightarrow a} \Phi_p(q) = \lim_{q \rightarrow a} \{G(p, q) - g_0(q)\} = \lim_{q \rightarrow a} \left\{ \sup_{[q, p]} g - g_0(q) \right\} = +\infty$$

where we have used the coercivity of g (as assumed in (3.1)) and the monotonicity of g_0 .

If $b = +\infty$, then

$$\lim_{q \rightarrow b} \Phi_p(q) = \lim_{q \rightarrow b} \{G(p, q) - g_0(q)\} \leq \lim_{q \rightarrow b} \{G(p, q) - g_-(q)\} = \lim_{q \rightarrow b} \left\{ \inf_{[p, q]} g - \inf_{[q, b]} g \right\} = -\infty$$

where we have used again the coercivity of g .

Step 1.2: remaining part of the argument

Because Φ_p is continuous nonincreasing, we deduce that there exists at least some $c \in [a, b]$ such that $\Phi_p(c) = 0$. Now even if c is non unique, the common value $g_0(c) = G(c, p)$ is unique, because of the monotonicities of g_0 and $G(p, \cdot)$. This defines uniquely the value $g_1(p)$ in (3.26). Consider $p' \geq p$. Then there exists $c, c' \in [a, b]$ such that

$$g_1(p) = g_0(c) = G(p, c) \leq G(p', c), \quad g_1(p') = g_0(c') = G(p', c')$$

Hence $\Phi_{p'}(c) \geq 0 = \Phi_p(c)$. This shows that we can always choose $c' \geq c$, and then $g_1(p') = g_0(c') \geq g_0(c) = g_1(p)$, which shows that g_1 is nonincreasing, and moreover, we can always choose a map $p \mapsto c = c(p)$ which is nondecreasing such that $\Phi_p(c(p)) = 0$. Moreover, the continuity of g_0, G and the uniqueness of the value $g_1(p)$ imply the continuity of g_1 .

Finally, by construction, for $p \in [a, b]$ we have $g_1(p) = g_0(c) = G(p, c) \in [G(p, b), G(p, a)] = [g_-(p), g_+(p)]$, which shows (3.25).

Step 2: local constancy of g_1

Assume that $g_1(p) \neq g(p)$ and recall that $g(p) \neq g_1(p) = g_0(c) = G(p, c) = \begin{cases} \sup_{[c,p]} g = \sup_{[c,p]} g & \text{if } c < p \\ \inf_{[p,c]} g = \inf_{(p,c]} g & \text{if } c > p \end{cases}$,

where we have used the fact that the inf / sup can not be reached at p , because $g(p) \neq g_1(p)$. Notice also that we can not have $c = p$, otherwise we would get $g_1(p) = g_0(p) = G(p, p) = g(p)$, which is impossible by assumption. This shows now that for p_ε close to p , we also have (by continuity of g, g_1, g_0)

$$g(p_\varepsilon) \neq g_1(p_\varepsilon) = g_0(c) = G(p_\varepsilon, c) = \begin{cases} \sup_{[c,p_\varepsilon]} g = \sup_{[c,p_\varepsilon]} g & \text{if } c < p_\varepsilon \\ \inf_{[p_\varepsilon,c]} g = \inf_{(p_\varepsilon,c]} g & \text{if } c > p_\varepsilon \end{cases} = g_1(p)$$

This justifies a posteriori that we can choose $c_\varepsilon := c$ in $g_1(p_\varepsilon) = g_0(c_\varepsilon) = G(p_\varepsilon, c_\varepsilon)$. Hence g_1 is locally constant on $\{g_1 \neq g\}$.

Step 3: conclusion for g_1

From Proposition 3.9, we deduce that \mathcal{G}_{g_1} is a Riemann germ on a junction $1 : 0$ and $g_1 = \hat{g}_{\mathcal{G}_{g_1}}$.

Step 4: conclusion for \hat{g}

Consider a generalized Riemann germ $\mathcal{G} \subset [a, b]$. Let $\hat{g} := \hat{g}_{\mathcal{G}}$ be the Godunov flux associated to \mathcal{G} on a junction $1 : 0$. Then we have $\mathcal{G} = \mathcal{G}_{\hat{g}} := \{\hat{g} = g\}$. From Propositions 3.6, 3.8, the function \hat{g} satisfies (3.25). Consider g_1 given by formula (3.26) for $g_0 := \hat{g}$. Now if $\hat{p} \in \mathcal{G}$, then we check immediately that $g_1(\hat{p}) = \hat{g}(\hat{p}) = g(\hat{p})$. Therefore $\mathcal{G} \subset \mathcal{G}_{g_1} := \{g_1 = g\}$. Because the family $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}}$ is already a partition of $[a, b]$, where \hat{g} and g_1 coincide, we deduce that $g_1 = \hat{g}$, and then $\mathcal{G}_{g_1} = \mathcal{G}_{\hat{g}} = \mathcal{G}$. Moreover, this shows that (3.26) holds true with $g_1 = g_0 = \hat{g}$. This ends the proof of the lemma.

3.6 Proposition 2.13, Theorem 2.14 and their proofs on a junction $1 : 0$

Proof of Proposition 2.13 for $1 : 0$ junction

For the proof we refer to the table of Subsection 2.4. Notice that for $1 : 0$ junctions, Riemann monotonicities of π and \hat{f} reduces to the fact that those functions are nondecreasing. Then the result of Proposition 2.13 for $1 : 0$ junction follows from Propositions 3.6, 3.8, 3.10 (respectively first properties of gen. Riemann germs, monotonicity, relaxation formula) and Lemma 3.4 (dissipation property).

Proof of Theorem 2.14 for $1 : 0$ junction

For the proof we refer to the table of Subsection 2.4. Point i) of Theorem 2.14 follows from ii) of Lemma 3.5 (first character. of gen. Riemann germs). Point ii) of Theorem 2.14 follows from Propositions 3.9 (second character. of (gen.) Riemann germs).

3.7 Weak stability of the Riemann problem

Lemma 3.11 (Weak stability of Riemann problem on a single branch $1 : 0$)

Assume (3.1). Let $p_n \in [a, b]$ and $\hat{p}_n \in BA(\hat{p}_n)$. We call $v_n = v_{p_n, \hat{p}_n}$ the unique entropy solution of (3.2) with $(p, \hat{p}) := (p_n, \hat{p}_n)$. Assume that $(p_n, \hat{p}_n) \rightarrow (p_\infty, \hat{p}_\infty) \in [a, b]^2$ as $n \rightarrow \infty$. Then, up to extraction of a subsequence (still denoted by v_n), there exists $\tilde{p}_\infty \in [a, b]$ such that $p_\infty \in BA(\tilde{p}_\infty)$ and

$$v_{p_n, \hat{p}_n} = v_n \rightarrow v_\infty := v_{p_\infty, \tilde{p}_\infty} \quad \text{in } L^1_{loc}([0, +\infty) \times (-\infty, 0])$$

Here \tilde{p}_∞ is the trace of the limit, while \hat{p}_∞ is the limit of the trace. Moreover, we have

$$(3.27) \quad g(\tilde{p}_\infty) = g(\hat{p}_\infty) \quad \text{with } \tilde{p}_\infty \in \{p_\infty, \hat{p}_\infty\}$$

(even if \hat{p}_n may not converge towards \tilde{p}_∞ , and p_∞ may not belong to $BA(\hat{p}_\infty)$).
In other words, for self-similar solutions on a half line, if the limit is not constant in space and time, then the trace of the limit is equal to the limit of the trace.

Proof of Lemma 3.11

Step 1: first results

Recall that $v_n(t, x) = V_n(x/t)$ with V_n monotone and bounded in $L^\infty((-\infty, 0]; [a, b])$. By classical Helly's theorem for monotone functions, we know that, up to extract a subsequence, we have $V_n \rightarrow V_\infty$ a.e. on $(-\infty, 0]$, and then $v_n \rightarrow v_\infty$ in $L^1_{loc}([0, +\infty) \times (-\infty, 0])$ with $v_\infty(t, x) := V_\infty(x/t)$. By stability of entropy solutions and of their initial data, we have that v_∞ is a solution of (3.2) with initial data $p := p_\infty$. Because V_∞ is monotone, it also has a trace $\tilde{p}_\infty \in [a, b]$ at $x = 0^-$. Hence $v_\infty = v_{p_\infty, \tilde{p}_\infty}$ with $p_\infty \in BA(\tilde{p}_\infty)$.

Moreover, we have
$$\begin{cases} \hat{p}_n := \hat{p}_{\hat{\lambda}_n, p_n}, & \text{with } \hat{\lambda}_n := g(\hat{p}_n) & \text{and } \hat{\lambda}_\infty := g(\hat{p}_\infty) \\ \tilde{p}_\infty := \tilde{p}_{\tilde{\lambda}_\infty, p_\infty} & \text{with } \tilde{\lambda}_\infty := g(\tilde{p}_\infty) \end{cases}.$$

Step 2: proof that $\tilde{\lambda}_\infty = \hat{\lambda}_\infty$

Case A: $p_\infty \neq \tilde{p}_\infty$

Because $BA(\tilde{p}_\infty) \setminus \{\tilde{p}_\infty\}$ is a relative open set of $[a, b]$, we deduce that $p_n \in BA(\tilde{p}_\infty) \setminus \{\tilde{p}_\infty\}$ for n large enough. Hence $\hat{p}_n = \tilde{p}_\infty$, and then $\tilde{p}_\infty = \hat{p}_\infty$ and $\tilde{\lambda}_\infty = \hat{\lambda}_\infty$.

Case B: $p_\infty = \tilde{p}_\infty$

If $\hat{p}_\infty = p_\infty$, then we get immediately that $\hat{\lambda}_\infty = \tilde{\lambda}_\infty$. Assume therefore that $\hat{p}_\infty \neq p_\infty$. Precisely assume that $\hat{p}_\infty < p_\infty$ (the case $\hat{p}_\infty > p_\infty$ is similar).

Because $(\hat{p}_n, p_n) \rightarrow (\hat{p}_\infty, p_\infty)$, this implies for n large enough that

$$(3.28) \quad g < \hat{\lambda}_n = g(\hat{p}_n) \quad \text{on} \quad (\hat{p}_n, p_n]$$

Let us call the interval $I_n := [\hat{p}_n, p_n]$. Then from the proof of Lemma 3.1, Step 2, Case 2, we have

$$\tilde{g}_n := \text{concave envelope of } g^{I_n} \quad \text{with} \quad g^{I_n}(a) := \begin{cases} g(a) & \text{if } a \in I_n \\ -\infty & \text{if } a \in \mathbb{R} \setminus I_n \end{cases}$$

Hence $\tilde{g}_n(\hat{p}_n) = g(\hat{p}_n) = \hat{\lambda}_n$ and $\tilde{g}_n(p_n) < \hat{\lambda}_n$, and from (3.28), we deduce that the concave function \tilde{g}_n satisfies

$$(3.29) \quad \tilde{g}_n < \hat{\lambda}_n \quad \text{on} \quad (\hat{p}_n, p_n]$$

We deduce (see Lemma 9.1) that

$$(3.30) \quad (\xi_L^n, \xi_R^n) := (\tilde{g}'_n(p_n^-), \tilde{g}'_n(\hat{p}_n^+)) \quad \text{satisfies} \quad \xi_L^n \leq \xi_R^n \leq 0$$

and get

$$(3.31) \quad V_n(\xi) = \begin{cases} p_n & \text{if } \xi < \xi_L^n \\ \hat{p}_n & \text{if } \xi_R^n < \xi \leq 0 \\ ((\tilde{g}_n|_{I_n})')^{-1}(\xi) & \text{if } \xi \in [\xi_L^n, \xi_R^n] \end{cases}$$

which is well defined for almost every $\xi > 0$. Moreover V_n is nonincreasing.

We deduce that $\hat{\lambda}_n > g(p_n) \rightarrow g(p_\infty)$ and then

$$(3.32) \quad \hat{\lambda}_\infty \geq g(p_\infty)$$

Assume by contradiction that

$$(3.33) \quad \hat{\lambda}_\infty > g(p_\infty)$$

Setting $I_\infty := [\hat{p}_\infty, p_\infty]$, we see that $\tilde{g}_n \rightarrow \tilde{g}_\infty$ locally uniformly on $[\hat{p}_\infty, p_\infty]$, where \tilde{g}_∞ is the concave envelope of g^{I_∞} . Then passing to the limit in (3.29), we get
$$\begin{cases} \tilde{g}_\infty \leq \hat{\lambda}_\infty = g(\hat{p}_\infty) & \text{on } [\hat{p}_\infty, p_\infty] \\ \tilde{g}_\infty = g & \text{on } \partial[\hat{p}_\infty, p_\infty] \end{cases}.$$
 Because of

(3.33) and the concavity of \tilde{g}_∞ , we deduce that \tilde{g}_∞ is above its chord on $[\hat{p}_\infty, p_\infty]$ and then

$$(3.34) \quad \tilde{g}_\infty > g(p_\infty) \quad \text{on} \quad [\hat{p}_\infty, p_\infty)$$

Then it is easy to check that V_n converges almost everywhere (by Helly's theorem for sequences of monotone functions) towards V_∞ which is given by (3.30)-(3.31) for $n = \infty$. Moreover (3.34) insures that $V_\infty \neq \text{const}$ a.e. on $(-\infty, 0)$. On the other hand, notice that $v_\infty = v_{p_\infty, \tilde{p}_\infty} = v_{p_\infty, p_\infty}$ and then this function is uniquely given by $v_\infty \equiv p_\infty$. Therefore $V_\infty \equiv p_\infty = \text{const}$. Contradiction. Hence (3.33) is false, and we conclude from (3.32) that $\hat{\lambda}_\infty = g(p_\infty) = g(\tilde{p}_\infty) = \tilde{\lambda}_\infty$, where we have used our assumption $p_\infty = \tilde{p}_\infty$. This shows (3.27). This ends the proof of the lemma.

4 Riemann problem on a junction with $N \geq 1$ branches

4.1 Two families of transformations and reduction to $N : 0$ junctions

Given assumption (2.2), there are naturally two families of transformations with actions on (J, f, \mathcal{G}) where J is a junction with $N \geq 1$ branches, $f^j : [a^j, b^j] \rightarrow \mathbb{R}$ are maps for $j = 1, \dots, N$ and $\mathcal{G} \subset [a, b]$ is a set. Those are I -inversions and I' -reversions where $I, I' \subset \{1, \dots, N\}$ are subsets of indices. The first family of I -inversions does not modify the junction J , but only modify (f, \mathcal{G}) . On the contrary the second family of I -reversions modifies (J, f) but does not modify the set \mathcal{G} .

4.1.1 I -inversions

We will need the following partial inversion transform defined for a subset I of indices.

Definition 4.1 (I -inversion)

Let (J, f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set. Given a subset $I \subset \{1, \dots, N\}$, we define the I -inversion with respect to I as the map $(\bar{\cdot}) : (J, f, \mathcal{G}) \mapsto (\bar{J}, \bar{f}, \bar{\mathcal{G}})$ defined for $\bar{\varepsilon}^j := \begin{cases} -1 & \text{if } j \in I \\ +1 & \text{otherwise} \end{cases}$ as

$$(4.1) \quad \begin{cases} \bar{J} := J \\ \bar{f}^j(\bar{p}^j) := \bar{\varepsilon}^j f^j(\bar{\varepsilon}^j \bar{p}^j) \quad \text{for } \bar{p}^j \in [\bar{a}^j, \bar{b}^j] = \bar{\varepsilon}^j [a^j, b^j] \\ \bar{\mathcal{G}} := \{\bar{p} \in [\bar{a}, \bar{b}], \quad p \in \mathcal{G}\} \quad \text{with } \bar{p}^j := \bar{\varepsilon}^j p^j \end{cases}$$

If $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ is any map, we also define the action of the I -inversion $(\bar{\cdot})$ on \hat{f} as

$$(4.2) \quad \bar{\hat{f}}^j(\bar{p}) := \bar{\varepsilon}^j \hat{f}^j(p) \quad \text{with } \bar{p} \text{ defined in (4.1)}$$

Remark 4.2 Notice that the notation for \bar{a}, \bar{b} is not consistent with the general definition of \bar{p} as a function of p . But this inconsistency will not bring any confusion.

Then we have the following result

Lemma 4.3 (I -inversion of a germ)

Let (J, f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set. Given a subset $I \subset \{1, \dots, N\}$, consider the I -inversion which maps (J, f, \mathcal{G}) to $(\bar{J}, \bar{f}, \bar{\mathcal{G}})$ with (\bar{J}, \bar{f}) satisfying (2.2).

i) (Case of a generalized Riemann germ)

Then \mathcal{G} is a generalized Riemann germ with respect to (J, f) , if and only if $\bar{\mathcal{G}}$ is also a generalized Riemann germ with respect to (\bar{J}, \bar{f}) . Moreover for \bar{p} defined in (4.1), we have

$$(4.3) \quad \begin{cases} BA^{(\bar{J}^j, \bar{f}^j)}(\bar{p}^j) = \bar{\varepsilon}^j BA^{(J^j, f^j)}(p^j) \\ \pi_{\bar{\mathcal{G}}}^j(\bar{p}) = \bar{\varepsilon}^j \pi_{\mathcal{G}}^j(p) \end{cases}$$

ii) (Case of a set with special expression)

Assume that there exists some function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ such that the set \mathcal{G} satisfies $\mathcal{G} = \{p \in [a, b], \hat{f}(p) = f(p)\}$.

Given $\bar{\hat{f}}$ in (4.2), we have $\bar{\mathcal{G}} = \{\bar{p} \in [\bar{a}, \bar{b}], \bar{\hat{f}}(\bar{p}) = \bar{f}(\bar{p})\}$.

Proof of Lemma 4.3

It is straightforward to check point ii). Let us now prove point i). Consider an entropy solution u of $u_t^j + \partial_x(f^j(u^j)) = 0$ on $(0, +\infty) \times J^j$. We set

$$\bar{u}^j(t, x) := \bar{\varepsilon}^j u^j(t, x)$$

Then \bar{u} is an entropy solution u of $\bar{u}_t^j + \partial_x(\bar{f}^j(\bar{u}^j)) = 0$ on $(0, +\infty) \times \bar{J}^j$. Moreover \bar{u} is self-similar if u is, and

$$u(t, 0) \in \mathcal{G} \quad \text{is equivalent to} \quad \bar{u}(t, 0) \in \bar{\mathcal{G}}$$

This implies that $\bar{\mathcal{G}}$ is generalized Riemann germ with respect to (\bar{J}, \bar{f}) , if \mathcal{G} is with respect to (J, f) . It is indeed an equivalence. Moreover, under I -inversion, the properties (4.3) of the basins of attraction and of the natural projection follow from the definitions. This ends the proof of the lemma.

4.1.2 I -reversions

We will need the following partial reversion transform defined for a subset I of indices.

Definition 4.4 (I -reversion)

Let (J, f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set. Given a subset $I \subset \{1, \dots, N\}$, we define the I -reversion with respect to I as the map $(\tilde{\cdot}) : (J, f, \mathcal{G}) \mapsto (\tilde{J}, \tilde{f}, \tilde{\mathcal{G}})$ defined for $\tilde{\varepsilon}^j := \begin{cases} -1 & \text{if } j \in I \\ +1 & \text{otherwise} \end{cases}$ as

$$\left\{ \begin{array}{l} \tilde{J} := \{0\} \cup \bigcup_{j=1, \dots, N} \tilde{J}^j \quad \text{with} \quad \tilde{J}^j := \tilde{\varepsilon}^j J^j \\ \tilde{f}^j(p^j) := \tilde{\varepsilon}^j f^j(p^j) \\ \tilde{\mathcal{G}} := \mathcal{G} \end{array} \right.$$

If $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ is any map, we also define the action of the I -reversion $(\tilde{\cdot})$ on \hat{f} as

$$(4.4) \quad \tilde{\hat{f}}^j(p) := \tilde{\varepsilon}^j \hat{f}^j(p)$$

We see that for indices in I , the I -reversion transform changes outgoing branches into ingoing branches, and vice versa.

Then we have the straightforward result.

Lemma 4.5 (I -reversion of germ)

Let (J, f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set. Given a subset $I \subset \{1, \dots, N\}$, consider the I -inversion which maps (J, f, \mathcal{G}) to $(\tilde{J}, \tilde{f}, \tilde{\mathcal{G}})$ with (\tilde{J}, \tilde{f}) satisfying (2.2).

i) (Case of a generalized Riemann germ)

Then \mathcal{G} is a generalized Riemann germ with respect to (J, f) , if and only if $\tilde{\mathcal{G}}$ is also a generalized Riemann germ with respect to (\tilde{J}, \tilde{f}) . Moreover, for all $q \in [a, b]$ we have

$$(4.5) \quad \left\{ \begin{array}{l} BA^{(\tilde{J}, \tilde{f})}(q) = BA^{(J, f)}(q) \\ \pi_{\tilde{\mathcal{G}}} = \pi_{\mathcal{G}} \end{array} \right.$$

ii) (Case of a set with special expression)

Assume that there exists some function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ such that the set \mathcal{G} satisfies $\mathcal{G} = \{p \in [a, b], \hat{f}(p) = f(p)\}$.

Given $\tilde{\hat{f}}$ in (4.4), we have $\tilde{\mathcal{G}} = \{p \in [a, b], \tilde{\hat{f}}(p) = \tilde{f}(p)\}$.

Proof of Lemma 4.5

It is straightforward to check point ii). Let us now prove point i). Consider an entropy solution u of $u_t^j + \partial_x(f^j(u^j)) = 0$ on $(0, +\infty) \times J^j$. We set

$$\tilde{u}^j(t, x) := u^j(t, \tilde{\varepsilon}^j x)$$

Then \tilde{u} is an entropy solution u of $\tilde{u}_t^j + \partial_x(\tilde{f}^j(\tilde{u}^j)) = 0$ on $(0, +\infty) \times \tilde{J}^j$. Moreover \tilde{u} is self-similar if u is, and

$$u(t, 0) \in \mathcal{G} \quad \text{is equivalent to} \quad \tilde{u}(t, 0) \in \tilde{\mathcal{G}}$$

This implies that $\tilde{\mathcal{G}}$ is generalized Riemann germ with respect to (\tilde{J}, \tilde{f}) , if \mathcal{G} is with respect to (J, f) . Moreover it is an equivalence. Moreover, under I -reversion, the properties (4.5) of the basins of attraction and of the natural projection follow from the definitions. This ends the proof of the lemma.

It is straightforward to check the following result.

Lemma 4.6 (Commutativity of inversions and reversions)

Let (J, f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set. Given two subsets $\bar{I}, \tilde{I} \subset \{1, \dots, N\}$, consider the \bar{I} -inversion $(\bar{\cdot})^{\bar{I}}$ and the \tilde{I} -reversion $(\tilde{\cdot})^{\tilde{I}}$. Then we have the commutativity $(\bar{\cdot})^{\bar{I}} \circ (\tilde{\cdot})^{\tilde{I}} = (\tilde{\cdot})^{\tilde{I}} \circ (\bar{\cdot})^{\bar{I}}$.

4.1.3 Reduction to $N : 0$ junctions

The following result is a straightforward corollary of Lemma 4.5.

Corollary 4.7 (Reduction to $N : 0$ junctions)

Let (J, f) satisfying (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ with respect to (J, f) . Set

$$I := \{j \in \{1, \dots, N\}, \quad \sigma^j = -1\}$$

The action of a I -reversion on (J, f, \mathcal{G}) defines $(\tilde{J}, \tilde{f}, \tilde{\mathcal{G}})$. Then $\tilde{\mathcal{G}}$ is a generalized Riemann germ with respect to (\tilde{J}, \tilde{f}) with $\tilde{J}^j \simeq (-\infty, 0) = \tilde{\sigma}^j \cdot (-\infty, 0)$ for all $j = 1, \dots, N$, i.e. with $\tilde{\sigma}^j = 1$ for all j . This means that \tilde{J} is a junction of type $N : 0$.

This way, we see that we reduce the problem of $n : m$ junctions with $N := n + m$, to the case $n = N$ and $m = 0$. Such junctions are then called $N : 0$ junctions. In the remaining part of the section, we will work with such junctions which have more symmetries. The result for the original problem can then be obtained easily by the inverse of the reverse transform.

4.2 Proof of Proposition 2.4: L^1 estimate for Riemann problem

We can do the proof of Proposition 2.4 in two steps.

Step 1: reduction

We first notice that the problem reduces to a single branch $N = 1$ and using a reversion, we can assume that we work on a $1 : 0$ junction.

Step 2: the computation

We now compute formally for $j = 1$ with $J^j = (-\infty, 0)$

$$\begin{aligned} \frac{d}{dt} \int_{J^j} (u^j(t, \cdot) - p^j) &= \int_{J^j} u_t^j(t, \cdot) \\ &= - \int_{J^j} \partial_x \{f^j(u^j(t, \cdot))\} dx \\ &= - [f^j]_{p^j}^{p^j} \\ &= f^j(p^j) - f^j(p^j) \end{aligned}$$

This computation can easily be justified (in the sense of distributions), and this implies the first equality of (2.7), because $u^j(0, \cdot) = p^j$. The second equality of (2.7) follows from the fact that the sign of $u^j - p^j$ only depends on j , for self-similar solutions (see Lemma 9.1). This ends the proof of the proposition.

4.3 First results on generalized Riemann germs

Assume (2.2) with $N \geq 1$ branches $\simeq (-\infty, 0)$. We recall briefly some definitions/notations. For $j \in \{1, \dots, N\}$, we recall that the Godunov flux $G^j : [a^j, b^j]^2 \rightarrow \mathbb{R}$ associated to f^j , is $G^j(q, r) :=$

$\left\{ \begin{array}{ll} \inf_{[q,r]} f^j & \text{if } q \leq r \\ \sup_{[r,q]} f^j & \text{if } q \geq r \end{array} \right. \cdot$ We also define the following nondecreasing functions of $p^j \in [a^j, b^j]$

$$(4.6) \quad f_-^j(p^j) := \inf_{[p^j, b^j]} f^j = G^j(p^j, b^j) \leq f_+^j(p^j) := \sup_{[a^j, p^j]} f^j = G^j(p^j, a^j)$$

and for $\lambda \in [f_-^j(p^j), f_+^j(p^j)]$, we define the following element of $[a^j, b^j]$

$$(4.7) \quad \hat{p}_{\lambda, p^j}^j := \begin{cases} p^j & \text{if } f^j(p^j) = \lambda \\ \sup \{q^j \in (p^j, b^j], f^j > \lambda \text{ on } (p^j, q^j)\} & \text{if } f^j(p^j) > \lambda \\ \inf \{q^j \in [a^j, p^j), f^j < \lambda \text{ on } (q^j, p^j)\} & \text{if } f^j(p^j) < \lambda \end{cases}$$

which is nonincreasing in λ . For $p = (p^1, \dots, p^N) \in [a, b]$, we also define the following subset of $[a, b]$

$$(4.8) \quad \hat{\mathcal{P}}_p := \prod_{j=1, \dots, N} \hat{\mathcal{P}}_{p^j}^j \quad \text{with} \quad \hat{\mathcal{P}}_{p^j}^j := \left\{ \hat{p}_{\lambda, p^j}^j \in [a^j, b^j], \lambda \in [f_-^j(p^j), f_+^j(p^j)] \right\}$$

We also recall the basins of attraction defined for all $p \in [a, b]$ by

$$(4.9) \quad BA(p) := \prod_{j=1, \dots, N} BA^j(p^j) \quad \text{with} \quad BA^j(p^j) := BA_-^j(p^j) \cup \{p^j\} \cup BA_+^j(p^j)$$

where $BA_{\pm}^j = BA_{\pm}^j(p^j)$ are given for $\lambda^j := f^j(p^j)$ by $\begin{cases} BA_+^j := \{q^j \in (p^j, b^j], f^j < \lambda^j \text{ on } (p^j, q^j)\} \\ BA_-^j := \{q^j \in [a^j, p^j), f^j > \lambda^j \text{ on } [q^j, p^j)\} \end{cases}$.

Then from Lemma 3.2 and the definitions, we get immediately

Lemma 4.8 (Inverse characterization of the map $p \mapsto \hat{\mathcal{P}}_p$)

Assume (2.2) with $N \geq 1$ and a $N : 0$ junction. Then for two arbitrary vectors $p, \hat{p} \in [a, b]$, we have

$$\hat{p} \in \hat{\mathcal{P}}_p \quad \text{if and only if} \quad p \in BA(\hat{p})$$

The following result follow immediately from the definitions and Lemma 3.1.

Lemma 4.9 (Generalized Riemann germ characterization, $N \geq 1$)

Assume (2.2) with $N \geq 1$ and let $\mathcal{G} \subset [a, b]$ be a set.

i) (First characterization)

The set \mathcal{G} is a generalized Riemann germ if and only if for all $p \in [a, b]$, we have the singleton property $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$, where $\hat{\mathcal{P}}_p$ is defined in (4.8).

ii) (Equivalent characterization)

The set \mathcal{G} is a generalized Riemann germ if and only if $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}}$ is a partition of $[a, b]$.

Proposition 4.10 (First properties of generalized Riemann germs, $N \geq 1$)

Assume (2.2) with $N \geq 1$ and let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ for a junction $N : 0$. Given $p \in [a, b]$, the unique \mathcal{G} -entropy solution of (2.6) writes $u_{p, \hat{p}}^{\mathcal{G}}$ with $\pi(p) := \pi_{\mathcal{G}}(p) := \hat{p} \in \mathcal{G}$ and $\pi : [a, b] \rightarrow \mathcal{G}$ which satisfies $\pi \circ \pi = \pi$. We set $\hat{f} := \hat{f}_{\mathcal{G}} := f \circ \pi$.

i) (Local constancy): The map \hat{f} is locally constant on $\{\hat{f} \neq f\}$.

ii) (Inverse of π): For all $\hat{p} \in \mathcal{G}$, we have $\pi^{-1}(\hat{p}) = BA(\hat{p})$.

iii) (Level set formulation of the germ): We have $\mathcal{G} = \mathcal{G}_{\hat{f}} := \{p \in [a, b], \hat{f}(p) = f(p)\}$.

iv) (Characterization of \hat{f}): Assume that $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ is continuous. Then \hat{f} is fully characterized as the continuous function which is locally constant on $\{\hat{f} \neq f\}$ such that $\mathcal{G} = \mathcal{G}_{\hat{f}}$.

v) (Monotone bounds): We have $f_- \leq \hat{f} \leq f_+$ with f_{\pm} defined in (4.6).

Remark 4.11 Notice that the monotonicity properties of \hat{f} are not proved in Proposition 4.10. Again it seems to be a delicate result. We will prove it later using as a key step a slicing lemma which reduces to germs for a single branch, for which we already have proved some monotonicity.

Proof of Proposition 4.10

Step 1: proof of ii)

From i) of Lemma 4.9 and Lemma 4.8, we have $\pi^{-1}(\hat{p}) = BA(\hat{p})$ for all $\hat{p} \in \mathcal{G}$.

Step 2: proof of iii)

From b) in hands, then Step 3 of the proof of Proposition 3.6 applies word by word.

Step 3: proof of i)

Let $p \in [a, b] \cap \{\hat{f} \neq f\}$ and let $\hat{p} = \pi(p) \in \mathcal{G}$ such that $p \in BA(\hat{p})$. We set

$$(4.10) \quad I_p(\hat{f}) := \left\{ j \in \{1, \dots, N\}, \quad \hat{f}^j(p) \neq f^j(p^j) \right\}$$

Because $\hat{f} = f \circ \pi$, we deduce that for all $j \in I_p(\hat{f})$ that $p^j \in BA(\hat{p}^j) \setminus \{\hat{p}^j\}$, which is a relative open set of $[a^j, b^j]$. We deduce the existence of some $\varepsilon > 0$ small enough such that $[a^j, b^j] \cap Q_{p^j}^j \subset BA^j(\hat{p}^j)$ with $Q_{p^j}^j := (p^j - \varepsilon, p^j + \varepsilon)$. This means that

$$(4.11) \quad [a, b] \cap Q_\varepsilon^{\hat{f}}(p) \subset BA(\hat{p}) \quad \text{with} \quad Q_\varepsilon^{\hat{f}}(p) := p + \sum_{j \in I_p(\hat{f})} (-\varepsilon, \varepsilon)e_j$$

with the convention that $Q_\varepsilon^{\hat{f}}(p) = \{p\}$ when $I_p(\hat{f}) = \emptyset$. Then $\hat{f} = \text{const} = \hat{f}(p) = f(\hat{p})$ on $[a, b] \cap Q_\varepsilon^{\hat{f}}(p)$, which means exactly that \hat{f} is locally constant on $\{\hat{f} \neq f\}$.

Step 4: proof of v)

Consider $p \in [a, b]$. We know that $p \in BA(\hat{p})$ with $\hat{p} \in \mathcal{G}$. Hence for each index j , we have $\hat{p}^j = \hat{p}_{\lambda^j, p^j}^j$ with $\hat{f}^j(p) = f^j(\hat{p}^j) = \lambda^j \in [f_-^j(p^j), f_+^j(p^j)]$ (by assumption). This gives the monotone bounds.

Step 5: proof of iv)

Assume that \hat{f} is continuous. Now consider some continuous function $\tilde{f} : [a, b] \rightarrow \mathbb{R}$ which is locally constant on $\{\tilde{f} \neq f\}$ such that $\mathcal{G} = \mathcal{G}_{\tilde{f}}$. We want to show that $\tilde{f} = \hat{f}$.

We already know that $\tilde{f} = f = \hat{f}$ on \mathcal{G} , where $\mathcal{G} = \mathcal{G}_{\tilde{f}}$ is a closed set, because \hat{f} is continuous. Moreover, we have for all $p \in [a, b] \setminus \mathcal{G}$, there exists $\varepsilon > 0$ such that

$$\frac{\partial \tilde{f}}{\partial p^j} = 0 \quad \text{in} \quad [a, b] \cap Q_\varepsilon^{\tilde{f}}(p) \quad \text{for all} \quad j \in I_p(\tilde{f})$$

where $I_p(\tilde{f})$ and $Q_\varepsilon^{\tilde{f}}(p)$ are defined respectively in (4.10) and (4.11). By continuity of f , \tilde{f} and the structure of each basin on attraction, we deduce that $\tilde{f} = f(\hat{p}) = \hat{f}$ on $BA(\hat{p})$ for all $\hat{p} \in \mathcal{G}$. Because the family of basins of attraction forms a partition of $[a, b]$, we deduce that $\tilde{f} = \hat{f}$.

This ends the proof of the proposition.

4.4 Slicing lemma and basic monotonicities

The following simple lemma is a key tool.

Lemma 4.12 (Slicing lemma)

Assume (2.2) with $N \geq 2$, and let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ for a $N : 0$ junction. Let $1 \leq n < N$. Then for $p \in [a, b]$, we write

$$p = (p', p'') \quad \text{with} \quad p' = (p^1, \dots, p^n) \in [a', b'] \quad \text{and} \quad p'' = (p^{n+1}, \dots, p^N) \in [a'', b''].$$

Given some $p_0'' \in [a'', b'']$, we set $\Pi := [a', b'] \times \{p_0''\}$ and define the slicing of the germ \mathcal{G} with respect to p_0'' :

$$\mathcal{G}_{p_0''} := \{p' \in [a', b'], \quad \text{s.t. there exists } \hat{p} = (\hat{p}', \hat{p}'') \in \mathcal{G} \text{ with } BA(\hat{p}) \cap \Pi \neq \emptyset\}$$

Then $\mathcal{G}_{p_0''} \subset [a', b']$ is a generalized Riemann germ for a $n : 0$ junction. Moreover

$$(4.12) \quad \mathcal{G}_{p_0''} = \left\{ p' = (p^1, \dots, p^n) \in [a', b'], \quad \hat{f}^j(p', p_0'') = f^j(p^j) \quad \text{for} \quad j = 1, \dots, n \right\}$$

and the effective flux function associated to $\mathcal{G}_{p_0''}$ is

$$(4.13) \quad \hat{f}_{\mathcal{G}_{p_0''}} = (\hat{f}^j(\cdot, p_0''))_{j=1, \dots, n}$$

Proof of Lemma 4.12

Recall that $BA(\hat{p})$ is given in (4.9) and that in the present case all branches satisfy $J^j \simeq (-\infty, 0)$. With $\mathcal{G}_\Pi := \{\hat{p} \in \mathcal{G}, BA(\hat{p}) \cap \Pi \neq \emptyset\}$, we can write

$$\mathcal{G}_{p_0''} := \{\hat{p}' \in [a', b'], \quad \text{s.t. there exists } \hat{p} = (\hat{p}', \hat{p}'') \in \mathcal{G}_\Pi\}$$

Because $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}}$ is a partition of $[a, b]$, we see that Π is covered exactly once by the family $(BA(\hat{p}))_{\hat{p} \in \mathcal{G}_\Pi}$. Hence $(\Pi \cap BA(\hat{p}))_{\hat{p} \in \mathcal{G}_\Pi}$ is a partition of Π . This implies that $\mathcal{G}_{p_0''} \subset [a', b']$ is a generalized Riemann germ. Now define

$$\tilde{\mathcal{G}}_{p_0''} := \left\{ p' = (p^1, \dots, p^n) \in [a', b'], \quad \hat{f}^j(p', p_0'') = f^j(p^j) \quad \text{for } j = 1, \dots, n \right\}$$

For any $\hat{p}' \in \mathcal{G}_{p_0''}$, we set $\hat{p} := \pi(\hat{p}', p_0'') \in \mathcal{G}_\Pi$, where $\pi = \pi_{\mathcal{G}} : [a, b] \rightarrow \mathcal{G}$ is the natural projection associated to \mathcal{G} . Then \hat{p} writes $\hat{p} = (\hat{p}', \hat{p}'')$. Moreover $\hat{f} = \text{const} = f(\hat{p})$ on $BA(\hat{p})$, and $BA(\hat{p}) \cap \Pi \ni (\hat{p}', p_0'')$. This shows in particular that $\mathcal{G}_{p_0''} \subset \tilde{\mathcal{G}}_{p_0''}$.

Conversely, consider $p' \in \tilde{\mathcal{G}}_{p_0''}$ and set $\pi(p', p_0'') =: \hat{p} = (\hat{p}', \hat{p}'') \in \mathcal{G}_\Pi$ with $p' \in BA'(\hat{p}')$ (with obvious notation). Hence $f^j(p^j) = \hat{f}^j(p', p_0'') = f^j(\hat{p}) = f^j(\hat{p}^j)$ for $j = 1, \dots, n$. Because we have $BA^j(\hat{p}^j) \cap \{f^j = f^j(\hat{p}^j)\} = \{\hat{p}^j\}$ for all $j = 1, \dots, N$, we deduce that $p' = \hat{p}'$ with $(\hat{p}', \hat{p}'') \in \mathcal{G}_\Pi$. This implies the reverse inclusion $\mathcal{G}_{p_0''} \supset \tilde{\mathcal{G}}_{p_0''}$. This shows (4.12). Moreover, we get that $\hat{f}_{\mathcal{G}_{p_0''}} = (\hat{f}^j(\cdot, p_0''))_{j=1, \dots, n}$, i.e. (4.13). This ends the proof of the lemma.

We now state the following result.

Proposition 4.13 (Basic monotonicities of \hat{f} , $N \geq 1$)

We work under assumptions of Proposition 4.10. Then we have on $[a, b]$

$$\left. \begin{array}{l} p \mapsto f^j(p) \\ p \mapsto \pi^j(p) \end{array} \right\} \text{ are nondecreasing in } p^j, \text{ for all } j \quad \text{(Basic monotonicities)}$$

Moreover we have

$$(4.14) \quad \left\{ \begin{array}{l} \text{for all index } j, \text{ the function } \hat{f}^j \text{ is locally Lipschitz continuous in the variable } p^j, \\ \text{and for all } q_0 \in [a, b] \text{ and for all } j, \text{ we have with } \iota_{q_0}^j(p^j) := (q_0^1, \dots, q_0^{j-1}, p^j, q_0^{j+1}, \dots, q_0^N) \\ \max(0, (f^j)') \geq (\hat{f}_{q_0}^j)' \geq 0 \quad \text{a.e. on } [a^j, b^j] \text{ with } \hat{f}_{q_0}^j := \hat{f}^j \circ \iota_{q_0}^j \end{array} \right.$$

and for all $q_0 \in [a, b]$ and all index j we have

$$\left((\hat{f}_{q_0}^j)' \in \{0, \min(0, (f^j)')\} \quad \text{and } (\hat{f}_{q_0}^j)' = (f^j)' > 0 \text{ implies } \hat{f}_{q_0}^j = f^j \right) \quad \text{holds a.e. on } [a^j, b^j]$$

Proof of Proposition 4.13

Applying the Slicing Lemma 4.12 with $n = 1$, we get for $j = 1$, that $\hat{f}_{q_0}^j$ is the flux function at the junction $1 : 0$ associated to a generalized Riemann germ. Applying vi) of Proposition 3.6 and Proposition 3.8, we deduce the result for $j = 1$. Up to relabel the indices, we get the result for all indices $j = 1, \dots, N$. This ends the proof of the lemma.

4.5 Theorem 2.14 and its proof: characterization of generalized Riemann germs

We first start with the following result and then give the proof of Theorem 2.14.

Proposition 4.14 (Generating generalized Riemann germs, $N \geq 1$)

Assume (2.2) for a $N : 0$ junction with $N \geq 1$ and let $\mathcal{G} \subset [a, b]$ be a set.

Then \mathcal{G} is a generalized Riemann germs if and only if $\mathcal{G} = \mathcal{G}_{\hat{f}} := \left\{ \hat{f} = f \right\}$ for some function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ which is locally constant on $\left\{ \hat{f} \neq f \right\}$ and satisfying for all j

$$(4.15) \quad \left\{ \begin{array}{l} p^j \mapsto \hat{f}^j(p) \text{ is nondecreasing on } [a, b] \\ f_- \leq \hat{f} \leq f_+ \end{array} \right. \quad \begin{array}{l} \text{(Basic monotonicities)} \\ \text{(Monotone bounds)} \end{array}$$

with f_{\pm} defined in (4.6).

Proof of Proposition 4.14

Part I: properties of generalized Riemann germs

Notice that if \mathcal{G} is a generalized Riemann germ, then Proposition 4.10 and Proposition 4.13 shows that $\hat{f} := \hat{f}_{\mathcal{G}}$ satisfies all the required conditions.

Part II: constructing a generalized Riemann germ

Conversely, assume that $\tilde{f} : [a, b] \rightarrow \mathbb{R}^N$ is locally constant on $\{\tilde{f} \neq f\}$ and satisfies (4.15) for all index j .

Then let $\mathcal{G} := \mathcal{G}_{\tilde{f}} := \{\tilde{f} = f\}$. Let us check that \mathcal{G} is a generalized Riemann germ, i.e. that \mathcal{G} satisfies the following singleton property $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$ for all $p \in [a, b]$.

Step 1: nonemptiness of $\mathcal{G} \cap \hat{\mathcal{P}}_p$

Recall that $f_- \leq \tilde{f} \leq f_+$. We first define $\pi(p) := \hat{p}$ for $p \in [a, b]$. For each index j , setting $\lambda := \tilde{f}^j(p)$, we define

$$\hat{p}^j := \pi^j(p) := \begin{cases} p^j & \text{if } f^j(p^j) = \lambda \\ \sup \{q^j \in [p^j, b^j], f^j > \lambda \text{ on } (p^j, q^j]\} & \text{if } f^j(p^j) > \lambda \\ \inf \{q^j \in [a^j, p^j], f^j < \lambda \text{ on } [q^j, p^j)\} & \text{if } f^j(p^j) < \lambda \end{cases}$$

Exactly as in Step 1 of the proof of Proposition 3.9, we conclude that $\mathcal{G} \cap \hat{\mathcal{P}}_p \supset \{\hat{p}\}$.

Step 2: $\mathcal{G} \cap \hat{\mathcal{P}}_p$ is reduced to a singleton

Exactly as in Step 2 of the proof of Proposition 3.9, we conclude that $\mathcal{G} \cap \hat{\mathcal{P}}_p = \{\hat{p}\}$. which shows that \mathcal{G} is a generalized Riemann germ. This ends the proof of the proposition.

Proof of Theorem 2.14

For the proof we refer to the table of Subsection 2.4. Point i) of Theorem 2.14 follows from ii) of Lemma 4.9, while point ii) of Theorem 2.14 follows from follows from Proposition 4.14. This ends the proof of the theorem.

4.6 Partial relaxation formula

Proposition 4.15 (Partial relaxation formula, $N \geq 1$)

Assume (2.2) for a $N : 0$ junction with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ and $\hat{f} := \hat{f}_{\mathcal{G}}$. For any $p \in [a, b]$ and index j , we define for all $q^j \in [a^j, b^j]$

$$\hat{f}_p^j := \hat{f}^j \circ \iota_p^j \quad \text{with} \quad \iota_p^j(q^j) := \hat{f}^j(p^1, \dots, p^{j-1}, q^j, p^{j+1}, \dots, p^N)$$

Then for all index j , the function \hat{f}_p^j satisfies

$$(4.16) \quad \begin{cases} \hat{f}_p^j : [a^j, b^j] \rightarrow \mathbb{R} \text{ is continuous nondecreasing} & \text{(Monotonicity)} \\ \hat{f}_{p,-}^j \leq \hat{f}_p^j \leq \hat{f}_{p,+}^j & \text{(Bounds)} \\ \hat{f}_{p,-}^j(q^j) = \inf_{[q^j, b^j]} \hat{f}_p^j, \\ \hat{f}_{p,+}^j(q^j) = \sup_{[a^j, q^j]} \hat{f}_p^j \end{cases}$$

and we have the following partial relaxation formula

$$(4.17) \quad \left\{ \hat{f}_p^j(p^j) \right\} = \bigcup_{q^j \in [a^j, b^j]} \left\{ G^j(p^j, q^j) \right\} \cap \left\{ \hat{f}_p^j(q^j) \right\}$$

Proof of Proposition 4.15

The result follows from the slicing lemma 4.12, considering $\mathcal{G}_p^j := \left\{ q^j \in [a^j, b^j], \hat{f}_p^j(q^j) = f^j(q^j) \right\}$, which is a generalized Riemann germ in $[a^j, b^j]$. Moreover the associated effective flux function is $\hat{f}_{\mathcal{G}_p^j} = \hat{f}_p^j$. Then (4.16) and (4.17) follow from Proposition 3.10. This ends the proof of the lemma.

4.7 Theorem 2.16 and its proof: characterization of Riemann germs

Before to start the proof of Theorem 2.16, recall that we consider Kruřkov entropy solution

$$u_{p,\hat{p}}^{\mathcal{G}} := (u_{p^1,\hat{p}^1}^1, \dots, u_{p^N,\hat{p}^N}^N) \in L_{loc}^1([0, +\infty) \times (-\infty, 0))^N \simeq L_{loc}^1([0, +\infty) \times J)$$

to the \mathcal{G} -Riemann problem (2.6).

Proof of Theorem 2.16

Part I: proof of i)

We only do the proof for $N : 0$ junctions. The general case of junctions $n : m$ is recovered from $N : 0$ with $N := n + m$, using suitable I -reversions (see Lemma 4.5).

Step 1: necessary condition

Assume that $\mathcal{G} \subset [a, b]$ is a Riemann germ. For $p_n \rightarrow p_\infty$, we set $\tilde{p}_\infty := \pi(p_\infty)$, and up to extract a subsequence (still denoted by p_n), we have $\hat{p}_n := \pi(p_n) \rightarrow \hat{p}_\infty \in [a, b]$. By Definition 2.5 of Riemann germs, we have

$$u_{p_n, \hat{p}_n}^{\mathcal{G}} \rightarrow u_{p_\infty, \tilde{p}_\infty}^{\mathcal{G}} \quad \text{in } L_{loc}^1([0, +\infty) \times J), \quad \text{as } n \rightarrow +\infty$$

Hence for $p_n \in BA(\hat{p}_n)$ and from the weak stability Lemma 3.11, we conclude that $f(\hat{p}_\infty) = f(\tilde{p}_\infty)$. By definition, this shows that $\hat{f}(p_n) = f(\hat{p}_n) \rightarrow f(\hat{p}_\infty) = f(\tilde{p}_\infty) = f(\pi(p_\infty)) = \hat{f}(p_\infty)$, which shows the continuity of \hat{f} .

Step 2: sufficient condition

Let \mathcal{G} be a generalized Riemann germ and assume that $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ is continuous. We want to show that \mathcal{G} is a Riemann germ. Because $\mathcal{G} = \{\hat{f} = f\}$, we deduce that \mathcal{G} is a closed set. Now if $p_n \in [a, b]$ satisfies $p_n \rightarrow p_\infty \in [a, b]$, then, up to extract a subsequence, we have $\mathcal{G} \ni \pi(p_n) =: \hat{p}_n \rightarrow \hat{p}_\infty \in \mathcal{G}$. From the weak stability Lemma 3.11, there exists $\tilde{p}_\infty \in [a, b]$ with $p_\infty \in BA(\tilde{p}_\infty)$ such that, up to extract a subsequence, we have

$$u_n := u_{p_n, \hat{p}_n}^{\mathcal{G}} \rightarrow u_\infty := u_{p_\infty, \tilde{p}_\infty}^{\mathcal{G}} \quad \text{in } L_{loc}^1([0, +\infty) \times J), \quad \text{as } n \rightarrow +\infty$$

and we want to show that $u_{p_\infty, \tilde{p}_\infty}^{\mathcal{G}} = u_{p_\infty, \pi(p_\infty)}^{\mathcal{G}}$, i.e. that $\tilde{p}_\infty = \pi(p_\infty)$. From Lemma 3.11, we also know that

$$(4.18) \quad f(\hat{p}_\infty) = f(\tilde{p}_\infty) \quad \text{with} \quad \tilde{p}_\infty \in \{p_\infty, \hat{p}_\infty\}$$

Up to extract a subsequence (still denoted by p_n), we can assume that there exists $s = (s^j)_{j=1, \dots, N} \in \{\pm 1\}^N$ such that for all index j , we have

$$(4.19) \quad s^j(\hat{p}_n^j - p_n^j) \leq 0 \quad \text{for all } n$$

We set $I := \{j \in \{1, \dots, N\}, \quad s^j = -1\}$. Then, up to use the I -inversion and (4.19), we can assume that $\hat{p}_n \leq p_n$. Hence

$$\hat{f} = \hat{\lambda}_n := f(\hat{p}_n) \quad \text{on} \quad BA(\hat{p}_n) \supset [\hat{p}_n, p_n)$$

Passing to the limit, we get from the continuity of \hat{f} that

$$\hat{f} = \hat{\lambda}_\infty = f(\hat{p}_\infty) \quad \text{on} \quad [\hat{p}_\infty, p_\infty] \supset \{\tilde{p}_\infty\}$$

where we have used (4.18) for the last inclusion. Using again (4.18), we get $f(\tilde{p}_\infty) = f(\hat{p}_\infty) = \hat{\lambda}_\infty = \hat{f}(\tilde{p}_\infty)$. This shows that $\tilde{p}_\infty \in \mathcal{G}$. Hence $p_\infty \in BA(\tilde{p}_\infty)$ implies $\tilde{p}_\infty \in \mathcal{G} \cap \hat{\mathcal{P}}_{p_\infty} = \{\pi(p_\infty)\}$, and then $\tilde{p}_\infty = \pi(p_\infty)$ which shows $u_\infty = u_{p_\infty, \tilde{p}_\infty}^{\mathcal{G}} = u_{p_\infty, \pi(p_\infty)}^{\mathcal{G}}$. This shows that the limit u_∞ is unique, and independent of the extracted subsequence. Therefore, the full sequence u_n converges towards u_∞ . This establishes that \mathcal{G} is a Riemann germ.

Part II: proof of ii)

The result follows from iv) of Proposition 4.10. This ends the proof of the proposition.

4.8 Dissipation properties for $N : 0$ junctions

Lemma 4.16 (Dissipative points are in the germ)

Assume (2.2) for $N : 0$ junction with $N \geq 1$. Assume that $\mathcal{G} \subset [a, b]$ is a generalized Riemann germ, and let \hat{f} be its associated effective flux function. Let $\hat{p}, \hat{q} \in \mathcal{G}$ and $p \in BA(\hat{p}), q \in BA(\hat{q})$. Recall that $\sigma^j = +1$,

$$D^{f^j}(p, q) := \text{sign}(p^j - q^j) \cdot \{f^j(p) - f^j(q)\} \quad \text{and} \quad D_+^{f^j}(p, q) := \text{sign}^+(p^j - q^j) \cdot \{f^j(p) - f^j(q)\}$$

i) (Dissipative points are in the germ)

We have for all index j

$$(4.20) \quad D^{f^j}(p, \hat{p}) \leq 0$$

and

$$\sum_{j=1, \dots, N} D^{f^j}(p, \hat{p}) =: D^f(p, \hat{p}) \geq 0 \quad \implies \quad p = \hat{p} \in \mathcal{G}$$

ii) (Properties of $D^{\hat{f}^j}$ and $D_+^{\hat{f}^j}$)

We have for all index j

$$(4.21) \quad D^{\hat{f}^j}(p, q) \geq D^{f^j}(\hat{p}, \hat{q}) \quad \text{and} \quad D_+^{\hat{f}^j}(p, q) \geq D_+^{f^j}(\hat{p}, \hat{q})$$

Proof of Lemma 4.16

Step 1: proof of i)

This is a corollary of Lemma 3.4.

Step 2: proof of ii)

We only do the proof for D_+ , because $D(p, q) = D_+(p, q) + D_+(q, p)$ then implies the result for D .

We set $\lambda_p := \hat{f}(p)$, $\lambda_q := \hat{f}(q)$, and $\hat{p}^j = \hat{p}_{\lambda_p^j, p^j}^j$, $\hat{q}^j = \hat{p}_{\lambda_q^j, q^j}^j$. Recall that (here with $\sigma^j = +1$)

$$D_+^j := D_+^{f^j}(p, q) = \{\text{sign}^+(p^j - q^j)\} \cdot \{f^j(p) - f^j(q)\}$$

and let us set

$$\hat{D}_+^j := D_+^{\hat{f}^j}(\hat{p}, \hat{q}) = \{\text{sign}^+(\hat{p}^j - \hat{q}^j)\} \cdot \{f^j(\hat{p}) - f^j(\hat{q})\}$$

Because $\hat{f}(p) = \hat{f}(\hat{p})$ and the same for q , we deduce that only the change of value of $\text{sign}^+(p^j - q^j) \neq \text{sign}^+(\hat{p}^j - \hat{q}^j)$ can affect the difference $D_+^j - \hat{D}_+^j$. We then distinguish the only two cases where this happens.

Case A: $p^j > q^j$ and $\hat{p}^j \leq \hat{q}^j$

This means that $\hat{p}^j = \hat{p}_{\lambda_p^j, p^j}^j \leq \hat{p}_{\lambda_q^j, q^j}^j = \hat{q}^j$. Then either $\hat{p}^j < \hat{q}^j$ or $\hat{p}^j = \hat{q}^j$, and from the monotonicities of the map $(\lambda^j, p^j) \mapsto \hat{p}_{\lambda^j, p^j}^j$ which are (\downarrow, \uparrow) , we deduce that

$$\lambda_p^j > \lambda_q^j \quad \text{or} \quad (\hat{p}^j = \hat{q}^j \quad \text{and} \quad \lambda_p^j = \lambda_q^j)$$

i.e.

$$D_+^j > 0 = \hat{D}_+^j \quad \text{or} \quad D_+^j = 0 = \hat{D}_+^j$$

Case B: $p^j \leq q^j$ and $\hat{p}^j > \hat{q}^j$

From the monotonicities, we deduce that $\lambda_p^j < \lambda_q^j$. Hence $D_+^j = 0 > \hat{D}_+^j$.

Conclusion

In all cases, we deduce that $D_+^j \geq \hat{D}_+^j$ which shows (4.21).

This ends the proof of the lemma.

4.9 Characterization of dissipative functions and technical approximations

Notice that while D^{f^j} is a continuous function, $D^{\hat{f}^j}$ is not continuous in general. For later use, we will need the following technical result.

Lemma 4.17 (Approximation of $D^{\hat{f}}$ and $D_+^{\hat{f}}$)

Assume (2.2) with $N \geq 1$. Consider the relative open set Ω of $[a, b]^2$ defined by

$$(4.22) \quad \Omega := \{(p, q) \in [a, b]^2, \quad p^j \neq q^j \quad \text{for all } j = 1, \dots, N\}$$

and any continuous function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$. Then for all $(p, q) \in [a, b]^2$, there exists a sequence $(p_\delta, q_\delta) \in \Omega$ such that $(p_\delta, q_\delta) \rightarrow (p, q)$ as $\delta \rightarrow 0^+$ and

$$(4.23) \quad D^{\hat{f}}(p, q) \geq \liminf_{\delta \rightarrow 0^+} D^{\hat{f}}(p_\delta, q_\delta)$$

and

$$(4.24) \quad D_+^{\hat{f}}(p, q) \geq \liminf_{\delta \rightarrow 0^+} D_+^{\hat{f}}(p_\delta, q_\delta)$$

Proof of Lemma 4.17

We only do the proof for $0 : N$ junctions (because the signs of the present computation are then more natural than for $N : 0$ junctions). The general case can be recovered using reversion transforms. We first notice that $D^{\hat{f}}$ is continuous on Ω . Assume now that $(p, q) \in [a, b]^2 \setminus \Omega$, and define the sets

$$I_\pm := \{j \in \{1, \dots, N\}, \quad \pm(p^j - q^j) > 0\}, \quad I_0 := \{j \in \{1, \dots, N\}, \quad (p^j - q^j) = 0\}$$

and

$$I_{0,\pm} := \left\{j \in I_0, \quad \pm \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} > 0 \right\}, \quad I_{0,0} := \left\{j \in I_0, \quad \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} = 0 \right\}$$

We set

$$h := p - q, \quad \bar{h} := \sum_{j \in I_{0,+} \cup I_{0,0}} e_j - \sum_{j \in I_{0,-}} e_j$$

Notice that for $\delta > 0$, we have

$$(4.25) \quad p - q + \delta \bar{h} = h + \delta \bar{h} \in E_K := E_{K_+} + E_{K_-} \subset \mathbb{R}^N \quad \text{with} \quad E_{K_\pm} := \pm \sum_{j \in K_\pm} (0, +\infty) e_j$$

and

$$K_+ \cap K_- = \emptyset, \quad K_+ \cup K_- = \{1, \dots, N\} \quad \text{with} \quad K_+ := I_+ \cup I_{0,+} \cup I_{0,0}, \quad K_- := I_- \cup I_{0,-}$$

If p, q belong to the interior of the box $[a, b]$, then we can consider the couple $(p + \delta \bar{h}, q) \in \Omega$ for $\delta > 0$ small enough. In general, we may have $a, b \in \partial[a, b]$. Nevertheless in all cases, we can find $\bar{p}, \bar{q} \in \mathbb{R}^N$ such that $\bar{p} - \bar{q} = \bar{h}$, and for $\delta > 0$ small enough, we have

$$p_\delta := p + \delta \bar{p}, \quad q_\delta := q + \delta \bar{q}, \quad \text{satisfy} \quad p_\delta, q_\delta \in [a, b]$$

Because $p_\delta - q_\delta = h + \delta \bar{h} \in E_K$, we deduce that $(p_\delta, q_\delta) \in \Omega$, and then

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} D^{\hat{f}}(p_\delta, q_\delta) &= - \left\{ \sum_{j \in K_+} \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} - \sum_{j \in K_-} \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} \right\} \\ &= D^{\hat{f}}(p, q) - \sum_{j \in I_{0,+} \cup I_{0,0}} [\hat{f}^j]_q^p + \sum_{j \in I_{0,-}} [\hat{f}^j]_q^p \\ &\leq D^{\hat{f}}(p, q) \end{aligned}$$

where we have used the definition of $I_{0,\pm}$ and $I_{0,0}$ in the last line. This shows (4.23).

Similarly, we have

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} D_+^{\hat{f}}(p_\delta, q_\delta) &= - \sum_{j \in K_+} \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\} \\ &= D_+^{\hat{f}}(p, q) - \sum_{j \in I_{0,+} \cup I_{0,0}} [\hat{f}^j]_q^p \\ &\leq D_+^{\hat{f}}(p, q) \end{aligned}$$

which shows (4.24). This ends the proof of the lemma.

Lemma 4.18 (A property of dissipative functions for $0 : N$ junctions)

Assume (2.2) with $N \geq 1$ for some $0 : N$ junction. Consider a continuous function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$.

i) (Dissipative functions)

Then \hat{f} satisfies

$$(4.26) \quad D^{\hat{f}} \geq 0 \quad \text{on} \quad [a, b]^2$$

if and only if in the sense of distributions in $\mathcal{D}'(a, b)$ with $(a, b) := \prod_{j=1, \dots, N} (a^j, b^j)$

$$(4.27) \quad \left\{ \begin{array}{l} \bar{S}_K := \sum_{j \in K} \hat{f}^j - \sum_{j \notin K} \hat{f}^j \\ \partial_k \bar{S}_K \leq 0 \\ \partial_k \bar{S}_K \geq 0 \end{array} \right. \begin{array}{l} \text{for all } k \in K \\ \text{for all } k \notin K \end{array} \Bigg|, \quad \text{for all } K \subset \{1, \dots, N\}$$

ii) (Semi-dissipative functions)

Then \hat{f} satisfies

$$(4.28) \quad D_+^{\hat{f}} \geq 0 \quad \text{on } [a, b]^2$$

if and only if in $\mathcal{D}'(a, b)$

$$(4.29) \quad \left\{ \begin{array}{l} S_K := \sum_{j \in K} \hat{f}^j \\ \partial_k S_K \leq 0 \\ \partial_k S_K \geq 0 \end{array} \right. \begin{array}{l} \text{for all } k \in K \\ \text{for all } k \notin K \end{array} \Bigg|, \quad \text{for all } K \subset \{1, \dots, N\}$$

Proof of Lemma 4.18

We first do the proof in part I, assuming more regularity on \hat{f} , and then in Part II for \hat{f} only continuous.

Part I: assuming \hat{f} locally Lipschitz continuous

Step 1: proof of i)

Step 1.1: (4.26) implies (4.27)

Let us consider two subsets $K_{\pm} \subset \{1, \dots, N\}$ such that $K_+ \cap K_- = \emptyset$, a point $p \in (a, b) := \prod_{j=1, \dots, N} (a^j, b^j)$

and $h \in E := E_{K_+} + E_{K_-} \subset \mathbb{R}^N$ with $E_{K_{\pm}} := \pm \sum_{j \in K_{\pm}} (0, +\infty) e_j$ and $\varepsilon > 0$ small enough such that we have

$$p_{\varepsilon} := p + \varepsilon h \in (a, b). \quad \text{Then we have } 0 \leq D^{\hat{f}}(p_{\varepsilon}, p) = - \left\{ \sum_{j \in K_+} \left\{ \hat{f}^j(p_{\varepsilon}) - \hat{f}^j(p) \right\} - \sum_{j \in K_-} \left\{ \hat{f}^j(p_{\varepsilon}) - \hat{f}^j(p) \right\} \right\}.$$

Setting $S_{K_+, K_-} := \sum_{j \in K_+} \hat{f}^j - \sum_{j \in K_-} \hat{f}^j$, we get

$$h \cdot DS_{K_+, K_-} \leq 0 \quad \text{a.e. on } (a, b), \quad \text{for all } h \in E$$

Now choosing

$$h := \pm e_k + \delta \left\{ \sum_{j \in K_+} e_j - \sum_{j \in K_-} e_j \right\} \quad \text{if } k \in K_{\pm}$$

then in the limit $\delta \rightarrow 0^+$, we get $\pm \partial_k S_{K_+, K_-} \leq 0$ for all $k \in K_{\pm}$. If moreover we choose K_{\pm} such that $K := K_+ = \{1, \dots, N\} \setminus K_-$, then we get relation (4.27).

Step 1.2: (4.27) implies (4.26)

Conversely, assume that (4.27) holds true with $K := K_+ := \{1, \dots, N\} \setminus K_-$. With the same notation as in Step 1 with $\varepsilon := 1$ and $q := p_{\varepsilon} = p + h$, and $h \in E := E_{K_+} + E_{K_-}$, we get

$$D^{\hat{f}}(q, p) = - \{ \bar{S}_K(q) - \bar{S}_K(p) \} = - \int_0^1 h \cdot D \bar{S}_K(p + th) dt \geq 0 \quad \text{with } h := q - p \in E = E_{K_+} + E_{K_-}$$

For Ω defined in (4.22), this implies $D^{\hat{f}} \geq 0$ on Ω . From Lemma 4.17, we deduce that $D^{\hat{f}} \geq 0$ on $\bar{\Omega} = [a, b]^2$, which is (4.26).

Step 2: proof of ii)

The proof follows the same lines as the one of point i).

Step 2.1: (4.28) implies (4.29)

As in Step 1.1, we consider a point $p \in (a, b)$, $h \in E := E_{K_+} + E_{K_-} \subset \mathbb{R}^N$ and $p_{\varepsilon} := p + \varepsilon h \in (a, b)$. Then we get

$$\left\{ \begin{array}{l} 0 \leq D_+^{\hat{f}}(p_{\varepsilon}, p) = - \sum_{j \in K_+} \left\{ \hat{f}^j(p_{\varepsilon}) - \hat{f}^j(p) \right\} \\ 0 \leq D_+^{\hat{f}}(p, p_{\varepsilon}) = - \sum_{j \in K_-} \left\{ \hat{f}^j(p) - \hat{f}^j(p_{\varepsilon}) \right\} \end{array} \right.$$

Hence for $S_{K_{\pm}} := \sum_{j \in K_{\pm}} \hat{f}^j$, we get

$$\pm h \cdot DS_{K_{\pm}} \leq 0 \quad \text{a.e. on } (a, b), \quad \text{for all } h \in E$$

Now choosing h as in Step 1.1, in the limit $\delta \rightarrow 0^+$, we get (4.29) with $K := K_+ = \{1, \dots, N\} \setminus K_-$.

Step 2.2: (4.29) implies (4.28)

Conversely, assume that (4.29) holds true with $K := K_+ = \{1, \dots, N\} \setminus K_-$. With the same notation as in Step 1.2 with $\varepsilon := 1$ and $q := p_{\varepsilon} = p + h$, we get

$$D_+^{\hat{f}}(q, p) = - \sum_{j \in K_+} \left\{ \hat{f}^j(q) - \hat{f}^j(p) \right\} = - \int_0^1 h \cdot D \left\{ \sum_{j \in K_+} \hat{f}^j(p + th) \right\} dt \geq 0 \quad \text{with } h := q - p \in E = E_{K_+} + E_{K_-}$$

For Ω defined in (4.22), this implies $D_+^{\hat{f}} \geq 0$ on Ω . From Lemma 4.17, we deduce that $D_+^{\hat{f}} \geq 0$ on $\bar{\Omega} = [a, b]^2$, which is (4.28).

Part II: general continuous \hat{f}

We now only assume \hat{f} continuous. Let $0 \leq \rho_{\varepsilon} := \varepsilon^{-N} \rho(\varepsilon^{-1} \cdot)$ be a mollifier with $\text{supp}(\rho) \subset [-1, 1]^N$. Extending \hat{f} (for instance by zero outside the box $[a, b]$), and for p, q in the interior of the box $[a, b]$, and integrating $D^{\hat{f}}(p + \xi, q + \xi) \geq 0$ (resp. $D_+^{\hat{f}}(p + \xi, q + \xi) \geq 0$) over the measure $\rho_{\varepsilon}(\xi) d\xi$ and setting $\hat{f}^{\varepsilon} := \hat{f} \star \rho_{\varepsilon}$, we get $D^{\hat{f}^{\varepsilon}} \geq 0$ (resp. $D_+^{\hat{f}^{\varepsilon}} \geq 0$) on $[a, b]_{\varepsilon} := [a + \varepsilon(1, \dots, 1), b - \varepsilon(1, \dots, 1)]$.

Defining \bar{S}_K^{ε} (resp. S_K^{ε}) as \bar{S}_K (resp. S_K) with \hat{f}^{ε} instead of \hat{f} , we see that Steps 1 and 2 do apply, and in the limit $\varepsilon \rightarrow 0$, we recover the desired result, using the continuity of \hat{f} .

This ends the proof of the lemma.

As a straightforward application of a reversion transform, we get

Corollary 4.19 (A property of dissipative functions for $N : 0$ junctions)

Assume (2.2) with $N \geq 1$ for some $N : 0$ junction. Consider a continuous function $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$.

i) (Dissipative functions)

Then \hat{f} satisfies

$$(4.30) \quad D^{\hat{f}} \geq 0 \quad \text{on } [a, b]^2$$

if and only if in $\mathcal{D}'(a, b)$ with $(a, b) := \prod_{j=1, \dots, N} (a^j, b^j)$

$$(4.31) \quad \left\{ \begin{array}{l} \bar{S}_K := \sum_{j \in K} \hat{f}^j - \sum_{j \notin K} \hat{f}^j \\ \partial_k \bar{S}_K \geq 0 \\ \partial_k \bar{S}_K \leq 0 \end{array} \quad \begin{array}{l} \text{for all } k \in K \\ \text{for all } k \notin K \end{array} \right\}, \quad \text{for all } K \subset \{1, \dots, N\}$$

ii) (Semi-dissipative functions)

Then \hat{f} satisfies

$$(4.32) \quad D_+^{\hat{f}} \geq 0 \quad \text{on } [a, b]^2$$

if and only if in $\mathcal{D}'(a, b)$

$$(4.33) \quad \left\{ \begin{array}{l} S_K := \sum_{j \in K} \hat{f}^j \\ \partial_k S_K \geq 0 \\ \partial_k S_K \leq 0 \end{array} \quad \begin{array}{l} \text{for all } k \in K \\ \text{for all } k \notin K \end{array} \right\}, \quad \text{for all } K \subset \{1, \dots, N\}$$

5 Gluing and Riemann monotonicity

5.1 Riemann monotonicity

Proposition 5.1 (The map π is Riemann monotone on $N : 0$ junctions)

Assume (2.2) for a $N : 0$ junction with $N \geq 1$. Let \mathcal{G} be a generalized Riemann germ, and let $\pi := \pi_{\mathcal{G}}$ its associated projection. Then the map π is Riemann monotone in the sense of Definition 2.11.

Proof of Proposition 5.1

For all $p, q \in [a, b]$, we set $\hat{p} := \pi(p)$, $\hat{q} := \pi(q)$ and $[\pi]_q^p := \pi(p) - \pi(q) = \hat{p} - \hat{q}$. Assume by contradiction that π is not Riemann monotone, i.e. that we have

$$(5.1) \quad (p - q) \diamond [\pi]_q^p \leq 0 \quad \text{and} \quad [\pi]_q^p \neq 0$$

Recall that

$$BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset \quad \implies \quad \hat{p} = \hat{q} \quad \iff \quad [\pi]_q^p = 0$$

From (5.1), we then deduce

$$(5.2) \quad BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$$

We set

$$I := \{k \in \{1, \dots, N, \quad p^k \geq q^k\}\}, \quad \bar{I} := \{k \in \{1, \dots, N, \quad p^k < q^k\}\}$$

and then $I \cup \bar{I} = \{1, \dots, N\}$. Up to apply a \bar{I} -inversion (see Definition 4.1), we can assume that

$$(5.3) \quad p \geq q, \quad \hat{p} \leq \hat{q}, \quad \hat{p} \neq \hat{q}$$

We now do the proof by recurrence on $N \geq 1$.

Step 1: Case $N = 1$

Then we get $p \geq q$, $\pi(p) \leq \pi(q)$. Because π is nondecreasing (see vi) of Proposition 3.6), we get $\hat{p} = \pi(p) = \pi(q) = \hat{q}$. Contradiction.

Step 2: Case $N \geq 2$, and Proposition 5.8 holds true for $N' := N - 1$

Step 2.1: Case of the hyperplane intersection

Step 2.1.1: setting of the problem

Assume by contradiction that there exists an index $k_0 \in \{1, \dots, N\}$ and $p_0^{k_0} \in [a^{k_0}, b^{k_0}]$ such that the "hyperplane" intersection with the box $[a, b]$ defined as $\Pi_{k_0} := \{p \in [a, b], \quad p^{k_0} = p_0^{k_0}\}$ satisfies

$$(5.4) \quad \begin{cases} \Pi_{k_0} \cap BA(\hat{p}) \neq \emptyset \\ \Pi_{k_0} \cap BA(\hat{q}) \neq \emptyset \end{cases}$$

Up to relabel the indices, we can assume that $k_0 = N$, and then (5.4) means

$$(5.5) \quad BA(\hat{p}^N) \cap BA(\hat{q}^N) \neq \emptyset$$

From the slicing Lemma 4.12, we know that the flux function $\hat{f}_{\mathcal{G}}$ restricted to $\Pi_N \cap [a, b]$ is associated to a generalized Riemann germ $\mathcal{G}_{p_0^N} \subset \tilde{Q} := \prod_{k=1, \dots, N-1} [a_k, b_k]$, and we set the associated projection map

$\tilde{\pi} : \tilde{Q} \rightarrow \mathcal{G}_{p_0^N}$ defined as $\tilde{\pi} := \pi_{\mathcal{G}_{p_0^N}} := \pi'(\cdot, p_0^N)$ for $\pi = (\pi', \pi^N)$. Moreover, writing

$$\hat{p} = (\hat{p}', \hat{p}^N), \quad \hat{q} = (\hat{q}', \hat{q}^N), \quad p = (p', p^N), \quad q = (q', q^N)$$

ans using $(p', p_0^N) \in BA(\hat{p})$ and $(q', p_0^N) \in BA(\hat{q})$, we get that $\hat{p}' = \tilde{\pi}(p')$, $\hat{q}' = \tilde{\pi}(q') \in \mathcal{G}_{p_0^N}$. Then (5.3) implies $(p' - q') \diamond [\tilde{\pi}]_{q'}^{p'} \leq 0$. By recurrence assumption, notice that $\tilde{\pi}$ is Riemann monotone, and then $[\tilde{\pi}]_{q'}^{p'} = 0$, i.e. $\hat{p}' = \hat{q}'$.

Step 2.1.2: consequences

Hence we get $BA(\hat{p}) = Q' \times BA^N(\hat{p}^N)$, $BA(\hat{q}) = Q' \times BA^N(\hat{q}^N)$ with $Q' := BA(\hat{p}') = BA(\hat{q}')$. Hence (5.2) means $BA^N(\hat{p}^N) \cap BA^N(\hat{q}^N) = \emptyset$. Contradiction with (5.5). This implies there is no index k_0 such that (5.4) holds true, which is the next case.

Step 2.2: Case of no hyperplane intersection

We now assume that there is no index k_0 such that (5.4) holds true. Then this implies that $BA(\hat{p})$ and $BA(\hat{q})$ are well separated in all directions, and then in particular we have $p \geq q$, $\hat{p} < \hat{q}$. Again, because $BA(\hat{p})$ and $BA(\hat{q})$ are well separated in all directions, we deduce that

$$p^k \in BA(\hat{p}^k) < BA(\hat{q}^k) \ni q^k$$

which implies $p^k < q^k$. Contradiction.

Step 3: conclusion

Therefore (5.1) is false and we conclude that π is Riemann monotone. This ends the proof of the proposition.

Notice that if \hat{f} is Riemann monotone on $[a, b]$, then f is in particular Riemann monotone on \mathcal{G} . Conversely, we also have

Lemma 5.2 (Transfer of Riemann monotonicity, $N : 0$ junctions)

Assume (2.2) for $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ.

Assume that f is Riemann monotone on \mathcal{G} . Then $\hat{f} := \hat{f}_{\mathcal{G}}$ is Riemann monotone on $[a, b]$.

Proof of Lemma 5.2

Assume that f is Riemann monotone on \mathcal{G} . This means that for all $\hat{p}, \hat{q} \in \mathcal{G}$, we have

$$(5.6) \quad (\hat{p} - \hat{q}) \diamond [f]_{\hat{q}}^{\hat{p}} \leq 0 \quad \implies \quad [f]_{\hat{q}}^{\hat{p}} = 0$$

Now consider any $p \in BA(\hat{p})$ and $q \in BA(\hat{q})$, and assume that $(p - q) \diamond [\hat{f}]_q^p \leq 0$. From ii) of Lemma 4.16 on dissipation properties, we deduce that $(\hat{p} - \hat{q}) \diamond [f]_{\hat{q}}^{\hat{p}} \leq 0$ and then (5.6) shows that $0 = [f]_{\hat{q}}^{\hat{p}} = [\hat{f}]_q^p$, i.e. that \hat{f} is Riemann monotone. This ends the proof of the lemma.

We finish this subsection with the following results.

Lemma 5.3 (Injectivity of perturbed Riemann monotone functions for $N : 0$ junctions)

Assume (2.2) for $N \geq 1$. Assume that $h : [a, b] \rightarrow \mathbb{R}^N$ is Riemann monotone.

Then for any $\varepsilon > 0$, the function $h_\varepsilon := h + \varepsilon Id : [a, b] \rightarrow \mathbb{R}^N$ is injective.

Proof of Lemma 5.3

First, up to a reversion transform (see Definition 4.4), we can assume that $\sigma^j = 1$ for all indices j . Let $\varepsilon > 0$ and consider $p, q \in [a, b]$ such that $h_\varepsilon(p) = h_\varepsilon(q)$. Set

$$I := \{j \in \{1, \dots, N\}, \quad p^j < q^j\}$$

Then up to a I -inversion (see Definition 4.1), we can assume that $p \geq q$ and $h_\varepsilon(p) = h_\varepsilon(q)$. Hence

$$(p^j - q^j) \cdot [h^j]_q^p = (p^j - q^j) \cdot [h_\varepsilon^j]_q^p - \varepsilon(p^j - q^j)^2 = -\varepsilon(p^j - q^j)^2 \leq 0$$

Therefore $(p - q) \diamond [h]_q^p \leq 0$ and the Riemann monotonicity of h implies that $[h]_q^p = 0$. Therefore

$$\varepsilon[Id]_q^p = [h_\varepsilon]_q^p - [h]_q^p = 0$$

i.e. $p = q$, which shows the injectivity of h_ε . This ends the proof of the lemma.

Remark 5.4 All results of Subsection 5.1 generalize to junctions of type $n : m$. This follows from the use of suitable I -reversions (see Definition 4.4 and Lemma 4.5).

5.2 Proposition 2.13 and its proof: properties of generalized Riemann germs

We are now ready to give the proof of Proposition 2.13.

Proof of Proposition 2.13

The result follows from Propositions 4.10, 4.13, 5.1, 4.10 and Lemma 3.4. This ends the proof of the proposition.

5.3 Subclasses of germs and effect of slicing

Lemma 5.5 (Flux properties of subclasses of germs)

Assume (2.2) for $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized germ. Let $\hat{f} := \hat{f}_{\mathcal{G}}$. Then the following holds.

i) (Kruřkov germs)

Then \mathcal{G} is a Kruřkov germ if and only if

$$(5.7) \quad D\hat{f} \geq 0 \quad \text{on} \quad [a, b]^2$$

with

$$D^{\hat{f}}(p, q) = \sum_{j=1, \dots, N} D^{\hat{f}^j}(p, q) \quad \text{with} \quad D^{\hat{f}^j}(p, q) = \sigma^j \cdot \text{sign}(p^j - q^j) \cdot \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\}$$

i') (D_+ -germs)

Then \mathcal{G} is a D_+ -germ if and only if

$$(5.8) \quad D_+^{\hat{f}} \geq 0 \quad \text{on} \quad [a, b]^2$$

with

$$D_+^{\hat{f}}(p, q) = \sum_{j=1, \dots, N} D_+^{\hat{f}^j}(p, q) \quad \text{with} \quad D_+^{\hat{f}^j}(p, q) = \sigma^j \cdot \text{sign}^+(p^j - q^j) \cdot \left\{ \hat{f}^j(p) - \hat{f}^j(q) \right\}$$

ii) (HJ germs)

Then \mathcal{G} is a HJ germ if and only if there exists some function $\hat{h} : [a, b] \rightarrow \mathbb{R}$ such that

$$(5.9) \quad \hat{f}^j = \hat{h} \quad \text{for all} \quad j = 1, \dots, N$$

iii) (Monotone germs)

Then \mathcal{G} is a monotone germ if and only if

$$(5.10) \quad p \mapsto \sigma^j \hat{f}^j(p) \quad \text{is nonincreasing in } p^k \text{ for all } k \neq j$$

iv) (Conservative germs)

Then \mathcal{G} is a conservative germ if and only if

$$(5.11) \quad \sum_{J^j \simeq (-\infty, 0)} \hat{f}^j = \sum_{J^j \simeq (0, +\infty)} \hat{f}^j \quad \text{on} \quad [a, b].$$

This means

$$RH^{\hat{f}} = 0 \quad \text{on} \quad [a, b], \quad \text{with} \quad RH^{\hat{f}}(p) := \sum_{j=1, \dots, N} \sigma^j \hat{f}^j(p) = IN - OUT.$$

v) (Effect of reversion transform)

Moreover Kruřkov germs, monotone germs and conservative germs are preserved by reversion transform of Definition 4.4, while HJ germs are not preserved in general by reversion transforms.

vi) (Effect of inversion transforms)

Moreover Kruřkov germs are preserved by any inversion of Definition 4.1, while HJ germs, monotone germs and conservative are not preserved in general by inversion transforms.

Proof of Lemma 5.5

We mainly to the proof in the case of $N : 0$ junctions, the general case following by reversion transforms.

Step 1: proof of i)

Recall that \mathcal{G} is a Kruřkov germ, i.e.

$$(5.12) \quad D^{\hat{f}} \geq 0 \quad \text{on} \quad \mathcal{G}^2$$

and $\hat{f} = f \circ \pi$. Hence for all $p, q \in [a, b]$, and $\hat{p} := \pi(p)$ and $\hat{q} := \pi(q)$, and $f^j(\hat{p}) = f^j(\hat{p}^j)$, we have from (4.21) that $D^{\hat{f}}(p, q) \geq D^{\hat{f}}(\hat{p}, \hat{q}) \geq 0$, which shows (5.7). Conversely it is straightforward that (5.7) implies (5.12). This shows point i).

Step 1': proof of i')

Recall that \mathcal{G} is a D_+ -germ, i.e.

$$(5.13) \quad D_+^{\hat{f}} \geq 0 \quad \text{on} \quad \mathcal{G}^2$$

and $\hat{f} = f \circ \pi$. Hence for all $p, q \in [a, b]$, and $\hat{p} := \pi(p)$ and $\hat{q} := \pi(q)$, and $f^j(\hat{p}) = f^j(\hat{p}^j)$, we have from (4.21) that $D_+^{\hat{f}}(p, q) \geq D_+^{\hat{f}}(\hat{p}, \hat{q}) \geq 0$, which shows (5.8). Conversely it is straightforward that (5.8) implies (5.13). This shows point i').

Step 2: proof of ii)

Recall that \mathcal{G} is a HJ germ, i.e. there exists some function $h : \mathcal{G} \rightarrow \mathbb{R}$ such that

$$(5.14) \quad f^j = h \quad \text{on} \quad \mathcal{G}, \quad \text{for all} \quad j = 1, \dots, N$$

Setting $\hat{h} := h \circ \pi$, we see that (5.14) is equivalent to (5.9) which shows point ii).

Step 3: proof of iii)

Recall that a generalized Riemann germ \mathcal{G} is monotone if and only if

$$(5.15) \quad \text{for all } p, q \in [a, b], \quad (p \geq q \text{ implies } \pi(p) \geq \pi(q))$$

and we want to show that it is equivalent to (5.10). Fix some $k \in \{1, \dots, N\}$, consider $p, q \in [a, b]$ such that $p - q = \varepsilon e_k$ for some $\varepsilon \geq 0$, and call $\hat{p} := \pi(p)$, $\hat{q} := \pi(q)$.

On the one hand, recall from Proposition 5.1 that π is Riemann monotone, and then that $\pi^k(p) \geq \pi^k(q)$, i.e. that $\hat{p}^k \geq \hat{q}^k$ is always satisfied. On the other hand, (even if it is not used in this proof), notice that the monotonicity in (2.14) shows that $\sigma^k \hat{f}^k(p) \geq \sigma^k \hat{f}^k(q)$ is also always satisfied.

Now in order to show equivalence between (5.15) and (5.10), we see that it is sufficient to show that for all $j \in \{1, \dots, N\} \setminus \{k\}$, inequality $\hat{p}^j \geq \hat{q}^j$ is equivalent to $\sigma^j \hat{f}^j(p) \leq \sigma^j \hat{f}^j(q)$. We now focus on the case $\sigma^j = 1$ (the reasoning is similar for $\sigma^j = -1$). Because $\hat{f}^j(p) = f^j \circ \pi(p) = f^j(\hat{p}^j)$, it remains to show that

$$(5.16) \quad \hat{p}^j \geq \hat{q}^j \text{ is equivalent to } f^j(\hat{p}^j) \leq f^j(\hat{q}^j), \text{ for } J^j \simeq (-\infty, 0)$$

Recall that $BA^j = BA^{(J^j, f^j)}$. Because $p^j = q^j$, we deduce that $BA^j(\hat{p}^j) \cap BA^j(\hat{q}^j) \neq \emptyset$. Then Definition 2.9 of BA^j for $J^j \simeq (-\infty, 0)$ (see also the associated figure) implies (5.16). This ends the proof of the equivalence of (5.15) with (5.10).

Step 4: proof of iv)

Recall that \mathcal{G} is a conservative germ, i.e. $\sum_{J^j \simeq (-\infty, 0)} f^j = \sum_{J^j \simeq (0, +\infty)} f^j$ on \mathcal{G} . For $\hat{f}^j = f^j \circ \pi$, this shows that

this is equivalent to (5.11), and this proves iv).

Step 5: proof of v) and vi)

The result is straightforward. This ends the proof of the lemma.

Lemma 5.6 (Nature of germs after slicing)

Assume (2.2) for $N \geq 2$. Let $\mathcal{G} \subset [a, b]$ be a generalized germ. Let $1 \leq n < N$. Then for $p \in [a, b]$, we write

$$p = (p', p'') \quad \text{with } p' = (p^1, \dots, p^n) \in [a', b'] \quad \text{and } p'' = (p^{n+1}, \dots, p^N) \in [a'', b''].$$

Given some $p''_0 \in [a'', b'']$, let $\mathcal{G}_{p''_0}$ be the sliced germ defined in Lemma 4.12.

If \mathcal{G} is a Kruřkov (resp. HJ, resp. monotone) germ, then the sliced germ $\mathcal{G}_{p''_0}$ is also a Kruřkov (resp. HJ, resp. monotone) germ.

Proof of Lemma 5.6

From the definitions and Lemma 5.5, the result follows from the following property

$$\mathcal{G}_{p''_0} = \left\{ p' = (p^1, \dots, p^n) \in [a', b'], \quad \hat{f}^j(p', p''_0) = f^j(p^j) \quad \text{for } j = 1, \dots, n \right\}$$

which is provided by the slicing Lemma 4.12. This ends the proof of the lemma.

Remark 5.7 In Lemma 5.6, if \mathcal{G} is a conservative germ, then $\mathcal{G}_{p''_0}$ satisfies the following relation

$$\sum_{J^j \simeq (-\infty, 0)} f^j(p) = \sum_{J^j \simeq (0, +\infty)} f^j(p) \quad \text{for all } p = (p', p''_0) \in \mathcal{G}_{p''_0} \times \{p''_0\}$$

which is the memory at the level of the sliced germ $\mathcal{G}_{p''_0}$ of the Rankine-Hugoniot relation satisfied by \mathcal{G} .

5.4 Theorem 2.17 and its proof: Riemann monotonicity of $\sigma \diamond \hat{f}$

The proof of Theorem 2.17 is done at the end of the subsection as a corollary of the following more general proposition.

Proposition 5.8 (The map $h \circ \pi$ is Riemann monotone on $N : 0$ junctions)

Assume (2.2) for a $N : 0$ junction with $N \geq 1$. Let \mathcal{G} be a generalized Riemann germ, and let $\pi := \pi_{\mathcal{G}}$ its associated projection. Let $h = (h^1, \dots, h^N) : [a, b] \rightarrow \mathbb{R}^N$ be a continuous map with for $k = 1, \dots, N$

$$(5.17) \quad \text{each component } h^k : [a^k, b^k] \rightarrow \mathbb{R} \text{ is non constant on any nondegenerate interval}$$

Assume that the map $\hat{h} := h \circ \pi : [a, b] \rightarrow \mathbb{R}^N$ is continuous and satisfies

$$(5.18) \quad \begin{cases} p \mapsto \hat{h}^k(p) \text{ is nondecreasing in } p^k, \text{ for each } k = 1, \dots, N \\ \hat{h} \text{ is locally constant on } \{\hat{h} \neq h\} \end{cases}$$

Then \hat{h} satisfies

$$(5.19) \quad (p - q) \diamond [\hat{h}]_q^p \leq 0 \implies \hat{h} = \text{const on } \text{co}(BA(\hat{p}) \cup BA(\hat{q}))$$

where $\text{co}(A)$ is the convex hull of a set $A \subset \mathbb{R}^N$. In particular \hat{h} is Riemann monotone.

Proof of Proposition 5.8

The proof follows the one of Proposition 5.1 with \hat{h} instead of π , and with some adaptations.

For all $p, q \in [a, b]$, we set $\hat{p} := \pi(p)$, $\hat{q} := \pi(q)$ and $[\hat{h}]_q^p := \hat{h}(p) - \hat{h}(q)$. Assume by contradiction that \hat{h} does not satisfy (5.19), i.e. that we have

$$(5.20) \quad (p - q) \diamond [\hat{h}]_q^p \leq 0 \quad \text{and} \quad \hat{h} \neq \text{const on } \text{co}(BA(\hat{p}) \cup BA(\hat{q}))$$

Recall that

$$BA(\hat{p}) \cap BA(\hat{q}) \neq \emptyset \implies \hat{p} = \hat{q} \implies \hat{h} = \text{const on } \text{co}(BA(\hat{p}) \cup BA(\hat{q})) = BA(\hat{p}) = BA(\hat{q})$$

From (5.20), we then deduce $BA(\hat{p}) \cap BA(\hat{q}) = \emptyset$ and up to apply an inversion, we can assume that $p \geq q$, $\hat{p} \leq \hat{q}$, $\hat{p} \neq \hat{q}$. We now do the proof by recurrence on $N \geq 1$.

Step 1: Case $N = 1$

Then (5.20) means

$$\hat{h}(p) \leq \hat{h}(q) \quad \text{and} \quad \hat{h} \neq \text{const on } \text{co}(BA(\hat{p}) \cup BA(\hat{q}))$$

Because \hat{h} is nondecreasing, we get $\hat{h}(p) = \hat{h}(q)$, and then $\hat{h} = \text{const}$ on $\text{co}(BA(\hat{p}) \cup BA(\hat{q}))$. Contradiction.

Step 2: Case $N \geq 2$, and Proposition 5.8 holds true for $N' := N - 1$

Step 2.1: Case of the hyperplane intersection

Step 2.1.1: setting of the problem

Assume by contradiction that there exists an index $k_0 \in \{1, \dots, N\}$ and $p_0^{k_0} \in [a^{k_0}, b^{k_0}]$ such that the "hyperplane" intersection with the box $[a, b]$, defined as $\Pi_{k_0} := \left\{ p \in [a, b], \quad p^{k_0} = p_0^{k_0} \right\}$, satisfies

$$(5.21) \quad \begin{cases} \Pi_{k_0} \cap BA(\hat{p}) \neq \emptyset \\ \Pi_{k_0} \cap BA(\hat{q}) \neq \emptyset \end{cases}$$

Up to relabel the indices, we can assume that $k_0 = N$, and then (5.21) means

$$(5.22) \quad BA(\hat{p}^N) \cap BA(\hat{q}^N) \neq \emptyset$$

From the slicing Lemma 4.12, we know that the flux function $\hat{f}_{\mathcal{G}}$ restricted to $\Pi_N \cap [a, b]$ is associated to a generalized Riemann germ $\mathcal{G}_{p_0^N} \subset \tilde{Q} := \prod_{k=1, \dots, N-1} [a_k, b_k]$, and we set the associated projection map

$\tilde{\pi} : \tilde{Q} \rightarrow \mathcal{G}_{p_0^N}$ defined as $\tilde{\pi} := \pi_{\mathcal{G}_{p_0^N}} := \pi'(\cdot, p_0^N)$ for $\pi = (\pi', \pi^N)$. Moreover, we write

$$\hat{p} = (\hat{p}', \hat{p}^N), \quad \hat{q} = (\hat{q}', \hat{q}^N), \quad p = (p', p^N), \quad q = (q', q^N), \quad h = (h', h^N), \quad \hat{h} = (\hat{h}', \hat{h}^N)$$

and (on the model of $\hat{f}_{\mathcal{G}_{p_0^N}} := \hat{f}_{\mathcal{G}}(\cdot, p_0^N)$) we set $\hat{h}' := \hat{h}'(\cdot, p_0^N) = h' \circ (\tilde{\pi}, p_0^N) = (h^k \circ \tilde{\pi}^k)_{k=1, \dots, N-1} = h' \circ \tilde{\pi}$, where we use our usual abuse of notation for the argument of h or h' , and where $p' \mapsto (h' \circ \tilde{\pi})^k(p')$ is

nondecreasing in p^k for every $k = 1, \dots, N - 1$.

Using $(p', p_0^N) \in BA(\hat{p})$ and $(q', p_0^N) \in BA(\hat{q})$, we get that $\hat{p}' = \tilde{\pi}(p')$, $\hat{q}' = \tilde{\pi}(q') \in \mathcal{G}_{p_0^N}$ and

$$\hat{h}'(p) = \hat{h}'(p', p^N) = h' \circ \pi(p', p^N) = h' \circ \pi(p', p_0^N) = \hat{h}(p') \quad \text{and} \quad \hat{h}'(q) = \hat{h}(q')$$

Hence (5.20) implies $(p' - q') \diamond [\hat{h}]_{q'}^{p'} \leq 0$. Notice that by construction, \hat{h} is locally constant on $\{\hat{h} \neq h'\}$ and such that $p \mapsto \hat{h}^k(p)$ is nondecreasing in p^k for $k = 1, \dots, N - 1$. This shows that \hat{h} satisfies (5.18) for $N' := N - 1$. Then by recurrence assumption, we know that $\hat{h} = h' \circ \tilde{\pi}$ satisfies (5.19) for $N' := N - 1$. We deduce that

$$h' \circ \tilde{\pi} = \hat{h} = \text{const} = \lambda' := h'(p') = h'(q') \quad \text{on} \quad Q' := \prod_{k=1, \dots, N-1} Q^k \quad \text{with} \quad Q^k := \text{co}(BA(\hat{p}^k) \cup BA(\hat{q}^k))$$

where we recall that $\text{co}(A)$ is the convex hull of A .

Step 2.1.2: consequences

Let $\Omega' := \prod_{k=1, \dots, N-1} \Omega^k$ with $\Omega^k := \{p^k \in [a^k, b^k], h^k(p^k) \neq \lambda'^k\}$. From (5.17), h^k is non constant on any

nondegenerate interval. Hence Ω^k is a relative open set of $[a^k, b^k]$, which is also dense in Q^k . By assumption \hat{h} is locally constant on $\{\hat{h} \neq h'\}$. This implies that $\hat{h}^N(\cdot, p_0^N)$ is locally constant on $\Omega' \subset \{\lambda' = \hat{h}' \neq h'\}$.

Notice that $\Omega' \cap Q'$ is a relative open set of Q' . Now $\Omega' \cap Q'$ is dense in Q' , and \hat{h}^N is continuous and locally constant on $\Omega' \cap Q'$. Therefore we deduce that $\hat{h}^N(\cdot, p_0^N) = \text{const}$ on Q' . Therefore

$$h^N(\hat{p}^N) = h^N(\pi^N(p', p_0^N)) = \hat{h}^N(p', p_0^N) = \text{const} = \hat{h}^N(q', p_0^N) = h^N(\hat{q}^N)$$

From (5.22), we deduce that $BA(\hat{p}^N) \cap BA(\hat{q}^N) \neq \emptyset$ with $h^N(\hat{p}^N) = h^N(\hat{q}^N)$, which from the structure of the Basins of Attraction, implies that $\hat{p}^N = \hat{q}^N$, and then we can choose any $p_0^N \in BA(\hat{p}^N) = BA(\hat{q}^N)$, which shows that

$$\hat{h} = \text{const} = h(\hat{p}) = h(\hat{q}) \quad \text{on} \quad \text{co}(BA(\hat{p}) \cup BA(\hat{q})) = Q' \times BA(\hat{p}^N)$$

Contradiction with (5.20). This implies there is no index k_0 such that (5.21) holds true, which is the next case.

Step 2.2: Case of no hyperplane intersection

We now assume that there is no index k_0 such that (5.21) holds true. Then this implies that $BA(\hat{p})$ and $BA(\hat{q})$ are well separated in all directions, and we get $\hat{p} < \hat{q}$ and then $p > q$, and also $p \in BA(\hat{p}) < BA(\hat{q}) \ni q$, which implies $p < q$. Contradiction.

Step 3: conclusion

Therefore (5.20) is false and this implies (5.19). This ends the proof of the proposition.

We will need the following proposition

Proposition 5.9 (Riemann monotonicity of dissipative functions)

Assume (2.2) with $N \geq 1$, and recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the branches. Assume that $h : [a, b] \rightarrow \mathbb{R}^N$ is a continuous map. Assume also that the map $p \mapsto h^j(p)$ is locally Lipschitz continuous in p^j uniformly in the other coordinates p^k for $k \neq j$, and for all $j = 1, \dots, N$.

If h satisfies

$$(5.23) \quad D^h(p, q) = \sum_{j=1, \dots, N} \sigma^j \cdot \text{sign}(p^j - q^j) \cdot \{h^j(p) - h^j(q)\} \geq 0 \quad \text{for all } p, q \in [a, b]$$

then $\sigma \diamond h$ is Riemann monotone on $[a, b]$.

and its straightforward corollary.

Corollary 5.10 (Riemann monotonicity for Kruřkov germs)

Assume (2.2) with $N \geq 1$, and recall that $\sigma \in \{\pm 1\}^N$ encodes the orientations of the branches. Let $\mathcal{G} \subset [a, b]$ be a Kruřkov germ, and $f_{\mathcal{G}}$ be its associated Godunov flux. Then the map $\sigma \diamond f_{\mathcal{G}}$ is Riemann monotone.

Proof of Proposition 5.9

Up to apply some suitable reversion, we can assume that we work with $\sigma^j = 1$ for all indices j , i.e. for $N : 0$ junctions. By assumption, we consider a continuous dissipative function $h : [a, b] \rightarrow \mathbb{R}^N$, i.e. satisfying

$$(5.24) \quad D^h(p, q) = \sum_{j=1, \dots, N} \text{sign}(p^j - q^j) \cdot [h^j]_q^p \geq 0 \quad \text{for all } p, q \in [a, b].$$

and such that the map $p \mapsto h^j(p)$ is locally Lipschitz continuous in p^j uniformly in the other coordinates p^k for $k \neq j$, and for all $j = 1, \dots, N$.

Now assume that $p, q \in [a, b]$ satisfy $(p - q) \diamond [h]_q^p \leq 0$. Then (5.24) implies $(p - q) \diamond [h]_q^p = 0$. Up to apply suitable inversions, we can assume that $(p - q) \geq 0$. We then set $I := \{i \in \{1, \dots, N\}, (p - q)^i > 0\}$, $\bar{I} := \{1, \dots, N\} \setminus I$. In particular, we get

$$(5.25) \quad \begin{cases} [h^i]_q^p = 0 & \text{for all } i \in I \\ (p - q)^i = 0 & \text{for all } i \in \bar{I} \end{cases}$$

Moreover, consider the set $\Omega := \{\xi \in \mathbb{R}^N, p + \xi, q + \xi \in [a, b]\}$ and consider the quantity

$$T^j(\xi, t) := \partial_j h^j(q + \xi + t(p - q)) - \sum_{i \in \{1, \dots, N\} \setminus \{j\}} |\partial_j h^i(q + \xi + t(p - q))| \geq 0$$

which is defined for a.e. ξ and is nonnegative from ii) of Proposition 6.4. Then we get for a.e. $\xi \in \Omega$

$$\begin{aligned} & \sum_{i \in I} [h^i]_{q+\xi}^{p+\xi} \\ &= \sum_{i, j \in I} \int_0^1 dt \partial_j h^i(q + \xi + t(p - q)) \cdot (p - q)^j \\ &= \sum_{j \in I} (p - q)^j \cdot \int_0^1 dt \left\{ \sum_{i \in I} \partial_j h^i(q + \xi + t(p - q)) \right\} \\ &\geq \sum_{j \in I} (p - q)^j \cdot \int_0^1 dt \left\{ \partial_j h^i(q + \xi + t(p - q)) - \sum_{i \in I \setminus \{j\}} |\partial_j h^i(q + \xi + t(p - q))| \right\} \\ &= \sum_{j \in I} (p - q)^j \cdot \int_0^1 dt \left\{ T^j(\xi, t) + \sum_{i \in \bar{I}} |\partial_j h^i(q + \xi + t(p - q))| \right\} \\ &\geq \sum_{j \in I} (p - q)^j \cdot \int_0^1 dt \left\{ \sum_{i \in \bar{I}} |\partial_j h^i(q + \xi + t(p - q))| \right\} \\ &= \sum_{i \in \bar{I}} \int_0^1 dt \left\{ \sum_{j \in I} (p - q)^j \cdot |\partial_j h^i(q + \xi + t(p - q))| \right\} \\ &\geq \sum_{i \in \bar{I}} \left| \int_0^1 dt \left\{ \sum_{j \in I} (p - q)^j \cdot \partial_j h^i(q + \xi + t(p - q)) \right\} \right| \\ &= \sum_{i \in \bar{I}} \left| \int_0^1 dt \left\{ \sum_{j \in \{1, \dots, N\}} (p - q)^j \cdot \partial_j h^i(q + \xi + t(p - q)) \right\} \right| \\ &= \sum_{i \in \bar{I}} |[h^i]_{q+\xi}^{p+\xi}| \end{aligned}$$

i.e.

$$\sum_{i \in I} [h^i]_{q+\xi}^{p+\xi} \geq \sum_{i \in \bar{I}} |[h^i]_{q+\xi}^{p+\xi}| \quad \text{for a.e. } \xi \in \Omega$$

By continuity of both sides, we deduce for $\xi = 0$ that $\sum_{i \in I} [h^i]_q^p \geq \sum_{i \in \bar{I}} |[h^i]_q^p|$. From (5.25), we deduce $[h]_q^p = 0$. Hence we have shown that $(p - q) \diamond [h]_q^p \leq 0$ implies $[h]_q^p = 0$, i.e. that h is Riemann monotone.

This ends the proof of the proposition.

Proof of Theorem 2.17

We want to show that $\sigma \diamond \hat{f}_{\mathcal{G}}$ is Riemann monotone. We only do it for $N : 0$ junctions, and then show that $\hat{f} := \hat{f}_{\mathcal{G}}$ is Riemann monotone. For the general case of junctions $n : m$ with $N := n + m$, we use suitable I -reversions (see Lemma 4.5).

Step 1: proof of i)

For junctions $N : 0$ and under nondegeneracy assumption (2.16), it follows from Proposition 5.8 with $h := f$ and $\hat{h} := \hat{f}$ which is continuous.

Step 2: proof of ii)

For Kružkov germs \mathcal{G} , we know from Lemma 5.5 that $h := \hat{f}_{\mathcal{G}}$ satisfies (5.23). We know moreover that $|\partial_j h^j(p)| \leq |f^{j'}(p^j)|$ for all j , as do all Godunov fluxes. Then the result follows from Corollary 5.10. This ends the proof of the theorem.

5.5 Theorem 2.19 and its proof: gluing Riemann germs

The proof of Theorem 2.19 is mainly based on Proposition 2.19, where we first glue the fluxes. Then in Corollary 5.12, we show that this is equivalent to glue the germs. Finally in Lemma 5.13, we check the associativity of the gluing. The proof of Theorem 2.19 is done at the end of this subsection, as a consequence of those three results.

Proposition 5.11 (Gluing of flux functions \hat{f}_{γ} for $n_{\gamma} : m_{\gamma}$ junctions)

For $\gamma = \alpha, \beta$, assume that f_{γ} satisfies (2.2) for $N_{\gamma} = n_{\gamma} + m_{\gamma}$ and $n_{\gamma} : m_{\gamma}$ junctions J_{γ} with $J_{\gamma}^j \simeq \sigma_{\gamma}^j \cdot (-\infty, 0)$ and $\sigma_{\gamma} \in \{\pm 1\}^{N_{\gamma}}$, and consider Riemann germs \mathcal{G}_{γ} with respect to (J_{γ}, f_{γ}) .

For $\gamma = \alpha$ or β , assume either that 1) f_{γ} satisfies nondegeneracy condition (2.16), or that 2) \mathcal{G}_{γ} is a Kružkov germ. We allow mixing cases for α and β .

We set $[a, b]_{\gamma}^j := [a_{\gamma}^j, b_{\gamma}^j]$. We assume that for each $\gamma = \alpha, \beta$, there exists one index $j_{\gamma} \in \{1, \dots, N_{\gamma}\}$ such that

$$(5.26) \quad f_{\alpha}^{j_{\alpha}} = f_{\beta}^{j_{\beta}} =: f^0 \quad \text{on} \quad [a, b]_{\alpha}^{j_{\alpha}} = [a, b]_{\beta}^{j_{\beta}} =: [a^0, b^0] \quad \text{with} \quad J_{\alpha}^{j_{\alpha}} \simeq (0, +\infty) \quad \text{and} \quad J_{\beta}^{j_{\beta}} \simeq (-\infty, 0)$$

and we glue those two branches. To simplify the notation, up to relabel the indices, we now assume that $j_{\alpha} = 0 = j_{\beta}$, and the indices now go through the values $\{0, \dots, N_{\gamma} - 1\}$. Hence we now have

$$\left\{ \begin{array}{l} f_{\gamma} = (f_{\gamma}^0, \dots, f_{\gamma}^{N_{\gamma}-1}) \\ \mathcal{G}_{\gamma} \subset [a, b]_{\gamma} := \prod_{i=0, \dots, N_{\gamma}-1} [a, b]_{\gamma}^i \\ [a, b]_{\gamma}' := \prod_{i=1, \dots, N_{\gamma}-1} [a, b]_{\gamma}^i \\ J_{\alpha}^0 \simeq (0, +\infty) \quad \text{and} \quad J_{\beta}^0 \simeq (-\infty, 0) \end{array} \right.$$

and consider $\hat{f}_{\gamma} = (\hat{f}_{\gamma}^0, \dots, \hat{f}_{\gamma}^{N_{\gamma}-1})$ the associated flux. Let the new junction after gluing be defined by

$$\underline{J} := \{0\} \cup \left(\bigcup_{j=1, \dots, N_{\alpha}-1} J_{\alpha}^j \right) \cup \left(\bigcup_{k=1, \dots, N_{\beta}-1} J_{\beta}^k \right)$$

and

$$\underline{\sigma}^I := \begin{cases} \sigma_{\alpha}^k & \text{if } I = (k, \alpha) \\ \sigma_{\beta}^k & \text{if } I = (k, \beta) \end{cases}$$

For $p_{\gamma} = (p_{\gamma}^1, \dots, p_{\gamma}^{n_{\gamma}-1}) \in [a, b]_{\gamma}'$ (avoiding notation p_{γ}' to keep light notations), let us consider the set

$$R := \left\{ r \in [a^0, b^0], \quad \hat{f}_{\alpha}^0(r, p_{\alpha}) = \hat{f}_{\beta}^0(r, p_{\beta}) \right\} \quad \text{with} \quad \hat{f}_{\alpha}^0(\downarrow, p_{\alpha}), \quad \hat{f}_{\beta}^0(\uparrow, p_{\beta})$$

Then R is non empty, and define the set

$$\Lambda := \left\{ \lambda = \tilde{f}(r, p_\alpha, p_\beta) \in \mathbb{R}^{N_\alpha + N_\beta - 2}, \quad r \in R \right\}$$

with

$$\tilde{f}(r, p_\alpha, p_\beta) := (\hat{f}_\alpha^1(r, p_\alpha), \dots, \hat{f}_\alpha^{N_\alpha - 1}(r, p_\alpha); \hat{f}_\beta^1(r, p_\beta), \dots, \hat{f}_\beta^{N_\beta - 1}(r, p_\beta)) \in \mathbb{R}^{N_\alpha + N_\beta - 2}$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, and this defines the following map

$$\begin{aligned} \underline{\hat{f}} : [a, b]'_\alpha \times [a, b]'_\beta &\rightarrow \mathbb{R}^{N_\alpha + N_\beta - 2} \\ (p_\alpha, p_\beta) &\mapsto \underline{\hat{f}}(p_\alpha, p_\beta) := \lambda \end{aligned}$$

and we set the map

$$\underline{f} := (f_\alpha^1, \dots, f_\alpha^{N_\alpha - 1}; f_\beta^1, \dots, f_\beta^{N_\beta - 1}) : [a, b]'_\alpha \times [a, b]'_\beta \rightarrow \mathbb{R}^{N_\alpha + N_\beta - 2}$$

0) (Gluing Riemann germs)

Then $\underline{\hat{f}}$ is continuous, the map $\underline{\sigma} \diamond \underline{\hat{f}}$ is Riemann monotone, and the set

$$\mathcal{G} := \left\{ P \in [a, b]'_\alpha \times [a, b]'_\beta, \quad \underline{\hat{f}}(P) = \underline{f}(P) \right\}$$

is a Riemann germ with respect to (J, \underline{f}) and $\underline{\hat{f}}$ is the associated Godonuv flux at the junction $(n_\alpha + n_\beta - 1) : (m_\alpha + m_\beta - 1)$, i.e.

$$\underline{\hat{f}} = \underline{\hat{f}}_{\mathcal{G}}$$

i) (Gluing Kruřkov germs)

Assume that \mathcal{G}_γ are Kruřkov germs for $\gamma = \alpha, \beta$. Then \mathcal{G} is also a Kruřkov germ.

ii) (Gluing HJ germs)

Assume that \mathcal{G}_γ is a HJ germ for $\gamma = \alpha, \beta$. Then \mathcal{G} is also a HJ germ.

iii) (Gluing monotone germs)

Assume that \mathcal{G}_γ are monotone germs for $\gamma = \alpha, \beta$. Then \mathcal{G} is also a monotone germ.

iv) (Gluing conservative germs)

Assume that \mathcal{G}_γ are conservative germs for $\gamma = \alpha, \beta$. Then \mathcal{G} is also a conservative germ.

Proof of Proposition 5.11

Step 1: nonemptyness of R

We set $p := p_\alpha, q := p_\beta$ and for $P := (p, q)$, and $r \in [a^0, b^0]$, we set

$$(5.27) \quad g : [a^0, b^0] \rightarrow \mathbb{R} \quad \text{with} \quad g(r) := \tilde{g}(r, p, q) := \hat{f}_\alpha^0(r, p) - \hat{f}_\beta^0(r, q) \quad \text{with} \quad \hat{f}_\alpha^0(\downarrow, p), \quad \hat{f}_\beta^0(\uparrow, q)$$

Recall that using (5.26) and (2.15), we get

$$g(a^0) = \hat{f}_\alpha^0(a^0, p) - \hat{f}_\beta^0(a^0, q) \geq (f_\alpha^0)_-(a^0) - (f_\beta^0)_+(a^0) = G^{f^0}(a^0, a^0) - G^{f^0}(a^0, a^0) = 0$$

$$g(b^0) = \hat{f}_\alpha^0(b^0, p) - \hat{f}_\beta^0(b^0, q) \leq (f_\alpha^0)_+(b^0) - (f_\beta^0)_-(b^0) = G^{f^0}(b^0, b^0) - G^{f^0}(b^0, b^0) = 0$$

Therefore $g(a^0) \geq 0 \geq g(b^0)$, and by continuity of g , we get $R \neq \emptyset$.

Step 2: Λ is a singleton

From the monotonicities of $\hat{f}_\gamma^0(\cdot, p_\gamma)$ given in (5.27), we deduce that g is nonincreasing. Assume that there exists $r, r' \in R$ with $r < r'$. Then this implies that g is constant (and vanishes) on $[r, r']$, and we deduce that R is a closed interval. Moreover, both maps $\hat{f}_\gamma^0(\cdot, p_\gamma)$ for $\gamma = \alpha, \beta$ are also constant on $R \supset [r, r']$. From nondegeneracy condition (2.16), we deduce that $\Omega^0 := \left\{ \hat{f}_\gamma^0(\cdot, p_\gamma) \neq f_\gamma^0 \right\}$ is (relatively) open and dense in the interval R . Because \hat{f}_γ^0 is locally constant on $\left\{ \hat{f}_\gamma \neq f_\gamma \right\}$ and is continuous on $[a, b]_\gamma$, we deduce that not only the 0-component of $\hat{f}_\gamma(\cdot, p_\gamma)$ is constant, but the full function satisfies $\hat{f}_\gamma(\cdot, p_\gamma) = \text{const}$ on R , and then $\tilde{f}(\cdot, p_\alpha, p_\beta)$ is constant on R . This shows that Λ is reduced to a singleton.

Step 3: Continuity of $\underline{\hat{f}}$

Consider a sequence $(r_n, \underline{p}_{\alpha,n}, \underline{p}_{\beta,n}) \rightarrow (r, p_\alpha, p_\beta)$ as $n \rightarrow +\infty$, such that

$$\begin{cases} \lambda_n := \underline{\hat{f}}(p_{\alpha,n}, p_{\beta,n}) = \tilde{f}(r_n, p_{\alpha,n}, p_{\beta,n}) \rightarrow \tilde{f}(r, p_\alpha, p_\beta) =: \lambda_0, \\ 0 = g(r_n) = \hat{f}_\alpha^0(r_n, p_{\alpha,n}) - \hat{f}_\beta^0(r_n, p_{\beta,n}) \rightarrow 0 = g(r) = \hat{f}_\alpha^0(r, p_\alpha) - \hat{f}_\beta^0(r, p_\beta) \end{cases}$$

The second line shows that $r \in R$, and then $\lambda_0 = \tilde{f}(r, p_\alpha, p_\beta) = \underline{\hat{f}}(p_\alpha, p_\beta)$. This shows the continuity of $\underline{\hat{f}}$.

Step 4: bounds on $\underline{\hat{f}}$

For $I \in \mathcal{I} := \{(1, \alpha), \dots, (n_\alpha - 1, \alpha); (1, \beta), \dots, (n_\beta - 1, \beta)\}$, we set

$$\underline{f}^I = \begin{cases} \hat{f}_\alpha^k & \text{if } I = (k, \alpha) \\ \hat{f}_\beta^k & \text{if } I = (k, \beta) \end{cases} \quad \text{and} \quad c^I = \begin{cases} c_\alpha^k & \text{if } I = (k, \alpha) \\ c_\beta^k & \text{if } I = (k, \beta) \end{cases} \quad \text{for } c = a, b$$

By definition of $\underline{\hat{f}}$, we still get that $\underline{f}_-^I \leq \underline{\hat{f}} \leq \underline{f}_+^I$ for all $I \in \mathcal{I}$, i.e. $\underline{f}_- \leq \underline{\hat{f}} \leq \underline{f}_+$.

Step 5: local constancy of $\underline{\hat{f}}$

Consider $P = (p_\alpha, p_\beta) \in [a, b]'_\alpha \times [a, b]'_\beta$ be such that $\underline{f}(P) \neq \underline{\hat{f}}(P) = \tilde{f}(r, P)$, and let us set $K_P := \{I \in \mathcal{I}, \underline{\hat{f}}^I(P) \neq \underline{f}^I(P)\}$. Recall that $\tilde{f}(\cdot, P) = \underline{\hat{f}}(P)$ on R . In particular, for $Q_\varepsilon := P + \sum_{I \in K_P} (-\varepsilon, \varepsilon)e_I$,

we get from the continuity of \tilde{f} that for $\varepsilon > 0$ small enough, we have $\tilde{f} = \text{const} = \underline{\hat{f}}(P)$ on $R \times \left(\left([a, b]'_\alpha \times [a, b]'_\beta \right) \cap Q_\varepsilon \right)$. Therefore $\underline{\hat{f}} = \text{const} = \underline{\hat{f}}(P)$ on $\left([a, b]'_\alpha \times [a, b]'_\beta \right) \cap Q_\varepsilon$, which shows the local constancy of $\underline{\hat{f}}$.

Step 6: Riemann monotonicity of $\underline{\sigma} \diamond \underline{\hat{f}}$

Step 6.1: preliminary

We know that $\sigma_\gamma \diamond \hat{f}_\gamma$ are Riemann monotone from Theorem 2.17 in Case 1) and from Corollary 5.10 in Case 2). Precisely, we know that for $P = (p_\alpha, p_\beta)$ and $p := p_\alpha, q := p_\beta$, we have

$$(5.28) \quad \begin{cases} ((\bar{r}, \bar{p}) - (r, p)) \diamond \sigma_\alpha \diamond [\hat{f}_\alpha]_{(r,p)}^{(\bar{r}, \bar{p})} \leq 0 & \implies [\hat{f}_\alpha]_{(r,p)}^{(\bar{r}, \bar{p})} = 0 \\ ((\bar{r}, \bar{q}) - (r, q)) \diamond \sigma_\beta \diamond [\hat{f}_\beta]_{(r,q)}^{(\bar{r}, \bar{q})} \leq 0 & \implies [\hat{f}_\beta]_{(r,q)}^{(\bar{r}, \bar{q})} = 0 \end{cases}$$

and consider (\bar{p}, \bar{q}) and (p, q) such that $((\bar{p}, \bar{q}) - (p, q)) \diamond \underline{\sigma} \diamond [\underline{\hat{f}}]_{(p,q)}^{(\bar{p}, \bar{q})} \leq 0$, and we want to show that $[\underline{\hat{f}}]_{(p,q)}^{(\bar{p}, \bar{q})} = 0$.

Here we set $\underline{\hat{f}}(\bar{p}, \bar{q}) = \tilde{f}(\bar{r}, \bar{p}, \bar{q})$, $\underline{\hat{f}}(p, q) = \tilde{f}(r, p, q)$. Hence

$$\begin{cases} \max_{k \in \{1, \dots, N_\alpha - 1\}} (\bar{p}^k - p^k) \cdot \sigma_\alpha^k \cdot \left\{ \hat{f}_\alpha^k(\bar{r}, \bar{p}) - \hat{f}_\alpha^k(r, p) \right\} \leq 0 \\ \max_{k \in \{1, \dots, N_\beta - 1\}} (\bar{q}^k - q^k) \cdot \sigma_\beta^k \cdot \left\{ \hat{f}_\beta^k(\bar{r}, \bar{q}) - \hat{f}_\beta^k(r, q) \right\} \leq 0 \end{cases}$$

Step 6.2: core of the argument

Recall that $\tilde{g}(\bar{r}, \bar{p}, \bar{q}) = 0 = \tilde{g}(r, p, q)$, and then

$$(5.29) \quad \left\{ \hat{f}_\alpha^0(\bar{r}, \bar{p}) - \hat{f}_\beta^0(\bar{r}, \bar{q}) \right\} - \left\{ \hat{f}_\alpha^0(r, p) - \hat{f}_\beta^0(r, q) \right\} = 0$$

which implies (multiplying by $-(\bar{r} - r)$)

$$(5.30) \quad -(\bar{r} - r) \cdot \left\{ \hat{f}_\alpha^0(\bar{r}, \bar{p}) - \hat{f}_\alpha^0(r, p) \right\} + (\bar{r} - r) \cdot \left\{ \hat{f}_\beta^0(\bar{r}, \bar{q}) - \hat{f}_\beta^0(r, q) \right\} = 0$$

Hence

$$(5.31) \quad -(\bar{r} - r) \cdot \left\{ \hat{f}_\alpha^0(\bar{r}, \bar{p}) - \hat{f}_\alpha^0(r, p) \right\} \leq 0 \quad \text{with} \quad \sigma_\alpha^0 = -1$$

or

$$(5.32) \quad (\bar{r} - r) \cdot \left\{ \hat{f}_\beta^0(\bar{r}, \bar{q}) - \hat{f}_\beta^0(r, q) \right\} \leq 0 \quad \text{with} \quad \sigma_\beta^0 = +1$$

If (5.31) holds true, then (5.28) implies $[\hat{f}_\alpha]_{(r,p)}^{(\bar{r}, \bar{p})} = 0$. Therefore equation (5.30) implies that inequality (5.32) also holds true, and then $[\hat{f}_\beta]_{(r,q)}^{(\bar{r}, \bar{q})} = 0$. This implies $[\underline{\hat{f}}]_{(p,q)}^{(\bar{p}, \bar{q})} = 0$. This shows that $\underline{\sigma} \diamond \underline{\hat{f}}$ is Riemann monotone.

Step 7: conclusion

In particular, the Riemann monotonicity of $\underline{\sigma} \diamond \hat{f}$ implies the directional monotonicity $\underline{\sigma} \diamond \underline{f}$, i.e. for $P := (p_\alpha, p_\beta)$, we deduce that $P^I \mapsto \underline{\sigma}^I \hat{f}^I(P)$ is nondecreasing on $[a, b]^I := [a^I, b^I]$. We conclude that $\mathcal{G} := \left\{ P \in [a, b]'_\alpha \times [a, b]'_\beta, \quad \hat{f}(P) = \underline{f}(P) \right\}$ is a Riemann germ (because \hat{f} is continuous), and $\hat{f} = \underline{f}_\mathcal{G}$.

Step 8: additional argument for Kruřkov germs

Recall from Lemma 5.5 that for Kruřkov germs \mathcal{G}_γ , we have

$$\text{IN} - \text{OUT} = \sum_{k=0, \dots, N_\gamma-1} \sigma_\gamma^k \cdot \text{sign}(p_\gamma^k - q_\gamma^k) \cdot \left\{ \hat{f}_\gamma^k(p_\gamma) - \hat{f}_\gamma^k(q_\gamma) \right\} \geq 0 \quad \text{for all } p_\gamma, q_\gamma \in [a, b]_\gamma$$

which means in particular that

$$\left\{ \begin{array}{l} -\text{sign}(\bar{r} - r) \cdot \left\{ \hat{f}_\alpha^0(\bar{r}, \bar{p}) - \hat{f}_\alpha^0(r, p) \right\} + S'_\alpha \geq 0 \quad \text{with } S'_\alpha := \sum_{k=1, \dots, N_\alpha-1} \sigma_\alpha^k \cdot \text{sign}(\bar{p}^k - p^k) \cdot \left\{ \hat{f}_\alpha^k(\bar{r}, \bar{p}) - \hat{f}_\alpha^k(r, p) \right\} \\ \text{sign}(\bar{r} - r) \cdot \left\{ \hat{f}_\beta^0(\bar{r}, \bar{q}) - \hat{f}_\beta^0(r, q) \right\} + S'_\beta \geq 0 \quad \text{with } S'_\beta := \sum_{k=1, \dots, N_\beta-1} \sigma_\beta^k \cdot \text{sign}(\bar{q}^k - q^k) \cdot \left\{ \hat{f}_\beta^k(\bar{r}, \bar{q}) - \hat{f}_\beta^k(r, q) \right\} \end{array} \right.$$

Because we have $\tilde{g}(\bar{r}, \bar{p}, \bar{q}) = 0 = \tilde{g}(r, p, q)$ which implies (5.29), we can now take the sum of both inequalities and get that $S'_\alpha + S'_\beta \geq 0$, i.e. for $\bar{P} := (\bar{p}, \bar{q})$ and $P = (p, q)$, that

$$\sum_{I \in \mathcal{I}} \underline{\sigma}^I \cdot \text{sign}(\bar{P}^I - P^I) \cdot \left\{ \underline{f}^I(\bar{P}) - \underline{f}^I(P) \right\} \geq 0$$

which shows exactly that \mathcal{G} is a Kruřkov germ.

Step 9: additional argument for HJ germs

Recall from Lemma 5.5 that for HJ germs \mathcal{G}_γ , there exists $\hat{h}_\gamma : [a, b]_\gamma \rightarrow \mathbb{R}$ such that $\hat{f}_\gamma^k = \hat{h}_\gamma$ for all $k = 0, \dots, N_\gamma - 1$. Recall that $0 = g(r) = \tilde{g}(r, p, q) = \hat{f}_\alpha^0(r, p) - \hat{f}_\beta^0(r, q)$ with

$$\left\{ \begin{array}{l} \hat{f}_\alpha^0(r, p) = \hat{h}_\alpha(r, p) = \hat{f}_\alpha^j(r, p) \quad \text{for all } j \in \{1, \dots, N_\alpha\} \\ \hat{f}_\beta^0(r, q) = \hat{h}_\beta(r, q) = \hat{f}_\beta^k(r, q) \quad \text{for all } k \in \{1, \dots, N_\beta\} \end{array} \right.$$

Hence defining $\hat{h}(p, q) := \hat{f}_\alpha^0(r, p) = \hat{f}_\beta^0(r, q)$, we see from Step 2 that this quantity is the same for r' such that $g(r') = 0$, and then is well defined. Moreover, we also have $\underline{f}^I = \hat{h}$ for all $I \in \mathcal{I}$. This shows that \mathcal{G} is a HJ germ.

Step 10: additional argument for monotone germs

Recall that \mathcal{G}_γ are monotone germs, i.e. that

$$(5.33) \quad p \mapsto \sigma_\gamma^j \hat{f}_\gamma^j(p) \quad \text{is nonincreasing in } p^k \text{ for all } k \neq j$$

Assume by contradiction that \mathcal{G} is not a monotone germ, i.e. that

$$(5.34) \quad P \mapsto \underline{\sigma}^I \underline{f}^I(P) \quad \text{is NOT nonincreasing in } p^K \text{ for all } K \neq I$$

i.e. that there exists P, \tilde{P} such that $P - \tilde{P} \in (0, +\infty) \cdot e_K$ and $\underline{\sigma}^I \underline{f}^I(P) > \underline{\sigma}^I \underline{f}^I(\tilde{P})$.

Case A: $I := (\alpha, j)$ and $K := (\alpha, k)$ with $k \neq j$

Then we can write

$$(5.35) \quad P = (p, q), \quad \tilde{P} = (\tilde{p}, q), \quad p - \tilde{p} \in (0, +\infty) \cdot e_k, \quad \sigma_\alpha^j \hat{f}_\alpha^j(r, p) > \sigma_\alpha^j \hat{f}_\alpha^j(\tilde{r}, \tilde{p})$$

with

$$\left\{ \begin{array}{l} 0 = g(r) = \hat{f}_\alpha^0(r, p) - \hat{f}_\beta^0(r, q) \\ 0 = g(\tilde{r}) = \hat{f}_\alpha^0(\tilde{r}, \tilde{p}) - \hat{f}_\beta^0(\tilde{r}, q) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \hat{f}_\alpha^0(\downarrow, \tilde{p}), \quad \hat{f}_\beta^0(\uparrow, q) \\ z \mapsto \sigma_\alpha^0 \hat{f}_\alpha^0(\tilde{r}, \tilde{p} + ze_k) \quad \text{nonincreasing with } \sigma_\alpha^0 = -1 \end{array} \right.$$

We deduce that $r \geq \tilde{r}$, $p \geq \tilde{p}$ with $p^j = \tilde{p}^j$, and then monotonicity (5.33) implies $\sigma_\alpha^j \hat{f}_\alpha^j(r, p) \leq \sigma_\alpha^j \hat{f}_\alpha^j(\tilde{r}, \tilde{p})$, which is in contradiction with (5.35).

Case A': $I := (\beta, j)$ and $K := (\beta, k)$ with $k \neq j$

This case is symmetric to case A (up to a reversion transform (see Definition (4.4)) on component f_β^0).

Case B: $I := (\alpha, j)$ and $K := (\beta, k)$

Then we can write

$$(5.36) \quad P = (p, q), \quad \tilde{P} = (p, \tilde{q}), \quad q - \tilde{q} \in (0, +\infty) \cdot e_k, \quad \sigma_\alpha^j \hat{f}_\alpha^j(r, p) > \sigma_\alpha^j \hat{f}_\alpha^j(\tilde{r}, \tilde{p})$$

with

$$\begin{cases} 0 = g(r) = \hat{f}_\alpha^0(r, p) - \hat{f}_\beta^0(r, q) \\ 0 = g(\tilde{r}) = \hat{f}_\alpha^0(\tilde{r}, p) - \hat{f}_\beta^0(\tilde{r}, \tilde{q}) \end{cases} \quad \text{and} \quad \begin{cases} \hat{f}_\alpha^0(\downarrow, p), \quad \hat{f}_\beta^0(\uparrow, \tilde{q}) \\ z \mapsto \sigma_\beta^0 \hat{f}_\beta^0(\tilde{r}, \tilde{q} + ze_k) \end{cases} \quad \text{nonincreasing with } \sigma_\beta^0 = +1$$

we get that $r \geq \tilde{r}$, $q \geq \tilde{q}$, and then monotonicity (5.33) implies $\sigma_\alpha^j \hat{f}_\alpha^j(r, p) \leq \sigma_\alpha^j \hat{f}_\alpha^j(\tilde{r}, p)$, which is in contradiction with (5.36).

Conclusion

Therefore (5.34) is false and this shows that \mathcal{G} is a monotone germ.

Step 11: additional argument for conservative germs

Recall from Lemma 5.5 that \mathcal{G}_γ are conservative germs, i.e.

$$\sum_{J_\alpha^j \simeq (-\infty, 0)} \hat{f}_\alpha^j(r, p) = \hat{f}_\alpha^0(r, p) + \sum_{j \neq 0, J_\alpha^j \simeq (0, +\infty)} \hat{f}_\alpha^j(r, p) \quad \text{for all } (r, p) \in [a^0, b^0] \times [a, b]'_\alpha$$

and

$$\hat{f}_\beta^0(r, q) + \sum_{j \neq 0, J_\beta^j \simeq (-\infty, 0)} \hat{f}_\beta^j(r, q) = \sum_{J_\beta^j \simeq (0, +\infty)} \hat{f}_\beta^j(r, q) \quad \text{for all } (r, q) \in [a^0, b^0] \times [a, b]'_\beta$$

For r such that $\hat{f}_\alpha^0(r, p) = \hat{f}_\beta^0(r, q)$, we deduce that $\sum_{J^I \simeq (-\infty, 0)} \hat{f}^I(p, q) = \sum_{J^I \simeq (0, +\infty)} \hat{f}^I(p, q)$, which shows that \mathcal{G} is conservative. This ends the proof of the lemma.

Then we have the following corollary of Proposition 5.11.

Corollary 5.12 (Gluing of Riemann germs \mathcal{G}_γ for $n_\gamma : m_\gamma$ junctions)

For $\gamma = \alpha, \beta$, assume that f_γ satisfies (2.2) for $N_\gamma = n_\gamma + m_\gamma$ and $n_\gamma : m_\gamma$ junctions J_γ with $J_\gamma^j \simeq \sigma_\gamma^j \cdot (-\infty, 0)$ and $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$, and consider Riemann germs \mathcal{G}_γ with respect to (J_γ, f_γ) .

For $\gamma = \alpha$ or β , assume either that 1) f_γ satisfies nondegeneracy condition (2.16), or that 2) \mathcal{G}_γ is a Kružkov germ. We allow mixing cases for α and β .

We set $[a, b]_\gamma^j := [a_\gamma^j, b_\gamma^j]$. We assume that for each $\gamma = \alpha, \beta$, there exists one index $j_\gamma \in \{1, \dots, N_\gamma\}$ such that

$$f_\alpha^{j_\alpha} = f_\beta^{j_\beta} =: f^0 \quad \text{on } [a, b]_\alpha^{j_\alpha} = [a, b]_\beta^{j_\beta} =: [a^0, b^0] \quad \text{with } J_\alpha^{j_\alpha} \simeq (0, +\infty) \quad \text{and} \quad J_\beta^{j_\beta} \simeq (-\infty, 0)$$

and we glue those two branches. We call $\hat{f}_\gamma : [a, b]_\gamma \rightarrow \mathbb{R}^{n_\gamma}$ the associated fluxes. We set

$$\begin{cases} f'_\gamma := (f_\gamma^1, \dots, f_\gamma^{j_\gamma-1}, f_\gamma^{j_\gamma+1}, \dots, f_\gamma^{N_\gamma}) \\ \hat{f}'_\gamma := (\hat{f}_\gamma^1, \dots, \hat{f}_\gamma^{j_\gamma-1}, \hat{f}_\gamma^{j_\gamma+1}, \dots, \hat{f}_\gamma^{N_\gamma}) \\ (r, p_\gamma) := (p_\gamma^1, \dots, p_\gamma^{j_\gamma-1}, r, p_\gamma^{j_\gamma+1}, \dots, p_\gamma^{N_\gamma}) \end{cases}$$

with an abuse of notation for the last line. Then we define

$$\hat{f} := \hat{f}_\alpha \# \hat{f}_\beta : [a, b]'_\alpha \times [a, b]'_\beta \rightarrow \mathbb{R}^{N_\alpha + N_\beta - 2} \quad \text{with } [a, b]'_\gamma := \prod_{j \in \{1, \dots, N_\gamma\} \setminus \{j_\gamma\}} [a, b]_\gamma^j$$

where for any $p_\alpha \in [a, b]'_\alpha$ and $p_\beta \in [a, b]'_\beta$ (avoiding notation p'_α, p'_β to keep light notations) and

$$\begin{cases} f(p_\alpha, p_\beta) := (f'_\alpha; f'_\beta)(p_\alpha, p_\beta) \\ \hat{f}(p_\alpha, p_\beta) := (\hat{f}'_\alpha; \hat{f}'_\beta)(r, p_\alpha, p_\beta) \quad \text{for some } r \in [a^0, b^0] \text{ such that } \hat{f}_\alpha^{j_\alpha}(r, p_\alpha) = \hat{f}_\beta^{j_\beta}(r, p_\beta) \end{cases}$$

Here such r does exist, and when it is not unique, it does not change the value of $\hat{f}(p_\alpha, p_\beta)$.

We define $\mathcal{G} := \mathcal{G}_\alpha \#_{j_\alpha : j_\beta} \mathcal{G}_\beta$, the gluing of germs \mathcal{G}_α with \mathcal{G}_β along branches $J_\alpha^{j_\alpha}$ and $J_\beta^{j_\beta}$, as

$$(5.37) \quad \mathcal{G} := \mathcal{G}_\alpha \#_{j_\alpha : j_\beta} \mathcal{G}_\beta := \left\{ P \in [a, b]'_\alpha \times [a, b]'_\beta, \quad \hat{f}(P) = f(P) \right\}$$

Moreover the following holds.

i) (Gluing germs)

Then we have

(5.38)

$$\mathcal{G} := \left\{ (\hat{p}_\alpha, \hat{p}_\beta) \in [a, b]'_\alpha \times [a, b]'_\beta, \quad \text{there exists } \hat{r}_\alpha, \hat{r}_\beta \in [a^0, b^0] \quad \text{s.t.} \quad \left\{ \begin{array}{l} (\hat{r}_\alpha, \hat{p}_\alpha) \in \mathcal{G}_\alpha \\ (\hat{r}_\beta, \hat{p}_\beta) \in \mathcal{G}_\beta \\ f^0(\hat{r}_\alpha) = G^{f^0}(\hat{r}_\alpha, \hat{r}_\beta) = f^0(\hat{r}_\beta) \end{array} \right\} \right\}$$

where we recall that the standard Godunov flux is given by

$$G^{f^0}(\hat{r}_\alpha, \hat{r}_\beta) = \begin{cases} \min_{[\hat{r}_\alpha, \hat{r}_\beta]} f^0 & \text{if } \hat{r}_\alpha \leq \hat{r}_\beta \\ \max_{[\hat{r}_\beta, \hat{r}_\alpha]} f^0 & \text{if } \hat{r}_\alpha \geq \hat{r}_\beta \end{cases}$$

ii) (identity element: gluing with standard Godunov flux)

a) Assume that $N_\beta = 2$ with $1 : 1$ junction J_β with $\hat{f}_\beta^j(r, s) = G^{f^0}(r, s)$ for $j = 0, 1$. Then $\hat{f} = \hat{f}_\alpha$ and $\mathcal{G} = \mathcal{G}_\alpha$.

b) Assume that $N_\alpha = 2$ with $1 : 1$ junction J_α with $\hat{f}_\alpha^j(r, s) = G^{f^0}(r, s)$ for $j = 0, 1$. Then $\hat{f} = \hat{f}_\beta$ and $\mathcal{G} = \mathcal{G}_\beta$.

Proof of Corollary 5.12

Step 1: proof of i)

From definition given in (5.37), we see that

$$(p_\alpha, p_\beta) \in \mathcal{G} \iff \exists r \in [a, b]_\alpha^{j_\alpha} = [a, b]_\beta^{j_\beta} \text{ s.t. } \left\{ \begin{array}{l} (\hat{r}_\alpha, \hat{p}_\alpha) = \pi_{\mathcal{G}_\alpha}(r, p_\alpha) \\ (\hat{r}_\beta, \hat{p}_\beta) = \pi_{\mathcal{G}_\beta}(r, p_\beta) \\ f'_\alpha(\hat{p}_\alpha) = f'_\alpha(\hat{r}_\alpha, \hat{p}_\alpha) = \hat{f}'_\alpha(r, p_\alpha) = f'_\alpha(p_\alpha) \\ f'_\beta(\hat{p}_\beta) = f'_\beta(\hat{r}_\beta, \hat{p}_\beta) = \hat{f}'_\beta(r, p_\beta) = f'_\beta(p_\beta) \\ f^0(\hat{r}_\alpha) = f_\alpha^{j_\alpha}(\hat{r}_\alpha) = \hat{f}_\alpha^{j_\alpha}(r, p_\alpha) = \hat{f}_\beta^{j_\beta}(r, p_\beta) = f_\beta^{j_\beta}(\hat{r}_\beta) = f^0(\hat{r}_\beta) \end{array} \right.$$

Notice that $p_\alpha \in BA(\hat{p}_\alpha)$, $p_\beta \in BA(\hat{p}_\beta)$, $r \in BA_\alpha^{((0, +\infty), f_\alpha^{j_\alpha})}(\hat{r}_\alpha) \cap BA_\beta^{((-\infty, 0), f_\beta^{j_\beta})}(\hat{r}_\beta)$. From the basins of attraction, we then deduce that

$$p_\alpha = \hat{p}_\alpha, \quad p_\beta = \hat{p}_\beta, \quad f^0(\hat{r}_\alpha) = G^{f^0}(\hat{r}_\alpha, \hat{r}_\beta) = f^0(\hat{r}_\beta), \quad \hat{r}_\alpha, \hat{r}_\beta \in [a^0, b^0]$$

and it is straightforward to check that conversely this implies that $(p_\alpha, p_\beta) \in \mathcal{G}$. Therefore this shows characterization (5.38) of \mathcal{G} .

Step 2: proof of ii)

We only prove a) (the proof of b) is similar). Assume that $\hat{f}_\beta^j = G^{f^0}$. Then

$$\hat{f}(p_\alpha, p_\beta) = \tilde{f}(r, p_\alpha, p_\beta) \quad \text{for any } r \in [a^0, b^0] \text{ s.t. } \hat{f}_\alpha^0(p_\alpha, r) = G^{f^0}(r, p_\beta)$$

In particular, if $P = (p_\alpha, p_\beta) \in \mathcal{G}_\alpha$, then we can choose $r = p_\beta$, and we see that $\hat{f}(P) = f(P)$, i.e. $P \in \mathcal{G}$. This shows that $\mathcal{G}_\alpha \subset \mathcal{G}$. Because both are generalized Riemann germs, we deduce that we have equality, i.e. $\mathcal{G}_\alpha = \mathcal{G}$, and then moreover $\hat{f} = \hat{f}_\mathcal{G} = \hat{f}_{\mathcal{G}_\alpha} = \hat{f}_\alpha$. This ends the proof.

Lemma 5.13 (Associativity of the gluing)

For $\delta = \alpha, \beta, \gamma$, assume that f_δ satisfies (2.2) for $N_\delta = n_\delta + m_\delta$ and $n_\delta : m_\delta$ junctions J_δ with $J_\delta^j \simeq \sigma_\delta^j \cdot (-\infty, 0)$ and $\sigma_\delta \in \{\pm 1\}^{N_\delta}$, and consider Riemann germs \mathcal{G}_δ with respect to (J_δ, f_δ) .

For $\delta = \alpha, \beta$ or γ , assume either 1) that f_δ satisfies nondegeneracy condition (2.16), or 2) that \mathcal{G}_δ is a Kruřkov germ. We allow mixing cases for α, β and γ .

We set $[a, b]_\delta^j := [a_\delta^j, b_\delta^j]$. We also assume that there exists $j_\alpha, j_\beta, k_\gamma, k_\beta$, with $j_\delta, k_\delta \in \{1, \dots, N_\delta\}$ such that

$$\begin{cases} f_\alpha^{j_\alpha} = f_\beta^{j_\beta} =: f^A, & [a, b]_\alpha^{j_\alpha} = [a, b]_\beta^{j_\beta} =: [a, b]^A & -\sigma_\alpha^{j_\alpha} = 1 = \sigma_\beta^{j_\beta}, & J_\alpha^{j_\alpha} \simeq (0, +\infty), & J_\beta^{j_\beta} \simeq (-\infty, 0), \\ f_\beta^{k_\beta} = f_\gamma^{k_\gamma} =: f^B, & [a, b]_\beta^{k_\beta} = [a, b]_\gamma^{k_\gamma} =: [a, b]^B & -\sigma_\beta^{k_\beta} = 1 = \sigma_\gamma^{k_\gamma}, & J_\beta^{k_\beta} \simeq (0, +\infty), & J_\gamma^{k_\gamma} \simeq (-\infty, 0), \end{cases} \quad k_\beta \neq j_\beta$$

With notation of Corollary 5.12, we glue

$$(\mathcal{G}_\alpha \# \mathcal{G}_\beta) \# \mathcal{G}_\gamma := \underset{j_\alpha: j_\beta}{(\mathcal{G}_\alpha \# \mathcal{G}_\beta)} \# \underset{k_\beta: k_\gamma}{\mathcal{G}_\gamma} \quad \text{gluing first } J_\alpha^{j_\alpha} \text{ with } J_\beta^{j_\beta}, \text{ and then } J_\beta^{k_\beta} \text{ with } J_\gamma^{k_\gamma}$$

and also glue

$$\mathcal{G}_\alpha \# (\mathcal{G}_\beta \# \mathcal{G}_\gamma) := \mathcal{G}_\alpha \# \underset{j_\alpha: j_\beta}{(\mathcal{G}_\beta \# \mathcal{G}_\gamma)} \quad \text{gluing first } J_\beta^{k_\beta} \text{ with } J_\gamma^{k_\gamma}, \text{ and then } J_\alpha^{j_\alpha} \text{ with } J_\beta^{j_\beta}$$

Then we have

$$(5.39) \quad (\mathcal{G}_\alpha \# \mathcal{G}_\beta) \# \mathcal{G}_\gamma = \mathcal{G}_\alpha \# (\mathcal{G}_\beta \# \mathcal{G}_\gamma)$$

Proof of Lemma 5.13

We set

$$\begin{cases} [a, b]_\delta' := \prod_{j \in \{1, \dots, N_\delta\} \setminus \{j_\delta\}} [a, b]_\delta^j & \text{for } \delta = \alpha, \beta \\ [a, b]_\delta^* := \prod_{k \in \{1, \dots, N_\delta\} \setminus \{k_\delta\}} [a, b]_\delta^k & \text{for } \delta = \beta, \gamma \\ [a, b]_\beta'' := \prod_{j \in \{1, \dots, N_\beta\} \setminus \{j_\beta, k_\beta\}} [a, b]_\beta^j \end{cases}$$

Step 1: first computation

We simply compute

$$\mathcal{G}_{\alpha\beta} := \mathcal{G}_\alpha \# \mathcal{G}_\beta := \left\{ \begin{array}{l} (\hat{p}_\alpha, \hat{p}_\beta) \in [a, b]_\alpha' \times [a, b]_\beta', \\ \text{there exists } \hat{r}_\alpha, \hat{r}_\beta \in [a, b]^A \text{ s.t. } \begin{cases} (\hat{p}_\alpha, \hat{r}_\alpha) \in \mathcal{G}_\alpha \\ (\hat{p}_\beta, \hat{r}_\beta) \in \mathcal{G}_\beta \\ f^A(\hat{r}_\alpha) = G^{f^A}(\hat{r}_\alpha, \hat{r}_\beta) = f^A(\hat{r}_\beta) \end{cases} \end{array} \right\}$$

with abuse of notation for $(\hat{p}_\alpha, \hat{r}_\alpha)$ and $(\hat{p}_\beta, \hat{r}_\beta)$, and similarly

$$\mathcal{G}_{\alpha\beta} \# \mathcal{G}_\gamma := \left\{ \begin{array}{l} (\hat{p}_{\alpha\beta}, \hat{p}_\gamma) \in ([a, b]_\alpha' \times [a, b]_\beta'') \times [a, b]_\gamma^*, \\ \text{there exists } \hat{r}'_\beta, \hat{r}_\gamma \in [a, b]^B \text{ s.t. } \begin{cases} (\hat{p}_{\alpha\beta}, \hat{r}'_\beta) \in \mathcal{G}_{\alpha\beta} \\ (\hat{p}_\gamma, \hat{r}_\gamma) \in \mathcal{G}_\gamma \\ f^B(\hat{r}'_\beta) = G^{f^B}(\hat{r}'_\beta, \hat{r}_\gamma) = f^B(\hat{r}_\gamma) \end{cases} \end{array} \right\}$$

Hence for $\hat{p}_{\alpha\beta} = (\hat{p}_\alpha, \hat{p}'_\beta)$ we get

$$\mathcal{G}_{\alpha\beta} \# \mathcal{G}_\gamma := \left\{ \begin{array}{l} (\hat{p}_\alpha, \hat{p}'_\beta, \hat{p}_\gamma) \in [a, b]_\alpha' \times [a, b]_\beta'' \times [a, b]_\gamma^*, \\ \left\{ \begin{array}{l} \text{there exists } \hat{r}_\alpha, \hat{r}_\beta \in [a, b]^A \\ \text{there exists } \hat{r}'_\beta, \hat{r}_\gamma \in [a, b]^B \end{array} \right. \text{ s.t. } \begin{cases} (\hat{p}_\alpha, \hat{r}_\alpha) \in \mathcal{G}_\alpha \\ (\hat{p}'_\beta, \hat{r}'_\beta, \hat{r}_\beta) \in \mathcal{G}_\beta \\ (\hat{p}_\gamma, \hat{r}_\gamma) \in \mathcal{G}_\gamma \\ f^A(\hat{r}_\alpha) = G^{f^A}(\hat{r}_\alpha, \hat{r}_\beta) = f^A(\hat{r}_\beta) \\ f^B(\hat{r}'_\beta) = G^{f^B}(\hat{r}'_\beta, \hat{r}_\gamma) = f^B(\hat{r}_\gamma) \end{cases} \end{array} \right\}$$

Step 2: second computation

We simply compute

$$\mathcal{G}_{\beta\gamma} := \mathcal{G}_\beta \# \mathcal{G}_\gamma := \left\{ \begin{array}{l} (\hat{p}_\beta, \hat{p}_\gamma) \in [a, b]_\beta^* \times [a, b]_\gamma^*, \\ \text{there exists } \hat{r}'_\beta, \hat{r}_\gamma \in [a, b]^B \text{ s.t. } \begin{cases} (\hat{p}_\beta, \hat{r}'_\beta) \in \mathcal{G}_\beta \\ (\hat{p}_\gamma, \hat{r}_\gamma) \in \mathcal{G}_\gamma \\ f^B(\hat{r}'_\beta) = G^{f^B}(\hat{r}'_\beta, \hat{r}_\gamma) = f^B(\hat{r}_\gamma) \end{cases} \end{array} \right\}$$

and then for $\hat{p}_{\beta\gamma} = (\hat{p}'_{\beta}, \hat{p}_{\gamma})$

$$\mathcal{G}_{\alpha} \# \mathcal{G}_{\beta\gamma} := \left\{ \begin{array}{l} (\hat{p}_{\alpha}, \hat{p}_{\beta\gamma}) \in [a, b]_{\alpha}' \times [a, b]_{\beta}'' \times [a, b]_{\gamma}^* \\ \text{there exists } \hat{r}_{\alpha}, \hat{r}_{\beta} \in [a, b]^A \text{ s.t. } \end{array} \left\{ \begin{array}{l} (\hat{p}_{\alpha}, \hat{r}_{\alpha}) \in \mathcal{G}_{\alpha} \\ (\hat{p}'_{\beta}, \hat{r}_{\beta}, \hat{p}_{\gamma}) \in \mathcal{G}_{\beta\gamma} \\ f^A(\hat{r}_{\alpha}) = G^{f^A}(\hat{r}_{\alpha}, \hat{r}_{\beta}) = f^A(\hat{r}_{\beta}) \end{array} \right\} \right\}$$

i.e.

$$\mathcal{G}_{\alpha} \# \mathcal{G}_{\beta\gamma} := \left\{ \begin{array}{l} (\hat{p}_{\alpha}, \hat{p}'_{\beta}, p_{\gamma}) \in [a, b]_{\alpha}' \times [a, b]_{\beta}'' \times [a, b]_{\gamma}^* \\ \left\{ \begin{array}{l} \text{there exists } \hat{r}_{\alpha}, \hat{r}_{\beta} \in [a, b]^A \\ \text{there exists } \hat{r}'_{\beta}, \hat{r}_{\gamma} \in [a, b]^B \end{array} \right. \text{ s.t. } \end{array} \left\{ \begin{array}{l} (\hat{p}_{\alpha}, \hat{r}_{\alpha}) \in \mathcal{G}_{\alpha} \\ (\hat{p}'_{\beta}, \hat{r}'_{\beta}, \hat{r}_{\beta}) \in \mathcal{G}_{\beta} \\ (\hat{p}_{\gamma}, \hat{r}_{\gamma}) \in \mathcal{G}_{\gamma} \\ f^A(\hat{r}_{\alpha}) = G^{f^A}(\hat{r}_{\alpha}, \hat{r}_{\beta}) = f^A(\hat{r}_{\beta}) \\ f^B(\hat{r}'_{\beta}) = G^{f^B}(\hat{r}'_{\beta}, \hat{r}_{\gamma}) = f^B(\hat{r}_{\gamma}) \end{array} \right\} \right\}$$

Step 3: conclusion

Hence $\mathcal{G}_{\alpha} \# \mathcal{G}_{\beta\gamma} = \mathcal{G}_{\alpha\beta} \# \mathcal{G}_{\gamma}$, which shows (5.39). This ends the proof of the lemma.

Proof of Theorem 2.19

For the proof, we refer to the table of Subsection 2.4. The result follows from Proposition 5.11, Corollary 5.12 and Lemma 5.13. This ends the proof.

5.6 Self-gluing of Kruřkov germs and more

Proposition 5.14 (Self-gluing of a flux function \hat{f}_{γ} for $n_{\gamma} : m_{\gamma}$ junction, for a Kruřkov germ)

Let γ be a fixed index. Assume that the function f_{γ} satisfies (2.2) with $N_{\gamma} = n_{\gamma} + m_{\gamma}$ with $n_{\gamma}, m_{\gamma} \geq 1$ and $N_{\gamma} \geq 3$. We consider some $n_{\gamma} : m_{\gamma}$ junction J_{γ} with $J_{\gamma}^j \simeq \sigma_{\gamma}^j \cdot (-\infty, 0)$ and $\sigma_{\gamma} \in \{\pm 1\}^{N_{\gamma}}$, and a Kruřkov germ \mathcal{G}_{γ} with respect to (J_{γ}, f_{γ}) . We set $[a, b]_{\gamma}^j := [a_{\gamma}^j, b_{\gamma}^j]$. Up to relabel the indices, we assume that the indices go through the values $\{0, \dots, N_{\gamma} - 1\}$.

We now assume that there exists two indices $j_1, j_2 \in \{0, \dots, N_{\gamma} - 1\}$ such that

$$(5.40) \quad f_{\gamma}^{j_1} = f_{\gamma}^{j_2} =: f^0 \quad \text{on} \quad [a, b]_{\gamma}^{j_1} = [a, b]_{\gamma}^{j_2} =: [a^0, b^0] \quad \text{with} \quad J_{\gamma}^{j_1} \simeq (0, +\infty) \quad \text{and} \quad J_{\gamma}^{j_2} \simeq (-\infty, 0)$$

and we glue those two branches. To simplify the notation, we also assume that $j_1 = 0$ and $j_2 = N_{\gamma} - 1$. Hence we now have

$$(5.41) \quad \left\{ \begin{array}{l} f_{\gamma} = (f_{\gamma}^0, \dots, f_{\gamma}^{N_{\gamma}-1}) \\ \mathcal{G}_{\gamma} \subset [a, b]_{\gamma} := \prod_{i=0, \dots, N_{\gamma}-1} [a, b]_{\gamma}^i \\ [a, b]_{\gamma}'' := \prod_{i=1, \dots, N_{\gamma}-2} [a, b]_{\gamma}^i \\ J_{\gamma}^0 \simeq (0, +\infty) \quad \text{and} \quad J_{\gamma}^{N_{\gamma}-1} \simeq (-\infty, 0) \\ f_{\gamma}^0 = f_{\gamma}^{N_{\gamma}-1} =: f^0 \quad \text{on} \quad [a, b]_{\gamma}^0 = [a, b]_{\gamma}^{N_{\gamma}-1} =: [a^0, b^0] \end{array} \right.$$

Let the new junction after gluing be defined by

$$\underline{J} := \{0\} \cup \left(\bigcup_{j=1, \dots, N_{\gamma}-2} J_{\gamma}^j \right) \quad \text{and} \quad \underline{\sigma}^k := \sigma_{\gamma}^k \quad \text{for} \quad k = 1, \dots, N_{\gamma} - 2$$

We consider $\hat{f}_{\gamma} : [a, b]_{\gamma} \rightarrow \mathbb{R}^{N_{\gamma}}$ the Godunov flux associated to the germ \mathcal{G}_{γ} , and with notation $p_{\gamma} = (p_{\gamma}^1, \dots, p_{\gamma}^{N_{\gamma}-2}) \in [a, b]_{\gamma}''$ (avoiding notation p_{γ}'' to keep light notations), we consider the set

$$R := \left\{ r \in [a^0, b^0], \quad \hat{f}_{\gamma}^0(r, p_{\gamma}, r) = \hat{f}_{\gamma}^{N_{\gamma}-1}(r, p_{\gamma}, r) \right\} \quad \text{with} \quad \hat{f}_{\gamma}^0(\downarrow, p_{\gamma}, r), \quad \hat{f}_{\gamma}^{N_{\gamma}-1}(r, p_{\gamma}, \uparrow)$$

Then R is non empty, and define the set

$$\Lambda := \left\{ \lambda = \tilde{f}(r, p_\gamma) \in \mathbb{R}^{N_\gamma-2}, \quad r \in R \right\} \quad \text{with} \quad \tilde{f}(r, p_\gamma) := (\hat{f}_\gamma^1(r, p_\gamma, r), \dots, \hat{f}_\gamma^{N_\gamma-2}(r, p_\gamma, r))$$

Then Λ is reduced to a singleton $\Lambda = \{\lambda\}$, and this defines the following map

$$\begin{aligned} \underline{\hat{f}} : [a, b]''_\gamma &\rightarrow \mathbb{R}^{N_\gamma-2} \\ p_\gamma &\mapsto \underline{\hat{f}}(p_\gamma) := \lambda \end{aligned}$$

and we set the map

$$(5.42) \quad \underline{f} := (f_\gamma^1, \dots, f_\gamma^{N_\gamma-2}) : [a, b]''_\gamma \rightarrow \mathbb{R}^{N_\gamma-2}$$

Moreover, the map $\underline{\hat{f}}$ is continuous and the set

$$\mathcal{G} := \left\{ P \in [a, b]''_\gamma, \quad \underline{\hat{f}}(P) = \underline{f}(P) \right\}$$

is a Kružkov germ with respect to $(\underline{J}, \underline{f})$ and $\underline{\hat{f}}$ is the associated Godunov flux at the junction $(n_\gamma - 1) : (m_\gamma - 1)$, i.e.

$$\underline{\hat{f}} = \underline{\hat{f}}_{\mathcal{G}}.$$

Proof of Proposition 5.14

Step 1: nonemptiness of R

We set $p := p_\gamma$, and for $r \in [a^0, b^0]$, we set (with some abuse of notation)

$$g : [a^0, b^0] \rightarrow \mathbb{R} \quad \text{with} \quad g(r) := \tilde{g}(r, p) := \hat{f}_\gamma^0(r, p, r) - \hat{f}_\gamma^{N_\gamma-1}(r, p, r)$$

Using the last line of (5.41) and (2.15), we get

$$g(a^0) = \hat{f}_\gamma^0(a^0, p, a^0) - \hat{f}_\gamma^{N_\gamma-1}(a^0, p, a^0) \geq f_{\gamma,-}^0(a^0) - f_{\gamma,+}^{N_\gamma-1}(a^0) = f_\gamma^0(a^0) - f_\gamma^{N_\gamma-1}(a^0) = 0$$

$$g(b^0) = \hat{f}_\gamma^0(b^0, p, b^0) - \hat{f}_\gamma^{N_\gamma-1}(b^0, p, b^0) \leq f_{\gamma,+}^0(b^0) - f_{\gamma,-}^{N_\gamma-1}(b^0) = f_\gamma^0(b^0) - f_\gamma^{N_\gamma-1}(b^0) = 0$$

Therefore $g(a^0) \geq 0 \geq g(b^0)$ and by continuity of g , we get $R \neq \emptyset$.

Step 2: Λ is a singleton

Step 2.1: getting g vanishing on $R \supset [r, \bar{r}]$

Because \mathcal{G}_γ is a Kružkov germ, recall that the map

$$(5.43) \quad [a^0, b^0]^2 \ni (r, r') \mapsto (\hat{f}_\gamma^0(r, p, r'), \hat{f}_\gamma^{N_\gamma-1}(r, p, r'))$$

satisfies for all $(\bar{r}, \bar{r}'), (r, r') \in [a^0, b^0]^2$ and all frozen $p \in [a, b]''_\gamma$ (with $\sigma_\gamma^0 = -1$ and $\sigma_\gamma^{N_\gamma-1} = 1$)

$$(5.44) \quad -\text{sign}(\bar{r} - r) \cdot \left\{ \hat{f}_\gamma^0(\bar{r}, p, \bar{r}') - \hat{f}_\gamma^0(r, p, r') \right\} + \text{sign}(\bar{r}' - r') \cdot \left\{ \hat{f}_\gamma^{N_\gamma-1}(\bar{r}, p, \bar{r}') - \hat{f}_\gamma^{N_\gamma-1}(r, p, r') \right\} \geq 0$$

For $\bar{r} = \bar{r}' > r = r'$, we get $g(\bar{r}) - g(r) \leq 0$, which shows that g is nonincreasing. Hence g is constant (and vanishes) on $[r, \bar{r}]$, if $r, \bar{r} \in R$. We deduce that R is closed interval where g vanishes.

Step 2.2: coincidence of \hat{f}_γ^0 and $\hat{f}_\gamma^{N_\gamma-1}$

For frozen p , inequality (5.44) shows that the map defined in (5.43) is itself associated to a Kružkov germ $1 : 1$, and then its flux is such that the matrix

$$\left(\begin{array}{cc} -\frac{\partial \hat{f}_\gamma^0}{\partial r} & -\frac{\partial \hat{f}_\gamma^0}{\partial r'} \\ \frac{\partial \hat{f}_\gamma^{N_\gamma-1}}{\partial r} & \frac{\partial \hat{f}_\gamma^{N_\gamma-1}}{\partial r'} \end{array} \right) (r, p, r')$$

has nonnegative diagonal and is diagonal column-dominant, i.e.

$$\begin{cases} \left(-\frac{\partial \hat{f}_\gamma^0}{\partial r} - \left| \frac{\partial \hat{f}_\gamma^{N_\gamma-1}}{\partial r} \right| \right) (r, p, r') \geq 0 \\ \left(-\left| \frac{\partial \hat{f}_\gamma^0}{\partial r'} \right| + \frac{\partial \hat{f}_\gamma^{N_\gamma-1}}{\partial r'} \right) (r, p, r') \geq 0 \end{cases} \quad \text{for a.e. } (r, r') \in [a^0, b^0]^2$$

Setting

$$\bar{g}(r, r') := \hat{f}_\gamma^0(r, p, r') - \hat{f}_\gamma^{N_\gamma-1}(r, p, r')$$

we deduce that the map $r \mapsto \bar{g}(r, r')$ is nonincreasing for a.e. r' and the map $r' \mapsto \bar{g}(r, r')$ is nonincreasing for a.e. r . By continuity of \bar{g} , we deduce that \bar{g} is nonincreasing everywhere in both variables r, r' . Now if $r < \bar{r}$ with $r, \bar{r} \in R$, we deduce that $\bar{g}(r, r) = \bar{g}(\bar{r}, \bar{r}) = 0$. From the monotonicities of \bar{g} , we deduce that $\bar{g} = 0$ on $[r, \bar{r}]^2$. This implies more generally that

$$(5.45) \quad \hat{f}_\gamma^0(\cdot, p, \cdot) = \hat{f}_\gamma^{N_\gamma-1}(\cdot, p, \cdot) \quad \text{on } R^2$$

Step 2.3: concluding that Λ is reduced to a singleton

Because \mathcal{G}_γ is a Kruřkov germ, we know that \hat{f}_γ satisfies $D^{\hat{f}_\gamma} \geq 0$ on $[a, b]_\gamma^2$. We deduce from Proposition 5.9, that $\sigma \diamond \hat{f}_\gamma$ is Riemann monotone. If R is reduced to a singleton than Λ is also a singleton. Then assume that $r, \bar{r} \in R$ with $\bar{r} > r$. Then from (5.45), we deduce that $\{(\bar{r}, p, \bar{r}) - (r, p, r)\} \diamond \sigma \diamond \left[\hat{f}_\gamma \right]_{(r, p, r)}^{(\bar{r}, p, \bar{r})} = 0$ and the Riemann monotonicity of $\sigma \diamond \hat{f}_\gamma$ implies $\left[\hat{f}_\gamma \right]_{(r, p, r)}^{(\bar{r}, p, \bar{r})} = 0$. This shows in particular that $\tilde{f}(\bar{r}, p) = \tilde{f}(r, p)$, and implies that Λ is reduced to a singleton.

Step 3: Continuity of $\underline{\hat{f}}$

Consider a sequence $(r_n, p_n) \rightarrow (r, p)$ such that

$$\begin{cases} \lambda_n := \underline{\hat{f}}(p_n) = \tilde{f}(r_n, p_n) \rightarrow \tilde{f}(r, p) =: \lambda_0, \\ 0 = g(r_n) = \hat{f}_\gamma^0(r_n, p_n, r_n) - \hat{f}_\gamma^{N_\gamma-1}(r_n, p_n, r_n) \rightarrow 0 = g(r) = \hat{f}_\gamma^0(r, p, r) - \hat{f}_\gamma^{N_\gamma-1}(r, p, r) \end{cases}$$

The second line shows that $r \in R$, and then $\lambda_0 = \tilde{f}(r, p) = \underline{\hat{f}}(p) =: \lambda$. This shows the continuity of $\underline{\hat{f}}$.

Step 4: bounds on $\underline{\hat{f}}$

We consider $\underline{\hat{f}}$ as defined in (5.42). By definition of $\underline{\hat{f}}$, we still get that $\underline{\hat{f}}_- \leq \underline{\hat{f}} \leq \underline{\hat{f}}_+$ for all $I \in \mathcal{I} := \{1, \dots, N_\gamma - 2\}$, i.e. $\underline{\hat{f}}_- \leq \underline{\hat{f}} \leq \underline{\hat{f}}_+$.

Step 5: local constancy of $\underline{\hat{f}}$

Let $P := p \in [a, b]''_\gamma$ be such that $\underline{\hat{f}}(P) \neq \hat{f}(P) = \tilde{f}(r, P)$, and let us set $K_P := \{I \in \mathcal{I}, \underline{\hat{f}}^I(P) \neq \hat{f}^I(P)\}$. Recall that $\tilde{f}(\cdot, P) = \underline{\hat{f}}(P)$ on R . In particular, for $Q_\varepsilon := P + \sum_{I \in K_P} (-\varepsilon, \varepsilon)e_I$, we get from the continuity and the local constancy of \hat{f}_γ that for $\varepsilon > 0$ small enough, we have $\tilde{f} = \text{const}$ on $R \times ([a, b]''_\gamma \cap Q_\varepsilon)$. Hence $\underline{\hat{f}} = \text{const}$ on $[a, b]''_\gamma \cap Q_\varepsilon$, which shows the local constancy of $\underline{\hat{f}}$.

Step 6: directional monotonicity of $\underline{\hat{f}}$

For $P := p$, we want to show that

$$(5.46) \quad P^I \mapsto \underline{\sigma}^I \underline{\hat{f}}^I(P) \quad \text{is nondecreasing on } [a, b]^I := [a^I, b^I]$$

To this end, we have to take into account the fact that $\underline{\hat{f}}(P) = \tilde{f}(r, P)$ with the dependence $r = r(P)$. To reach our goal, it is more convenient to use directly that \mathcal{G}_γ is a Kruřkov germ. This means that

$$(5.47) \quad \begin{aligned} 0 \leq & -\text{sign}(\bar{r} - r) \cdot \left\{ \hat{f}_\gamma^0(\bar{r}, \bar{p}, \bar{r}) - \hat{f}_\gamma^0(r, p, r) \right\} + \text{sign}(\bar{r} - r) \cdot \left\{ \hat{f}_\gamma^{N_\gamma-1}(\bar{r}, \bar{p}, \bar{r}) - \hat{f}_\gamma^{N_\gamma-1}(r, p, r) \right\} \\ & + \sum_{k=1, \dots, N_\gamma-2} \sigma_\gamma^k \cdot \text{sign}(\bar{p}^k - p^k) \cdot \left\{ \hat{f}_\gamma^k(\bar{r}, \bar{p}, \bar{r}) - \hat{f}_\gamma^k(r, p, r) \right\} \end{aligned}$$

Using $\tilde{g}(\bar{r}, \bar{p}) = 0 = \tilde{g}(r, p)$, we see that (5.47) implies

$$\sum_{k=1, \dots, N_\gamma - 2} \sigma^I \cdot \text{sign}(\bar{p}^k - p^k) \cdot \left\{ \hat{f}^k(\bar{p}) - \hat{f}^k(p) \right\} \geq 0$$

which means that \hat{f} is associated to a Kruřkov germ. In particular, this implies the directional monotonicity (5.46) of \hat{f} .

Step 7: conclusion

We conclude that $\mathcal{G} := \left\{ P \in [a, b]''_\gamma, \hat{f}(P) = f(P) \right\}$ is a Riemann germ (because \hat{f} is continuous), and $\hat{f} = \hat{f}_\mathcal{G}$. Moreover \mathcal{G} is a Kruřkov germ. This ends the proof of the Proposition.

Corollary 5.15 (Self-gluing of a Kruřkov germ for a $n_\gamma : m_\gamma$ junction)

Let γ be a fixed index. Assume that the function f_γ satisfies (2.2) with $N_\gamma = n_\gamma + m_\gamma$ with $n_\gamma, m_\gamma \geq 1$ and $N_\gamma \geq 3$. We consider some $n_\gamma : m_\gamma$ junction J_γ with $J_\gamma^j \simeq \sigma_\gamma^j \cdot (-\infty, 0)$ and $\sigma_\gamma \in \{\pm 1\}^{N_\gamma}$, and a **Kruřkov** germ \mathcal{G}_γ with respect to (J_γ, f_γ) . To simplify the presentation, assume that the indices go through $i = 0, \dots, N_\gamma - 1$, i.e. $f_\gamma = (f_\gamma^0, \dots, f_\gamma^{N_\gamma - 1})$ and $[a, b]_\gamma := \prod_{i=0, \dots, N_\gamma - 1} [a, b]_\gamma^i$ with $[a, b]_\gamma^i := [a_\gamma^i, b_\gamma^i]$. We call

$\hat{f}_\gamma : [a, b]_\gamma \rightarrow \mathbb{R}^{N_\gamma}$ the Godunov flux associated to \mathcal{G}_γ . Assume also that

$$\begin{cases} f_\gamma^{j_1} = f_\gamma^{j_2} =: f^0 & \text{defined on } [a, b]_\gamma^{j_1} = [a, b]_\gamma^{j_2} = [a^0, b^0] \\ J_\gamma^{j_1} = (0, +\infty), & J_\gamma^{j_2} = (-\infty, 0) \end{cases}$$

in order to glue branch $J_\gamma^{j_1}$ with branch $J_\gamma^{j_2}$. Then we define for $j_1 := 0$ and $j_2 := N_\gamma - 1$

$$\hat{f} := (\hat{f}_\gamma)^{\sharp_{j_1: j_2}} : [a, b]''_\gamma \rightarrow \mathbb{R}^{N_\gamma - 2} \quad \text{with} \quad [a, b]''_\gamma := \prod_{i=1, \dots, N_\gamma - 2} [a, b]_\gamma^i$$

where for any $p \in [a, b]''_\gamma$, we have

$$\hat{f}(p) := (\hat{f}_\gamma^1, \dots, \hat{f}_\gamma^{N_\gamma - 2})(r, p, r) \quad \text{for some } r \in [a, b]_\gamma^0 = [a, b]_\gamma^{N_\gamma - 1} =: [a^0, b^0] \text{ such that } \hat{f}_\gamma^0(r, p, r) = \hat{f}_\gamma^{N_\gamma - 1}(r, p, r)$$

where such r does exist, and when it is not unique, it does not change the value of $\hat{f}(p)$.

We also define

$$f(p) = (f_\gamma^1, \dots, f_\gamma^{N_\gamma - 2})(p)$$

We define

$$(5.48) \quad \mathcal{G} := (\mathcal{G}_\gamma)^{\sharp_{j_1: j_2}} := \left\{ P \in [a, b]''_\gamma, \hat{f}(P) = f(P) \right\}$$

Then \mathcal{G} is a Kruřkov germ and satisfies

$$(5.49) \quad \mathcal{G} := \left\{ \hat{p} \in [a, b]''_\gamma, \text{ there exists } \hat{r}, \hat{r}' \in [a^0, b^0] \text{ s.t. } \begin{cases} (\hat{r}, \hat{p}, \hat{r}') \in \mathcal{G}_\gamma \\ f^0(\hat{r}) = G^{f^0}(\hat{r}, \hat{r}') = f^0(\hat{r}') \end{cases} \right\}$$

where we recall that the standard Godunov flux is given by

$$G^{f^0}(\hat{r}, \hat{r}') = \begin{cases} \min_{[\hat{r}, \hat{r}']} f^0 & \text{if } \hat{r} \leq \hat{r}' \\ \max_{[\hat{r}', \hat{r}]} f^0 & \text{if } \hat{r} \geq \hat{r}' \end{cases}$$

Proof of Corollary 5.15

Notice that by definition, the function \hat{f} is equal to $\hat{f}_\mathcal{G}$ given by Proposition 5.14.

From definition (5.48), we have

$$p \in \mathcal{G} \iff \left\{ \begin{array}{l} \text{there exists } r \in [a, b]_\gamma^0 = [a, b]_\gamma^{N_\gamma - 1} \text{ s.t.} \\ \left\{ \begin{array}{l} \hat{f}_\gamma^i(r, p, r) = f_\gamma^i(p), \quad i = 1, \dots, N_\gamma - 2 \\ \hat{f}_\gamma^0(r, p, r) = \hat{f}_\gamma^{N_\gamma - 1}(r, p, r) \end{array} \right. \end{array} \right\}$$

and then (using the fact that $f_\gamma(\hat{r}, \hat{p}, \hat{r}') = \hat{f}_\gamma(\hat{r}, \hat{p}, \hat{r}') = \hat{f}_\gamma(r, p, r)$ for $(\hat{r}, \hat{p}, \hat{r}') = \pi_{\mathcal{G}_\gamma}(r, p, r)$)

$$p \in \mathcal{G} \iff \left\{ \begin{array}{l} \text{there exists } r \in [a, b]_\gamma^0 = [a, b]_\gamma^{N_\gamma-1} \text{ s.t.} \\ \left\{ \begin{array}{l} (\hat{r}, \hat{p}, \hat{r}') = \pi_{\mathcal{G}_\gamma}(r, p, r) \\ f(\hat{p}) = f(p) \\ f^0(\hat{r}) = f_\gamma^0(\hat{r}) = f_\gamma^{N_\gamma-1}(\hat{r}') = f^0(\hat{r}') \end{array} \right. \end{array} \right\}$$

Notice that

$$p \in BA(\hat{p}), \quad r \in BA^{((0, +\infty), f_\gamma^0)}(\hat{r}) \cap BA^{((-\infty, 0), f_\gamma^{N_\gamma-1})}(\hat{r}')$$

From the basins of attraction with $f_\gamma^0 = f_\gamma^{N_\gamma-1} = f^0$, we then deduce that

$$(5.50) \quad p = \hat{p}, \quad f^0(\hat{r}) = G^{f^0}(\hat{r}, \hat{r}') = f^0(\hat{r}'), \quad \hat{r}, \hat{r}' \in [a^0, b^0], \quad \text{with } (\hat{r}, \hat{p}, \hat{r}') \in \mathcal{G}_\gamma$$

and it is straightforward to check that conversely (5.50) implies that $p \in \mathcal{G}$ in the sense of definition (5.48). Therefore this shows characterization (5.49) of \mathcal{G} . This ends the proof.

Similarly to Lemma 5.13, we show the following results

Lemma 5.16 (Associativity of self-gluing for Kruřkov germs)

Let γ be a fixed index. Assume that f_γ satisfies (2.2) for $N_\gamma = n_\gamma + m_\gamma \geq 5$ with $n_\gamma, m_\gamma \geq 2$ and some $n_\gamma : m_\gamma$ junction J_γ . Assume that \mathcal{G}_γ is a Kruřkov germ with respect to (J_γ, f_γ) . We also assume that there exists four distinct indices $j_1, j_2, k_1, k_2 \in \{1, \dots, N_\gamma\}$ such that

$$\left\{ \begin{array}{lll} f_\gamma^{j_1} = f_\gamma^{j_2} =: f^A, & J_\gamma^{j_1} \simeq (0, +\infty), & J_\gamma^{j_2} \simeq (-\infty, 0) \\ f_\gamma^{k_1} = f_\gamma^{k_2} =: f^B, & J_\gamma^{k_1} \simeq (0, +\infty), & J_\gamma^{k_2} \simeq (-\infty, 0) \end{array} \right.$$

Then with notation of Corollary 5.15, we have the germ equality

$$\left((\mathcal{G}_\gamma)_{j_1:j_2}^\# \right)_{k_1:k_2}^\# = \left((\mathcal{G}_\gamma)_{k_1:k_2}^\# \right)_{j_1:j_2}^\#$$

which is associated to a $(n_\gamma - 2) : (m_\gamma - 2)$ junction.

and

Lemma 5.17 (Associativity of the gluing and the self-gluing for Kruřkov germs)

For $\gamma = \alpha, \beta$, assume that f_γ satisfies (2.2) for $N_\gamma = n_\gamma + m_\gamma$ for junctions J_γ of type $n_\gamma : m_\gamma$. We consider Kruřkov germs \mathcal{G}_γ with respect to (J_γ, f_γ) .

We also assume that there exists $j_1 \in \{1, \dots, N_\alpha\}$ and three distinct indices $j_2, k_1, k_2 \in \{1, \dots, N_\beta\}$ such that

$$\left\{ \begin{array}{lll} f_\alpha^{j_1} = f_\beta^{j_2} =: f^A, & J_\alpha^{j_1} \simeq (0, +\infty), & J_\beta^{j_2} \simeq (-\infty, 0) \\ f_\beta^{k_1} = f_\beta^{k_2} =: f^B, & J_\beta^{k_1} \simeq (0, +\infty), & J_\beta^{k_2} \simeq (-\infty, 0) \end{array} \right.$$

Then with notation of Corollaries 5.12 and 5.15, we have the germ equality

$$\mathcal{G}_\alpha \#_{j_1:j_2} \left((\mathcal{G}_\beta)_{k_1:k_2}^\# \right) = \left(\mathcal{G}_\alpha \#_{j_1:j_2} \mathcal{G}_\beta \right)_{k_1:k_2}^\#$$

which is associated to a $(n_\alpha + n_\beta - 2) : (m_\alpha + m_\beta - 2)$ junction.

6 Applications

6.1 Restriction of Riemann germs to bounded boxes - a priori L^∞ bounds

We also have the following result (which can also be used to derive a priori L^∞ bounds on solutions with initial values in a bounded set K).

Proposition 6.1 (Restriction of Riemann germs to bounded boxes)

For $N \geq 1$, assume (2.2) and nondegeneracy condition (2.16), and let $\mathcal{G} \subset [a, b]$ be a Riemann germ with respect to (J, f) . Let $K \subset [a, b] \cap \mathbb{R}^N$ be a compact set.

i) (Bounded box)

Then there exists a box $[\bar{a}, \bar{b}]$ such that

$$(6.1) \quad K \subset [\bar{a}, \bar{b}] \subset [a, b] \quad \text{with } [\bar{a}, \bar{b}] \text{ bounded and } \pi_{\mathcal{G}}([\bar{a}, \bar{b}]) \subset [\bar{a}, \bar{b}].$$

Moreover, for the inclusion there exists a minimal box $[\bar{a}_*, \bar{b}_*]$ satisfying (6.1).

ii) (Restricted Riemann germ)

For any box $[\bar{a}, \bar{b}]$ satisfying (6.1), then the set $\mathcal{G}' := \mathcal{G} \cap [\bar{a}, \bar{b}]$ is a Riemann germ with respect to $(J, f|_{[\bar{a}, \bar{b}]})$.

Proof of Lemma 6.1

Up to use reversion transforms, we can assume that the junction is of type $0 : N$. Up to use inversion transforms, we can also assume that $\theta^k = +1$ in (2.2) for each index k . Moreover, up to increase the compact set K , we can assume that $K = [a, b] \cap \mathbb{R}^N$ with $\bar{a}, \bar{b} \in \mathbb{R}^N$ with $\bar{a} < \bar{b}$.

Part 1: case $b = (+\infty, \dots, +\infty)$

Step 1: construction of \bar{a}

For each index j , we have the following dichotomy. Either $a^j > -\infty$, and then we set $\bar{a}^j := a^j$. Or we have $a^j = -\infty$, and using the coercivity of f^j at $-\infty$, we choose $\bar{a}^j \in (-\infty, \underline{a}^j]$ such that $f^j(\bar{a}^j) = \inf_{(-\infty, \bar{a}^j]} f^j$.

Then (2.14) implies for all index j that

$$\hat{f}^j(p) \geq f_-^j(p) = \inf_{(-\infty, p^j]} f^j = \inf_{[\bar{a}^j, p^j]} f^j \quad \text{for all } p \in [\bar{a}, b]$$

Hence, up to replace \mathcal{G} by $\mathcal{G} \cap [\bar{a}, b]$ which is again a Riemann germ (from ii) of Theorem 2.14 and the continuity of \hat{f}), we can assume that $a = \bar{a} \in \mathbb{R}^N$.

Step 2: bound on \mathcal{G} towards $+\infty$

We claim that there exists some

$$(6.2) \quad \text{there exists some } b_* \in [b, b] \cap \mathbb{R}^N \text{ such that } \mathcal{G} \cap (b_* + [0, +\infty)^N) = \emptyset$$

Assume by contradiction that (6.2) is false. Then we deduce that we can construct a sequence $(p_n)_{n \in \mathbb{N}}$ with

$$(6.3) \quad p_n \in \mathcal{G}, \quad p_n < p_{n+1}, \quad p_n^j \rightarrow +\infty \quad \text{as } n \rightarrow +\infty \quad \text{for all index } j$$

Setting $\hat{p}_n := \pi(p_n) = p_n$, and using the fact that each f^j is coercive, we see that up to extract a subsequence (still denoted by n), we can assume that $\hat{p}_n \leq \hat{p}_{n+1}$ and $\hat{f}(\hat{p}_n) < \hat{f}(\hat{p}_{n+1})$ (using also that $\hat{f} = f$ on \mathcal{G}). This means that

$$(6.4) \quad (\hat{p}_{n+1} - \hat{p}_n) \cdot [\hat{f}]_{\hat{p}_n}^{\hat{p}_{n+1}} \geq 0 \quad \text{with} \quad [\hat{f}]_{\hat{p}_n}^{\hat{p}_{n+1}} := \hat{f}^j(\hat{p}_{n+1}) - \hat{f}^j(\hat{p}_n) > 0$$

From Theorem 2.17, we know that $\sigma \diamond \hat{f} = -\hat{f}$ is Riemann monotone with $\sigma^j = -1$ for all j . Therefore

$$-(\hat{p}_{n+1} - \hat{p}_n) \cdot [\hat{f}]_{\hat{p}_n}^{\hat{p}_{n+1}} \leq 0 \quad \text{implies} \quad [\hat{f}]_{\hat{p}_n}^{\hat{p}_{n+1}} = 0$$

Contradiction with (6.4). Therefore (6.3) is false, and this implies (6.2).

Step 3: bound from above on \hat{f} on $[\bar{a}, b]$

We claim that

$$(6.5) \quad \sup_{j=1, \dots, N} \sup_{[\bar{a}, b]} \hat{f}^j < +\infty$$

We do the proof by recurrence on the number $N \geq 1$ of branches, using (6.2).

Step 3.1: case $N = 1$

For b_* given in (6.2) for $N = 1$, recall that $b_* \in [b, +\infty)$, and consider $\hat{b}_* := \pi_{\mathcal{G}}(b_*) \in \mathcal{G} \subset [a, b]$. Then (6.2) implies that $\hat{b}_* < b_*$ and $BA(\hat{b}_*) \supset \hat{b}_* + [0, +\infty) \ni b_*$. Hence $\hat{f} = \text{const}$ on $[b_*, +\infty)$. By assumption, \mathcal{G} is a Riemann germ, and then \hat{f} is continuous, and then bounded on $[\bar{a}, b_*]$. This implies that $\sup_{[\bar{a}, b]} \hat{f} < +\infty$,

which shows (6.5) for $N = 1$.

Step 3.2: case $N = n + 1$ for $n \geq 1$

Given $n \geq 1$, assume that (6.5) is true for $N - 1 = n$, and let us show it for $N := n + 1$. To this end, we use the slicing lemma 4.12.

We first freeze the last coordinate \bar{a}^N of $\bar{a} = (\bar{a}', \bar{a}^N)$. We consider the germ

$$\mathcal{G}'_{\bar{a}^N} := \left\{ p' := (p^1, \dots, p^{N-1}) \in [a', b'], \quad \hat{f}'_{\bar{a}^N}(p') = f'(p') \right\} \quad \text{with} \quad \begin{cases} f' & := (f^1, \dots, f^{N-1}) \\ \hat{f}'_{\bar{a}^N}(p') & := (\hat{f}^1, \dots, \hat{f}^{N-1})(p', \bar{a}^N), \end{cases}$$

Then from assumption (6.5) at the level $N - 1$, we deduce that

$$(6.6) \quad \sup_{j=1, \dots, N-1} \sup_{[\bar{a}', b']} \hat{f}'_{\bar{a}^N}{}^j < +\infty$$

Because $\mathcal{G}'_{\bar{a}^N} = \left\{ p' \in [\bar{a}', b'], \quad \hat{f}'_{\bar{a}^N}(p') = f'(p') \right\}$ and f' is coercive, we deduce from (6.6) that there exists some $R > 0$ such that $\mathcal{G}'_{\bar{a}^N} \subset \bar{a}' + [0, R]^{N-1}$. Now for any $p \in [\bar{a}', b'] \times \{\bar{a}^N\}$ and $\hat{p} := \pi_{\mathcal{G}}(p)$, using the fact that the function \hat{f}^N is nonincreasing in p^N for $0 : N$ junctions, we deduce that $\hat{f}^N(p) = \hat{f}^N(\hat{p}) \leq \hat{f}^N(\hat{p}', \bar{a}^N) = \hat{f}'_{\bar{a}^N}(\hat{p}')$ with $\hat{p}' \in \mathcal{G}'_{\bar{a}^N}$. Therefore

$$(6.7) \quad \hat{f}^N(p) \leq \sup_{[\bar{a}', \bar{a}'+(R, \dots, R)]} \hat{f}'_{\bar{a}^N}{}^N \leq C_R$$

Hence, using again the monotonicity of \hat{f}^N , we get $\sup_{[\bar{a}, b]} \hat{f}^N \leq \sup_{[\bar{a}', b'] \times \{\bar{a}^N\}} \hat{f}^N \leq C_R$. Similarly, up to increase the constant C_R , we deduce

$$\sup_{j=1, \dots, N} \sup_{[\bar{a}, b]} \hat{f}^j \leq C_R.$$

Step 4: construction of \bar{b}

Using the coercivity of f^j at $+\infty$, we choose $\bar{b}^j \in [b^j, +\infty)$ such that $\sup_{[\bar{a}, b]} \hat{f}^j \leq f^j(\bar{b}^j) = \sup_{[\bar{a}^j, \bar{b}^j]} f^j$. Then we

have

$$\hat{f}^j(p) \leq \sup_{[\bar{a}, b]} \hat{f}^j \leq \sup_{[\bar{a}, b]} \hat{f}^j \leq f^j(\bar{b}^j) \leq \sup_{[p^j, \bar{b}^j]} f^j \quad \text{for all } p \in [\bar{a}, \bar{b}]$$

We conclude that

$$\inf_{[\bar{a}^j, p^j]} f^j \leq \hat{f}^j(p) \leq \sup_{[p^j, \bar{b}^j]} f^j \quad \text{for all } p \in [\bar{a}, \bar{b}]$$

and from ii) of Theorem 2.14, we conclude that $\mathcal{G}' := \mathcal{G} \cap [\bar{a}, \bar{b}]$ is a generalized germ with respect to $(J, f|_{[\bar{a}, \bar{b}]})$ with $\hat{f}_{\mathcal{G}'} = (\hat{f}_{\mathcal{G}})|_{[\bar{a}, \bar{b}]}$. Because $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous, we also deduce that \mathcal{G}' is a Riemann germ.

Step 5: a further property of $[\bar{a}, \bar{b}]$

Consider any $p \in [\bar{a}, \bar{b}]$, and define $\tilde{p} := \pi_{\mathcal{G}'}(p) \in \mathcal{G}' \subset \mathcal{G}$. Then we have (from the definition of the basins of attraction) $p \in BA^{J, f|_{[\bar{a}, \bar{b}]}}(\tilde{p}) = BA^{J, f}(\tilde{p}) \cap [\bar{a}, \bar{b}]$, and then $p \in BA^{J, f}(\tilde{p})$ with $\tilde{p} \in \mathcal{G}$. Hence $\tilde{p} = \hat{p} := \pi_{\mathcal{G}}(p)$. In other words, this shows that $(\pi_{\mathcal{G}})|_{[\bar{a}, \bar{b}]} = \pi_{\mathcal{G}'}$ with $\mathcal{G}' = \mathcal{G} \cap [\bar{a}, \bar{b}]$. In particular, we see that if $\mathcal{G} \cap [\bar{a}, \bar{b}]$ is a Riemann germ, then the box $[\bar{a}, \bar{b}]$ satisfies $\pi_{\mathcal{G}}([\bar{a}, \bar{b}]) \subset [\bar{a}, \bar{b}]$.

Step 6: converse property

We now want to show that, under our assumption (2.2), for any bounded box $[\bar{a}, \bar{b}] \subset [a, b]$ such that $\pi_{\mathcal{G}}([\bar{a}, \bar{b}]) \subset [\bar{a}, \bar{b}]$, then $\mathcal{G} \cap [\bar{a}, \bar{b}]$ is a Riemann germ. From ii) of Theorem 2.14, we only have to show that for all j

$$(6.8) \quad \inf_{[\bar{a}^j, p^j]} f^j \leq \hat{f}^j(p) \leq \sup_{[p^j, \bar{b}^j]} f^j \quad \text{for all } p \in [\bar{a}, \bar{b}]$$

Indeed, let $p \in [\bar{a}, \bar{b}]$. By assumption, we have $\hat{p} := \pi_{\mathcal{G}}(p) \in [\bar{a}, \bar{b}]$. From the definition of the basin of attraction $BA(\hat{p}) \ni p \in [\bar{a}, \bar{b}]$, and for a junction of type $0 : N$, we have $\begin{cases} f^j(p^j) \geq f^j(\hat{p}^j) & \text{if } p^j \geq \hat{p}^j \\ f^j(p^j) \leq f^j(\hat{p}^j) & \text{if } p^j \leq \hat{p}^j \end{cases}$.

Because $p^j \in BA(\hat{p}^j) \cap [\bar{a}^j, \bar{b}^j] \ni \hat{p}^j$, we deduce that

$$\sup_{[p^j, \bar{b}^j]} f^j \geq \begin{cases} f^j(p^j) \geq f^j(\hat{p}^j) & \text{if } p^j \geq \hat{p}^j \\ f^j(\hat{p}^j) & \text{if } p^j \leq \hat{p}^j \end{cases}, \quad \text{and} \quad \inf_{[\bar{a}^j, p^j]} f^j \leq \begin{cases} f^j(\hat{p}^j) & \text{if } p^j \geq \hat{p}^j \\ f^j(p^j) \leq f^j(\hat{p}^j) & \text{if } p^j \leq \hat{p}^j \end{cases}$$

which implies that $\inf_{[\bar{a}^j, \bar{p}^j]} f^j \leq f^j(\hat{p}^j) = \hat{f}^j(\hat{p}) = \hat{f}^j(p) \leq \sup_{[p^j, \bar{b}^j]} f^j$, which is exactly (6.8).

Step 7: minimal box

Given the compact set K , consider the set $S := \{[\bar{a}, \bar{b}] \subset \mathbb{R}^N, \quad K \subset [\bar{a}, \bar{b}] \quad \text{and} \quad \pi_{\mathcal{G}}([\bar{a}, \bar{b}]) \subset [\bar{a}, \bar{b}]\}$. Then define the set $K^{\mathcal{G}} := \bigcap_{[\bar{a}, \bar{b}] \in S} [\bar{a}, \bar{b}]$. By construction, this set is closed and is a box which contains K , i.e. we

have $K^{\mathcal{G}} = [\bar{a}_*, \bar{b}_*] \supset K$. Moreover, we have $\pi_{\mathcal{G}}(K^{\mathcal{G}}) \subset K^{\mathcal{G}}$, and then $K^{\mathcal{G}} \in S$ which shows that $K^{\mathcal{G}}$ is the minimal element of S for the inclusion.

Part 2: case $b \neq (+\infty, \dots, +\infty)$

This part is an easy adaptation of Part 1. Step 1 is unchanged. If $b \in \mathbb{R}^N$, then we can choose $\bar{b} := b$. Assume now that $b \notin \mathbb{R}^N$. Then, in Step 2, relation (6.2) has to be replaced by

$$(6.9) \quad \text{there exists some } b_* \in [b, b] \cap \mathbb{R}^N \text{ such that } \mathcal{G} \cap \left(b_* + \sum_{b^j = +\infty} [0, +\infty) e_j \right) = \emptyset$$

where (e_1, \dots, e_N) is the canonical basis of \mathbb{R}^N . Step 3.1 is unchanged. In Step 3.2, relation (6.7) has to be modified in

$$(6.10) \quad \hat{f}^N(p) \leq \sup_{[\bar{a}', \bar{a}' + (R, \dots, R)] \cap [\bar{a}', b']} \hat{f}_{\bar{a}'}^N \leq C_R$$

Finally in Step 4, we have to redefine $\bar{b}^j := b^j$ only when $b^j \in \mathbb{R}$. The remaining part of the proof is unchanged. This ends the proof of the proposition.

6.2 Theorem 2.21 and its proof: Kruřkov germs

Theorem 2.21 will be a corollary of the following two lemmata.

Lemma 6.2 (D -maximality of Kruřkov germs)

Assume (2.2) with $N \geq 1$ with $\mathcal{G} \subset [a, b]$ a generalized Riemann germ. Consider the set

$$\mathcal{G}' := \{p \in [a, b], \quad D^f(p, \hat{q}) \geq 0 \quad \text{for all } \hat{q} \in \mathcal{G}\} \quad \text{for } D^f \text{ defined in (2.8)}$$

If \mathcal{G} is a Kruřkov germ, then it satisfies the following D -maximality property: $\mathcal{G}' = \mathcal{G}$.

Proof of Lemma 6.2

Recall that by definition of the Kruřkov germ \mathcal{G} , it satisfies $D^f \geq 0$ on $\mathcal{G} \times \mathcal{G}$. Hence $\mathcal{G} \subset \mathcal{G}'$. Assume that there exists $p \in \mathcal{G}' \setminus \mathcal{G}$. From Lemma 3.4, we deduce that $D^f(p, \hat{p}) < 0$ for $\hat{p} := \pi(p)$. Contradiction. Therefore $\mathcal{G}' = \mathcal{G}$, and this ends the proof.

Lemma 6.3 (Stability of Kruřkov germs)

Assume (2.2). Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ.

If \mathcal{G} is a Kruřkov germ, then $\hat{f}_{\mathcal{G}}$ is continuous.

Proof of Lemma 6.3

We will show the result using Theorem 2.16. We will indeed show that the \mathcal{G} -Riemann problem is stable. Let $[a, b] \ni p_n \rightarrow p_\infty$ as $n \rightarrow +\infty$. From Lemma 3.11, we have for $\hat{p}_n = \pi(p_n)$, that there exists \tilde{p}_∞ such that $p_\infty \in BA(\tilde{p}_\infty)$ and

$$u_n := u_{p_n, \hat{p}_n}^{\mathcal{G}} \rightarrow u_\infty := u_{p_\infty, \tilde{p}_\infty} \quad \text{in } L_{loc}^1([0, +\infty) \times J)$$

where we do not know yet that \tilde{p}_∞ belongs to \mathcal{G} . This is what we want to prove. For any $0 \leq \varphi^k \in C_c^1([0, +\infty) \times \bar{J}^k)$ with $\bar{J}^k := \{0\} \cup J^k \simeq [0, +\infty)$, we define

$$I^k(\varphi, u_n, c) := \int_{(0, +\infty) \times J^k} \{\eta^k(u_n^k, c^k) \varphi_t^k + \psi^k(u_n^k, c^k) \varphi_x^k\} \, dt dx + \int_{\{0\} \times J^k} \eta^k(p_n, c^k) \varphi^k \, dx$$

From Definition 2.32 of Kruřkov entropy solutions, we have for any $c \in \mathcal{G}$ (which can be seen as a constant \mathcal{G} -entropy solution), we have

$$I^k(\phi, u_n, c) \geq 0 \quad \text{for any } 0 \leq \phi^k \in C_c^1([0, +\infty) \times J^k)$$

Now for $0 \leq \varphi^k \in C_c^1([0, +\infty) \times \bar{J}^k)$, and using the continuity of $\psi^k(\cdot, c^k)$, and the notion of trace at $x = 0$ (which is automatically well-defined for Riemann problem, because of the monotonicity of u_n^k in x), we get (which can be justified, by approximation from C^1 to Lipschitz continuous, with $\phi_\varepsilon^k := \varphi^k - \varphi^k(t, 0)\theta_\varepsilon(x)$ with $\theta_\varepsilon(x) := \max\{0, 1 - \varepsilon^{-1}x\}$ in the limit $\varepsilon \rightarrow 0$), that

$$I^k(\varphi, u_n, c) \geq \int_{(0, +\infty) \times \{0\}} -\psi^k(\hat{p}_n, c^k)\varphi^k$$

Choosing $\varphi^k = \varphi^j =: \varphi^0$ on $(0, +\infty) \times \{0\}$, we get

$$I(\varphi, u_n, c) := \sum_{k=1, \dots, N} I^k(\varphi, u_n, c) \geq \int_{(0, +\infty) \times \{0\}} D^f(p_n, c)\varphi^0 \geq 0$$

where the last inequality follows from $D^f(p_n, c) \geq 0$ because $p_n, c \in \mathcal{G}$ and \mathcal{G} is a Kruřkov germ. Passing to the limit, we get $I(\varphi, u_\infty, c) \geq 0$. Now choosing $\varphi_\varepsilon^j(t, x) = \varphi^0(t, 0)\theta_\varepsilon(x)$ for all index j (up to use again approximations from C^1 to Lipschitz continuous), we get in the limit $\varepsilon \rightarrow 0$ that

$$0 \leq I(\varphi_\varepsilon, u_\infty, c) \rightarrow \int_{(0, +\infty) \times \{0\}} D(\tilde{p}_\infty, c)\varphi^0 = D(\tilde{p}_\infty, c) \left\{ \int_{(0, +\infty) \times \{0\}} \varphi^0 \right\}$$

Because this is true for all $0 \leq \varphi^0 \in C_c^1([0, +\infty))$, we get $D(\tilde{p}_\infty, c) \geq 0$ for all $c \in \mathcal{G}$. From Lemma 6.2, we deduce that $\tilde{p}_\infty \in \mathcal{G}$ with $u_\infty = u_{p_\infty, \tilde{p}_\infty}$. Hence $\tilde{p}_\infty \in \mathcal{G} \cap \hat{\mathcal{P}}_{p_\infty} = \{\pi(p_\infty)\}$, i.e. $\tilde{p}_\infty = \pi(p_\infty)$ and $u_\infty = u_{p_\infty, \pi(p_\infty)}^{\mathcal{G}}$. This shows the stability of \mathcal{G} -Riemann problem. Hence from Theorem 2.16, we deduce that \hat{f} is continuous. This ends the proof of the lemma.

Proposition 6.4 (Characterization of Kruřkov germs among generalized Riemann germs)

Assume (2.2) with $N \geq 1$.

i) (Kruřkov germs)

Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ, and let $\hat{f} := \hat{f}_{\mathcal{G}}$ be its associated Godunov germ. Then \mathcal{G} is a Kruřkov germ if and only if \hat{f} is locally Lipschitz continuous on $[a, b]$ and its Jacobian matrix is **diagonally dominant**, i.e. precisely

$$(6.11) \quad \sigma^i \partial_i \hat{f}^i \geq \sum_{j \in \{1, \dots, N\} \setminus \{i\}} |\partial_i \hat{f}^j| \quad \text{a.e. on } [a, b], \quad i = 1, \dots, N$$

ii) (General continuous dissipative functions)

Assume that $h : [a, b] \rightarrow \mathbb{R}^N$ is a continuous map. Assume also that the map $p \mapsto h^j(p)$ is locally Lipschitz continuous in p^j uniformly in the other coordinates p^k for $k \neq j$, and for all $j = 1, \dots, N$.

Then $D^h \geq 0$ on $[a, b]^2$ if and only if h is locally Lipschitz continuous and satisfies

$$(6.12) \quad \sigma^i \partial_i h^i \geq \sum_{j \in \{1, \dots, N\} \setminus \{i\}} |\partial_i h^j| \quad \text{a.e. on } [a, b], \quad i = 1, \dots, N$$

Proof of Proposition 6.4

Part I: proof of i)

We first recall that from Lemma 5.5, the generalized Riemann germ \mathcal{G} is a Kruřkov germ if and only if it satisfies $D^{\hat{f}} \geq 0$ on $[a, b]^2$ for $\hat{f} := \hat{f}_{\mathcal{G}}$. We only do the proof for $N : 0$ junctions. The general case then follows by reversion transforms.

Step 1: necessary condition

Step 1.1: interior regularization

Consider a Kružkov germ \mathcal{G} . Then recall that $D^{\hat{f}} \geq 0$ implies for all points p, q in the interior of the box $[a, b]$ with $\sigma^j = 1$ for all j

$$(6.13) \quad D^{\hat{f}}(p + \xi, q + \xi) = \sum_{j=1, \dots, N} \text{sign}(p^j - q^j) \cdot \left\{ \hat{f}^j(p + \xi) - \hat{f}^j(q + \xi) \right\} \geq 0$$

Let $0 \leq \rho_\varepsilon := \varepsilon^{-N} \rho(\varepsilon^{-1} \cdot)$ be a mollifier with $\text{supp}(\rho) \subset [-1, 1]^N$. Extending \hat{f} (for instance by zero outside the box $[a, b]$), and integrating (6.13) over the measure $\rho_\varepsilon(\xi) d\xi$ and setting $\hat{f}^\varepsilon := \hat{f} \star \rho_\varepsilon$, we get $D^{\hat{f}^\varepsilon} \geq 0$ on $[a, b]_\varepsilon := [a + \varepsilon(1, \dots, 1), b - \varepsilon(1, \dots, 1)]$.

Step 1.2: application

Recall that Corollary 4.19 holds for $N : 0$ junctions, and (4.31) means for \hat{f}^ε that a.e. on $[a, b]_\varepsilon$

$$(6.14) \quad \left. \begin{array}{l} \bar{S}_K := \sum_{j \in K} (\hat{f}^\varepsilon)^j - \sum_{j \notin K} (\hat{f}^\varepsilon)^j \\ \partial_k \bar{S}_K \geq 0 \\ \partial_k \bar{S}_K \leq 0 \end{array} \right\} \begin{array}{l} \text{for all } k \in K \\ \text{for all } k \notin K \end{array} \Bigg|, \quad \text{for all } K \subset \{1, \dots, N\}$$

Fix a point $p \in [a, b]$ where \hat{f}^ε has a derivative, and fix the index 1. Let $K_-(p) := \{j \in \{2, \dots, N\}, \partial_1(\hat{f}^\varepsilon)^j(p) \leq 0\}$. Then for $K := K(p) := K_-(p) \cup \{1\}$, we get from (6.14)

$$\partial_1(\hat{f}^\varepsilon)^1(p) + \sum_{j \in K_-(p)} \partial_1(\hat{f}^\varepsilon)^j(p) - \sum_{j \in \{2, \dots, N\} \setminus K_-(p)} \partial_1(\hat{f}^\varepsilon)^j(p) \geq 0$$

i.e.

$$(6.15) \quad \partial_1(\hat{f}^\varepsilon)^1 \geq \sum_{j \in \{2, \dots, N\}} |\partial_1(\hat{f}^\varepsilon)^j| \quad \text{a.e. on } [a, b]_\varepsilon$$

The same result holds true for all other index than 1, which shows that \hat{f}^ε satisfies (6.11) on $[a, b]_\varepsilon$.

Step 1.3: the limit $\varepsilon \rightarrow 0$

We also know that $0 \leq \partial_j \hat{f}^j \leq |(f^j)'|$ a.e. for $N : 0$ junctions. Joint to (6.15), this shows that \hat{f}^ε is locally Lipschitz continuous, uniformly in ε in the interior of the box $[a, b]$. From Lemma 6.3, we know that \hat{f} is continuous on $[a, b]$. Hence $\hat{f}^\varepsilon \rightarrow \hat{f}$ locally uniformly in the interior of the box $[a, b]$, and we conclude that \hat{f} is locally Lipschitz continuous in the interior of the box $[a, b]$. Then, inside the interior of the box $[a, b]$, we can apply Corollary 4.19 to \hat{f} as in Step 1.2. This shows (6.11) for $\sigma^j = 1$ for all j . Finally, the continuity of \hat{f} on $[a, b]$ implies that \hat{f} is locally Lipschitz continuous on the whole box $[a, b]$.

Step 2: sufficient condition

Assume now that \hat{f} is locally Lipschitz continuous on $[a, b]$ and satisfies (6.11) for $\sigma^j = 1$ for all j . Then this implies immediately (6.14), which from Corollary 4.19 is a characterization of $D^{\hat{f}} \geq 0$ for a $N : 0$ junctions. Finally recall that Lemma 5.5 shows that \mathcal{G} is a Kružkov germ.

Part II: proof of ii)

The same argument as in Part I applies and give the result. This ends the proof of the proposition.

Proof of Theorem 2.21:

For the proof, we refer to the table of Subsection 2.4. This follows from Lemmata 6.2 and Proposition 6.4. This ends the proof of the theorem.

We end this subsection with the following result.

Lemma 6.5 (Convergence of Godunov flux for Kružkov germs)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G}_n \subset [a, b]$ for $n \in \mathbb{N}$ be a sequence of Kružkov germs with respect to (J, f) , with associated Godunov flux $\hat{f}_n := \hat{f}_{\mathcal{G}_n}$. Then, up to extract a subsequence (still denoted by n), there exists a Kružkov germ $\mathcal{G}_\infty \subset [a, b]$ with respect to (J, f) and $\hat{f}_\infty := \hat{f}_{\mathcal{G}_\infty}$ such that

$$\hat{f}_{\mathcal{G}_n} \rightarrow \hat{f}_{\mathcal{G}_\infty} \quad \text{as } n \rightarrow +\infty$$

Proof of Lemma 6.5

We know from i) of Proposition 6.4 and vii) of Proposition 2.13 that the sequence \hat{f}_n is locally uniformly

Lipschitz. Hence from Ascoli-Arzelà theorem, we can extract a convergent subsequence and call \hat{f}_∞ the limit. We define the set $\mathcal{G}_\infty := \{\hat{f}_\infty = f\}$, which is a closed subset of $[a, b]$. Notice that we have $D^{\hat{f}_n} \geq 0$ on $[a, b]^2$. Passing to the limit, we recover $D^{\hat{f}_\infty} \geq 0$ on $[a, b]^2$. Hence $D^f \geq 0$ on $\mathcal{G}_\infty \times \mathcal{G}_\infty$, which shows that \mathcal{G}_∞ is a Kružkov germ, once we know that \mathcal{G}_∞ is a generalized Riemann germ.

Now recall that \hat{f}_n satisfy (2.14), and also (2.12) which is

$$(6.16) \quad \left\{ \begin{array}{l} \text{For all } p \in [a, b] \text{ and } K_{n,p} := \{j \in \{1, \dots, N\}, \hat{f}_n^j(p) \neq f^j(p)\}, \quad \text{there exists } \varepsilon_n > 0 \\ \text{such that for } Q_{n,\varepsilon_n}(p) := p + \sum_{j \in K_{n,p}} (-\varepsilon_n, \varepsilon_n) e_j, \text{ we have} \\ \hat{f}_n = \text{const} \quad \text{on} \quad [a, b] \cap Q_{n,\varepsilon_n}(p) \end{array} \right.$$

Then it is easy to see that condition (6.16) is closed. This follows from the Lipschitz continuity of f which allows to bound from below ε_n for n large enough as $\hat{f}_n \rightarrow \hat{f}_\infty$, starting from $K_{\infty,p}$ defined similarly for \hat{f}_∞ . We conclude that \hat{f}_∞ satisfies (2.12) and (2.14). Then Theorem 2.14 ii) implies that \mathcal{G}_∞ is a generalized Riemann germ and that \hat{f}_∞ is the Godunov flux associated to \mathcal{G}_∞ . We conclude that \mathcal{G}_∞ is a Kružkov germ with $\hat{f}_{\mathcal{G}_\infty} = \hat{f}_\infty$. This ends the proof of the lemma.

6.3 Theorem 2.23 and its proof: D_+ -germs

Proposition 6.6 (Characterization of D_+ -germs among generalized Riemann germs)

Assume (2.2) with $N \geq 1$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ, and let $\hat{f} := \hat{f}_{\mathcal{G}}$ be its associated Godunov germ. Then \mathcal{G} is a D_+ -germ if and only if \hat{f} is locally Lipschitz continuous on $[a, b]$ and satisfies a.e. on $[a, b]$

$$(6.17) \quad \partial_i(\sigma^i \hat{f}^i) \geq \sum_{j \in \{1, \dots, N\} \setminus \{i\}} |\partial_i \hat{f}^j|, \quad \text{for all } i = 1, \dots, N$$

and

$$(6.18) \quad \partial_i(\sigma^j \hat{f}^j) \leq 0, \quad \text{for all } i \neq j$$

Proof of Proposition 6.6

The proof follows part of the lines of the proof of Proposition 6.4.

We first recall that from Lemma 5.5, the generalized Riemann germ \mathcal{G} is a D_+ -germ if and only if it satisfies $D_+^{\hat{f}} \geq 0$ on $[a, b]^2$ for $\hat{f} := \hat{f}_{\mathcal{G}}$. We only do the proof for $N : 0$ junctions. The general case then follows by reversion transforms.

Step 1: necessary condition

Step 1.1: preliminary

We first notice that a D_+ -germ is in particular a Kružkov germ (i.e. a D -germ) because $D(p, q) = D_+(p, q) + D_+(q, p)$. Then Proposition 6.4 implies that \hat{f} is locally Lipschitz continuous on $[a, b]$ and satisfies (6.17).

Step 1.2: application

Recall that Corollary 4.19 holds for $N : 0$ junctions, and (4.33) means for \hat{f} that a.e. on $[a, b]$ we have

$$(6.19) \quad \left. \begin{array}{l} S_K := \sum_{j \in K} \hat{f}^j \\ \partial_k S_K \geq 0 \\ \partial_k S_K \leq 0 \end{array} \right| \quad \text{for all } k \in K \quad \left. \vphantom{\begin{array}{l} S_K := \sum_{j \in K} \hat{f}^j \\ \partial_k S_K \geq 0 \\ \partial_k S_K \leq 0 \end{array}} \right\}, \quad \text{for all } K \subset \{1, \dots, N\}$$

Hence for $K := \{j\}$, this gives (6.18) with $\sigma^\ell = 1$ for all $\ell = 1, \dots, N$.

Step 2: sufficient condition

Assume now that \hat{f} is locally Lipschitz continuous on $[a, b]$ and satisfies (6.17) and (6.18) for $\sigma^j = 1$ for all j . Then (6.17) implies immediately the second line of (6.19), while (6.18) implies the third line of (6.19).

Therefore (6.19) holds true, and then Corollary 4.19 implies that $D_+^{\hat{f}} \geq 0$ for a $N : 0$ junctions. Finally recall that Lemma 5.5 shows that \mathcal{G} is a D_+ -germ. This ends the proof of the proposition.

Proof of Theorem 2.23

For the proof, we refer to the table of Subsection 2.4. The result follows from Proposition 6.6 and i') and iii) of Lemma 5.5. This ends the proof of the theorem.

6.4 Theorem 2.24 and its proof: conservative Riemann germs

Proof of Theorem 2.24

Recall that we assume that \mathcal{G} is a Riemann germ which is conservative. From Lemma 5.5, this means $\hat{f} := \hat{f}_{\mathcal{G}}$ is continuous and satisfies

$$RH^{\hat{f}} = 0 \quad \text{with} \quad RH^{\hat{f}}(p) := \sum_{j=1, \dots, N} \sigma^j \hat{f}^j(p)$$

Still from Lemma 5.5, recall that \mathcal{G} is monotone if and only if

$$(6.20) \quad p \mapsto \sigma^j \hat{f}^j(p) \quad \text{is nonincreasing in } p^k \text{ for all } k \neq j, \text{ and nondecreasing in } p^j$$

and \mathcal{G} is Kruřkov if and only if $D^{\hat{f}} \geq 0$. We first notice that up to apply a suitable reversion transform, we can assume that the junction is of type $N : 0$, i.e. that $J^j \simeq (-\infty, 0)$ with $\sigma^j = 1$ for all j .

Step 1: monotone implies Kruřkov

Assume that \hat{f} satisfies (6.20) and let us show that it satisfies (4.33). To this end, consider a set $K \subset \{1, \dots, N\}$, and call $S_K := \sum_{j \in K} \hat{f}^j$. Then (6.20) implies

$$(6.21) \quad \partial_{\ell} S_K \leq 0 \quad \text{for all } \ell \notin K$$

We also have $S_K = RH^{\hat{f}} - \sum_{j \notin K} \hat{f}^j = -\sum_{j \notin K} \hat{f}^j$. Hence for all $\ell \in K$ and $j \notin K$, we have $\partial_{\ell} \hat{f}^j \leq 0$, and then

$$(6.22) \quad \partial_{\ell} S_K \geq 0 \quad \text{for all } \ell \in K$$

Therefore (6.21)-(6.22) show (4.33). From ii) of Lemma 4.18, this implies $D_+^{\hat{f}} \geq 0$, and then $D^{\hat{f}} \geq 0$ and \mathcal{G} is a Kruřkov germ.

Step 2: Kruřkov implies monotone

Assume that \mathcal{G} is Kruřkov. Then $D^{\hat{f}} \geq 0$, and from i) of Lemma 4.18, this implies that \bar{S}_K satisfies (6.21)-(6.22), with

$$\bar{S}_K := \sum_{j \in K} \hat{f}^j - \sum_{j \notin K} \hat{f}^j = -RH^{\hat{f}} + 2 \sum_{j \in K} \hat{f}^j = 2 \sum_{j \in K} \hat{f}^j = 2S_K$$

Therefore S_K also satisfies (6.21)-(6.22). Given an index k and $K := \{k\}$ and $\ell \notin K$ or $\ell \in K$, we deduce from (6.21)-(6.22) applied to S_K that $\partial_{\ell} \hat{f}^k \leq 0$ for all $\ell \neq k$ and $\partial_k \hat{f}^k \geq 0$, which is nothing else than condition (6.20). Therefore \mathcal{G} is monotone.

Step 3: equivalence with D_+ -germs

Using Theorem 2.23, we deduce that monotone is equivalent to Kruřkov which is also equivalent to D_+ -germ. This ends the proof of the lemma.

6.5 Theorem 2.25 and its proof: Hamilton-Jacobi germs and HJ-relaxation operator

The proof of Theorem 2.25 requires two intermediate results: first Lemma 6.7 shows the existence of some HJ-relaxation operator, which in particular has an action on locally constant functions. Then Proposition 6.9 presents important properties of HJ germs. Finally, we conclude the section with the proof of Theorem 2.25.

Lemma 6.7 (HJ-relaxation operator)

Assume (2.2) with $N \geq 1$. Let h_0 be such that

$$(6.23) \quad \begin{cases} h_0 : [a, b] \rightarrow \mathbb{R} \quad \text{is continuous} \\ p \mapsto \sigma^j h_0(p) \quad \text{is nondecreasing in } p^j \text{ for all } j = 1, \dots, N, \\ f_0 := (h_0, \dots, h_0) \text{ satisfies the monotone bounds given in the second line of (2.14)}. \end{cases}$$

Assume moreover that either $[a, b] \subset \mathbb{R}^N$ is compact, or $[a, b] \cap \mathbb{R}^N = \mathbb{R}^N$, or

$$(6.24) \quad f_0 := (h_0, \dots, h_0) \text{ is locally constant on } \{f_0 \neq f\}$$

Then the following formula defines uniquely $h_1(p)$ for each $p \in [a, b]$

$$(6.25) \quad \{h_1(p)\} = \bigcup_{q \in [a, b]} \left(\{h_0(q)\} \cap \bigcap_{J^j \simeq (-\infty, 0)} \{G^j(p^j, q^j)\} \cap \bigcap_{J^j \simeq (0, +\infty)} \{G^j(q^j, p^j)\} \right).$$

Then we set

$$(6.26) \quad \mathfrak{R}h_0 := h_1$$

and h_1 satisfies (6.23) and moreover $f_1 := (h_1, \dots, h_1)$ is locally constant on $\{f_1 \neq f\}$.

Remark 6.8 Notice that in Lemma 6.7 and in the special case of assumption (6.24), we do not have to require the continuity of h_0 in (6.23). Indeed the continuity of h_0 is then automatic, as it will be independently shown later in *i*) of Proposition 6.9, for the HJ germ $\mathcal{G}_{f_0} = \{f_0 = f\}$.

Proof of Lemma 6.7

Step 1: h_1 is well-defined with good properties

Step 1.1: non emptiness of the intersection

We consider h_0 satisfying (6.23). In particular, for $f_0 := (h_0, \dots, h_0)$ we have $f_- \leq f_0 \leq f_+$, which means

$$(6.27) \quad \begin{cases} f_-^j(q) = G^{f^j}(q^j, b^j) \leq h_0(q) \leq f_+^j(q) = G^{f^j}(q^j, a^j) & \text{if } J^j \simeq (-\infty, 0) \\ f_-^j(q) = G^{f^j}(a^j, q^j) \leq h_0(q) \leq f_+^j(q) = G^{f^j}(b^j, q^j) & \text{if } J^j \simeq (0, +\infty) \end{cases}$$

Recall that the monotonicities of h_0 are opposite for ingoing and outgoing branches. It is more convenient to have the same monotonicities on each variable. To this end, we define

$$\bar{p}^j := \sigma^j p^j, \quad [\bar{a}^j, \bar{b}^j] := \sigma^j [a^j, b^j] \quad \bar{f}^j(\bar{p}) = f^j(p), \quad \bar{J}^j := \sigma^j(-\infty, 0) \quad \text{for } j = 1, \dots, N \quad \text{and} \quad \bar{h}_0(\bar{p}) := h_0(p)$$

(which can be seen as a composition of some inversion transform and the reversion transform) which satisfies $\bar{h}_0(\uparrow, \dots, \uparrow)$ and

$$(6.28) \quad \bar{f}_-^j(\bar{q}) = G^{\bar{f}^j}(\bar{q}^j, \bar{b}^j) \leq \bar{h}_0(\bar{q}) \leq \bar{f}_+^j(\bar{q}) = G^{\bar{f}^j}(\bar{q}^j, \bar{a}^j) \quad \text{and} \quad \bar{J}^j \simeq (-\infty, 0) \quad \text{for all } j$$

We also set $\bar{J} := \{0\} \cup \bigcup_{j=1, \dots, N} \bar{J}^j$. Then (\bar{J}, \bar{f}) satisfies (2.2), with possible coercivity assumption for some

$\bar{\theta}^k \in \{\pm 1\}$. Up to apply some inversion, we can furthermore assume that $\bar{\theta}^k = 1$.

We distinguish cases.

Case A: $\bar{a}, \bar{b} \in \mathbb{R}^N$

Then $\bar{f}_-^j(\bar{a}) = \inf_{[\bar{a}^j, \bar{b}^j]} \bar{f}^j$ and $\bar{f}_+^j(\bar{b}) = \sup_{[\bar{a}^j, \bar{b}^j]} \bar{f}^j$, $j = 1, \dots, N$. We define the functions

$$\Phi_{\bar{p}}^j(\bar{q}) := G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) - \bar{h}_0(\bar{q}), \quad j = 1, \dots, N$$

For any $\bar{p} \in [\bar{a}, \bar{b}]$ and using the monotonicity of $\Phi_{\bar{p}}^j$, we get for $\iota_{\bar{q}}^j(\bar{a}^j) := (\bar{q}^1, \dots, \bar{q}^{j-1}, \bar{a}^j, \bar{q}^{j+1}, \dots, \bar{q}^N)$ and for all $\bar{q} \in [\bar{a}, \bar{b}]$ that

$$\Phi_{\bar{p}}^j(\iota_{\bar{q}}^j(\bar{a}^j)) = G^{\bar{f}^j}(\bar{p}^j, \bar{a}^j) - \bar{h}_0(\iota_{\bar{q}}^j(\bar{a}^j)) \geq G^{\bar{f}^j}(\bar{p}^j, \bar{a}^j) - \bar{f}_+^j(\bar{a}^j) = \sup_{[\bar{a}^j, \bar{p}^j]} \bar{f}^j - \bar{f}^j(\bar{a}^j) \geq 0$$

We also get

$$\Phi_{\bar{p}}^j(\iota_{\bar{q}}^j(\bar{b}^j)) = G^{\bar{f}^j}(\bar{p}^j, \bar{b}^j) - \bar{h}_0(\iota_{\bar{q}}^j(\bar{b}^j)) \leq G^{\bar{f}^j}(\bar{p}^j, \bar{b}^j) - \bar{f}_-^j(\bar{b}^j) = \inf_{[\bar{p}^j, \bar{b}^j]} \bar{f}^j - \bar{f}^j(\bar{b}^j) \leq 0$$

Recall that the map $\bar{q} \mapsto \Phi_{\bar{p}}(\bar{q})$ is continuous. Hence we deduce from Poincaré-Miranda theorem (see [17]) that there exists $\bar{q} \in [\bar{a}, \bar{b}]$ such that

$$(6.29) \quad \Phi_{\bar{p}}(\bar{q}) = 0$$

This implies that

$$(6.30) \quad \{\bar{h}_0(\bar{q})\} \cap \bigcap_{j=1, \dots, N} \{G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j)\} \neq \emptyset$$

Case B: $-\bar{a}^j = \bar{b}^j = +\infty$ for all j

Here we do not use Poincaré-Miranda theorem, but make a direct proof. We first notice that (6.28) implies for the special choice $\bar{q} := \bar{p}$ $G^{\bar{f}^j}(\bar{p}^j, \bar{b}^j) \leq \bar{h}_0(\bar{p}) \leq G^{\bar{f}^j}(\bar{p}^j, \bar{a}^j)$ for all j , i.e. (using the coercivity of \bar{f}^j to bound $\underline{\lambda}_{\bar{p}}$ from below)

$$-\infty < \underline{\lambda}_{\bar{p}} := \max_{j=1, \dots, N} G^{\bar{f}^j}(\bar{p}^j, \bar{b}^j) = \max_{j=1, \dots, N} \bar{f}_-^j(\bar{p}^j) \leq \bar{h}_0(\bar{p}) \leq \min_{j=1, \dots, N} \bar{f}_+^j(\bar{p}^j) = \min_{j=1, \dots, N} G^{\bar{f}^j}(\bar{p}^j, \bar{a}^j) =: \bar{\lambda}_{\bar{p}} = +\infty$$

Consider now for $\varepsilon > 0$, the functions

$$G_\varepsilon^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) := G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) - \varepsilon(\bar{q}^j - \bar{p}^j), \quad j = 1, \dots, N$$

Then $\bar{q}^j \mapsto G_\varepsilon^{\bar{f}^j}(\bar{p}^j, \bar{q}^j)$ is locally Lipschitz continuous and satisfies $\frac{\partial}{\partial \bar{q}^j} G_\varepsilon^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) \leq -\varepsilon$ a.e.. Moreover $G_\varepsilon^{\bar{f}^j}(\bar{p}^j, \bar{a}^j) \geq \bar{\lambda}_{\bar{p}} - \varepsilon(\bar{a}^j - \bar{p}^j) \geq \bar{\lambda}_{\bar{p}} = +\infty$ and $G_\varepsilon^{\bar{f}^j}(\bar{p}^j, \bar{b}^j) \leq \underline{\lambda}_{\bar{p}} - \varepsilon(\bar{b}^j - \bar{p}^j) \leq \underline{\lambda}_{\bar{p}}$. Hence for any $\lambda \in \mathbb{R}$, there exists a unique $\bar{q}_\varepsilon^j(\lambda) \in \mathbb{R}$ such that

$$(6.31) \quad G_\varepsilon^{\bar{f}^j}(\bar{p}^j, \bar{q}_\varepsilon^j(\lambda)) = \lambda$$

and moreover the map $\lambda \mapsto \bar{q}_\varepsilon^j(\lambda)$ is nonincreasing and continuous. Now define $\Theta_\varepsilon(\lambda) := \lambda - \bar{h}_0(\bar{q}_\varepsilon(\lambda))$, which by construction is also nondecreasing in λ , and satisfies $\Theta_\varepsilon(\underline{\lambda}_{\bar{p}}) \leq 0 \leq \Theta_\varepsilon(\bar{\lambda}_{\bar{p}}) = +\infty$. By continuity of Θ_ε , we deduce that there exists $\lambda^\varepsilon \in [\underline{\lambda}_{\bar{p}}, +\infty)$ such that

$$(6.32) \quad \Theta_\varepsilon(\lambda^\varepsilon) = 0$$

Up to extract a subsequence assume that $\lambda^\varepsilon \rightarrow \lambda^* \in [\underline{\lambda}_{\bar{p}}, +\infty]$ as $\varepsilon \rightarrow 0$. Assume by contradiction that $\lambda^* = +\infty$. From (6.31) and (6.32), we have

$$\bar{h}_0(\bar{q}^\varepsilon) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^{\varepsilon, j}) - \varepsilon(\bar{q}^{\varepsilon, j} - \bar{p}^j) = \lambda^\varepsilon \rightarrow +\infty \quad \text{with} \quad \bar{q}^{\varepsilon, j} := \bar{q}_\varepsilon^j(\lambda^\varepsilon)$$

where the second equality implies that $\bar{q}^{\varepsilon, j} \rightarrow -\infty$. This is in contradiction with the first equality and the monotonicity of \bar{h}_0 . We conclude that $\lambda^* \in [\underline{\lambda}_{\bar{p}}, +\infty)$ and then $\bar{q}^\varepsilon \rightarrow \bar{q}^*$ which satisfies $\bar{h}_0(\bar{q}^*) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^{*, j})$ for all j , which is (6.30) for $\bar{q} := \bar{q}^*$.

Case C: $\bar{f}_0 := (\bar{h}_0, \dots, \bar{h}_0)$ is locally constant on $\{\bar{f}_0 \neq \bar{f}\}$

From ii) of Theorem 2.14, we deduce that $\mathcal{G}_{\hat{f}_0} := \{\hat{f}_0 = \bar{f}\}$ is a generalized Riemann germ. Setting $\pi := \pi_{\mathcal{G}_{\hat{f}_0}}$, we define

$$\hat{p} := \pi(\bar{p}), \quad \bar{p} \in BA^{(\bar{J}, \bar{f})}(\hat{p}), \quad \lambda_0 := \bar{h}_0(\hat{p}) = \bar{h}_0(\bar{p})$$

where in the last equality we have used the fact that \bar{h}_0 is constant on $BA^{(\bar{J}, \bar{f})}(\hat{p})$. Notice now that $\hat{p} \in \mathcal{G}_{\hat{f}_0}$ and then $G^{\bar{f}^j}(\bar{p}^j, \hat{p}^j) = \bar{f}^j(\hat{p}^j) = \bar{h}_0(\hat{p}) = \lambda_0$ which is (6.30) with $\bar{q} := \hat{p}$.

Step 1.2: uniqueness of the common value

Now assume that there exist two values $\bar{q}, \bar{q}' \in [\bar{a}, \bar{b}]$ such that $\begin{cases} \lambda = \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) \\ \lambda' = \bar{h}_0(\bar{q}') = G^{\bar{f}^j}(\bar{p}^j, \bar{q}'^j) \end{cases}$. Assume by contradiction that $\lambda \neq \lambda'$. Up to exchange \bar{q} and \bar{q}' , we can assume $\lambda' > \lambda$. From the monotonicities of the standard Godunov fluxes, we deduce that $\bar{q}'^j < \bar{q}^j$. From the monotonicities of $\bar{h}_0(\uparrow, \dots, \uparrow)$, we deduce that $\lambda' = \bar{h}_0(\bar{q}') \leq \bar{h}_0(\bar{q}) = \lambda$. Contradiction. Hence $\lambda' = \lambda$, and we have uniqueness of the value λ and we call $\bar{h}_1(\bar{p}) := \lambda$. We also define $h_1(p) := \bar{h}_1(\bar{p})$.

Step 2: h_1 satisfies condition (6.23)

It is sufficient to check that \bar{h}_1 satisfies condition (6.23) with h_0 replaced by \bar{h}_0 and $\sigma^j = 1$ for all j .

Step 2.1: continuity of \bar{h}_1

The continuity of \bar{h}_1 follows from the compactness of the set of solutions \bar{q} to (6.29). Indeed consider a sequence $[\bar{a}, \bar{b}] \ni \bar{p}_n \rightarrow \bar{p} \in [\bar{a}, \bar{b}]$ as $n \rightarrow +\infty$, and an associated sequence $\bar{q}_n \in [\bar{a}, \bar{b}]$ such that

$$(6.33) \quad \bar{h}_1(\bar{p}_n) = \bar{h}_0(\bar{q}_n) = G^{\bar{f}^j}(\bar{p}_n^j, \bar{q}_n^j) \quad \text{for all } j$$

If $[\bar{a}, \bar{b}] \subset \mathbb{R}^N$ is compact, then the result follows from the continuity of \bar{h}_0, \bar{f} .

We will use coercivity in assumption (2.2) in order to show compactness of the sequence \bar{q}_n . Then assume by contradiction that $|\bar{q}_n^k| \rightarrow +\infty$ as $n \rightarrow +\infty$ for at least some index k . Notice that

$$G^{\bar{f}^k}(\bar{p}_n^k, \bar{q}_n^k) = \inf_{[\bar{p}_n^k, \bar{q}_n^k]} \bar{f}^k \leq \bar{f}^k(\bar{p}_n^k) \quad \text{and} \quad \bar{h}_0(\bar{q}_n) \geq \bar{f}_-^k(\bar{q}_n) = \inf_{[\bar{q}_n^k, \bar{b}^k]} \bar{f}^k \rightarrow +\infty \quad \text{if} \quad \bar{q}_n^k \rightarrow +\infty$$

and this leads to a contradiction with equality (6.33). Therefore we deduce that $\bar{q}_n^k \rightarrow -\infty$, and then $\bar{h}_0(\bar{q}_n) = G^{\bar{f}^k}(\bar{p}_n^k, \bar{q}_n^k) = \sup_{[\bar{q}_n^k, \bar{p}_n^k]} \bar{f}^k \rightarrow +\infty$. Now (6.33) implies that $\bar{q}_n^j \rightarrow -\infty$ for all $j = 1, \dots, N$.

Consider any $\bar{c} \in [\bar{a}, \bar{b}]$. The monotonicity of \bar{h}_0 then implies that $\bar{h}_0(\bar{c}) \geq \bar{h}_0(\bar{q}_n) \rightarrow +\infty$. Contradiction. Therefore we conclude that \bar{q}_n stays bounded, and we can extract a convergent subsequence that we still denote $(\bar{q}_n)_n$ such that $\bar{q}_n \rightarrow \bar{q}$. Using the continuities of \bar{h}, \bar{f} , we can pass to the limit in (6.33) which gives $\bar{h}_1(\bar{p}_n) \rightarrow \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) = \bar{h}_1(\bar{p})$ for all j , which shows the continuity of \bar{h}_1 .

Step 2.2: checking other properties of \bar{h}_1

Let us now show the monotonicities of \bar{h}_1 . We write $\bar{h}_1(\bar{p}) = \lambda = \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j)$, $j = 1, \dots, N$, and consider \bar{p}' with $\bar{p}'^1 = \bar{p}^1$ and $\bar{p}'^j \geq \bar{p}^j$, and $\bar{h}_1(\bar{p}') = \lambda' = \bar{h}_0(\bar{q}') = G^{\bar{f}^j}(\bar{p}'^j, \bar{q}'^j)$.

Assume by contradiction that $\lambda' < \lambda$. Then the monotonicities of the standard Godunov fluxes imply $\bar{q}'^j > \bar{q}^j$ and the monotonicity of \bar{h}_0 implies $\lambda' = \bar{h}_0(\bar{q}') \geq \bar{h}_0(\bar{q}) = \lambda$. Contradiction. We deduce that $\lambda' \geq \lambda$, and then \bar{h}_1 is nonincreasing in \bar{p}^1 . Similarly, we show that \bar{h}_1 is nondecreasing in each coordinate \bar{p}^j .

We now check that \bar{h}_1 satisfies the bounds given in the second line of (2.14) for h_0 replaced by \bar{h}_0 and $\sigma^j = 1$ for all j . Recall that \bar{h}_0 satisfies (6.28), i.e.

$$\bar{f}_-^j(\bar{q}) = G^{\bar{f}^j}(\bar{q}^j, \bar{b}^j) \leq \bar{h}_0(\bar{q}) \leq \bar{f}_+^j(\bar{q}) = G^{\bar{f}^j}(\bar{q}^j, \bar{a}^j) \quad \text{for all } j$$

Then for $\bar{h}_1(\bar{p}) = \lambda = \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j)$ for all j , we get

$$\bar{f}_-^j(\bar{p}^j) = G^{\bar{f}^j}(\bar{p}^j, \bar{b}^j) \leq \lambda = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) \leq G^{\bar{f}^j}(\bar{p}^j, \bar{a}^j) = \bar{f}_+^j(\bar{p}^j)$$

which shows exactly that \bar{h}_1 satisfies (6.27) which are the bounds given in the second line of (2.14).

Step 3: local constancy of $f_1 := (h_1, \dots, h_1)$

Again, we show it for $\bar{f}_1 := (\bar{h}_1, \dots, \bar{h}_1)$ with respect to \bar{f} and this gives the result for $f_1 = (h_1, \dots, h_1)$ with respect to f . Assume that $\bar{h}_1(\bar{p}) \neq \bar{f}^j(\bar{p})$ for all $j \in I \subset \{1, \dots, N\}$, and recall the argument of the proof of Proposition 3.10, in Step 2. We have here

$$\bar{f}^j(\bar{p}) \neq \bar{h}_1(\bar{p}) = \bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}^j, \bar{q}^j) = \begin{cases} \sup_{[\bar{q}^j, \bar{p}^j]} \bar{f}^j = \sup_{[\bar{q}^j, \bar{p}^j]} \bar{f}^j & \text{if } \bar{q}^j < \bar{p}^j \\ \inf_{[\bar{p}^j, \bar{q}^j]} \bar{f}^j = \inf_{[\bar{p}^j, \bar{q}^j]} \bar{f}^j & \text{if } \bar{q}^j > \bar{p}^j \end{cases}$$

where we have used the fact that the inf / sup can not be reached at \bar{p}^j , because $\bar{f}^j(\bar{p}^j) \neq \bar{h}_1(\bar{p})$. Notice also that we can not have $\bar{q}^j = \bar{p}^j$, otherwise we would get $\bar{h}_1(\bar{p}) = \bar{h}_0(\bar{p}) = G^{\bar{f}^j}(\bar{p}^j, \bar{p}^j) = \bar{f}^j(\bar{p}^j) = \bar{f}^j(\bar{p})$, which

is impossible by assumption. This shows now that for \bar{p}_ε close to \bar{p} in $\left(\bar{p} + \sum_{j \in I} \mathbb{R}e_j\right) \cap [\bar{a}, \bar{b}]$, we also have

(by continuity of \bar{f}, \bar{h}_0)

$$\bar{h}_0(\bar{q}) = G^{\bar{f}^j}(\bar{p}_\varepsilon^j, \bar{q}^j) \begin{cases} = \begin{cases} \sup_{[\bar{q}^j, \bar{p}_\varepsilon^j]} \bar{f}^j = \sup_{[\bar{q}^j, \bar{p}_\varepsilon^j]} \bar{f}^j & \text{if } \bar{q}^j < \bar{p}_\varepsilon^j \\ \inf_{[\bar{p}_\varepsilon^j, \bar{q}^j]} \bar{f}^j = \inf_{[\bar{p}_\varepsilon^j, \bar{q}^j]} \bar{f}^j & \text{if } \bar{q}^j > \bar{p}_\varepsilon^j \end{cases} & \text{for all } j \in I \\ = \bar{h}_1(\bar{p}) \quad \text{because } \bar{p}_\varepsilon^j = \bar{p}^j & \text{for all } j \notin I \end{cases}$$

This justifies a posteriori that we can choose $\bar{q}_\varepsilon := \bar{q}$ in $\bar{h}_1(\bar{p}_\varepsilon) = \bar{h}_0(\bar{q}_\varepsilon) = G^{\bar{f}^j}(\bar{p}_\varepsilon^j, \bar{q}_\varepsilon^j)$. Therefore $\bar{f}^j(\bar{p}_\varepsilon) \neq \bar{h}_1(\bar{p}_\varepsilon) = \bar{h}_0(\bar{q}) = \bar{h}_1(\bar{p})$ for all $j \in I$. Hence \bar{f}_1 is locally constant on $\{\bar{f}_1 \neq \bar{f}\}$.

This ends the proof of the lemma. Then we have the following result.

Proposition 6.9 (Properties of HJ germs)

Assume (2.2) with $N \geq 1$. Let \mathcal{G} be a generalized Riemann germ which is a HJ germ.

i) (Regularity properties of \hat{f})

Then \mathcal{G} is a Riemann germ, i.e. $\hat{f} = \hat{f}_{\mathcal{G}}$ is continuous, and \hat{f} is locally Lipschitz continuous on $[a, b]$.

ii) (HJ-relaxation formula)

Moreover there exists some continuous function $\hat{h} : [a, b] \rightarrow \mathbb{R}$ such that for all index j , we have $\hat{f}^j = \hat{h}$. And the function \hat{h} satisfies the following HJ-relaxation formula

$$(6.34) \quad \text{for all } p \in [a, b], \text{ there exists } q \in [a, b] \text{ s.t. } \hat{h}(p) = \hat{h}(q) = \begin{cases} G^{f^j}(p^j, q^j) & \text{if } J^j \simeq (-\infty, 0) \\ G^{f^j}(q^j, p^j) & \text{if } J^j \simeq (0, +\infty) \end{cases}$$

Proof of Proposition 6.9

Step 1: proof of i)

By assumption, we know that there exists $h : \mathcal{G} \rightarrow \mathbb{R}$ such that $f = (h, \dots, h)$ on \mathcal{G} . Define $\hat{h} := h \circ \pi_{\mathcal{G}} : [a, b] \rightarrow \mathbb{R}$. This implies that $\hat{f} = (\hat{h}, \dots, \hat{h}) : [a, b] \rightarrow \mathbb{R}^N$. Up to apply a reversion transform, we can assume that we work on $N : 0$ junction. From (4.14), we deduce that for each $j \in \{1, \dots, N\}$, the map $p \mapsto \hat{h}(p)$ is locally Lipschitz continuous in p^j , uniformly in p^k for $k \neq j$. Because this is true for any index j , we deduce that \hat{h} and then \hat{f} is continuous. Moreover, from (4.14) we have $|\partial_{p^j} \hat{h}(p)| \leq |(f^j)'(p^j)|$ for a.e. $p \in [a, b]$.

Step 2: proof of ii)

From Step 1, we know that \hat{f} is locally constant on $\{\hat{f} \neq f\}$ and $\mathcal{G} = \{\hat{f} = f\}$. Hence any $\hat{p} \in \mathcal{G}$ satisfies $\hat{h}(\hat{p}) = G^{f^j}(\hat{p}^j, \hat{p}^j)$ for all $j = 1, \dots, N$. Therefore, setting $h_0 := \hat{h}$, we see that the function h_1 given by relaxation formula (6.25) satisfies

$$h_1(\hat{p}) = h_0(q) = G^{f^j}(\hat{p}^j, q^j) = \hat{h}(\hat{p}) \quad \text{with } q = \hat{p}$$

i.e. $h_1 = \hat{h}$ on \mathcal{G} . Therefore $f_1 = (h_1, \dots, h_1)$ satisfies $f_1 = f$ on \mathcal{G} , and from Lemma 6.7, we know that f_1 is locally constant on $\{f_1 \neq f\}$ and also satisfies (2.14). Therefore ii) of Theorem 2.14 shows that $f_1 = \hat{f}_{\mathcal{G}} = \hat{f}$. Moreover we have $(\hat{h}, \dots, \hat{h}) = \hat{f} = f_1 = (h_1, \dots, h_1)$ with $h_1 = \mathfrak{R}h_0 = \mathfrak{R}\hat{h}$, where $\mathfrak{R}h_0$ is defined in (6.26). This shows that $\hat{h} = \mathfrak{R}\hat{h}$ which means exactly (6.34). This ends the proof of the lemma.

Proof of Theorem 2.25

For the proof, we refer to the table of Subsection 2.4. Theorem 2.25 follows from Proposition 6.9, and the two last lines of (2.18) follow from ii) of Theorem 2.14. This ends the proof of the theorem.

6.6 Theorem 2.28 and its proof: HJ germ \mathcal{G} determined by $\chi\mathcal{G}$

We now start with the following result.

Lemma 6.10 (Key reduction for HJ germs)

Assume (2.2) with $N \geq 1$. Let \mathcal{G} be a generalized Riemann germ which is a HJ germ with $\hat{f}_{\mathcal{G}} = (\hat{h}, \dots, \hat{h})$. We consider any $y = (y^1, \dots, y^N)$ with $y^j \in J^j$ for all j . Let $p \in [a, b]$ be such that there exists $\lambda \in \mathbb{R}$ with

$$(6.35) \quad f^j(p^j) = \lambda \quad \text{for all } j = 1, \dots, N$$

Assume the following

$$(6.36) \quad \text{for all } q \in \underline{\chi}\mathcal{G}, \quad (q \diamond y \geq p \diamond y \implies \hat{h}(q) \leq \lambda)$$

and

$$(6.37) \quad \text{for all } q \in \overline{\chi}\mathcal{G}, \quad (q \diamond y \leq p \diamond y \implies \hat{h}(q) \geq \lambda)$$

with sub/super characteristic sets $\underline{\chi}\mathcal{G}$ and $\overline{\chi}\mathcal{G}$ given in Definition 2.27 and Hadamard product \diamond given in Definition 2.10. Then

$$\lambda = \hat{h}(p)$$

Proof of Lemma 6.10

Let $p \in [a, b]$ satisfying (6.35) and set

$$v^j(t, x) = -\lambda t + p^j x, \quad j = 1, \dots, N$$

Then v^j is a viscosity solution of $v_t^j + f^j(v_x^j) = 0$ on $\mathbb{R} \times J^j$, $j = 1, \dots, N$. We claim that $v = (v^1, \dots, v^N)$ is a \hat{h} -viscosity solution on $\{x = 0\}$ if $v^0(t, 0) := v^j(t, 0)$ satisfies

$$(6.38) \quad v_t^0 + \hat{h}(v_x^1, \dots, v_x^N)(t, 0) = 0 \quad \text{for all } t \in \mathbb{R}$$

and we show it in the next steps. To this end, for $q \in [a, b]$, we set $w = (w^1, \dots, w^N)$ with

$$w^j(t, x) := -\lambda t + q^j x, \quad j = 1, \dots, N$$

Step 1: checking that v is a \hat{h} -viscosity subsolution on $\{x = 0\}$

Notice that for all $q \in [a, b]$, inequality $q \diamond y \geq p \diamond y$ is equivalent to

$$(6.39) \quad w \geq v \quad \text{on } \mathbb{R} \times J \quad \text{with equality on } \mathbb{R} \times \{0\}$$

Set $I := \{j \in \{1, \dots, N\}, \sigma^j = 1\}$. Now from Lemma 9.4 (and using I -reversion composed with I -inversion), we get that v is a \hat{h} -viscosity subsolution on $\{x = 0\}$ if and only if for all $q \in \underline{\chi}\mathcal{G}$, relation (6.39) implies the subsolution viscosity inequality $-\lambda + \hat{h}(q) \leq 0$. In other words, condition (6.36) is equivalent to the fact that v is a \hat{h} -viscosity subsolution on $\{x = 0\}$.

Step 2: checking that v is a \hat{h} -viscosity supersolution on $\{x = 0\}$

Similarly notice that for all $q \in [a, b]$, inequality $q \diamond y \leq p \diamond y$ is equivalent to

$$(6.40) \quad w \leq v \quad \text{on } \mathbb{R} \times J \quad \text{with equality on } \mathbb{R} \times \{0\}$$

Now from Lemma 9.4 (and using I -reversion composed with I -inversion), we get that v is a \hat{h} -viscosity supersolution on $\{x = 0\}$ if and only if for all $q \in \overline{\chi}\mathcal{G}$, relation (6.40) implies the supersolution viscosity inequality $-\lambda + \hat{h}(q) \geq 0$. In other words, condition (6.37) is equivalent to the fact that v is a \hat{h} -viscosity supersolution on $\{x = 0\}$.

Step 3: conclusion

We conclude that v is both a \hat{h} -viscosity subsolution and supersolution on $\{x = 0\}$. Therefore v is a \hat{h} -viscosity solution and satisfies (6.38), i.e. $-\lambda + \hat{h}(p) = 0$, which ends the proof of the lemma.

Proof of Theorem 2.28

Let $\hat{f}_{\mathcal{G}} = (\hat{h}, \dots, \hat{h})$ and $\hat{f}_{\mathcal{G}_0} = (\hat{h}_0, \dots, \hat{h}_0)$ be the associated Godunov fluxes to the germs \mathcal{G} and \mathcal{G}_0 . Let $p \in \mathcal{G}_0 \supset \underline{\chi}\mathcal{G}$ and let us show that $p \in \mathcal{G}$. We set $\lambda := \hat{h}_0(p) = f^j(p^j)$.

Step 1: proof of (6.36)

Let q be such that

$$(6.41) \quad q \in \underline{\chi}\mathcal{G} \quad \text{and} \quad q \diamond y \geq p \diamond y$$

Because $\underline{\chi}\mathcal{G} \subset \mathcal{G}_0$, we deduce that $q \in \mathcal{G}_0$ and then $\hat{h}(q) = f^j(q) = \hat{h}_0(q) \leq \hat{h}_0(p) = \lambda$, where we have used the monotonicities of \hat{h}_0 and inequality in (6.41). This shows (6.36).

Step 2: proof of (6.37)

The proof is similar to the one of Step 1.

Step 3: conclusion

From Lemma 6.10, we deduce that $f^j(p^j) = \hat{h}_0(p) = \lambda = \hat{h}(p)$. Therefore $p \in \mathcal{G}$. This ends the proof of the lemma.

6.7 Theorem 2.29 and its proof: conservative 1 : 1 germs

The proof of Theorem 2.29 requires the following intermediate result.

Lemma 6.11 (Conservative 1 : 1 junctions)

Assume (2.2) for $N = 2$ and 1 : 1 junctions. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ.

i) (Equivalence of conservative and HJ germs)

Then \mathcal{G} is conservative, i.e. satisfies

$$(6.42) \quad f^1 = f^2 \quad \text{on} \quad \mathcal{G}$$

if and only if it is a HJ germ.

ii) (Further properties of the germ)

When \mathcal{G} is conservative, then \mathcal{G} is also a Riemann germ which is at the same time a Kruřkov germ, a HJ germ and a monotone germ.

Proof of Lemma 6.11

Point i) follows from the definitions. We now focus on the proof of point ii).

Relation (6.42) shows that \mathcal{G} is a HJ germ, and then a Riemann germ from Proposition 6.9. Defining $h : \mathcal{G} \rightarrow \mathbb{R}$ by $h(p) := f^1(p) = f^2(p)$ for $p \in \mathcal{G}$, and setting $\hat{h} := h \circ \pi_{\mathcal{G}}$, we see that $\hat{f} = (\hat{f}^1, \hat{f}^2) = (\hat{h}, \hat{h})$ with monotonicities $\hat{h}(\uparrow, \downarrow)$. This shows that \mathcal{G} is a monotone germ.

We now set for $\sigma^1 = 1 = -\sigma^2$

$$D^{\hat{f}}(p, q) := \sum_{k=1,2} D^{\hat{f}^k}(p, q) = \text{IN} - \text{OUT} \quad \text{with} \quad D^{\hat{f}^k}(p, q) := \sigma^k \cdot \text{sign}(p^k - q^k) \cdot \left\{ \hat{f}^k(p) - \hat{f}^k(q) \right\}$$

Hence we have $D^{\hat{f}}(p, q) = \left\{ \text{sign}(p^1 - q^1) - \text{sign}(p^2 - q^2) \right\} \cdot \left\{ \hat{h}(p) - \hat{h}(q) \right\}$ and the monotonicities of \hat{h} show that $D^{\hat{f}} \geq 0$ on $[a, b]^2$. Because $D^{\hat{f}}$ and D^f coincide on $\mathcal{G} \times \mathcal{G}$, we deduce that $D^f \geq 0$ on $\mathcal{G} \times \mathcal{G}$. Therefore \mathcal{G} is a Kruřkov germ. This ends the proof of the lemma.

Proof of Theorem 2.29

For the proof, we refer to the table of Subsection 2.4. The result follows from Lemma 6.11, from ii) of Proposition 6.9 for the relaxation formula and from Theorem 2.28 for the characterization of the germ by $\chi\mathcal{G}$. Notice that inequalities (2.22) then follow automatically from ii) of Theorem 2.14. The independent proof of relation (2.20) (which gives explicitly a way to recover the full germ from its characteristic subset) is postponed to Lemma 7.17. This ends the proof of the theorem.

6.8 Proposition 2.30 and its proof: germ product property for conservative lines

Proof of Proposition 2.30

We do the proof by recurrence on $n \geq 1$.

Step 1: case $n = 1$

Then we can apply Theorem 2.29 to get that $\mathcal{G} = \mathcal{G}^1$ is a Riemann germ which is Kruřkov, HJ and monotone. From [14] (and also Theorem 2.28), we know that all HJ germs with all convex fluxes (and then also with all concave fluxes) are classified by a flux limiter $A^j \leq A_*^j := \min_{\alpha=L,R} \max f^{j\alpha}$ for $j = 1$ with

$$\mathcal{G}^j := \mathcal{G}_{A^j}^j := \left\{ (p^{jL}, p^{jR}) \in Q^j, \quad \min \left\{ A^j, f^{1L,+}(p^{jL}), f^{jR,-}(p^{jR}) \right\} = f^{jL}(p^{jL}) = f^{jR}(p^{jR}) \right\}$$

with $Q^j := [a^{jL}, b^{jL}] \times [a^{jR}, b^{jR}]$ and

$$f^{j\alpha,+}(z) := \begin{cases} f^{j\alpha}(z) & \text{for } z \in [a^{j\alpha}, c^{j\alpha}] \\ f^{j\alpha}(c^{j\alpha}) & \text{for } z \in [c^{j\alpha}, b^{j\alpha}] \end{cases} \quad \text{and} \quad f^{j\alpha,-}(z) := \begin{cases} f^{j\alpha}(c^{j\alpha}) & \text{for } z \in [a^{j\alpha}, c^{j\alpha}] \\ f^{j\alpha}(z) & \text{for } z \in [c^{j\alpha}, b^{j\alpha}] \end{cases}$$

Moreover, for $A^j \leq A_*^j$, let us denote $U^j(A^j) = (u^{jL}, u^{jR}) \in Q^j$ such that $f^{j\alpha}(u^{j\alpha}) = A^j$, $\alpha = L, R$, and

$$U_*^j(A^j) := (u_-^{jL}(A^j), u_+^{jR}(A^j)) \quad \text{with} \quad \chi\mathcal{G}_{A^j}^j = \{U_*^j(A^j)\}$$

where the characteristic subset $\chi\mathcal{G}^j$ of the HJ germ \mathcal{G}^j is defined in Definition 2.27 and with $u_{\pm}^{j\alpha}(A^j) \in [a^{j\alpha}, b^{j\alpha}]$ defined by

$$f^{j\alpha,\pm}(u_{\pm}^{j\alpha}(A^j)) = f^{j\alpha}(u_{\pm}^{j\alpha}(A^j)) = A^j$$

Because $BA(U_*^j(A^j)) = \left((u_+^{jL}(A^j), +\infty) \times (-\infty, u_-^{jR}(A^j)) \right) \cap Q^j$, we see that if $U_*^j(B) \in \mathcal{G}_{A^j}^j$ then $U_*^j(B) \in \mathcal{G}_{A^j}^j \cap BA(U_*^j(A^j))$ and then (using for instance Theorem 2.28)

$$U_*^j(B) \in \mathcal{G}_{A^j}^j \implies B = A^j$$

Step 2: from $n \geq 1$ to $n + 1$

Step 2.1: first property of maximal A

Assume that the result is true up to level n , and let us show that it is true at the level $n + 1$. Now consider any $A = (A^1, \dots, A^{n+1}) \in \prod_{j=1, \dots, n+1} (-\infty, A_*^j]$ such that there exists $U^A := (U^1, \dots, U^{n+1}) \in \mathcal{G}$ with $U^j = U^j(A^j)$ for $j = 1, \dots, n + 1$, and which is maximal in the following sense:

$$\text{there exist no } \bar{A} \geq A \text{ with } \bar{A} \neq A \text{ such that } U^{\bar{A}} \in \mathcal{G}$$

Now for $U' := (U^1(A^1), \dots, U^n(A^n)) \in \mathbb{R}^{2n}$, consider the sliced germ (see the slicing Lemma 4.12)

$$\mathcal{G}_{U'} \subset Q^j \cap \{f^{jL}(p^{jL}) = f^{jR}(p^{jR})\} \quad \text{for } j := n + 1$$

From Step 1, we know that there exists $A_0^{n+1} \leq A_*^{n+1}$ such that $U^{n+1}(A^{n+1}) \in \mathcal{G}_{U'} = \mathcal{G}_{A_0^{n+1}}^{n+1}$. Because A is maximal, this implies that $A^{n+1} = A_0^{n+1}$, and then we can assume (without loss of generality) that $U^{n+1}(A^{n+1}) = U_*^{n+1}(A^{n+1}) =: U_*^{n+1}$.

Conversely, the sliced germ satisfies $U' \in \mathcal{G}_{U_*^{n+1}} = \prod_{j=1, \dots, n} \mathcal{G}_{A_0^j}^j$ for some $A_0^j \leq A_*^j$ for $j = 1, \dots, n$. Because (A^1, \dots, A^n) is also maximal, we deduce that $A^n = A_0^n$, and by recurrence $(A^1, \dots, A^n) = (A_0^1, \dots, A_0^n)$. Therefore, we conclude that

$$A = (A^1, \dots, A^{n+1}) \text{ maximal } \left. \begin{array}{l} U^A \in \mathcal{G} \\ U^A \text{ maximal} \end{array} \right\} \implies \left\{ \begin{array}{l} \tilde{U} = (U', U_*^{n+1}) \in \mathcal{G}, \quad \text{for } U^A =: (U', U^{n+1}) \quad \text{and} \quad U_*^{n+1} := U_*^{n+1}(A^{n+1}) \\ \mathcal{G}_{U_*^{n+1}} = \prod_{j=1, \dots, n} \mathcal{G}_{A_0^j}^j \end{array} \right.$$

and then

$$A = (A^1, \dots, A^{n+1}) \text{ maximal} \implies U_*^A \in \mathcal{G}, \quad \text{with } U_*^A := (U_*^1, \dots, U_*^{n+1}) \quad \text{and} \quad U_*^j := U_*^j(A^j) \quad \text{for all } j$$

Step 2.2: uniqueness of the maximal A

Now consider A and \bar{A} maximal. Then $U_*^A, U_*^{\bar{A}} \in \mathcal{G}$ and $\pi_{\mathcal{G}} : \emptyset \neq BA(U_*^A) \cap BA(U_*^{\bar{A}}) \rightarrow \{U_*^A\} = \{U_*^{\bar{A}}\}$ implies $\bar{A} = A$. Therefore any maximal A (which always exists) is then unique, and let us call it A_0 . We have in particular $U_*^{A_0} \in \mathcal{G}$.

Step 2.3: going further for component $n + 1$

We want to show that $\mathcal{G} \subset \prod_{j=1, \dots, n+1} \mathcal{G}_{A_0^j}^j$. Consider any $U^A \in \mathcal{G}$ with A not necessarily maximal. In particular, we have then $A \leq A_0$. We write $U^A = (U', U^{n+1})$ with $U^{n+1} = U^{n+1}(A^{n+1})$.

Case 1: $A^{n+1} < A_0^{n+1}$

We want to show that $U^{n+1} \neq U_*^{n+1}(A^{n+1})$. Assume by contradiction that

$$(6.43) \quad U^{n+1} = U_*^{n+1}(A^{n+1})$$

Then, up to increase A^1, \dots, A^n , we can assume that (A^1, \dots, A^n) is maximal for the sliced germ $\mathcal{G}_{U_*^{n+1}}$. Hence Step 2.1 shows that $(U_*^1(A^1), \dots, U_*^n(A^n)) \in \mathcal{G}_{U_*^{n+1}}$, and then $U_*^A = (U_*^1(A^1), \dots, U_*^{n+1}(A^{n+1})) \in \mathcal{G}$. Then the argument of Step 2.2 shows that $A = A_0$. Contradiction with $A^{n+1} < A_0^{n+1}$. Therefore (6.43) is false, and we get $U^{n+1} \in \mathcal{G}_{A_0^{n+1}}^{n+1} \setminus \{U_*^{n+1}(A_0^{n+1})\}$.

Case 2: $A^{n+1} = A_0^{n+1}$

Then $U^{n+1} \in \mathcal{G}_{A_0^{n+1}}^{n+1}$.

Conclusion

We deduce that in all cases, we have: $U = (U^1, \dots, U^{n+1}) \in \mathcal{G} \implies U^{n+1} \in \mathcal{G}_{A_0^{n+1}}^{n+1}$.

Step 2.4: final step

Repeating Step 2.3 for all components $j = 1, \dots, n + 1$, we conclude that

$$U = (U^1, \dots, U^{n+1}) \in \mathcal{G} \implies U^j \in \mathcal{G}_{A_0^j}^j \quad \text{for all } j = 1, \dots, n + 1$$

i.e. $\mathcal{G} \subset \prod_{j=1, \dots, n+1} \mathcal{G}_{A_0^j}^j$. Because both sides are (complete) generalized Riemann germs, we deduce that we have equality. This ends the proof of the lemma.

Remark 6.12 Notice that from [14], the result of Proposition 2.30 stays true if strict concavity of each $f^{j\alpha}$ is replaced by the fact that continuous $f^{j\alpha} : [a^{j\alpha}, b^{j\alpha}] \rightarrow \mathbb{R}$ is increasing on $(a^{j\alpha}, c^{j\alpha})$ and decreasing on $(c^{j\alpha}, b^{j\alpha})$.

7 Complementary results

7.1 More on Riemann monotonicity

Our goal is now to show that for locally Lipschitz continuous maps $h : \mathbb{R}^N \rightarrow \mathbb{R}^N$ which are Riemann monotone, their Jacobian matrices have a particular monotonicity property, called P_0 -monotonicity. In order to describe this property, it is useful to focus first on the linear case $h(x) = Mx$, and to transfer our notions of monotonicity for the functions $h = \hat{f}_{\mathcal{G}}$, to the square matrices M .

Definition 7.1 (P_0 -monotone, Riemann monotone, Kruřkov monotone)

A principal minor of a $n \times n$ matrix $M = (M_{ij})_{i,j \in \{1, \dots, n\}}$ is the determinant of a submatrix $M_I := (M_{ij})_{i,j \in I}$ for a subset $\emptyset \neq I \subset \{1, \dots, n\}$, obtained by depletion of columns and of lines with the same labels. Let M be a real $n \times n$ matrix.

i) (P_0 -monotone)

A matrix M is said to be a P_0 -monotone, if all its principal minors are nonnegative.

ii) (Riemann monotone)

A matrix M is said to be a Riemann monotone, if for all $x \in \mathbb{R}^n$, it satisfies

$$x \diamond Mx \leq 0 \implies Mx = 0$$

iii) (Kruřkov monotone)

A matrix M is said to be Kruřkov monotone if it satisfies

$$M_{jj} \geq \sum_{i \in \{1, \dots, n\} \setminus \{j\}} |M_{ij}| \quad \text{for all } j = 1, \dots, n$$

(i.e. if M is (non-negatively) diagonally column-dominant).

Remark 7.2 We refer the reader to the book of JOHNSON, SMITH, TSATSOMEROS [15], for the properties of P_0 -(monotone) matrices.

Then we have the following result.

Lemma 7.3 (Hierarchy of monotonicities)

For a real $n \times n$ matrix M , we have the following hierarchy

$$\text{i) } M \text{ is Kruřkov monotone} \implies \text{ii) } M \text{ is Riemann monotone} \implies \text{iii) } M \text{ is } P_0\text{-monotone}$$

Proof of Lemma 7.3

Step 1: i) implies ii)

Assume that M is Kruřkov monotone in the sense of Definition 7.1, and consider the linear map $h(x) := Mx$. Then ii) of Proposition 6.4 shows that $D^h \geq 0$. Then Proposition 5.9 implies that h is Riemann monotone. Because h is linear, this means that M is Riemann monotone in the sense of Definition 7.1.

Step 2: ii) implies iii)

We only show that the determinant of the full matrix M is nonnegative, and the argument is the same for

all principal submatrices. The argument of the proof of Lemma 5.3 shows that the linear map of matrix $M + \varepsilon id_{\mathbb{R}^n}$ is injective for all $\varepsilon > 0$. For large $\varepsilon > 0$, the determinant is then positive, hence by continuity in $\varepsilon > 0$, we deduce that $\det(M + \varepsilon id_{\mathbb{R}^n}) > 0$. In the limit $\varepsilon \rightarrow 0^+$, this shows that $\det(M) \geq 0$. This ends the proof of the lemma.

Then we have the following result.

Proposition 7.4 (Jacobian of Riemann monotone maps)

Assume (2.2) for $N \geq 1$. Assume that $h : [a, b] \rightarrow \mathbb{R}^N$ is Riemann monotone and locally Lipschitz continuous. Then the Jacobian matrix $(\partial_j h^i)_{i,j \in \{1, \dots, N\}}$ is a P_0 -monotone matrix a.e. on $[a, b]$.

Proof of Proposition 7.4

The proof follows closely the proof for matrices (see Step 2 of the proof of Lemma 7.3).

We only show that the determinant of the full Jacobian matrix is nonnegative, and the argument is the same for all principal submatrices. Assume by contradiction that there exists a Lebesgue point $p_0 \in [a, b]$ such that

$$(7.1) \quad \det(A) < 0 \quad \text{with} \quad A := Dh(p_0)$$

Then Lemma 5.3 shows that $h + \varepsilon id_{\mathbb{R}^n}$ is injective for all $\varepsilon > 0$. For large $\varepsilon > 0$, the determinant $\det(A + \varepsilon id_{\mathbb{R}^n})$ is positive, hence by continuity in $\varepsilon > 0$, we deduce that $\det(A + \varepsilon id_{\mathbb{R}^n}) \geq 0$ for all $\varepsilon > 0$. In the limit $\varepsilon \rightarrow 0^+$, this gives a contradiction with (7.1). Therefore (7.1) is false, and we conclude that $\det(Dh) \geq 0$ a.e. on $[a, b]$. This ends the proof of the proposition.

The following result shows that P_0 -monotonicity is indeed very close to Riemann monotonicity.

Lemma 7.5 (A property of P_0 -monotone matrices)

Let M be a P_0 -monotone $n \times n$ matrix. Then for all $\varepsilon > 0$, the matrix $M^\varepsilon := M + \varepsilon Id$ is Riemann monotone.

Proof of Lemma 7.5

Consider some $x \in \mathbb{R}^n$ such that $x \diamond M^\varepsilon x \leq 0$. We now use an argument introduced in FIEDLER AND PTÁK [10], in the proof of their Theorem 1.3. We deduce that there exists some diagonal matrix $\Delta \geq 0$, with nonnegative diagonal such that $M^\varepsilon x = -\Delta x$. Let us denote $\Delta^\varepsilon := \Delta + \varepsilon Id$, and M_I is the principal submatrix of M of indices I , and similarly for Δ_I^ε the submatrix of Δ^ε of indices $\bar{I} := \{1, \dots, N\} \setminus I$. If $x \neq 0$, then x is a 0-eigenvector of the matrix $M + \Delta^\varepsilon$, and using the fact that M is P_0 -monotone, we get

$$0 = \det(M + \Delta^\varepsilon) = \sum_{I \subset \{1, \dots, N\}} \det(M_I) \det(\Delta_I^\varepsilon) \geq \det(\Delta^\varepsilon) \geq \varepsilon^n > 0$$

Contradiction. Hence we deduce that $x = 0$ and then $M^\varepsilon x = 0$. Hence we have shown that $x \diamond M^\varepsilon x \leq 0$ implies $M^\varepsilon x = 0$, i.e. that M^ε is Riemann monotone. This ends the proof of the lemma.

We finish this subsection with two counter-examples, which show that we can not hope the Jacobian matrix of Riemann monotone maps, to be Riemann monotone everywhere.

Lemma 7.6 (Counter-example for matrices)

For $n = 2$, consider $A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then A is P_0 -monotone, but is not Riemann monotone.

Proof of Lemma 7.6

Consider $p := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we have $p \diamond Ap = 0_{\mathbb{R}^2}$ and $Ap \neq 0_{\mathbb{R}^2}$. This ends the proof of the lemma.

Lemma 7.7 (Counter-example for maps)

For $n = 2$ and $x, y \in \mathbb{R}$, consider the map $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$h = (h^x, h^y) \quad \text{with} \quad \begin{cases} h^x(x, y) = x^3 + y \\ h^y(x, y) = -x + y^3 \end{cases}$$

Then the map h is Riemann monotone, but its Jacobian matrix $Dh(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not Riemann monotone as a matrix (i.e. in the sense of Definition 7.1).

Proof of Lemma 7.7

For $p = (x, y)$ and $\bar{p} := (\bar{x}, \bar{y})$, we see that $(\bar{p} - p) \diamond [h]_p^{\bar{p}} \leq 0$ means

$$\begin{cases} (\bar{x} - x) \cdot \{(\bar{x}^3 - x^3) + (\bar{y} - y)\} \leq 0 \\ (\bar{y} - y) \cdot \{-(\bar{x} - x) + (\bar{y}^3 - y^3)\} \leq 0 \end{cases}$$

Taking the sum, we get $0 \leq (\bar{x} - x)(\bar{x}^3 - x^3) + (\bar{y} - y)(\bar{y}^3 - y^3) \leq 0$, which implies $\bar{x} - x = 0 = \bar{y} - y$, i.e. $\bar{p} = p$ and then $[h]_p^{\bar{p}} = 0$. Hence $(\bar{p} - p) \diamond [h]_p^{\bar{p}} \leq 0$ implies $[h]_p^{\bar{p}} = 0$, which shows that h is Riemann monotone. This ends the proof of the lemma.

7.2 Adding an $(N + 1)$ -th branch to get conservative germs

In what follows, RH refers to the Rankine-Hugoniot relation.

Lemma 7.8 (Adding an $(N + 1)$ -th branch)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set. Define

$$(RH^f)(p) := \sum_{j=1, \dots, N} \sigma^j f^j(p^j)$$

Consider the smallest one-dimensional box $I_{N+1} := [a_{N+1}, b_{N+1}] \subset \mathbb{R}$ such that

$$I_{N+1} \supset -RH^f(\mathcal{G})$$

Let us consider any locally Lipschitz continuous decreasing bijective function $f^{N+1} : I_{N+1} \rightarrow I_{N+1}$. We then define

$$(7.2) \quad \tilde{f} := (f, f^{N+1}), \quad \tilde{a} := (a, a_{N+1}), \quad \tilde{b} := (b, b_{N+1}), \quad \tilde{J} := J \cup J^{N+1} \quad \text{with} \quad J^{N+1} \simeq (-\infty, 0), \quad \sigma^{N+1} = 1$$

and the set

$$(7.3) \quad \tilde{\mathcal{G}} := \left\{ \tilde{p} := (p, p^{N+1}) \in [\tilde{a}, \tilde{b}], \quad p^{N+1} := (f^{N+1})^{-1}(-RH^f(p)) \quad \text{with} \quad p \in \mathcal{G} \right\}$$

i) (Construction of a conservative germ $\tilde{\mathcal{G}}$)

Then $\tilde{\mathcal{G}} \subset [\tilde{a}, \tilde{b}]$ is a generalized Riemann germ with respect to (\tilde{J}, \tilde{f}) if and only if \mathcal{G} is a generalized Riemann germ with respect to (J, f) . When it is the case, then $\tilde{\mathcal{G}}$ is moreover conservative. More precisely if $\hat{f} = \hat{f}_{\mathcal{G}}$ (resp. $\hat{f} := \hat{f}_{\tilde{\mathcal{G}}}$) is the flux associated to the Riemann germ \mathcal{G} (resp. $\tilde{\mathcal{G}}$), then

$$\hat{f} = \hat{f}_{\tilde{\mathcal{G}}} = (\hat{f}, -RH^{\hat{f}}) \quad \text{and} \quad RH^{\hat{f}} = 0 \quad \text{on} \quad [\tilde{a}, \tilde{b}]$$

ii) (Case of Riemann germs)

Then $\tilde{\mathcal{G}} \subset [\tilde{a}, \tilde{b}]$ is a Riemann germ if and only if $\mathcal{G} \subset [a, b]$ is a Riemann germ.

iii) (Case of D_+ -germs)

Then \mathcal{G} is a D_+^f -germ if and only if $\tilde{\mathcal{G}}$ is a $D_+^{\tilde{f}}$ -germ.

iv) (Case of conservative dissipative sets)

Let $\mathcal{G} \subset [a, b] \subset \mathbb{R}^N$ be a set and $\tilde{\mathcal{G}} \subset [\tilde{a}, \tilde{b}] \subset \mathbb{R}^{1+N}$ be the set defined in (7.3). Then

$$(7.4) \quad \begin{cases} D^f \geq 0 & \text{on } \mathcal{G} \times \mathcal{G} \\ RH^f = 0 & \text{on } \mathcal{G} \end{cases}$$

implies

$$(7.5) \quad \begin{cases} D^{\tilde{f}} \geq 0 & \text{on } \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \\ RH^{\tilde{f}} = 0 & \text{on } \tilde{\mathcal{G}} \end{cases}$$

for \tilde{f} defined in (7.2).

Proof of Lemma 7.8

Points i) and ii) of the lemma are straightforward. We now prove iii). Up to apply a suitable reversion transform, we can assume that J is of type $N : 0$.

Step 1: from $D_+^{\hat{f}}$ -germ to D_+^f -germ

Consider $\tilde{p} = (p, p^{N+1}), \tilde{q} = (q, q^{N+1}) \in [\tilde{a}, \tilde{b}]$.

We have

$$D_+^{\hat{f}}(\tilde{p}, \tilde{q}) = D_+^{\hat{f}}(p, q) + \{\text{sign}^+(p^{N+1} - q^{N+1})\} \cdot \left\{ -RH^{\hat{f}}(p) - \left(-RH^{\hat{f}}(q) \right) \right\}$$

with $D_+^{\hat{f}}(p, q) = \sum_{j=1, \dots, N} \{\text{sign}^+(p^j - q^j)\} \cdot [\hat{f}^j]_q^p$. Either $p^{N+1} \leq q^{N+1}$ and then $D_+^{\hat{f}}(\tilde{p}, \tilde{q}) = D_+^{\hat{f}}(p, q)$. Or $p^{N+1} > q^{N+1}$ and (using $1 - \text{sign}^+(x) = -\text{sign}^-(x)$ if $x \neq 0$), we get, if $p^j \neq q^j$ for all $j = 1, \dots, N$, that

$$\begin{aligned} D_+^{\hat{f}}(\tilde{p}, \tilde{q}) &= D_+^{\hat{f}}(p, q) - [RH^{\hat{f}}]_q^p \\ &= \sum_{j=1, \dots, N} \{\text{sign}^-(p^j - q^j)\} \cdot [\hat{f}^j]_q^p \\ &= \sum_{j=1, \dots, N} \{\text{sign}^+(q^j - p^j)\} \cdot [\hat{f}^j]_p^q \\ &= D_+^{\hat{f}}(q, p) \end{aligned}$$

For Ω defined in (4.22), this means $D_+^{\hat{f}}(\tilde{p}, \tilde{q}) = D_+^{\hat{f}}(q, p)$ if $(q, p) \in \Omega$. Hence if $D_+^{\hat{f}} \geq 0$, then $\tilde{\mathcal{G}}$ is Kružkov and then \hat{f} is continuous and f is also continuous. Moreover we have $D_+^{\hat{f}} \geq 0$ on Ω . From Lemma 4.17, we deduce that $D_+^{\hat{f}} \geq 0$ on $\bar{\Omega} = [a, b]^2$. Therefore, if $\tilde{\mathcal{G}}$ is a $D_+^{\hat{f}}$ -germ, then \mathcal{G} is a D_+^f -germ.

Step 2: from D_+^f -germ to $D_+^{\hat{f}}$ -germ

Conversely, if $D_+^f \geq 0$, then

$$0 \leq D_+^{\hat{f}}(\tilde{p}, \tilde{q}) = \begin{cases} D_+^f(p, q) & \text{if } p^{N+1} < q^{N+1} \\ D_+^f(q, p) & \text{if } p^{N+1} > q^{N+1} \end{cases}$$

Hence for

$$\tilde{\Omega} := \left\{ (\tilde{p}, \tilde{q}) \in [\tilde{a}, \tilde{b}], \quad \tilde{p}^j \neq \tilde{q}^j \quad \text{for all } j = 1, \dots, N+1 \right\}$$

we see that $D_+^{\hat{f}} \geq 0$ on $\tilde{\Omega}$. From Lemma 4.17, we deduce that $D_+^{\hat{f}} \geq 0$ on $\bar{\tilde{\Omega}} = [\tilde{a}, \tilde{b}]^2$. Therefore, if \mathcal{G} is a D_+^f -germ, then $\tilde{\mathcal{G}}$ is a $D_+^{\hat{f}}$ -germ.

Step 3: proof of iv)

Assume that the set \mathcal{G} satisfies (7.4). For all $p, q \in \mathcal{G}$, consider $p^{N+1} := (f^{N+1})^{-1}(-RH^f(p)) = (f^{N+1})^{-1}(0)$ and $q^{N+1} := (f^{N+1})^{-1}(-RH^f(q)) = (f^{N+1})^{-1}(0)$. Then for $\tilde{p} = (p, p^{N+1}), \tilde{q} = (q, q^{N+1}) \in \tilde{\mathcal{G}} \subset [\tilde{a}, \tilde{b}]$, we have

$$D^{\hat{f}}(\tilde{p}, \tilde{q}) = D^f(p, q) + \{\text{sign}(p^{N+1} - q^{N+1})\} \cdot \{-RH^f(p) - (-RH^f(q))\} = D^f(p, q) \geq 0$$

From the definition of $\tilde{\mathcal{G}}$, we deduce (7.5).

This ends the proof of the lemma.

7.3 Duality for D_+ -germs

In this section, we study the following notion of duality, in particular useful for D_+ -germs.

Definition 7.9 (Left-dual and right-dual)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set.

Then we define the left-dual of the set \mathcal{G} as

$${}^* \mathcal{G} := \mathcal{G}^* := \left\{ p' \in [a, b], \quad D_+^f(p', p) \geq 0 \quad \text{for all } p \in \mathcal{G} \right\}$$

and its right-dual

$$\mathcal{G}^* := \mathcal{G}^* := \left\{ p' \in [a, b], \quad D_+^f(p, p') \geq 0 \quad \text{for all } p \in \mathcal{G} \right\}$$

where D_+^f is defined in (2.9).

Remark 7.10 (A model case)

Before to start with duality, it is instructive to keep in mind the following model case for $1 : 0$ junctions. For $N = 1$, we consider $f^1(u) = g(u) = u(1 - u)$ on $[0, 1]$, with $A_{max} := \sup_{[0,1]} g = \frac{1}{4} = g(u_0)$ with $u_0 := \frac{1}{2}$. We set the monotone envelopes of g

$$g^+(p) := \begin{cases} g(u) & \text{for } p \in [0, u_0] \\ g(u_0) & \text{for } p \in [u_0, 1] \end{cases} \quad \text{and} \quad g^-(p) := \begin{cases} g(u_0) & \text{for } p \in [0, u_0] \\ g(u) & \text{for } p \in [u_0, 1] \end{cases}$$

and the monotone inverse functions $u_+(\lambda) \leq u_-(\lambda)$ defined by

$$[0, u_0] \ni u_+(\lambda) := \begin{cases} (g_+)^{-1}(\lambda) & \text{for } \lambda \in [0, A_{max}) \\ u_0 & \text{for } \lambda = A_{max} \end{cases} \quad \text{and} \quad [u_0, 1] \ni u_-(\lambda) := \begin{cases} (g_-)^{-1}(\lambda) & \text{for } \lambda \in [0, A_{max}) \\ u_0 & \text{for } \lambda = A_{max} \end{cases}$$

For a $1 : 0$ junction, and for the parameter $A \in [0, A_{max}]$, we can consider the following germ $\mathcal{G}_A \subset [0, 1]$ defined by

$$\mathcal{G}_A := \{p \in [0, 1], \hat{g}_A(p) = g(p)\} \quad \text{with flux} \quad \hat{g}_A(p) := \min\{A, g^+(p)\}$$

Then for $D_+^g(p, q) := \text{sign}^+(p - q) \cdot \{g(p) - g(q)\}$, a direct computation gives

$$\begin{cases} \mathcal{G}_A & = [0, u_+(A)] \cup \{u_-(A)\} \\ * (\mathcal{G}_A) & = [0, u_-(A)] \\ (\mathcal{G}_A)^* & = [0, u_+(A)] \cup [u_-(A), 1] \end{cases}$$

Here the left-dual behaves like a sort of left completion of \mathcal{G}_A , while the right dual behaves like a sort of right completion.

We now start with the following straightforward result (with $D_+^f(p, q) := \sum_{j=1, \dots, N} \sigma^j \cdot \text{sign}^+(p^j - q^j) \cdot \{f^j(p^j) - f^j(q^j)\}$).

Lemma 7.11 (Explicit characterization of duality)

Assume (2.2) with $N \geq 1$. Then for any sets $P, Q \subset [a, b]$, we have $D_+^f(P, Q) \geq 0$ if and only if

$$(7.6) \quad \sum_{j \in K} \sigma^j \cdot \{f^j(p) - f^j(q)\} \geq 0 \quad \text{for all } (p, q) \in P \times Q, \quad \text{for all } K \subset \{1, \dots, N\} \quad \text{such that } p - q \in \tilde{E}_K,$$

with

$$(7.7) \quad \tilde{E}_K := \sum_{j \in K} (0, +\infty) e_j - \sum_{j \notin K} [0, +\infty) e_j.$$

Remark 7.12 Notice that quantity \tilde{E}_K shares some similarities with quantity E_K defined in (4.25).

We also notice the following straightforward result about left and right duals.

Lemma 7.13 (Exchanging left and right-duals by inversion transform)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set.

Consider the full inversion transform

$$\bar{p} := -p, \quad \bar{f}(\bar{p}) := -f(-\bar{p}), \quad \bar{\mathcal{G}} := -\mathcal{G}$$

Then we have

$$D_+^f(p, q) = D_+^{\bar{f}}(\bar{q}, \bar{p})$$

and

$${}^f \bar{\mathcal{G}} = \overline{{}^f \mathcal{G}^*} \quad \text{and} \quad {}^* \bar{\mathcal{G}} = \overline{{}^* \mathcal{G}^f}$$

We now have the following result.

Lemma 7.14 (Characterization of the duals of D_+ -germs)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a D_+ -germ. Then we have for $\hat{f} = \hat{f}_{\mathcal{G}}$

$$(7.8) \quad {}^*\mathcal{G} = \mathcal{G}^{SUB} := \left\{ p \in [a, b], \quad \sigma \diamond (\hat{f} - f)(p) \leq 0 \right\} \quad \text{and} \quad D_+^f(\mathcal{G}^{SUB}, \mathcal{G}) \geq 0$$

and

$$(7.9) \quad \mathcal{G}^* = \mathcal{G}^{SUP} := \left\{ p \in [a, b], \quad \sigma \diamond (\hat{f} - f)(p) \geq 0 \right\} \quad \text{and} \quad D_+^f(\mathcal{G}, \mathcal{G}^{SUP}) \geq 0$$

Proof of Lemma 7.14

We only prove (7.8), because (7.9) follows from (7.8) and Lemma 7.13.

Step 1: proof that $\mathcal{G}^{SUB} \subset {}^*\mathcal{G}$

Fix some $p \in \mathcal{G}^{SUB}$, and let us consider any $q \in \mathcal{G}$ such that $p - q \in \check{E}_K$ with \check{E}_K defined in (7.7). Then we get

$$\begin{aligned} D_+^f(p, q) &= \sum_{j \in K} \sigma^j \cdot \{f^j(p) - f^j(q)\} \\ &= \sum_{j \in K} \sigma^j \cdot \{f^j(p) - \hat{f}^j(q)\} \\ &\geq \sum_{j \in K} \sigma^j \cdot \{\hat{f}^j(p) - \hat{f}^j(q)\} \\ &= D_+^{\hat{f}}(p, q) \\ &\geq 0 \end{aligned}$$

because \mathcal{G} is a D_+ -germ. Therefore $p \in {}^*\mathcal{G}$.

Step 2: proof that $\mathcal{G}^{SUB} \supset {}^*\mathcal{G}$

Conversely, consider $p \in {}^*\mathcal{G}$ and for $\hat{p} := \pi_{\mathcal{G}}(p)$, let us set

$$(7.10) \quad K := \left\{ k \in \{1, \dots, N\}, \quad \sigma^k \cdot \{f^k(p^k) - f^k(\hat{p}^k)\} < 0 \right\}.$$

Because p^k belongs to the basin of attraction $BA^k(\hat{p}^k)$ on the branch $J^k \simeq \sigma^k \cdot (-\infty, 0)$, we deduce that

$$p^k > \hat{p}^k \quad \text{for all } k \in K$$

Similarly, for all $k \in \{1, \dots, N\} \setminus K$, we have $\sigma^k \cdot \{f^k(p^k) - f^k(\hat{p}^k)\} \geq 0$ and then

$$p^k \leq \hat{p}^k \quad \text{for all } k \in \{1, \dots, N\} \setminus K$$

Therefore $\hat{p} \in \mathcal{G}$ with $p - \hat{p} \in \check{E}_K$. Because $p \in {}^*\mathcal{G}$, we deduce that

$$0 \leq D_+^f(p, \hat{p}) = \sum_{j \in K} \sigma^j \cdot \{f^j(p) - f^j(\hat{p})\} \stackrel{(7.10)}{<} 0$$

if $K \neq \emptyset$. Contradiction. Therefore $K = \emptyset$, and $\sigma \diamond (f - \hat{f})(p) \geq 0$, i.e. $p \in \mathcal{G}^{SUB}$.

This shows (7.8) and ends the proof of the lemma.

Corollary 7.15 (Key dissipation of the duals)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a D_+ -germ. Then we have

$$D_+({}^*\mathcal{G}, \mathcal{G}^*) \geq 0, \quad \text{i.e.} \quad D_+(\mathcal{G}^{SUB}, \mathcal{G}^{SUP}) \geq 0$$

and

$${}^*\mathcal{G} \cap \mathcal{G}^* = \mathcal{G}$$

Proof of Corollary 7.15

Let $p \in {}^*\mathcal{G} = \mathcal{G}^{SUB} = \left\{ \sigma \diamond (\hat{f} - f) \leq 0 \right\}$ and $q \in \mathcal{G}^* = \mathcal{G}^{SUP} = \left\{ \sigma \diamond (\hat{f} - f) \geq 0 \right\}$ such that $p - q \in \check{E}_K$.

Then we have

$$\begin{aligned} D_+^f(p, q) &= \sum_{j \in K} \sigma^j \cdot \{f^j(p) - f^j(q)\} \\ &\geq \sum_{j \in K} \sigma^j \cdot \{\hat{f}^j(p) - \hat{f}^j(q)\} \\ &= D_+^{\hat{f}}(p, q) \\ &\geq 0 \end{aligned}$$

because \mathcal{G} is a D_+ -germ. This ends the proof of the corollary.

Lemma 7.16 (Max and Min of duals for D_+ -germs)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a D_+ -germ. Then we have componentwisely

$$\max \{ \mathcal{G}^{SUB}, \mathcal{G}^{SUB} \} \subset \mathcal{G}^{SUB}$$

and

$$\min \{ \mathcal{G}^{SUP}, \mathcal{G}^{SUP} \} \subset \mathcal{G}^{SUP}$$

Proof of Lemma 7.16

For $\gamma = \alpha, \beta$, let us consider $p_\gamma \in \mathcal{G}^{SUB}$, which then satisfies $\sigma \diamond (\hat{f} - f)(p_\gamma) \leq 0$. Then consider $p := \max \{ p_\alpha, p_\beta \}$ componentwisely, i.e.

$$p^j := \max \{ p_\alpha^j, p_\beta^j \}, \quad j = 1, \dots, N$$

Recall that we have the monotonicities $(\sigma^1 \hat{f}^1)(\uparrow, \downarrow, \dots, \downarrow)$. Let $\gamma_1 \in \{\alpha, \beta\}$ be such that $p^1 = p_{\gamma_1}^1$. Then, using the monotonicities of $\sigma^1 \hat{f}^1$, we deduce that

$$\sigma^1 \hat{f}^1(p) = \sigma^1 \hat{f}^1(p_{\gamma_1}^1, p') \leq \sigma^1 \hat{f}^1(p_{\gamma_1}) = \sigma^1 f^1(p_{\gamma_1}^1) = \sigma^1 f^1(p)$$

Similarly, we get $\sigma \diamond (\hat{f} - f)(p) \leq 0$, which shows that $p \in \mathcal{G}^{SUB}$. The proof for the minimum is similar. This ends the proof of the lemma.

7.4 Duality and characteristic subsets for conservative 1 : 1 germs

Lemma 7.17 (Duality and characteristic subsets for conservative 1 : 1 germs)

Assume (2.2) with $N = 2$ for 1 : 1 junction with $f = (f^L, f^R)$ for indices $j = L, R$ (for left and right) with $\sigma^L = 1$ and $\sigma^R = -1$. Let $\mathcal{G} \subset [a, b]$ be a conservative Kruzkov germ. Following Definition 2.27, we recall the following sets of characteristic points of \mathcal{G} (the sub-characteristic set $\underline{\chi}\mathcal{G}$ and super-characteristic set $\overline{\chi}\mathcal{G}$)

$$\left\{ \begin{array}{l} \underline{\chi}\mathcal{G} := \left\{ \hat{p} = (\hat{p}^L, \hat{p}^R) \in \mathcal{G}, \quad \left| \begin{array}{ll} f^L < f^L(\hat{p}^L) & \text{on } (\hat{p}^L, \hat{p}^L + \varepsilon) \subset [a^L, b^L] \\ f^R < f^R(\hat{p}^R) & \text{on } (\hat{p}^R - \varepsilon, \hat{p}^R) \subset [a^R, b^R] \end{array} \right| \quad \text{for some } \varepsilon > 0 \right\} \\ \overline{\chi}\mathcal{G} := \left\{ \hat{p} = (\hat{p}^L, \hat{p}^R) \in \mathcal{G}, \quad \left| \begin{array}{ll} f^L > f^L(\hat{p}^L) & \text{on } (\hat{p}^L - \varepsilon, \hat{p}^L) \subset [a^L, b^L] \\ f^R > f^R(\hat{p}^R) & \text{on } (\hat{p}^R, \hat{p}^R + \varepsilon) \subset [a^R, b^R] \end{array} \right| \quad \text{for some } \varepsilon > 0 \right\} \\ \chi\mathcal{G} := \underline{\chi}\mathcal{G} \cup \overline{\chi}\mathcal{G} \end{array} \right.$$

For $p = (p^L, p^R)$ and $q = (q^L, q^R)$, we recall

$$RH^f(p) := f^L(p^L) - f^R(p^R), \quad D_+^f(p, q) = \text{sign}^+(p^L - q^L) \cdot \{f^L(p^L) - f^L(q^L)\} - \text{sign}^+(p^R - q^R) \cdot \{f^R(p^R) - f^R(q^R)\}$$

Then we have

$$(7.11) \quad \left\{ \begin{array}{l} (\underline{\chi}\mathcal{G})'' := \left\{ p \in [a, b], \quad RH^f(p) \geq 0, \quad D_+^f(p, q) \geq 0 \quad \text{for all } q \in \underline{\chi}\mathcal{G} \right\} = \mathcal{G}^{SUB} = \left\{ \sigma \diamond (\hat{f} - f) \leq 0 \right\} \\ (\overline{\chi}\mathcal{G})'' := \left\{ p \in [a, b], \quad RH^f(p) \leq 0, \quad D_+^f(q, p) \geq 0 \quad \text{for all } q \in \overline{\chi}\mathcal{G} \right\} = \mathcal{G}^{SUP} = \left\{ \sigma \diamond (\hat{f} - f) \geq 0 \right\} \end{array} \right.$$

and

$$(7.12) \quad (\chi\mathcal{G})' := \left\{ p \in [a, b], \quad RH^f(p) = 0, \quad D^f(p, q) \geq 0 \quad \text{for all } q \in \chi\mathcal{G} \right\} = \mathcal{G}$$

Proof of Lemma 7.17

Step 1: proof of (7.11)

We prove the first line of (7.11) (the proof of the second line is similar). Consider $p \in \text{"}(\underline{\chi}\mathcal{G})$. Then we have

$$(7.13) \quad f^L(p^L) - f^R(p^R) \geq 0$$

and

$$(7.14) \quad \text{sign}^+(p^L - q^L) \cdot \{f^L(p^L) - f^L(q^L)\} - \text{sign}^+(p^R - q^R) \cdot \{f^R(p^R) - f^R(q^R)\} \geq 0 \quad \text{for all } q \in \underline{\chi}\mathcal{G}$$

Assume by contradiction that $p \notin \mathcal{G}^{SUB} = \{\hat{f}^L - f^L \leq 0, -(\hat{f}^R - f^R) \leq 0\}$.

Case A: $f^L(p^L) < f^L(\hat{p}^L)$

Then we get $p^L \in BA^L(\hat{p}) \cap (\hat{p}^L, +\infty)$. Moreover, using (7.13), we get

$$f^R(p^R) \leq f^L(p^L) < f^L(\hat{p}^L) = f^R(\hat{p}^R)$$

and then $p^R \in BA^R(\hat{p}) \cap (-\infty, \hat{p}^R)$, which shows that $\hat{p} \in \underline{\chi}\mathcal{G}$. Then the first term in (7.14) leads to a contradiction for the choice of $q := \hat{p}$.

Case B: $f^R(p^R) > f^R(\hat{p}^R)$

Similarly to Case A, we get $p^R \in BA^R(\hat{p}) \cap (\hat{p}^R, +\infty)$. Moreover, using (7.13), we get now

$$f^L(p^L) \geq f^R(p^R) > f^R(\hat{p}^R) = f^L(\hat{p}^L)$$

and then $p^L \in BA^L(\hat{p}) \cap (-\infty, \hat{p}^L)$, which shows that $\hat{p} \in \bar{\chi}\mathcal{G}$. Then the second term in (7.14) leads to a contradiction for the choice of $q := \hat{p}$.

Conclusion and consequences

Therefore $p \in \mathcal{G}^{SUB}$. Hence we have shown that

$$(7.15) \quad \text{"}(\bar{\chi}\mathcal{G}) \subset \mathcal{G}^{SUB} = *\mathcal{G}$$

But, by duality, the inclusion $\underline{\chi}\mathcal{G} \subset \mathcal{G}$ implies the reverse inclusion

$$(7.16) \quad \text{"}(\underline{\chi}\mathcal{G}) \supset \text{"}\mathcal{G} := *\mathcal{G} \cap \{RH^f \geq 0\}$$

Moreover using the fact that $RH^{\hat{f}} = 0$, we deduce that

$$\{RH^f \geq 0\} \supset \{\sigma \diamond (\hat{f} - f) \leq 0\} = \mathcal{G}^{SUB} = *\mathcal{G}$$

Hence (7.16) shows that $\text{"}(\underline{\chi}\mathcal{G}) \supset *\mathcal{G}$ and the reverse inclusion (7.15) implies the equality, i.e. the first line of (7.11).

Step 2: proof of (7.12)

Step 2.1: preliminaries

Let $p \in (\underline{\chi}\mathcal{G})'$, i.e. satisfying

$$(7.17) \quad f^L(p^L) - f^R(p^R) = 0$$

and

$$(7.18) \quad \text{sign}(p^L - q^L) \cdot \{f^L(p^L) - f^L(q^L)\} - \text{sign}(p^R - q^R) \cdot \{f^R(p^R) - f^R(q^R)\} \geq 0 \quad \text{for all } q \in \underline{\chi}\mathcal{G}$$

Again assume by contradiction that $p \notin \mathcal{G}^{SUB} = \{\hat{f}^L - f^L \leq 0, -(\hat{f}^R - f^R) \leq 0\}$.

Case A: $f^L(p^L) < f^L(\hat{p}^L)$

Because (7.17) implies (7.13), then Case A of Step 1 shows that $\hat{p} \in \underline{\chi}\mathcal{G}$ with

$$\left\{ \begin{array}{l} f^L(p^L) < f^L(\hat{p}^L) \\ p^L > \hat{p}^L, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} f^R(p^R) < f^R(\hat{p}^R) \\ p^R < \hat{p}^R \end{array} \right. .$$

Then both terms in (7.18) lead to a contradiction for $q := \hat{p}$.

Case B: $f^R(p^R) > f^R(\hat{p}^R)$

Similarly, because (7.17) implies (7.13), then Case B of Step 1 shows that $\hat{p} \in \overline{\chi\mathcal{G}}$ with

$$\left\{ \begin{array}{l} f^L(p^L) > f^L(\hat{p}^L) \\ p^L < \hat{p}^L \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} f^R(p^R) > f^R(\hat{p}^R) \\ p^R > \hat{p}^R \end{array} \right.$$

and both terms (7.18) lead to a contradiction for $q := \hat{p}$.

Conclusion: $p \in \mathcal{G}^{SUB}$

Step 2.2: Further conclusion

Similarly we show that $p \in \mathcal{G}^{SUP}$. Hence $p \in \mathcal{G}^{SUB} \cap \mathcal{G}^{SUP} = \mathcal{G}$. This shows that

$$(7.19) \quad (\chi\mathcal{G})' \subset \mathcal{G}$$

Conversely, notice that $\chi\mathcal{G} \subset \mathcal{G}$, and then by duality, we get

$$(7.20) \quad (\chi\mathcal{G})' \supset \mathcal{G}' = \mathcal{G}^D \cap \{RH^f = 0\}$$

with $\mathcal{G}^D := \{p \in [a, b], D^f(p, q) \geq 0 \text{ for all } q \in \mathcal{G}\}$. From Lemma 6.2 on the D -maximality of Kruřkov germs, we deduce that

$$\mathcal{G}^D = \mathcal{G} \subset \{RH^f = 0\}$$

where the last inclusion follows from the fact that \mathcal{G} is a conservative (Kruřkov) germ. Then (7.20) gives $(\chi\mathcal{G})' \supset \mathcal{G}' = \mathcal{G}$, and the reverse inclusion (7.19) implies the equality, i.e. (7.12). This ends the proof of the lemma.

Remark 7.18 For $N = 1$ and for a junction of type $1 : 0$ or of type $0 : 1$ (where no Rankine-Hugoniot relation is required), results similar to Lemma 7.17 still hold. Precisely, we have

$$\left\{ p \in [a, b], \quad D_+^f(p, q) \geq 0 \quad \text{for all } q \in \underline{\chi\mathcal{G}} \right\} = \mathcal{G}^{SUB}$$

$$\left\{ p \in [a, b], \quad D_+^f(q, p) \geq 0 \quad \text{for all } q \in \overline{\chi\mathcal{G}} \right\} = \mathcal{G}^{SUP}$$

and

$$\left\{ p \in [a, b], \quad D^f(p, q) \geq 0 \quad \text{for all } q \in \chi\mathcal{G} \right\} = \mathcal{G}$$

7.5 Polar decomposition of Godunov flux for bell-shaped fluxes

In the special case of bell-shaped fluxes, it is possible to show that Godunov flux at the junction has a unique polar decomposition in preflux composed with a capacity. The capacity is explicit, while the preflux encodes the structure of the Godunov flux. The preflux is unique on the image of the capacity.

Definition 7.19 (Preflux)

Let $N \geq 1$ and $\sigma \in \{\pm 1\}^N$.

0) (Preflux)

We say that $\hat{\gamma}$ is a **preflux** if it satisfies the following set of conditions

$$(7.21) \quad \left\{ \begin{array}{ll} \hat{\gamma} : [0, +\infty)^N \rightarrow [0, +\infty)^N \quad \text{is continuous} & \text{(Continuity)} \\ 0 \leq \hat{\gamma} \leq id_{[0, +\infty)^N} & \text{(Bounds)} \\ \gamma \mapsto \hat{\gamma}^j(\gamma) \text{ is nondecreasing in } \gamma^j, \text{ for all } j = 1, \dots, N & \text{(Basic monotonicity)} \\ \hat{\gamma} \text{ is locally constant on } \{\hat{\gamma} \neq id_{[0, +\infty)^N}\} \text{ in the sense of Definition 2.12} & \text{(Local constancy)} \end{array} \right.$$

Recall that the local constancy of $\hat{\gamma}$ means that for all $\gamma_* \in [0, +\infty)^N$, and $I := \{j \in \{1, \dots, N\}, \hat{\gamma}^j(\gamma_*) \neq \gamma_*^j\}$, there exists $\varepsilon > 0$ such that

$$\hat{\gamma} = \text{const} = \hat{\gamma}(\gamma_*) \quad \text{on} \quad Q_\varepsilon(\gamma_*) := \left(\gamma_* + \sum_{j \in I} (-\varepsilon, \varepsilon) e_j \right) \cap [0, +\infty)^N$$

i) (HJ preflux)

We say that the preflux $\hat{\gamma}$ is HJ if

$$\hat{\gamma}^j = g \quad \text{for all } j = 1, \dots, N, \quad \text{for some function } g : [0, +\infty)^N \rightarrow [0, +\infty)$$

ii) (Kruřkov preflux)

We say that the preflux $\hat{\gamma}$ is Kruřkov if

$$0 \leq D_*^{\hat{\gamma}}(\bar{\gamma}, \gamma) := \sum_{j=1, \dots, N} \text{sign}(\bar{\gamma}^j - \gamma^j) \cdot \{\hat{\gamma}^j(\bar{\gamma}) - \hat{\gamma}^j(\gamma)\} \quad \text{for all } \bar{\gamma}, \gamma \in [0, +\infty)^N$$

iii) (σ -monotone Kruřkov preflux)

We say that the preflux $\hat{\gamma}$ is σ -monotone Kruřkov if

$$0 \leq D_{*+}^{\hat{\gamma}}(\bar{\gamma}, \gamma) := \sum_{j=1, \dots, N} \text{sign}^{\sigma^j}(\bar{\gamma}^j - \gamma^j) \cdot \{\hat{\gamma}^j(\bar{\gamma}) - \hat{\gamma}^j(\gamma)\} \quad \text{for all } \bar{\gamma}, \gamma \in [0, +\infty)^N$$

(where we make some abuse of notation for sign^{σ^j}).

iv) (σ -monotone preflux)

We say that the preflux $\hat{\gamma}$ is σ -monotone if

$$\text{the maps } \gamma \mapsto \sigma^j \hat{\gamma}^j(\gamma) \text{ are nonincreasing in the variable } \sigma^k \gamma^k \text{ for all } k \neq j.$$

v) (conservative preflux)

We say that the preflux $\hat{\gamma}$ is conservative if

$$\sum_{j=1, \dots, N} \sigma^j \cdot \hat{\gamma}^j = 0$$

Remark 7.20 Notice that condition (7.21) defining a preflux, means exactly that $\hat{\gamma}$ is a Godunov flux for a junction $N : 0$ with function $f := \text{id}_{[0, +\infty)^N}$ associated to a Riemann germ $\mathcal{G}_{\hat{\gamma}} := \{\hat{\gamma} = \text{id}_{[0, +\infty)^N}\} \subset [0, +\infty)^N$. This follows immediately from Theorems 2.14 and 2.16.

Definition 7.21 (Capacity, for bell-shaped fluxes)

Assume (2.2) for $N \geq 1$.

i) (Bell-shaped)

We say that $f = (f^1, \dots, f^N)$ is bell-shaped, if each scalar function $f^k : [a^k, b^k] \rightarrow [0, +\infty)$ is continuous, satisfies $f^k(a^k) = 0 = f^k(b^k)$, has a maximum value $f_{\max}^k := f^k(c^k)$ at $c^k \in (a^k, b^k)$ and is increasing on (a^k, c^k) and decreasing on (c^k, b^k) for $k = 1, \dots, N$. We set the monotone functions

$$f^{k,+}(q) = \begin{cases} f^k(q) & \text{for } q \in [a^k, c^k] \\ f^k(c^j) & \text{for } q \in [c^k, b^k] \end{cases} \quad \text{and} \quad f^{k,-}(q) = \begin{cases} f^k(c^k) & \text{for } q \in [a^k, c^k] \\ f^k(q) & \text{for } q \in [c^k, b^k] \end{cases}$$

ii) (Capacity)

We recall that the orientations of each branch J^k is encoded in $\sigma^k = +1$ if $J^k \simeq (-\infty, 0)$ and $\sigma^k = -1$ if $J^k \simeq (0, +\infty)$. We define the capacity $\bar{\gamma} = (\bar{\gamma}^1, \dots, \bar{\gamma}^N) : [a, b] \rightarrow [0, +\infty)^N$ as the following function for $p \in [a, b]$

$$(7.22) \quad \bar{\gamma}^k(p) := \bar{\gamma}^k(p^k) := f^{k, \sigma^k}(p^k) \quad \text{for all } k = 1, \dots, N$$

(with a slight abuse of notation).

Remark 7.22 As an example, consider some $1 : 1$ junction with indices $j = L, R$ (for left and right) with $\sigma^L = 1$ and $\sigma^R = -1$. Then the standard Godunov flux $\hat{f} = (\hat{f}^L, \hat{f}^R)$ associated to some HJ germ with flux limiter A , is given by $\hat{f}^L(p) = \hat{f}^R(p) = \min \{A, f^{L,+}(p^L), f^{R,-}(p^R)\}$. Then the capacity is $\bar{\gamma}(p) = (f^{L,+}(p^L), f^{R,-}(p^R))$ for $p = (p^L, p^R)$ and the preflux is $\hat{\gamma} = (\hat{\gamma}^L, \hat{\gamma}^R)$ with $\hat{\gamma}^L(\gamma) = \hat{\gamma}^R(\gamma) = \min \{A, \gamma^L, \gamma^R\}$ for $\gamma = (\gamma^L, \gamma^R)$. Here the preflux $\hat{\gamma}$ has all the properties: $\hat{\gamma}$ is HJ, conservative, σ -monotone, Kruřkov, and σ -monotone Kruřkov.

Proposition 7.23 (Polar decomposition of Godunov flux, for bell-shaped fluxes)

Assume (2.2) for $N \geq 1$ for a junction (J, f) . Assume that f is bell-shaped, and call $\bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N$ the capacity given by definition 7.21.

i) (Polar decomposition)

Let $\mathcal{G} \subset [a, b]$ be a Riemann germ with respect to (J, f) . Then the Godunov flux $\hat{f}_{\mathcal{G}}$ associated to \mathcal{G} has the following polar decomposition

$$\hat{f}_{\mathcal{G}} = \hat{\gamma} \circ \bar{\gamma}$$

where $\hat{\gamma} : [0, +\infty)^N \rightarrow [0, +\infty)^N$ is a preflux (as in Definition 7.19) and $\bar{\gamma}$ is the capacity. Moreover the preflux $\hat{\gamma}$ is unique on the image of the capacity.

ii) (Riemann germ construction)

Given any preflux $\hat{\gamma} : [0, +\infty)^N \rightarrow [0, +\infty)^N$, we define

$$(7.23) \quad \mathcal{G} := \mathcal{G}_{\hat{f}} := \left\{ p \in [a, b], \quad \hat{f}(p) = f(p) \right\} \quad \text{with} \quad \hat{f} := \hat{\gamma} \circ \bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N.$$

Then \mathcal{G} is a Riemann germ.

iii) (Further properties of the germ)

In the previous construction i)-ii), the preflux $\hat{\gamma}$ is HJ (resp. σ -monotone, conservative, conservative Kruřkov), if and only if the germ \mathcal{G} is HJ (resp. monotone, conservative, conservative Kruřkov).

Moreover, if the Riemann germ \mathcal{G} is Kruřkov (resp. monotone Kruřkov), then the preflux $\hat{\gamma}$ is also Kruřkov (resp. σ -monotone Kruřkov).

iv) (Counter-example: Kruřkov preflux $\not\Rightarrow$ Kruřkov germ)

There are examples where the preflux $\hat{\gamma}$ is Kruřkov (resp. σ -monotone Kruřkov), and where the associated Riemann germ $\mathcal{G}_{\hat{f}}$ given in (7.23) is not Kruřkov (resp. not monotone Kruřkov).

Remark 7.24 Notice that Kruřkov property for preflux $\hat{\gamma}$ is not transferable in general to the germ $\mathcal{G}_{\hat{f}}$, because the Kruřkov property does not behave well by composition by functions in general, contrarily to monotonicity properties.

Proof of Proposition 7.23

Step 1: proof of ii)

Step 1.1: continuity and basic monotonicity

We notice that the map $\bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N$ defined in (7.22) is continuous and each map $p^j \mapsto \sigma^j \bar{\gamma}^j(p^j)$ is nondecreasing. This implies that \hat{f} is continuous and the map $p \mapsto \sigma^j \hat{f}^j(p)$ is nondecreasing in p^j for all indices j .

Step 1.2: local constancy

Let us now check that \hat{f} is locally constant on $\left\{ \hat{f} \neq f \right\}$. Fix some $p \in [a, b]$, and let

$$\tilde{I} := \left\{ j \in \{1, \dots, N\}, \quad \hat{f}^j(p) \neq f^j(p) \right\} \quad \text{and} \quad I := \left\{ j \in \{1, \dots, N\}, \quad \hat{\gamma}^j(\bar{\gamma}(p)) \neq \bar{\gamma}^j(p) \right\}$$

Consider $q \in [a, b]$ such that

$$q^j \quad \begin{cases} = p^j & \text{if } j \notin \tilde{I} \\ \in (p^j - \varepsilon, p^j + \varepsilon) \cap [a^j, b^j] & \text{if } j \in \tilde{I} \end{cases}$$

Now, consider some $j \in \tilde{I}$ and assume that $\sigma^j = +1$ (the case $\sigma^j = -1$ is similar). We distinguish two cases.

Case A: $p^j \in [a^j, c^j]$

Then we have $\hat{\gamma}^j(\bar{\gamma}(p)) = \hat{f}^j(p) \neq f^j(p) = f^{j,+}(p^j) = \bar{\gamma}^j(p)$, which shows that $j \in I$. Moreover $\bar{\gamma}^j(q) \in (\bar{\gamma}^j(p) - \delta, \bar{\gamma}^j(p) + \delta) \cap [0, +\infty)$ for some $\delta > 0$ small enough.

Case B: $p^j \in (c^j, b^j]$

Then for $\varepsilon > 0$ small enough, we have $q^j > c^j$ and then $\bar{\gamma}^j(q) = f^{j,+}(q^j) = f^j(c^j) = \bar{\gamma}^j(p)$.

Conclusion

Using both cases A and B, and the local constancy of $\hat{\gamma}$ on $\{\hat{\gamma} \neq id_{[0,+\infty)^N}\}$, we deduce that for $\varepsilon > 0$ small enough, we have $\hat{f}(q) = \hat{\gamma}(\bar{\gamma}(q)) = \hat{\gamma}(\bar{\gamma}(p)) = \hat{f}(p)$, which means exactly that \hat{f} is locally constant on $\{\hat{f} \neq f\}$.

Step 1.3: bounds

We also have for $p \in [a, b]$

$$\left\{ \begin{array}{l} f_-^j(p^j) = \inf_{[p^j, b^j]} f^j = 0 \leq \hat{f}^j(p) \leq \bar{\gamma}^j(p) = f^{j,+}(p^j) = \sup_{[a^j, p^j]} f^j = f_+^j(p^j) \quad \text{if } J^j \simeq (-\infty, 0) \\ f_-^j(p^j) = \inf_{[a^j, p^j]} f^j = 0 \leq \hat{f}^j(p) \leq \bar{\gamma}^j(p) = f^{j,-}(p^j) = \sup_{[p^j, b^j]} f^j = f_+^j(p^j) \quad \text{if } J^j \simeq (0, +\infty) \end{array} \right.$$

From the first three steps and the characterization of generalized Riemann germs (see ii) of Theorem 2.14), and of Riemann germs (see i) of Theorem 2.16), we deduce that \mathcal{G} is a Riemann germ.

Step 2: proof of i)

Let \mathcal{G} be a Riemann germ, and set $\hat{f} := \hat{f}_{\mathcal{G}}$ which is known to be continuous.

Step 2.1: unique decomposition of \hat{f}

Notice that the capacity $\bar{\gamma}$ genuinely varies on the set

$$K := \prod_{j=1, \dots, N} K^j \quad \text{with} \quad K^j := \begin{cases} [a^j, c^j] & \text{if } \sigma^j = 1 \\ [c^j, b^j] & \text{if } \sigma^j = -1 \end{cases}$$

We then define $\rho = (\rho^1, \dots, \rho^N) : [a, b] \rightarrow K$ as

$$\rho^j(p^j) = \begin{cases} \max\{p^j, c^j\} & \text{if } \sigma^j = 1 \\ \min\{p^j, c^j\} & \text{if } \sigma^j = -1 \end{cases}, \quad \text{for all } p^j \in [a^j, b^j]$$

which is such that $f^{j, \sigma^j} = f^j \circ \rho^j$, and then $\bar{\gamma} = f \circ \rho$. Because ρ is a projection, we deduce in particular that

$$(7.24) \quad \bar{\gamma} = \bar{\gamma}|_K \circ \rho.$$

Now for $p \in [a, b]$, we first distinguish the first index and set $p = (p^1, p')$, and consider $g := \hat{f}^1(\cdot, p') : [a^1, b^1] \rightarrow \mathbb{R}$. Assume also that $\sigma^1 = 1$ (the case $\sigma^1 = -1$ is similar). From Theorem 2.14 and the slicing Lemma 4.12, we know that g is nondecreasing and is locally constant on $\{g \neq f^1\}$. Because f^1 is decreasing on (c^1, b^1) , we deduce that g is constant on $[c^1, b^1]$. Hence $g \neq f^1$ a.e. on $[c^1, b^1]$. Because \hat{f} is locally constant on $\{\hat{f} \neq f\}$, we deduce that the whole function $\hat{f}(\cdot, p') : [a^1, b^1] \rightarrow \mathbb{R}^N$ is locally constant on $[c^1, b^1]$. Therefore, we have $\hat{f}(\cdot, p') = \hat{f}(\cdot, p') \circ \rho^1$. The the same reasoning with all indices j , shows that

$$(7.25) \quad \hat{f} = \hat{f} \circ \rho$$

We now define

$$(7.26) \quad \hat{\gamma} := \hat{f} \circ (\bar{\gamma}|_K)^{-1} : \Gamma_0(c) \rightarrow \Gamma_0(c) \quad \text{with} \quad \Gamma_0(c) := [0, \bar{\gamma}(c)] = \prod_{j=1, \dots, N} [0, f^j(c^j)].$$

Hence we get $\hat{f}|_K = \hat{\gamma} \circ \bar{\gamma}|_K$ and then using (7.25), we get

$$\begin{aligned} \hat{f} &= \hat{f}|_K \circ \rho \\ &= \hat{\gamma} \circ \bar{\gamma}|_K \circ \rho \\ &\stackrel{(7.24)}{=} \hat{\gamma} \circ \bar{\gamma}. \end{aligned}$$

Notice that the invertibility of $\bar{\gamma}|_K$ shows that the function $\hat{\gamma}$ is unique on the image K of the capacity $\bar{\gamma}$ and given by (7.26).

Step 2.2: properties of the preflux

The fact that $\hat{\gamma}$ is preflux in the sense of point 0) of Definition 7.19 (i.e. continuity, bounds, basic monotonicity and local constancy) follows immediately from its expression (7.26), and from the similar properties of \hat{f} .

Step 3: proof of iii)

Step 3.1: σ -monotone preflux

For a σ -monotone preflux, we know for $k \neq j$ that $\sigma^j \hat{\gamma}^j$ is nonincreasing in $\sigma^k \bar{\gamma}^k$, which is itself nondecreasing in p^k , by definition of $\bar{\gamma}$. By composition, we deduce that $\sigma^j \hat{f}^j$ is nonincreasing in p^k for all $k \neq j$. This shows that if the preflux $\hat{\gamma}$ is σ -monotone then (from Lemma 5.5) the Riemann germ \mathcal{G} is monotone. Conversely, we similarly get that if \mathcal{G} is monotone, then $\hat{\gamma}$ is σ -monotone.

Step 3.2: HJ and conservative properties

It is straightforward to check that the preflux $\hat{\gamma}$ is HJ (resp. conservative) if and only if \hat{f} satisfies the same properties, which from Lemma 5.5 is equivalent to the similar properties for the germ \mathcal{G} .

Step 3.3: Kruřkov germ \implies Kruřkov preflux

The result follows from the restriction $\hat{f}|_K = \hat{\gamma} \circ \bar{\gamma}|_K$, and the fact that $\bar{\gamma}|_K : K \rightarrow [0, \bar{\gamma}(c)]$ is bijective. We also use the change of variables $\text{sign}(\bar{\gamma}^j(\bar{p}^j) - \bar{\gamma}^j(p^j)) = \sigma^j \text{sign}(\bar{p}^j - p^j)$ for $\bar{p}, p \in K$. The similar result holds true for monotone Kruřkov germ which implies σ -monotone Kruřkov preflux, using the change of variables $\text{sign}^{\sigma^j}(\bar{\gamma}^j(\bar{p}^j) - \bar{\gamma}^j(p^j)) = \sigma^j \text{sign}^+(\bar{p}^j - p^j)$.

Step 3.4: conservative Kruřkov

If $\mathcal{G}_{\hat{f}}$ is conservative Kruřkov, then Steps 3.2 and 3.3 imply that the preflux $\hat{\gamma}$ is conservative Kruřkov. Conversely, if the preflux $\hat{\gamma}$ is conservative Kruřkov, we can only indirectly show that it transfers to the germ. We first show the following result.

Step 3.4.1: $\hat{\gamma}$ is conservative Kruřkov $\implies \hat{\gamma}$ is σ -monotone

We consider a (partial) inversion-reversion and set

$$\tilde{\gamma}(\gamma) := \hat{\gamma}(\sigma \diamond \gamma) : \Omega \rightarrow [0, +\infty)^N \quad \text{with} \quad \Omega := \prod_{j=1, \dots, N} \bar{J}^j, \quad \text{and} \quad \bar{J}^j = \begin{cases} [0, +\infty) & \text{if } \sigma^j = 1 \\ (-\infty, 0] & \text{if } \sigma^j = -1 \end{cases}$$

Then it is easy to check that $\tilde{\gamma}$ is a Godunov flux for the Riemann germ

$$\mathcal{G}_{\tilde{\gamma}} := \left\{ \tilde{\gamma} = \tilde{f} \right\} \subset \Omega \quad \text{with} \quad \tilde{f} := \sigma \diamond id_{\Omega}$$

with respect to (J, \tilde{f}) . Then the fact that $\hat{\gamma}$ is conservative Kruřkov implies that the germ $\mathcal{G}_{\tilde{\gamma}}$ is conservative Kruřkov. From Theorem 2.24, we know that conservative Kruřkov implies that $\mathcal{G}_{\tilde{\gamma}}$ is monotone, which means that $\hat{\gamma}$ is σ -monotone.

Step 3.4.2: core of the proof

Now we know that $\hat{\gamma}$ is σ -monotone and conservative. Because those properties are transferable in general to the germ $\mathcal{G}_{\hat{f}}$ (from Steps 3.1 and 3.2), we deduce that $\mathcal{G}_{\hat{f}}$ is monotone conservative, and then from Theorem 2.24, $\mathcal{G}_{\hat{f}}$ is also conservative Kruřkov. This ends the proof of Step 3.4 for the equivalence of conservative Kruřkov preflux $\hat{\gamma}$ and conservative Kruřkov germ $\mathcal{G}_{\hat{f}}$.

Step 4: proof of iv), a counter-example

We build a counter-example for some 1 : 1 junction, with indices $j = L, R$ (for left and right). We build some preflux $\hat{\gamma}$ which is Kruřkov, but non conservative. We will then show that $\mathcal{G}_{\hat{f}}$ is not Kruřkov for $\hat{f} := \hat{\gamma} \circ \bar{\gamma}$, with the capacity $\bar{\gamma}(p) = (f^{L,+}(p^L), f^{R,-}(p^R))$ for $p = (p^L, p^R) \in [a, b]$, where $f = (f^L, f^R)$ is bell-shaped. Precisely, consider $\phi : [0, +\infty) \rightarrow [0, +\infty)$ which is a continuous increasing and bijective function. We set

$$\hat{\gamma} : [0, +\infty)^2 \rightarrow [0, +\infty)^2 \quad \text{with} \quad \hat{\gamma}(\gamma) := (\min \{ \gamma^L, \phi(\gamma^R) \}, \min \{ \gamma^R, \phi^{-1}(\gamma^L) \}) \quad \text{for} \quad \gamma := (\gamma^L, \gamma^R)$$

Then it is easy to check that $\hat{\gamma} = (\hat{\gamma}^L, \hat{\gamma}^R)$ is a preflux. Moreover for $\sigma = (\sigma^L, \sigma^R) = (1, -1)$, the preflux $\hat{\gamma}$ is σ -monotone, Kruřkov, and σ -monotone Kruřkov. Moreover $\hat{\gamma}$ is conservative (or equivalently HJ) if and only if $\phi = id_{[0, +\infty)}$. Now assume that $\phi \neq id_{[0, +\infty)}$, such that $\hat{\gamma}$ is not conservative, say with $\phi(\gamma_0^R) > \gamma_0^R$ for some $\gamma_0^R > 0$. Then assume that the maxima of the bell-shaped functions satisfy $f^L(c^L) > \phi(\gamma_0^R)$ and $f^R(c^R) > \gamma_0^R$. Then consider $p_0^R \in (c^R, b^R)$ such that $f^R(p_0^R) = \gamma_0^R$, and $p_0^L \in (a^L, c^L)$ such that $f^L(p_0^L) = \phi(\gamma_0^R)$, and call $p_0 := (p_0^L, p_0^R)$. Then we get

$$\bar{\gamma}(b) = (f^{L,+}(b^L), f^{R,-}(b^R)) = (f^L(c^L), 0), \quad \bar{\gamma}(p_0) = (f^{L,+}(p_0^L), f^{R,-}(p_0^R)) = (\phi(\gamma_0^R), \gamma_0^R)$$

and using $\hat{f} := \hat{\gamma} \circ \bar{\gamma}$, we get $\hat{f}(b) = (0, 0)$ and $\hat{f}(p_0) = (\phi(\gamma_0^R), \gamma_0^R)$. Hence

$$\begin{aligned} D^{\hat{f}}(b, p_0) &= \text{sign}(b^L - p_0^L) \cdot \left\{ \hat{f}^L(b) - \hat{f}^L(p_0) \right\} - \text{sign}(b^R - p_0^R) \cdot \left\{ \hat{f}^R(b) - \hat{f}^R(p_0) \right\} \\ &= \left\{ \hat{f}^L(b) - \hat{f}^L(p_0) \right\} - \left\{ \hat{f}^R(b) - \hat{f}^R(p_0) \right\} \\ &= \gamma_0^R - \phi(\gamma_0^R) \\ &< 0 \end{aligned}$$

The case $\phi(\gamma_0^R) < \gamma_0^R$ leads similarly to $D^{\hat{f}}(a, p_0) = \phi(\gamma_0^R) - \gamma_0^R < 0$, using $\hat{f}(a) = (0, 0)$. Then from Lemma 5.5, we deduce that the Riemann germ $\mathcal{G}_{\hat{f}} \subset [a, b]$ is not Kruřkov. A fortiori, $\mathcal{G}_{\hat{f}}$ is neither monotone Kruřkov. This ends the proof of the proposition.

Lemma 7.25 (Characterization of prefluxes)

Assume (2.2) for $N \geq 1$ for a junction (J, f) with $\sigma \in \{\pm 1\}^N$. Let $\tilde{f}^j : \sigma^j[0, +\infty) \rightarrow [0, +\infty)$ with $\tilde{f}^j(p^j) := \sigma^j p^j$.

i) (Characterization of prefluxes)

Then a map $\hat{\gamma} : [0, +\infty)^N \rightarrow [0, +\infty)^N$ is a preflux if and only if $\hat{f} := \hat{\gamma} \circ \tilde{f}$ is a Godunov flux associated to

$$\text{a Riemann germ } \mathcal{G} := \left\{ p \in \prod_{j=1, \dots, n} \sigma^j[0, +\infty), \quad \hat{f}(p) = \tilde{f}(p) \right\} \text{ with respect to } (J, \tilde{f}).$$

ii) (Characterization of conservative σ -monotone prefluxes)

Moreover, the preflux $\hat{\gamma}$ is conservative σ -monotone if and only if the germ \mathcal{G} is conservative monotone, or equivalently if and only if $\hat{\gamma}$ is Lipschitz continuous and satisfies

$$(7.27) \quad \begin{cases} \sigma^k \sigma^j \partial_k \hat{\gamma}^j \leq 0 & \text{for all } k \neq j \\ 1 \geq \sigma^j \partial_j \hat{\gamma}^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{\gamma}^k| & \text{for all indices } j \end{cases}$$

Proof of Lemma 7.25

The proof of i) is easy and similar to the proof of i) of Proposition 7.23 on polar decomposition. Hence we skip it. The proof of ii) is also similar and follows from the relation $(\partial_{p^k} \sigma^j \tilde{f}^j)(p) = (\partial_{\sigma^k p^k} (\sigma^j \hat{\gamma}^j))(\sigma \diamond p)$. Notice that (7.27) follows from the fact that conservative monotone germs are conservative Kruřkov germs (see Theorem 2.24), and both characterizations of monotone germs (Theorem 2.22) and of Kruřkov germs (Theorem 2.21). Notice that the bound by 1 in the second line of (7.27) follows from vii) of Proposition 2.13. This ends the proof of the lemma.

We finish this section with examples.

Lemma 7.26 (Example of the truncation preflux)

Assume $N \geq 1$, and let $\bar{\lambda} = (\bar{\lambda}^1, \dots, \bar{\lambda}^N) \in [0, +\infty)^N$, and the truncation function

$$(7.28) \quad T_{\bar{\lambda}} : [0, +\infty)^N \rightarrow [0, +\infty)^N \quad \text{with} \quad T_{\bar{\lambda}}(\gamma) = (\min \{\gamma^1, \bar{\lambda}^1\}, \dots, \min \{\gamma^N, \bar{\lambda}^N\})$$

Then $T_{\bar{\lambda}}$ is a (Kruřkov) preflux.

Proof of Lemma 7.26

The function $T_{\bar{\lambda}}$ is continuous. We obviously have $0 \leq T_{\bar{\lambda}}(\gamma) \leq \gamma$. Moreover if $T_{\bar{\lambda}}^j(\gamma) < \gamma^j$, then all coordinates of $T_{\bar{\lambda}}$ are locally independent on γ^j , which means that $T_{\bar{\lambda}}$ is locally constant on $\{T_{\bar{\lambda}} \neq \text{id}_{[0, +\infty)^N}\}$. Finally each map $\gamma \mapsto T_{\bar{\lambda}}^j(\gamma)$ is nondecreasing in γ^j . This shows that $T_{\bar{\lambda}}$ is a preflux. Moreover its is straightforward to check that it is a Kruřkov preflux. This ends the proof of the lemma.

Lemma 7.27 (Composition of prefluxes)

Assume $N \geq 1$, and let $\hat{\gamma}_0 : [0, +\infty)^N \rightarrow [0, +\infty)^N$ be a preflux, and for $\bar{\lambda} \in [0, +\infty)^N$, let $T_{\bar{\lambda}} : [0, +\infty)^N \rightarrow [0, +\infty)^N$ be the truncation preflux defined in (7.28). Then $\hat{\gamma} := \hat{\gamma}_0 \circ T_{\bar{\lambda}}$ is also a preflux.

Moreover, if $\hat{\gamma}_0$ is HJ (resp. σ -monotone, conservative), then $\hat{\gamma}$ has the same property.

Proof of Lemma 7.26

Step 1: $\hat{\gamma}$ is a preflux

The function $\hat{\gamma}$ is continuous by composition. Recall that $0 \leq \hat{\gamma}_0(\gamma), T_{\bar{\lambda}}(\gamma) \leq \gamma$, an inequality which is also preserved by composition. Assume that for some $\gamma_* \in [0, +\infty)^N$, we have $\hat{\gamma}_0^j(T_{\bar{\lambda}}(\gamma_*)) = \hat{\gamma}^j(\gamma_*) < \gamma_*^j$. We know that $T_{\bar{\lambda}}^j(\gamma_*) \leq \gamma_*^j$.

Case A: $T_{\bar{\lambda}}^j(\gamma_*) = \gamma_*^j$

Recall that $\hat{\gamma}_0(\gamma)$ is independent on γ^j close to $T_{\bar{\lambda}}^j(\gamma_*) = \gamma_*^j$. Now because

$$(7.29) \quad \text{the components of } T_{\bar{\lambda}}^k(\gamma) \text{ do not depend on } \gamma^j \text{ for } k \neq j,$$

this implies that $\hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$ is also independent on γ^j close to γ_*^j .

Case B: $T_{\bar{\lambda}}^j(\gamma_*) < \gamma_*^j$

Then the whole vector $T_{\bar{\lambda}}$ is independent on γ^j for γ^j close to γ_*^j , and then $\hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$ is also independent on γ^j for γ^j close to γ_*^j .

This shows that $\hat{\gamma}$ is locally constant on $\{\hat{\gamma} \neq id_{[0, +\infty)^N}\}$. Finally the fact that $\gamma \mapsto \hat{\gamma}_0^j(\gamma)$ is nondecreasing in γ^j , and the same property for $T_{\bar{\lambda}}$ and property (7.29) imply that $\gamma \mapsto (\hat{\gamma}_0^j \circ T_{\bar{\lambda}})(\gamma)$ is nondecreasing in γ^j . Therefore $\hat{\gamma}$ is also a preflux.

Step 2: further properties of $\hat{\gamma}$

Notice that if $\hat{\gamma}_0$ is HJ (resp. σ -monotone, conservative), then it is directly transferable by composition to $\hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$. This ends the proof of the lemma.

8 Examples and counter-examples

8.1 A monotone germ which is not a Riemann germ

Lemma 8.1 (A monotone germ which is not a Riemann germ on a 1 : 1 junction)

Assume (2.2) with $N = 2$ for $j = L, R$ with $J^L \simeq (-\infty, 0)$ and $J^R \simeq (0, +\infty)$ and $[a, b] = [0, 1]^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \rightarrow \mathbb{R}$ for $j = L, R$ with

$$\left\{ \begin{array}{l} 0 < \hat{p}^L < \hat{q}^L < 1 \\ f^L \text{ decreasing} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} 0 < \hat{p}^R < \hat{q}^R < 1 \\ f^R(\hat{p}^R) = f^R(\hat{q}^R) = 0 \\ f^R < 0 \\ f^R > 0 \end{array} \right. \quad \begin{array}{l} \text{on } [0, \hat{p}^R) \\ \text{on } (\hat{p}^R, \hat{q}^R) \cup (\hat{q}^R, 1] \end{array}$$

We set

$$\mathcal{G} = \{\hat{p}, \hat{q}\} \quad \text{with} \quad \hat{p} = (\hat{p}^L, \hat{p}^R) \quad \text{and} \quad \hat{q} = (\hat{q}^L, \hat{q}^R)$$

Then $\mathcal{G} \subset [a, b]$ is a generalized Riemann germ with respect to (J, f) , which is also a monotone germ. Moreover $\hat{f} = \hat{f}_{\mathcal{G}}$ satisfies

$$\left\{ \begin{array}{l} \hat{f}^R = 0 \\ \hat{f}^L = \begin{cases} f^L(\hat{p}^L) \\ f^L(\hat{q}^L) \end{cases} \end{array} \right. \quad \begin{array}{l} \text{on } [a, b] = [0, 1]^2 \\ \text{on } BA(\hat{p}) = [0, 1] \times [0, \hat{q}^R) \\ \text{on } BA(\hat{q}) = [0, 1] \times [\hat{q}^R, 1] \end{array}$$

Hence \hat{f}^L is discontinuous on $[0, 1] \times \{\hat{q}^R\}$ with $\partial_{p^R} \hat{f}^L \leq 0$ and \mathcal{G} is not a Riemann germ.

8.2 A monotone germ with \hat{f} continuous but not locally Lipschitz

Lemma 8.2 (A monotone germ on a 2 : 0 junction, with \hat{f} continuous but not locally Lipschitz)

Set $N = 2$ and $[a, b] = [0, 1]^2$ and three C^1 functions

$$\left\{ \begin{array}{l} g : [0, 1] \rightarrow [0, 1] \text{ an increasing bijection (hence with } g(0) = 0 \text{ and } g(1) = 1) \\ f^1 : [0, 1] \rightarrow \mathbb{R} \text{ decreasing with } (f^1)' \leq -\delta < 0 \text{ on } [0, 1] \\ f^2 : [0, 1] \rightarrow \mathbb{R} \text{ increasing with } (f^2)' \geq \delta > 0 \text{ on } [0, 1] \end{array} \right.$$

Then the set

$$\mathcal{G} := \{p = (p^1, p^2) \in [a, b], \quad p^2 = g(p^1)\}$$

is a Riemann germ, and satisfies $\mathcal{G} = \{\hat{f} = f\}$ with continuous function $\hat{f} = \hat{f}_{\mathcal{G}}$ given by

$$\begin{cases} \hat{f}^2(p) := f^2(p^2) \\ \hat{f}^1(p) := f^1(g^{-1}(p^2)) \end{cases}$$

Then \mathcal{G} is a monotone germ for a $2:0$ junction.

Moreover \mathcal{G} is a Kruřkov germ (for f) if and only if

$$(8.1) \quad \text{the map } (f^1 + f^2 \circ g) : [0, 1] \rightarrow \mathbb{R} \text{ is nondecreasing}$$

In particular, if $g'(0) = 0$, then \mathcal{G} is a monotone germ, is not a Kruřkov germ and \hat{f} is not locally Lipschitz.

Proof of Lemma 8.2

We just compute $D^f(p, q) = \text{sign}(p^1 - q^1) \cdot \{f^1(p^1) - f^1(q^1)\} + \text{sign}(p^2 - q^2) \cdot \{f^2(p^2) - f^2(q^2)\}$. Then for $p, q \in \mathcal{G}$, i.e. for $p^2 = g(p^1)$ and $q^2 = g(q^1)$, we get (with g increasing) that $D^f(p, q) = \text{sign}(p^1 - q^1) \cdot [f^1 + f^2 \circ g]_{q^1}^{p^1}$, and then $D^f \geq 0$ on \mathcal{G}^2 if and only if (8.1) holds true. Moreover, if $g'(0) = 0$, then $(f^1 + f^2 \circ g)'(0) = (f^1)'(0) \leq -\delta$ and (8.1) does not hold true, which shows that \mathcal{G} is not a Kruřkov germ. This ends the proof of the lemma.

8.3 Counter-example to Riemann monotonicity for degenerate f

Lemma 8.3 (Counter-example to Riemann monotonicity for degenerate f on a $2:0$ junction)

Assume (2.2) with $N = 2$, $J^j \simeq (-\infty, 0)$ for $j = 1, 2$ and $[a, b] := [0, 1]^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \rightarrow \mathbb{R}$ for $j = 1, 2$ with

$$\begin{cases} f^2 = 0 \\ f^1 \text{ decreasing} \\ h : [0, 1] \rightarrow [0, 1] \text{ continuous increasing and bijective} \end{cases}$$

We set

$$\mathcal{G} := \{p = (p^1, p^2) \in [a, b], \quad p^1 = h(p^2)\}$$

Then \mathcal{G} is a Riemann germ with respect to (J, f) with

$$\begin{cases} \hat{f}^2 = 0 \\ \hat{f}^1(p) = (f^1 \circ h)(p^2) \end{cases}$$

Here \hat{f} is continuous but not Riemann monotone. Moreover \mathcal{G} is a monotone germ.

Proof of Lemma 8.3

Consider $p, q \in [a, b]$ with $p > q$, $\hat{f}^1(p) < \hat{f}^1(q)$, $\hat{f}^2(p) = 0 = \hat{f}^2(q)$. Hence $(p - q) \diamond [\hat{f}]_q^p \leq 0$ but $[\hat{f}]_q^p \neq 0$. This ends the proof of the lemma.

8.4 Counter-example to gluing without Riemann monotonicity

Lemma 8.4 (Counter-example to gluing Riemann germs without Riemann monotonicity)

Assume (2.2) with $N = 2$, $J^j \simeq (-\infty, 0)$ for $j = 1, 2$ and $[a, b] := [0, 1]^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \rightarrow \mathbb{R}$ for $j = 1, 2$ with

$$\begin{cases} f^2 = 0 \\ f^1 \text{ decreasing} \\ h : [0, 1] \rightarrow [0, 1] \text{ continuous increasing and bijective} \end{cases}$$

We set

$$\mathcal{G} := \{p = (p^1, p^2) \in [a, b], \quad p^1 = h(p^2)\} =: \bar{\mathcal{G}}$$

Let us define

$$\bar{f} = (\bar{f}^1, \bar{f}^2) := (-f^1, -f^2) \quad \text{and} \quad \bar{J} := \{0\} \cup \bigcup_{j=1,2} \bar{J}^2 \quad \text{with} \quad \bar{J}^j := -J^j \quad \text{for} \quad j = 1, 2$$

Then \mathcal{G} is a monotone Riemann germ with respect to (J, f) and $\bar{\mathcal{G}}$ is a monotone Riemann germ with respect to (\bar{J}, \bar{f}) with $\hat{f} = -\hat{f}$. Moreover both \mathcal{G} and $\bar{\mathcal{G}}$ are not Riemann monotone and the set

$$\mathcal{G}_0 := \bar{\mathcal{G}} \sharp \mathcal{G} \quad \text{for the gluing of branch } \bar{J}^2 \simeq (0, +\infty) \text{ with } J^2 \simeq (-\infty, 0)$$

defined as in (5.38) is such that \mathcal{G}_0 is not a generalized Riemann germ.

Proof of Lemma 8.4

From Lemma 8.3, we know that \mathcal{G} is a monotone Riemann germ with $\begin{cases} \hat{f}^2 = 0 \\ \hat{f}^1(p) = (f^1 \circ h)(p^2) \end{cases}$. Moreover, by construction, $\bar{\mathcal{G}}$ is obtained by reversion transform of \mathcal{G} (see Definition 4.4), and then is also a monotone Riemann germ. Define the set for $f^0 := 0 = f^2 = \bar{f}^2$

$$\mathcal{G}_0 := \left\{ (p^1, \bar{p}^1) \in [a^1, b^1]^2, \quad \text{there exists } (\bar{p}^2, p^2) \in [a^2, b^2]^2 \text{ s.t.} \quad \left| \begin{array}{l} (\bar{p}^1, \bar{p}^2) \in \bar{\mathcal{G}} \\ (p^1, p^2) \in \mathcal{G} \\ f^0(\bar{p}^2) = G^{f^0}(\bar{p}^2, p^2) = f^0(p^2) \end{array} \right. \right\}$$

By definition, we have $\mathcal{G}_0 = \bar{\mathcal{G}} \sharp \mathcal{G}$ for the gluing of branch $\bar{J}^2 \simeq (0, +\infty)$ with $J^2 \simeq (-\infty, 0)$. If \hat{f} and $\hat{\bar{f}}$ would be Riemann monotone, then $\mathcal{G}_0 \subset [a^1, b^1]^2$ would be a Riemann germ with respect to (J_0, f_0) with $f_0 := (f^1, \bar{f}^1)$ and $J_0 := \{0\} \cup J^1 \cup \bar{J}^1$. But we have $\mathcal{G}_0 = [a^1, b^1]^2$, and for any $q \in [a^1, b^1]^2$, we have

$$BA^{(J_0, f_0)}(q) = [a^1, b^1]^2$$

Therefore $(BA(\hat{q}))_{\hat{q} \in \mathcal{G}_0}$ is not a partition of $[a^1, b^1]^2$, which shows that \mathcal{G}_0 is not a generalized Riemann germ. This ends the proof of the lemma.

8.5 Counter-example to self-gluing of HJ germs

Lemma 8.5 (Counter-example to self-gluing of HJ germs)

Consider $g : [0, 1] \rightarrow \mathbb{R}$ with $g(u) := u(1-u)$. For $N = 3$, consider $f^0 = f^1 = f^2 := g$, $[a, b] := [0, 1]^3$ and branches $J^0 \simeq (-\infty, 0)$ and $J^1 \simeq J^2 \simeq (0, +\infty)$ and $J := \{0\} \cup \bigcup_{j=0,1,2} J^j$. We set the nondecreasing and nonincreasing envelopes of g

$$g^+(u) := \max_{[0, u]} g \quad \text{and} \quad g^-(u) := \max_{[u, 1]} g \quad \text{for all} \quad u \in [0, 1]$$

Then the set

$$\mathcal{G} := \{p = (p^0, p^1, p^2) \in [0, 1]^3, \quad \min \{g^+(p^0), g^-(p^1), g^-(p^2)\} = g(p^0) = g(p^1) = g(p^2)\}$$

is a HJ germ with respect to (J, f) , which is not a Kruřkov germ.

Let $\mathcal{G}^{2:0} \sharp$ be the self-gluing of \mathcal{G} along branches J^2 and J^0 for the flux $f^2 = f^0 = g$. Then $\mathcal{G}^{2:0} \sharp$ is not a generalized Riemann germ with respect to (J^1, f^1) .

Proof of Lemma 8.5

Here $g : [0, 1] \rightarrow \mathbb{R}$ is a bell-shaped flux. For $\gamma = (\gamma^0, \gamma^1, \gamma^2)$ and $N = 3$, define the function $\gamma : [0, +\infty)^N \rightarrow [0, +\infty)^N$ as $\hat{\gamma}^j(\gamma) := \min \{\gamma^0, \gamma^1, \gamma^2\}$ for $j = 0, 1, 2$. Then it is easy to check that $\hat{\gamma}$ is a HJ preflux in the sense of Definition 7.19. For the capacity $\bar{\gamma}(p) := (g^+(p^0), g^-(p^1), g^-(p^2))$, we then deduce that $\gamma := \hat{\gamma} \circ \bar{\gamma}$ is a Godunov flux, and that \mathcal{G} is the associated HJ germ. We set $G^g : [0, 1]^2 \rightarrow \mathbb{R}$ for the standard Godunov flux associated to the flux g , which satisfies here $G^g(p^L, p^R) = \min(g^+(p^L), g^-(p^R))$. Then recall that the standard Godunov germ is defined as $\mathcal{G}^g := \{p = (p^L, p^R) \in [0, 1]^2, \quad g(p^L) = G^g(p^L, p^R) = g(p^R)\}$. Then, by definition, we have

$$\mathcal{G}^{2:0} \sharp = \{p^1 \in [0, 1], \quad \text{there exists } (p^2, p^0) \in \mathcal{G}^g \text{ such that } p = (p^0, p^1, p^2) \in \mathcal{G}\}$$

For $\lambda \in [0, \frac{1}{4}]$, we define $u_{\pm}(\lambda) := (g^{\pm})^{-1}(\lambda)$. Now for any $p^1 \in [0, 1]$, set $\lambda := g(p^1)$ and $p^0 := u_+(\lambda) =: p^2$. Hence it is straightforward to check that $(p^2, p^0) \in \mathcal{G}^g$ and $(p^0, p^1, p^2) \in \mathcal{G}$, which implies that $p^1 \in \mathcal{G}^{0:2}$. Therefore $\mathcal{G}^{2:0} = [0, 1]$ and $(BA^{(J^1, f^1)}(\hat{p}^1))_{\hat{p}^1 \in \mathcal{G}^{0:2}}$ is not a partition of $[0, 1]$, by definition of the Basin of Attraction. Therefore Theorem 2.14 implies that $\mathcal{G}^{2:0}$ is not a generalized Riemann germ (with respect to (J^1, f^1)). Finally Corollary 5.15 implies that \mathcal{G} is not a Kruřkov germ. This ends the proof of the lemma.

8.6 Strange germs for 2 : 0 junctions and classification

Lemma 8.6 (Strange germs for 2 : 0 junctions and classification; $f^1 \downarrow, f^2 \uparrow$)

Assume (2.2) for a with $N = 2$, $J^j \simeq (-\infty, 0)$ for $j = 1, 2$ and $[a, b] := [0, 1]^2$. We set $J := \{0\} \cup J^1 \cup J^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \rightarrow \mathbb{R}$ for $j = 1, 2$ with

$$f^1 \text{ decreasing and } f^2 \text{ increasing.}$$

i) (Classification of Riemann germs)

Then the set $\mathcal{G} \subset [a, b]$ is a Riemann germ with respect to (J, f) if and only if there exists $h : [0, 1] \rightarrow [0, 1]$ continuous such that

$$(8.2) \quad \mathcal{G} = \mathcal{G}_h \quad \text{with} \quad \mathcal{G}_h := \{p = (p^1, p^2) \in [a, b], \quad p^1 = h(p^2)\}$$

We also have $\mathcal{G}_h = \{\hat{f}_h = f\}$ with continuous function $\hat{f}_h = \hat{f}_{\mathcal{G}_h}$ given by

$$(8.3) \quad \begin{cases} \hat{f}_h^2(p) := f^2(p^2) \\ \hat{f}_h^1(p) := f^1(h(p^2)) \end{cases}$$

ii) (Existence of strange germs, nonconservative Kruřkov germs)

If h is not monotone, then the Riemann germ \mathcal{G} is not monotone, not HJ , neither conservative. Moreover, if f^1, f^2, h are Lipschitz continuous, then the Riemann germ \mathcal{G} is Kruřkov if and only if

$$(8.4) \quad (f^2)' + \{(f^1)' \circ h\} \cdot |h'| \geq 0 \quad \text{a.e. on } [0, 1]$$

with the convention that $g \cdot |h'| = 0$ if $h' = 0$, even where g is not defined. In particular \mathcal{G} is not Kruřkov if $|h'|$ is large enough.

On the contrary, if $(f^2)' \geq \delta > 0$ a.e. on $[0, 1]$, and for $|h'|$ small enough with h non monotone, then \mathcal{G} is a nonmonotone nonconservative Kruřkov germ.

Moreover if h is nondecreasing such that the function $f^2 + f^1 \circ h$ is nondecreasing and non identically equal to zero, then \mathcal{G} is a monotone nonconservative Kruřkov germ.

Proof of Lemma 8.6

Step 1: proof of i)

Step 1.1: necessary inclusion of \mathcal{G}

From the slicing lemma 4.12, notice that if $\mathcal{G} \subset [a, b]$ is a Riemann germ with respect to (J, f) with $\hat{f} := \hat{f}_{\mathcal{G}}$, then for any fixed $p^2 \in [0, 1]$, the set

$$\mathcal{G}_{p^2} := \left\{ p^1 \in [0, 1], \quad \hat{f}^1(p^1, p^2) = f^1(p^1) \right\}$$

is a generalized Riemann germ with respect to (J^1, f^1) , which is also a Riemann germ because \hat{f} is continuous. Because f^1 is decreasing and the map $\hat{f}_{p^2}^1 : p^1 \mapsto \hat{f}^1(p^1, p^2)$ is nondecreasing and locally constant on $\{\hat{f}_{p^2}^1 \neq f^1\}$, we deduce that $\hat{f}_{p^2}^1$ is constant on $[0, 1]$. Moreover, we know that we have $\hat{f}_{p^2}^1 = f^1 \circ \pi_{\mathcal{G}_{p^2}}$. Because f^1 is decreasing, we deduce that $\pi_{\mathcal{G}_{p^2}} : [0, 1] \rightarrow [0, 1]$ is constant and set $h(p^2) := \pi_{\mathcal{G}_{p^2}}(p^1)$ for any $p^1 \in [0, 1]$. Hence we have $\hat{f}^1(p^1, p^2) := f^1(h(p^2))$. Still because f^1 is decreasing, and \hat{f}^1 is continuous, we deduce that $h : [0, 1] \rightarrow [0, 1]$ is continuous. Therefore

$$\mathcal{G} \subset \left\{ \hat{f}^1 = f^1 \right\} = \mathcal{G}_h \quad \text{with continuous } h : [0, 1] \rightarrow [0, 1].$$

It is also easy to check that $\mathcal{G}_h = \{\hat{f}_h = f\}$.

Step 1.2: property of \mathcal{G}_h

Conversely, consider $\mathcal{G}_h = \{\hat{f}_h = f\}$ with \hat{f}_h given in (8.3) for continuous $h : [0, 1] \rightarrow [0, 1]$. Notice also that because f^2 is increasing, we see that \hat{f}_h^j has expected monotonicities in p^j . Moreover it is easy to check that \hat{f}_h is locally constant on $\{\hat{f}_h \neq f\}$. Now from (2.15) and from the monotonicities of the f^j 's, we get

$$f_-^1 = f^1 \leq f^1(0) \equiv f_+^1 \quad \text{and} \quad f_-^2 \equiv f^2(0) \leq f^2 = f_+^2$$

Hence from ii) of Theorem 2.14, we deduce that \mathcal{G}_h is a generalized Riemann germ, and from i) of Theorem 2.16, we deduce that \mathcal{G}_h is a Riemann germ.

Step 1.3: conclusion for \mathcal{G}

Now from Step 1.1, we have $\mathcal{G} \subset \mathcal{G}_h \subset [a, b]$ with both \mathcal{G} and \mathcal{G}_h generalized Riemann germs (in $[a, b]$) with respect to (J, f) . From i) of Theorem 2.14, we conclude that $\mathcal{G} = \mathcal{G}_h$.

Step 2: proof of ii)

For conservative, HJ and monotone germs, we use the definitions of the classes of germs and also Lemma 5.5 for their characterization in term of their Godunov fluxes \hat{f} .

Hence the Riemann germ \mathcal{G} is conservative if and only if $\hat{f}^1 + \hat{f}^2 = 0$, i.e. $-f^1(h(p^2)) = f^2(p^2)$, i.e. the (continuous) function $h := (-f^1)^{-1} \circ f^2$ is increasing.

Similarly, the Riemann germ \mathcal{G} is HJ if and only if $\hat{f}^1 = \hat{f}^2$ and is nondecreasing in each coordinate p^j , i.e. $h := (f^1)^{-1} \circ f^2$ is decreasing.

Similarly, the Riemann germ \mathcal{G} is monotone if and only if $p^2 \mapsto \hat{f}^1(p)$ and $p^1 \mapsto \hat{f}^2(p)$ are nonincreasing, i.e. h is nondecreasing.

By definition, the Riemann germ \mathcal{G} is Kruřkov if and only if we have $D^f(\bar{p}, p) \geq 0$ for all $p, \bar{p} \in \mathcal{G}$, i.e.

$$(8.5) \quad \text{sign}(\bar{p}^1 - p^1) \cdot \{f^1(\bar{p}^1) - f^1(p^1)\} + \text{sign}(\bar{p}^2 - p^2) \cdot \{f^2(\bar{p}^2) - f^2(p^2)\} \geq 0 \quad \text{for all } p^1 := h(p^2), \quad \bar{p}^1 = h(\bar{p}^2), \quad p^2, \bar{p}^2 \in [0, 1]$$

Because the composition of Lipschitz functions is Lipschitz, and using Rademacher's theorem, we deduce (8.4) in the limit $|\bar{p}^2 - p^2| \rightarrow 0$. Conversely, the integration of (8.4) implies (8.5).

In particular, if h is nondecreasing such that the function $f^2 + f^1 \circ h$ is also nondecreasing and non identically equal to zero, then \mathcal{G} is a monotone nonconservative Kruřkov germ. This ends the proof of the lemma.

Lemma 8.7 (Classification for $2 : 0$ junctions; $f^1 \downarrow, f^2 \downarrow$)

Assume (2.2) for a with $N = 2$, $J^j \simeq (-\infty, 0)$ for $j = 1, 2$ and $[a, b] := [0, 1]^2$. We set $J := \{0\} \cup J^1 \cup J^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \rightarrow \mathbb{R}$ for $j = 1, 2$ with

$$f^1 \text{ and } f^2 \text{ decreasing.}$$

Then the set $\mathcal{G} \subset [a, b]$ is a Riemann germ with respect to (f, J) if and only if there exists $\hat{p} \in [a, b]$ such that $\mathcal{G} = \{\hat{p}\}$. When it is the case, then we also have $\mathcal{G} = \{\hat{f} = f\}$ with the function $\hat{f} = \hat{f}_{\mathcal{G}}$ given by $\hat{f} = \text{const} = f(\hat{p})$.

Proof of Lemma 8.7

Consider some $p \in [a, b]$, and set $\hat{p} := \pi(p)$ with $\pi := \pi_{\mathcal{G}} : [a, b] \rightarrow \mathcal{G}$ the natural projection map. Because f^1, f^2 are both decreasing for a $2 : 0$ junction, we deduce that $BA(\hat{p}) = [a, b]$, and then $\pi = \text{const} = \hat{p}$, which shows that $\mathcal{G} = \{\hat{p}\}$. Moreover $\hat{f}_{\mathcal{G}} = f(\hat{p})$. Conversely, for any $\hat{p} \in [a, b]$, it is straightforward to check that $\mathcal{G} = \{\hat{p}\}$ is a Riemann germ. This ends the proof of the lemma.

Lemma 8.8 (Classification for $2 : 0$ junctions; $f^1 \uparrow, f^2 \uparrow$)

Assume (2.2) for a with $N = 2$, $J^j \simeq (-\infty, 0)$ for $j = 1, 2$ and $[a, b] := [0, 1]^2$. We set $J := \{0\} \cup J^1 \cup J^2$. We consider Lipschitz continuous functions $f^j : [0, 1] \rightarrow \mathbb{R}$ for $j = 1, 2$ with

$$f^1 \text{ and } f^2 \text{ increasing.}$$

i) (Classification)

Then the set $\mathcal{G} \subset [a, b]$ is a Riemann germ with respect to (f, J) if and only if there exist continuous maps $h_{\pm}^j : [0, 1] \rightarrow [0, 1]$ for $j = 1, 2$ such that $0 \leq h_-^j \leq h_+^j \leq 1$, and for $\bar{j} \in \{1, 2\} \setminus \{j\}$, we have

$$\mathcal{G} = K^1 \cap K^2 \quad \text{with} \quad K^j := \left\{ p \in [a, b], \quad h_-^j(p^{\bar{j}}) \leq p^j \leq h_+^j(p^{\bar{j}}) \right\}$$

with moreover $\mathcal{G} = \left\{ \hat{f} = f \right\}$, with \hat{f} locally constant on $\left\{ \hat{f} \neq f \right\}$, where $\hat{f} : [a, b] \rightarrow \mathbb{R}^2$ is continuous and satisfies for $j = 1, 2$

$$(8.6) \quad \hat{f}^j(p) = f^j(T_{h_-^j(p^{\bar{j}})}^{h_+^j(p^{\bar{j}})}(p^{\bar{j}})) \quad \text{with} \quad T_x^y(z) = x \vee z \wedge y \quad \text{for} \quad x \leq y$$

and $x \vee z := \max(x, z)$, $y \wedge z := \min(y, z)$.

ii) (Further properties)

Moreover, we have the following properties for any $p \in [0, 1]^2$ with $\hat{p} := \pi_{\mathcal{G}}(p) \in [0, 1]^2$, we have

$$(8.7) \quad \left\{ \begin{array}{ll} \left((\hat{f} - f)^1(p) > 0 \quad \text{and} \quad (\hat{f} - f)^2(p) > 0 \right) & \text{implies} \quad BA(\hat{p}) \supset [(0, 0), \hat{p}] \quad \ni p \\ \left((\hat{f} - f)^1(p) > 0 \quad \text{and} \quad (\hat{f} - f)^2(p) < 0 \right) & \text{implies} \quad BA(\hat{p}) \supset [(0, \hat{p}^2), (\hat{p}^1, 1)] \quad \ni p \\ \left((\hat{f} - f)^1(p) < 0 \quad \text{and} \quad (\hat{f} - f)^2(p) > 0 \right) & \text{implies} \quad BA(\hat{p}) \supset [(\hat{p}^1, 0), (1, \hat{p}^2)] \quad \ni p \\ \left((\hat{f} - f)^1(p) < 0 \quad \text{and} \quad (\hat{f} - f)^2(p) < 0 \right) & \text{implies} \quad BA(\hat{p}) \supset [\hat{p}, (1, 1)] \quad \ni p \end{array} \right.$$

Proof of Lemma 8.8

Part 1: proof of i)

Step 1: necessary conditions on \mathcal{G}

Step 1.1: freezing p^2

From the slicing lemma 4.12, notice that if $\mathcal{G} \subset [a, b]$ is a Riemann germ with respect to (J, f) with $\hat{f} := \hat{f}_{\mathcal{G}}$, then for any fixed $p^2 \in [0, 1]$, the set

$$\mathcal{G}_{p^2} := \left\{ p^1 \in [0, 1], \quad \hat{f}^1(p^1, p^2) = f^1(p^1) \right\}$$

is a generalized Riemann germ with respect to (J^1, f^1) , which is also a Riemann germ because \hat{f} is continuous. Because f^1 is increasing and the map $\hat{f}_{p^2}^1 : p^1 \mapsto \hat{f}^1(p^1, p^2)$ is nondecreasing and locally constant on $\left\{ \hat{f}_{p^2}^1 \neq f^1 \right\}$, we deduce that $\hat{f}_{p^2}^1$ coincides with f^1 only on a subinterval $[h_-^1(p^2), h_+^1(p^2)] \subset [0, 1]$, and satisfies

$$\hat{f}_{p^2}^1(p^1) = f^1(T_{z_-}^{z_+}(p^1)) \quad \text{with} \quad \begin{cases} z_- := h_-^1(p^2) \\ z_+ := h_+^1(p^2) \end{cases}$$

Because \hat{f}^1 is continuous, we deduce that $\left\{ \hat{f}^1 = f^1 \right\}$ is a closed set, and then h_-^1 is lower semicontinuous and h_+^1 is upper semicontinuous. Moreover, it is easy to see that the strict monotonicity of f^1 and the continuity of \hat{f}^1 also imply the continuity of h_{\pm}^1 .

Step 1.2: freezing p^1 and first consequences

By symmetry from Step 1, we get a similar result, exchanging indices 1 and 2. This shows (8.6) with continuous maps $h_{\pm}^j : [0, 1] \rightarrow [0, 1]$. Moreover, this implies that $\mathcal{G} = \left\{ \hat{f} = f \right\} = K^1 \cap K^2$, and \hat{f} is locally constant on $\left\{ \hat{f} \neq f \right\}$ and $\hat{f} : [a, b] \rightarrow \mathbb{R}^2$ is continuous from i) of Theorem 2.16, because \mathcal{G} is a Riemann germ.

Step 2: sufficient conditions for \mathcal{G}

From ii) of Theorem 2.14, we only have to check the second line of (2.14). Recall that

$$f^j(p^j) = \inf_{[p^j, 1]} f^j = f_-^j(p^j) \leq f_+^j(p^j) = \sup_{[0, p^j]} f^j = f^j(1)$$

and because f^j is increasing. We deduce from the expression of \hat{f}^j that $f_-^j \leq \hat{f}^j \leq f_+^j$, and Theorem 2.14 implies that $\mathcal{G} = \left\{ \hat{f} = f \right\}$ is a generalized Riemann germ. Because \hat{f} is continuous, \mathcal{G} is then a Riemann

germ.

Part 2: proof of ii)

We only do the proof for the first line of (8.7) (the other cases are similar). Assume that $(\hat{f}^1 - f^1)(p) > 0$, $(\hat{f}^2 - f^2)(p) > 0$. Because f^1, f^2 are increasing, we deduce that $BA(\hat{p}) \cap [0_{\mathbb{R}^2}, \hat{p}] = [0_{\mathbb{R}^2}, \hat{p}]$, which shows the result with moreover $\hat{f} = f(\hat{p})$ on $[0_{\mathbb{R}^2}, \hat{p}]$. This ends the proof of the lemma.

8.7 An explicit example of gluing without cancellation property

Lemma 8.9 (A 1 : 1 explicit example of gluing without cancellation property)

Assume (2.2) with $N = 2$, $J^L \simeq (-\infty, 0)$ and $J^R \simeq (0, +\infty)$ and $f^j = g$ with $[a^j, b^j] = [0, 1]$ for $j = L, R$, with $g : [0, 1] \rightarrow \mathbb{R}$ strictly concave with $g(0) = 0 = g(1)$. Let $A_0 := \max_{[0,1]} g = g(p_0) > 0$ with $p_0 \in (0, 1)$ and $A \in [0, A_0]$. We define

$$g^+(x) := \begin{cases} g(x) & \text{for } x \in [0, p_0] \\ g(p_0) & \text{for } x \in (p_0, 1] \end{cases} \quad \text{and} \quad g^-(x) := \begin{cases} g(p_0) & \text{for } x \in [0, p_0] \\ g(x) & \text{for } x \in (p_0, 1] \end{cases}$$

and

$$\mathcal{G}_A := \{(p^L, p^R) \in [0, 1]^2, \quad \min \{A, g^+(p^L), g^-(p^R)\} = g(p^L) = g(p^R)\}$$

Then for any $A, B \in [0, A_0]$, we have

$$\mathcal{G}_A \# \mathcal{G}_B = \mathcal{G}_{\min\{A, B\}} \quad \text{for the gluing of } J_{\mathcal{G}_A}^R \simeq (0, +\infty) \text{ with } J_{\mathcal{G}_B}^L \simeq (-\infty, 0)$$

In particular we always have

$$\mathcal{G}_{A_0} \# \mathcal{G}_A = \mathcal{G}_A \# \mathcal{G}_A \quad \text{and} \quad \mathcal{G}_A \# \mathcal{G}_{A_0} = \mathcal{G}_A \# \mathcal{G}_A$$

which does not imply the cancellation property that $\mathcal{G}_A = \mathcal{G}_{A_0}$.

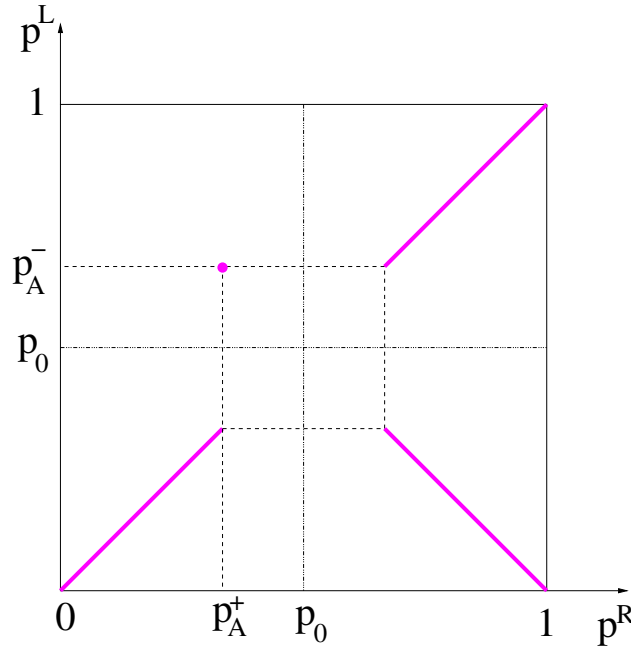


Figure 2: The germ \mathcal{G}_A for $A \in (0, A_0)$

Proof of Lemma 8.9

For $\lambda \in [0, A_0]$, let $p_\lambda^\pm \in [0, 1]$ be such that $g(p_\lambda^\pm) = \lambda = g^\pm(p_\lambda^\pm)$. Notice that \mathcal{G}_λ is a HJ germ. From [14], we know that all HJ germs with all convex fluxes (and then also with all concave fluxes) are classified by a

flux limiter, which is λ , and particularly by a single point $(p_\lambda^-, p_\lambda^+) \in \mathcal{G}_\lambda$, because $\chi\mathcal{G}_\lambda = \{(p_\lambda^-, p_\lambda^+)\}$, where $\chi\mathcal{G}_\lambda$ is the characteristic subset of the HJ germ \mathcal{G}_λ (see Theorem 2.28). Let

$$\tilde{\mathcal{G}}_0 := \mathcal{G}_A \# \mathcal{G}_B = \left\{ (p^L, \tilde{p}^R) \in [0, 1]^2, \quad \text{there exists } (\tilde{p}^L, p^R) \in [0, 1]^2 \text{ s.t. } \begin{cases} (p^L, p^R) \in \mathcal{G}_A \\ (\tilde{p}^L, \tilde{p}^R) \in \mathcal{G}_B \\ g(p^R) = G^g(p^R, \tilde{p}^L) = g(\tilde{p}^L) \end{cases} \right\}$$

From Theorem 2.19, we know that $\tilde{\mathcal{G}}_0$ is a HJ germ, and then of the form $\tilde{\mathcal{G}}_0 = \mathcal{G}_\lambda$ for some $\lambda \in [0, A_0]$. Moreover $(p^L, \tilde{p}^R) \in \tilde{\mathcal{G}}_0$ if and only if (using $G^g(p^R, \tilde{p}^L) = \min\{g^+(p^R), g^-(\tilde{p}^L)\}$)

$$\begin{cases} \min\{A, g^+(p^L), g^-(p^R)\} = g(p^L) = g(p^R) \\ \min\{B, g^+(\tilde{p}^L), g^-(\tilde{p}^R)\} = g(\tilde{p}^L) = g(\tilde{p}^R) \\ g(p^R) = \min\{g^+(p^R), g^-(\tilde{p}^L)\} = g(\tilde{p}^L) \end{cases}$$

For $C := \min\{A, B\}$, we have (with obvious notation for the gluing $(p^L, p^R) \# (\tilde{p}^L, \tilde{p}^R) := (p^L, \tilde{p}^R)$)

$$\tilde{\mathcal{G}}_0 \ni (p_C^-, p_C^+) = \begin{cases} (p_C^-, p_C^+) \# (p_C^+, p_C^+) & \text{if } C = A \\ (p_C^-, p_C^-) \# (p_C^-, p_C^+) & \text{if } C = B \end{cases}$$

We deduce that $\lambda = C$ and $\tilde{\mathcal{G}}_0 = \mathcal{G}_C$ which ends the proof of the lemma.

8.8 An example of non commutativity of the gluing

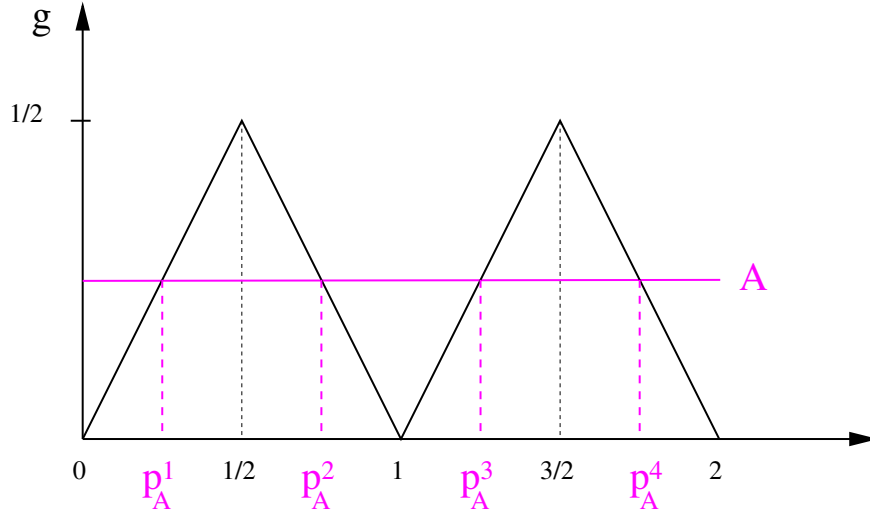


Figure 3: Graph of g

Lemma 8.10 (Explicit example of a HJ germ for 1 : 1 junction)

We set $J := \{0\} \cup J^L \cup J^R$ with $J^L \simeq (-\infty, 0)$ and $J^R \simeq (0, +\infty)$. We also set $f = (f^L, f^R) := (g, g)$ and $[a, b] := [0, 2]^2$, with

$$g : [0, 2] \rightarrow \mathbb{R} \quad \text{with} \quad g(u) := g_0(u) + g_0(u-1) \quad \text{and} \quad g_0(u) := \max\{0, \min\{u, 1-u\}\}$$

Given $\mathcal{A} := (A, B, C, D)$ with $\frac{1}{2} > A > B > C > D > 0$, we want to define a germ $\mathcal{G}_\mathcal{A}$. To this end, given $\lambda \in [0, \frac{1}{2}]$, we set

$$p_\lambda^1 := \lambda, \quad p_\lambda^2 = 1 - \lambda, \quad p_\lambda^3 := 1 + \lambda, \quad p_\lambda^4 := 2 - \lambda \quad \text{with} \quad \begin{cases} g(p_\lambda^k) = \lambda \quad \text{for } k = 1, \dots, 4 \\ 0 \leq p_\lambda^1 \leq p_\lambda^2 \leq p_\lambda^3 \leq p_\lambda^4 \leq 2 \end{cases}$$

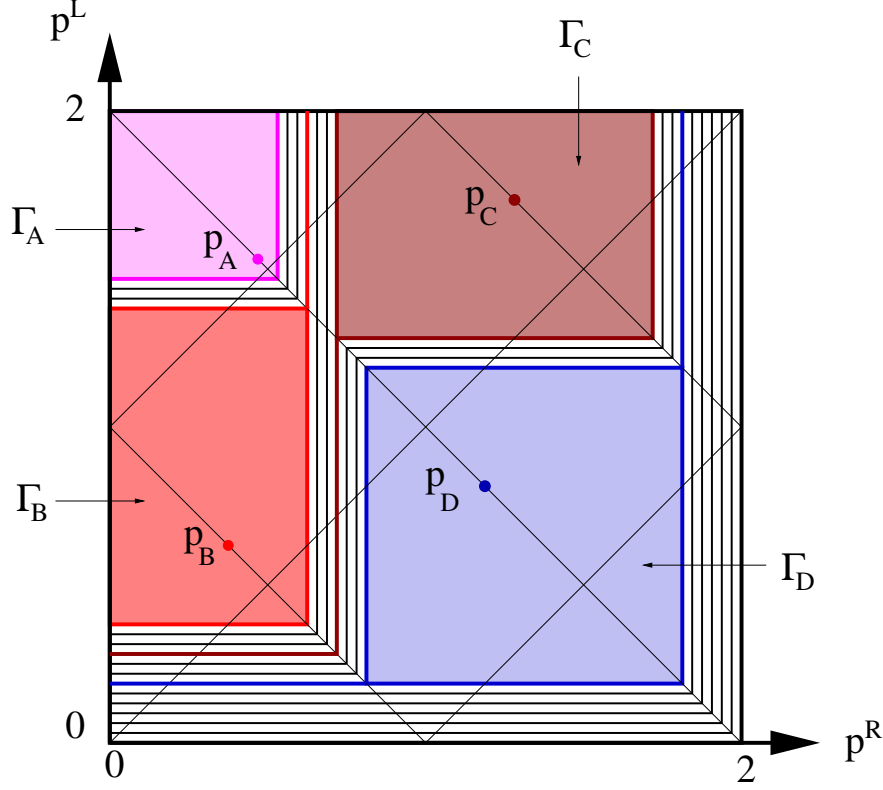


Figure 4: Level sets of \hat{h} with four plateaux (see (8.8))

We also set for $p = (p^L, p^R)$

$$\begin{cases} p_A := (p_A^4, p_A^1) \\ p_B := (p_B^2, p_B^1) \\ p_C := (p_C^4, p_C^3) \\ p_D := (p_D^2, p_D^3) \end{cases}$$

We then define for $p = (p^L, p^R) \in [a, b] := [0, 2]^2$

$$(8.8) \quad \hat{h}(p) := \lambda \quad \text{for } p \in \Gamma_\lambda$$

with

$$\Gamma_\lambda := \begin{cases} [p_A^3, 2] \times [0, p_A^1] & \text{for } \lambda = A \\ \Gamma_\lambda^{AB} := (\{p_\lambda^3\} \times [0, p_\lambda^2]) \cup ([p_\lambda^3, 2] \times \{p_\lambda^2\}) & \text{for } \lambda \in (A, B) \\ ([p_B^1, p_B^3] \times [0, p_B^2]) \cup \Gamma_B^{AB} & \text{for } \lambda = B \\ \Gamma_\lambda^{BC} := (\{p_\lambda^1\} \times [0, p_\lambda^2]) \cup ([p_\lambda^1, 2] \times \{p_\lambda^2\}) & \text{for } \lambda \in (B, C) \\ ([p_C^3, 2] \times [p_C^2, p_C^4]) \cup \Gamma_C^{BC} & \text{for } \lambda = C \\ \Gamma_\lambda^{CD} := (\{p_\lambda^1\} \times [0, p_\lambda^2]) \cup ([p_\lambda^1, p_\lambda^3] \times \{p_\lambda^2\}) \cup (\{p_\lambda^3\} \times [p_\lambda^2, p_\lambda^4]) \cup ([p_\lambda^3, 2] \times \{p_\lambda^4\}) & \text{for } \lambda \in (C, D) \\ ([p_\lambda^1, p_\lambda^3] \times [p_\lambda^2, p_\lambda^4]) \times \Gamma_D^{CD} & \text{for } \lambda = D \\ \Gamma_\lambda^{D0} := (\{p_\lambda^1\} \times [p_\lambda^1, p_\lambda^4]) \cup ([p_\lambda^1, 2] \times \{p_\lambda^4\}) & \text{for } \lambda \in (D, 0) \end{cases}$$

which is a continuous function $\hat{h} : [0, 2]^2 \rightarrow \mathbb{R}$. Let $\hat{f} = (\hat{h}, \hat{h})$. Then

$$\mathcal{G}_A := \{p \in [a, b], \quad \hat{f}(p) = f(p)\}$$

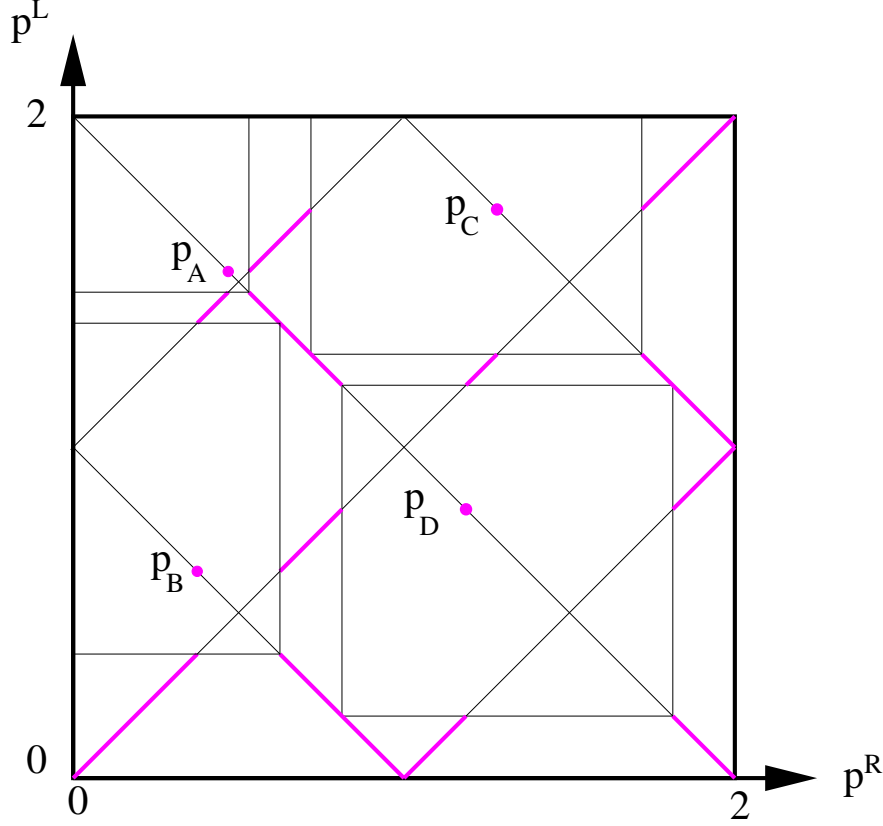


Figure 5: HJ Germ $\mathcal{G}_A \subset \{g(p^L) = g(p^R)\}$ and characteristic subset $\chi\mathcal{G}_A = \{p_A, p_B, p_C, p_D\}$

is a generalized Riemann germ which is a HJ germ (and then also a conservative germ and a Kružkov germ) with respect to (J, f) . Moreover we have

$$(8.9) \quad \chi\mathcal{G}_A = \{p_A, p_B, p_C, p_D\}$$

Proof of Lemma 8.10

It is easy to check that $\hat{h} : [0, 2]^2 \rightarrow \mathbb{R}$ is continuous. Moreover by construction $\hat{f} := (\hat{h}, \hat{h})$ is locally constant on $\{\hat{f} \neq f\}$, with \hat{h} nonincreasing in p^R and nondecreasing in p^L . Moreover, we have the following monotone bounds

$$\max \{f_-^L(p^L), f_-^R(p^R)\} = 0 \leq \hat{h}(p) \leq \min \{f_+^L(p^L), f_+^R(p^R)\} = \min \left\{ \frac{1}{2}, p^L, 2 - p^R \right\}$$

with

$$\begin{cases} f_+^L(p^L) := \sup_{[0, p^L]} g = \min \left\{ p^L, \frac{1}{2} \right\} \\ f_-^L(p^L) := \inf_{[p^L, 2]} g = 0 \\ f_-^R(p^R) := \inf_{[0, p^R]} g = 0 \\ f_+^R(p^R) := \sup_{[p^R, 2]} g = \min \left\{ 2 - p^R, \frac{1}{2} \right\} \end{cases}$$

Then from ii) of Theorem 2.14, we deduce that \mathcal{G}_A is a generalized Riemann germ with respect to (J, f) . Moreover by construction it is a HJ germ. Finally it is easy to check (8.9). This ends the proof of the lemma.

Lemma 8.11 (An example of non commutativity of the gluing)

We work with notation of Lemma 8.10. Let

$$A := (A, B, C, D) \quad \text{with} \quad \frac{1}{2} > A > B > C > D > 0$$

and

$$A' := (A', B', C, D) \quad \text{with} \quad \frac{1}{2} > A' > B' > C > D > 0$$

and

$$A' > A > B' > B > C > D$$

Then the gluing satisfies

$$(8.10) \quad \tilde{\mathcal{G}}' := \mathcal{G}_{A'} \# \mathcal{G}_{A'} = \mathcal{G}_{(B', B, C, D)} \neq \mathcal{G}_{(B, B, C, D)} = \mathcal{G}_{A'} \# \mathcal{G}_A =: \tilde{\mathcal{G}}$$

Proof of Lemma 8.11

First notice that by gluing HJ germs are preserved. Hence both $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ are HJ germs, and are then characterized by their characteristic subsets that we expect to satisfy

$$(8.11) \quad \chi_{\tilde{\mathcal{G}}} \supset \{p_{\tilde{A}}, p_{\tilde{B}}, p_{\tilde{C}}, p_{\tilde{D}}\} \quad \text{and} \quad \chi_{\tilde{\mathcal{G}}'} \supset \{p_{\tilde{A}'}, p_{\tilde{B}'}, p_{\tilde{C}'}, p_{\tilde{D}'}\}$$

Precisely we compute (with obvious notation for $p = (p^L, p^R)$ and the gluing $(p^L, p^R) \# (p^{L'}, p^{R'}) := (p^L, p^{R'})$)

$$\left\{ \begin{array}{llll} p_{\tilde{A}'} := (p_{B'}^4, p_{B'}^1) & = (p_{B'}^4, p_{B'}^2) \# (p_{B'}^2, p_{B'}^1) & \in \mathcal{G}_{A'} \# \mathcal{G}_{A'} & \text{with } \tilde{A}' := B' \\ p_{\tilde{B}'} := (p_{B'}^2, p_{B'}^1) & = (p_{B'}^2, p_{B'}^1) \# (p_{B'}^1, p_{B'}^1) & \in \mathcal{G}_{A'} \# \mathcal{G}_{A'} & \text{with } \tilde{B}' := B \\ p_{\tilde{C}'} := (p_{C'}^4, p_{C'}^3) & = (p_{C'}^4, p_{C'}^4) \# (p_{C'}^4, p_{C'}^3) & \in \mathcal{G}_{A'} \# \mathcal{G}_{A'} & \text{with } \tilde{C}' := C \\ p_{\tilde{D}'} := (p_{D'}^2, p_{D'}^3) & = (p_{D'}^2, p_{D'}^2) \# (p_{D'}^2, p_{D'}^3) & \in \mathcal{G}_{A'} \# \mathcal{G}_{A'} & \text{with } \tilde{D}' := D \\ \\ p_{\tilde{A}} := (p_B^4, p_B^1) & = (p_B^4, p_B^2) \# (p_B^2, p_B^1) & \in \mathcal{G}_{A'} \# \mathcal{G}_A & \text{with } \tilde{A} := B \\ p_{\tilde{B}} := (p_B^2, p_B^1) & = (p_B^2, p_B^2) \# (p_B^2, p_B^1) & \in \mathcal{G}_{A'} \# \mathcal{G}_A & \text{with } \tilde{B} := B \\ p_{\tilde{C}} := (p_C^4, p_C^3) & = (p_C^4, p_C^4) \# (p_C^4, p_C^3) & \in \mathcal{G}_{A'} \# \mathcal{G}_A & \text{with } \tilde{C} := C \\ p_{\tilde{D}} := (p_D^2, p_D^3) & = (p_D^2, p_D^2) \# (p_D^2, p_D^3) & \in \mathcal{G}_{A'} \# \mathcal{G}_A & \text{with } \tilde{D} := D \end{array} \right.$$

This shows (8.11), and then Theorem 2.28 shows (8.10). This ends the proof of the lemma.

8.9 D -maximality does not imply completeness

We give two examples. The first example is explicit for $2 : 0$ junction, while the second is less explicit for junctions $0 : 3$.

Lemma 8.12 (Explicit conservative D -maximal set, which is not complete, for $2:0$ junctions)

Let $g(x) := |x| - 1$ with $g : \mathbb{R} \rightarrow \mathbb{R}$, and a $2 : 0$ junction J with $f^1 = f^2 := g$. There exists an explicit set $\mathcal{G} \subset \mathbb{R}^2$ satisfying

$$(8.12) \quad \begin{cases} D^f \geq 0 & \text{on } \mathcal{G} \times \mathcal{G} \\ f^1 + f^2 = 0 & \text{on } \mathcal{G} \end{cases}$$

Then the set \mathcal{G} is conservative D -maximal in the following sense: if some set $\mathcal{G}' \subset \mathbb{R}^2$ satisfies (8.12) and $\mathcal{G} \subset \mathcal{G}'$, then $\mathcal{G}' = \mathcal{G}$. Moreover \mathcal{G} is not complete, i.e. $\bigcup_{\hat{p} \in \mathcal{G}} BA(\hat{p}) \neq \mathbb{R}^2$.

Proof of Lemma 8.12

Step 0: preliminaries

We consider the following three points $A := (2, 0)$, $B := (0, 2)$, $B' := (0, -2)$, and define the subset $\mathcal{G} \subset \mathbb{R}^2$ as $\mathcal{G} := S_{AB} \cup S_{AB'}$, where S_{AB} is the closed segment joining A to B in \mathbb{R}^2 . For all $p, q \in \mathbb{R}^2$, recall the dissipation

$$D^f := D^f(p, q) := \sum_{j=1,2} \text{sign}(p^j - q^j) \cdot [f^j]_{q^j}^{p^j} \quad \text{with} \quad [f^j]_{q^j}^{p^j} := f^j(p^j) - f^j(q^j)$$

Step 1: proof that $D^f \geq 0$ on $\mathcal{G} \times \mathcal{G}$

We have

$$D^f = \begin{cases} [f^1]_{q^1}^{p^1} + [f^2]_{q^2}^{p^2} \geq 0 & \text{if } p, q \in S_{AB}, & \text{with } p^2 = p^1 \geq q^1 = q^2, \\ [f^1]_{q^1}^{p^1} - [f^2]_{q^2}^{p^2} \geq 0 & \text{if } p, q \in S_{AB'}, & \text{with } -p^2 = p^1 \geq q^1 = -q^2, \\ [f^1]_{q^1}^{p^1} + [f^2]_{q^2}^{p^2} \geq 0, & \text{if } p \in S_{AB}, \quad q \in S_{AB'}, & \text{with } p^2 = p^1 \geq q^1 = -q^2 \geq 0, \\ -[f^1]_{q^1}^{p^1} + [f^2]_{q^2}^{p^2} \geq 0, & \text{if } p \in S_{AB}, \quad q \in S_{AB'}, & \text{with } 0 \leq p^2 = p^1 < q^1 = -q^2. \end{cases}$$

Step 2: maximality of \mathcal{G}

Notice first that it is straightforward to check that $f^1 + f^2 = 0$ on \mathcal{G} .

Assume now that there exists $p \in \mathbb{R}^2$ such that $\mathcal{G}' := \{p\} \cup \mathcal{G}$ satisfies (8.12). Then we have $(f^1 + f^2)(p) = 0$, i.e. $|p^1| + |p^2| = 2$. If $p^1 < 0$, then $|p^2| < 2$ and we get

$$D^f(B', p) = \{f^1(0) - f^1(p^1)\} - \{f^2(-2) - f^2(p^2)\} = -|p^1| - \{2 - |p^2|\} = 2|p^2| - 4 < 0$$

Contradiction. Therefore $p^1 \geq 0$ and $p \in \mathcal{G}$, i.e. $\mathcal{G}' = \mathcal{G}$. We conclude that $\mathcal{G} \subset \mathbb{R}^2$ satisfying (8.12) is maximal for the inclusion.

Step 3: basin of attraction of \mathcal{G}

We have

$$\begin{cases} BA(p) = (-\infty, 0] \times (-\infty, 2), & \text{for } p = B' \\ BA(p) = \{p^1\} \times (\infty, -p^2) & \text{for all } p \in S_{AB'} \setminus \{B'\} \\ BA(p) = \{p\} & \text{for all } p \in S_{AB} \setminus \{A, B\} \\ BA(p) = (-\infty, 0] \times \{2\}, & \text{for } p = B \end{cases}$$

Then

$$\bigcup_{\hat{p} \in \mathcal{G}} BA(\hat{p}) = \mathbb{R}^2 \setminus \Omega_0 \quad \text{with the open set } \Omega_0 := \{p = (p^1, p^2) \in \mathbb{R}^2, \quad p^2 > h(p^1)\} \neq \emptyset$$

where $h(p^1) := \min\{2, 2 - p^1, -\infty \cdot 1_{\{p^1 > 2\}}\}$. This shows that \mathcal{G} is not complete and ends the proof of the lemma.

Lemma 8.13 (A conservative D -maximal set, which is not complete, for 3:0 junctions)

There exists a 3 : 0 junction J with $f^j = g$ for $j = 1, 2, 3$ for some Lipschitz continuous function $g : [\alpha, \beta] \rightarrow \mathbb{R}$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. We set $a := (\alpha, \alpha, \alpha)$ and $b := (\beta, \beta, \beta)$. Then there exists a set $\mathcal{G} \subset [a, b] \subset \mathbb{R}^N$ for $N = 3$ satisfying

$$(8.13) \quad \begin{cases} D^f \geq 0 & \text{on } \mathcal{G} \times \mathcal{G} \\ \sum_{j=1, \dots, N} f^j = 0 & \text{on } \mathcal{G} \end{cases}$$

Moreover \mathcal{G} is conservative D -maximal in the following sense: for every set $\mathcal{G}' \subset [a, b]$ satisfying (8.13) such that $\mathcal{G} \subset \mathcal{G}'$, then $\mathcal{G}' = \mathcal{G}$. Moreover \mathcal{G} is not complete, i.e. $\bigcup_{\hat{p} \in \mathcal{G}} BA(\hat{p}) \neq [a, b]$.

Proof of Lemma 8.13

Step 1: properties of \mathcal{G}_0

Let $\varepsilon, \delta, \eta \in (0, 1)$ be such that $\alpha < -1 < \varepsilon + \delta < \beta$ and g such that

$$g(\alpha) = 0, \quad g(-1) = \eta, \quad g(\varepsilon) = -(1 + \eta), \quad g(\varepsilon + \delta) = 1, \quad g(\beta) = 0$$

We set

$$\begin{cases} U_1 := -e_1 + \varepsilon e_2 + (\varepsilon + \delta)e_3 \\ U_2 := -e_2 + \varepsilon e_3 + (\varepsilon + \delta)e_1 \\ U_3 := -e_3 + \varepsilon e_1 + (\varepsilon + \delta)e_2 \end{cases}$$

and consider the set

$$\mathcal{G}_0 := \{a, b, U_1, U_2, U_3\} \subset [a, b].$$

Using $g(-1) + g(\varepsilon) + g(\varepsilon + \delta) = 0$, it is straightforward to check that \mathcal{G}_0 satisfies the second line of (8.13). Now let us check that we have

$$(8.14) \quad D_+^f \geq 0 \quad \text{on } \mathcal{G}_0 \times \mathcal{G}_0$$

which implies that \mathcal{G}_0 satisfies the first line of (8.13), because $D^f(p, q) = D_+^f(p, q) + D_-^f(q, p)$. In order to check (8.14), now consider the matrix E whose lines are vectors U_1, U_2, U_3 , i.e.

$$E = \begin{pmatrix} U_1^1 & U_1^2 & U_1^3 \\ U_2^1 & U_2^2 & U_2^3 \\ U_3^1 & U_3^2 & U_3^3 \end{pmatrix} = \begin{pmatrix} -1 & \varepsilon & \varepsilon + \delta \\ \varepsilon + \delta & -1 & \varepsilon \\ \varepsilon & \varepsilon + \delta & -1 \end{pmatrix}$$

Then we consider $B = g(E)$ componentwise, we get with $g_i^j := g(U_i^j)$

$$B = \begin{pmatrix} g_1^1 & g_1^2 & g_1^3 \\ g_2^1 & g_2^2 & g_2^3 \\ g_3^1 & g_3^2 & g_3^3 \end{pmatrix} = \begin{pmatrix} \eta & -(1+\eta) & 1 \\ 1 & \eta & -(1+\eta) \\ -(1+\eta) & 1 & \eta \end{pmatrix}$$

Recall that

$$D_+(U_1, U_2) = \text{sign}^+(U_1^1 - U_2^1) \cdot \{g_1^1 - g_2^1\} + \text{sign}^+(U_1^2 - U_2^2) \cdot \{g_1^2 - g_2^2\} + \text{sign}^+(U_1^3 - U_2^3) \cdot \{g_1^3 - g_2^3\}$$

We get

$$\begin{cases} D_+(U_1, U_2) = \{g_1^2 - g_2^2\} + \{g_1^3 - g_2^3\} = -\{g_1^1 - g_2^1\} = D_+(U_2, U_1) \\ D_+(U_2, U_3) = \{g_2^1 - g_3^1\} + \{g_2^2 - g_3^2\} = -\{g_2^3 - g_3^3\} = D_+(U_3, U_2) \\ D_+(U_3, U_1) = \{g_3^1 - g_1^1\} + \{g_3^2 - g_1^2\} = -\{g_3^3 - g_1^3\} = D_+(U_1, U_3) \end{cases}$$

Hence $D_+(U_i, U_j) \geq 0$ for $i, j = 1, \dots, 3$ if and only if $g_1^1 \leq g_2^1, g_2^2 \leq g_3^2, g_3^3 \leq g_1^3$, which is the case because $g_1^1 = g_2^2 = g_3^3 = \eta \leq 1 = g_2^1 = g_3^2 = g_1^3$. We also have $D_+(a, b) = 0 = D_+(b, a)$ because $g(\alpha) = 0 = g(\beta)$. Moreover Rankine-Hugoniot relation $f^1 + f^2 + f^3 = 0$ implies

$$D_+(U_i, a) = 0 = D_+(a, U_i), \quad D_+(U_i, b) = 0 = D_+(b, U_i), \quad \text{for } i = 1, 2, 3$$

Hence (8.14) holds true, and then \mathcal{G}_0 satisfies (8.13).

Step 2: definition of $\tilde{\mathcal{G}}$

We now consider a set $\tilde{\mathcal{G}} \subset [a, b]$ satisfying (8.13) with $\mathcal{G}_0 \subset \tilde{\mathcal{G}}$, and such that $\tilde{\mathcal{G}}$ is maximal for the inclusion.

Step 3: uncompleteness of $\tilde{\mathcal{G}}$

Assume by contradiction that $\tilde{\mathcal{G}}$ is complete, i.e. that $\bigcup_{\hat{p} \in \tilde{\mathcal{G}}} BA(\hat{p}) = [a, b]$. From Lemma 3.4, we deduce that

$(BA(\hat{p}))_{\hat{p} \in \tilde{\mathcal{G}}}$ is a partition of $[a, b]$, and then $\tilde{\mathcal{G}}$ is a generalized Riemann germ. Because $\tilde{\mathcal{G}}$ satisfies (8.13), we deduce that $\tilde{\mathcal{G}}$ is a conservative Kruřkov germ, hence $\hat{f} = \hat{f}_{\tilde{\mathcal{G}}}$ satisfies for $N = 3$

$$(8.15) \quad \begin{cases} D^{\hat{f}} \geq 0 & \text{on } [a, b]^2 \\ \sum_{j=1, \dots, N} \hat{f}^j = 0 & \text{on } [a, b] \end{cases}$$

We set $\hat{f}_0^j := \hat{f}^j(0)$ for $j = 1, 2, 3$ and $0 = 0_{\mathbb{R}^3}$. We compute

$$\begin{cases} D^{\hat{f}}(U_1, 0) = \{g_1^2 - \hat{f}_0^2\} + \{g_1^3 - \hat{f}_0^3\} = -\{g_1^1 - \hat{f}_0^1\} = D^{\hat{f}}(0, U_1) \\ D^{\hat{f}}(U_2, 0) = \{g_2^1 - \hat{f}_0^1\} + \{g_2^2 - \hat{f}_0^2\} = -\{g_2^3 - \hat{f}_0^3\} = D^{\hat{f}}(0, U_2) \\ D^{\hat{f}}(U_3, 0) = \{g_3^1 - \hat{f}_0^1\} + \{g_3^2 - \hat{f}_0^2\} = -\{g_3^3 - \hat{f}_0^3\} = D^{\hat{f}}(0, U_3) \end{cases}$$

From the first line of (8.15), we deduce $g_1^1 \leq \hat{f}_0^1, g_2^2 \leq \hat{f}_0^2, g_3^3 \leq \hat{f}_0^3$, which gives

$$0 < 3\eta = g_1^1 + g_2^2 + g_3^3 \leq \hat{f}_0^1 + \hat{f}_0^2 + \hat{f}_0^3 = 0$$

where the last equality follows from the second line of (8.15). Contradiction. Hence $\tilde{\mathcal{G}}$ is not complete and this ends the proof.

Corollary 8.14 (Counter-example to completeness for $N \geq 3$)

Let $N \geq 3$. Then there exists a $N : 0$ junction and particular Lipschitz continuous functions $(f^j)_{j=1, \dots, N}$ and a set $\mathcal{G} \subset [a, b] \subset \mathbb{R}^N$ satisfying (8.13). Moreover \mathcal{G} is conservative D -maximal in the following sense: for every set $\mathcal{G}' \subset [a, b]$ satisfying (8.13) such that $\mathcal{G} \subset \mathcal{G}'$, then $\mathcal{G}' = \mathcal{G}$. Moreover \mathcal{G} is not complete, i.e.

$$\bigcup_{\hat{p} \in \mathcal{G}} BA(\hat{p}) \neq [a, b].$$

Proof of Corollary 8.14

For $N = 3$, the result follows from Lemma 8.13 for some set \mathcal{G}_3 satisfying (8.13) which is maximal for the inclusion, and the contradiction to the completeness of \mathcal{G}_3 precisely follows from the evaluation of

$$(8.16) \quad D^{\hat{f}_{\mathcal{G}_3}}(0_{\mathbb{R}^3}, U_j) \geq 0, \quad j = 1, 2, 3.$$

For $N \geq 3$, we set

$$(8.17) \quad p'_N = (p_N^4, \dots, p_N^N) \quad \text{with} \quad p_N^k := (f^k)^{-1}(0), \quad k = 4, \dots, N.$$

Now fix some $N \geq 3$, and assume that there exists some set \mathcal{G}_N satisfying (8.13) which is maximal for the inclusion. Let us now show that we can transfer the result to the level $N + 1$. Indeed, applying Lemma 7.8, we see that we can construct a set $\tilde{\mathcal{G}}_N \subset [\tilde{a}, \tilde{b}] \subset \mathbb{R}^{N+1}$ which satisfies (8.13), hence at the level $N + 1$. Then consider a set \mathcal{G}_{N+1} with $\tilde{\mathcal{G}}_N \subset \mathcal{G}_{N+1} \subset [\tilde{a}, \tilde{b}] \subset \mathbb{R}^{N+1}$ satisfying (8.13) at the level $N + 1$, and which is maximal for the inclusion.

Assume now by contradiction that the set \mathcal{G}_{N+1} is complete. Then the argument of Step 3 of the proof of Lemma 8.13 applies and shows that \mathcal{G}_{N+1} is indeed a conservative Kruřkov germ. In particular $\hat{f}_{\mathcal{G}_{N+1}}$ is continuous, and using definition (8.17) at level $N + 1$, we get

$$D^{\hat{f}_{\mathcal{G}_3}}(0_{\mathbb{R}^3}, U_j) = D^{\hat{f}_{\mathcal{G}_{N+1}}}((0_{\mathbb{R}^3}, p'_{N+1}), (U_j, p'_{N+1})) \geq 0, \quad j = 1, 2, 3$$

which leads to the same contradiction as (8.16) did. Therefore $\mathcal{G}_{N+1} \subset \mathbb{R}^{N+1}$ must be not complete. This ends the proof of the corollary.

9 Appendix of Part I

9.1 Standard Riemann problem

We consider the entropy solution $u = u(t, x)$ to the following Riemann problem

$$(9.1) \quad \begin{cases} u_t + (g(u))_x = 0 & \text{on } (0, +\infty) \times \mathbb{R} \\ u(0, x) = u_0(x) := \begin{cases} p_L & \text{if } x < 0 \\ p_R & \text{if } x > 0 \end{cases} \end{cases}$$

Lemma 9.1 (Explicit solution to Riemann's problem on the real line)

Assume that $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, locally Lipschitz. Then for any $p_L, p_R \in \mathbb{R}$, there exists a unique entropy solution u to (9.1). It satisfies $u(t, x) = U(x/t)$ for all $t > 0$. Moreover, let us define $I := [\min(p_L, p_R), \max(p_L, p_R)]$ and

$$\tilde{g} := \begin{cases} \text{convex envelop of } g^I & \text{if } p_L < p_R \\ \text{concave envelop of } g^I & \text{if } p_L > p_R \end{cases} \quad \text{with} \quad g^I(a) := \begin{cases} g(a) & \text{if } a \in I \\ +\infty & \text{if } a \in \mathbb{R} \setminus I, \quad p_L < p_R \\ -\infty & \text{if } a \in \mathbb{R} \setminus I, \quad p_L > p_R \end{cases}$$

and set

$$\xi_L \leq \xi_R \quad \text{with} \quad (\xi_L, \xi_R) := \begin{cases} (\tilde{g}'(p_L^+), \tilde{g}'(p_R^-)) & \text{if } p_L < p_R \\ (0, 0) & \text{if } p_L = p_R \\ (\tilde{g}'(p_L^-), \tilde{g}'(p_R^+)) & \text{if } p_L > p_R \end{cases}$$

Then

$$(9.2) \quad U(\xi) = \begin{cases} p_L & \text{if } \xi < \xi_L \\ p_R & \text{if } \xi > \xi_R \\ ((\tilde{g}_I)')^{-1}(\xi) & \text{if } \xi \in [\xi_L, \xi_R] \end{cases}$$

where the map $U : \mathbb{R} \rightarrow \mathbb{R}$ is monotone and is uniquely defined outside a countable set.

Remark 9.2 Notice that $((\tilde{g}_I)')^{-1}$ is not defined for $\xi_L = 0 = \xi_R$, which only arises when $p_L = p_R$. When $p_L \neq p_R$, as a help, the function $((\tilde{g}_I)')^{-1}$ is better understood as the inverse maximal monotone graph of the maximal monotone graph $(\tilde{g}_I)'$.

proof of Lemma 9.1

For the proof, we refer to the textbook SERRE [19], where it is done for C^∞ functions g . Indeed, only the regularity C^2 is used there. This can easily be extended to the case of g continuous and locally Lipschitz, by approximation, and stability of entropy solutions. On the same topic, the reader can also consult textbooks DAFERMOS [9] and HOLDEN, RISEBRO [13]. This ends the proof of the lemma.

Remark 9.3 Notice that the result of Lemma 9.1 also follows from HJ-SCL relations (for instance justified using vanishing viscosity method, and BV bounds for scalar conservation laws). Indeed it is also a straightforward consequence of Hopf formula (see Theorem 3.1 in [5]) for convex initial data of Hamilton-Jacobi equations. Here the convex initial data is $W_0(x) = p_L x \cdot 1_{\{x < 0\}} + p_R x \cdot 1_{\{x > 0\}}$ when $p_L < p_R$, for non convex Hamiltonian g .

9.2 Reduction of test functions for viscosity solutions on junctions

Consider the problem

$$(9.3) \quad \begin{cases} v_t^j + f^j(v_x^j) = 0 & \text{on } \mathbb{R} \times J^j, \quad j = 1, \dots, N \\ v^0(t, 0) := v^j(t, 0) & \text{on } \mathbb{R} \times \{0\}, \quad j = 1, \dots, N \\ v_t^0 + \hat{h}(v_x^1, \dots, v_x^N) = 0 & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

We also define the half-relaxation operators

$$(9.4) \quad \begin{cases} (\underline{R}\hat{h})(p) := \sup_{q \in [p, b]} \min \{ \hat{h}, f_{\min} \} (q) & \text{with } f_{\min}(q) := \min_{j=1, \dots, N} f^j(q^j) \\ (\overline{R}\hat{h})(p) := \inf_{q \in [a, p]} \max \{ \hat{h}, f_{\max} \} (q) & \text{with } f_{\max}(q) := \max_{j=1, \dots, N} f^j(q^j) \end{cases}$$

Lemma 9.4 (Reducing the set of test functions)

Assume (2.2) with $N \geq 1$ with a junction J of type $0 : N$. Let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ which is a HJ germ. Then the associated Godunov flux is $\hat{f}_{\mathcal{G}} = (\hat{h}, \dots, \hat{h})$ with $\hat{h} : [a, b] \rightarrow \mathbb{R}$ given by i) of Theorem 2.25. For any $p \in [a, b]$, consider the function $v = (v^1, \dots, v^N)$ defined by

$$v^j(t, x) := -\lambda t + p^j x \quad \text{for all } (t, x) \in \mathbb{R} \times J^j, \quad j = 1, \dots, N$$

i) (Viscosity subsolutions)

Then \hat{h} satisfies

$$(9.5) \quad \hat{h} = \underline{R}\hat{h} \quad \text{with } \underline{R} \text{ defined in (9.4)}$$

Moreover v is a viscosity subsolution of (9.3) if and only if $-\lambda + f^j(p^j) \leq 0$ and

$$\text{for all } q \in \underline{\chi}\mathcal{G}, \quad \left(q \geq p \implies -\lambda + \hat{h}(q) \leq 0 \right)$$

ii) (Viscosity supersolutions)

Then \hat{h} satisfies

$$(9.6) \quad \hat{h} = \overline{R}\hat{h} \quad \text{with } \overline{R} \text{ defined in (9.4)}$$

Moreover v is a viscosity supersolution of (9.3) if and only if $-\lambda + f^j(p^j) \geq 0$ and

$$\text{for all } q \in \overline{\chi}\mathcal{G}, \quad \left(q \leq p \implies -\lambda + \hat{h}(q) \geq 0 \right)$$

Proof of Lemma 9.4

Step 1: proof of (9.5)

By definition of $\underline{R}\hat{h}$ and by monotonicity of \hat{h} , we have $\underline{R}\hat{h} \leq \hat{h}$. Now for $p \in [a, b]$, let $\hat{p} := \pi_{\mathcal{G}}(p)$. Then

$$(\underline{R}\hat{h})(p) := \sup_{q \in [p, b]} \min \{ \hat{h}, f_{\min} \} (q) \geq \sup_{q \in [p, b] \cap BA(\hat{p})} \min \{ \hat{h}, f_{\min} \} (q)$$

By construction, there exists $q \in [p, b] \cap BA(\hat{p})$ such that $f^j(q^j) \geq f^j(\hat{p}^j) = \hat{h}(\hat{p}) = \hat{h}(q)$, $j = 1, \dots, N$. Therefore $(\underline{R}\hat{h})(p) \geq \hat{h}(\hat{p}) = \hat{h}(p)$, which implies the equality and then shows (9.5).

Step 2: proof of (9.6)

The proof is similar to Step 1.

Step 3: proof of i)

The proof of i) is a simple variant of the proof of Theorem 2.7 in [14]. An easy adaptation of the proof is done for instance in [11] in the case $[a, b] \cap \mathbb{R}^N = \mathbb{R}^N$, in the subsection on the reduction of test functions. The adaptation to the general case $[a, b]$ is straightforward (and indeed easier).

Step 4: proof of ii)

The proof is similar to Step 3. This ends the proof of the lemma.

9.3 Gluing of matrices

In Subsection 5.5, we have studied the gluing of two germs $\mathcal{G}_\alpha \sharp \mathcal{G}_\beta$. We have also seen that naturally is associated the gluing of their Godunov fluxes $\hat{f} = \hat{f}_\alpha \sharp \hat{f}_\beta$.

In this section, we are interested in the algebra giving the Jacobian matrix $D\hat{f}$ in terms of the two Jacobian matrices $D\hat{f}_\alpha$ and $D\hat{f}_\beta$. This is given by the following result.

Lemma 9.5 (Formal Jacobian matrix after gluing)

For $\gamma = \alpha, \beta$, let some integers $n_\gamma \geq 2$. By abuse of notation, let us also allow the indices α, β to denote two sets of indices with $\alpha \simeq \{1, \dots, n_\alpha - 1\}$ and $\beta \simeq \{1, \dots, n_\beta - 1\}$. Now let $\hat{f}_\gamma = (\hat{f}_\gamma^0, \hat{f}_\gamma^1, \dots, \hat{f}_\gamma^{n_\gamma-1}) : \mathbb{R}^{n_\gamma} \supset [a, b]_\gamma \rightarrow \mathbb{R}^{n_\gamma}$. For $p_\gamma = (p_\gamma^1, \dots, p_\gamma^{n_\gamma-1}) \in \mathbb{R}^{n_\gamma-1}$, let us set $\hat{f} = (-\hat{f}_\alpha^0, \hat{f}_\alpha^1, \dots, \hat{f}_\alpha^{n_\alpha-1}) \sharp \hat{f}_\beta$ defined formally by

$$(9.7) \quad \hat{f}^j(p_\alpha, p_\beta) := \begin{cases} \hat{f}_\alpha^j(r, p_\alpha) & \text{if } j \in \alpha \\ \hat{f}_\beta^j(r, p_\beta) & \text{if } j \in \beta \end{cases} \quad \text{with } r \text{ satisfying } -\hat{f}_\alpha^0(r, p_\alpha) = \hat{f}_\beta^0(r, p_\beta)$$

(having in mind $J_\alpha^0 \simeq (0, +\infty)$ and $J_\beta^0 \simeq (-\infty, 0)$, and then here $(-\hat{f}_\alpha^0)(\downarrow, p_\alpha)$ and $\hat{f}_\beta^0(\uparrow, p_\beta)$, i.e. $\hat{f}_\alpha^0(\uparrow, p_\alpha)$). We set (with index i for the line and j for the column)

$$\hat{B}' := (\partial_j \hat{f}_\alpha^i)_{i,j \in \{0\} \cup \alpha} = \begin{pmatrix} \partial_0 \hat{f}_\alpha^0 & (\partial_j \hat{f}_\alpha^0)_{j \in \alpha} \\ (\partial_0 \hat{f}_\alpha^i)_{i \in \alpha} & (\partial_j \hat{f}_\alpha^i)_{i,j \in \alpha} \end{pmatrix} = \begin{pmatrix} B'_0 & D' \\ C' & B' \end{pmatrix}$$

and

$$\hat{B} := (\partial_j \hat{f}_\beta^i)_{i,j \in \{0\} \cup \beta} = \begin{pmatrix} \partial_0 \hat{f}_\beta^0 & (\partial_j \hat{f}_\beta^0)_{j \in \beta} \\ (\partial_0 \hat{f}_\beta^i)_{i \in \beta} & (\partial_j \hat{f}_\beta^i)_{i,j \in \beta} \end{pmatrix} = \begin{pmatrix} B_0 & D \\ C & B \end{pmatrix}$$

Then we have formally

$$\hat{B}' \star \hat{B} := \lambda_0 (\partial_j \hat{f}^i)_{i,j \in \alpha \cup \beta} = \lambda_0 \begin{pmatrix} B' & 0 \\ 0 & B \end{pmatrix} - \begin{pmatrix} C' \\ C \end{pmatrix} \cdot (D' \quad D) \quad \text{with } \lambda_0 := B'_0 + B_0$$

When $\lambda_0 > 0$, we set $\hat{B}' \sharp \hat{B} := \lambda_0^{-1} (\hat{B}' \star \hat{B})$.

Proof of Lemma 9.5

Taking the derivative of the last equation of (9.7), we get easily

$$\begin{cases} \partial_j \hat{f}_\alpha^0 + \lambda_0 \partial_j r = 0 & \text{if } j \in \alpha \\ \partial_j \hat{f}_\beta^0 + \lambda_0 \partial_j r = 0 & \text{if } j \in \beta \end{cases}$$

The elimination of $\partial_j r$ then gives

$$\lambda_0 (\partial_j \hat{f}^i)_{i,j \in \alpha \cup \beta} = \begin{pmatrix} (\lambda_0 \partial_j \hat{f}_\alpha^i - \partial_0 \hat{f}_\alpha^i \partial_j \hat{f}_\alpha^0)_{i,j \in \alpha} & (0 - \partial_0 \hat{f}_\alpha^i \partial_j \hat{f}_\beta^0)_{(i,j) \in \beta \times \alpha} \\ (0 - \partial_0 \hat{f}_\beta^i \partial_j \hat{f}_\alpha^0)_{(i,j) \in \alpha \times \beta} & (\lambda_0 \partial_j \hat{f}_\beta^i - \partial_0 \hat{f}_\beta^i \partial_j \hat{f}_\beta^0)_{i,j \in \beta} \end{pmatrix}$$

and the result follows. This ends the proof of the lemma.

Then we have the following result about the new algebra of gluing of matrices.

Proposition 9.6 (Properties of the gluing of matrices)

Let M_n denote the set of $n \times n$ matrices, with the convention that $M_n = \{0\}$ for $n \leq 0$. For $m, n \geq 0$ and $m + n \geq 1$, the gluing map

$$\begin{aligned} \star : M_{m+1} \times M_{n+1} &\rightarrow M_{m+n} \\ (\hat{B}', \hat{B}) &\mapsto \hat{B}' \star \hat{B} \end{aligned}$$

is quadratic. Moreover if the square matrices \hat{B}', \hat{B} are P_0 -monotone (resp. Riemann monotone, resp. Kružkov monotone, in the sense of Definition 7.1), then $\hat{B}' \star \hat{B}$ is also a P_0 -monotone (resp. Riemann monotone, resp. Kružkov monotone).

For the proof of Proposition 9.6, we need the following easy result.

Lemma 9.7 (A property of Riemann monotone matrices)

Assume that the following matrix $\hat{B} = \begin{pmatrix} B_0 & D \\ C & B \end{pmatrix}$ is Riemann monotone (in the sense of Definition 7.1), where the block decomposition is for $B_0 \in \mathbb{R}$ and $B \in \mathbb{R}^n$. We have $B_0 \geq 0$. Moreover $B_0 = 0$ implies $C = 0$.

Proof of Lemma 9.7

Recall that by assumption, for all $x \in \mathbb{R}^{1+n}$, we know that $x \diamond (\hat{B} \cdot x) \leq 0$ implies $\hat{B} \cdot x = 0$. Now for $x = (a, 0, \dots, 0)^T$ with $a \neq 0$, we get $x \diamond (\hat{B} \cdot x) = 0$ with $B_0 = 0$, and then $0 = \hat{B} \cdot x = a \begin{pmatrix} 0 \\ C \end{pmatrix}$ implies $C = 0$. The fact that $B_0 \geq 0$ is general and follows from the fact that \hat{B} is in particular P_0 -monotone. This ends the proof of the lemma.

Proof of Proposition 9.6

Step 1: Proof of P_0 -monotonicity

We claim that

$$(9.8) \quad \delta := \det(\hat{B}' \star \hat{B}) = \lambda_0^{(m+n-1)} \left\{ \det(B') \det(\hat{B}) + \det(\hat{B}') \det(B) \right\}$$

Notice that $\delta \geq 0$ if both matrices \hat{B}', \hat{B} are P_0 -monotone. This property also passes to minors of $\hat{B}' \star \hat{B}$, because they are then expressed as functions of minors of \hat{B}' and of \hat{B} .

Now let us show (9.8). Denoting by \check{B}'_l the matrix B' whose column l has been suppressed (and similarly

\check{B}_q the matrix B whose column q has been suppressed), we get

$$\begin{aligned}
\lambda_0^{-(m+n-1)}\delta &:= \lambda_0^{-(m+n-1)}\det(\hat{B}' \star \hat{B}) \\
&= \lambda_0^{-(m+n-1)} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda_0 B' - C' D' & -C' D \\ 0 & -C D' & \lambda_0 B - C D \end{vmatrix} \\
&= \lambda_0^{-(m+n-1)} \begin{vmatrix} 1 & 0 & 0 \\ C'/\lambda_0 & \lambda_0 B' - C' D' & -C' D \\ C/\lambda_0 & -C D' & \lambda_0 B - C D \end{vmatrix} \\
&= \lambda_0^{-(m+n-1)} \begin{vmatrix} 1 & \lambda_0 D' & \lambda_0 D \\ C'/\lambda_0 & \lambda_0 B' & 0 \\ C/\lambda_0 & 0 & \lambda_0 B \end{vmatrix} \\
&= \begin{vmatrix} \lambda_0 & D' & D \\ C' & B' & 0 \\ C & 0 & B \end{vmatrix} \\
&= \lambda_0 \det(B') \det(B) + \det(B) \cdot \sum_{l=1}^m (-1)^l D'_l \det(C', \check{B}'_l) + (-1)^m \sum_{q=1}^n (-1)^q D_q \begin{vmatrix} C' & B' & 0 \\ C & 0 & \check{B}_q \end{vmatrix} \\
&= \lambda_0 \det(B') \det(B) + \det(B) \cdot \sum_{l=1}^m (-1)^l D'_l \det(C', \check{B}'_l) + \sum_{q=1}^n (-1)^q D_q \begin{vmatrix} B' & C' & 0 \\ 0 & C & \check{B}_q \end{vmatrix} \\
&= \lambda_0 \det(B') \det(B) + \det(B) \cdot \sum_{l=1}^m (-1)^l D'_l \det(C', \check{B}'_l) + \det(B') \cdot \sum_{q=1}^n (-1)^q D_q \det(C, \check{B}_q)
\end{aligned}$$

where, in the fifth line we have factorized the first column by λ_0^{-1} and the other columns by λ_0 , in the sixth line we have used expansion along the first row. Again using expansion of the determinant on the first row, recall that

$$\begin{cases} \det(\hat{B}') = B'_0 \det(B') + \sum_{l=1}^m (-1)^l D'_l \det(C', \check{B}'_l) \\ \det(\hat{B}) = B_0 \det(B) + \sum_{q=1}^n (-1)^q D_q \det(C, \check{B}_q) \\ \lambda_0 = B'_0 + B_0 \end{cases}$$

Hence $\lambda_0^{-(m+n-1)}\delta = \det(\hat{B}')\det(B) + \det(B')\det(\hat{B})$ which shows (9.8).

Step 2: Proof of Riemann monotonicity

Case A: $\lambda_0 = 0$

The case $\lambda_0 = B'_0 + B_0 = 0$ implies $\hat{B}' \star \hat{B} = -\check{C}\check{D}$ with $\check{C} := \begin{pmatrix} C' \\ C \end{pmatrix}$ and $\check{D} := (D' \ D)$.

Recall that $B'_0, B_0 \geq 0$. Hence $\lambda_0 = 0$ also implies $B'_0 = 0 = B_0$. Moreover Lemma 9.7 shows that $\check{C} = 0$. Hence $\hat{B}' \star \hat{B} = 0$ which is in particular Riemann monotone.

Case B: $\lambda_0 := B'_0 + B_0 > 0$

Then in statement of Proposition 5.11, we can consider the functions

$$\begin{cases} \hat{f}_\alpha^0(r, p_\alpha) &:= B'_0 r + D' \cdot p_\alpha \\ \hat{f}_\alpha^i(r, p_\alpha) &:= (C')^i r + (B' \cdot p_\alpha)^i \\ \hat{f}_\beta^0(r, p_\beta) &:= B_0 r + D \cdot p_\beta \\ \hat{f}_\beta^i(r, p_\beta) &:= C^i r + (B \cdot p_\beta)^i \end{cases}$$

They are not locally constant, but there is a unique solution $r \in \mathbb{R}$ of

$$-\hat{f}_\alpha^0(r, p_\alpha) = \hat{f}_\beta^0(r, p_\beta).$$

When \hat{B}', \hat{B} are Riemann monotone matrices, then $\hat{f}_\alpha, \hat{f}_\beta$ are Riemann monotone maps. Then Step 6 of the proof of Proposition 5.11 applies and shows that $\hat{f}(p_\alpha, p_\beta)$ is Riemann monotone, which means precisely that the matrix $\hat{B}' \# \hat{B} := \lambda_0^{-1}(\hat{B}' \star \hat{B})$ is a Riemann monotone matrix.

Step 3: Proof of Kruřkov monotonicity

If $\lambda_0 = 0$, then Step 2 shows that $\hat{B}' \star \hat{B} = 0$ which is in particular Kruřkov monotone. If $\lambda_0 > 0$, the proof follows the lines of Step 2, replacing Riemann monotonicity by Kruřkov monotonicity, and using Step 8 of the proof of Proposition 5.11 instead of Step 6. This ends the proof of the lemma.

Remark 9.8 *Our calculations are related to the Schur complement (see [15]) for $\hat{B} := \begin{pmatrix} B_0 & D \\ C & B \end{pmatrix}$, which is $\hat{B}/B_0 := B - CB_0^{-1}D$ and satisfies classically $\det \hat{B} = \det B_0 \cdot \det(\hat{B}/B_0)$.*

Lemma 9.9 (Formal Jacobian matrix after self-gluing)

Let γ be a fixed index, and some integers $n_\gamma \geq 3$. Let $I_\gamma := \{0, \dots, n_\gamma - 1\}$ and $I'' := \{1, \dots, n_\gamma - 2\}$. Now let $\hat{f}_\gamma = (\hat{f}_\gamma^0, \hat{f}_\gamma^1, \dots, \hat{f}_\gamma^{n_\gamma-1}) : \mathbb{R}^{n_\gamma} \supset [a, b]_\gamma \rightarrow \mathbb{R}^{n_\gamma}$. For $p_\gamma = (p_\gamma^1, \dots, p_\gamma^{n_\gamma-2}) \in \mathbb{R}^{n_\gamma-2}$, let us set $\hat{f} = (-\hat{f}_\gamma^{k_1}, \hat{f}_\gamma^1, \dots, \hat{f}_\gamma^{n_\gamma-2}, \hat{f}_\gamma^{k_2})^{\#}_{k_1, k_2}$ with $k_1 := 0$ and $k_2 := n_\gamma - 1$, defined formally for $j \in I''$ by

$$(9.9) \quad \hat{f}^j(p_\gamma) := \hat{f}_\gamma^j(r, p_\gamma, r) \quad \text{with } r \text{ satisfying} \quad -\hat{f}_\gamma^0(r, p_\gamma, r) = \hat{f}_\gamma^{n_\gamma-1}(r, p_\gamma, r).$$

(having in mind $J_\gamma^{k_1} \simeq (0, +\infty)$ and $J_\gamma^{k_2} \simeq (-\infty, 0)$, and then here $(-\hat{f}_\gamma^{k_1})(\downarrow, p_\gamma, r)$ and $\hat{f}_\gamma^{k_2}(r', p_\gamma, \uparrow)$, i.e. $\hat{f}_\gamma^{k_1}(\uparrow, p_\gamma, r)$). We set (with index i for the line and j for the column)

$$\hat{B} := (\partial_j \hat{f}_\gamma^i)_{i,j \in \{k_1\} \cup I'' \cup \{k_2\}} = \begin{pmatrix} (\partial_{k_1} \hat{f}_\gamma^{k_1}) & (\partial_j \hat{f}_\gamma^{k_1})_{j \in I''} & (\partial_{k_2} \hat{f}_\gamma^{k_1}) \\ (\partial_{k_1} \hat{f}_\gamma^i)_{i \in I''} & (\partial_j \hat{f}_\gamma^i)_{i,j \in I''} & (\partial_{k_2} \hat{f}_\gamma^i)_{i \in I''} \\ (\partial_{k_1} \hat{f}_\gamma^{k_2}) & (\partial_j \hat{f}_\gamma^{k_2})_{j \in I''} & (\partial_{k_2} \hat{f}_\gamma^{k_2}) \end{pmatrix} = \begin{pmatrix} B'_0 & D' & F'_0 \\ C' & B & C \\ E'_0 & D & B_0 \end{pmatrix}$$

Then we have formally

$$\hat{B}^{j_1 \star j_2} := \lambda_0 (\partial_j \hat{f}_\gamma^i)_{i,j \in I''} = \lambda_0 B - (C' + C) \cdot (D + D') \quad \text{with} \quad \lambda_0 := \{B'_0 + F'_0\} + \{E'_0 + B_0\}$$

When $\lambda_0 > 0$, we moreover set $\hat{B}^{j_1 \star j_2} := \lambda_0^{-1} \hat{B}^{j_1 \star j_2}$.

Proof of Lemma 9.9

Taking the derivative of the last equality of (9.9), we get easily

$$-\left\{ \partial_j \hat{f}_\gamma^{j_1} + (\partial_j r) \left\{ \partial_{j_1} \hat{f}_\gamma^{j_1} + \partial_{j_2} \hat{f}_\gamma^{j_1} \right\} \right\} = \partial_j \hat{f}_\gamma^{j_2} + (\partial_j r) \left\{ \partial_{j_1} \hat{f}_\gamma^{j_2} + \partial_{j_2} \hat{f}_\gamma^{j_2} \right\}$$

i.e.

$$\partial_j \hat{f}_\gamma^{j_1} + \partial_j \hat{f}_\gamma^{j_2} + \lambda_0 \partial_j r = 0$$

The elimination of $\partial_j r$ then gives

$$\lambda_0 (\partial_j \hat{f}_\gamma^i)_{i,j \in I''} = \left(\lambda_0 \partial_j \hat{f}_\gamma^i - \left\{ \partial_j \hat{f}_\gamma^{j_1} + \partial_j \hat{f}_\gamma^{j_2} \right\} \cdot \left\{ \partial_{j_1} \hat{f}_\gamma^i + \partial_{j_2} \hat{f}_\gamma^i \right\} \right)_{i,j \in I''}$$

and the result follows. This ends the proof of the lemma.

Proposition 9.10 (Properties of the self-gluing of a matrix)

Let M_n denote the set of $n \times n$ matrices, with the convention that $M_n = \{0\}$ for $n \leq 0$. For $n \geq 3$, the self-gluing map

$$\begin{aligned} \star : M_n &\rightarrow M_{n-2} \\ \hat{B} &\mapsto \hat{B}^\star := \hat{B}^{j_1 \star j_2} \quad \text{with} \quad j_1 := 0, \quad j_2 := n - 1 \end{aligned}$$

is quadratic. Moreover if the square matrix \hat{B} is Kruřkov monotone (in the sense of Definition 7.1), then \hat{B}^\star is also Kruřkov monotone.

Proof of Proposition 9.10

Case A: $\lambda_0 := \{B'_0 + F'_0\} + \{E'_0 + B_0\} = 0$

Because \hat{B} is Kruřkov monotone, we know with $I'' := \{1, \dots, n-2\}$ that

$$\begin{cases} B'_0 \geq |E'_0| + \sum_{i \in I''} |(C')^i| \\ B_0 \geq |F'_0| + \sum_{i \in I''} |C^i| \end{cases}$$

Hence $\lambda_0 = 0$ implies $C' = 0 = C$ and then $\hat{B}^* = 0$, which is in particular a Kruřkov monotone matrix.

Case B: $\lambda_0 > 0$

Then in statement of Proposition 5.14, for $n_\gamma := n$, we can consider the functions for $p_\gamma = (p_\gamma^1, \dots, p_\gamma^{n_\gamma-2}) \in \mathbb{R}^{n_\gamma-2}$ with $k_1 := 0$, $k_2 := n_\gamma - 1$

$$\begin{cases} \hat{f}_\gamma^{k_1}(r', p_\gamma, r) & := B'_0 r' + D' \cdot p_\gamma + F'_0 r \\ \hat{f}_\gamma^i(r', p_\gamma, r) & := (C')^i r' + (B \cdot p_\gamma)^i + C^i r, \quad i \in I'' \\ \hat{f}_\gamma^{k_2}(r', p_\gamma, r) & := E'_0 r' + D \cdot p_\gamma + B_0 r \end{cases}$$

They are not locally constant, but there is a unique solution $r \in \mathbb{R}$ of

$$-\hat{f}_\gamma^{k_1}(r, p_\gamma, r) = \hat{f}_\gamma^{k_2}(r, p_\gamma, r).$$

The proof follows the lines of Step 6 of the proof of Proposition 5.11, which shows that \hat{f} defined in (9.9) satisfies $D^{\hat{f}} \geq 0$, which means exactly that $\lambda_0^{-1} \hat{B}^*$ is Kruřkov monotone. This ends the proof of the lemma.

Part II

Existence and uniqueness theory for Kruřkov germs

10 Properties of semisolutions

10.1 Stability

Lemma 10.1 (Solutions versus sub/supersolutions)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a generalized Riemann germ. Then u is a \mathcal{G} -entropy solution of (2.4) if and only if it is a \mathcal{G} -entropy subsolution and supersolution of (2.4).

Proof of Lemma 10.1

We know that $\mathcal{G}^{SUB} \cap \mathcal{G}^{SUP} = \mathcal{G}$. Hence the desired property is true at the junction point for \mathcal{G} -entropy.

We now want to check (it is probably very classical) that on each branch a function is an entropy solution if and only if it is an entropy subsolutions and supersolution. To this end, we consider the case $N = 1$ and drop the index j . We get with notation ψ^j in (2.24) and ψ_\pm^j in (2.25)

$$\begin{cases} |u - k| = |u - k|_+ + |u - k|_- \quad \text{and} \quad |u - k|_- = |k - u|_+ \\ \psi(u, k) = \psi_+(u, k) + \psi_-(u, k) \end{cases}$$

and this implies that if a function u is both an entropy subsolution and supersolution, then it is an entropy solution. Conversely for an entropy solution (hence bounded), we have for all $k \in \mathbb{R}$

$$(10.1) \quad \begin{cases} |u - k| + |u - a| = 2|u - k|_+ + |k - a| + 2|u - a|_- \\ \psi(u, k) + \psi(u, a) = 2\psi_+(u, k) + \{f(k) - f(a)\} + 2\psi_-(u, a) \end{cases}$$

and

$$(10.2) \quad \begin{cases} |u - k| + |u - b| = 2|u - k|_- + |k - b| + 2|u - b|_+ \\ \psi(u, k) + \psi(u, b) = 2\psi_-(u, k) - \{f(k) - f(b)\} + 2\psi_+(u, b) \end{cases}$$

Notice that $u \in [a, b]$ implies $0 = 2|u - a|_- = 2\psi_-(u, a) = 2|u - b|_+ = 2\psi_+(u, b)$. Hence this shows in the integral formulation that every standard entropy solution u satisfying $a \leq u \leq b$ also satisfies both conditions of standard entropy subsolution and supersolution for $k \in [a, b]$, and then for all $k \in \mathbb{R}$. This ends the proof of the lemma.

In general, we do not expect to have stability of \mathcal{G} -solutions for all Riemann germs \mathcal{G} . Here we present stability for certain subclasses.

Lemma 10.2 (Stability of solutions and of sub/supersolutions)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set.

We consider a sequence of functions $(u_n)_{n \in \mathbb{N}}$ with $u_n : [0, +\infty) \times J \rightarrow \mathbb{R}$, such that

$$u_n \rightarrow u_\infty \quad \text{in} \quad L^1_{loc}([0, +\infty) \times J)$$

i) (Solutions for Kruřkov germs)

If \mathcal{G} is a Kruřkov germ and if each function u_n is a \mathcal{G} -entropy solution of (2.4), then the limit u is also a \mathcal{G} -entropy solution of (2.4).

ii) (Subsolutions/supersolutions for monotone Kruřkov germs)

If \mathcal{G} is a monotone Kruřkov germ and if each function u_n is a \mathcal{G} -entropy subsolution (resp. supersolution) of (2.4), then the limit u is also a \mathcal{G} -entropy subsolution (resp. supersolution) of (2.4).

Proof of Lemma 10.2

First recall that the stability of Kruřkov entropy solutions/subsolutions/supersolutions is classical outside the junction point, and follows from the very definition of Kruřkov entropy solutions. Hence it remains to show the stability of the boundary condition at the junction point.

Step 1: proof of i)

We simply use the integral formulation of (2.4), which is recalled in Proposition 2.39. Forgetting the initial data, and focusing on the junction point, this means the following for u_n . For all test functions $0 \leq \varphi^j \in C^1_c((0, +\infty) \times \bar{J}^j)$, and with $\bar{J}^j = \{0\} \cup J^j \simeq [0, +\infty)$ or $(-\infty, 0]$ with $\varphi^j(t, 0) = \varphi^k(t, 0)$ for all $t \in [0, +\infty)$ and all index j, k , we have

$$(10.3) \quad \sum_k \left\{ \int_{(0, +\infty) \times J^k} \{ \eta^k(u_n, c) \varphi^k_t + \psi^k(u_n, c) \varphi^k_x \} dt dx \right\} \geq 0 \quad \text{for all elements } c = (c^1, \dots, c^N) \in \mathcal{G}$$

At the limit $n \rightarrow +\infty$, the function u_∞ still satisfies (10.3). Choosing

$$\varphi^j(t, x) = \beta(t) \alpha_\varepsilon^j(x) \quad \text{with} \quad \alpha_\varepsilon^j(x) = \alpha^j(\varepsilon^{-1}x) \quad \text{with} \quad 0 \leq \beta \in C^1_c(0, +\infty), \quad 0 \leq \alpha^j \in C^1_c(\bar{J}^j), \quad \alpha^j(0) = 1$$

and using the existence of strong Panov's traces, we get at the limit $\varepsilon \rightarrow 0$:

$$\int_{(0, +\infty) \times \{0\}} \beta D(u_\infty, c) \geq 0 \quad \text{with} \quad D(u_\infty, c) := \sum_{J^k \simeq (-\infty, 0)} \psi^k(u_\infty, c) - \sum_{J^k \simeq (0, +\infty)} \psi^k(u_\infty, c)$$

We deduce that for a.e. time $t \in (0, +\infty)$, we have $D(u_\infty(t, 0), c) \geq 0$ for all $c \in \mathcal{G}$. Because \mathcal{G} is a D -germ, it is D -maximal, and this implies that $u_\infty(t, 0) \in \mathcal{G}$ for a.e. time $t \in (0, +\infty)$, which shows that u_∞ is a \mathcal{G} -entropy solution of (2.4).

Step 2: proof of ii)

The proof follows the same lines as in Step 1. Recall that at the junction point, \mathcal{G} -entropy subsolutions (resp supersolutions) u_n of (2.4) satisfy the following. For all test functions $0 \leq \varphi^j \in C^1_c((0, +\infty) \times \bar{J}^j)$ with $\bar{J}^j = \{0\} \cup J^j \simeq [0, +\infty)$ or $(-\infty, 0]$, and with $\varphi^j(t, 0) = \varphi^k(t, 0)$ for all $t \in [0, +\infty)$ and all index j, k , we have the following inequality

$$(10.4) \quad \sum_k \left\{ \int_{(0, +\infty) \times J^k} \{ \eta^k_+(u_n, c) \varphi^k_t + \psi^k_+(u_n, c) \varphi^k_x \} dt dx \right\} \geq 0 \quad \text{for all } c \in \mathcal{G}^{SUB}$$

(respectively the same relation with $(\eta^k_-, \psi^k_-, \mathcal{G}^{SUP})$ instead of $(\eta^k_+, \psi^k_+, \mathcal{G}^{SUB})$, with $\psi^k_-(u_n, c) = \psi^k_+(c, u_n)$). Then we deduce $D_+(u_\infty(t, 0), c) \geq 0$ for all $c \in \mathcal{G}^{SUB}$. Using the fact that the left-dual satisfies $*\mathcal{G}^{SUB} = \mathcal{G}^{SUB}$, we deduce that $u_\infty(t, 0) \in \mathcal{G}^{SUB}$ for a.e. time $t \in (0, +\infty)$, which shows that u_∞ is a \mathcal{G} -entropy subsolution (resp. supersolution) of (2.4). This ends the proof of the lemma.

10.2 L^1 -contraction, uniqueness and comparison

Lemma 10.3 (L^1 -contraction, uniqueness and comparison)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a set. Let us consider two initial data $u_0, v_0 \in p_0 + L^1(J)$ for some constant $p_0 \in \mathbb{R}^N$.

i) (L^1 -contraction and uniqueness)

Assume that \mathcal{G} is a Kruřkov germ. Let u, v be two \mathcal{G} -entropy solutions of (2.4) with respective initial data u_0, v_0 . Then we have

$$\int_{\{t\} \times J} |u - v| \leq \int_{\{0\} \times J} |u - v| \quad \text{for all } t \geq 0$$

In particular, if $u_0 = v_0$, we get $u = v$, i.e. we have uniqueness of the solution.

ii) (Comparison)

Assume that \mathcal{G} is a monotone Kruřkov germ. Let u (resp. v) be a \mathcal{G} -entropy subsolution (resp. supersolution) of (2.4), with respective initial data u_0, v_0 . Then we have

$$\int_{\{t\} \times J} |u - v|_+ \leq \int_{\{0\} \times J} |u - v|_+ \quad \text{for all } t \geq 0$$

Proof of Lemma 10.3

Step 1: proof of i)

Recall that the doubling of variable method introduced by Kruřkov, allows to claim for inequalities on $\mathcal{D}' = \mathcal{D}'((0, +\infty) \times J^j)$ that

$$(10.5) \quad \left. \begin{array}{l} \partial_t \eta^j(u, k) + \partial_x \psi^j(u, k) \leq 0 \\ \partial_t \eta^j(k, v) + \partial_x \psi^j(k, v) \leq 0 \end{array} \right\} \text{ for all } k \in \mathbb{R}^N, \quad \text{implies} \quad \partial_t \eta^j(u, v)_t + \partial_x \psi^j(u, v) \leq 0$$

Notice that in the original paper [16], Kruřkov uses the Lipschitz continuity of the fluxes f^j (that we also assume), even if it also works for continuous fluxes (using the Lebesgue Dominated Convergence theorem). Inspired by the integral formulation of (2.4) given in Proposition 2.39 at the junction point, we consider the following. For all test functions $0 \leq \varphi^j \in C_c^1((0, +\infty) \times \bar{J}^j)$ with $\bar{J}^j = \{0\} \cup J^j \simeq [0, +\infty)$ or $(-\infty, 0]$ with

$$(10.6) \quad \varphi^j(t, 0) = \varphi^k(t, 0) =: \varphi(t, 0) \quad \text{for all } t \in [0, +\infty) \text{ and all index } j, k$$

we set

$$(10.7) \quad I(\varphi) := \sum_k \left\{ \int_{(0, +\infty) \times J^k} \{ \eta^k(u, v) \varphi_t^k + \psi^k(u, v) \varphi_x^k \} dt dx \right\}$$

Now set

$$\tilde{\varphi}_\varepsilon^j(t, x) := \varphi(t, 0) \alpha_\varepsilon^j(x) \quad \text{with} \quad \alpha_\varepsilon^j(x) := \alpha^j(\varepsilon^{-1}(x)), \quad 0 \leq \alpha^j \in C_c^1(\bar{J}^j), \quad \alpha^j(0) = 1$$

Then we write $I(\varphi) = I(\tilde{\varphi}_\varepsilon) + I(\varphi - \tilde{\varphi}_\varepsilon)$. Because $\varphi - \tilde{\varphi}_\varepsilon$ is the limit of functions in $C_c^1((0, +\infty) \times (J \setminus \{0\}))$, we deduce from (10.5) that $I(\varphi - \tilde{\varphi}_\varepsilon) \geq 0$. In the other hand, we have using Panov's traces that

$$I(\tilde{\varphi}_\varepsilon) \rightarrow I_0 \quad \text{as } \varepsilon \rightarrow 0, \quad \text{with} \quad I_0 := \int_{(0, +\infty) \times \{0\}} \varphi D(u, v)$$

Because $\varphi \geq 0$ and $D(\mathcal{G}, \mathcal{G}) \geq 0$, we deduce that $I_0 \geq 0$. Hence $I(\varphi) \geq 0$. Using Panov's traces at time $t = 0^+$, and the same argument as before but in time instead of space, we get for all test functions $0 \leq \varphi^j \in C_c^1([0, +\infty) \times \bar{J}^j)$ satisfying (10.6) that

$$\sum_k \left\{ \int_{(0, +\infty) \times J^k} \{ \eta^k(u, v) \varphi_t^k + \psi^k(u, v) \varphi_x^k \} dt dx + \int_{\{0\} \times J^k} \eta^k(u, v) \varphi^k dx \right\} \geq 0$$

Using the fact that the fluxes f^j are locally Lipschitz continuous (say with constant L), we know that we have finite propagation. Now, given any compact set $K_0 \subset J$, and for $t \geq 0$, let $K_t := J \setminus \Omega_t$ with $\Omega_t := B_{tL} + \Omega_0$

and $\Omega_0 := J \setminus K_0$. Then we can find a sequence of test functions $\varphi_n(t, x)$ approximating $1_{K_t}(x)1_{[0, s]}(t)$ for any $s > 0$. Hence we get

$$\int_{\{s\} \times K_s} |u - v| \leq \int_{\{0\} \times K_0} |u - v| = \int_{\{0\} \times K_0} |u_0 - v_0|$$

This property implies that $u, v \in C^0([0, +\infty); L^1_{loc}(J))$ (for instance testing u with a function v which is locally constant on $K_0 \not\equiv 0$, we can repeat it for all such K_0 and all translations in time). Now assuming that $u_0, v_0 \in p_0 + L^1(J)$, we deduce from monotone convergence theorem, that we can pass to the limit where K_0 tends to the whole junction J , which gives

$$\int_{\{s\} \times J} |u - v| \leq \int_J |u_0 - v_0| = \int_{\{0\} \times J} |u - v|$$

Step 2: proof of ii)

The proof follows the lines of Step 1. The method of doubling of variables of Kruřkov also works for Kruřkov semi-entropies, and gives that

$$\left. \begin{array}{l} \partial_t \eta_+^j(u, k) + \partial_x \psi_+^j(u, k) \leq 0 \\ \partial_t \eta_+^j(k, v) + \partial_x \psi_+^j(k, v) \leq 0 \end{array} \right\} \text{ for all } k \in \mathbb{R}^N, \quad \text{implies} \quad \partial_t(\eta_+^j(u, v) + \partial_x \psi_+^j(u^j, v^j)) \leq 0$$

Then for all test functions $0 \leq \varphi^j \in C_c^1((0, +\infty) \times \bar{J}^j)$ with $\bar{J}^j = \{0\} \cup J^j \simeq [0, +\infty)$ or $(-\infty, 0]$ satisfying (10.6), we set

$$(10.8) \quad I_+(\varphi) := \sum_k \left\{ \int_{(0, +\infty) \times J^k} \{ \eta_+^k(u, v) \varphi_t^k + \psi_+^k(u, v) \varphi_x^k \} dt dx \right\}$$

and using the fact that $D_+(\mathcal{G}^{SUB}, \mathcal{G}^{SUP}) \geq 0$, we get that $I_+(\varphi) \geq 0$ and also deduce that

$$\sum_k \left\{ \int_{(0, +\infty) \times J^k} \{ \eta_+^k(u, v) \varphi_t^k + \psi_+^k(u, v) \varphi_x^k \} dt dx + \int_{\{0\} \times J^k} \eta_+^k(u, v) \varphi^k dx \right\} \geq 0$$

Panov's traces also work for subsolutions and supersolutions, and this is also the case for the finite propagation behaviour. We get in particular that $|u - v|_+ \in C^0([0, +\infty); L^1(J))$, and that

$$\int_{\{s\} \times J} |u - v|_+ \leq \int_J |u_0 - v_0|_+ = \int_{\{0\} \times J} |u - v|_+$$

This ends the proof of the lemma.

10.3 Maximum and minimum of semisolutions

Lemma 10.4 (Maximum/minimum of sub/supersolutions)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a monotone Kruřkov germ.

i) (Maximum of two subsolutions)

Let u, w be two \mathcal{G} -entropy subsolutions of (2.4). Then $\max(u, w)$ is a \mathcal{G} -entropy subsolutions of (2.4).

ii) (Minimum of two supersolutions)

Let u, w be two \mathcal{G} -entropy supersolutions of (2.4). Then $\min(u, w)$ is a \mathcal{G} -entropy supersolutions of (2.4).

Proof of Lemma 10.4

We prove point i) (the proof of ii) is similar).

Step 1: checking that $\max(u, w)$ is an entropy subsolution, outside the junction point

The result of point i) should be standard, but we are not aware of a direct proof (see nevertheless Bianca, Dogbe [24] for an indirect proof). Because we only want to check that each component $\max(u^j, w^j)$ is an entropy solution on each branch J^j , we can consider the case $N = 1$ and drop the index j in all expressions.

We define the entropy $\eta_0(\alpha, \beta; \gamma) := \eta_+(\alpha \vee \beta, \gamma) = |\alpha \vee \beta - \gamma|_+$, which is clearly symmetric in α, β , and satisfies (using $\alpha \vee \beta = \beta + |\alpha - \beta|_+$)

$$\eta_0(\alpha, \beta; \gamma) = |\beta + |\alpha - \beta|_+ - \gamma|_+ = \begin{cases} |\alpha - \beta|_+ + \beta - \gamma & \text{if } \beta \geq \gamma \\ |\alpha - \gamma|_+ & \text{if } \beta \leq \gamma \end{cases} = |\alpha - \beta \vee \gamma|_+ + \text{sign}^+(\beta - \gamma) \cdot (\beta - \gamma)$$

For $\psi_+(\beta, \gamma) = \text{sign}^+(\beta - \gamma) \cdot \{f(\beta) - f(\gamma)\}$, we now define $q_0(\alpha, \beta; \gamma) := \psi_+(\alpha, \beta \vee \gamma) + \psi_+(\beta, \gamma)$. At this stage, it is not clear if q_0 is symmetric or not in α, β , and we set

$$\delta q_0(\alpha, \beta; \gamma) := q_0(\alpha, \beta; \gamma) - q_0(\beta, \alpha; \gamma) = \psi_+(\alpha, \beta \vee \gamma) + \psi_+(\beta, \gamma) - \psi_+(\beta, \alpha \vee \gamma) - \psi_+(\alpha, \gamma)$$

which is antisymmetric in α, β . We now only consider the case $\alpha \leq \beta$, (because the other case is symmetric).

$\alpha \leq \beta$	$\psi_+(\alpha, \beta \vee \gamma)$	$\psi_+(\beta, \alpha \vee \gamma)$	$\psi_+(\alpha, \gamma)$	$\psi_+(\beta, \gamma)$	$2\psi_+(\alpha \vee \beta, \gamma)$	$\delta q_0(\alpha, \beta; \gamma)$
$\alpha \leq \beta \leq \gamma$	$\psi_+(\alpha, \gamma) = 0$	$\psi_+(\beta, \gamma) = 0$	$\psi_+(\alpha, \gamma) = 0$	$\psi_+(\beta, \gamma) = 0$	$2\psi_+(\beta, \gamma) = 0$	0
$\alpha \leq \gamma \leq \beta$	$\psi_+(\alpha, \beta) = 0$	$\psi_+(\beta, \gamma)$	$\psi_+(\alpha, \gamma) = 0$	$\psi_+(\beta, \gamma)$	$2\psi_+(\beta, \gamma)$	0
$\gamma \leq \alpha \leq \beta$	$\psi_+(\alpha, \beta) = 0$	$\psi_+(\beta, \alpha)$	$\psi_+(\alpha, \gamma)$	$\psi_+(\beta, \gamma)$	$2\psi_+(\beta, \gamma)$	0

We deduce from the table that q_0 is symmetric in α, β and that $2\psi_+(\alpha \vee \beta; \gamma) = q_0(\alpha, \beta; \gamma) + q_0(\beta, \alpha; \gamma)$, which shows that $q_0(\alpha, \beta; \gamma) = \psi_+(\alpha \vee \beta; \gamma)$. Hence we can apply the method of doubling of variables of Kruřkov, which gives that

$$\left. \begin{aligned} \partial_t \eta_0(u, k; c) + \partial_x q_0(u, k; c) &\leq 0 \\ \partial_t \eta_0(w, \ell; c) + \partial_x q_0(w, \ell; c) &\leq 0 \end{aligned} \right\} \text{ for all } k, \ell \in \mathbb{R}^N, \text{ implies } \partial_t \eta_0(u, w; c) + \partial_x q_0(u, w; c) \leq 0$$

i.e. $\partial_t \eta_+(u \vee w, c) + \partial_x \psi_+(u \vee w, c) \leq 0$ for all $c \in \mathbb{R}$. This shows that $\max(u, w)$ is a Kruřkov entropy subsolution (outside the origin).

Step 2: checking that $\max(u, w)$ is a \mathcal{G} -entropy subsolution, at the junction point

Here this is the simplest part. We just have to check that $\max(u, w)(t, 0) \in \mathcal{G}^{SUB}$ for a.e. time $t > 0$, which follows from Lemma 7.16. This ends the proof of the lemma.

10.4 Proof of Theorem 2.38: properties of semisolutions for monotone Kruřkov germs

Proof of Theorem 2.38

For the proof we refer to the table of Subsection 2.4. The result follows from Lemmata 10.2 (stability), 10.3 (Max/min) and 10.4 (L^1 -comparison).

11 Existence via vanishing viscosity for Kruřkov germs

In this section, our goal is to get existence (and indeed uniqueness) of \mathcal{G} -entropy solutions to problem (2.23), which describes scalar conservation laws on a junction. We will be able to reach the end of this program only in the special case of Kruřkov germs \mathcal{G} .

11.1 General strategy for the proof of existence

Here we present very briefly the general strategy of the proof of existence. We only give the heuristics, without justification. The key idea is to get a priori bounds on the solution, and then to justify them using some method of approximation. To simplify, we assume that J is a junction of type $0 : N$, i.e. that $J^j \simeq (0, +\infty)$ for all j . We consider solutions $u = (u^1, \dots, u^N)$ of

$$\begin{cases} u_t + (f(u))_x = 0 & \text{on } (0, +\infty) \times J \\ u(t, 0) \in \mathcal{G} & \text{a.e. on } (0, +\infty) \times \{0\} \\ u = u_0 & \text{on } \{0\} \times J \end{cases}$$

0. L^∞ estimate

Assuming u_0 bounded, we first construct a bounded box $[\bar{a}, \bar{b}] \supset u_0(J)$ such that $\mathcal{G} \cap [\bar{a}, \bar{b}]$ is still a Riemann

germ on $[\bar{a}, \bar{b}]$, which is not a straightforward result. We then show by the maximum principle (up to the boundary) that $u(t, \cdot) \in [\bar{a}, \bar{b}]$ for all $t > 0$.

1. u_t estimate

Then for $J^* := J \setminus \{0\}$, for two solutions u, v we have $|u - v|_t + (\psi^f(u, v))_x \leq 0$ with $\psi^{f^j}(u, v) := \text{sign}(u^j - v^j) \cdot \{f^j(u^j) - f^j(v^j)\}$. The integration by parts on $(0, t) \times J^*$ gives the contraction estimate

$$\int_{(0,t) \times \{0\}} D^f(u, v) + \int_{\{t\} \times J} |u - v| \leq \int_{\{0\} \times J} |u - v| \quad \text{where} \quad D^f(u, v) := \sum_{j=1, \dots, N} -\psi^{f^j}(u, v) \geq 0$$

for $u(t, 0), v(t, 0) \in \mathcal{G}$, because \mathcal{G} is a Kruřkov germ. In particular for $v(t, x) := u(t + h, x)$ and dividing by $h \rightarrow 0^+$, we get formally

$$\int_{\{t\} \times J} |u_t| \leq \int_{\{0\} \times J} |u_t|$$

Up to a boundary layer correction term when $u_0(0) \notin \mathcal{G}$, we can show that such inequality holds true.

2. u_x estimate

We write $(u_x)_t + (f'(u)u_x)_x = 0$. Multiplying by $\text{sign}(u_x)$, we get $|u_x|_t + (f'(u)|u_x|)_x \leq 0$. Integrating on $(0, t) \times (\delta, +\infty)$, we get

$$\begin{aligned} \int_{\{t\} \times (\delta, +\infty)} |u_x| &\leq \int_{\{0\} \times (\delta, +\infty)} |u_x| + \int_{(0,t) \times \{\delta\}} |(f(u))_x| \\ &\leq \int_{\{0\} \times (\delta, +\infty)} |u_x| + \int_{(0,t) \times \{\delta\}} |u_t| \end{aligned}$$

which is the boundary BLN estimate. Integrating on δ , and using the u_t estimate, we get what we call the interior BLN estimate on u_x .

3. Recovering the junction condition

The previous bounds give some a priori BV_{loc} estimates, which are sufficient for any reasonable approximation process. Still in such approximation process, the key point is to recover the junction condition $u(t, 0) \in \mathcal{G}$, at the limit. This is done using a weak version of the boundary condition. Precisely, this is the following for $\varphi = (\varphi^1, \dots, \varphi^N)$ with $0 \leq \varphi^j \in C^1((0, +\infty) \times \bar{J}^j)$, and for all stationary constant solution $c \in \mathcal{G}$

$$0 \leq \int_{(0, +\infty) \times J^*} \{ |u - c| \varphi_t + \psi^f(u, c) \varphi_x \}$$

Focusing φ on the junction point $x = 0$, we get for all $0 \leq \phi \in C_c^1(0, +\infty)$

$$0 \leq \int_{(0, +\infty) \times \{0\}} D^f(u, c) \phi(t), \quad \text{i.e.} \quad 0 \leq D^f(u, c) \quad \text{for all} \quad c \in \mathcal{G}$$

which implies $u(t, 0) \in \mathcal{G}$, because \mathcal{G} is maximal, due to the fact that \mathcal{G} is Kruřkov.

11.2 Strategy of the proof by vanishing viscosity

Here we present briefly the strategy of the proof of existence by vanishing viscosity. Given a Kruřkov germ \mathcal{G} , we call $\hat{f} := \hat{f}_{\mathcal{G}}$ its associated Godunov flux. In a first step, the existence is reached using several approximations of the problem, and in a second step we relax all those approximations. We list below our set of approximations:

1) (Boundedness)

We assume that the box $[a, b]$ is bounded. We also work on a truncated junction $J_R := J \cap B_R(0)$ with bounded branches of length R , with suitable boundary condition for $x = \pm R$, encoded in some suitable germ \mathcal{G}_R and some associated flux \hat{f}_R .

2) (Regularized fluxes)

We consider smooth flux functions $f_{\eta}, \hat{f}_{\eta}, \hat{f}_{R, \eta}$, instead of functions f, \hat{f}, \hat{f}_R which are only locally Lipschitz continuous.

3) (Regularization of the solution by vanishing viscosity)

We introduce some viscosity $\varepsilon > 0$. We also assume that initial data is smooth, bounded and satisfies some

compatibility conditions.

With those approximations in hands, we will be able to justify the following estimates on the solution u^ε

$$(11.1) \quad \left\{ \begin{array}{ll} u^{\varepsilon,j}(t,x) \in [a^j, b^j] \quad \text{for all } (t,x) \in [0, +\infty) \times J_R^j, \quad j = 1, \dots, N & \text{(Box estimate)} \\ \int_{\{t\} \times J_R} |u_t^\varepsilon| \leq \int_{\{0\} \times J_R} |u_t^\varepsilon| & \text{(Contraction estimate)} \\ \int_{\{t\} \times J_{\delta, R-\delta}} |u_x^\varepsilon| \leq \int_{\{0\} \times J_{\delta, R-\delta}} |u_x^\varepsilon| + \int_{(0,t) \times \partial J_{\delta, R-\delta}} |u_t^\varepsilon| & \text{(Boundary BLN estimate)} \end{array} \right.$$

for all small $\delta > 0$ with

$$J_{\delta, R-\delta} := (J \cap B_{R-\delta}) \setminus \bar{B}_\delta$$

Once we have estimates (11.1), it is sufficient to get BV bounds on the solution u^ε , locally outside the origin $x = 0$ and outside $x \in \partial J_R$. We can then remove fluxes approximations 2). Then notice that all $c \in \mathcal{G}$ are solutions for $x \notin \partial J_R$, while all $c_R \in \mathcal{G}_R$ are solutions for all $x \neq 0$. This allows us show the following property with $\varphi \in \text{Lip}([0, +\infty) \times \bar{J}_R; \mathbb{R}_+)$

$$(11.2) \quad \left\{ \begin{array}{l} \int_{\{0\} \times J_R} |u^\varepsilon - c| \varphi + \int_{(0, +\infty) \times J_R} \{ |u^\varepsilon - c| \varphi_t + \psi^f(u^\varepsilon, c) \varphi_x \} \geq -\varepsilon C_\varphi \\ \text{for all } \varphi \in C_c([0, +\infty) \times J_R) \quad \text{and all } c \in \mathcal{G} \\ \\ \int_{\{0\} \times J_R} |u^\varepsilon - c_R| \varphi + \int_{(0, +\infty) \times J_R} \{ |u^\varepsilon - c_R| \varphi_t + \psi^f(u^\varepsilon, c_R) \varphi_x \} \geq -\varepsilon C_\varphi \\ \text{for all } \varphi \in C_c([0, +\infty) \times (\bar{J}_R \setminus \{0\})) \quad \text{and all } c_R \in \mathcal{G}_R \end{array} \right.$$

We can then remove the viscosity approximation 3) with $\varepsilon \rightarrow 0$. We end up with entropy solutions with zero viscosity on the bounded junction J_R . The boundary conditions at $x = 0$ and $x \in \partial J_R$ then follow from (11.2).

In a final step, we can consider the limit $R \rightarrow +\infty$, and recover the desired solution on the full junction J .

As a guide for the remaining part of this section, let us indicate the flux \hat{f}_R that we choose. For $\sigma = (-1, \dots, -1)$, and given some $p_0 \in [a, b]$, and $p = (p^1, \dots, p^N) \in [a, b]$, we set for all $j = 1, \dots, N$

$$(11.3) \quad \hat{f}_R^j(p) := G^{f^j}(p^j, p_0^j) \quad \text{and} \quad J_R^j \simeq (0, R)$$

where G^{f^j} is the standard Godunov flux associated to f^j . Then the associated germ is $\mathcal{G}_R := \{p \in [a, b], \hat{f}_R(p) = f(p)\}$.

In order to simplify the presentation, we will also use extensively the following result. It is a reduction result for a problem with viscosity $\varepsilon \geq 0$ on the full junction J . Its proof is straightforward.

Lemma 11.1 (Reduction from $n : m$ junction to $0 : n + m$ junction)

Assume (2.2) with $N \geq 1$, and let $\mathcal{G} \subset [a, b]$ be a Riemann germ with respect to (J, f) of orientation $\sigma \in \{\pm 1\}^N$ with $J^j \simeq \sigma^j \cdot (-\infty, 0)$, with Godunov flux $\hat{f} := \hat{f}_{\mathcal{G}} : [a, b] \rightarrow \mathbb{R}^N$. Let u^ε be a solution of

$$(11.4) \quad \left\{ \begin{array}{lll} u_t^{\varepsilon,j} + (f^j(u^{\varepsilon,j}))_x & = \varepsilon u_{xx}^{\varepsilon,j} & \text{on } (0, +\infty) \times J^j \\ f^j(u^{\varepsilon,j}) - \varepsilon u_x^{\varepsilon,j} & = \hat{f}^j(u^\varepsilon) & \text{on } (0, +\infty) \times \{0\} \\ u^{\varepsilon,j} & = u_0^j & \text{on } \{0\} \times J \end{array} \right.$$

Define the following type of reversion transform

$$(\tilde{u}_0^j(x), \tilde{u}^{\varepsilon,j}(t, x), \tilde{J}^j, \tilde{f}^j(p^j), \hat{\tilde{f}}^j(p)) := \begin{cases} (\tilde{u}_0^j(x), u^{\varepsilon,j}(t, x), J^j, f^j(p^j), \hat{f}^j(p)) & \text{if } \sigma^j = -1 \\ (\tilde{u}_0^j(-x), u^{\varepsilon,j}(t, -x), -J^j, -f^j(p^j), -\hat{f}^j(p)) & \text{if } \sigma^j = 1 \end{cases}$$

Then u^ε solves (11.4) if and only if $\tilde{u}^\varepsilon := (\tilde{u}^{\varepsilon,1}, \dots, \tilde{u}^{\varepsilon,N})$ solves

$$\begin{cases} \tilde{u}_t^{\varepsilon,j} + (\hat{f}^j(\tilde{u}^{\varepsilon,j}))_x &= \varepsilon \tilde{u}_{xx}^{\varepsilon,j} & \text{on } (0, +\infty) \times \tilde{J}^j \\ \hat{f}^j(\tilde{u}^{\varepsilon,j}) - \varepsilon \tilde{u}_x^{\varepsilon,j} &= \hat{f}^j(\tilde{u}^\varepsilon) & \text{on } (0, +\infty) \times \{0\} \\ \tilde{u}^{\varepsilon,j} &= \tilde{u}_0^j & \text{on } \{0\} \times \tilde{J} \end{cases}$$

and junction \tilde{J} is of type $0 : N$ with $\tilde{J}^j \simeq (0, +\infty)$ for all j .

11.3 Viscous regime for truncated junction with regularized fluxes

Lemma 11.2 (Existence in the viscous regime for truncated junction with regularized fluxes)

Assume (2.2) with $N \geq 1$ with junction J of type $0 : N$ and bounded box $[a, b] \subset \mathbb{R}^N$. Let $R > 0$, $J_R := J \cap B_R$ and $\sigma = (-1, \dots, -1) = -\sigma_R \in \mathbb{R}^N$ and let us consider functions for all $j = 1, \dots, N$ and $p \in \mathbb{R}^N$

$$(11.5) \quad \left\{ \begin{array}{l} \hat{f}_\eta, \hat{f}_{R,\eta} : \mathbb{R}^N \rightarrow \mathbb{R}^N, \\ f_\eta, \hat{f}_\eta, \hat{f}_{R,\eta} \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N) \\ f_\eta = (f_\eta^1, \dots, f_\eta^N) \text{ with } f_\eta^j : \mathbb{R} \rightarrow \mathbb{R} \\ p^j \mapsto \hat{f}_\eta^j(p) \text{ is nonincreasing} \\ p^j \mapsto \hat{f}_{R,\eta}^j(p) \text{ is nondecreasing} \\ \left. \begin{array}{l} \sigma^j \cdot (\hat{f}_\eta^j(p))|_{p^j=a^j} \leq \sigma^j \cdot f_\eta^j(a^j) \\ \sigma_R^j \cdot (\hat{f}_{R,\eta}^j(p))|_{p^j=a^j} \leq \sigma_R^j \cdot f_\eta^j(a^j) \\ \sigma^j \cdot (\hat{f}_\eta^j(p))|_{p^j=b^j} \geq \sigma^j \cdot f_\eta^j(b^j) \\ \sigma_R^j \cdot (\hat{f}_{R,\eta}^j(p))|_{p^j=b^j} \geq \sigma_R^j \cdot f_\eta^j(b^j) \end{array} \right\} \text{ for all } p \in [a, b] \end{array} \right.$$

Assume that the initial data $u_0 = (u_0^1, \dots, u_0^N)$ satisfies $u_0^j \in C^\infty(\bar{J}_R^j; [a^j, b^j])$ with $J_R^j \simeq (0, R)$. Let $\varepsilon > 0$.

i) (Existence)

Then there exists $u^\varepsilon = (u^{\varepsilon,j})_{j=1,\dots,N}$ with

$$(11.6) \quad u^{\varepsilon,j} : [0, +\infty) \times \bar{J}_R^j \rightarrow [a^j, b^j]$$

solution of

$$(11.7) \quad \begin{cases} u_t^{\varepsilon,j} + (f_\eta^j(u^{\varepsilon,j}))_x &= \varepsilon u_{xx}^{\varepsilon,j} & \text{on } (0, +\infty) \times J_R^j \\ f_\eta^j(u^{\varepsilon,j}) - \varepsilon u_x^{\varepsilon,j} &= \hat{f}_\eta^j(u^\varepsilon) & \text{on } (0, +\infty) \times \{0\} \\ f_\eta^j(u^{\varepsilon,j}) - \varepsilon u_x^{\varepsilon,j} &= \hat{f}_{R,\eta}^j(u^\varepsilon) & \text{on } (0, +\infty) \times \{R\} \end{cases}$$

with initial condition

$$(11.8) \quad u^{\varepsilon,j} = u_0^j \quad \text{on } \{0\} \times \bar{J}_R^j$$

In particular we have $u^{\varepsilon,j} \in C_{t,x}^{\frac{\alpha}{2}, \alpha}([0, +\infty) \times \bar{J}_R^j)$ for all $\alpha \in (0, 1)$.

ii) (Further regularity)

Assume moreover that the initial data u_0 satisfies the following compatibility conditions for all $j = 1, \dots, N$

$$(11.9) \quad \begin{cases} f_\eta^j(u_0^j) - \varepsilon (u_0^j)_x = \hat{f}_\eta^j(u_0) & \text{for } x = 0 \\ f_\eta^j(u_0^j) - \varepsilon (u_0^j)_x = \hat{f}_{R,\eta}^j(u_0) & \text{for } x = R. \end{cases}$$

and that we have the following additional regularity

$$(11.10) \quad f_\eta, \hat{f}_\eta, \hat{f}_{R,\eta} \in W_{loc}^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N)$$

Then the solution has regularity $u^{\varepsilon,j} \in C_{t,x}^{\frac{2+\alpha}{2}, 2+\alpha}([0, +\infty) \times \bar{J}_R^j)$ for all $\alpha \in (0, 1)$.

Proof of Lemma 11.2

First notice that, up to rescale the PDE in space and redefine properly the functions $f_\eta, \hat{f}_\eta, \hat{f}_{R,\eta}$, we can assume that $\varepsilon = 1$.

Step 1: first global existence result

We can apply Lemma 15.7 to $h(u) := -f_\eta(u)$, $g_0(u) := \hat{f}_\eta(u) - f_\eta(u)$ and $g_R(u) := \hat{f}_{R,\eta}(u) - f_\eta(u)$. We get the existence of a global solution $u = (u^1, \dots, u^N)$ with $u^j \in C_{t,x}^{\frac{\alpha}{2},\alpha}([0, +\infty) \times \bar{J}_R^j)$ for any $\alpha \in (0, 1)$, of

$$(11.11) \quad \begin{cases} u_t + (f_\eta(u))_x = u_{xx} & \text{on } (0, +\infty) \times J_R^* \\ f_\eta(u) - u_x = \hat{f}_\eta(u) & \text{on } (0, +\infty) \times \{0\} \\ f_\eta(u) - u_x = \hat{f}_{R,\eta}(u) & \text{on } (0, +\infty) \times (J \cap \partial B_R) \\ u = u_0 & \text{on } \{0\} \times J_R^* \end{cases}$$

Step 2: Box bounds (11.6)

In order to show that the solution u stays in the box $[a, b]$ if the initial data is, we want to use the maximum principle. Recall that $\hat{f}_\eta, \hat{f}_{R,\eta}$ satisfy the barrier bounds (the two last lines of (11.5)). Given some index j , we want to show that the solutions satisfy for instance

$$(11.12) \quad u^j \leq b^j$$

Given the PDE on the branch J_R^j , i.e. $(u^j - b^j)_t + (f_\eta^j(u^j) - f_\eta^j(b^j))_x = (u^j - b^j)_{xx}$ and multiplying it by $\text{sign}^+(u^j - b^j)$ (or an approximation of it), we get (with $\psi_+^{f_\eta^j}$ defined in (2.25))

$$(|u^j - b^j|_+)_t + (\psi_+^{f_\eta^j}(u^j, b^j))_x \leq (|u^j - b^j|_+)_{xx}$$

and integrating over $J_R^j \simeq (0, R)$, we get

$$\frac{d}{dt} \int_{\{t\} \times J_R^j} |u^j - b^j|_+ + \left[\psi_+^{f_\eta^j}(u^j, b^j) - \text{sign}^+(u^j - b^j) u_x^j \right]_{x=0}^{x=R} (t) \leq 0, \quad \text{i.e.} \quad \frac{d}{dt} \int_{\{t\} \times J_R^j} |u^j - b^j|_+ + \mathcal{D}^j \leq 0$$

with

$$\mathcal{D}^j := \text{sign}^+(u^j - b^j) \cdot \left\{ \hat{f}_{R,\eta}^j(u^j) - f_\eta^j(b^j) \right\}_{|x=R} - \text{sign}^+(u^j - b^j) \cdot \left\{ \hat{f}_\eta^j(u) - f_\eta^j(b^j) \right\}_{|x=0} =: \mathcal{D}_R^j + \mathcal{D}_0^j$$

Recall that at $x = 0$ with $J^j \simeq (0, +\infty)$, we have $\hat{f}_\eta^j(p)|_{p^j=b^j} \leq f_\eta^j(b^j)$, where we have used the two last lines of (11.5) for \hat{f}_η and $\sigma^j = -1$. Recall also that the map $p^j \mapsto \hat{f}_\eta^j(p)$ is nonincreasing. This property implies that $\mathcal{D}_0^j \geq 0$. Similarly, at $x = R$ with $J_R^j \simeq (0, R)$ (and $\sigma_R^j = 1$), we have $\hat{f}_{R,\eta}^j(p)|_{p^j=b^j} \geq f_\eta^j(b^j)$, where we have used the two last lines of (11.5) for $\hat{f}_{R,\eta}$ and $\sigma_R^j = 1$. This shows that $\mathcal{D}_R^j \geq 0$. Hence

$$\frac{d}{dt} \int_{\{t\} \times J_R^j} |u^j - b^j|_+ \leq 0$$

Because the initial data satisfies $u_0^j \leq b^j$, this shows that $u^j \leq b^j$, i.e. the maximum principle. All those arguments can be made rigorous, as it is classical. Similarly, we show the other bounds, and get that $u(t, x) \in [a, b]$ for all $(t, x) \in [0, +\infty) \times \bar{J}_R$. Notice that regularity $C_{t,x}^{\frac{\alpha}{2},\alpha}$ of the solution u^ε follows from i) of Lemma 15.7.

Step 3: further regularity

Notice that point ii) of the lemma follows immediately from assumption (11.9), additional regularity (14.7) and from regularity results ii) and iii) of Lemma 15.7. Finally, changing the variables back from $\varepsilon = 1$ to the original ε , we get the results for the solution u^ε of (11.7). This ends the proof of the lemma.

Lemma 11.3 (Contraction-dissipation in the viscous regime for 0 : N junction J)

We work under assumptions of Lemma 11.2 i), and assume that $u^\varepsilon, v^\varepsilon$ are two solutions of (11.7) with respective initial data u_0, v_0 . Then we have the following Kato inequality

$$(11.13) \quad \int_{\{t\} \times J_R} |u^\varepsilon - v^\varepsilon| + \int_{(0,t) \times \{0\}} D^{\hat{f}_\eta}(u^\varepsilon, v^\varepsilon) + \int_{(0,t) \times \{R\}} D^{\hat{f}_{R,\eta}}(u^\varepsilon, v^\varepsilon) \leq \int_{\{0\} \times J_R} |u^\varepsilon - v^\varepsilon|$$

with

$$D^{\hat{f}_\eta}(u^\varepsilon, v^\varepsilon) := \sum_{j=1}^N \sigma^j \psi^{\hat{f}_\eta^j}(u^\varepsilon, v^\varepsilon) \quad \text{and} \quad \psi^{\hat{f}_\eta^j}(u^\varepsilon, v^\varepsilon) := \text{sign}(u^{\varepsilon,j} - v^{\varepsilon,j}) \cdot \{f_\eta^j(u^\varepsilon) - \hat{f}_\eta^j(v^\varepsilon)\} \quad \text{with} \quad \sigma^j = -1$$

and

$$D^{\hat{f}_{R,\eta}}(u^\varepsilon, v^\varepsilon) := \sum_{j=1}^N \sigma_R^j \psi^{\hat{f}_{R,\eta}^j}(u^\varepsilon, v^\varepsilon) \quad \text{with} \quad \sigma_R^j = 1.$$

Moreover, for every test function $0 \leq \varphi^j \in C_c^1([0, +\infty) \times \bar{J}^j)$ such that

$$(11.14) \quad \begin{cases} \varphi^j(t, 0) = \varphi^k(t, 0) =: \varphi(t, 0) & \text{for all indices } j, k \\ \varphi^j(t, R) = \varphi^k(t, R) =: \varphi(t, R) & \text{for all indices } j, k \end{cases}$$

we have

$$(11.15) \quad \begin{aligned} & \int_{(0, +\infty) \times \{0\}} D^{\hat{f}_\eta}(u^\varepsilon, v^\varepsilon) \varphi + \int_{(0, +\infty) \times \{R\}} D^{\hat{f}_{R,\eta}}(u^\varepsilon, v^\varepsilon) \varphi \\ & \leq \varepsilon C_{R,\varphi} + \int_{\{0\} \times J_R} |u^\varepsilon - v^\varepsilon| \varphi + \int_{(0, +\infty) \times J_R} \{|u^\varepsilon - v^\varepsilon| \varphi_t + \psi^{\hat{f}_\eta}(u^\varepsilon, v^\varepsilon) \varphi_x \} \end{aligned}$$

where $C_{R,\varphi}$ depends on R and φ and is independent on ε .

Proof of Lemma 11.3

Step 1: preparation

Consider $\theta_\nu : \mathbb{R} \rightarrow \mathbb{R}$ a (symmetric) approximation of the absolute value with

$$\theta'_\nu(y) := \begin{cases} 1 & \text{if } y > \nu \\ \nu^{-1}y & \text{if } |y| \leq \nu \\ -1 & \text{if } y < -\nu \end{cases}$$

Now multiplying by $\theta'_\nu(u^{\varepsilon,j} - v^{\varepsilon,j})$ the difference of the two PDEs satisfied by $u^{\varepsilon,j}$ and $v^{\varepsilon,j}$, we get the very classical estimate (at least in the sense of distributions)

$$(11.16) \quad \partial_t \theta_\nu(u^{\varepsilon,j} - v^{\varepsilon,j}) + \partial_x \psi_\nu^{f_\eta^j}(u^\varepsilon, v^\varepsilon) + A_\nu^j = \varepsilon (\theta_\nu(u^{\varepsilon,j} - v^{\varepsilon,j}))_{xx} - \varepsilon \theta_\nu''(u^{\varepsilon,j} - v^{\varepsilon,j}) |(u^{\varepsilon,j} - v^{\varepsilon,j})_x|^2$$

where $\psi_\nu^{f_\eta^j}(u^\varepsilon, v^\varepsilon) := \theta'_\nu(u^{\varepsilon,j} - v^{\varepsilon,j}) \cdot \{f_\eta^j(u^\varepsilon) - f_\eta^j(v^\varepsilon)\}$ with $f_\eta^j(u^\varepsilon) := f_\eta^j(u^{\varepsilon,j})$ and

$$A_\nu^j := A_\nu^j[u^\varepsilon, v^\varepsilon] := -\theta_\nu''(u^{\varepsilon,j} - v^{\varepsilon,j}) \cdot \{f_\eta^j(u^\varepsilon) - f_\eta^j(v^\varepsilon)\} \cdot (u^{\varepsilon,j} - v^{\varepsilon,j})_x$$

Now, following the ideas of [23], we have the monotone convergence for any radius $R > 0$

$$(11.17) \quad \int_{[0,R] \cap \{|w| \leq \nu\}} |w_x| dx \rightarrow \int_{[0,R] \cap \{w=0\}} |w_x| dx = 0 \quad \text{as } \nu \rightarrow 0$$

where the equality to zero is classical for $w \in W^{1,2}([0,R])$. This implies that $A_\nu^j \rightarrow 0$ in the sense of distributions as $\nu \rightarrow 0$, and using $\theta_\nu'' \geq 0$, as $\nu \rightarrow 0$, we recover the standard result in the sense of distributions

$$\partial_t |u^{\varepsilon,j} - v^{\varepsilon,j}| + \partial_x \psi^{f_\eta^j}(u^\varepsilon, v^\varepsilon) \leq \varepsilon |u^{\varepsilon,j} - v^{\varepsilon,j}|_{xx}$$

where $\psi^{f_\eta^j}(u^\varepsilon, v^\varepsilon) := \text{sign}(u^{\varepsilon,j} - v^{\varepsilon,j}) \cdot \{f_\eta^j(u^\varepsilon) - f_\eta^j(v^\varepsilon)\}$. To simplify the notation, we set

$$\begin{cases} \theta_\nu^j(u^\varepsilon - v^\varepsilon) := \theta_\nu(u^{\varepsilon,j} - v^{\varepsilon,j}) \\ \theta_\nu^{j'}(u^\varepsilon - v^\varepsilon) := \theta'_\nu(u^{\varepsilon,j} - v^{\varepsilon,j}) \\ \theta_\nu^{j''}(u^\varepsilon - v^\varepsilon) := \theta''_\nu(u^{\varepsilon,j} - v^{\varepsilon,j}) \end{cases}$$

Multiplying first (11.16) by a smooth nonnegative test function $0 \leq \varphi^j \in C_c^\infty([0, +\infty) \times \bar{J}_R^j)$, we get by integration by parts on $J_R^j \simeq (0, R)$

$$\begin{aligned}
& - \int_{\{0\} \times J_R^j} \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi^j - \int_{(0, +\infty) \times \{0\}} \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi^j + \int_{(0, +\infty) \times \{R\}} \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi^j \\
& - \int_{(0, +\infty) \times J_R^j} \left\{ \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_t^j + \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi_x^j \right\} + \int_{(0, +\infty) \times J_R^j} A_\nu^j \varphi^j \\
& = \int_{(0, +\infty) \times \{R\}} \left\{ \varepsilon (\theta_\nu^j(u^\varepsilon - v^\varepsilon))_x \varphi^j - \varepsilon \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_x^j \right\} - \int_{(0, +\infty) \times \{0\}} \left\{ \varepsilon (\theta_\nu^j(u^\varepsilon - v^\varepsilon))_x \varphi^j - \varepsilon \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_x^j \right\} \\
& + \int_{(0, +\infty) \times J_R^j} \varepsilon \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_{xx}^j - \int_{(0, +\infty) \times J_R^j} \varepsilon \theta_\nu^{j''}(u^\varepsilon - v^\varepsilon) |(u^{\varepsilon, j} - v^{\varepsilon, j})_x|^2 \varphi^j
\end{aligned}$$

Using the fact that the solutions satisfy

$$\begin{cases} \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) - \varepsilon (\theta_\nu^j(u^\varepsilon - v^\varepsilon))_x = \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) & \text{on } x = 0 \\ \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) - \varepsilon (\theta_\nu^j(u^\varepsilon - v^\varepsilon))_x = \psi_\nu^{f_{R, \nu}^j}(u^\varepsilon, v^\varepsilon) & \text{on } x = R \end{cases}$$

(which is already satisfied by the initial data), we get

$$\begin{aligned}
& \int_{(0, +\infty) \times \{R\}} \psi_\nu^{f_{R, \nu}^j}(u^\varepsilon, v^\varepsilon) \varphi^j - \int_{(0, +\infty) \times \{0\}} \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi^j \\
& \leq \int_{(0, +\infty) \times \{0\}} \varepsilon \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_x^j - \int_{(0, +\infty) \times \{R\}} \varepsilon \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_x^j + \int_{(0, +\infty) \times J_R^j} \varepsilon \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_{xx}^j \\
& + \int_{\{0\} \times J_R^j} \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi^j + \int_{(0, +\infty) \times J_R^j} \left\{ \theta_\nu^j(u^\varepsilon - v^\varepsilon) \varphi_t^j + \psi_\nu^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi_x^j \right\} \\
& - \int_{(0, +\infty) \times J_R^j} A_\nu^j \varphi^j
\end{aligned}$$

In the limit $\nu \rightarrow 0$, this gives with the special convention $|u^\varepsilon - v^\varepsilon|^j := |u^{\varepsilon, j} - v^{\varepsilon, j}|$ (11.18)

$$\begin{aligned}
& \left\{ \int_{(0, +\infty) \times \{R\}} \psi^{f_{R, \nu}^j}(u^\varepsilon, v^\varepsilon) \varphi^j + \varepsilon |u^\varepsilon - v^\varepsilon|^j \varphi_x^j \right\} - \left\{ \int_{(0, +\infty) \times \{0\}} \psi^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi^j + \varepsilon |u^\varepsilon - v^\varepsilon|^j \varphi_x^j \right\} \\
& \leq \int_{(0, +\infty) \times J_R^j} \varepsilon |u^\varepsilon - v^\varepsilon|^j \varphi_{xx}^j + \int_{\{0\} \times J_R^j} |u^\varepsilon - v^\varepsilon|^j \varphi^j + \int_{(0, +\infty) \times J_R^j} \left\{ |u^\varepsilon - v^\varepsilon|^j \varphi_t^j + \psi^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi_x^j \right\}
\end{aligned}$$

Step 2: proof of (11.13)

For $s > 0$ and $\varphi^j(t, x) := 1_{[0, s]}(t)$, and summing on j in (11.18), we get (11.13) with t replaced by s .

Step 3: proof of (11.15)

We come back to (11.18) with general test function φ^j satisfying (11.14). We now consider a constant $M \geq \max_j \max(|a^j|, |b^j|)$. Summing over indices j , we get

$$\begin{aligned}
& \int_{(0, +\infty) \times \{0\}} D^{f_\nu}(u^\varepsilon, v^\varepsilon) \varphi + \int_{(0, +\infty) \times \{R\}} D^{f_{R, \nu}}(u^\varepsilon, v^\varepsilon) \varphi \\
& \leq 2\varepsilon M C'_{R, \varphi} + \sum_{j=1}^N \int_{\{0\} \times J^j} |u^\varepsilon - v^\varepsilon|^j \varphi^j + \sum_{j=1}^N \int_{(0, +\infty) \times J^j} \left\{ |u^\varepsilon - v^\varepsilon|^j \varphi_t^j + \psi^{f_\nu^j}(u^\varepsilon, v^\varepsilon) \varphi_x^j \right\}
\end{aligned}$$

where $C'_{R,\varphi}$ is a constant only depending on R and φ , and we set $C_{R,\varphi} := 2MC'_{R,\varphi}$. This is (11.15). This ends the proof of the lemma.

Corollary 11.4 (True contraction in the viscous regime)

We work under assumptions of Lemma 11.2 ii). We assume that u^ε is a solution of (11.7) with initial data u_0 satisfying in particular compatibility conditions (11.9). We assume moreover that \hat{f}_η and $\hat{f}_{R,\eta}$ satisfy for all $j = 1, \dots, N$

$$(11.19) \quad \left\{ \begin{array}{l} -\partial_j \hat{f}_\eta^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}_\eta^k| \\ \partial_j \hat{f}_{R,\eta}^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}_{R,\eta}^k| \end{array} \right. \quad \text{on } [a, b]$$

Then we have

$$(11.20) \quad \int_{\{t\} \times J_R} |u^\varepsilon - v^\varepsilon| \leq \int_{\{0\} \times J_R} |u^\varepsilon - v^\varepsilon|$$

Moreover we have

$$(11.21) \quad \int_{\{t\} \times J_R} |u_t^\varepsilon| \leq \int_{\{0\} \times J_R} |u_t^\varepsilon|$$

Proof of Corollary 11.4

Notice that assumption (11.19) implies $D\hat{f}_\eta \geq 0$, $D\hat{f}_{R,\eta} \geq 0$ on $[a, b]^2$. Together with (11.13), this implies (11.20). Second inequality (11.21) follows from the choice of $v^\varepsilon(t, x) := u^\varepsilon(t + \tau, x)$, and dividing the difference by τ and passing to the limit. Here, we use in particular regularity of the solution up to the time $t = 0$ with $u^{\varepsilon,j} \in C_{t,x}^{1+\frac{\beta}{2}, 2+\beta}([0, +\infty) \times \bar{J}_R^j)$, where we have $(u^{\varepsilon,j})_t(0, \cdot) = \varepsilon(u_0^j)_{xx} - (f_\eta^j)'(u_0^j) \cdot (u_0^j)_x$. This ends the proof of the corollary.

Lemma 11.5 (Boundary and interior BLN estimates for a single truncated branch)

Assume (2.2) with $N = 1$, and denote $J^1 = (0, +\infty)$, and the truncated branch $J_R^1 := (0, R)$ for some $R > 0$ and a bounded interval $[a, b] \subset \mathbb{R}$. Consider $f_\eta = f_\eta^1 \in (W^{1,\infty} \cap W_{loc}^{2,\infty})(\mathbb{R}; \mathbb{R})$. For $\varepsilon > 0$, we consider a solution $u^\varepsilon \in C_{t,x}^{1+\frac{\beta}{2}, 2+\beta}([0, +\infty) \times \bar{J}_R^1; [a, b])$ for some $\beta \in (0, 1)$ of

$$u_t^\varepsilon + (f_\eta(u^\varepsilon))_x = \varepsilon u_{xx}^\varepsilon \quad \text{on } (0, +\infty) \times J_R^1$$

Then for any $\delta \in (0, R/4)$ and $t > s \geq 0$, we have

$$\int_{\{t\} \times (\delta, R-\delta)} |u_x^\varepsilon| \leq \int_{\{s\} \times (\delta, R-\delta)} |u_x^\varepsilon| + \left\{ \int_{(s,t) \times \{\delta\}} |u_t^\varepsilon| + \int_{(s,t) \times \{R-\delta\}} |u_t^\varepsilon| \right\} \quad \text{(Boundary BLN estimate)}$$

where the lateral integral lies in the bracket.

Moreover, we have

$$(11.22) \quad \int_{\{t\} \times (2\delta, R-2\delta)} |u_x^\varepsilon| \leq \int_{\{s\} \times (\delta, R-\delta)} |u_x^\varepsilon| + \delta^{-1} \left\{ \int_{(s,t) \times (\delta, 2\delta)} |u_t^\varepsilon| + \int_{(s,t) \times (R-2\delta, R-\delta)} |u_t^\varepsilon| \right\} \quad \text{(Interior BLN estimate)}$$

Proof of Lemma 11.5

Step 1: Boundary BLN estimate

Recall that the classical BLN estimate (14) in Bardos, Leroux, Nedelec [23] holds for zero Dirichlet boundary conditions. Here we do not assume any boundary conditions, but get a sort of BLN estimate up to the boundary, that we simply call boundary BLN estimate. We follow quite closely the ideas of the proof in [23]. Precisely, we adapt the estimate on u_x . We take the x -derivative of the PDE for $u := u^\varepsilon$

$$\partial_t u_x + \partial_x F = \varepsilon (u_x)_{xx} \quad \text{with } F := (f_\eta(u))_x$$

and multiply by $\theta'_\nu(u_x)$, we get

$$(11.23) \quad \partial_t \theta_\nu(u_x) + \partial_x \Phi_\nu + B_\nu = \varepsilon(\theta_\nu(u_x))_{xx} - \varepsilon \theta''_\nu(u_x) |u_{xx}|^2$$

with $\Phi_\nu := F \cdot \theta'_\nu(u_x)$, $B_\nu := -F \cdot \theta''_\nu(u_x) u_{xx}$ and $|F| \leq L|u_x|$, where L is the Lipschitz constant of f_η . Hence we have $B_\nu \rightarrow 0$ as $\nu \rightarrow 0$ (like in (11.17)). Therefore, as $\nu \rightarrow 0$, it gives $\partial_t |u_x| + \partial_x \{F \cdot \text{sign}(u_x)\} - 0 \leq \varepsilon(|u_x|)_{xx}$, which is not good enough. Indeed, we must use the PDE, in some sense on the artificial boundary $x \in \partial(\delta, R - \delta)$.

We now do the proof formally and explain later how to make it rigorous. Integrating (11.23), we get

$$\int_{J_{R,\delta}} \partial_t \theta_\nu(u_x) + [\Phi_\nu]_{\partial J_{R,\delta}} + \int_{J_{R,\delta}} B_\nu = \varepsilon[(\theta_\nu(u_x))_x]_{\partial J_{R,\delta}} - \int_{J_{R,\delta}} \varepsilon \theta''_\nu(u_x) |u_{xx}|^2 \quad \text{with } J_{R,\delta} := (\delta, R - \delta)$$

Now the PDE is valid up on the boundary, i.e. $u_t + F = \varepsilon u_{xx}$ on $(0, +\infty) \times \partial J_{R,\delta}$. Hence, multiplying by $\theta'_\nu(u_x)$, we get

$$\begin{aligned} [\Phi_\nu]_{\partial J_{R,\delta}} &= [\{\varepsilon u_{xx} - u_t\} \cdot \theta'_\nu(u_x)]_{\partial J_{R,\delta}} \\ &= [\{\varepsilon(\theta_\nu(u_x))_x - u_t \cdot \theta'_\nu(u_x)\}]_{\partial J_{R,\delta}} \end{aligned}$$

Hence this gives

$$\int_{J_{R,\delta}} \partial_t \theta_\nu(u_x) + [-u_t \cdot \theta'_\nu(u_x)]_{\partial J_{R,\delta}} + \int_{J_{R,\delta}} B_\nu = - \int_{J_{R,\delta}} \varepsilon \theta''_\nu(u_x) |u_{xx}|^2$$

Integrating in time on (s, t) , and using $|\theta'_\nu| \leq 1$, we get

$$\int_{\{t\} \times J_{R,\delta}} \theta_\nu(u_x) + \int_{(s,t) \times J_{R,\delta}} B_\nu \leq \int_{\{s\} \times J_{R,\delta}} \theta_\nu(u_x) + \int_{(s,t) \times \partial J_{R,\delta}} |u_t|$$

Passing to the limit $\nu \rightarrow 0$, we have $B_\nu \rightarrow 0$ and get the boundary BLN estimate

$$(11.24) \quad \int_{\{t\} \times J_{R,\delta}} |u_x| \leq \int_{\{s\} \times J_{R,\delta}} |u_x| + \int_{(s,t) \times \partial J_{R,\delta}} |u_t|$$

Now this formal proof of (11.24) can easily be justified using test functions as in the proof of Lemma 11.3.

Step 2: interior BLN estimate

From the boundary BLN estimate, we can always deduce an interior BLN estimate as follows. We follow an idea introduced in Lemma 4.2 in [25]. Replacing δ by $h > 0$ in (11.24), and integrating (11.24) for $h \in (\delta, 2\delta)$, we get (dividing by δ)

$$\delta^{-1} \int_{(\delta, 2\delta)} dh \int_{\{t\} \times J_{R,h}} |u_x| \leq \delta^{-1} \int_{(\delta, 2\delta)} dh \int_{\{s\} \times J_{R,h}} |u_x| + \delta^{-1} \int_{(\delta, 2\delta)} dh \int_{(s,t) \times \partial J_{R,h}} |u_t|$$

and then

$$\int_{\{t\} \times J_{R,2\delta}} |u_x| \leq \int_{\{s\} \times J_{R,\delta}} |u_x| + \delta^{-1} \left\{ \int_{(s,t) \times (\delta, 2\delta)} |u_t| + \int_{(s,t) \times (R-2\delta, R-\delta)} |u_t| \right\}$$

which is (11.22). This ends the proof of Lemma.

11.4 Removing regularization of the fluxes

Proposition 11.6 (Existence in the viscous regime for truncated junctions)

Assume (2.2) with $N \geq 1$ with junction J of type $0 : N$ and bounded box $[a, b] \subset \mathbb{R}^N$. Let $R > 0$ and

$\sigma = (-1, \dots, -1) = -\sigma_R \in \mathbb{R}^N$ and let us consider functions for all $j = 1, \dots, N$ and all $p \in [a, b]$

$$(11.25) \quad \left\{ \begin{array}{l} \hat{f}, \hat{f}_R : [a, b] \rightarrow \mathbb{R}^N, \\ f, \hat{f}, \hat{f}_R \in W^{1, \infty}([a, b]; \mathbb{R}^N) \\ \\ f = (f^1, \dots, f^N) \quad \text{with} \quad f^j : [a^j, b^j] \rightarrow \mathbb{R} \\ \\ \sigma^j \cdot (\hat{f}^j(p))|_{p^j=a^j} \leq \sigma^j \cdot f^j(a^j) \\ \sigma_R^j \cdot (\hat{f}_R^j(p))|_{p^j=a^j} \leq \sigma_R^j \cdot f^j(a^j) \\ \\ \sigma^j \cdot (\hat{f}^j(p))|_{p^j=b^j} \geq \sigma^j \cdot f^j(b^j) \\ \sigma_R^j \cdot (\hat{f}_R^j(p))|_{p^j=b^j} \geq \sigma_R^j \cdot f^j(b^j) \end{array} \right.$$

We assume moreover that \hat{f} and \hat{f}_R satisfy for all $j = 1, \dots, N$ and a.e. on $[a, b]$

$$(11.26) \quad \left\{ \begin{array}{l} -\partial_j \hat{f}^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}^k| \\ \\ \partial_j \hat{f}_R^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}_R^k| \end{array} \right.$$

Assume that the initial data $u_0 = (u_0^1, \dots, u_0^N)$ satisfies

$$(11.27) \quad u_0^j \in C^\infty(\bar{J}_R^j; [a^j, b^j]) \quad \text{with} \quad J_R^j \simeq (0, R).$$

Let $\varepsilon > 0$.

i) (Existence)

Then there exists $u^\varepsilon = (u^{\varepsilon, j})_{j=1, \dots, N}$ with

$$(11.28) \quad u^{\varepsilon, j} : [0, +\infty) \times \bar{J}_R^j \rightarrow [a^j, b^j]$$

solution of

$$(11.29) \quad \left\{ \begin{array}{ll} u_t^{\varepsilon, j} + (f^j(u^{\varepsilon, j}))_x = \varepsilon u_{xx}^{\varepsilon, j} & \text{on} \quad (0, +\infty) \times J_R^j \\ f^j(u^{\varepsilon, j}) - \varepsilon u_x^{\varepsilon, j} = \hat{f}^j(u^\varepsilon) & \text{on} \quad (0, +\infty) \times \{0\} \\ f^j(u^{\varepsilon, j}) - \varepsilon u_x^{\varepsilon, j} = \hat{f}_R^j(u^\varepsilon) & \text{on} \quad (0, +\infty) \times \{R\} \end{array} \right.$$

where the boundary conditions are satisfied in a weak sense (i.e. against test functions) and with initial condition

$$(11.30) \quad u^{\varepsilon, j} = u_0^j \quad \text{on} \quad \{0\} \times \bar{J}_R^j$$

In particular we have $u^{\varepsilon, j} \in C_{t,x}^{\frac{\alpha}{2}, \alpha}([0, +\infty) \times \bar{J}_R^j)$ for all $\alpha \in (0, 1)$.

ii) (Additional bounds)

Assume moreover that the initial data u_0 satisfies the following compatibility conditions for all $j = 1, \dots, N$

$$(11.31) \quad \left\{ \begin{array}{ll} f^j(u_0^j) - \varepsilon (u_0^j)_x = \hat{f}^j(u_0) & \text{for} \quad x = 0 \\ f^j(u_0^j) - \varepsilon (u_0^j)_x = \hat{f}_R^j(u_0) & \text{for} \quad x = R. \end{array} \right.$$

and set

$$(11.32) \quad \left\{ \begin{array}{l} u_t^\varepsilon := \varepsilon (u_0)_{xx} - (f(u_0))_x \\ u_x^\varepsilon := (u_0)_x \end{array} \right. \quad \text{on} \quad \{0\} \times J_R^* \quad \text{with} \quad J_R^* := J_R \setminus \{0\}$$

Then we have the following bounds for all $t > 0$

$$(11.33) \quad \int_{\{t\} \times J_R} |u_t^\varepsilon| \leq \int_{\{0\} \times J_R} |u_t^\varepsilon|$$

and for all $\delta \in (0, R/4)$

$$(11.34) \quad \int_{\{t\} \times (2\delta, R-2\delta)} |u_x^\varepsilon| \leq \int_{\{0\} \times (\delta, R-\delta)} |u_x^\varepsilon| + \delta^{-1} \left\{ \int_{(0,t) \times (\delta, 2\delta)} |u_t^\varepsilon| + \int_{(0,t) \times (R-2\delta, R-\delta)} |u_t^\varepsilon| \right\}$$

Proof of Proposition 11.6

We first notice that (11.26) implies on $[a, b]$ and for all j , the map $p^j \mapsto \hat{f}^j(p)$ is nonincreasing, while the map $p^j \mapsto \hat{f}_R^j(p)$ is nondecreasing.

Step 1: Extension of the fluxes to the whole space

Let us simply extend $f^j : [a^j, b^j] \rightarrow \mathbb{R}$ by continuity to a function still denoted by $f^j : \mathbb{R} \rightarrow \mathbb{R}$, with constant value $f^j(a^j)$ on the left, and constant value $f^j(b^j)$ on the right of the interval $[a^j, b^j]$. This automatically extends to a bounded and Lipschitz continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ (with our standard abuse of notation).

We now want to extend the bounded and Lipschitz vectorial functions $\hat{f}, \hat{f}_R : [a, b] \rightarrow \mathbb{R}^N$ to the whole \mathbb{R}^N . We only do it for \hat{f} (this is similar for \hat{f}_R). We first notice that the first line of (11.26) for j means

$$(11.35) \quad -\partial_j \hat{f}^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} \omega_{jk} \partial_j \hat{f}^k \quad \text{for all } \omega_{jk} \in \{\pm 1\}$$

which is equivalent to the fact that the map

$$p^j \mapsto h_\omega^j(p) := \hat{f}^j(p) + \sum_{k \in \{1, \dots, N\} \setminus \{j\}} \omega_{jk} \hat{f}^k(p) \quad \text{is nonincreasing.}$$

Step 1.1: extension along p^1

We now first extend \hat{f} along the first coordinate p^1 as follows for $p' = (p^2, \dots, p^N) \in \Pi'_2 := \prod_{j=2, \dots, N} [a^j, b^j]$

$$\hat{f}(p^1, p') := \begin{cases} \hat{f}(a^1, p') & \text{for } p^1 < a^1 \\ \hat{f}(p^1, p') & \text{for } p^1 \in [a^1, b^1] \\ \hat{f}(b^1, p') & \text{for } p^1 > b^1 \end{cases}$$

Notice in particular that $p^1 \mapsto h_\omega^1(p)$ is also nonincreasing on $\Pi_1 := \mathbb{R} \times \Pi'_2$, because \hat{f} is constant in p^1 on $\{p^1 < a^1\}$ and also on $\{p^1 > b^1\}$. For $j \neq 1$, the maps $p^j \mapsto h_\omega^j(p)$ are also nonincreasing on Π_1 , because this is already the case for any fixed $p^1 \in [a^1, b^1]$, and in particular for $p^1 \in \partial[a^1, b^1]$. Moreover, for all j , the maps \hat{f}^j are also bounded and Lipschitz continuous on Π_1 .

Step 1.2: extension along p^2

Now for $p^1 \in \mathbb{R}$ and $p'' := (p^3, \dots, p^N) \in \Pi'_3 := \prod_{j=3, \dots, N} [a^j, b^j]$, we set

$$\hat{f}(p^1, p^2, p'') := \begin{cases} \hat{f}(p^1, a^2, p'') & \text{for } p^2 < a^2 \\ \hat{f}(p^1, p^2, p'') & \text{for } p^2 \in [a^2, b^2] \\ \hat{f}(p^1, b^2, p'') & \text{for } p^2 > b^2 \end{cases}$$

which extends \hat{f} to $\Pi_2 := \mathbb{R}^2 \times \Pi'_3$. Exactly the same reasoning as in Step 1.1 applies and shows that $p^j \mapsto h_\omega^j(p)$ is nonincreasing on Π_2 for all j , and that \hat{f} is bounded and Lipschitz continuous on Π_2 .

Step 1.3: induction and conclusion

We do the proof by induction on j , and finally get that $p^j \mapsto h_\omega^j(p)$ is nonincreasing on $\Pi_N := \mathbb{R}^N$ for all ω and for all j . For all indices j , this shows that $-\partial_j \hat{f}^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}^k|$ a.e. on \mathbb{R}^N . This implies

in particular that $p^j \mapsto \hat{f}^j(p)$ is nonincreasing for all j . Moreover the map \hat{f} is bounded and Lipschitz continuous over \mathbb{R}^N . We conclude that the extensions $f, \hat{f}, \hat{f}_R : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are bounded and globally Lipschitz continuous. Moreover \hat{f}, \hat{f}_R satisfy (11.26) a.e. on \mathbb{R}^N , and

$$\begin{cases} \text{Lip}(f^j; \mathbb{R}) &= \text{Lip}(f^j; [a^j, b^j]) \\ \text{Lip}(\hat{f}; \mathbb{R}^N) &= \text{Lip}(\hat{f}; [a, b]) \\ \text{Lip}(\hat{f}_R; \mathbb{R}^N) &= \text{Lip}(\hat{f}_R; [a, b]) \end{cases}$$

Step 2: Regularizations

Step 2.1: Regularization of the fluxes

We now consider some nonnegative function $\rho \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \rho = 1$. For some $\eta \in (0, 1]$ and for $p = (p^1, \dots, p^N)$, we set $\bar{\rho}_\eta(p) := \varepsilon^{-N} \bar{\rho}(\eta^{-1}p)$ with the product $\bar{\rho}(p) := \rho(p^1) \dots \rho(p^N)$. We set the mollifications $\hat{g}_\eta := \hat{f} \star_{\mathbb{R}^N} \bar{\rho}_\eta$ and $\hat{g}_{R, \eta} := \hat{f}_R \star_{\mathbb{R}^N} \bar{\rho}_\eta$. Applying mollification in (11.35), we get

$$-\partial_j \hat{g}_\eta^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} \omega_{jk} \partial_j \hat{g}_\eta^k \quad \text{for all } \omega_{jk} \in \{\pm 1\}$$

which shows that $-\partial_j \hat{g}_\eta^j \geq \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{g}_\eta^k|$ on \mathbb{R}^N . This implies that $\hat{g}_\eta, \hat{g}_{R,\eta}$ satisfy (11.26).

We now set the regularized fluxes $\hat{f}_\eta(p) := \hat{g}_\eta(p) - \mu \{p - m\}$ and $\hat{f}_{R,\eta}(p) := \hat{g}_\eta(p) + \mu \{p - m\}$ with $m := \frac{a+b}{2}$ and $\mu := \sqrt{\eta}$. Then $\hat{f}_\eta, \hat{f}_{R,\eta}$ still satisfy (11.26). They satisfy moreover the four last lines of (11.25) for our choice of μ and for $\eta > 0$ small enough. We also consider $f_\eta := f \star_{\mathbb{R}^N} \bar{\rho}_\eta$, which satisfies $f_\eta^j = f^j \star_{\mathbb{R}} \rho_\eta$ with $\rho_\eta(x) := \eta^{-1} \rho(\eta^{-1}x)$. Moreover we have

$$\begin{cases} \text{Lip}(f_\eta^j; \mathbb{R}) & \leq \text{Lip}(f^j; [a^j, b^j]) \\ \text{Lip}(\hat{f}_\eta; \mathbb{R}^N) & \leq \text{Lip}(\hat{f}; [a, b]) \\ \text{Lip}(\hat{f}_{R,\eta}; \mathbb{R}^N) & \leq \text{Lip}(\hat{f}_R; [a, b]) \end{cases}$$

Step 2.2: contraction-regularization of the initial data

For $\nu \in (0, 1)$, we contract u_0 , defining $u_{0,\nu} := m + (1 - \nu)(u_0 - m)$ with $m := \frac{a+b}{2}$. Let $0 < \beta \ll \nu$. Then for $\nu > 0$ small enough, we have $u_{0,\nu}(x) \in [a, b]_{-\beta} := \prod_{j=1, \dots, N} [a^j + \beta, b^j - \beta]$ for all $x \in [0, R]$. Then we extend $u_{0,\nu}$ outside the interval $[0, R]$ in a C^∞ function such that $u_{0,\nu}(x) \in [a, b]_{-\beta/2}$ for all $x \in \mathbb{R}$, and such that $u_{0,\nu}$ takes a constant value $u_{0,\nu}(-1)$ for $x < -1$ and the constant value $u_{0,\nu}(R+1)$ for $x > R+1$. Now, consider the mollification $v_{0,\nu,\eta} := u_{0,\nu} \star_{\mathbb{R}} \rho_\eta$. It satisfies

$$(11.36) \quad v_{0,\nu,\eta}(x) \subset [a, b]_{-\beta/2} \quad \text{for all } x \in \mathbb{R}.$$

We want our approximation to satisfy compatibility conditions (11.31). To this end, we now consider a function $\psi \in C_c^\infty([0, R_*])$ with $R_* := \min\{1, R\}$ and $\eta \in (0, 1]$ such that $\psi_\eta(x) = \eta \psi(\eta^{-1}x)$, $\psi(0) = 0$ and $\psi'(0) = 1$. Setting $\lambda := \varepsilon^{-1} \{f_\eta - \hat{f}_\eta\}(v_{0,\nu,\eta}(0)) - v'_{0,\nu,\eta}(0)$ and $\lambda_R := \varepsilon^{-1} \{f_\eta - \hat{f}_{R,\eta}\}(v_{0,\nu,\eta}(R)) - v'_{0,\nu,\eta}(R)$, we see that the function

$$w_0(x) := v_{0,\nu,\eta}(x) + \lambda \psi_\eta(x) - \lambda_R \psi_\eta(R - x) \quad \text{for } x \in [0, R]$$

satisfies

$$\begin{cases} f_\eta^j(w_0^j) - \varepsilon(w_0^j)_x = \hat{f}_\eta^j(w_0) & \text{for } x = 0 \\ f_\eta^j(w_0^j) - \varepsilon(w_0^j)_x = \hat{f}_{R,\eta}^j(w_0) & \text{for } x = R. \end{cases}$$

From (11.36), we deduce that for η small enough, we have $w_0(x) \subset [a, b]_{-\beta/4}$ for all $x \in \mathbb{R}$. Hence $f_\eta, \hat{f}_\eta, \hat{f}_{R,\eta} \in W_{loc}^{2,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ satisfy (11.25)-(11.26) and w_0 satisfies (11.27), (11.31).

Step 3: A priori estimates

Let us call $U = u_{\nu,\eta}^\varepsilon$ the solution given by Lemma 11.2 for fluxes $(f_\eta, \hat{f}_\eta, \hat{f}_{R,\eta})$ and initial data $u_{0,\nu,\eta} := w_0$.

We have in particular $U^j \in C_{t,x}^{\frac{2+\alpha}{2}, 2+\alpha}([0, +\infty) \times \bar{J}_R^j)$ for all $\alpha \in (0, 1)$. From (11.21) and (11.22), we get

$$(11.37) \quad \left\{ \begin{array}{l} U^j(t, x) \in [a^j, b^j] \quad \text{for all } (t, x) \in [0, +\infty) \times \bar{J}_R^j \\ \int_{\{t\} \times J_R} |U_t| \leq \int_{\{0\} \times J_R} |U_t| \\ \int_{\{t\} \times (2\delta, R-2\delta)} |U_x| \leq \int_{\{0\} \times (\delta, R-\delta)} |U_x| + \delta^{-1} \left\{ \int_{(0,t) \times (\delta, 2\delta)} |U_t| + \int_{(0,t) \times (R-2\delta, R-\delta)} |U_t| \right\} \end{array} \right. \quad \text{for all } \delta \in (0, R/4)$$

where

$$\left\{ \begin{array}{l} U_t(0, \cdot) = \varepsilon(w_0)_{xx} - (f_\eta(w_0))_x \\ U_x(0, \cdot) = (w_0)_x \end{array} \right. \quad \text{with } w_0 = u_{0,\nu,\eta}$$

We make a discussion for $U_t(0, \cdot)$ (the discussion is similar and simpler for $U_x(0, \cdot)$, and we skip it). Notice that we have $U_t(0, x) = A(x) + B(x)$ with $A(x) := \varepsilon(v_{0,\nu,\eta})_{xx} - (f_\eta(v_{0,\nu,\eta}))_x$ and $B(x) := U_t^j(0, x) - A(x)$. Moreover, we have

$$\limsup_{(\nu,\eta) \rightarrow (0,0)} |A^j(x)| \leq \varepsilon |(u_0^j)_{xx}(x)| + \text{Lip}(f^j; [a, b]) \cdot |(u_0^j)_x(x)| \quad \text{for all } x \in (0, R)$$

and

$$|A^j|_{L^\infty(0,R)} \leq 1 + \varepsilon |(u_0^j)_{xx}|_{L^\infty(0,R)} + \text{Lip}(f^j; [a, b]) \cdot |(u_0^j)_x|_{L^\infty(0,R)} \quad \text{for } (\nu, \eta) \text{ close enough to } (0, 0)$$

The contribution to B only arises through the functions $\psi_\eta(x)$ and $\psi_\eta(R-x)$. Because $|\psi_\eta''|_{L^1(0,R)} \leq C$ and $|\text{supp}(\psi_\eta)| \rightarrow 0$ and $\lambda, \lambda_R \rightarrow 0$ as $(\nu, \eta) \rightarrow (0, 0)$, we deduce that $|B|_{L^1(0,R)} \rightarrow 0$ as $(\nu, \eta) \rightarrow (0, 0)$.

Step 4: Removing regularization of the fluxes

Setting $K[u_0] := \sum_{j=1, \dots, N} \left\{ \varepsilon |(u_0^j)_{xx}|_{L^1(0,R)} + \text{Lip}(f^j; [a, b]) \cdot |(u_0^j)_x|_{L^1(0,R)} \right\}$ and $K'[u_0] = 1 + K[u_0]$, and for (ν, η) close enough to $(0, 0)$, from Step 3 we get

$$(11.38) \quad \left\{ \begin{array}{l} U^j(t, x) \in [a^j, b^j] \quad \text{for all } (t, x) \in [0, +\infty) \times \bar{J}_R^j \\ \int_{\{t\} \times J_R} |U_t| \leq K'[u_0] \\ \int_{\{t\} \times (2\delta, R-2\delta)} |U_x| \leq \int_{\{0\} \times (\delta, R-\delta)} |(u_0)_x| + \delta^{-1} K'[u_0] \quad \text{for all } \delta \in (0, R/4) \end{array} \right.$$

We conclude from the compactness of the inclusion $BV_{loc} \subset L^1_{loc}$, that up to extract a subsequence (and using a diagonal extraction argument to cover $[0, +\infty) \times \bar{J}_R$), that we have

$$U = u_{\nu, \eta}^\varepsilon \rightarrow u^\varepsilon \quad \text{in } L^1_{loc}([0, +\infty) \times \bar{J}_R) \quad \text{as } (\nu, \eta) \rightarrow (0, 0)$$

From (11.37), we deduce that

$$\left\{ \begin{array}{l} u^{\varepsilon, j}(t, x) \in [a^j, b^j] \quad \text{for all } (t, x) \in [0, +\infty) \times \bar{J}_R^j \\ \int_{\{t\} \times J_R} |u_t^\varepsilon| \leq \int_{\{0\} \times J_R} |u_t^\varepsilon| \\ \int_{\{t\} \times (2\delta, R-2\delta)} |u_x^\varepsilon| \leq \int_{\{0\} \times (\delta, R-\delta)} |u_x^\varepsilon| + \delta^{-1} \left\{ \int_{(0,t) \times (\delta, 2\delta)} |u_t^\varepsilon| + \int_{(0,t) \times (R-2\delta, R-\delta)} |u_t^\varepsilon| \right\} \quad \text{for all } \delta \in (0, R/4) \end{array} \right.$$

where u^ε satisfies (11.32). Moreover we can pass to the limit in the PDE (and its weak formulation). Therefore u^ε solves (11.29) with initial data as in (11.30) with u_0 solving (11.31). Finally Lemma 11.2 shows that $u^{\varepsilon, j} \in C_{t,x}^{\frac{\alpha}{2}, \alpha}([0, +\infty) \times \bar{J}_R^j)$ for all $\alpha \in (0, 1)$. This ends the proof of the proposition.

11.5 Removing the viscosity

Proposition 11.7 (Existence for truncated junctions)

Assume (2.2) with $N \geq 1$, nondegeneracy condition (2.16), for a junction J of type $0 : N$ and bounded box $[a, b] \subset \mathbb{R}^N$. Let $R > 0$, $J_R := J \cap B_R$ with $J_R^j \simeq (0, R)$ for all j and $\sigma = (-1, \dots, -1) = -\sigma_R \in \mathbb{R}^N$.

Assume that f, \hat{f}, \hat{f}_R satisfy (11.25) and (11.26). Assume also that $\mathcal{G} := \left\{ p \in [a, b], \quad \hat{f}(p) = f(p) \right\}$ and

$\mathcal{G}_R := \left\{ p \in [a, b], \quad \hat{f}_R(p) = f(p) \right\}$ are two Kruřkov germs respectively at the junction point $x = 0$, and at the junction point $x = R$.

Assume that the initial data $u_0 = (u_0^1, \dots, u_0^N)$ is of bounded variations, i.e. satisfies $u_0^j \in BV(J_R^j; [a^j, b^j])$.

i) (Existence and uniqueness)

Then there exists a unique $u = (u^j)_{j=1, \dots, N}$ with

$$(11.39) \quad u^j : [0, +\infty) \times J_R^j \rightarrow [a^j, b^j]$$

entropy solution of

$$(11.40) \quad \left\{ \begin{array}{lll} u_t^j + (f^j(u^j))_x = 0 & \text{on} & (0, +\infty) \times J_R^j \\ f^j(u^j) = \hat{f}^j(u) & \text{a.e. on} & (0, +\infty) \times \{0\} \\ f^j(u^j) = \hat{f}_R^j(u) & \text{a.e. on} & (0, +\infty) \times \{R\} \end{array} \right.$$

where the boundary conditions are satisfied in the sense of strong traces and with initial condition

$$(11.41) \quad u^j = u_0^j \quad \text{on} \quad \{0\} \times J_R^j$$

Moreover we have

$$u \in \text{Lip}([0, +\infty); \mathcal{M}(J_R^*))$$

with $J_R^* := J_R \setminus \{0\}$ and $\mathcal{M}(J_R^*)$ denotes the set of measures on J_R^* . For $w = (w^1, \dots, w^N)$, where each w^j denotes a finite measure on J_R^j , the associated norm is defined as $|w|_{\mathcal{M}(J_R^*)} := \sum_{j=1, \dots, N} |w^j|_{\mathcal{M}(J_R^j)}$, where

$\mathcal{M}(J_R^j)$ is the set of (real valued) measures on the open interval $J_R^j \simeq (0, R)$ and $|\cdot|_{\mathcal{M}(J_R^j)}$ is the total variation of the measure.

ii) (Additional bounds)

Then we have the following bounds for a.e. $t > 0$

$$(11.42)$$

$$|u_t(t, \cdot)|_{\mathcal{M}(J_R^*)} \leq L|(u_0)_x|_{\mathcal{M}(J_R^*)} + K_0 \quad \text{with} \quad \left\{ \begin{array}{l} L := \sup_{j=1, \dots, N} \text{Lip}(f^j; [a^j, b^j]) \\ K_0 := \sum_{j=1, \dots, N} \left\{ |(f^j - \hat{f}^j)(u_0(0))| + |(f^j - \hat{f}^j)(u_0(R))| \right\} \end{array} \right\}$$

and

$$(11.43) \quad |u_x(t, \cdot)|_{\mathcal{M}(J_{R-2\delta} \setminus \bar{B}_{2\delta})} \leq (1 + \delta^{-1}L)|(u_0)_x|_{\mathcal{M}(J_R^*)} + \delta^{-1}K_0 \quad \text{for all} \quad \delta \in (0, R/4)$$

Proof of Proposition 11.7

Consider the solution u^ε given in Proposition 11.6 for the fluxes f, \hat{f}, \hat{f}_R and some initial data u_0^ε which approximates the function u_0 and which is described below.

Step 1: definition of the smooth initial data u_0^ε

Precisely, because $u_0 \in BV(J_R^*)$, we deduce that u_0^j has a two limits $u_0^j(0)$ and $u_0^j(R)$. We then extend u_0^j by the value $u_0^j(0)$ for $x \leq 0$ and by the value $u_0^j(R)$ for $x \geq R$. We get the BV semi-norms $[u_0^j]_{BV(\mathbb{R})} = [u_0^j]_{BV(J_R^j)} := |(u_0^j)_x|_{\mathcal{M}(J_R^j)}$. For $\nu \in (0, 1)$, we then contract u_0 , defining $u_{0,\nu} := m + (1 - \nu)(u_0 - m)$ with $m := \frac{a+b}{2}$. Let

$$(11.44) \quad 0 < \beta \ll \nu$$

Then for $\nu > 0$ small, we have $u_{0,\nu}(x) \in [a, b]_{-\beta} := \prod_{j=1, \dots, N} [a^j + \beta, b^j - \beta]$ for all $x \in \mathbb{R}$. For $\eta, \mu > 0$, using the notation $\rho_\eta = \eta^{-1}\rho(\eta^{-1}\cdot)$ and $\psi_\mu = \mu\psi(\mu^{-1}\cdot)$ of Step 2 of the proof of Proposition 11.6, we set $v_{0,\nu,\eta} := u_{0,\nu} \star_{\mathbb{R}} \rho_\eta : \mathbb{R} \rightarrow [a, b]_{-\beta}$. Setting $\lambda := \varepsilon^{-1} \left\{ f - \hat{f} \right\} (v_{0,\nu,\eta}(0)) - v'_{0,\nu,\eta}(0)$ and $\lambda_R := \varepsilon^{-1} \left\{ f - \hat{f}_R \right\} (v_{0,\nu,\eta}(R)) - v'_{0,\nu,\eta}(R)$, we see that

$$u_0^\varepsilon(x) := v_{0,\nu,\eta}(x) + \lambda\psi_\mu(x) - \lambda_R\psi_\mu(R-x) \quad \text{for} \quad x \in [0, R]$$

satisfies

$$\begin{cases} f(u_0^\varepsilon) - \varepsilon(u_0^\varepsilon)_x = \hat{f}(u_0^\varepsilon) & \text{for} \quad x = 0 \\ f(u_0^\varepsilon) - \varepsilon(u_0^\varepsilon)_x = \hat{f}_R(u_0^\varepsilon) & \text{for} \quad x = R. \end{cases}$$

Notice that for

$$(11.45) \quad \mu \ll \eta \ll \varepsilon$$

we have in particular $u_{0,\nu,\eta,\mu}(x) \in [a, b]_{-\beta/2}$ for all $x \in [0, R]$.

Step 2: a priori bounds

Using $(v_{0,\nu,\eta})_{xx} = (u_{0,\nu})_x \star_{\mathbb{R}} (\rho_\eta)_x$ and (11.44)-(11.45), we get

$$\varepsilon|(v_{0,\nu,\eta})_{xx}|_{L^1(J_R^j)} \leq \varepsilon|(u_0^j)_x|_{\mathcal{M}(\mathbb{R})}|(\rho_\eta)_x|_{L^1(\mathbb{R})} = \varepsilon\eta^{-1}[u_0^j]_{BV(J_R^j)}|\rho_x|_{L^1(\mathbb{R})} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0$$

and

$$\varepsilon |(\lambda^j \psi_\mu)_{xx}|_{L^1(J_R^j)} = \varepsilon |\lambda^j| |\psi''|_{L^1(0,+\infty)} \rightarrow |(f^j - \hat{f}^j)(u_0(0))| \cdot |\psi''|_{L^1(0,+\infty)} \quad \text{as } (\varepsilon, \nu) \rightarrow (0, 0)$$

and similarly

$$|(v_{0,\nu,\eta}^j)_x|_{L^1(J_R^j)} \leq |(u_0^j)_x|_{\mathcal{M}(\mathbb{R})} |\rho_\eta|_{L^1(\mathbb{R})} = [u_0^j]_{BV(J_R^j)}$$

and

$$|(\lambda^j \psi_\mu)_x|_{L^1(J_R^j)} = |\lambda^j \mu| \cdot |\psi'|_{L^1(0,+\infty)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

Notice that $\psi'(0) = 1$ and $\psi \in C_c^\infty([0, R_*])$. Hence we can find a sequence of such ψ^n such that $|(\psi^n)''|_{L^1(0,+\infty)} = [(\psi^n)']_{BV(0,+\infty)} \rightarrow 1$, where the limit is the BV semi-norm of Heaviside function. Setting

$$K[u_0] := \sum_{j=1,\dots,N} \left\{ |(f^j - \hat{f}^j)(u_0(0))| + |(f^j - \hat{f}_R^j)(u_0(R))| + \text{Lip}(f^j; [a, b]) \cdot [u_0^j]_{BV(0,R)} \right\}, \quad K'[u_0] = 1 + K[u_0]$$

and for (ε, ν) close enough to $(0, 0)$, we deduce from (11.38) that for all time $t > 0$

$$\begin{cases} u^{\varepsilon,j}(t, x) \in [a^j, b^j] & \text{for all } (t, x) \in [0, +\infty) \times J_R^j \\ |u_t^\varepsilon(t, \cdot)|_{\mathcal{M}(J_R^*)} \leq K'[u_0] \\ |u_x^\varepsilon(t, \cdot)|_{\mathcal{M}(J_{R-2\delta} \setminus \bar{B}_{2\delta})} \leq [u_0]_{BV(J_R^*)} + \delta^{-1} K'[u_0] & \text{for all } \delta \in (0, R/4) \end{cases}$$

From compactness of $BV_{loc} \subset L^1_{loc}$, we get that, up to extract a subsequence, we have

$$\left\{ \begin{array}{l} u^\varepsilon \rightarrow u \quad \text{in } L^1_{loc}([0, +\infty) \times J_R) \\ u_0^\varepsilon \rightarrow u_0 \quad \text{in } L^1(J_R) \end{array} \right\} \quad \text{as } (\varepsilon, \nu) \rightarrow (0, 0)$$

From the semi-continuity of BV norms/total variations of measures, this implies in particular bounds (11.42) and (11.43).

Step 3: test function formulations

Recall that (11.15) holds true for two solutions $u^\varepsilon, v^\varepsilon$ and fluxes satisfying assumption (11.5).

Step 3.1: interior formulations

In the special case where the test function satisfies $\varphi = (0, \dots, 0, \varphi^j, 0, \dots, 0)$ with $0 \leq \varphi^j \in C_c^\infty([0, +\infty) \times J_R^j)$, relation (11.15) still holds true if the solutions are only solutions on $(0, +\infty) \times J_R^j$. In particular, we can choose $v^\varepsilon = c := (0, \dots, 0, c^j, 0, \dots, 0)$ for any $c^j \in [a^j, b^j]$. This gives

$$0 \leq \varepsilon C_{R,\varphi} + \int_{\{0\} \times J_R^j} |u_0^{\varepsilon,j} - c^j| \varphi^j + \int_{(0,+\infty) \times J_R^j} \left\{ |u^\varepsilon - c^j| \varphi_t^j + \psi^{f^j}(u^\varepsilon, c) \varphi_x^j \right\}$$

In the limit $\varepsilon \rightarrow 0$, we recover for all $c^j \in [a^j, b^j]$

$$0 \leq \int_{\{0\} \times J_R^j} |u_0^j - c^j| \varphi^j + \int_{(0,+\infty) \times J_R^j} \left\{ |u - c^j| \varphi_t^j + \psi^{f^j}(u, c) \varphi_x^j \right\}$$

which means exactly that u^j is an entropy solution of (11.40) only on $(0, +\infty) \times J_R^j$ and with initial data u_0^j .

Step 3.2: boundary formulation at $x = 0$

In the special case where the test function satisfies $\varphi = (\varphi^1, \dots, \varphi^N)$ with $0 \leq \varphi^j \in C_c^\infty([0, +\infty) \times (J_R^j \cup \{0\}))$ with $\varphi^j(t, 0) = \varphi^k(t, 0)$ for all k, j and all times $t \geq 0$, then relation (11.15) still holds true if the solutions are only solutions on $(0, +\infty) \times J_R$, where we recall that $0 \in J_R$. In particular, we have for all $v^\varepsilon := c \in \mathcal{G}$

$$0 \leq \int_{(0,+\infty) \times \{0\}} D\hat{f}(u^\varepsilon, c) \varphi \leq \varepsilon C_{R,\varphi} + \int_{\{0\} \times J_R} |u_0^\varepsilon - c| \varphi + \int_{(0,+\infty) \times J_R} \left\{ |u^\varepsilon - c| \varphi_t + \psi^f(u^\varepsilon, c) \varphi_x \right\}$$

In the limit $\varepsilon \rightarrow 0$, we recover $0 \leq \int_{\{0\} \times J_R} |u_0 - c| \varphi + \int_{(0,+\infty) \times J_R} \left\{ |u - c| \varphi_t + \psi^f(u, c) \varphi_x \right\}$. Now we can find a sequence of test functions approximating the functions $\varphi_\gamma^j(t, x) = \zeta(t) \max\{0, 1 - \gamma^{-1}x\}$ for all j and

$\gamma > 0$. Using nondegeneracy condition (2.16), we deduce from Theorem 2.33 that u has a strong trace at $x = 0$ for a.e. time $t > 0$. Therefore, in the limit $\gamma \rightarrow 0$ focusing on the junction point $x = 0$, we get

$$0 \leq \int_{(0,+\infty) \times \{0\}} \left\{ \sum_{j=1,\dots,N} -\psi^{f^j}(u, c) \right\} \zeta = \int_{(0,+\infty) \times \{0\}} D^f(u, c) \zeta$$

Because this is true for any function $0 \leq \zeta \in C_c^\infty([0, +\infty))$, we deduce that for a.e. time $t > 0$, we have $0 \leq D^f(u(t, 0), c)$ for all $c \in \mathcal{G}$. Because $u(t, 0) \in [a, b]$ and \mathcal{G} is a Kruřkov germ, we deduce from Theorem 2.21 that \mathcal{G} is maximal and then $u(t, 0) \in \mathcal{G}$ for a.e. time $t > 0$. This shows the second line of (11.40).

Step 3.3: boundary formulation at $x = R$

The proof is similar to Step 3.2, and shows the third line of (11.40).

Step 4: uniqueness

Finally the uniqueness of the solution u follows as usual, from the L^1 -contraction as in i) of Lemma 10.3. We do not repeat the details here. This ends the proof of the proposition.

11.6 Proof of Theorem 2.36: removing the truncations

Proposition 11.8 (Existence on the infinite junction)

Assume (2.2) with $N \geq 1$, nondegeneracy condition (2.16), and let $\mathcal{G} \subset [a, b]$ be a Kruřkov germ. Consider an initial data u_0 with $u_0(J) \subset [a, b]$ such that $u_0 - p_0 \in (BV \cap L^1)(J)$ for some $p_0 \in [a, b] \cap \mathbb{R}^N$. Then there exists a unique \mathcal{G} -entropy solution u of (2.4) with initial data u_0 .

Proof of Proposition 11.8 and of Theorem 2.36

Step 1: reduction to a bounded box

Because $u_0 \in BV(J)$, we deduce that u_0 is bounded. From Proposition 6.1, there exists a bounded box $[\bar{a}, \bar{b}] \subset [a, b]$ such that $u_0(J) \subset [\bar{a}, \bar{b}]$ and $\mathcal{G} \cap [\bar{a}, \bar{b}]$ is a Riemann germ on the bounded box $[\bar{a}, \bar{b}]$.

Step 2: construction of the germ \mathcal{G}_R

Let $R > 0$ and $J_R := J \cap B_R$ and $J_R^j := J^j \cap B_R$. We set

$$\hat{f}_R^j(u^j) := \begin{cases} G^{f^j}(u^j, p_0^j) & \text{if } J_R^j \simeq (0, R) \\ G^{f^j}(p_0^j, u^j) & \text{if } J^j \simeq (-R, 0) \end{cases}$$

where $G^{f^j}(\uparrow, \downarrow)$ is the standard Godunov flux associated to f^j . Indeed, using Lemma 11.1 it is simpler to work with junctions J of type $0 : N$, and at the very end of the proof to use Lemma 11.1 to come back to the original problem. Hence in the remaining of the proof, we assume that J is of type $0 : N$, and then $J_R^j \simeq (0, R)$ for all j .

Here we have

$$\mathcal{G}_R := \left\{ p \in [a, b], \quad \hat{f}_R(p) = f(p) \right\} = \prod_{j=1,\dots,N} \mathcal{G}_R^j \quad \text{with} \quad \mathcal{G}_R^j := \left\{ p^j \in [a^j, b^j], \quad \hat{f}_R^j(p) = f^j(p^j) \right\}$$

Notice that each \mathcal{G}_R^j is a Kruřkov germ (as any germ for a single branch), and then it is easy to check that the product \mathcal{G}_R is also a Kruřkov germ.

Step 3: properties of the Godunov fluxes

From now on, and to simplify the presentation, let us assume without loss of generality that $[\bar{a}, \bar{b}] = [a, b]$ is a bounded box. Notice now that the four last lines of (11.5) are automatically satisfied for the Godunov fluxes $\hat{f} = \hat{f}_{\mathcal{G}}$ and $\hat{f}_R := \hat{f}_{\mathcal{G}_R}$, because of the monotone bounds (second line of (2.14)) satisfied by the Godunov fluxes.

Step 4: construction of a solution on J_R and the limit $R \rightarrow +\infty$

Therefore Proposition 11.7 applies with $p_0 := u_0(R^-)$ and gives the existence of a solution u_R of (11.40) with initial data $(u_0)|_{J_R}$. In particular we have $\hat{f}_R(u_0) = f(u_0)$ at $x = R$. From the a priori bounds given in Proposition 11.7, we can pass to the limit with $u_R \rightarrow u$ as $R \rightarrow +\infty$. This leads to the existence of a solution u with bounds.

Step 5: uniqueness and L^1 -contraction

The uniqueness and L^1 -contraction follow from i) of Lemma 10.3.

Finally, in the special case $p_0 = 0$, we recover the results of Theorem 2.36, and for general p_0 , this proves Proposition 11.8. This ends the proof.

12 Existence via semi-discretized schemes for Kruřkov germs

12.1 The semi-discretized scheme

Assume (2.2) with $N \geq 1$ and junction (J, f) with $\sigma \in \{\pm 1\}^N$. We set the disjoint union

$$(12.1) \quad J_{\mathbb{N}} := \coprod_{\alpha=1, \dots, N} J_{\mathbb{N}}^{\alpha} \quad \text{with} \quad J_{\mathbb{N}}^{\alpha} := -\sigma^{\alpha} \mathbb{N}, \quad \text{for} \quad \alpha = 1, \dots, N$$

For $R \in \mathbb{N}$ with $R \geq 3$, we can also consider truncated branches

$$J_R := \coprod_{\alpha=1, \dots, N} J_R^{\alpha} \quad \text{with} \quad J_R^{\alpha} := \{k \in -\sigma^{\alpha} \mathbb{N}, \quad |k| \leq R\}, \quad \text{for} \quad \alpha = 1, \dots, N$$

We set $R^{\alpha} := \sigma^{\alpha} R$. Given some $p_0 \in [a, b]$, we also consider the following flux

$$\hat{f}_R^{\alpha}(p) := \begin{cases} G^{\alpha}(p^{\alpha}, p_0^{\alpha}) & \text{if } \sigma^{\alpha} = -1 \\ G^{\alpha}(p_0^{\alpha}, p^{\alpha}) & \text{if } \sigma^{\alpha} = 1 \end{cases}$$

where $G^{\alpha}(\uparrow, \downarrow)$ is the standard Godunov flux associated to the function f^{α} . Recall that $\mathcal{G}_R := \{p \in [a, b], \hat{f}_R(p) = f(p)\}$ is a Kruřkov germ with respect to junction $(-J, f)$.

We consider functions $u : [0, +\infty) \times J_R \rightarrow \mathbb{R}$ with $u = (u_k^{\alpha}(t))$ for $t \geq 0$, $k \in J_R^{\alpha}$ and $\alpha = 1, \dots, N$ solution of the following semi-discretized scheme for $u_k := (u_k^1, \dots, u_k^N)$ and $\Delta x > 0$

$$(12.2) \quad \left\{ \begin{array}{lll} \partial_t u_k^{\alpha} + \frac{G^{\alpha}(u_k^{\alpha}, u_{k+1}^{\alpha}) - G^{\alpha}(u_{k-1}^{\alpha}, u_k^{\alpha})}{\Delta x} = 0 & \text{for} & k \in J_{\mathbb{N}}^{\alpha} \setminus \{0, R^{\alpha}\} \\ \partial_t u_k^{\alpha} + \frac{G^{\alpha}(u_k^{\alpha}, u_{k+1}^{\alpha}) - \hat{f}^{\alpha}(u_k)}{\Delta x} = 0 & \text{for} & k = 0 \quad \text{with } \sigma^{\alpha} = -1 \\ \partial_t u_k^{\alpha} + \frac{\hat{f}^{\alpha}(u_k) - G^{\alpha}(u_{k-1}^{\alpha}, u_k^{\alpha})}{\Delta x} = 0 & \text{for} & k = 0 \quad \text{with } \sigma^{\alpha} = 1 \\ \partial_t u_k^{\alpha} + \frac{\hat{f}_R^{\alpha}(u_k) - G^{\alpha}(u_{k-1}^{\alpha}, u_k^{\alpha})}{\Delta x} = 0 & \text{for} & k = R^{\alpha} \quad \text{with } \sigma^{\alpha} = -1 \\ \partial_t u_k^{\alpha} + \frac{G^{\alpha}(u_k^{\alpha}, u_{k+1}^{\alpha}) - \hat{f}_R^{\alpha}(u_k)}{\Delta x} = 0 & \text{for} & k = R^{\alpha} \quad \text{with } \sigma^{\alpha} = 1 \end{array} \right.$$

12.2 Preparation

Then we have the following result.

Lemma 12.1 (Existence of a semi-discrete solution)

Assume (2.2) with $N \geq 1$ and junction (J, f) with $\sigma \in \{\pm 1\}^N$ and $[a, b]$ bounded. Assume that $\mathcal{G} \subset [a, b]$ is a Kruřkov germ, whose associated flux is $\hat{f} := \hat{f}_{\mathcal{G}}$. With notation (12.1), assume that the initial data $u(0)$ satisfies

$$(12.3) \quad u_k^{0, \alpha}(t) \in [a^{\alpha}, b^{\alpha}] \quad \text{for all } k \in J_R^{\alpha}, \quad \alpha = 1, \dots, N$$

at the initial time $t = 0$. For any $\Delta x > 0$, there exists a unique solution $u : [0, +\infty) \times J_R \rightarrow \mathbb{R}$ of (12.2) with initial data $u(0)$. Moreover $u(t)$ satisfies (12.3) for all $t \geq 0$ and $u \in C^{1,1}([0, +\infty); L^{\infty}(J_R))$.

Proof of Lemma 12.1

We first extend the fluxes f, \hat{f}, \hat{f}_R from the box $[a, b]$ to the whole space \mathbb{R}^N as in Step 1 of the proof of Proposition 11.6. We also define the extended Godunov flux $G^{\alpha} = G^{f^{\alpha}}$ from the extended flux f^{α} .

Then the result follows easily from the fact that \hat{f}, \hat{f}_R and f and $(G^{\alpha})_{\alpha}$ are globally Lipschitz continuous functions. Then the classical Cauchy-Lipschitz theorem applies to the Banach space $L^{\infty}(J_R)$, and gives the existence and uniqueness of a solution in $C^{1,1}$. Moreover, the fluxes \hat{f}, \hat{f}_R satisfy the second line of (2.14)

which implies that f, \hat{f}, \hat{f}_R satisfy (11.25). Let us now show that $u_k^\alpha(t)$ remains in the box $\in [a^\alpha, b^\alpha]$. Let us show for instance that $u_k^\alpha(t) \leq b^\alpha$. Formally, if the supremum of $u_k^\alpha(t_0)$ on $k \in J_R^\alpha$ is reached at some index k_0 for some time t_0 with value $u_{k_0}^\alpha(t_0) = b^\alpha$, then we get for $k = k_0$

$$\partial_t u_{k_0}^\alpha(t_0) = \begin{cases} \frac{G^\alpha(u_{k_0-1}^\alpha, b^\alpha) - G^\alpha(b^\alpha, u_{k_0+1}^\alpha)}{\Delta x} \leq 0 & \text{if } k_0 \in J_R^\alpha \setminus \{0, R^\alpha\} \\ \frac{\hat{f}^\alpha(u_k) - G^\alpha(b^\alpha, u_{k+1}^\alpha)}{\Delta x} \leq 0 & \text{if } k_0 = 0 \quad \text{and } \sigma^\alpha = -1 \\ \frac{G^\alpha(u_{k_0-1}^\alpha, b^\alpha) - \hat{f}^\alpha(u_k)}{\Delta x} \leq 0 & \text{if } k_0 = 0 \quad \text{and } \sigma^\alpha = 1 \\ \frac{G^\alpha(u_{k_0-1}^\alpha, b^\alpha) - \hat{f}_R^\alpha(u_k)}{\Delta x} \leq 0 & \text{if } k_0 = R^\alpha \quad \text{and } \sigma^\alpha = -1 \\ \frac{\hat{f}_R^\alpha(u_k) - G^\alpha(b^\alpha, u_{k+1}^\alpha)}{\Delta x} \leq 0 & \text{if } k_0 = R^\alpha \quad \text{and } \sigma^\alpha = 1 \end{cases}$$

where in the first line we have used the monotonicities of the Godunov flux $G^\alpha(\uparrow, \downarrow)$, while in the four last lines, we have also used the four last lines of (11.25). This shows formally that $u_k^\alpha(t) \leq b^\alpha$ for all times $t \geq 0$ and $k \in J_R^\alpha$. As it is usual, this can be made rigorous, using for instance classical technics proving the comparison principle for viscosity solutions. This ends the proof of the lemma.

We will need the following easy result.

Lemma 12.2 (Comparison of entropy fluxes)

We define the entropy flux for $x', y', x, y \in [a^\alpha, b^\alpha]$ as

$$\Psi^\alpha(x', y'; x, y) := G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y)$$

where $G^\alpha : [a^\alpha, b^\alpha]^2 \rightarrow \mathbb{R}$ is the standard Godunov flux associated to the function f^α . Then we have

$$(12.4) \quad \text{sign}(y - y') \cdot \{G^\alpha(x, y) - G^\alpha(x', y')\} \leq \Psi^\alpha(x', y'; x, y) \leq \text{sign}(x - x') \cdot \{G^\alpha(x, y) - G^\alpha(x', y')\}$$

Proof of Lemma 12.2

We only prove the right inequality, i.e.

$$(12.5) \quad G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y) \leq \text{sign}(x - x') \cdot \{G^\alpha(x, y) - G^\alpha(x', y')\}$$

Notice that the proof of the left inequality is similar.

Case A: $x - x' \geq 0$

Subcase A.1: $y \geq y'$

Then (12.5) means $G(x, y) - G(x', y') \leq G(x, y) - G(x', y')$, which is indeed an equality.

Subcase A.2: $y \leq y'$

Then (12.5) means $G(x, y') - G(x', y) \leq G(x, y) - G(x', y')$, which is true because of the monotonicity of $G(x'', \downarrow)$.

Case B: $x - x' \leq 0$

Subcase B.1: $y \geq y'$

Then (12.5) means $G(x', y) - G(x, y') \leq -\{G(x, y) - G(x', y')\}$, which is true because of the monotonicity of $G(x'', \downarrow)$.

Subcase B.2: $y \leq y'$

Then (12.5) means $G(x', y') - G(x, y) \leq -\{G(x, y) - G(x', y')\}$, which is again an equality.

This ends the proof of the lemma.

For later use, we will need the following.

Definition 12.3 (The approximate solution u^ε given by the semi-discrete scheme)

We set $\varepsilon := \Delta x > 0$. Given a solution u of the semi-discretized scheme (12.2), we define $u^\varepsilon = (u^{\varepsilon,1}, \dots, u^{\varepsilon,N})$ as

$$(12.6) \quad u^{\varepsilon,\alpha}(t, x) := \sum_{k \in J_R^\alpha} u_k^\alpha(t) \chi_k^{\varepsilon,\alpha}(x) \quad \text{with} \quad \chi_k^{\varepsilon,\alpha}(x) := \begin{cases} 1_{[k\varepsilon, (k+1)\varepsilon)}(x) & \text{if } \sigma^\alpha = -1 \\ 1_{((k-1)\varepsilon, k\varepsilon]}(x) & \text{if } \sigma^\alpha = 1 \end{cases} \quad \text{for } \alpha = 1, \dots, N$$

Lemma 12.4 (Entropy inequalities)

We work under assumptions of Lemma 12.1 and recall that $\sigma \in \{\pm\}^N$. Let $u(t), v(t)$ two solutions of the semi-discretized scheme (12.2) respectively with initial data $u(0), v(0)$. Then we have the following

i) (Pointwise entropy inequality)

$$(12.7) \quad \partial_t W_k^\alpha + \frac{\Psi_{k+\frac{1}{2}}^\alpha - \Psi_{k-\frac{1}{2}}^\alpha}{\Delta x} \leq 0 \quad \text{for all } k \in J_R^\alpha, \quad \alpha = 1, \dots, N$$

with for $k \in \mathbb{Z}$

$$\left\{ \begin{array}{l} W_k^\alpha := |u_k^\alpha - v_k^\alpha| \quad \text{for } k \in J_R^\alpha \\ \\ \Psi_{k+\frac{1}{2}}^\alpha := \begin{cases} \begin{array}{l} \Psi^\alpha(u_k^\alpha, u_{k+1}^\alpha; v_k^\alpha, v_{k+1}^\alpha) & \text{if } k \in J_R^\alpha \setminus \{R\} \\ \Psi_R^\alpha(u_k, v_k) & \text{if } k = R \end{array} & \text{if } \sigma^\alpha = -1 \\ \\ \Psi_0^\alpha(u_{k+1}, v_{k+1}) & \text{if } k = -1 \\ \\ \begin{array}{l} \Psi^\alpha(u_k^\alpha, u_{k+1}^\alpha; v_k^\alpha, v_{k+1}^\alpha) & \text{if } k \in J_R^\alpha \setminus \{0\} \\ \Psi_0^\alpha(u_k, v_k) & \text{if } k = 0 \end{array} & \text{if } \sigma^\alpha = 1 \\ \\ \Psi_R^\alpha(u_{k+1}, v_{k+1}) & \text{if } k = -R-1 \end{cases} \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \Psi^\alpha(x', y'; x, y) := G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y) \\ \Psi_0^\alpha(z', z) = \text{sign}(z'^\alpha - z^\alpha) \cdot \left\{ \hat{f}^\alpha(z') - \hat{f}^\alpha(z) \right\} \\ \Psi_R^\alpha(z', z) = \text{sign}(z'^\alpha - z^\alpha) \cdot \left\{ \hat{f}_R^\alpha(z') - \hat{f}_R^\alpha(z) \right\} \end{array} \right.$$

ii) (Integral entropy inequality)

Let $u^\varepsilon, v^\varepsilon$ as in Definition 12.3 with $\varepsilon := \Delta x$. Define also

$$(12.8) \quad \Psi^{\varepsilon,\alpha}(t, x) := \sum_{k \in J_R^\alpha \setminus \{R^\alpha\}} \Psi_{k+\frac{1}{2}}^\alpha(t) \chi_k^{\varepsilon,\alpha}(x) \quad \text{with} \quad \chi_k^{\varepsilon,\alpha}(x) := \begin{cases} 1_{[k\varepsilon, (k+1)\varepsilon)}(x) & \text{if } \sigma^\alpha = -1 \\ 1_{((k-1)\varepsilon, k\varepsilon]}(x) & \text{if } \sigma^\alpha = 1 \end{cases} \quad \text{for } \alpha = 1, \dots, N$$

Let us consider a smooth test function $0 \leq \varphi_k^\alpha(t)$ with compact support in $t \in [0, +\infty)$ for $k \in J_R^\alpha$ which satisfies $\varphi_0^\alpha(t) = \varphi_0^\beta(t)$ for all α, β . Then we have the following integral entropy inequality

$$(12.9) \quad \int_{\{0\} \times J} |u^\varepsilon - v^\varepsilon| \varphi^\varepsilon + \int_{(0, +\infty) \times J} \left\{ |u^\varepsilon - v^\varepsilon| \varphi_t^\varepsilon - \sigma \Psi^\varepsilon \cdot \left\{ \frac{\varphi^\varepsilon(t, \cdot - \sigma\varepsilon) - \varphi^\varepsilon(t, \cdot)}{\varepsilon} \right\} \right\} \geq 0$$

where φ^ε is defined from φ by the same formula defining u^ε from u .

Proof of Lemma 13.3

Step 1: proof of i)

We start with the semi-discretized scheme satisfied by u and v , and multiply the difference of the schemes by $\text{sign}(u_k^\alpha - v_k^\alpha)$ (starting with an approximation of it). We get

$$\partial_t |u_k^\alpha - v_k^\alpha| + \frac{\Psi_k^{\alpha,+} - \Psi_k^{\alpha,-}}{\Delta x} = 0 \quad \text{for } k \in J_R^\alpha$$

with

$$\Psi_k^{\alpha,+} := \begin{cases} \begin{cases} \text{sign}(u_k^\alpha - v_k^\alpha) \cdot \{G^\alpha(u_k^\alpha, u_{k+1}^\alpha) - G^\alpha(v_k^\alpha, v_{k+1}^\alpha)\} & \text{if } k \in J_R^\alpha \setminus \{R\} \\ \Psi_R^\alpha(u_k, v_k) & \text{if } k = R \end{cases} & \text{if } \sigma^\alpha = -1 \\ \begin{cases} \text{sign}(u_k^\alpha - v_k^\alpha) \cdot \{G^\alpha(u_k^\alpha, u_{k+1}^\alpha) - G^\alpha(v_k^\alpha, v_{k+1}^\alpha)\} & \text{if } k \in J_R^\alpha \setminus \{0\} \\ \Psi_0^\alpha(u_k, v_k) & \text{if } k = 0 \end{cases} & \text{if } \sigma^\alpha = 1 \end{cases}$$

and

$$\Psi_k^{\alpha,-} := \begin{cases} \begin{cases} \text{sign}(u_k^\alpha - v_k^\alpha) \cdot \{G^\alpha(u_{k-1}^\alpha, u_k^\alpha) - G^\alpha(v_{k-1}^\alpha, v_k^\alpha)\} & \text{if } k \in J_R^\alpha \setminus \{0\} \\ \Psi_0^\alpha(u_k, v_k) & \text{if } k = 0 \end{cases} & \text{if } \sigma^\alpha = -1 \\ \begin{cases} \text{sign}(u_k^\alpha - v_k^\alpha) \cdot \{G^\alpha(u_{k-1}^\alpha, u_k^\alpha) - G^\alpha(v_{k-1}^\alpha, v_k^\alpha)\} & \text{if } k \in J_R^\alpha \setminus \{-R\} \\ \Psi_R^\alpha(u_k, v_k) & \text{if } k = -R \end{cases} & \text{if } \sigma^\alpha = 1 \end{cases}$$

Then Lemma 12.2 means $\Psi_{k+\frac{1}{2}}^\alpha \leq \Psi_k^{\alpha,+}$ and $\Psi_{k-\frac{1}{2}}^\alpha \geq \Psi_k^{\alpha,-}$ for all $k \in J_R^\alpha$, which implies (12.7).

Step 2: proof of ii)

Now, in order to simplify the presentation, let us assume that $\sigma^\alpha = -1$, say for all α (which can be done using a suitable change of variables like in Lemma 11.1). Consider any test function $0 \leq \varphi_k^\alpha(t)$ which is with compact support for $t \in [0, +\infty)$. Multiplying (12.7) by φ_k^α for $k = 0, \dots, R$, and summing over k and integrating in time over $(0, +\infty)$, we get

$$\begin{aligned} 0 &\geq \varepsilon \sum_{k=0, \dots, R} \int_{(0, +\infty)} \left\{ \{\partial_t W_k^\alpha\} \varphi_k^\alpha + \left\{ \frac{\Psi_{k+\frac{1}{2}}^\alpha - \Psi_{k-\frac{1}{2}}^\alpha}{\varepsilon} \right\} \cdot \varphi_k^\alpha \right\} \\ &= \int_{(0, +\infty)} \left\{ \varepsilon \sum_{k=0, \dots, R} \{\partial_t W_k^\alpha\} \varphi_k^\alpha + \left\{ \Psi_R^\alpha(u_R, v_R) \varphi_R^\alpha - \Psi_0^\alpha(u_0, v_0) \varphi_0^\alpha + \sum_{k=0, \dots, R-1} \Psi_{k+\frac{1}{2}}^\alpha \varphi_k^\alpha - \sum_{k=1, \dots, R} \Psi_{k-\frac{1}{2}}^\alpha \varphi_k^\alpha \right\} \right\} \\ &= \int_{(0, +\infty)} \left\{ \Psi_R^\alpha(u_R, v_R) \varphi_R^\alpha - \Psi_0^\alpha(u_0, v_0) \varphi_0^\alpha \right\} + \int_{(0, +\infty)} \left\{ \varepsilon \sum_{k=0, \dots, R} \{\partial_t W_k^\alpha\} \varphi_k^\alpha + \left\{ \sum_{k=0, \dots, R-1} \Psi_{k+\frac{1}{2}}^\alpha \{\varphi_k^\alpha - \varphi_{k+1}^\alpha\} \right\} \right\} \\ &= \int_{(0, +\infty)} \left\{ \Psi_R^\alpha(u_R, v_R) \varphi_R^\alpha - \Psi_0^\alpha(u_0, v_0) \varphi_0^\alpha \right\} + \int_{(0, +\infty) \times J^\alpha} \left\{ \{\partial_t |u^{\varepsilon, \alpha} - v^{\varepsilon, \alpha}|\} \varphi^{\varepsilon, \alpha} - \Psi^{\varepsilon, \alpha} \left\{ \frac{\varphi^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - \varphi^{\varepsilon, \alpha}}{\varepsilon} \right\} \right\} \end{aligned}$$

An integration by parts in time gives

$$\begin{aligned} &\int_{\{0\} \times J^\alpha} |u^{\varepsilon, \alpha} - v^{\varepsilon, \alpha}| \varphi^{\varepsilon, \alpha} + \int_{(0, +\infty) \times J^\alpha} \left\{ |u^{\varepsilon, \alpha} - v^{\varepsilon, \alpha}| \varphi_t^{\varepsilon, \alpha} + \Psi^{\varepsilon, \alpha} \left\{ \frac{\varphi^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - \varphi^{\varepsilon, \alpha}}{\varepsilon} \right\} \right\} \\ (12.10) \quad &\geq \int_{(0, +\infty)} \left\{ \Psi_R^\alpha(u_R, v_R) \varphi_R^\alpha - \Psi_0^\alpha(u_0, v_0) \varphi_0^\alpha \right\} \end{aligned}$$

Summing over α and using $\varphi^\varepsilon \geq 0$, we get

$$\begin{aligned} &\int_{\{0\} \times J} |u^\varepsilon - v^\varepsilon| \varphi^\varepsilon + \int_{(0, +\infty) \times J} \left\{ |u^\varepsilon - v^\varepsilon| \varphi_t^\varepsilon + \Psi^\varepsilon \left\{ \frac{\varphi^\varepsilon(\cdot, \cdot + \varepsilon) - \varphi^\varepsilon}{\varepsilon} \right\} \right\} \\ &\geq \int_{(0, +\infty) \times (J \cap \partial B_{\varepsilon R})} D\hat{f}^R(u^\varepsilon, v^\varepsilon) \varphi^\varepsilon + \int_{(0, +\infty) \times \{0\}} D\hat{f}(u^\varepsilon, v^\varepsilon) \varphi^\varepsilon \end{aligned}$$

with

$$D\hat{f}(u^\varepsilon, v^\varepsilon) := \sum_{\alpha=1}^N \sigma^\alpha \psi^{\hat{f}^\alpha}(u^\varepsilon, v^\varepsilon) \quad \text{and} \quad \psi^{\hat{f}^\alpha}(u^\varepsilon, v^\varepsilon) := \text{sign}(u^{\varepsilon, \alpha} - v^{\varepsilon, \alpha}) \cdot \left\{ \hat{f}^\alpha(u^\varepsilon) - \hat{f}^\alpha(v^\varepsilon) \right\}$$

and

$$D^{\hat{f}_R}(u^\varepsilon, v^\varepsilon) := \sum_{\alpha=1}^N \sigma_R^\alpha \psi^{\hat{f}_R^\alpha}(u^\varepsilon, v^\varepsilon) \quad \text{with} \quad \sigma_R^\alpha = -\sigma^\alpha.$$

Because both \mathcal{G} and \mathcal{G}_R are Kruřkov germs respectively to (J, f) and $(-J, f)$, we deduce that $D^{\hat{f}}, D^{\hat{f}_R} \geq 0$. Joint to the fact that $\varphi^\varepsilon \geq 0$, we see that (12.10) implies

$$\int_{\{0\} \times J} |u^\varepsilon - v^\varepsilon| \varphi^\varepsilon + \int_{(0, +\infty) \times J} \left\{ |u^\varepsilon - v^\varepsilon| \varphi_t^\varepsilon + \Psi^\varepsilon \left\{ \frac{\varphi^\varepsilon(\cdot, \cdot + \varepsilon) - \varphi^\varepsilon}{\varepsilon} \right\} \right\} \geq 0$$

More generally, for arbitrary signs $\sigma^\alpha \in \{\pm 1\}$ with $\sigma_R := -\sigma$, we get (12.9). This ends the proof of the lemma.

Corollary 12.5 (Contraction estimates)

We work under assumptions of Lemma 12.1. Let u, v be two solutions of the semi-discretized scheme (12.2). We use notation of Definition 12.3 for $u^\varepsilon, v^\varepsilon$. Then for all $t > 0$, we have for $R_\varepsilon := (R + 1)\varepsilon$

$$(12.11) \quad \int_{\{t\} \times (J \cap B_{R_\varepsilon})} |u^\varepsilon - v^\varepsilon| \leq \int_{\{0\} \times (J \cap B_{R_\varepsilon})} |u^\varepsilon - v^\varepsilon|$$

We also have

$$(12.12) \quad \int_{\{t\} \times (J \cap B_{R_\varepsilon})} |\partial_t u^\varepsilon| \leq \int_{\{0\} \times (J \cap B_{R_\varepsilon})} |\partial_t u^\varepsilon|$$

Proof of Corollary 13.4

On the one hand, inequality (12.11) follows from the integral entropy inequality (12.9) for $\varphi^\varepsilon \equiv 1$. On the other hand, inequality (12.12) follows from (12.11) choosing $v^\varepsilon = u^\varepsilon(\cdot + \tau)$, dividing by τ and passing to the limit $\tau \rightarrow 0^+$. This ends the proof of the lemma.

Lemma 12.6 (BNL estimates)

We work under assumptions of Lemma 12.1. Given a solution u of the semi-discretized scheme (12.2), we consider u^ε as in Definition 12.3. Fix α such that $\sigma^\alpha = -1$. Let $k_i \in J_R^\alpha$ and $x_i = k_i \varepsilon$ for $i = 1, 2$ with $x_1 < x_2$ and $2 \leq k_i \leq R - 2$. Then we have boundary BLN estimate

$$(12.13) \quad \int_{\{t\} \times (x_1, x_2)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| - \int_{\{0\} \times (x_1, x_2)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| \leq \sum_{i=1,2} \varepsilon^{-1} \int_{(0,t) \times (x_i, x_i + \varepsilon)} |\partial_t u^{\varepsilon, \alpha}|$$

Moreover for $R_\varepsilon := (R + 1)\varepsilon$ and $x_1 = 2\delta = R_\varepsilon - x_2 + \varepsilon$ with $\delta \in \varepsilon(\mathbb{N} \setminus \{0, 1\})$, we have the interior BLN estimate

$$(12.14) \quad \int_{\{t\} \times (2\delta, R_\varepsilon - 2\delta)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| \\ \leq \int_{\{0\} \times (\delta, R_\varepsilon - \delta)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| + \delta^{-1} \left\{ \int_{(0,t) \times (\delta, 2\delta)} |\partial_t u^{\varepsilon, \alpha}| + \int_{(0,t) \times (R_\varepsilon - 2\delta, R_\varepsilon - \delta)} |\partial_t u^{\varepsilon, \alpha}| \right\}$$

Proof of Lemma 13.5

Recall that $\sigma^\alpha = -1$, and define $\tilde{u}^{\varepsilon, \alpha}(t, x) := u^{\varepsilon, \alpha}(t, x + \varepsilon)$. For $i = 1, 2$, assume $x_i = k_i \varepsilon \in J^\alpha \simeq (0, +\infty)$ for $i = 1, 2$ with $k_i \in J_R^\alpha \setminus \{0, R\}$ with $2 \leq k_i \leq R - 2$ and $x_1 < x_2$. Integrating (12.7) over (x_1, x_2) , we get

$$\int_{\{t\} \times (x_1, x_2)} \left\{ \partial_t |\tilde{u}^{\varepsilon, \alpha} - u^{\varepsilon, \alpha}| + \sum_{k \in J_R^\alpha} \left\{ \frac{\Psi_{k+\frac{1}{2}}^\alpha - \Psi_{k-\frac{1}{2}}^\alpha}{\varepsilon} \right\} \chi_k^{\varepsilon, \alpha} \right\} \leq 0$$

with $\Psi_{k+\frac{1}{2}}^\alpha := \Psi^\alpha(\tilde{u}_k^\alpha, \tilde{u}_{k+1}^\alpha; u_k^\alpha, u_{k+1}^\alpha) = \Psi^\alpha(u_{k+1}^\alpha, u_{k+2}^\alpha; u_k^\alpha, u_{k+1}^\alpha)$. From (12.4), it satisfies

$$|\Psi_{k+\frac{1}{2}}^\alpha| \leq |G^\alpha(u_{k+1}^\alpha, u_{k+2}^\alpha) - G^\alpha(u_k^\alpha, u_{k+1}^\alpha)| = \varepsilon |\partial_t u_{k+1}^\alpha|$$

Hence

$$\begin{aligned} \partial_t \int_{\{t\} \times (x_1, x_2)} |\tilde{u}^{\varepsilon, \alpha} - u^{\varepsilon, \alpha}| &\leq - \left\{ \Psi_{k_2 - \frac{1}{2}}^\alpha - \Psi_{k_1 - \frac{1}{2}}^\alpha \right\} \\ &\leq |\Psi_{k_2 - \frac{1}{2}}^\alpha| + |\Psi_{k_1 - \frac{1}{2}}^\alpha| \\ &= \varepsilon \left\{ |\partial_t u_{k_2}^\alpha| + |\partial_t u_{k_1}^\alpha| \right\} \\ &= \sum_{i=1,2} \int_{\{t\} \times (x_1, x_1 + \varepsilon)} |\partial_t u^{\varepsilon, \alpha}| \end{aligned}$$

Integrating on $(0, t)$, we get

$$\int_{\{t\} \times (x_1, x_2)} |\tilde{u}^{\varepsilon, \alpha} - u^{\varepsilon, \alpha}| - \int_{\{0\} \times (x_1, x_2)} |\tilde{u}^{\varepsilon, \alpha} - u^{\varepsilon, \alpha}| \leq \sum_{i=1,2} \int_{(0,t) \times (x_i, x_i + \varepsilon)} |\partial_t u^{\varepsilon, \alpha}|$$

which implies (12.6). Alternatively, this inequality can also be obtained from the integral entropy inequality for a suitable test function, which is a characteristic function of the rectangle $(0, t) \times (x_1, x_2)$.

In particular, for $x_1 = h$ and $x_2 = R_\varepsilon - h - \varepsilon$, we get that (12.6) means

$$\begin{aligned} &\int_{\{t\} \times (h, R_\varepsilon - h)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| \\ &\leq \int_{\{0\} \times (h, R_\varepsilon - h)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| + \varepsilon^{-1} \left\{ \int_{(0,t) \times (h, h + \varepsilon)} |\partial_t u^{\varepsilon, \alpha}| + \int_{(0,t) \times (R_\varepsilon - h - \varepsilon, R_\varepsilon - h)} |\partial_t u^{\varepsilon, \alpha}| \right\} \end{aligned}$$

Now for $\Delta \in \mathbb{N} \setminus \{0, 1\}$ and $\delta := \varepsilon \Delta$, summing over $\varepsilon^{-1} h = \Delta, \dots, 2\Delta - 1$, we get

$$\begin{aligned} &\Delta \int_{\{t\} \times (2\delta, R_\varepsilon - 2\delta)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| \\ &\leq \Delta \int_{\{0\} \times (\delta, R_\varepsilon - \delta)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| + \varepsilon^{-1} \left\{ \int_{(0,t) \times (\delta, 2\delta)} |\partial_t u^{\varepsilon, \alpha}| + \int_{(0,t) \times (R_\varepsilon - 2\delta, R_\varepsilon - \delta)} |\partial_t u^{\varepsilon, \alpha}| \right\} \end{aligned}$$

which implies (12.14). This ends the proof of the lemma.

12.3 Convergence

Then we have the following result.

Proposition 12.7 (Convergence of the semi-discretized solution on the truncated junction)

Assume (2.2) with $N \geq 1$ and junction (J, f) with $\sigma \in \{\pm 1\}^N$ and $[a, b]$ bounded. Assume that $\mathcal{G} \subset [a, b]$ is a Kruřkov germ, whose associated flux is $\hat{f} := \hat{f}_{\mathcal{G}}$. With notation (12.1), assume that the initial data $u(0)$ satisfies (12.3). For any $\varepsilon := \Delta x > 0$ and $R \geq 3$, let u be the unique solution $u : [0, +\infty) \times J_R \rightarrow \mathbb{R}$ of (12.2) with initial data $u(0)$, and let $u^\varepsilon : [0, +\infty) \times (J \cap B_{R_\varepsilon}) \rightarrow \mathbb{R}$ with $R_\varepsilon := (R + 1)\varepsilon$ given by Definition 12.3. Assume $\bar{U}_0 = (\bar{U}_0^1, \dots, \bar{U}_0^N)$ with $\bar{U}_0^\alpha \in (BV \cap L^1)(J^\alpha; [a^\alpha, b^\alpha])$. Define for $\alpha = 1, \dots, N$

$$u_k^\alpha(0) := \varepsilon^{-1} \int_{J^\alpha} \bar{U}_0^\alpha \chi_k^{\varepsilon, \alpha} \quad \text{with } \chi_k^{\varepsilon, \alpha} \text{ given in Definition 12.3}$$

Then we have

$$u^\varepsilon \rightarrow \bar{U} \quad \text{in } L_{loc}^1([0, +\infty) \times (J \cap B_{\bar{R}})) \quad \text{as } (\varepsilon, R) \rightarrow (0, +\infty) \quad \text{with } R_\varepsilon \rightarrow \bar{R} > 0$$

where \bar{U} is the unique solution of (11.40) (where R is replaced by \bar{R}) and with initial data $\bar{U}(0, \cdot) = \bar{U}_0$.

Proof of Proposition 12.7

Recall that $u^\varepsilon(t, x)$ belongs to the box $[a, b]$ which gives an L^∞ estimate, and we have both estimate (12.12) on u_t^ε , and interior BLN estimate (13.5) on $\frac{u^\varepsilon(\cdot, \cdot + \varepsilon) - u^\varepsilon}{\varepsilon}$, which approximates the x -derivative of u^ε . This shows that we have uniform bounds on u^ε in $BV_{loc; t, x}^\varepsilon$. The compactness of $BV_{loc} \subset L_{loc}^1$ for bounded functions implies the convergence of u^ε in L_{loc}^1 to some function \bar{U} , up to extraction of a subsequence. Now our integral entropy inequalities (12.9) also passes to the limit and gives (for any constant V both solution of the limit problem and of the discretized problem)

$$(12.15) \quad \int_{\{0\} \times (J \cap B_{\bar{R}})} |\bar{U} - V| \varphi + \int_{(0, +\infty) \times (J \cap B_{\bar{R}})} \{|\bar{U} - V| \varphi_t + \bar{\Psi} \cdot \varphi_x\} \geq 0$$

with

$$\Psi^{\varepsilon, \alpha} \rightarrow \bar{\Psi}^\alpha = \Psi^\alpha(\bar{U}^\alpha, \bar{U}^\alpha, V^\alpha, V^\alpha)$$

where we recall that $\Psi^\alpha(x', y'; x, y) := G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y)$. This gives

$$\bar{\Psi}^\alpha = f^\alpha(\bar{U}^\alpha \vee V^\alpha) - f^\alpha(\bar{U}^\alpha \wedge V^\alpha) = \text{sign}(\bar{U}^\alpha - V^\alpha) \{f^\alpha(\bar{U}^\alpha) - f^\alpha(V^\alpha)\} = \psi^{f^\alpha}(\bar{U}^\alpha, V^\alpha)$$

which is exactly the flux of Kruřkov entropy. Choosing the test function φ^α with compact support in the interior of each branch $J^\alpha \cap B_{\bar{R}}$ and any constant $V^\alpha \in [a^\alpha, b^\alpha]$, we recover that \bar{U} is an entropy solution on each (truncated) branch. Now choosing $V \in \mathcal{G}$ and a test function φ with compact support in $J \cap B_{\bar{R}}$, with moreover φ focusing on the origin, we recover that

$$D^f(\bar{U}(t, 0), V) \geq 0 \quad \text{for a.e. } t > 0 \text{ and all } V \in \mathcal{G}$$

Because \mathcal{G} is Kruřkov, we recover that $\bar{U}(t, 0) \in \mathcal{G}$ for a.e. $t > 0$.

Similarly, we show that $(\bar{U}(t, \cdot))_{J \cap \partial B_{\bar{R}}} \in \mathcal{G}_R$. This shows that \bar{U} is solution of (11.40) (where R is replaced by \bar{R}) and with initial data $U(0, \cdot) = U_0$. The uniqueness of the solution shows the uniqueness of any accumulation point of the sequence u^ε . This implies the convergence of the full sequence u^ε towards \bar{U} . This ends the proof of the proposition.

13 Existence via fully discretized schemes for monotone Kruřkov germs

13.1 The fully discretized scheme

Assume (2.2) with $N \geq 1$ and junction (J, f) with $\sigma \in \{\pm 1\}^N$. We recall notation (12.1), namely

$$J_{\mathbb{N}} := \prod_{\alpha=1, \dots, N} J_{\mathbb{N}}^\alpha \quad \text{with} \quad J_{\mathbb{N}}^\alpha := -\sigma^\alpha \mathbb{N}, \quad \text{for } \alpha = 1, \dots, N$$

We consider functions $u : \mathbb{N} \times J_{\mathbb{N}} \rightarrow \mathbb{R}$ with $u = (u_k^{n, \alpha})$ for $n \in \mathbb{N}$, $k \in J_{\mathbb{N}}^\alpha$ and $\alpha = 1, \dots, N$ solution of the following fully discretized explicit (finite volume) scheme for $u_0^n := (u_0^{n, 1}, \dots, u_0^{n, N})$ and $\Delta x > 0$, $\Delta t > 0$ and $n \in \mathbb{N}$

$$(13.1) \quad \left\{ \begin{array}{ll} \frac{u_k^{n+1, \alpha} - u_k^{n, \alpha}}{\Delta t} + \frac{G^\alpha(u_k^{n, \alpha}, u_{k+1}^{n, \alpha}) - G^\alpha(u_{k-1}^{n, \alpha}, u_k^{n, \alpha})}{\Delta x} = 0 & \text{for } k \in J_{\mathbb{N}}^\alpha \setminus \{0\} \\ \frac{u_k^{n+1, \alpha} - u_k^{n, \alpha}}{\Delta t} + \frac{G^\alpha(u_k^{n, \alpha}, u_{k+1}^{n, \alpha}) - \hat{f}^\alpha(u_k^n)}{\Delta x} = 0 & \text{for } k = 0 \quad \text{with } \sigma^\alpha = -1 \\ \frac{u_k^{n+1, \alpha} - u_k^{n, \alpha}}{\Delta t} + \frac{\hat{f}^\alpha(u_k^n) - G^\alpha(u_{k-1}^{n, \alpha}, u_k^{n, \alpha})}{\Delta x} = 0 & \text{for } k = 0 \quad \text{with } \sigma^\alpha = 1 \end{array} \right.$$

where $G^\alpha = G^{f^\alpha}$ is the standard Godunov flux associated to the function $f^\alpha : [a^\alpha, b^\alpha] \rightarrow \mathbb{R}$. We also assume that $\hat{f} = \hat{f}_{\mathcal{G}} : [a, b] \rightarrow \mathbb{R}^N$ is the Godunov flux associated to a Kruřkov germ \mathcal{G} . When it will be useful, we will furthermore assume that \mathcal{G} is monotone.

13.2 Preparation

Then we have the following result.

Lemma 13.1 (Existence of a fully discrete solution)

Assume (2.2) with $N \geq 1$ and junction (J, f) with $\sigma \in \{\pm 1\}^N$ and $[a, b]$ bounded. We also assume that $\hat{f} = \hat{f}_{\mathcal{G}} : [a, b] \rightarrow \mathbb{R}^N$ is the Godunov flux associated to a Kruřkov germ \mathcal{G} . With notation (12.1), assume that the initial data $u^0 : J_{\mathbb{N}} \rightarrow \mathbb{R}$ (with $u^0 = (u_k^{0, \alpha})$ with $k \in J_{\mathbb{N}}^{\alpha}$) satisfies

$$(13.2) \quad u_k^{n, \alpha} \in [a^{\alpha}, b^{\alpha}] \quad \text{for all } k \in J_{\mathbb{N}}^{\alpha}, \quad \alpha = 1, \dots, N$$

at the initial time $n = 0$. For any $\Delta x > 0$, $\Delta t > 0$, satisfying the following CFL condition

$$(13.3) \quad L \frac{\Delta t}{\Delta x} \leq \frac{1}{2} \quad \text{with } L := \sup_{\alpha=1, \dots, N} \text{Lip}(f^{\alpha}; [a^{\alpha}, b^{\alpha}])$$

there exists a unique solution $u : \mathbb{N} \times J_{\mathbb{N}} \rightarrow \mathbb{R}$ of (13.1) with initial data u^0 . Moreover u satisfies (??) for all $n \geq 0$.

Furthermore, if the germ \mathcal{G} is monotone, then the scheme is also monotone.

Proof of Lemma 13.1

Recall that $\hat{f} = \hat{f}_{\mathcal{G}}$ satisfies on $[a, b]$

$$\sigma^j \partial_j \hat{f}^j \geq \sum_{\alpha \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}^{\alpha}|, \quad \text{for all } j = 1, \dots, N$$

because \mathcal{G} is Kruřkov. Furthermore, when \mathcal{G} is monotone, we also have on $[a, b]$

$$(13.4) \quad \partial_j(\sigma^j \hat{f}^j) \geq 0, \quad \partial_k(\sigma^j \hat{f}^j) \leq 0 \quad \text{for all } k \neq j$$

Step 1: extension, existence and uniqueness

We first extend the fluxes f, \hat{f} from the box $[a, b]$ to the whole space \mathbb{R}^N as in Step 1 of the proof of Proposition 11.6. Furthermore, when \mathcal{G} is monotone, it is important to notice that monotonicities (13.4) are still true for the extension \hat{f} on \mathbb{R}^N , as it follows easily from the method of extension. We also define the extended Godunov flux $G^{\alpha} = G^{f^{\alpha}}$ from the extended flux f^{α} .

Then the functions \hat{f}, f and $(G^{\alpha})_{\alpha}$ are defined on the whole \mathbb{R}^N , and are globally Lipschitz continuous satisfying in particular for all indices j, α

$$(13.5) \quad 0 \leq \sigma^j \partial_j \hat{f}^j(p) \leq |(f^j)'(p^j)| \leq L, \quad |\partial_{u^L} G^{\alpha}(u^L, u^R)|, |\partial_{u^R} G^{\alpha}(u^L, u^R)| \leq L.$$

Therefore there is existence and uniqueness of a solution u to the scheme (13.1).

Step 2: monotonicity of the scheme when \mathcal{G} is monotone

We have

$$u_k^{n+1, \alpha} = H_k^{\alpha}[u^n] := \begin{cases} H_k^{\alpha}(u_{k-1}^{n, \alpha}, u_k^{n, \alpha}, u_{k+1}^{n, \alpha}) & \text{if } k \in J_{\mathbb{N}}^{\alpha} \setminus \{0\} \\ H_k^{\alpha}(u_k^n, u_{k+1}^{n, \alpha}) & \text{if } k = 0 \quad \text{and } \sigma^{\alpha} = -1 \\ H_k^{\alpha}(u_{k-1}^{n, \alpha}, u_k^n) & \text{if } k = 0 \quad \text{and } \sigma^{\alpha} = 1 \end{cases}$$

with

$$H_k^{\alpha}[u^n] := \begin{cases} u_k^{n, \alpha} + \frac{\Delta t}{\Delta x} \cdot \{G^{\alpha}(u_{k-1}^{n, \alpha}, u_k^{n, \alpha}) - G^{\alpha}(u_k^{n, \alpha}, u_{k+1}^{n, \alpha})\} & \text{if } k \in J_{\mathbb{N}}^{\alpha} \setminus \{0\} \\ u_k^{n, \alpha} + \frac{\Delta t}{\Delta x} \cdot \{\hat{f}^{\alpha}(u_k^n) - G^{\alpha}(u_k^{n, \alpha}, u_{k+1}^{n, \alpha})\} & \text{if } k = 0 \quad \text{and } \sigma^{\alpha} = -1 \\ u_k^{n, \alpha} + \frac{\Delta t}{\Delta x} \cdot \{G^{\alpha}(u_{k-1}^{n, \alpha}, u_k^{n, \alpha}) - \hat{f}^{\alpha}(u_k^n)\} & \text{if } k = 0 \quad \text{and } \sigma^{\alpha} = 1 \end{cases}$$

Under the CFL condition (13.3), and from estimates (13.5) and the monotonicities (13.4) on \mathbb{R}^N and the natural monotonicities of the standard Godunov fluxes G^{α} 's, we deduce that H_k^{α} is monotonous in its

arguments.

Step 3: bounds in the box (even when \mathcal{G} is not monotone)

Moreover, the flux \hat{f} satisfies the second line of (2.14) which implies that f, \hat{f} satisfy (11.25). Let us now show that $u_k^{n,\alpha}$ remains in the box $\in [a^\alpha, b^\alpha]$. Let us show for instance that $u_k^{n,\alpha} \leq b^\alpha$. Formally, if the supremum of $(u_k^{n,\alpha})$'s on $k \in J_{\mathbb{N}}^\alpha$ is reached at some index k_0 for some time index n_0 with value $u_{k_0}^{n_0,\alpha} = b^\alpha$, then we get $(n, k) = (n_0, k_0)$

$$\frac{u_k^{n+1,\alpha} - u_k^{n,\alpha}}{\Delta t} = \begin{cases} \frac{G^\alpha(u_{k-1}^{n,\alpha}, b^\alpha) - G^\alpha(b^\alpha, u_{k+1}^{n,\alpha})}{\Delta x} \leq 0 & \text{if } k_0 \in J_{\mathbb{N}}^\alpha \setminus \{0\} \\ \frac{\hat{f}^\alpha(u_k^n) - G^\alpha(b^\alpha, u_{k+1}^{n,\alpha})}{\Delta x} \leq 0 & \text{if } k_0 = 0 \quad \text{and } \sigma^\alpha = -1 \\ \frac{G^\alpha(u_{k-1}^{n,\alpha}, b^\alpha) - \hat{f}^\alpha(u_k^n)}{\Delta x} \leq 0 & \text{if } k_0 = 0 \quad \text{and } \sigma^\alpha = 1 \end{cases}$$

where in the first line we have used the monotonicities of the Godunov flux $G^\alpha(\uparrow, \downarrow)$, while in the two last lines, we have also used the four last lines of (11.25). This shows formally that $u_k^{n,\alpha} \leq b^\alpha$ for all times $n \geq 0$ and $k \in J_{\mathbb{N}}^\alpha$. As it is usual, this can be made rigorous, using for instance classical technics proving the comparison principle for viscosity solutions. This ends the proof of the lemma.

For later use, we will need the following.

Definition 13.2 (The approximate solution u^ε given by the fully discrete scheme)

We set $\tau := \Delta t > 0$, $\varepsilon := \Delta x > 0$ and $\tilde{\varepsilon} := (\tau, \varepsilon)$. Given a solution u of the semi-discretized scheme (12.2), we define $u^{\tilde{\varepsilon}} = (u^{\tilde{\varepsilon},1}, \dots, u^{\tilde{\varepsilon},N})$ for $\alpha = 1, \dots, N$

$$(13.6) \quad u^{\tilde{\varepsilon},\alpha}(t, x) := \sum_{n \in \mathbb{N}, k \in J_R^\alpha} u_k^{n,\alpha}(t) \chi_k^{\tilde{\varepsilon},n,\alpha}(x) \quad \text{with} \quad \chi_k^{\tilde{\varepsilon},n,\alpha}(t, x) := \begin{cases} 1_{[n\tau, (n+1)\tau)}(t) \cdot 1_{[k\varepsilon, (k+1)\varepsilon)}(x) & \text{if } \sigma^\alpha = -1 \\ 1_{[n\tau, (n+1)\tau)}(t) \cdot 1_{((k-1)\varepsilon, k\varepsilon]}(x) & \text{if } \sigma^\alpha = 1 \end{cases}$$

Lemma 13.3 (Entropy inequalities for monotone Kruřkov germs)

We work under assumptions of Lemma 13.1 and recall that $\sigma \in \{\pm\}^N$. Let u, v two solutions of the fully discretized scheme (13.1) respectively with initial data u^0, v^0 . We assume that \mathcal{G} is **monotone Kruřkov**. Then we have the following

i) (Pointwise entropy inequality)

$$(13.7) \quad \frac{W_k^{n+1,\alpha} - W_k^{n,\alpha}}{\Delta t} + \frac{\Psi_{k+\frac{1}{2}}^{n,\alpha} - \Psi_{k-\frac{1}{2}}^{n,\alpha}}{\Delta x} \leq 0 \quad \text{for all } k \in J_{\mathbb{N}}^\alpha, \quad \alpha = 1, \dots, N$$

with for $k \in \mathbb{Z}$, $n \in \mathbb{N}$

$$\left\{ \begin{array}{l} W_k^{n,\alpha} := |u_k^{n,\alpha} - v_k^{n,\alpha}| \quad \text{for } k \in J_{\mathbb{N}}^\alpha \\ \Psi_{k+\frac{1}{2}}^{n,\alpha} := \begin{cases} \Psi^\alpha(u_k^{n,\alpha}, u_{k+1}^{n,\alpha}; v_k^{n,\alpha}, v_{k+1}^{n,\alpha}) & \text{if } k \in J_{\mathbb{N}}^\alpha \\ \Psi_0^\alpha(u_{k+1}^n, v_{k+1}^n) & \text{if } k = -1 \end{cases} & \text{if } \sigma^\alpha = -1 \\ \Psi_{k+\frac{1}{2}}^{n,\alpha} := \begin{cases} \Psi^\alpha(u_k^{n,\alpha}, u_{k+1}^{n,\alpha}; v_k^{n,\alpha}, v_{k+1}^{n,\alpha}) & \text{if } k \in J_{\mathbb{N}}^\alpha \setminus \{0\} \\ \Psi_0^\alpha(u_k^n, v_k^n) & \text{if } k = 0 \end{cases} & \text{if } \sigma^\alpha = 1 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \Psi^\alpha(x', y'; x, y) := G^\alpha(x' \vee x, y' \vee y) - G^\alpha(x' \wedge x, y' \wedge y) \\ \Psi_0^\alpha(z', z) = \hat{f}^\alpha(z' \vee z) - \hat{f}^\alpha(z' \wedge z) \\ \text{with } (z' \vee z)^\alpha := z'^\alpha \vee z^\alpha \quad \text{and} \quad (z' \wedge z)^\alpha := z'^\alpha \wedge z^\alpha \end{array} \right.$$

ii) (Integral entropy inequality)

Let $u^{\tilde{\varepsilon}}, v^{\tilde{\varepsilon}}$ as in Definition 13.2. Define also

$$(13.8) \quad \Psi^{\tilde{\varepsilon},\alpha}(t, x) := \sum_{n \in \mathbb{N}, k \in J_{\mathbb{N}}^\alpha} \Psi_{k+\frac{1}{2}}^{n,\alpha} \chi_k^{\tilde{\varepsilon},n,\alpha}(t, x) \quad \text{for } \alpha = 1, \dots, N$$

with $\chi^{\bar{\varepsilon}}$ defined in (13.6). Let us consider a test function $0 \leq \varphi_k^{n,\alpha}$ for $n \in \mathbb{N}$ for $k \in J_{\mathbb{N}}^{\alpha}$ which satisfies $\varphi_0^{n,\alpha} = \varphi_0^{n,\beta}$ for all α, β and $n \in \mathbb{N}$. For $\varphi^{\bar{\varepsilon}}$ is defined from φ by the same formula defining $u^{\bar{\varepsilon}}$ from u , and assuming that $\varphi^{\bar{\varepsilon}}$ has compact support on $[0, +\infty) \times J$, we have the following integral entropy inequality (13.9)

$$\int_{\{0\} \times J} |u^{\bar{\varepsilon}} - v^{\bar{\varepsilon}}| \varphi^{\bar{\varepsilon}} + \int_{(0, +\infty) \times J} \left\{ |u^{\bar{\varepsilon}} - v^{\bar{\varepsilon}}| (\cdot + \tau, \cdot) \cdot \left\{ \frac{\varphi^{\bar{\varepsilon}}(\cdot + \tau, \cdot) - \varphi^{\bar{\varepsilon}}}{\tau} \right\} - \sigma \Psi^{\bar{\varepsilon}} \left\{ \frac{\varphi^{\bar{\varepsilon}}(\cdot, \cdot - \sigma \varepsilon) - \varphi^{\bar{\varepsilon}}}{\varepsilon} \right\} \right\} \geq 0$$

Proof of Lemma 13.3

Step 1: proof of i)

We first notice that for $x', x \in \mathbb{R}$, we have $|x' - x| := x' \vee x - x' \wedge x$. Because we assume that \mathcal{G} is monotone, we know that the scheme is monotone. Hence

$$\left\{ \begin{array}{l} H_k^{\alpha}[u^n] \vee H_k^{\alpha}[v^n] \leq H_k^{\alpha}[u^n \vee v^n] \\ H_k^{\alpha}[u^n] \wedge H_k^{\alpha}[v^n] \geq H_k^{\alpha}[u^n \wedge v^n] \\ \text{with } (u^n \vee v^n)_k^{\alpha} := u_k^{n,\alpha} \vee v_k^{n,\alpha} \quad \text{and } (u^n \wedge v^n)_k^{\alpha} := u_k^{n,\alpha} \wedge v_k^{n,\alpha} \end{array} \right.$$

Therefore, starting with the fully discretized scheme satisfied by u and v , and get (with obvious notation)

$$u^{n+1} \vee v^{n+1} - u^{n+1} \wedge v^{n+1} \leq H[u^n \vee v^n] - H[u^n \wedge v^n]$$

Hence $W_k^{n,\alpha} := |u_k^{n,\alpha} - v_k^{n,\alpha}|$ satisfies

$$\frac{W_k^{n+1,\alpha} - W_k^{n,\alpha}}{\Delta t} + \frac{\Psi_{k+\frac{1}{2}}^{n,\alpha} - \Psi_{k-\frac{1}{2}}^{n,\alpha}}{\Delta x} \leq 0 \quad \text{for all } k \in J_{\mathbb{N}}^{\alpha}, \quad \alpha = 1, \dots, N$$

with $\Psi_{k \pm \frac{1}{2}}^{n,\alpha}$ as in the statement of i) of Lemma 13.3. This shows (13.7).

Step 2: proof of ii)

Now, in order to simplify the presentation, let us assume that $\sigma^{\alpha} = -1$, say for all α (which can be done using a suitable change of variables like in Lemma 11.1). Consider any test function $0 \leq \varphi_k^{n,\alpha}$. Multiplying (13.7) by $\varphi_k^{n,\alpha}$ for $k \in J_{\mathbb{N}}^{\alpha}$, and summing over k and integrating in time over $n \in \mathbb{N}$, we get for $\varphi^{\bar{\varepsilon}}$ with compact support in $[0, +\infty) \times J$

$$\begin{aligned} 0 &\geq \varepsilon \tau \sum_{n \in \mathbb{N}, k \in J_{\mathbb{N}}^{\alpha}} \left\{ \left\{ \frac{W_k^{n+1,\alpha} - W_k^{n,\alpha}}{\tau} \right\} \varphi_k^{n,\alpha} + \left\{ \frac{\Psi_{k+\frac{1}{2}}^{n,\alpha} - \Psi_{k-\frac{1}{2}}^{n,\alpha}}{\varepsilon} \right\} \cdot \varphi_k^{n,\alpha} \right\} \\ &= \left\{ \begin{array}{l} \varepsilon \sum_{k \in J_{\mathbb{N}}^{\alpha}} \left\{ -W_k^{0,\alpha} \varphi_k^{0,\alpha} + \sum_{n \in \mathbb{N}} W_k^{n+1,\alpha} \varphi_k^{n,\alpha} - \sum_{n \in \mathbb{N} \setminus \{0\}} W_k^{n,\alpha} \varphi_k^{n,\alpha} \right\} \\ \tau \sum_{n \in \mathbb{N}} \left\{ -\Psi_0^{\alpha}(u_0^n, v_0^n) \varphi_0^{n,\alpha} + \sum_{k \in J_{\mathbb{N}}^{\alpha}} \Psi_{k+\frac{1}{2}}^{n,\alpha} \varphi_k^{n,\alpha} - \sum_{k \in J_{\mathbb{N}}^{\alpha} \setminus \{0\}} \Psi_{k-\frac{1}{2}}^{n,\alpha} \varphi_k^{n,\alpha} \right\} \end{array} \right\} \\ &= \left\{ \begin{array}{l} \varepsilon \sum_{k \in J_{\mathbb{N}}^{\alpha}} \left\{ -W_k^{0,\alpha} \varphi_k^{0,\alpha} - \sum_{n \in \mathbb{N}} W_k^{n+1,\alpha} \left\{ \varphi_k^{n+1,\alpha} - \varphi_k^{n,\alpha} \right\} \right\} \\ \tau \sum_{n \in \mathbb{N}} \left\{ -\Psi_0^{\alpha}(u_0^n, v_0^n) \varphi_0^{n,\alpha} + \sum_{k \in J_{\mathbb{N}}^{\alpha}} \Psi_{k+\frac{1}{2}}^{n,\alpha} \left\{ \varphi_k^{n,\alpha} - \varphi_{k+1}^{n,\alpha} \right\} \right\} \end{array} \right\} \\ &= \left\{ \begin{array}{l} - \int_{\{0\} \times J^{\alpha}} W^{\bar{\varepsilon},\alpha} \varphi^{\bar{\varepsilon},\alpha} - \int_{(0, +\infty) \times J^{\alpha}} W^{\bar{\varepsilon},\alpha}(\cdot + \tau, \cdot) \cdot \left\{ \frac{\varphi^{\bar{\varepsilon},\alpha}(\cdot + \tau, \cdot) - \varphi^{\bar{\varepsilon},\alpha}}{\tau} \right\} \\ - \int_{(0, +\infty) \times \{0\}} \Psi_0^{\alpha}(u^{\bar{\varepsilon}}, v^{\bar{\varepsilon}}) \varphi^{\bar{\varepsilon},\alpha} - \int_{(0, +\infty) \times J^{\alpha}} \Psi^{\bar{\varepsilon},\alpha} \left\{ \frac{\varphi^{\bar{\varepsilon},\alpha}(\cdot, \cdot + \varepsilon) - \varphi^{\bar{\varepsilon},\alpha}}{\varepsilon} \right\} \end{array} \right\} \end{aligned}$$

i.e.

$$(13.10) \quad \int_{\{0\} \times J^\alpha} |u^\varepsilon - v^\varepsilon| \varphi^\varepsilon + \int_{(0,+\infty) \times J^\alpha} \left\{ |u^\varepsilon - v^\varepsilon|(\cdot + \tau, \cdot) \cdot \left\{ \frac{\varphi^\varepsilon(\cdot + \tau, \cdot) - \varphi^\varepsilon}{\tau} \right\} + \Psi^\varepsilon \left\{ \frac{\varphi^\varepsilon(\cdot, \cdot + \varepsilon) - \varphi^\varepsilon}{\varepsilon} \right\} \right\} \\ \geq - \int_{(0,+\infty) \times \{0\}} \Psi_0^\alpha(u^\varepsilon, v^\varepsilon) \varphi^\varepsilon$$

Summing over α and using $\varphi^\varepsilon \geq 0$, we get

$$\int_{\{0\} \times J} |u^\varepsilon - v^\varepsilon| \varphi^\varepsilon + \int_{(0,+\infty) \times J} \left\{ |u^\varepsilon - v^\varepsilon|(\cdot + \tau, \cdot) \cdot \left\{ \frac{\varphi^\varepsilon(\cdot + \tau, \cdot) - \varphi^\varepsilon}{\tau} \right\} + \Psi^\varepsilon \left\{ \frac{\varphi^\varepsilon(\cdot, \cdot + \varepsilon) - \varphi^\varepsilon}{\varepsilon} \right\} \right\} \\ \geq \int_{(0,+\infty) \times \{0\}} D^{\hat{f}}(u^\varepsilon \vee v^\varepsilon, u^\varepsilon \wedge v^\varepsilon) \varphi^\varepsilon$$

with

$$D^{\hat{f}}(p, q) := \sum_{\alpha=1}^N \sigma^\alpha \psi^{\hat{f}^\alpha}(p, q) \quad \text{and} \quad \psi^{\hat{f}^\alpha}(p, q) := \text{sign}(p^\alpha - q^\alpha) \cdot \left\{ \hat{f}^\alpha(p) - \hat{f}^\alpha(q) \right\}$$

Because \mathcal{G} is a Kruřkov germ with respect to (J, f) , we deduce that $D^{\hat{f}} \geq 0$. Joint to the fact that $\varphi^\varepsilon \geq 0$, we see that (13.10) implies

$$\int_{\{0\} \times J} |u^\varepsilon - v^\varepsilon| \varphi^\varepsilon + \int_{(0,+\infty) \times J} \left\{ |u^\varepsilon - v^\varepsilon|(\cdot + \tau, \cdot) \cdot \left\{ \frac{\varphi^\varepsilon(\cdot + \tau, \cdot) - \varphi^\varepsilon}{\tau} \right\} + \Psi^\varepsilon \left\{ \frac{\varphi^\varepsilon(\cdot, \cdot + \varepsilon) - \varphi^\varepsilon}{\varepsilon} \right\} \right\} \geq 0$$

More generally, for arbitrary signs $\sigma^\alpha \in \{\pm 1\}$, we get (13.9). This ends the proof of the lemma.

Corollary 13.4 (Contraction estimates)

We work under assumptions of Lemma 13.1 with \mathcal{G} monotone Kruřkov. Let u, v be two solutions of the fully discretized scheme (13.1). We use notation of Definition 13.2 for $u^\varepsilon, v^\varepsilon$. Then for all $t > 0$, we have

$$(13.11) \quad \int_{\{t\} \times J} |u^\varepsilon - v^\varepsilon| \leq \int_{\{0\} \times J} |u^\varepsilon - v^\varepsilon|$$

where the right hand side may be infinite.

We also have

$$(13.12) \quad \int_{\{t\} \times J} \left| \frac{u^\varepsilon(\cdot + \tau, \cdot) - u^\varepsilon}{\varepsilon} \right| \leq \int_{\{0\} \times J} \left| \frac{u^\varepsilon(\cdot + \tau, \cdot) - u^\varepsilon}{\varepsilon} \right|$$

Proof of Corollary 13.4

On the one hand, inequality (13.11) follows from the integral entropy inequality (13.9) for $\varphi^\varepsilon \equiv 1$. On the other hand, inequality (13.12) follows from (13.11) choosing $v^\varepsilon = u^\varepsilon(\cdot + \tau, \cdot)$, dividing by τ . This ends the proof of the lemma.

Lemma 13.5 (BNL estimates)

We work under assumptions of Lemma 13.1. Given a solution u of the fully discretized scheme (13.1), we consider u^ε as in Definition 13.2. Fix α such that $\sigma^\alpha = -1$. Let $k_i \in J_{\mathbb{N}}^\alpha$ and $x_i = k_i \varepsilon$ for $i = 1, 2$ with $x_1 < x_2$ and $k_i \geq 2$. Then we have boundary BLN estimate for $t \in \tau \mathbb{N} \setminus \{0, 1\}$

$$(13.13) \quad \int_{\{t\} \times (x_1, x_2)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| - \int_{\{0\} \times (x_1, x_2)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| \leq \sum_{i=1,2} \varepsilon^{-1} \int_{(0,t) \times (x_i, x_i + \varepsilon)} \left| \frac{u^{\varepsilon, \alpha}(\cdot + \tau, \cdot) - u^{\varepsilon, \alpha}}{\tau} \right|$$

Moreover for $x_1 = 2\delta$ with $\delta \in \varepsilon(\mathbb{N} \setminus \{0, 1\})$, we have the interior BLN estimate for $t \in \tau \mathbb{N} \setminus \{0, 1\}$

$$(13.14) \quad \int_{\{t\} \times (2\delta, +\infty)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| \leq \int_{\{0\} \times (\delta, +\infty)} \left| \frac{u^{\varepsilon, \alpha}(\cdot, \cdot + \varepsilon) - u^{\varepsilon, \alpha}}{\varepsilon} \right| + \delta^{-1} \int_{(0,t) \times (\delta, 2\delta)} \left| \frac{u^{\varepsilon, \alpha}(\cdot + \tau, \cdot) - u^{\varepsilon, \alpha}}{\tau} \right|$$

Proof of Lemma 13.5

Recall that $\sigma^\alpha = -1$, and define $v^{\tilde{\varepsilon},\alpha}(t, x) := u^{\tilde{\varepsilon},\alpha}(t, x + \varepsilon)$. For $i = 1, 2$, assume $x_i = k_i \varepsilon \in J^\alpha \simeq (0, +\infty)$ for $i = 1, 2$ with $k_i \in J_{\mathbb{N}}^\alpha \setminus \{0\}$ with $k_i \geq 2$ and $x_1 < x_2$. For $t \in \tau \mathbb{N} \setminus \{0, 1\}$, consider the (limit) test function $\varphi^{\tilde{\varepsilon},\alpha}(s, x) = 1_{[0,t)}(s) \cdot 1_{[x_1, x_2)}(x)$, with $\varphi^{\tilde{\varepsilon},\beta} = 0$ for $\beta \neq \alpha$. Plugging this test function in the integral entropy inequality (13.9), we get

$$\int_{\{0\} \times (x_1, x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| - \int_{\{t\} \times (x_1, x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| + \varepsilon^{-1} \sum_{i=1,2} \int_{(0,t) \times (x_i - \varepsilon, x_i)} -(-1)^i \Psi^{\tilde{\varepsilon},\alpha} \geq 0$$

Recall that $\Psi_{k+\frac{1}{2}}^{n,\alpha} := \Psi^\alpha(v_k^{n,\alpha}, v_{k+1}^{n,\alpha}; u_k^{n,\alpha}, u_{k+1}^{n,\alpha}) = \Psi^\alpha(u_{k+1}^{n,\alpha}, u_{k+2}^{n,\alpha}; u_k^{n,\alpha}, u_{k+1}^{n,\alpha})$. From (12.4), it satisfies

$$|\Psi_{k+\frac{1}{2}}^{n,\alpha}| \leq |G^\alpha(u_{k+1}^{n,\alpha}, u_{k+2}^{n,\alpha}) - G^\alpha(u_k^{n,\alpha}, u_{k+1}^{n,\alpha})| = \varepsilon \left| \frac{u_{k+1}^{n+1,\alpha} - u_{k+1}^{n,\alpha}}{\tau} \right|$$

Hence for $t = n\tau$

$$\begin{aligned} \int_{\{t\} \times (x_1, x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| - \int_{\{0\} \times (x_1, x_2)} |u^{\tilde{\varepsilon},\alpha} - v^{\tilde{\varepsilon},\alpha}| &\leq \sum_{i=1,2} \tau \sum_{m=0, \dots, n-1} |\Psi_{k_i - \frac{1}{2}}^{m,\alpha}| \\ &\leq \varepsilon \sum_{i=1,2} \tau \sum_{m=0, \dots, n-1} \left| \frac{u_{k_i}^{m+1,\alpha} - u_{k_i}^{m,\alpha}}{\tau} \right| \\ &= \sum_{i=1,2} \int_{(0,t) \times (x_i, x_i + \varepsilon)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot + \tau, \cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right| \end{aligned}$$

Hence we get

$$\int_{\{t\} \times (x_1, x_2)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot, \cdot + \varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| - \int_{\{0\} \times (x_1, x_2)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot, \cdot + \varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| \leq \varepsilon^{-1} \sum_{i=1,2} \int_{(0,t) \times (x_i, x_i + \varepsilon)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot + \tau, \cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right|$$

which is (13.13).

In particular, for $x_1 = h$ and $x_2 = +\infty$, we get that (13.13) means

$$\begin{aligned} &\int_{\{t\} \times (h, +\infty)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot, \cdot + \varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| \\ &\leq \int_{\{0\} \times (h, +\infty)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot, \cdot + \varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| + \varepsilon^{-1} \int_{(0,t) \times (h, h + \varepsilon)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot + \tau, \cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right| \end{aligned}$$

Now for $\Delta \in \mathbb{N} \setminus \{0, 1\}$ and $\delta := \varepsilon \Delta$, summing over $\varepsilon^{-1}h = \Delta, \dots, 2\Delta - 1$, we get

$$\begin{aligned} &\Delta \int_{\{t\} \times (2\delta, +\infty)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot, \cdot + \varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| \\ &\leq \Delta \int_{\{0\} \times (\delta, +\infty)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot, \cdot + \varepsilon) - u^{\tilde{\varepsilon},\alpha}}{\varepsilon} \right| + \varepsilon^{-1} \int_{(0,t) \times (\delta, 2\delta)} \left| \frac{u^{\tilde{\varepsilon},\alpha}(\cdot + \tau, \cdot) - u^{\tilde{\varepsilon},\alpha}}{\tau} \right| \end{aligned}$$

which implies (13.14). This ends the proof of the lemma.

13.3 Convergence

Then we have the following result.

Proposition 13.6 (Convergence of the fully discretized solution for monotone Kruřkov germs)
Assume (2.2) with $N \geq 1$ and junction (J, f) with $\sigma \in \{\pm 1\}^N$ and $[a, b]$ bounded. Assume that $\mathcal{G} \subset [a, b]$ is

a **monotone** Kruřkov germ, whose associated flux is $\hat{f} := \hat{f}_G$. Assume that $\tau := \Delta t > 0$ and $\varepsilon := \Delta x > 0$ satisfy the CFL condition (13.3), and let $\tilde{\varepsilon} := (\tau, \varepsilon)$.

Let u be the unique solution $u : \mathbb{N} \times J_{\mathbb{N}} \rightarrow \mathbb{R}$ of (13.1) with initial data u^0 satisfying (13.2). Let $u^{\tilde{\varepsilon}} : [0, +\infty) \times J \rightarrow \mathbb{R}$ given by Definition 13.2. Assume $\bar{U}_0 = (\bar{U}_0^1, \dots, \bar{U}_0^N)$ with $\bar{U}_0^\alpha \in (BV \cap L^1)(J^\alpha; [a^\alpha, b^\alpha])$. Define for $\alpha = 1, \dots, N$

$$u_k^{0,\alpha} := \varepsilon^{-1} \int_{J^\alpha} \bar{U}_0^\alpha \chi_k^{\varepsilon,\alpha} \quad \text{with } \chi_k^{\varepsilon,\alpha} \text{ given in Definition 13.2}$$

Then we have

$$u^{\tilde{\varepsilon}} \rightarrow \bar{U} \quad \text{in } L_{loc}^1([0, +\infty) \times J) \quad \text{as } \tilde{\varepsilon} \rightarrow (0, 0)$$

where \bar{U} is the unique solution of (2.23) and with initial data $\bar{U}(0, \cdot) = \bar{U}_0$.

Proof of Proposition 13.6

Recall that $u^{\tilde{\varepsilon}}(t, x)$ belongs to the box $[a, b]$ which gives an L^∞ estimate, and we have both contraction estimate (13.12) on $\frac{u^{\tilde{\varepsilon}}(\cdot + \tau, \cdot) - u^{\tilde{\varepsilon}}}{\tau}$ which approximates the t -derivative of $u^{\tilde{\varepsilon}}$, and interior BLN estimate (13.14) on $\frac{u^{\tilde{\varepsilon}}(\cdot, \cdot + \varepsilon) - u^{\tilde{\varepsilon}}}{\varepsilon}$, which approximates the x -derivative of $u^{\tilde{\varepsilon}}$. This shows that we have uniform bounds on $u^{\tilde{\varepsilon}}$ in $BV_{loc;t,x}^\varepsilon$. Again, the compactness of $BV_{loc} \subset L_{loc}^1$ for bounded functions implies the convergence of $u^{\tilde{\varepsilon}}$ in L_{loc}^1 to some function \bar{U} , up to extraction of a subsequence. The remaining part of the proof is similar to the one of the proof of Proposition 12.7. This ends the proof of the proposition.

14 Examples of conservative Kruřkov germs and their Godunov fluxes

14.1 Vanishing viscosity germ revisited

The germ \mathcal{G}_{VV} obtained by vanishing viscosity on a junction has been studied in several works. We can cite ANDREIANOV, KARLSEN, RISEBRO [3] for 1 : 1 junctions, and for $n : m$ junctions COCLITE, GARAVELLO [26] for existence, ANDREIANOV, COCLITE, DONADELLO [22] for existence and uniqueness for bell-shaped fluxes, and more recently MUSCH, FJORDHOLM, RISEBRO [18] for monotone fluxes and FJORDHOLM, MUSCH, RISEBRO [27] for fluxes with finite number of extrema.

Here we do not try to justify that \mathcal{G}_{VV} is obtained in the vanishing viscosity limit. On the contrary, we take \mathcal{G}_{VV} (or more precisely its associated Godunov flux) as a definition, and show that it is a conservative Kruřkov germ, removing in particular the technical condition of a finite number of extrema of the fluxes on each branch.

We have the following result.

Proposition 14.1 (Vanishing viscosity germ and its Godunov flux)

Assume (2.2) with $N \geq 1$ and junction (J, f) . Assume that $[a^j, b^j] = [a^0, b^0]$ for all $j = 1, \dots, N$. Assume moreover that either

$$(14.1) \quad \begin{cases} f^j \geq 0 & \text{on} \\ f^j = 0 & \text{on} \end{cases} \quad \begin{cases} [a^j, b^j] \\ \partial[a^j, b^j] \end{cases} \subset \mathbb{R} \quad \text{for all indices } j = 1, \dots, N$$

or

$$(14.2) \quad [a^0, b^0] \cap \mathbb{R} = \mathbb{R} \quad \text{and } N \geq 2 \text{ with } \theta^{j+} = \sigma^{j+}, \quad \theta^{j-} = -\sigma^{j-} \quad \text{for two indices } j_-, j_+ \text{ in (2.2)}.$$

Now for any $p \in [a, b] \cap \mathbb{R}^N$, consider

$$R_p := \left\{ r \in [a^0, b^0], \quad \sum_{\sigma^j=1} G^j(p^j, r) = \sum_{\sigma^j=-1} G^j(r, p^j) \right\}$$

and

$$\Lambda_p := \left\{ g(r) := (g^1, \dots, g^N)(r) \quad \text{with} \quad g^j(r) := \begin{cases} G^j(p^j, r) & \text{if } \sigma^j = 1 \\ G^j(r, p^j) & \text{if } \sigma^j = -1 \end{cases} \quad \Bigg| \quad \text{for all } r \in R_p \right\}$$

Then $\Lambda_p = \{\hat{f}(p)\}$ is reduced to a singleton. Moreover $\hat{f} : [a, b] \rightarrow \mathbb{R}^N$ defines a map such that the set

$$\mathcal{G}_{VV} := \left\{ p \in [a, b], \quad \hat{f}(p) = f(p) \right\}$$

is a conservative Kruřkov germ. Moreover \hat{f} is the associated Godunov flux $\hat{f} = \hat{f}_{\mathcal{G}_{VV}}$.

Proof of Proposition 14.1

Step 1: $R_p \neq \emptyset$

We recall that

$$G^j(\uparrow, \downarrow) = G^j(p, q) = \begin{cases} \min_{[p, q]} f^j & \text{if } p \leq q \\ \max_{[q, p]} f^j & \text{if } p \geq q \end{cases}$$

Then we set

$$g(r) := \sum_{\sigma^j=1} G^j(p^j, r) - \sum_{\sigma^j=-1} G^j(r, p^j)$$

which is nonincreasing in r .

Step 1.1: under assumption (14.1)

Inequality $f^j \geq 0$ implies $G^j \geq 0$ and then

$$\begin{cases} g(b^0) = \sum_{\sigma^j=1} 0 - \sum_{\sigma^j=-1} (\geq 0) \leq 0 \\ g(a^0) = \sum_{\sigma^j=1} (\geq 0) - \sum_{\sigma^j=-1} 0 \geq 0 \end{cases}$$

Because g is continuous, we deduce the existence of some $r \in [a^0, b^0]$ such that $g(r) = 0$, i.e. $r \in R_p$.

Step 1.2: under assumption (14.2)

Recall that the function $\theta^j f^j$ is coercive, i.e. $\liminf_{|p^j| \rightarrow +\infty} \theta^j f^j(p^j) = +\infty$. Hence

$$G^j(p^j, r) \rightarrow \begin{cases} -\infty & \text{if } r \rightarrow +\infty \text{ and } \theta^j = -1 = -\sigma^j \\ +\infty & \text{if } r \rightarrow -\infty \text{ and } \theta^j = 1 = \sigma^j \\ \min_{[p^j, +\infty)} f^j & \text{if } r \rightarrow +\infty \text{ and } \theta^j = 1 \\ \max_{(-\infty, p^j]} f^j & \text{if } r \rightarrow -\infty \text{ and } \theta^j = -1 \end{cases}$$

while

$$G^j(r, p^j) \rightarrow \begin{cases} +\infty & \text{if } r \rightarrow +\infty \text{ and } \theta^j = 1 = -\sigma^j \\ -\infty & \text{if } r \rightarrow -\infty \text{ and } \theta^j = -1 = \sigma^j \\ \max_{[p^j, +\infty)} f^j & \text{if } r \rightarrow +\infty \text{ and } \theta^j = -1 \\ \min_{(-\infty, p^j]} f^j & \text{if } r \rightarrow -\infty \text{ and } \theta^j = 1 \end{cases}$$

Hence under assumption (14.2), we deduce that

$$\begin{cases} g(r) \rightarrow -\infty & \text{as } r \rightarrow +\infty & \text{because of } j_- \\ g(r) \rightarrow +\infty & \text{as } r \rightarrow -\infty & \text{because of } j_+ \end{cases}$$

By continuity of g , we deduce the existence of some $r \in \mathbb{R}$ such that $g(r) = 0$, i.e. $r \in R_p$.

Step 2: Λ_p is reduced to a singleton

Assume that $r, \bar{r} \in R_p$ with $r < \bar{r}$. Hence $g(r) = 0 = g(\bar{r})$. Because g is a sum of N nonincreasing functions, we deduce that each function is constant on $[r, \bar{r}]$, i.e. $G^j(p^j, r) = G^j(p^j, \bar{r})$ for $\sigma^j = 1$ and $G^j(r, p^j) = G^j(\bar{r}, p^j)$ for $\sigma^j = -1$. Up to redefine r, \bar{r} , we can choose such elements such that $[r, \bar{r}]$ is the

maximal interval in $[a^0, b^0]$ where g vanishes. This implies that $g = \text{const}$ on $[r, \bar{r}]$, and then that $\Lambda_p = \{g(r)\}$ is a singleton. We set $\hat{f}(p) := g(r)$.

Step 3: continuity of \hat{f}

As usual the continuity of \hat{f} follows from the singleton property of Λ_p .

Step 4: bounds on \hat{f}

We want to check that $f_- \leq \hat{f} \leq f_+$. We only do it for components j such that $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar). This follows from $f_-^j(p^j) = G^j(p^j, b^j) \leq \hat{f}^j(p) = G^j(p^j, r) \leq G^j(p^j, a^j) = f_+^j(p^j)$.

Step 5: local constancy

Consider $p_* \in [a, b] \cap \mathbb{R}^N$ and assume that $K_{p_*} := \left\{ j \in \{1, \dots, N\}, \hat{f}^j(p_*) \neq f^j(p_*) \right\} \neq \emptyset$. Let $j \in K_{p_*}$. Assume that $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar). Then we have with $r_* \in R_{p_*}$

$$f^j(p_*^j) \neq \hat{f}^j(p_*) = G^j(p_*^j, r_*) = \begin{cases} \min_{[p_*^j, r_*^j]} f^j < f^j(p_*^j) & \text{if } p_*^j < r_* \\ \max_{[r_*, p_*^j]} f^j > f^j(p_*^j) & \text{if } p_*^j > r_* \end{cases}$$

Then there exists $\varepsilon > 0$ such that for

$$Q_\varepsilon(p_*) := \left(p_* + \sum_{j \in K_{p_*}} (-\varepsilon, \varepsilon) e_j \right) \cap [a, b]$$

and all $p \in Q_\varepsilon(p_*)$, we have $G^j(p^j, r_*) = G^j(p_*^j, r_*)$ for $\sigma^j = 1$ and $G^j(r_*, p^j) = G^j(r_*, p_*^j)$ for $\sigma^j = -1$. Hence $r^* \in R_p$ and moreover $\hat{f}(p) = \hat{f}(p_*)$. This shows that \hat{f} is locally constant on $Q_\varepsilon(p_*)$ and then on $\left\{ \hat{f} \neq f \right\}$.

Step 6: basic monotonicities

Consider some $p \in [a, b] \cap \mathbb{R}^N$, and fix some index j_0 . Assume also that $\sigma^{j_0} = 1$ (the case $\sigma^{j_0} = -1$ is similar). Then consider

$$[a, b] \ni \bar{p} = p + qe_{j_0} \quad \text{with } q \in [0, +\infty).$$

Step 6.1: monotonicity in r

Consider the minimal $r \in \mathbb{R}$ such that $r \in R_p$ and any $\bar{r} \in R_{\bar{p}}$. Then we have

$$g(\bar{p}, \bar{r}) := G^{j_0}(\bar{p}^{j_0}, \bar{r}) + \sum_{\sigma^j=1, j \neq j_0} G^j(p^j, \bar{r}) - \sum_{\sigma^j=-1} G^j(\bar{r}, p^j) = 0$$

Because of the monotonicities $G^j(\uparrow, \downarrow)$ and the fact that $\bar{p}^{j_0} \geq p^{j_0}$, we deduce that $0 = g(\bar{p}, \bar{r}) \geq g(p, \bar{r})$. Then the monotonicity of $g(p, \cdot)$ implies that $r \leq \bar{r}$.

Step 6.2: monotonicity of \hat{f}

We have

$$\begin{aligned} \hat{f}^{j_0}(\bar{p}) &= G^{j_0}(\bar{p}^{j_0}, \bar{r}) \\ &= \sum_{\sigma^j=-1} G^j(\bar{r}, p^j) - \sum_{\sigma^j=1, j \neq j_0} G^j(p^j, \bar{r}) \\ &\geq \sum_{\sigma^j=-1} G^j(r, p^j) - \sum_{\sigma^j=1, j \neq j_0} G^j(p^j, r) \\ &= G^{j_0}(p^{j_0}, r) \\ &= \hat{f}^{j_0}(p) \end{aligned}$$

which shows the monotonicity of $\sigma^j \hat{f}^j$ in p^j for $j = j_0$, i.e. the basic monotonicities of \hat{f} .

Step 7: conservative Riemann germ

From the previous steps, we deduce that \mathcal{G}_{VV} is a Riemann germ and $\hat{f} = \hat{f}_{\mathcal{G}_{VV}}$. Moreover by construction, the germ is conservative.

Step 8: monotone germ

We start as in Step 6 with index j_0 such that $\sigma^{j_0} = 1$ (the case $\sigma^{j_0} = -1$ is similar), and $\bar{p} = p + qe_j$ with $q \geq 0$. Using $\bar{r} \in R_{\bar{p}}$ and minimal $r \in R_p$, we get $\bar{r} \geq r$.

Case A: $\sigma^j = 1$ with $j \neq j_0$

We have

$$\hat{f}^j(\bar{p}) = G^j(p^j, \bar{r}) \leq G^j(p^j, r) = \hat{f}^j(p)$$

Case B: $\sigma^j = -1$

We have

$$\sigma^j \hat{f}^j(\bar{p}) = \sigma^j G^j(\bar{r}, p^j) \leq \sigma^j G^j(r, p^j) = \sigma^j \hat{f}^j(p)$$

Conclusion

In all cases, we get that the map $p \mapsto \sigma^j \hat{f}^j(p)$ is nonincreasing in p^{j_0} for all $j \neq j_0$. Because this is true for all indices j_0 , this implies that the germ \mathcal{G}_{VV} is monotone. But a conservative monotone germ is a conservative Kruřkov germ. This ends the proof of the proposition.

14.2 Holden-Risebro theory revisited

We revisit the pioneering work of HOLDEN AND RIESEBRO [31].

Lemma 14.2 (Convex optimization)

Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a strictly convex function and some fixed vector $\mathbb{R}^N \ni \bar{\gamma} = (\bar{\gamma}^1, \dots, \bar{\gamma}^N) \geq 0$ and $\sigma = (\sigma^1, \dots, \sigma^N) \in \{\pm 1\}^N$ with $\sigma \neq (1, \dots, 1)$ and $\sigma \neq (-1, \dots, -1)$. Let $L : \mathbb{R}^N \rightarrow \mathbb{R}$ be the linear map defined by $L(\gamma) := \sum_{j=1}^N \sigma^j \gamma^j$ for $\gamma \in \mathbb{R}^N$. Let

$$\hat{\gamma}(\bar{\gamma}) := \underset{K(\bar{\gamma})}{\text{Argmin}} \Psi \quad \text{with the convex} \quad K(\bar{\gamma}) := \{\gamma \in \Gamma(\bar{\gamma}), \quad L(\gamma) = 0\} \quad \text{and the box} \quad \Gamma(\bar{\gamma}) := [0, \bar{\gamma}] := \prod_{j=1}^N [0, \bar{\gamma}^j]$$

i) (Preflux properties)

Then $\hat{\gamma} : [0, +\infty)^N \rightarrow [0, +\infty)^N$ is a conservative preflux in the sense of Definition 7.19.

ii) (Holden-Risebro preflux: σ -monotonicity, when Ψ has separated variables)

Assume moreover that there exists N strictly convex functions $\Psi_j : \mathbb{R} \rightarrow \mathbb{R}$ such that $\Psi(\gamma) = \sum_{j=1}^N \Psi_j(\gamma^j)$ for $\gamma \in \mathbb{R}^N$. Then $\hat{\gamma}$ is σ -monotone in the sense of Definition 7.19, and we denote it $\hat{\gamma}^{HR}$ and call it a Holden-Risebro preflux..

Remark 14.3 In Lemma 14.2, $\bar{\gamma}$ denotes a vector of $[0, +\infty)^N$ which must not be confused with the capacity of Definition 7.21.

Proof of Lemma 14.2

Step 1: proof of continuity, basic monotonicity, conservation

Step 1.1: continuity

Notice that $0_{\mathbb{R}^N} \in K(\bar{\gamma})$ for all $\bar{\gamma} \geq 0$ and then $K(\bar{\gamma})$ is always non empty and compact convex. Then the strict convexity of Ψ implies the uniqueness of the minimizer $\hat{\gamma}(\bar{\gamma})$, which is then well-defined. Moreover this uniqueness also implies the continuity of the map $\bar{\gamma} \mapsto \hat{\gamma}(\bar{\gamma})$.

Step 1.2: basic monotonicity

Consider $\bar{\gamma}, \bar{\gamma}_* \in [0, +\infty)^N$ such that $\bar{\gamma} - \bar{\gamma}_* = \varepsilon e_k$ with $\varepsilon > 0$ and $k = 1$ to fix the ideas. Assume moreover that $\sigma^1 = 1$ (the case $\sigma^1 = -1$ is similar).

Assume by contradiction that

$$(14.3) \quad \hat{\gamma}^1(\bar{\gamma}) \leq \hat{\gamma}^1(\bar{\gamma}_*) \quad \text{and} \quad \hat{\gamma}(\bar{\gamma}) \neq \hat{\gamma}(\bar{\gamma}_*)$$

This implies that $\hat{\gamma}(\bar{\gamma}) \in [0, \bar{\gamma}_*]$. Because $L(\hat{\gamma}(\bar{\gamma})) = 0$, we deduce that $\hat{\gamma}(\bar{\gamma}) \in K(\bar{\gamma}_*) \subset K(\bar{\gamma})$. This implies that

$$\inf_{K(\bar{\gamma})} \Psi = \Psi(\hat{\gamma}(\bar{\gamma})) \geq \inf_{K(\bar{\gamma}_*)} \Psi = \Psi(\hat{\gamma}(\bar{\gamma}_*)) \geq \inf_{K(\bar{\gamma})} \Psi$$

Hence $\Psi(\hat{\gamma}(\bar{\gamma})) = \Psi(\hat{\gamma}(\bar{\gamma}_*))$, and because Ψ is strictly convex, we deduce that $\hat{\gamma}(\bar{\gamma}) = \hat{\gamma}(\bar{\gamma}_*)$. Contradiction with (14.3). We conclude that $\hat{\gamma}^1(\bar{\gamma}) > \hat{\gamma}^1(\bar{\gamma}_*)$ or $\hat{\gamma}(\bar{\gamma}) = \hat{\gamma}(\bar{\gamma}_*)$. This shows in particular that the map $\bar{\gamma} \mapsto \hat{\gamma}^j(\bar{\gamma})$ is nondecreasing in $\bar{\gamma}^j$, for $j = 1$, and then similarly for all j .

Step 1.3: conservation

By construction, we have $L(\hat{\gamma}(\bar{\gamma})) = 0$, and then $\hat{\gamma}$ is conservative.

Step 2: proof of local constancy

To simplify the notation, let us introduce $\hat{\gamma}_* := \hat{\gamma}(\bar{\gamma}_*)$.

Step 2.1: preliminary

Now, given $\bar{\gamma}_* \in [0, +\infty)^N$, the function Ψ is minimal at $\hat{\gamma}_*$ on $K(\bar{\gamma}_*)$ with $L(\hat{\gamma}_*) = 0$. Looking at the subdifferential $\partial(\Psi_{|\Gamma(\bar{\gamma}_*)})(\hat{\gamma}_*)$ of the convex function Ψ , it is easy to see that there exists some $\lambda \in \mathbb{R}$ (that can be interpreted as a Lagrange multiplier), such that the function $\Phi := \{\Psi - \lambda L\}_{|\Gamma(\bar{\gamma}_*)}$ is minimal on $\Gamma(\bar{\gamma}_*)$ at $\hat{\gamma}_*$. Hence the convex function Φ satisfies

$$(14.4) \quad \inf_{\Gamma(\bar{\gamma}_*)} \Phi \geq \Phi(\hat{\gamma}_*) \quad \text{and} \quad \xi \cdot \partial\Phi(\hat{\gamma}_*) \geq 0 \quad \text{for all} \quad \xi \in T_{\hat{\gamma}_*}\Gamma(\bar{\gamma}_*) := \lim_{\delta \rightarrow 0^+} \delta^{-1}(\Gamma(\bar{\gamma}_*) - \hat{\gamma}_*)$$

where $\xi \cdot \partial\Phi(x_0) \geq 0$ means $\xi \cdot v \geq 0$ for all $v \in \partial\Phi(x_0)$.

Step 2.2: variations and local constancy

Now assume that $I := \{j \in \{1, \dots, N\}, \hat{\gamma}^j(\bar{\gamma}_*) \neq \bar{\gamma}_*^j\} \neq \emptyset$, and for $\varepsilon > 0$, consider $\bar{\gamma} \in [0, +\infty)^N$ such that

$$(14.5) \quad \bar{\gamma}^j \begin{cases} = \bar{\gamma}_*^j & \text{if } j \notin I \\ \in (\bar{\gamma}_*^j - \varepsilon, \bar{\gamma}_*^j + \varepsilon) \cap [0, +\infty) & \text{if } j \in I \end{cases}$$

Assume that $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar). Then for ε small enough, we have $\begin{cases} \bar{\gamma}^j = \hat{\gamma}_*^j & \text{if } \bar{\gamma}_*^j = \hat{\gamma}_*^j \\ \bar{\gamma}^j > \hat{\gamma}_*^j & \text{if } \bar{\gamma}_*^j > \hat{\gamma}_*^j \end{cases}$.

Hence $[0, \bar{\gamma}^j] \subset [0, \hat{\gamma}_*^j] + T_{\hat{\gamma}_*^j}[0, \bar{\gamma}_*^j]$, and we conclude that

$$(14.6) \quad \hat{\gamma}_*, \hat{\gamma}(\bar{\gamma}) \in \Gamma(\bar{\gamma}) \subset \Gamma(\bar{\gamma}_*) + T_{\hat{\gamma}_*}\Gamma(\bar{\gamma}_*)$$

Therefore

$$\Phi(\hat{\gamma}_* + \xi) \geq \Phi(\hat{\gamma}_*) + \xi \cdot v \geq \Phi(\hat{\gamma}_*) \quad \text{for all } \xi \in T_{\hat{\gamma}_*}\Gamma(\bar{\gamma}_*), \quad v \in \partial\Phi(\hat{\gamma}_*)$$

where we have used the second part of (14.4) in the last inequality. Therefore, using now the first part of (14.4) and also (14.6), we get $\inf_{\Gamma(\bar{\gamma})} \Phi \geq \Phi(\hat{\gamma}_*)$. Because $L = 0$ on $K(\bar{\gamma})$, and $L(\hat{\gamma}_*) = 0$, we deduce

$\inf_{K(\bar{\gamma})} \Psi \geq \Psi(\hat{\gamma}_*)$. Because Ψ is strictly convex and $\hat{\gamma}_* \in K(\bar{\gamma})$, we conclude that $\hat{\gamma}(\bar{\gamma}) = \hat{\gamma}_*$. Then Definition (14.5) of $\bar{\gamma}$ shows exactly that $\hat{\gamma}$ is locally constant on $\{\hat{\gamma} \neq id_{[0, +\infty)^N}\}$.

Step 3: proof of ii)

We assume that $\Psi(\gamma) = \sum_{j=1}^N \Psi_j(\gamma^j)$, with $\Psi_j : \mathbb{R} \rightarrow \mathbb{R}$ strictly convex. We fix the index $k = 1$ and assume that $\sigma^1 = 1$ (the other cases are similar for $\sigma^1 = -1$ or other indices). We want to show that $\bar{\gamma}^1 \mapsto \sigma^j \hat{\gamma}^j(\bar{\gamma})$ is nonincreasing for $j \neq 1$.

Step 3.1: a simplified approximate situation

For $j \neq 1$, fix $\bar{\gamma}^j \geq 0$, and approximate each function Ψ_j , by a smooth convex coercive function $\Psi_j^\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ such that pointwisely, we have

$$\lim_{\varepsilon \rightarrow 0^+} \Psi_j^\varepsilon(x) = \begin{cases} \Psi_j(x) & \text{if } x \in [0, \bar{\gamma}^j] \\ +\infty & \text{if } x \notin [0, \bar{\gamma}^j] \end{cases}$$

We set $\Psi^\varepsilon(\gamma) := \Psi_1(\gamma^1) + \sum_{j=2}^N \Psi_j^\varepsilon(\gamma^j)$, $K^\varepsilon(\bar{\gamma}^1) := \{\gamma \in \Gamma^\varepsilon(\bar{\gamma}^1), L(\gamma) = 0\}$, $\Gamma^\varepsilon(\bar{\gamma}^1) = [0, \bar{\gamma}^1] \times \mathbb{R}^{N-1}$ and consider the approximate problem

$$\hat{\gamma}^\varepsilon(\bar{\gamma}^1) := \underset{K^\varepsilon(\bar{\gamma}^1)}{\text{Argmin}} \Psi^\varepsilon$$

Now it is easy to analyse the optimization problem for which we have suppressed the constraints for $j \neq 1$. We deduce that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ of the constraint $L = 0$, such that for $\bar{\gamma}' := (\bar{\gamma}^2, \dots, \bar{\gamma}^N)$, we have $D_{\bar{\gamma}'}(\Psi^\varepsilon - \lambda L)(\hat{\gamma}^\varepsilon(\bar{\gamma}^1)) = 0$, i.e.

$$\begin{pmatrix} (\Psi_2^\varepsilon)'(\hat{\gamma}^{\varepsilon,2}(\bar{\gamma}^1)) \\ \dots \\ (\Psi_N^\varepsilon)'(\hat{\gamma}^{\varepsilon,N}(\bar{\gamma}^1)) \end{pmatrix} = \lambda \begin{pmatrix} \sigma^2 \\ \dots \\ \sigma^N \end{pmatrix}$$

Similarly for some $\bar{\gamma}_*^1 > \bar{\gamma}^1$, we get the existence of some $\lambda_* \in \mathbb{R}$ such that

$$\begin{pmatrix} (\Psi_2^\varepsilon)'(\hat{\gamma}^{\varepsilon,2}(\bar{\gamma}_*^1)) \\ \dots \\ (\Psi_N^\varepsilon)'(\hat{\gamma}^{\varepsilon,N}(\bar{\gamma}_*^1)) \end{pmatrix} = \lambda_* \begin{pmatrix} \sigma^2 \\ \dots \\ \sigma^N \end{pmatrix}$$

Because each function $(\Psi_j^\varepsilon)'$ is nondecreasing, we deduce that

$$(14.7) \quad (\lambda_* - \lambda) \cdot \sigma^j \{ \hat{\gamma}^{\varepsilon,j}(\bar{\gamma}_*^1) - \hat{\gamma}^{\varepsilon,j}(\bar{\gamma}^1) \} \geq 0 \quad \text{for all } j \neq 1$$

Moreover relation $L(\hat{\gamma}^\varepsilon(\bar{\gamma}_*^1) - \hat{\gamma}^\varepsilon(\bar{\gamma}^1)) = 0$ shows that

$$\{ \hat{\gamma}^{\varepsilon,1}(\bar{\gamma}_*^1) - \hat{\gamma}^{\varepsilon,1}(\bar{\gamma}^1) \} + \sum_{j=2}^N \sigma^j \{ \hat{\gamma}^{\varepsilon,j}(\bar{\gamma}_*^1) - \hat{\gamma}^{\varepsilon,j}(\bar{\gamma}^1) \} = 0.$$

Notice that (14.7) shows that all terms in the sum $\sum_{j=2}^N$ have the same sign. Moreover the reasoning of Step 1.2 shows that $\{ \hat{\gamma}^{\varepsilon,1}(\bar{\gamma}_*^1) - \hat{\gamma}^{\varepsilon,1}(\bar{\gamma}^1) \} \geq 0$. We deduce that

$$(14.8) \quad \sigma^j \{ \hat{\gamma}^{\varepsilon,j}(\bar{\gamma}_*^1) - \hat{\gamma}^{\varepsilon,j}(\bar{\gamma}^1) \} \leq 0 \quad \text{for all } j \neq 1$$

Step 3.2: the limit $\varepsilon \rightarrow 0$

Using the strict convexity of Ψ , it is easy to show that $\lim_{\varepsilon \rightarrow 0^+} \hat{\gamma}^\varepsilon(\bar{\gamma}^1) = \hat{\gamma}(\bar{\gamma})$. We deduce from (14.8) that

$$\sigma^j \{ \hat{\gamma}^j(\bar{\gamma}_*^1, \bar{\gamma}') - \hat{\gamma}^j(\bar{\gamma}^1, \bar{\gamma}') \} \leq 0 \quad \text{for all } j \neq 1, \quad \text{with } \bar{\gamma}_*^1 > \bar{\gamma}^1 \quad \text{and } \sigma^1 = 1$$

which shows exactly that the maps $\bar{\gamma} \mapsto \sigma^j \hat{\gamma}^j(\bar{\gamma})$ are nonincreasing in $\sigma^1 \bar{\gamma}^1$ for all $j \neq 1$. Similarly, doing the reasoning for all indices, this implies the σ -monotonicity of $\hat{\gamma}$. This ends the proof of the lemma.

For later use, we will also need the following adaptation of the previous lemma.

Lemma 14.4 (Convex optimization, local variation)

Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a strictly convex function and some fixed vector $\mathbb{R}^N \ni \bar{\gamma}_* = (\bar{\gamma}_*^1, \dots, \bar{\gamma}_*^N) \geq 0$ and $\sigma = (\sigma^1, \dots, \sigma^N) \in \{\pm 1\}^N$. Let $L : \mathbb{R}^N \rightarrow \mathbb{R}$ be the linear map defined by $L(\gamma) := \sum_{j=1}^N \sigma^j \gamma^j$ for $\gamma \in \mathbb{R}^N$.

Assume that $L\bar{\gamma}_* \neq 0$, and let $\kappa \in (\min\{0, L\bar{\gamma}_*\}, \max\{0, L\bar{\gamma}_*\})$. Let

$$\hat{\gamma}(\bar{\gamma}) := \underset{K_\kappa(\bar{\gamma})}{\text{Argmin}} \Psi \quad \text{with the convex } K_\kappa(\bar{\gamma}) := \{ \gamma \in \Gamma(\bar{\gamma}), \quad L(\gamma) = \kappa \} \quad \text{and the box } \Gamma(\bar{\gamma}) := [0, \bar{\gamma}] := \prod_{j=1}^N [0, \bar{\gamma}^j]$$

Then there exists $\varepsilon > 0$ small enough such that the following holds for

$$Q_\varepsilon := \left(\bar{\gamma}_* + \sum_{j=1, \dots, N} (-\varepsilon, \varepsilon) e_j \right) \cap [0, +\infty)^N.$$

i) (Preflux properties)

Then $\hat{\gamma} : Q_\varepsilon \rightarrow [0, +\infty)^N$ is well defined and is (locally on Q_ε) a preflux in the sense of Definition 7.19.

ii) (σ -monotonicity, when Ψ has separated variables)

Assume moreover that there exists N strictly convex functions $\Psi_j : \mathbb{R} \rightarrow \mathbb{R}$ such that $\Psi(\gamma) = \sum_{j=1}^N \Psi_j(\gamma^j)$ for

$\gamma \in \mathbb{R}^N$. Then $\hat{\gamma}$ is (locally on Q_ε) σ -monotone in the sense of Definition 7.19.

Proof of Lemma 14.4

The proof follows exactly the same lines of the proof of Lemma 14.2, but replacing the linear map L by the affine map $\gamma \mapsto L\gamma - \kappa$. This ends the proof of the lemma.

Proposition 14.5 (Holden-Risebro germ)

Assume (2.2) for some $n : m$ junction (J, f) with $N = n + m$ and $n, m \geq 1$. Recall that $\sigma \in \{\pm 1\}^N$ and σ^j denotes the orientation of the branch J^j . Assume that $\sigma \neq (1, \dots, 1)$ and $\sigma \neq (-1, \dots, -1)$. Assume also

that f is bell-shaped, and call $\bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N$ the capacity $\bar{\gamma}^j := f^{j, \sigma^j}$ given in Definition 7.21. For $p \in [a, b]$, consider the convex set

$$K_0(p) := \{\gamma \in \Gamma_0(p), \quad L(\gamma) = 0\} \quad \text{with} \quad L(\gamma) := \sum_{k=1}^N \sigma^k \gamma^k \quad \text{and the box} \quad \Gamma_0(p) := [0, \bar{\gamma}(p)] = \prod_{k=1, \dots, N} [0, \bar{\gamma}^k(p)]$$

Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a strictly convex function.

i) (Riemann germ for strictly convex Ψ)

Then the set

$$\mathcal{G} = \mathcal{G}_\Psi := \left\{ p \in [a, b], \quad \hat{f}(p) = f(p) \right\} \quad \text{with} \quad \hat{f}(p) := \underset{K_0(p)}{\text{Argmin}} \Psi$$

is a conservative Riemann germ.

ii) (Conservative Kruřkov germ when Ψ has separated variables)

If $\Psi(\gamma) = \sum_{j=1}^N \Psi_j(\gamma^j)$ with each map $\Psi_j : \mathbb{R} \rightarrow \mathbb{R}$ is strictly convex, then \mathcal{G}_Ψ is a conservative Kruřkov germ, which is called a Holden-Risebro germ.

Remark 14.6 Notice that the original Holden-Risebro germ only concerns point ii) of Proposition 14.5. The reader can also consult HOLLE [32] for a different discussion of the conservative Kruřkov property of Holden-Risebro germs. Here point i) is a natural generalization, with weaker properties.

Proof of Proposition 14.5

We notice that the capacity $\bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N$ given by Definition 7.21 is continuous and each map $p^j \mapsto \sigma^j \bar{\gamma}^j(p^j)$ is nondecreasing. Moreover with notation $\hat{\gamma}, K, \Gamma$ of Lemma 14.2, we have

$$\hat{f} = \hat{\gamma} \circ \bar{\gamma}, \quad \Gamma_0 = \Gamma \circ \bar{\gamma}, \quad K_0 = K \circ \bar{\gamma}.$$

Step 1: Riemann germ property

Then i) of Lemma 14.2 and Proposition 7.23 show that \mathcal{G}_Ψ is a conservative Riemann germ.

Step 2: additional monotonicities for Ψ with separated variables

Assume that $\Psi(\gamma) = \sum_{j=1}^N \Psi_j(\gamma^j)$. Then ii) of Lemma 14.2 shows that the preflux $\hat{\gamma}$ is σ -monotone. Then iii) of Proposition 7.23 shows that \mathcal{G}_Ψ is monotone. We know (see Theorem 2.24) that any conservative monotone germ is a Kruřkov germ, and this ends the proof of the proposition.

14.3 Garavello-Piccoli solver for convergent junctions/data networks

We need to start with the following result.

Lemma 14.7 (Preflux by orthogonal projection in \mathbb{R}^N)

Let $N \geq 1$ and $\tilde{\sigma} = (\tilde{\sigma}^1, \dots, \tilde{\sigma}^{N+1}) = (1, \dots, 1, -1)$. For $y = (y^1, \dots, y^N) \in [0, +\infty)^N$, define

$$Ly := \sum_{j=1, \dots, N} y^j$$

Fix $\theta \in (0, 1]^N$ such that $L\theta = 1$. For $\gamma = (\gamma^1, \dots, \gamma^N) \in [0, +\infty)^N$ and $\gamma^{N+1} \in [0, +\infty)$, define the following function

$$\Psi(\cdot, \gamma^{N+1}) : [0, +\infty)^N \rightarrow [0, +\infty) \quad \text{with} \quad \Psi(y, \gamma^{N+1}) := |y - \gamma^{N+1} \theta|^2 \quad \text{and the box} \quad \Gamma(\gamma) := [0, \gamma] := \prod_{j=1}^N [0, \gamma^j]$$

We set

$$(14.9) \quad \hat{\gamma}(\gamma, \gamma^{N+1}) := \underset{K(\gamma, \gamma^{N+1})}{\text{Argmin}} \Psi(\cdot, \gamma^{N+1}) \quad \text{with} \quad K(\gamma, \gamma^{N+1}) := \{y \in \Gamma(\gamma), \quad Ly = \gamma^{N+1}\}$$

Fix some $\tilde{\gamma}_* = (\gamma_*, \gamma_*^{N+1}) = (\gamma_*^1, \dots, \gamma_*^{N+1}) \in [0, +\infty)^{N+1}$ such that $L\gamma_* > \gamma_*^{N+1}$. Then there exists $\varepsilon > 0$ small enough such that the following holds for

$$\tilde{Q}_\varepsilon(\tilde{\gamma}_*) := \left(\tilde{\gamma}_* + \sum_{j=1, \dots, N+1} (-\varepsilon, \varepsilon) e_j \right) \cap [0, +\infty)^{N+1}.$$

The function $\tilde{\gamma} : \tilde{Q}_\varepsilon(\tilde{\gamma}_*) \rightarrow [0, +\infty)^{N+1}$ given by $\tilde{\gamma}(\gamma, \gamma^{N+1}) := (\hat{\gamma}(\gamma, \gamma^{N+1}), \gamma^{N+1})$ is well defined and is (locally on $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$) a $\tilde{\sigma}$ -conservative $\tilde{\sigma}$ -monotone preflux in the sense of Definition 7.19.

Remark 14.8 Notice that $\hat{\gamma}$ is the orthogonal projection of $\gamma^{N+1}\theta$ onto the closed convex set $K(\gamma, \gamma^{N+1})$.

Proof of Lemma 14.7

Because $\Psi(\cdot, \gamma^{N+1})$ is strictly convex and $\left\{ \frac{\gamma_*^{N+1}}{L\gamma_*} \right\} \cdot \gamma_* \in K(\gamma_*, \gamma_*^{N+1}) \neq \emptyset$, we deduce that $\hat{\gamma}$ (and then $\tilde{\gamma}$) is well-defined and is continuous.

Step 1: bounds

By construction, we have $\tilde{\gamma} \in [0, \tilde{\gamma}]$ and then $0 \leq \tilde{\gamma}^j(\tilde{\gamma}) \leq \tilde{\gamma}^j$ for all indices $j = 1, \dots, N+1$.

Step 2: local constancy

By assumption, we have $L\gamma_* > \gamma_*^{N+1}$ and $L\hat{\gamma}(\gamma_*, \gamma_*^{N+1}) = \gamma_*^{N+1}$, which forces to have $I := \{j \in \{1, \dots, N\}, \hat{\gamma}^j(\gamma_*, \gamma_*^{N+1}) < \gamma_*^j\} \neq \emptyset$. Moreover for $\varepsilon > 0$, consider

$$Q_\varepsilon(\gamma_*) := \left(\gamma_* + \sum_{j \in I} (-\varepsilon, \varepsilon) e_j \right) \cap [0, +\infty)^N$$

Then for $\varepsilon > 0$ small enough and $\gamma \in Q_\varepsilon(\gamma_*)$, we still have $L\gamma > \gamma_*^{N+1}$. For fixed $\gamma_*^{N+1} \geq 0$, and using Lemma 14.4, and the fact that $\Psi(\cdot, \gamma_*^{N+1})$ is a sum of strictly convex functions in separated variables, we deduce that locally around γ_* , the map $\gamma \mapsto \hat{\gamma}(\gamma, \gamma_*^{N+1})$ is a preflux. Therefore the map $\gamma \mapsto \hat{\gamma}(\gamma, \gamma_*^{N+1})$ is locally constant in $\{\gamma, \hat{\gamma}(\gamma, \gamma_*^{N+1}) \neq \gamma\}$ close to γ_* . This shows the desired local constancy of $\hat{\gamma}$, and then the local constancy of $\tilde{\gamma}$ on $\{\tilde{\gamma}, \tilde{\gamma}(\tilde{\gamma}) \neq \tilde{\gamma}\}$, close to $\tilde{\gamma}_*$.

Step 3: basic monotonicity

For fixed γ_* and γ^{N+1} close enough to γ_*^{N+1} , consider $\gamma \in Q_\varepsilon(\gamma_*)$. Then the argument of Step 2 shows that $\gamma \mapsto \hat{\gamma}(\gamma, \gamma^{N+1})$ is a preflux. In particular the j component of $\hat{\gamma}$ is nondecreasing in γ^j for each $j = 1, \dots, N$. With $\hat{\gamma}^{N+1}(\gamma, \gamma^{N+1}) = \gamma^{N+1}$, we conclude that $\tilde{\gamma}$ satisfies the basic monotonicities, and $\tilde{\gamma}$ is then a preflux on $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$. Moreover it is $\tilde{\sigma}$ -conservative by construction.

Step 4: $\tilde{\sigma}$ -monotonicity

Because we have seen that $\gamma \mapsto \hat{\gamma}(\gamma, \gamma^{N+1})$ is a preflux, we know that for fixed γ^{N+1} it is σ -monotone in γ . Because $\tilde{\gamma}^{N+1}(\gamma, \gamma^{N+1}) = \gamma^{N+1}$, it only remains to show that the maps $\gamma^{N+1} \mapsto \hat{\gamma}^j(\gamma, \gamma^{N+1})$ are nondecreasing for all $j = 1, \dots, N$. Having (14.9) in mind, we see that the new difficulty here is that the convex function $\Psi(\cdot, \gamma^{N+1})$ itself does depend on γ^{N+1} .

Step 4.1: Generic case

Assume that $\tilde{\gamma}_* = (\gamma_*, \gamma_*^{N+1}) \in (0, +\infty)^N \times (0, +\infty)$ satisfies $L\gamma_* > \gamma_*^{N+1}$. Notice that the unique minimizer $\hat{\gamma}_* \in K(\gamma_*, \gamma_*^{N+1})$ is the orthogonal projection of $\gamma_*^{N+1}\theta$ onto the convex set $K(\gamma_*, \gamma_*^{N+1}) \ni \left\{ \frac{\gamma_*^{N+1}}{L\gamma_*} \right\} \cdot \gamma_*$, which is characterized by the following variational inequality

$$(14.10) \quad (\gamma_*^{N+1}\theta - \hat{\gamma}_*, y - \hat{\gamma}_*) \leq 0 \quad \text{for all } y \in K(\gamma_*, \gamma_*^{N+1}) = \{y \in \Gamma(\gamma_*), \quad Ly = \gamma_*^{N+1}\}$$

Step 4.1.1: decomposition

Define

$$I^\alpha := \{j \in \{1, \dots, N\}, 0 < \hat{\gamma}_*^j < \gamma_*^j\}, \quad I^\beta := \{j \in \{1, \dots, N\}, \hat{\gamma}_*^j = \gamma_*^j\}, \quad I^\delta := \{j \in \{1, \dots, N\}, \hat{\gamma}_*^j = 0\}$$

We set $W := V^\alpha \oplus V^\beta \oplus V^\delta$ with $V^c := \bigoplus_{k \in I^c} \mathbb{R} e_k$ for $c = \alpha, \beta, \delta$, and for any $\gamma \in W$, we write $\gamma = \gamma^\alpha + \gamma^\beta + \gamma^\delta$ with $\gamma^c \in V^c$ for $c = \alpha, \beta, \delta$. Notice that the minimizer $\hat{\gamma}_*$ is uniquely determined by the existence of Lagrange multipliers $\lambda_*^\alpha \in \mathbb{R}$ and $\mu_*^\beta \in V^\beta \cap [0, +\infty)^N$ and $\mu_*^\delta \in V^\delta \cap [0, +\infty)^N$ such that

$$(14.11) \quad \hat{\gamma}_* - \gamma_*^{N+1}\theta = D_y \Psi(\hat{\gamma}_*, \gamma_*^{N+1}) = \lambda_*^\alpha DL - \mu_*^\beta \diamond DL|_{V^\beta} + \mu_*^\delta \diamond DL|_{V^\delta}$$

Indeed for $\Phi(y) := \Psi(y, \gamma_*^{N+1}) - \lambda_*^\alpha (Ly - \gamma_*^{N+1})$, we have $\Phi(y) \geq \Phi(\hat{\gamma}_*) + D_y \Phi(\hat{\gamma}_*) \cdot (y - \hat{\gamma}_*) = \Phi(\hat{\gamma}_*) + (\mu_*^\delta - \mu_*^\beta, y - \hat{\gamma}_*) \geq \Phi(\hat{\gamma}_*)$ for $y \in \Gamma(\gamma_*)$, and with $Ly = \gamma_*^{N+1} = L\hat{\gamma}_*$, we get

$$\Psi(y, \gamma_*^{N+1}) \geq \Psi(\hat{\gamma}_*, \gamma_*^{N+1}) \quad \text{for all } y \in K(\gamma_*, \gamma_*^{N+1})$$

which shows that this characterizes the unique minimizer $\hat{\gamma}_*$ of the strictly convex function $\Psi(\cdot, \gamma_*^{N+1})$ on the convex set $K(\gamma_*, \gamma_*^{N+1})$ (this can also be seen as a special case of Karush-Kuhn-Tucker theorem). We deduce

$$(14.12) \quad \begin{cases} \hat{\gamma}_*^\delta - \gamma_*^{N+1} \theta^\delta = \lambda_*^\alpha DL|_{V^\delta} + \mu_*^\delta \geq \lambda_*^\alpha DL|_{V^\delta} \\ \hat{\gamma}_*^\beta - \gamma_*^{N+1} \theta^\beta = \lambda_*^\alpha DL|_{V^\beta} - \mu_*^\beta \leq \lambda_*^\alpha DL|_{V^\beta} \\ \hat{\gamma}_*^\alpha - \gamma_*^{N+1} \theta^\alpha = \lambda_*^\alpha DL|_{V^\alpha} \end{cases}$$

Step 4.1.2: proof that $I^\delta = \emptyset$ and $\lambda_*^\alpha \geq 0$

The fact that $\hat{\gamma}_*^\delta = 0$ implies with the first line of (14.12) that $I^\delta = \emptyset$ or $\lambda_*^\alpha < 0$. Assume by contradiction that $I^\delta \neq \emptyset$ and then that $\lambda_*^\alpha < 0$.

Case A: $I^\beta \neq \emptyset$

Then the second and third lines of (14.12) show that $\gamma_*^{N+1} \theta^\beta > \hat{\gamma}_*^\beta$ in V^β , and $\gamma_*^{N+1} \theta^\alpha \geq \hat{\gamma}_*^\alpha$ in V^α . Then $\gamma_*^{N+1} \theta \geq \hat{\gamma}_*$ with $\gamma_*^{N+1} \theta \neq \hat{\gamma}_*$. Hence $\gamma_*^{N+1} = L(\gamma_*^{N+1} \theta) \geq L\hat{\gamma}_* = \gamma_*^{N+1}$ with $L(\gamma_*^{N+1} \theta) \neq L\hat{\gamma}_*$. Contradiction.

Case B: $I^\beta = \emptyset$

Then the third line of (14.12) shows that

$$(14.13) \quad \begin{aligned} \lambda_*^\alpha |I^\alpha| &= L(\hat{\gamma}_*^\alpha - \gamma_*^{N+1} \theta^\alpha) \\ &= \left\{ \gamma_*^{N+1} - L(\hat{\gamma}_*^\beta + \hat{\gamma}_*^\delta) \right\} - \gamma_*^{N+1} L\theta^\alpha \\ &= L(\gamma_*^{N+1}(\theta^\beta + \theta^\delta) - \gamma_*^\beta) \\ &= \gamma_*^{N+1} L\theta^\delta > 0 \end{aligned}$$

where in the second line we have used $L\hat{\gamma}_* = \gamma_*^{N+1}$, in the third line we have used $L\theta = 1$, $\hat{\gamma}_*^\delta = 0$, $\hat{\gamma}_*^\beta = \gamma_*^\beta$, and in the last line we have used $I^\beta = \emptyset$ and $I^\delta \neq \emptyset$. Contradiction with $\lambda_*^\alpha < 0$.

Conclusion

We conclude that $\lambda_*^\alpha \geq 0$ and $I^\delta = \emptyset$.

Step 4.1.3: proof that $I^\alpha \neq \emptyset$

Assume by contradiction that $I^\alpha = \emptyset$. Then this implies that $\gamma_*^{N+1} \theta \geq \hat{\gamma}_* = \gamma_*$ and then $\gamma_*^{N+1} = L\gamma_*^{N+1} \theta \geq L\gamma_*$. Contradiction with $L\gamma_* > \gamma_*^{N+1}$. Hence $I^\alpha \neq \emptyset$. Setting $e^\alpha := |I^\alpha|^{-1} \sum_{j \in I^\alpha} e_j$ which satisfies $Le^\alpha = 1$,

we deduce from the third line of (14.12) that

$$(14.14) \quad \hat{\gamma}_*^\alpha = \gamma_*^{N+1} \theta^\alpha + \lambda_* e^\alpha \quad \text{with} \quad 0 \leq \lambda_* := \lambda_*^\alpha |I^\alpha| = L(\gamma_*^{N+1} \theta^\beta - \gamma_*^\beta)$$

where in the last equality, we have used the third line of (14.13) with $I^\delta = \emptyset$. Hence we get

$$(14.15) \quad \hat{\gamma}(\gamma_*, \gamma_*^{N+1}) = \hat{\gamma}_* = \gamma_*^{N+1}(\theta^\alpha + (L\theta^\beta)e^\alpha) - (L\gamma_*^\beta)e^\alpha + \gamma_*^\beta$$

From (14.11), we also have

$$(14.16) \quad \hat{\gamma}_* - \gamma_*^{N+1} \theta = \lambda_*^\alpha DL - \mu_*^\beta \quad \text{with} \quad \begin{cases} 0 < \hat{\gamma}_*^\alpha < \gamma_*^\alpha & \text{in } V^\alpha \\ \hat{\gamma}_*^\beta = \gamma_*^\beta & \text{and } \mu_*^\beta \geq 0 \end{cases}$$

which characterizes $\hat{\gamma}_* \in K(\gamma_*, \gamma_*^{N+1})$.

We now discuss a complement of independent interest. Because $\hat{\gamma}_*^\alpha \leq \gamma_*^\alpha$ and $\lambda_* \geq 0$, we deduce from (14.14) that $\gamma_*^{N+1} \theta^\alpha \in \Gamma(\gamma_*^\alpha)$. Now assume by contradiction that there exists $j \in I^\alpha$ such that $\gamma_*^{N+1} \theta^j = \gamma_*^j$. Then $\lambda_* \geq 0$ implies that $\hat{\gamma}_*^j = \gamma_*^j$. Contradiction with the definition of I^α . Hence we get moreover $\gamma_*^{N+1} \theta^\alpha < \gamma_*^\alpha$ in V^α .

Step 4.2: Generic case and variations

Now consider $\gamma_*^{N+1} := \gamma_*^{N+1} + q$ with $q \in (0, +\infty)$. Following (14.15), let us consider as a candidate for

$\hat{\gamma}(\gamma_*, \gamma_*^{N+1} + q)$, the vector $\xi := \hat{\gamma}_* + q \{ \theta^\alpha + (L\theta^\beta)e^\alpha \}$. Following (14.16), we now compute

$$\begin{aligned}
\xi - \gamma^{N+1}\theta &= \{ \hat{\gamma}_* - \gamma_*^{N+1}\theta \} + q \{ (L\theta^\beta)e^\alpha - \theta^\beta \} \\
&= \{ \lambda_*^\alpha DL - \mu_*^\beta \} + q \{ (L\theta^\beta)e^\alpha - \theta^\beta \} \\
&= \{ \lambda_*^\alpha DL - \mu_*^\beta \} + q \left\{ |I^\alpha|^{-1}(L\theta^\beta) \sum_{j \in I^\alpha} e_j - \theta^\beta \right\} \\
&= \{ \lambda_*^\alpha DL - \mu_*^\beta \} + q \left\{ |I^\alpha|^{-1}(L\theta^\beta) \left\{ DL - \sum_{j \in I^\beta} e_j \right\} - \theta^\beta \right\} \\
&= \lambda^\alpha DL - \mu^\beta \quad \text{with} \quad \begin{cases} \lambda^\alpha = \lambda_*^\alpha + q|I^\alpha|^{-1}L\theta^\beta \geq 0 \\ \mu^\beta = \mu_*^\beta + q \left\{ \theta^\beta + |I^\alpha|^{-1}(L\theta^\beta) \sum_{j \in I^\beta} e_j \right\} \geq 0 \quad \text{and} \quad \xi^\beta = \gamma_*^\beta \end{cases}
\end{aligned}$$

which satisfies $\xi^\alpha < \gamma_*^\alpha$ for $q > 0$ small enough. This shows that

$$\xi - \gamma^{N+1}\theta = \lambda^\alpha DL - \mu^\beta \quad \text{with} \quad \begin{cases} 0 < \xi^\alpha < \gamma_*^\alpha & \text{in } V^\alpha \\ \xi^\beta = \gamma_*^\beta & \text{and } \mu^\beta \geq 0 \end{cases}$$

As in (14.11), this relation characterizes $\hat{\gamma}(\gamma_*, \gamma^{N+1}) = \xi$. Because $\xi \geq \hat{\gamma}_*$, we deduce that $\hat{\gamma}(\gamma_*, \gamma^{N+1}) \geq \hat{\gamma}(\gamma_*, \gamma_*^{N+1})$. Hence for each $j = 1, \dots, N$, this shows that the map $\gamma^{N+1} \mapsto \hat{\gamma}^j(\gamma_*, \gamma^{N+1})$ is locally nondecreasing.

Step 4.3: General case

Let $\tilde{\gamma}_* = (\gamma_*, \gamma_*^{N+1}) \in [0, +\infty)^{N+1}$ such that $L\gamma_* > \gamma_*^{N+1}$. Using the continuity of $\hat{\gamma}$, we get (by perturbations) that $\hat{\gamma}^j$ is nondecreasing in γ^{N+1} on $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$ for all $j = 1, \dots, N$. With the other monotonicities, we deduce that $\tilde{\gamma}$ is $\tilde{\sigma}$ -monotone on $\tilde{Q}_\varepsilon(\tilde{\gamma}_*)$.

This ends the proof of the lemma.

Remark 14.9 Notice that Step 4 of the proof of Lemma 14.7 provides a proof of the monotonicity of the map $\gamma^{N+1} \mapsto \hat{\gamma}(\gamma_*, \gamma^{N+1})$ stated in Lemma 4 in [30], with more precise justifications.

The next result presents the preflux used by Garavello and Piccoli for \mathcal{RS}_2 in [30].

Lemma 14.10 (Data networks preflux)

Let $N \geq 1$ and $\sigma = (\sigma^1, \dots, \sigma^N) \in \{\pm 1\}^N$ with $\sigma \neq (1, \dots, 1)$ and $\sigma \neq (-1, \dots, -1)$. For $\gamma = (\gamma^1, \dots, \gamma^N) \in [0, +\infty)^N$, define

$$L^+(\gamma) := \sum_{\sigma^j=1} \gamma^j, \quad L^-(\gamma) := \sum_{\sigma^j=-1} \gamma^j$$

Let $\theta \in (0, 1)^N$ such that $L^+(\theta) = 1 = L^-(\theta)$, fix some $\bar{\gamma} = (\bar{\gamma}^1, \dots, \bar{\gamma}^N) \in [0, +\infty)^N$, and define the passing flux

$$\bar{\gamma}_0(\bar{\gamma}) := \bar{\gamma}_0 := \min \{ L^+(\bar{\gamma}), L^-(\bar{\gamma}) \}$$

and

$$\begin{cases} K^+(\bar{\gamma}) := \{ \gamma \in \Gamma^+(\bar{\gamma}), \quad L^+(\gamma) = \bar{\gamma}_0 \}, & \text{with } \Gamma^+(\bar{\gamma}) := \sum_{\sigma^j=1} [0, \bar{\gamma}^j] e_j \\ K^-(\bar{\gamma}) := \{ \gamma \in \Gamma^-(\bar{\gamma}), \quad L^-(\gamma) = \bar{\gamma}_0 \}, & \text{with } \Gamma^-(\bar{\gamma}) := \sum_{\sigma^j=-1} [0, \bar{\gamma}^j] e_j \end{cases}$$

Let us consider the convex compact set

$$K(\bar{\gamma}) := \{ \gamma \in [0, +\infty)^N, \quad \gamma = \gamma^+ + \gamma^-, \quad (\gamma^+, \gamma^-) \in K^+(\bar{\gamma}) \times K^-(\bar{\gamma}) \}$$

and the orthogonal projection on it

$$\hat{\gamma}^{DN}(\bar{\gamma}) := Proj_{K(\bar{\gamma})}^\perp(\bar{\gamma}_0(\bar{\gamma}) \cdot \theta)$$

Then $\hat{\gamma}^{DN} : [0, +\infty)^N \rightarrow [0, +\infty)^N$ is a conservative σ -monotone preflux in the sense of Definition 7.19.

Remark 14.11 *The convergent junction case in Section 5.2.2 of GARAVELLO, PICCOLI [12] is a special case of data networks (Riemann solver \mathcal{RS}_2 in [30]), which corresponds to $N = n + 1$ with n ingoing branches $\sigma^j = 1$ for $j = 1, \dots, n$ and a single outgoing branch $\sigma^{n+1} = -1$.*

Proof of Lemma 14.10

Step 1: preflux

By construction, the function $\hat{\gamma} := \hat{\gamma}^{DN}$ is continuous and conservative.

Moreover, we see that

$$\hat{\gamma} = \hat{\gamma}^+ + \hat{\gamma}^- \quad \text{with} \quad \begin{cases} \hat{\gamma}^+ := \text{Proj}_{|K^+(\bar{\gamma})}^\perp(\bar{\gamma}_0\theta^+) & \text{with } \theta^+ := \sum \theta^j e_j \\ \hat{\gamma}^- := \text{Proj}_{|K^-(\bar{\gamma})}^\perp(\bar{\gamma}_0\theta^-) & \text{with } \theta^- := \sum_{\sigma^j=-1}^{\sigma^j=1} \theta^j e_j \end{cases}$$

Step 1.1: bounds

By construction, we have $\hat{\gamma}^\pm \in K^\pm(\bar{\gamma}) \subset \Gamma^\pm(\bar{\gamma})$ and then $0 \leq \hat{\gamma}^j(\bar{\gamma}) \leq \bar{\gamma}^j$ for all indices j .

Step 1.2: local constancy

For $\gamma \in [0, +\infty)^N$, we consider the general splitting

$$\gamma = \gamma^+ + \gamma^- \quad \text{with} \quad (\gamma^+, \gamma^-) \in \Gamma^+(\gamma) \times \Gamma^-(\gamma)$$

For some $\bar{\gamma}_* \in [0, +\infty)^N$, assume that $I := \{j \in \{1, \dots, N\}, \hat{\gamma}^j(\bar{\gamma}_*) < \bar{\gamma}_*^j\} \neq \emptyset$. Assume by contradiction that there exists $j, k \in I$ such that $\sigma^j = 1$ and $\sigma^k = -1$. Then this implies that $L^\pm \hat{\gamma}(\bar{\gamma}_*) < L^\pm \bar{\gamma}_*$, and then $L^\pm \hat{\gamma}(\bar{\gamma}_*) < \bar{\gamma}_0(\bar{\gamma}_*) := \min\{L^+ \bar{\gamma}_*, L^- \bar{\gamma}_*\}$, i.e. $\hat{\gamma}^\pm(\bar{\gamma}_*) \notin K^\pm(\bar{\gamma}_*)$. Contradiction. Therefore we conclude that we have either $\sigma^j = 1$ for all indices $j \in I$, or $\sigma^j = -1$ for all indices $j \in I$.

Without loss of generality, let us assume that $\sigma^j = 1$ for all $j \in I$ (the case $\sigma^j = -1$ is similar). Then $L^+ \bar{\gamma}_*^+ > L^+ \hat{\gamma}_*^+ = L^- \bar{\gamma}_*^- = \bar{\gamma}_0(\bar{\gamma}_*)$. Therefore $K^-(\bar{\gamma}_*) = \{\bar{\gamma}_*^-\}$ and $\hat{\gamma}^-(\bar{\gamma}_*) = \bar{\gamma}_*^-$. Moreover for $\varepsilon > 0$, consider

$$Q_\varepsilon(\bar{\gamma}_*) := \left(\bar{\gamma}_* + \sum_{j \in I} (-\varepsilon, \varepsilon) e_j \right) \cap [0, +\infty)^N$$

Then for $\varepsilon > 0$ small enough and $\bar{\gamma} \in Q_\varepsilon(\bar{\gamma}_*)$, we have $\bar{\gamma}_0(\bar{\gamma}) = \bar{\gamma}_0(\bar{\gamma}_*)$. Therefore we still have $K^-(\bar{\gamma}) = \{\bar{\gamma}^-\} = \{\bar{\gamma}_*^-\}$, and then $\hat{\gamma}^-(\bar{\gamma}) = \bar{\gamma}_*^-$. Because $\hat{\gamma}^+$ is the orthogonal projection on $K^+(\bar{\gamma})$ of $\bar{\gamma}_0\theta^+$, we deduce that

$$(14.17) \quad \hat{\gamma}^+(\bar{\gamma}) := \underset{K^+(\bar{\gamma})}{\text{Argmin}} \Psi(\cdot, \bar{\gamma}_0) \quad \text{with} \quad \Psi(y, \bar{\gamma}_0) := |y - \bar{\gamma}_0\theta^+|^2$$

Because $\bar{\gamma}_0(\bar{\gamma}) = \bar{\gamma}_0(\bar{\gamma}_*) =: \bar{\gamma}_0^*$ is fixed, we can use Lemma 14.4, and the fact that $\Psi(\cdot, \bar{\gamma}_0^*)$ is a sum of strictly convex functions in separated variables, to deduce that for $\bar{\gamma}$ locally around $\bar{\gamma}_*$, the map $\bar{\gamma}^+ \mapsto \hat{\gamma}^+(\bar{\gamma}^+ + \bar{\gamma}_*^-) = \hat{\gamma}^+(\bar{\gamma})$ is a preflux. Therefore the map $\bar{\gamma} \mapsto \hat{\gamma}(\bar{\gamma}) = (\hat{\gamma}^+(\bar{\gamma}), \bar{\gamma}_*^-)$ is locally constant in $\{\bar{\gamma}, \hat{\gamma}(\bar{\gamma}) \neq \bar{\gamma}\}$ close to $\bar{\gamma}_*$. This is the desired local constancy of $\hat{\gamma}$.

Step 1.3: basic monotonicities and σ -monotonicity

Consider $\bar{\gamma}_*, \bar{\gamma} \in [0, +\infty)^N$, such that for some index j , say with $\sigma^j = 1$ (the case $\sigma^j = -1$ is similar), we have

$$(14.18) \quad \bar{\gamma} - \bar{\gamma}_* = qe_j \quad \text{with} \quad q \in (0, +\infty)$$

Case A: $L^+ \bar{\gamma}_* \geq L^- \bar{\gamma}_*$

Then $L^+ \bar{\gamma}_* \geq L^- \bar{\gamma}_* = \bar{\gamma}_0(\bar{\gamma}_*)$. Hence $L^+ \bar{\gamma} > L^- \bar{\gamma} = L^- \bar{\gamma}_* = \bar{\gamma}_0(\bar{\gamma})$. Therefore as in Step 1.2, we get (14.17), and Lemma 14.4 shows that $\bar{\gamma}^+ \mapsto \hat{\gamma}^+(\bar{\gamma}^+ + \bar{\gamma}_*^-)$ is a $(1, \dots, 1)$ -monotone preflux. In particular, for $\sigma^k = 1$, the component $\sigma^k \hat{\gamma}^k$ is nonincreasing in $\sigma^j \bar{\gamma}^j$ for $k \neq j$, and nondecreasing in $\sigma^j \bar{\gamma}^j$ for $k = j$. Moreover, we have $K^-(\bar{\gamma}) = \{\bar{\gamma}^-\} = \{\bar{\gamma}_*^-\}$, and then $\hat{\gamma}^-(\bar{\gamma}) = \bar{\gamma}_*^-$. Hence for $\sigma^k = -1$, each component $\sigma^k \hat{\gamma}^k$ is constant in $\sigma^j \bar{\gamma}^j$, and it is in particular nonincreasing.

Case B: $L^+ \bar{\gamma}_* < L^- \bar{\gamma}_*$

Then $\bar{\gamma}_0^* := \bar{\gamma}_0(\bar{\gamma}_*) = L^+ \bar{\gamma}_* < L^- \bar{\gamma}_*$, and then $K^+(\bar{\gamma}_*) = \{\bar{\gamma}_*^+\}$. Then for $q > 0$ small enough such that $\bar{\gamma}_0^* < \bar{\gamma}_0(\bar{\gamma}) = L^+ \bar{\gamma} < L^- \bar{\gamma} = L^- \bar{\gamma}_*$, we have $K^+(\bar{\gamma}) = \{\bar{\gamma}^+\}$ and $\hat{\gamma}^+(\bar{\gamma}) = \bar{\gamma}^+$ and in particular $\hat{\gamma}^k(\bar{\gamma}) = \bar{\gamma}^k = \bar{\gamma}_*^k$ for all k such that $\sigma^k = 1$ (for $k \neq j$ and for $k = j$). For $q > 0$ not small, notice that there

exists some unique $q_* > 0$, such that we get equality $\bar{\gamma}_0(\bar{\gamma}_* + q_* e_j) = L^+(\bar{\gamma}_* + q_* e_j) = L^-\bar{\gamma}_*$, and then we are back to Case A with $\bar{\gamma}_*$ replaced by $\bar{\gamma}_* + q_* e_j$.

Now notice that $\hat{\gamma}^-$ is the orthogonal projection on $K^-(\bar{\gamma})$ of $\bar{\gamma}_0 \theta^-$, and we deduce that

(14.19)
$$\hat{\gamma}^-(\bar{\gamma}) := \underset{K^-(\bar{\gamma})}{\text{Argmin}} \Psi(\cdot, \bar{\gamma}_0(\bar{\gamma})) \quad \text{with} \quad \Psi(y, \bar{\gamma}_0) := |y - \bar{\gamma}_0 \theta^-|^2 \quad \text{with} \quad K^-(\bar{\gamma}) = \{\gamma \in \Gamma^-(\bar{\gamma}), \quad L^-\gamma = L^-(\bar{\gamma}_0 \theta^-)\}$$

with $L^-(\bar{\gamma}_0 \theta^-) = \bar{\gamma}_0 = L^+\bar{\gamma} = L^+\bar{\gamma}^+$. Then Lemma 14.7 shows that the map $(\bar{\gamma}^-, \bar{\gamma}_0) \mapsto (\hat{\gamma}^-(\bar{\gamma}), \bar{\gamma}_0)$ is a $(1, 1, \dots, 1, -1)$ -monotone preflux. In particular each component of $\hat{\gamma}^-$ is nondecreasing in the agglomerated variable $\bar{\gamma}_0(\bar{\gamma}) = L^+\bar{\gamma}^+$. Therefore $\sigma^k \hat{\gamma}^k$ is nonincreasing in $\sigma^j \bar{\gamma}^j$ for all k such that $\sigma^k = -1$. This shows that $\hat{\gamma}$ is σ -monotone, and has the basic monotonicities ($\sigma^j \hat{\gamma}^j$ is nondecreasing in $\sigma^j \bar{\gamma}^j$). This ends the proof of the lemma.

Corollary 14.12 (Data networks germ; $\mathcal{R}S_2$ in [30])

Assume (2.2) for some $n : m$ junction (J, f) with $N = n + m$ and $n, m \geq 1$. Recall that $\sigma \in \{\pm 1\}^N$ and σ^j denotes the orientation of the branch J^j . Assume that $\sigma \neq (1, \dots, 1)$ and $\sigma \neq (-1, \dots, -1)$. Assume also that f is bell-shaped, and call $\bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N$ the capacity $\bar{\gamma}^j := f^{j, \sigma^j}$ given in Definition 7.21. Then the set

(14.20)
$$\mathcal{G} = \mathcal{G}_{\hat{f}} := \left\{ p \in [a, b], \quad \hat{f}(p) = f(p) \right\} \quad \text{with} \quad \hat{f}(p) := \hat{\gamma}^{DN} \circ \bar{\gamma}$$

is a conservative Kruřkov germ, where $\hat{\gamma}^{DN}$ is the data network preflux given in Lemma 14.10.

Proof of Corollary 14.12

From Lemma 14.10, we know that the preflux $\hat{\gamma}^{DN}$ is conservative and σ -monotone. This implies that \mathcal{G} is conservative monotone germ, hence conservative Kruřkov. This ends the proof of the corollary.

In [30], the authors provide an important result that gives us an enlighting heuristic for the proof of Corollary 14.12. We now state this result.

Lemma 14.13 (A contraction result, Lemma 6 in [30])

We work under the assumptions of Corollary 14.12, with furthermore the specific case of $f^j = f^0$ for $j = 1, \dots, N$. For \mathcal{G} defined in (14.20), and for all fixed $j \in \{1, \dots, N\}$, all $\hat{p} \in \mathcal{G}$, and $q \in [a, b]$ such that $q - \hat{p} \in \mathbb{R}^* e_j$ satisfies

$$\sigma^j c^j \geq 0 \quad \text{with} \quad c^j := \frac{f^j(q^j) - f^j(\hat{p}^j)}{q^j - \hat{p}^j} \quad \text{and} \quad \sigma^j = \begin{cases} 1 & \text{for ingoing road} \\ -1 & \text{for outgoing road} \end{cases}$$

then we have (recalling that $\hat{f} = f \circ \pi_{\mathcal{G}}$ and $\pi_{\mathcal{G}} \circ \pi_{\mathcal{G}} = \pi_{\mathcal{G}}$)

$$|\hat{f}^j(q) - f^j(q^j)| + \sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\hat{f}^k(q) - \hat{f}^k(\hat{p})| = |\hat{f}^j(\hat{p}) - f^j(q^j)|$$

Heuristic motivating the proof of Corollary 14.12

We now propose an heuristic to motivate the proof of Corollary 14.12 in the special case $f^j = f^0$ for $j = 1, \dots, N$, using Lemma 14.13. Using the fact that the map $x \mapsto |x - c|$ is 1-Lipschitz, we deduce that $|x - c| - |y - c| \leq |x - y|$ and get from Lemma 14.13 that
$$\sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\hat{f}^k(q) - \hat{f}^k(\hat{p})| \leq |\hat{f}^j(q) - \hat{f}^j(\hat{p})|.$$

In the limit $q \rightarrow \hat{p}$, this gives formally
$$\sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}^k| \leq |\partial_j \hat{f}^j| \quad \text{at } \hat{p} \quad \text{when} \quad \sigma^j (f^j)'(\hat{p}^j) \geq 0.$$
 Using vii)

of Proposition 2.13, if we know that \hat{f} is a Godunov flux associated to a Riemann germ \mathcal{G} , then we expect $\sigma^j \partial_j \hat{f}^j(\hat{p}) \in \{0, \max\{0, \sigma^j (f^j)'(\hat{p}^j)\}\}$. Then, in the very best case, we may expect to have on \mathcal{G}

$$\sum_{k \in \{1, \dots, N\} \setminus \{j\}} |\partial_j \hat{f}^k| \leq \sigma^j \partial_j \hat{f}^j$$

Because any Godunov flux \hat{f} is locally constant on $\{f \neq f\}$, we may expect that this inequality is indeed true not only on \mathcal{G} , but a.e. on $[a, b]$. If it is the case, then i) of Theorem 2.21 implies that \mathcal{G} is a Kruřkov germ. Because by construction \mathcal{G} is conservative, we get that \mathcal{G} is a conservative Kruřkov germ. This ends the heuristic.

Remark 14.14 (Comments on other Riemann solvers in [30])

In [30], Riemann solver \mathcal{RS}_3 is not associated to a (conservative) Kruřkov germ, because $\Gamma_i := \min(\gamma_i^{max}, \gamma_{i+n}^{max}) = \min(\bar{\gamma}^i, \bar{\gamma}^{i+n})$ is symmetric in ingoing index i and outgoing index $i + n$, which is not compatible with σ -monotonicity of the associated preflux for $\sigma^i = 1$ and $\sigma^{i+n} = -1$.

Similarly in [30], Riemann solver \mathcal{RS}_1 is not associated to a (conservative) Kruřkov germ for some $n : m$ junction with $1 \leq n \leq m$ and $m \geq 2$. Indeed if the ingoing branches $j = 2, \dots, n$ are empty with $\bar{\gamma}^j = 0$, then the problem is simply described by some $1 : m$ junction with effective passing flux equal to

$\min_{k=n+1, \dots, n+m} \left\{ \bar{\gamma}^1, \frac{\bar{\gamma}^k}{\theta^k} \right\}$ where $\theta^k := \alpha_{1k}$ and where $\bar{\gamma}$ is the capacity. As explained in Section 2.3 of [33], this corresponds to a HJ problem (hence HJ germ) for functions $f^1 := \frac{f^0}{1}$ and $f^k = \frac{f^0}{\theta^k}$. But monotonicities of the Godunov flux for HJ germs (with $\hat{f}^k = h(\uparrow, \downarrow, \dots, \downarrow) = h(p^1, p^{n+1}, \dots, p^{n+m})$) and for monotone germs (with $\partial_{p^{k'}}(\sigma^{k'} \hat{f}^k) \leq 0$ for $k' \neq k$, and $\partial_{p^k}(\sigma^k \hat{f}^k) \geq 0$) are indeed incompatible for $m \geq 2$.

14.4 Traffic lights germs revisited

We revisit the traffic light germs discovered in CARDALIAGUET, FORCADEL, MONNEAU [7] (see also TOWERS [36] for a special case), and give as a new result the explicit expression of their associated Godunov flux.

Consider a divergent junction $1 : 2$ with bell-shaped function $f = (f^0; f^1, f^2)$ in the sense of Definition 7.21 (with definitions of $f^{j,\pm}$), with ingoing flux f^0 and outgoing fluxes f^1, f^2 . We consider the following assumptions

$$(14.21) \quad \left\{ \begin{array}{ll} \bar{\lambda}^j \in [0, f_{max}^j] & \text{for } j = 0, 1, 2 \\ \bar{\lambda}^0 = \bar{\lambda}^1 + \bar{\lambda}^2 & \\ \text{the maps } \hat{\lambda}^k : [0, f_{max}^0] \rightarrow [0, \bar{\lambda}^k] \text{ are continuous nondecreasing} & \text{for } k = 1, 2 \\ \hat{\lambda}^k(0) = 0, \quad \hat{\lambda}^k(\bar{\lambda}^0) = \bar{\lambda}^k & \text{for } k = 1, 2 \\ \hat{\lambda}^1(\lambda) + \hat{\lambda}^2(\lambda) = \min(\lambda, \bar{\lambda}^0) & \text{for } \lambda \in [0, f_{max}^0] \end{array} \right.$$

For a divergent junction $1 : 2$, and for $\Lambda := (\bar{\lambda}, \hat{\lambda}) = (\bar{\lambda}^0, \bar{\lambda}^1, \bar{\lambda}^2, \hat{\lambda}^1, \hat{\lambda}^2)$, we consider the following Traffic Light germ, which is known to be a Kruřkov germ (see [7])

$$(14.22) \quad \mathcal{G}_\Lambda^{1:2} := \left\{ U = (u^0, u^1, u^2) \in \mathbb{R}^3, \left\{ \begin{array}{ll} a^j \leq u^j \leq b^j, & j = 0, 1, 2 \\ 0 \leq f^j(u^j) \leq \bar{\lambda}^j, & j = 0, 1, 2 \\ f^0(u^0) = f^1(u^1) + f^2(u^2) \\ f^{k,+}(u^k) \geq \hat{\lambda}^k(f^{0,+}(u^0)), & k = 1, 2 \end{array} \right. \right\}$$

Lemma 14.15 (Godunov flux of germ $\mathcal{G}_\Lambda^{1:2}$)

For $N = 3$, assume (2.2) for some $1 : 2$ junction (J, f) with $f = (f^0, f^1, f^2)$, with incoming branch denoted by J^0 ($\sigma^0 = 1$) and outgoing branches J^k ($\sigma^k = -1$) for indices $k = 1, 2$. Assume also that f is bell-shaped in the sense of Definition 7.21, and call $\bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N$ the capacity $\bar{\gamma}^j := f^{j,\sigma^j}$ given in Definition 7.21. We set $f_{max}^j := f^j(c^j)$ for $j = 0, 1, 2$ and assume (14.21) with $\bar{\lambda} = (\bar{\lambda}^0, \bar{\lambda}^1, \bar{\lambda}^2) \in [0, +\infty)^N$ and

$$T_{\bar{\lambda}} : [0, +\infty)^N \rightarrow [0, +\infty)^N, \quad \text{with } T_{\bar{\lambda}}(\gamma) = (\min\{\gamma^0, \bar{\lambda}^0\}, \min\{\gamma^1, \bar{\lambda}^1\}, \min\{\gamma^2, \bar{\lambda}^2\})$$

We define $\hat{\gamma}_0 = (\hat{\gamma}_0^0, \hat{\gamma}_0^1, \hat{\gamma}_0^2)$ for $\gamma = (\gamma^0, \gamma^1, \gamma^2) \in [0, +\infty)^N$

$$\left\{ \begin{array}{l} \hat{\gamma}_0^1(\gamma) = \min \left\{ \gamma^1, \max \left\{ \hat{\lambda}^1(\gamma^0), \gamma^0 - \gamma^2 \right\} \right\}, \\ \hat{\gamma}_0^2(\gamma) = \min \left\{ \gamma^2, \max \left\{ \hat{\lambda}^2(\gamma^0), \gamma^0 - \gamma^1 \right\} \right\}, \\ \hat{\gamma}_0^0 = \hat{\gamma}_0^1 + \hat{\gamma}_0^2 \end{array} \right.$$

Then

$$\mathcal{G} = \mathcal{G}_{\hat{f}} := \left\{ p \in [a, b], \quad \hat{f}(p) = f(p) \right\} \quad \text{with} \quad \hat{f} := \hat{\gamma} \circ \bar{\gamma} : [a, b] \rightarrow [0, +\infty)^N \quad \text{with} \quad \hat{\gamma} := \hat{\gamma}_0 \circ T_{\bar{\lambda}}$$

is a conservative Kruřkov germ $\mathcal{G} \subset [a, b]$ with respect to (J, f) , and $\hat{\gamma}$ is a preflux. Moreover we have $\mathcal{G} = \mathcal{G}_{\Lambda}^{1:2}$.

Proof of Lemma 14.15

Step 1: properties of $\hat{\gamma}$

Step 1.1: $\hat{\gamma}_0$ is a conservative preflux

We first notice that by construction, we have the function $\hat{\gamma}_0 : [0, +\infty)^N \rightarrow [0, +\infty)^N$ is continuous, and is conservative because $\hat{\gamma}_0^0 = \hat{\gamma}_0^1 + \hat{\gamma}_0^2$.

Step 1.2: preliminaries

We consider the following four sets (\hat{T} like triangle, \hat{Q} like sQuare)

$$\begin{cases} \hat{T}_0 & := \left\{ \gamma \in [0, +\infty)^N, \quad \gamma^1 + \gamma^2 \leq \gamma^0 \right\} \\ \hat{T}_1 & := \left\{ \gamma \in [0, +\infty)^N, \quad \gamma^1 + \gamma^2 \geq \gamma^0, \quad \gamma^1 \leq \hat{\lambda}^1(\gamma^0) \right\} \\ \hat{T}_2 & := \left\{ \gamma \in [0, +\infty)^N, \quad \gamma^1 + \gamma^2 \geq \gamma^0, \quad \gamma^2 \leq \hat{\lambda}^2(\gamma^0) \right\} \\ \hat{Q} & := \left\{ \gamma \in [0, +\infty)^N, \quad \gamma^k \geq \hat{\lambda}^k(\gamma^0), \quad k = 1, 2 \right\} \end{cases}$$

and we have (using the fact that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for $x \vee y := \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$)

$$(14.23) \quad (\hat{\gamma}_0^1, \hat{\gamma}_0^2) = \begin{cases} (\gamma^1, \gamma^2) & \text{if } \gamma \in \hat{T}_0 \\ (\gamma^1, \gamma^0 - \gamma^1) & \text{if } \gamma \in \hat{T}_1 \\ (\gamma^0 - \gamma^2, \gamma^2) & \text{if } \gamma \in \hat{T}_2 \\ (\hat{\lambda}^1(\gamma^0), \hat{\lambda}^2(\gamma^0)) & \text{if } \gamma \in \hat{Q} \quad \text{and} \quad \gamma^0 \leq \bar{\lambda}^0 \\ (\gamma^1, \gamma^2) & \text{if } \gamma \in \hat{Q} \quad \text{and} \quad \gamma^0 > \bar{\lambda}^0 \quad \text{and} \quad \gamma^1 + \gamma^2 \leq \gamma^0 \\ (\gamma^0 - \gamma^2, \gamma^0 - \gamma^1) & \text{if } \gamma \in \hat{Q} \quad \text{and} \quad \gamma^0 > \bar{\lambda}^0 \quad \text{and} \quad \gamma^1 + \gamma^2 \geq \gamma^0 \end{cases}$$

Step 1.3: bounds

By construction, we have $\hat{\gamma}_0 \geq 0$ and $\hat{\gamma}_0^k(\gamma) \leq \gamma^k$ for $k = 1, 2$. Moreover from (14.23), we get $\hat{\gamma}_0^0(\gamma) = \hat{\gamma}_0^1 + \hat{\gamma}_0^2 \leq \gamma^0$.

Step 1.4: local constancy

We see that $\hat{\gamma}_0$ is not locally constant on $\{\hat{\gamma}_0 \neq id_{[0, +\infty)^N}\}$ for $(\hat{\gamma}_0^1, \hat{\gamma}_0^2) = (\gamma^0 - \gamma^2, \gamma^0 - \gamma^1)$. On the contrary the restriction of $\hat{\gamma}_0$ to the box $K := [0, \bar{\lambda}]$ avoids this behaviour, and we see easily from the four first lines of (14.23), and from the possible combinations, that $(\hat{\gamma}_0)|_K$ is locally constant on $\{(\hat{\gamma}_0)|_K \neq id_K\}$.

Step 1.5: basic monotonicity

It is easy to check that $\gamma \mapsto \hat{\gamma}_0^j(\gamma)$ is nondecreasing in γ^j for $j = 0, 1, 2$.

Step 1.6: σ -monotonicity

In terms of $\gamma = (\gamma^0, \gamma^1, \gamma^2)$, we now want to check the σ -monotonicity of $\hat{\gamma}_0$, i.e. that $\sigma^j \hat{\gamma}_0^j$ is nonincreasing in $\sigma^k \gamma^k$ for $j \neq k$, i.e. the following monotonicities (with indeed $* = \uparrow$)

$$\begin{cases} \hat{\gamma}_0^0(*, \uparrow, \uparrow) \\ \hat{\gamma}_0^1(\uparrow, *, \downarrow) \\ \hat{\gamma}_0^2(\uparrow, \downarrow, *) \end{cases}$$

which is the case.

Step 1.7: conclusion

We now conclude by composition (similarly to the proof of Lemma 7.27) that $\hat{\gamma} = (\hat{\gamma}_0)|_K \circ T_{\bar{\lambda}}$ is a conservative σ -monotone preflux.

Step 2: properties of \hat{f}

From Proposition 7.23, we deduce that $\mathcal{G}_{\hat{f}}$ is a monotone conservative Riemann germ, hence a conservative Kruřkov germ (as follows from Theorem 2.24).

Step 3: identification of the germ

We now want to show that $\mathcal{G} := \mathcal{G}_{\hat{f}} = \mathcal{G}_{\Lambda}^{1:2}$. It is known from Lemma 1.5 in [7] that there is a generating set

$E_\Lambda \subset \mathcal{G}_\Lambda^{1:2}$ such that $E_\Lambda \subset \mathcal{G}$ implies $\mathcal{G} = \mathcal{G}_\Lambda^{1:2}$. Here the generating set is given by $E_\Lambda := \Gamma \cup \{P_1, P_2, P_3\}$, with

$$\begin{cases} \Gamma := \{P(\gamma^0), \quad \gamma^0 \in [0, \bar{\lambda}^0]\} & \text{with } P(\gamma^0) := (u_+^0(\gamma^0), u_+^1(\hat{\lambda}^1(\gamma^0)), u_+^2(\hat{\lambda}^2(\gamma^0))) \\ P_1 := (u_-^0(\bar{\lambda}^1), u_+^1(\bar{\lambda}^1), u_-(0)) \\ P_2 := (u_-^0(\bar{\lambda}^2), u_-^1(0), u_+^2(\bar{\lambda}^2)) \\ P_3 := (u_-^0(0), u_-^1(0), u_-^2(0)) \end{cases}$$

and

$$f^j(u_\pm^j(\lambda)) = \lambda, \quad u_\pm^j(\lambda) \in [a^j, c^j], \quad u_-^j(\lambda) \in [c^j, b^j], \quad \lambda \in [0, f^j(c^j)]$$

Step 3.1: checking Γ

Notice that $\bar{\gamma}(P(\gamma^0)) = (\gamma^0, f_{max}^1, f_{max}^2)$, and then for $\gamma^0 \in [0, \bar{\lambda}^0]$, we get $\hat{f}(P(\gamma^0)) = (\gamma^0, \hat{\lambda}^1(\gamma^0), \hat{\lambda}^2(\gamma^0)) = f(P(\gamma^0))$.

Step 3.2: checking P_1

We have $\bar{\gamma}(P_1) = (f_{max}^0, f_{max}^1, 0)$, and then $(T_K \circ \bar{\gamma})(P_1) = (\bar{\lambda}^0, \bar{\gamma}^1, 0)$. Therefore $\hat{f}(P_1) = (\bar{\lambda}^1, \bar{\lambda}^1, 0) = f(P_1)$.

Step 3.3: checking P_2

Symmetrically, we get $\hat{f}(P_2) = f(P_2)$.

Step 3.4: checking P_3

We have $\bar{\gamma}(P_3) = (f_{max}^0, 0, 0)$, and $(T_K \circ \bar{\gamma})(P_3) = (\bar{\lambda}^0, 0, 0)$. Therefore $\hat{f}(P_3) = (0, 0, 0) = f(P_3)$.

We conclude that $E_\Lambda \subset \mathcal{G}$, and then $\mathcal{G} = \mathcal{G}_\Lambda^{1:2}$. This ends the proof of the lemma.

Remark 14.16 (Convergent junctions 2 : 1)

Notice that for convergent junctions 2 : 1, the germ $\mathcal{G}_\Lambda^{2:1}$ has the same expression as $\mathcal{G}_\Lambda^{1:2}$ in (14.22) with only $f^{j,+}$ naturally changed in $f^{j,-}$ for $j = 0, 1, 2$. Similarly the expression of Godunov flux is still given by $\hat{f} = \hat{\gamma} \circ \bar{\gamma}$, where the preflux $\hat{\gamma} = \hat{\gamma}_0 \circ T_{\bar{\lambda}}$ is unchanged, but only the capacity $\bar{\gamma}^j = f^{j,\sigma^j}$ is naturally changed under the transformation $\sigma \rightarrow -\sigma$, for $\sigma = (\sigma^0, \sigma^1, \sigma^2)$.

15 Appendix of Part II

15.1 Tool box: some standard parabolic estimates

In this subsection, we recall some standard parabolic estimates. We consider the following problem for $T > 0$

$$(15.1) \quad \begin{cases} v_t - v_{xx} = f & \text{on } (0, T) \times I_R =: Q_{T,R}, \\ v = \varphi & \text{on } (0, T) \times \partial I_R \\ v = v_0 & \text{on } \{0\} \times \bar{I}_R \end{cases}$$

where for $R > 0$, we assume either that

$$I_R := (0, R) \quad \text{with} \quad \partial I_R = \{0, R\} \quad \text{and} \quad \bar{I}_R = [0, R]$$

or

$$I_R := \mathbb{R}/R\mathbb{Z} \quad \text{with} \quad \partial I_R = \emptyset \quad \text{and} \quad \bar{I}_R = I_R$$

In the following parabolic estimates, when it will be useful, we will make explicit the dependence of the constants in terms of T, R . We have the following results.

Lemma 15.1 (Calderon-Zygmund estimates)

Assume $T > 0$ and $I_R = \mathbb{R}/R\mathbb{Z}$ for some $R > 0$ and let $p \in (1, +\infty)$. Let v be a solution of (15.1) with $v_0 = 0$. Then we have

$$|v_t, v_{xx}|_{p, Q_{T,R}} \leq c |v_t - v_{xx}|_{p, Q_{T,R}} \quad \text{with} \quad |\cdot|_{p, Q_{T,R}} := |\cdot|_{L^p(Q_{T,R})}$$

for $c = c(p)$ independent on T, R .

Proof of Lemma 15.1

First notice that the solution v can be extended by periodicity to the whole line \mathbb{R} . We first rescale the problem from (T, R) to $(1, R/\sqrt{T})$. Then we can consider $\tilde{v}_n = v\phi_n$ with $\phi_n = 1$ on $[0, n]$ and $\phi_n = 0$ on $\mathbb{R} \setminus [-1, (n+1)]$. Applying standard Calderon-Zygmund estimate to \tilde{v}_n (see Theorem 9.1 in [34]), and

taking the limit $n \rightarrow +\infty$, we recover the result with some constant c only depending on p . Scaling back the problem to (T, R) , we get the result. This ends the proof of the lemma.

Lemma 15.2 (Precised Calderon-Zygmund estimates)

Assume $T > 0$ and $I_R = (0, R)$ for some $R > 0$ and let $p \in (1, +\infty)$. Let v be a solution of (15.1) with $v_0 = 0 = \varphi$. Then we have

$$(15.2) \quad \frac{|v|_{p, Q_{T,R}}}{T} + \frac{|v_x|_{p, Q_{T,R}}}{\sqrt{T}} + |v_t, v_{xx}|_{p, Q_{T,R}} \leq c |v_t - v_{xx}|_{p, Q_{T,R}}$$

for $c = c(p)$ independent on T, R .

Proof of Lemma 15.2

Up to extend v by antisymmetry $v(t, -x) = -v(t, x)$, and then by $2R$ -periodicity in x , we come back to problem covered by Lemma 15.1 with R replaced by $2R$. Now from Lemma 4.5 in [34] (on page 305), we have

$$|v|_{p, Q_{T,R}} \leq \tilde{c} T |v_t, v_{xx}|_{p, Q_{T,R}}$$

and

$$|v_x|_{p, Q_{T,R}} \leq \tilde{c} \sqrt{T} |v_t, v_{xx}|_{p, Q_{T,R}}$$

which implies (15.2) from Lemma 15.1. Notice that (15.2) also follows directly from the standard parabolic $W_{x,t}^{2,1;p}$ estimate, and by scaling in time. This ends the proof of the lemma.

We also have the following result.

Lemma 15.3 (Sobolev embedding in Hölder spaces)

Let $p > 3$ and $\alpha := 1 - 3/p \in (0, 1)$. For any $v \in W_{x,t}^{2,1;p}(Q_{T,R})$ with $I_R := (0, R)$. Then $v \in C_{x,t}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R})$ with $v_x \in C_{x,t}^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R})$. Moreover we have

$$(15.3) \quad \frac{|v_x|_{\infty, Q_{T,R}}}{T^{\frac{\alpha}{2}}} + \langle v_x \rangle_{Q_{T,R}}^{(\alpha)} \leq c \left\{ |v_t, v_{xx}|_{p, Q_{T,R}} + \frac{|v|_{p, Q_{T,R}}}{T} \right\} \quad \text{with} \quad \begin{cases} |\cdot|_{\infty, Q_{T,R}} := |\cdot|_{L^\infty(Q_{T,R})} \\ \langle \cdot \rangle_{Q_{T,R}}^{(\alpha)} := [\cdot]_{C_{x,t}^{\alpha, \frac{\alpha}{2}}(Q_{T,R})} \end{cases}$$

with $c = c(p)$ independent on $T, R > 0$, if T/R^2 is small enough. Here $\langle \cdot \rangle^{(\alpha)}$ is the standard parabolic Hölder semi-norm of parameter $\alpha \in (0, 1)$.

Proof of Lemma 15.3

Up to a scaling argument, we can change (T, R) into $(T', R') := (T/R^2, 1)$. Then Lemma 3.3 in [34] implies that there exists $c = c(p)$ such that

$$|v_x|_{\infty, Q_{T',R'}} \leq \frac{1}{2} c \left\{ \delta^\alpha |v_t, v_{xx}|_{p, Q_{T',R'}} + \delta^{\alpha-2} |v|_{p, Q_{T',R'}} \right\}$$

and

$$\langle v_x \rangle_{Q_{T',R'}}^{(\alpha)} \leq \frac{1}{2} c \left\{ |v_t, v_{xx}|_{p, Q_{T',R'}} + \delta^{-2} |v|_{p, Q_{T',R'}} \right\}$$

with $\delta := \min \left\{ \frac{1}{2}, \sqrt{T'} \right\}$, $R' = 1$ and $c = c(p)$ independent on T', R' . Now for T' small enough, we get

$$\frac{|v_x|_{\infty, Q_{T',R'}}}{T'^{\frac{\alpha}{2}}} + \langle v_x \rangle_{Q_{T',R'}}^{(\alpha)} \leq c \left\{ |v_t, v_{xx}|_{p, Q_{T',R'}} + \frac{|v|_{p, Q_{T',R'}}}{T'} \right\}$$

Then scaling back to (T, R) , this gives (15.3). This ends the proof of the lemma.

Corollary 15.4 (Sobolev parabolic estimates)

Assume $T > 0$ and $I_R = (0, R)$ for some $R > 0$ and let $p \in (3, +\infty)$ with $\alpha := 1 - 3/p \in (0, 1)$. Let v be a solution of (15.1) with $v_0 = 0 = \varphi$. Then, with notation of Lemma 15.3, we have

$$(15.4) \quad \frac{|v_x|_{p, Q_{T,R}}}{\sqrt{T}} + \frac{|v_x|_{\infty, Q_{T,R}}}{T^{\frac{\alpha}{2}}} + \langle v_x \rangle_{Q_{T,R}}^{(\alpha)} + |v_{xx}|_{p, Q_{T,R}} \leq c |v_t - v_{xx}|_{p, Q_{T,R}}$$

with $c = c(p)$ independent on $T, R > 0$, if T/R^2 is small enough.

Proof of Corollary 15.4

The result follows from Lemmata 15.3 and 15.2. This ends the proof of the corollary.

Lemma 15.5 (Schauder parabolic estimates)

Assume $T > 0$ and $I_R = (0, R)$ for some $R > 0$ and let $\alpha \in (0, 1)$. Let v be a solution of (15.1) with $v_0 = 0 = \varphi$. Assume that

$$(15.5) \quad f = 0 \quad \text{on} \quad \{0\} \times \partial I_R$$

If $f \in C_{x,t}^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R})$, then $v \in C_{x,t}^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q}_{T,R})$.

If $f \in C_{x,t}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R})$, then $v \in C_{x,t}^{3+\alpha, 1+\frac{1+\alpha}{2}}(\bar{Q}_{T,R})$.

Proof of Lemma 15.5

The result follows from Theorem 5.2 in [34] (on page 320). Notice that on the singular boundary $\Gamma_0 := \{0\} \times \partial I_R$, the two compatibility conditions (of order zero and of order one) are satisfied. Recall that the conditions are the following

$$\begin{cases} v_0 = \varphi & \text{on } \Gamma_0, & \text{(order zero)} \\ \varphi_t - (v_0)_{xx} = f & \text{on } \Gamma_0, & \text{(order one)} \end{cases}$$

The first condition is satisfied because $v_0 = 0 = \varphi$, and the second condition is also satisfied because of (15.5). This ends the proof of the lemma.

15.2 Existence of a PDE solution by fixed point

The goal of this subsection is to find a fixed point solution to a semi-linear problem (15.13) on

$$Q_{T,R} := (0, T) \times (0, R) \quad \text{for } T, R > 0.$$

As it is usual, we first start with a linear problem and insists on a priori estimates in short time, with sufficiently explicit dependence of the constants.

Precisely, we consider the following scalar linear problem

$$(15.6) \quad \begin{cases} u_t - u_{xx} = h_x & \text{on } (0, T) \times (0, R) \\ -u_x = g_0(t) & \text{on } (0, T) \times \{0\} \\ -u_x = g_R(t) & \text{on } (0, T) \times \{R\} \\ u = u_0 & \text{on } \{0\} \times (0, R) \end{cases}$$

with

$$(15.7) \quad \begin{cases} h = 0 & \text{on } (0, T) \times \{0\} \\ -u'_0(0) = g_0(0) \\ -u'_0(R) = g_R(0) \end{cases}$$

For problem (15.6), we will need several types of parabolic estimates on $Q_{T,R}$. To apply later on a fixed point method (with some contraction map say in $L^\infty(Q_{T,R})$), we will need estimates i) with explicit dependence in T , with constants small when T is small.

For later use, we will need to get controlled estimates as $R \rightarrow +\infty$. Hence for fixed $\rho > 0$, we will need local in space estimates on $Q_{T,\rho} := (0, T) \times (0, \rho)$ with constants depending only on T, ρ , but not on R .

For estimates ii) on the solution u in $C_{t,x}^{\frac{\alpha}{2}, \alpha}(\bar{Q}_{T,\rho})$ and on u_x in $L^p(Q_{T,\rho})$, we will take advantage of a priori bounds in L^∞ on the fixed point solution, and then of u, h, g_0, g_R .

For estimates iii) on the solution u in $W_{t,x}^{1,2;p}(Q_{T,\rho})$, we will take advantage of a priori L^p estimates on h_x .

Lemma 15.6 (A priori parabolic estimates)

Assume that u solves (15.6) with the first line of (15.7). Let $p \in (3, +\infty)$ and $\alpha := 1 - 3/p \in (0, 1)$. Assume that $u'_0 \in L^p(0, R)$.

i) (Global space estimates on $Q_{T,R}$)

Then there exists a constant $c = c(p)$ independent on T, R , such that for T/R^2 small enough, we have

$$(15.8) \quad \frac{|U|_p}{\sqrt{T}} + \frac{|U|_\infty}{T^{\frac{\alpha}{2}}} + \langle U \rangle_{Q_{T,R}}^{(\alpha)} + |U_x|_p \leq c|W|_p \quad \text{with} \quad \begin{cases} W := (u'_0, h, g_0, g_R, h(\cdot, R)) \\ |\cdot|_p := |\cdot|_{L^p(Q_{T,R})} \\ \langle \cdot \rangle_{Q_{T,R}}^{(\alpha)} := [\cdot]_{C_{x,t}^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R})} \end{cases}$$

and with

$$U := u - u_0 - R^{-1}G_R(t) \quad \text{where} \quad G'_R(t) := g_0(t) - g_R(t) + h(t, R) \quad \text{with} \quad G_R(0) = 0$$

ii) (Higher regularity)

Assume (15.7). Let $g := (g_0, g_R)$.

Moreover, if $u_0 \in C_x^{1+\alpha}([0, R])$ and $h, g \in C_{x,t}^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R})$, then $u \in C_{x,t}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R})$.

iii) (Further regularity)

Assume furthermore that $u_0 \in C_x^{2+\alpha}([0, R])$ and $h, g \in C_{x,t}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R})$; then $u \in C_{x,t}^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q}_{T,R})$.

Proof of Lemma 15.6

Step 0: preliminaries

We first symmetrize u in x and set (with the same notation) $u(t, x) := u(t, -x)$ for $x \in (-R, 0)$, and also symmetrize similarly $u_0(x) = u_0(-x)$. From the first line of (15.7), we have $h(t, 0) = 0$, and we can antisymmetrize h , setting $\check{h}(t, x) := -h(t, -x)$ for $x \in (-R, 0)$ such that $\check{h}_x(t, x) = (h_x)(t, -x)$. We also set $g_{-R} := -g_R$. Then u satisfies

$$\begin{cases} u_t - u_{xx} = 2g_0(t)\delta_0(x) + \check{h}_x & \text{on } (0, T) \times (-R, R) \\ -u_x = g_{\pm R}(t) & \text{on } (0, T) \times \{\pm R\} \\ u = u_0 & \text{on } \{0\} \times (-R, R) \end{cases}$$

Now we set $u = v_x$ and $u_0 = (v_0)_x$ with $v_0(0) = 0$, and setting $v_0(x) = -v_0(-x)$, we consider $v(t, x) = -v(t, -x)$ solution of

$$(15.9) \quad \begin{cases} v_t - v_{xx} = g_0(t)\text{sign}(x) + \check{h} & \text{on } (0, T) \times (-R, R) \\ v = 0 & \text{on } (0, T) \times \{0\} \\ v_t = g_0(t)\text{sign}(x) + \check{h} - g_{\pm R}(t) & \text{on } (0, T) \times \{\pm R\} \\ v = v_0 & \text{on } \{0\} \times (-R, R) \end{cases}$$

where the boundary condition in the third line comes from condition on $u_x = v_{xx}$, using the PDE satisfied by v up to the boundary (for instance in the case where all functions would be smooth, to fix the ideas). Setting (with $\check{h}(t, R) = h(t, R)$)

$$G'_R(t) := g_0(t) - g_R(t) + h(t, R) \quad \text{with} \quad G_R(0) = 0$$

we see that $\tilde{v}(t, x) := v(t, x) - G_R(t) \cdot R^{-1}x$ satisfies

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = \check{h} & \text{on } (0, T) \times (0, R) \\ \tilde{v} = 0 & \text{on } (0, T) \times \{0\} \\ \tilde{v} = c_0 & \text{on } (0, T) \times \{R\} \\ \tilde{v} = v_0 & \text{on } \{0\} \times (0, R) \end{cases}$$

with $\check{h}(t, x) := g_0(t) + h(t, x) - G'_R(t) \cdot R^{-1}x$ and the constant $c_0 := v_0(R)$.

Step 1: global space results i)

Then setting $\bar{v} := \tilde{v} - v_0$, we see that

$$\bar{v}(t, x) = v(t, x) - v_0(x) - G_R(t) \cdot R^{-1}x$$

solves

$$(15.10) \quad \begin{cases} \bar{v}_t - \bar{v}_{xx} = \bar{h} & \text{on } (0, T) \times (0, R) \\ \bar{v} = 0 & \text{on } (0, T) \times \{0\} \\ \bar{v} = 0 & \text{on } (0, T) \times \{R\} \\ \bar{v} = 0 & \text{on } \{0\} \times (0, R) \end{cases}$$

with $\bar{h} = \tilde{h} + v_0''$, i.e.

$$\bar{h}(t, x) = g_0(t) + h(t, x) - G'_R(t) \cdot R^{-1}x + u'_0(x).$$

From Corollary 15.4, we deduce that $U := \bar{v}_x = u(t, x) - u_0(x) - R^{-1}G_R(t)$ satisfies

$$(15.11) \quad \begin{aligned} \frac{|U|_p}{\sqrt{T}} + \frac{|U|_\infty}{T^{\frac{\alpha}{2}}} + \langle U \rangle^{(\alpha)} + |U_x|_p &\leq c|\bar{h}|_p \\ &\leq cK|W|_p \quad \text{with } W := (u'_0, h, g_0, g_R, h(\cdot, R)) \end{aligned}$$

where K is a numerical constant. This shows (15.8).

Step 2: complementary regularity results

Notice that for parabolic problem (15.10), compatibility conditions on the singular boundary $\{0\} \times \partial(0, R)$ are satisfied at order zero and order one. Indeed the condition at order zero is trivial, because it corresponds to the continuity of the solution \bar{v} on the singular boundary. The condition at order one means that $\bar{v}_t = \bar{v}_{xx} + \bar{h}$ with $\bar{v}_{xx} = 0 = \bar{v}_t$, i.e. $\bar{h} = 0$ on $\{0\} \times \partial(0, R)$. It is then easy to check that this is true when (15.7) holds true.

Step 2.1: higher regularity ii)

Hence, if $u_0 \in C^{1+\alpha}([0, R])$ and $h, g \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R})$, then using Schauder parabolic estimate (Lemma 15.5 ii)), we deduce that $\bar{v} \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q}_{T,R})$. Then we have $U = \bar{v}_x \in C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R})$. Because $U = u(t, x) - u_0(x) - R^{-1}G_R(t)$, we deduce that $u \in C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R})$.

Step 2.2: further regularity iii)

This step is similar to Step 2.1. If $u_0 \in C^{2+\alpha}([0, R])$ and $h, g \in C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R})$, then using Schauder parabolic estimate (Lemma 15.5 iii)), we deduce that $u \in C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q}_{T,R})$. This ends the proof of the lemma.

Given $w = (w^1, \dots, w^N)$, and functions $g = (g^1, \dots, g^N) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $h = (h^j)_{j=1, \dots, N}$ with $h^j : \mathbb{R} \rightarrow \mathbb{R}$, we now consider solutions $u = (u^1, \dots, u^N)$ to the system for $J_R := J \cap B_R$ and $J_R^* := J_R \setminus \{0\}$, with u^j defined on $[0, T) \times \bar{J}_R^j$ and $J_R^j := J^j \cap B_R \simeq -\sigma^j(0, R)$ for $j = 1, \dots, N$, and $\sigma^j \in \{\pm 1\}$.

$$(15.12) \quad \begin{cases} u_t - u_{xx} = (h(w) - h(w)(t, 0))_x & \text{on } (0, T) \times J_R^* \\ -u_x = g_0(w) & \text{on } (0, T) \times \{0\} \\ -u_x = g_R(w) & \text{on } (0, T) \times (J \cap \partial B_R) \\ u = u_0 & \text{on } \{0\} \times J_R^* \end{cases}$$

We also consider solutions to the fixed point problem $u \equiv w$ for $T = +\infty$

$$(15.13) \quad \begin{cases} u_t - u_{xx} = (h(u) - h(u)(t, 0))_x & \text{on } (0, +\infty) \times J_R^* \\ -u_x = g_0(u) & \text{on } (0, +\infty) \times \{0\} \\ -u_x = g_R(u) & \text{on } (0, +\infty) \times (J \cap \partial B_R) \\ u = u_0 & \text{on } \{0\} \times J_R^* \end{cases}$$

Consider also the following initial compatibility condition

$$(15.14) \quad \begin{cases} -(u_0)_x = g_0(u_0) & \text{on } \{0\} \times \{0\} \\ -(u_0)_x = g_R(u_0) & \text{on } \{0\} \times (J \cap \partial B_R) \end{cases}$$

Lemma 15.7 (Existence by fixed point)

Let $R > 0$ be fixed. Assume that $h, g_0, g_R : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are bounded functions which are globally Lipschitz continuous of Lipschitz constant $L_0 > 0$. Assume also that each component of the initial data satisfies $u_0^j \in C_x^{1+\alpha}(\bar{J}_R^j)$ for $j = 1, \dots, N$ and every $\alpha \in (0, 1)$.

i) (Existence)

Then there exists a unique global solution u with $u^j \in L_{loc}^\infty([0, +\infty); L^\infty(\bar{J}_R^j)) \cap C_{t,x}^{\frac{\alpha}{2}, \alpha}([0, +\infty) \times \bar{J}_R^j)$ of (15.13) for all $\alpha \in (0, 1)$ (where the boundary conditions have to be understood in a weak sense).

ii) (Higher regularity)

Moreover, if the initial compatibility condition (15.14) is satisfied, then $u^j \in C_{t,x}^{\frac{1+\alpha}{2}, 1+\alpha}([0, +\infty) \times \bar{J}_R^j)$ for any $\alpha \in (0, 1)$.

iii) (Further regularity)

Furthermore, if

$$\left\{ \begin{array}{l} h, g_0, g_R \in W_{loc}^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N) \\ u_0^j \in C_x^{2+\beta}(\bar{J}_R^j) \end{array} \right. \quad \text{for } j = 1, \dots, N, \quad \left| \quad \text{for some } \beta \in (0, 1) \right.$$

then $u^j \in C_{t,x}^{\frac{2+\beta}{2}, 2+\beta}([0, +\infty) \times \bar{J}_R^j)$.

Proof of Lemma 15.7

Step 1: contraction for short time

Because u_0 is in particular Lipschitz continuous, we have

$$(15.15) \quad u'_0 \in L^p(J_R)$$

and then $u'_0 \in L^p(Q_{T,R})$ with $Q_{T,R} := (0, T) \times J_R$. Let u_a (resp. u_b) be the solution to system (15.12) for $w = w_a$ (resp. $w = w_b$). We set $\bar{u} := u_a - u_b$, and $\bar{w} := w_a - w_b$ and

$$|u|_p := |u|_{L^p(Q_{T,R})} := \sum_{j=1}^N |u^j|_{L^p(Q_{T,R}^j)} \quad \text{with } Q_{T,R}^j := (0, T) \times J_R^j$$

From (15.8) applied to \bar{u} , we get in particular for $(\bar{w}(t, \sigma R))^j := \bar{w}^j(t, \sigma^j R)$ (by linearity on the initial data which is zero for \bar{u}) the following bound on a quantity \bar{U}

$$\begin{aligned} \frac{|\bar{U}|_\infty}{T^{\frac{\alpha}{2}}} &\leq c' L_0 \{ |\bar{w}|_p + |\bar{w}(\cdot, 0)|_p + |\bar{w}(\cdot, \sigma R)|_p \} \\ &\leq 3c' L_0 N (TR)^{\frac{1}{p}} |\bar{w}|_\infty \end{aligned}$$

where

$$\bar{U} := \bar{u} - R^{-1} \bar{G}_R(t), \quad \bar{G}_R(0) = 0$$

with

$$\bar{G}'_R(t) = F[w_a](t) - F[w_b](t) \quad \text{and} \quad F[w_a](t) := g_0(w_a(t, 0)) - g_R(w_a(t, \sigma R)) + h(w_a(t, \sigma R)) - h(w_a(t, 0)).$$

Hence

$$|R^{-1} \bar{G}_R|_\infty \leq 4L_0 TR^{-1} |\bar{w}|_\infty$$

and then

$$|\bar{u}|_\infty \leq \beta |\bar{w}|_\infty \quad \text{for some } \beta \in (0, 1)$$

for any $T \in (0, T_0]$ for some $T_0 > 0$ small enough (depending on R). This shows that the map $L^\infty(Q_{T,R}) \ni w \mapsto \Phi(w) := u \in L^\infty(Q_{T,R})$ is a contraction. Hence this map has a unique fixed point u , which is then solution of (15.12). Moreover we have $u = u_0 + \{\Phi(u_0 - u_0)\} + \{\Phi^2(u_0) - \Phi(u_0)\} + \dots$, and then

$$(15.16) \quad |u|_\infty \leq |u_0|_\infty + \frac{1}{1-\beta} |\Phi(u_0) - u_0|_\infty$$

Notice that T_0 (and also β) are independent on the Lipschitz norm of the initial data u_0 , nor on the $L^\infty(Q_{T,R})$ norm of the fixed point solution u .

Step 2: how to restart the time interval

Now if we want to restart the problem from a time $t_1 \in (T/2, T)$, as in (15.15), we then need to insure that $u_x(t_1, \cdot) \in L^p(J_R^*)$. Indeed, estimate (15.8) shows that

$$(15.17) \quad \frac{|U|_p}{\sqrt{T}} + \frac{|U|_\infty}{T^{\frac{\alpha}{2}}} + [U]_{C_{x,t}^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R})} + |U_x|_p \leq c'' |W|_p \quad \text{with} \quad W := (u'_0, h(u), h(u)(\cdot, 0), g_0(u)(\cdot, 0), g_R(u)(\cdot, R), h(u)(\cdot, R))$$

with

$$U := u - u_0 - R^{-1}G_R(t) \quad \text{with} \quad G'_R(t) = F[u](t) \quad \text{and} \quad G_R(0) = 0$$

and then $U_x = u_x - u'_0$. From (15.16), we know that $u \in L^\infty(Q_{T,R})$, and then the RHS of (15.17) is finite, and this implies the existence of some time t_1 such that $u_x(t_1, \cdot) \in L^p(J_R^*)$.

We can now iterate the procedure, and get the existence of a global solution u with $u^j \in L^\infty_{loc}([0, +\infty); L^\infty(\bar{J}_R^j))$. Moreover we see that $u^j \in C_{t,x}^{\frac{\alpha}{2}, \alpha}([0, +\infty) \times \bar{J}_R^j)$ for all $\alpha \in (0, 1)$.

Step 3: complementar regularity results

Step 3.1: proof of ii)

Moreover, notice that (15.17) implies that $u^j \in C_{x,t}^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R}^j)$. Then, using the fact that h, g_0, g_R are globally Lipschitz, we deduce that $h^j(u), g_0^j(u)(\cdot, 0), g_R^j(u)(\cdot, \sigma^j R) \in C_{x,t}^{\alpha, \frac{\alpha}{2}}(\bar{Q}_{T,R}^j)$. If u_0 satisfies relations (15.14), then Lemma 15.6 implies that $u^j \in C_{x,t}^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_{T,R}^j)$. In particular, relations (15.14) propagate in time. Precisely, this means that we can replace u_0 by $u(t_1, \cdot)$ in relations (15.14). Therefore regularity also propagates and we get that $u^j \in C_{x,t}^{1+\alpha, \frac{1+\alpha}{2}}([0, +\infty) \times \bar{J}_R^j)$.

Step 3.2: proof of iii)

The result follows similarly to Step 3.1. Notice in particular that $h \in W_{loc}^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ and $u \in C_{x,t}^{1+\beta, \frac{1+\beta}{2}}$ imply $h(u) \in C_{x,t}^{1+\beta, \frac{1+\beta}{2}}$ for $\beta \in (0, 1)$. We get similar regularity for $g_0(u)$ and $g_R(u)$. Therefore further regularity in iii) of Lemma 15.6 implies that $u^j \in C_{x,t}^{2+\alpha, \frac{2+\alpha}{2}}([0, +\infty) \times \bar{J}_R^j)$. This ends the proof of the lemma.

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