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Thermodynamics and maximal entropy of a monomer-dimer system on a ladder geometry

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Abstract. We consider a mixture of dimers and monomers, or defects, on a ladder lattice, with different binding energies, and study the thermodynamics of this system in the canonical ensemble, at a fixed density of monomers. We use Grassmannian techniques in order to construct an effective fermionic action, and evaluate exactly the entropy, specific heat, and correlation functions between monomers and pairs of monomers. We show in particular that, at low temperature, the residual entropy is maximal and equal to the full dimer configurational entropy for a density at which the chemical potential of the monomers is quasi independent of the temperature. The correlation functions present two correlation lengths that explain their oscillatory behavior. The characteristics of the entropy and specific heat are compared with dilute spin-ice systems and discussed.

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1. Introduction

Statistical models of dimers on square lattices or regular graphs are well known and representative of combinatorial problems that have exact solutions using independent algebraic methods. The different ways of arranging dimers in fully packed configurations are called perfect matching, and this results in a macroscopic number of possibilities, exponentially proportional to the number of sites, more commonly on planar lattices.

Originally, these models came from the interest in the thermodynamics related to the adsorption of gaseous diatomic molecules on crystalline surfaces after condensation [1], in the case of athermal mixtures where there is no energy involved in the mixing. Earlier experiments were performed on the adsorption of hydrogen and oxygen molecules on tungsten surfaces [2]. Those have also an interest in the quantum models of resonating valence bond wavefunctions on bipartite lattices [3]. The close packing problem, when all the lattice surface is covered by dimers, has to take into account the non-isotropic geometry of the molecules and responsible for the macroscopic configurational entropy. The ground state is therefore highly degenerate and the computation of all the configurations is a non-trivial mathematical problem related to the constraints imposed by

ice rules and their derivatives, in the sense that a single site can not share more than one dimer. For the full packing problem on the square, it is known since long ago that the configurational entropy per site, or the logarithm of the configurational partition function divided by the number of sites or surface area, after summation over all the possible close-packed arrangements, is equal to $G/\pi \simeq 0.2916$, where G is the Catalan constant [4, 5].

The usual techniques to solve these combinatorial problems, on bipartite or non-bipartite lattices, are closely related to fermionic methods, which involve the use of Pfaffians of skew symmetric matrices [6, 7, 5, 8, 9]. For example, the problem of finding the number of closed-packing arrangements on a typical square lattice can be mapped onto a problem of finding the Pfaffian of some adjacency matrix K , which is called Kasteleyn matrix, where, for simplification, the sign of the elements $K_{ij} = \pm 1$ is given by the orientation of the edges (ij) on the lattice, which constitutes an oriented graph, and solution of the Kasteleyn construction [5, 8]. This construction constraints the product of the K_{ij} s on each unit square to be equal to minus one. This trick is performed to compensate for the extra permutation sign in the Pfaffian formula, since each valid configuration must have a positive combinatorial weight. Furthermore, finite size effects on a square lattice were studied in the context of conformal field theory [10].

From these techniques, it is natural to look at a more sophisticated case where some sites are not covered by dimers, but rather by monomers, which physically represent defects or monoatomic particles mixed with dimers. Some possible applications concern mixtures of benzene-biphenyl [11], where the biphenyl molecules occupy twice the volume of the benzene molecules.

The number of monomers should be even in order for the remaining sites, not occupied by monomers, to be packed with dimers, when it is possible. This can also be viewed as removing an integer number of dimers, leaving an even number of empty sites. The Kasteleyn construction is still valid if we remove the edges joining the vertices occupied by monomers and reorient the remaining edges correctly following the Kasteleyn prescription. The modified matrix, build by deleting the lines and columns associated with the monomer sites, is then treated using perturbative methods in order to find, in the simplest case, the correlation function between one pair of monomers, see, for example, [12, 13]. For the case of only two monomers located on a diagonal line, it was shown that the correlation function, or more specifically the relative number of valid configurations, decays like the inverse square root of their distance, whereas the correlation between two pairs of dimers decreases like the inverse square of their distance. The correlation function in these particular cases can be expressed exactly using Toeplitz determinants [12, 9]. An extension in terms of closed loops, or general dimer model including monomers representing the weight of vertices, has been proposed [14], using properties and a generalization of the Kasteleyn construction. This provides exact results on rectangular lattice, such as the entropy as a function of the monomer density.

Exact combinatorial results were given for a fixed concentration of dimers, or dumbbells, on a one-dimensional array [16], and for a ladder geometry [17] using a recursive technique. Results in the case of parallel dimers on the ladder geometry [18] were further generalized on a square

lattice [19]. Interestingly, it was shown that the maximum number of arrangements of dimers on a ladder is reached for a dimer coverage approximately equal to 60.65% [17].

In more than two dimensions, polynomial time approximation algorithms were developed to obtain the number of valid configurations for a fixed density of monomers [15].

The question of solving the monomer-dimer partition function for a given configuration of monomers on a quadratic lattice was addressed previously, see, for example, [20, 21]. The solution can be expressed using Grassmann variables, and the resulting partition function is factorized into two Pfaffians [20]. It was indeed demonstrated that the corresponding effective Grassmannian action is still quadratic if we use extra local fermionic magnetic field variables coupled to each monomer, at the price of additional sums of quadratic Grassmannian terms in the effective action, running from each monomer position to the boundary of the lattice, in the case of open boundaries, or to another arbitrary monomer when periodic conditions are applied. These terms are necessarily present because of a local change in the sign of the coupling constants along a path connecting monomers. This is reminiscent of the disorder operators introduced for the Ising model, whose action is to change the sign of the ferromagnetic coupling constants along a similar path [22, 23].

In the case of boundary monomers on the square lattice, the partition function was calculated using the modified Kasteleyn matrix [24], in the scaling limit. It was also shown that, for $2n$ consecutive monomers, the relative partition function decreases like $n^{-1/4}e^{-2nG/\pi}$ for n large.

It has also been proved that the free energy of a general monomer-dimer system, in absence of physical interaction between dimers or monomers, is analytic in monomer density and temperature, and, as a consequence, there is no phase transition [25, 26]. Phase transitions were subsequently studied in the presence of short range attractive interactions [27, 28] between dimers, leading to nematic transitions. However, it is hard to obtain exact results in this case.

In the present manuscript, we consider a model of thermal dimers on a ladder, with a fixed concentration of monomers controlled by two chemical potentials, one for each branch, and the dimers are characterized by two different binding energies in the horizontal and vertical directions. In the absence of monomers, this model displays a finite entropy per site at zero temperature due to the macroscopic number of dimer configurations. The value of this entropy, for equal energy weights, is slightly less than the two-dimensional entropy of the full packing problem on a square lattice. We do not consider other possible geometries where the entropy can take a different value. One of our motivations is then to study the general thermodynamics of this mixture, and compute the exact correlation functions between monomers in order to obtain, in particular, the correlation lengths as functions of the relative values of the binding energies and temperature. We also explain the oscillating behavior of the correlations due to the competition between two correlation lengths. When the binding energies are equal, the entropy is constant for all temperatures, which leads to a vanishing specific heat. The value of this entropy in this case will therefore depend only on the concentration of monomers. One question that we address in this manuscript is how to estimate the concentration for which the entropy is maximal. As we shall see, the addition of a small amount of monomers will increase the degree of disorder, whereas a high concentration of

monomers will decrease the entropy to zero.

In order to analyze the model, we introduce Grassmann techniques that are applied to deal with the strong constraint on every site. The problem is then rewritten in terms of Pfaffians with an effective fermionic action, as presented previously in [20]. This formulation allows us to study in details the thermodynamics of a gas of monomers.

The manuscript is organized as follows: In part 2, we introduce the model on a lattice with a ladder geometry, and evaluate the exact partition function for an arbitrary configuration of monomers on the ladder. We then give numerical examples of relative partition functions for a set of two or four monomers, which are characterized by oscillations as functions of their distance, and determine the numerical values of the decay exponent accurately using numerical methods. In part 3, we introduce the general partition function in the grand canonical ensemble after summing over all the possible monomer configurations using the previous results by introducing a chemical potential that controls their number on each branch of the ladder. From this, we evaluate exactly in part 4 the free energy in the canonical ensemble after a Legendre transformation, entropy and specific heat, as well as the exact correlation functions, when all the monomers are located on one branch of the ladder, which is the integrable case. In view of these results, an analogy is proposed with spin-ice systems. In part 5, we give the exact expressions of the relative partition functions that were found numerically in part 2. Finally, in the last part 6, we discuss how the insertion of one monomer in one of the two branches of the ladder affects the value of the partition function in the configuration discussed in part 4.

2. Dimer model with monomers

We consider a quasi-one-dimensional or ladder lattice of size L , which can be odd or even, with periodic conditions. Each site either shares a dimer with another neighboring site, or is occupied by a defect, more precisely a monomer. For example, one typical configuration with two monomers or defects is displayed in figure 1. Not all configurations of monomers are allowed because of the constraints imposed by the dimers. We would like in this section to investigate the partition function for a given set of monomers located at fixed locations on the lattice. One of the technical aspect of studying dimers on a lattice is to introduce nilpotent variables that can be then treated using Grassmann integrals. The nilpotent variables are useful as they prevent more than one attachment on a given site by another dimer. This is very similar to the mapping of two-state spin models into Grassmannian representations, using anticommuting nilpotent variables. The advantage of using anticommuting Grassmann variables is the possibility to evaluate the partition function of some non-trivial model, which is reduced to the computation of a Grassmann Gaussian integral or Pfaffian. The Pfaffian is the square root of the determinant associated with the matrix of the model Hamiltonian, in contrast to Gaussian integrals of commuting nilpotent variables representing permanents of a similar matrix but which cannot be expressed as a determinant, and therefore are difficult to compute efficiently.

We first consider the partition function of a dimer system on a ladder, in the absence of

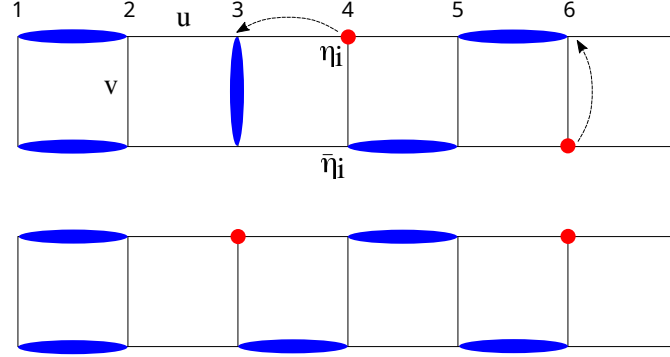


Figure 1: Dimer configurations with two monomers located on a ladder of size $L = 6$, with periodic conditions. Two allowed configurations are represented. The dimer configurations are described by nilpotent commuting variables η_i and $\bar{\eta}_i$ on each site of the upper and lower branch respectively, see the text for details. Each horizontal and vertical dimer has a Boltzmann weight equal to u and v respectively.

monomers, by a multiple integral over commuting nilpotent variables η_i and $\bar{\eta}_i$ [29, 30]

$$Z_0 = \int \prod_{i=1}^L d\bar{\eta}_i d\eta_i (1 + u\eta_i \eta_{i+1})(1 + u\bar{\eta}_i \bar{\eta}_{i+1})(1 + v\eta_i \bar{\eta}_i), \quad (1)$$

with $\eta_i^2 = \bar{\eta}_i^2 = 0$, and where u and v are the Boltzmann weights of the dimer bonds in the horizontal and vertical directions, respectively. This expression counts all the possible valid dimer configurations that exclude more than one dimer on a given site. We introduce energy parameters that represent the binding energies for each dimer, ϵ_u for a horizontal dimer and ϵ_v for a vertical dimer. In this case we have the Boltzmann weights $u = e^{-\beta\epsilon_u}$ and $v = e^{-\beta\epsilon_v}$, where β is the inverse of the temperature $k_B T$. Z_0 is therefore the sum of all the possible fully packed configurations of a dimer system with their corresponding weight. We remind the following rules $\int d\eta_i = 0$, and $\int d\eta_i \eta_i = 1$. We then introduce three pairs of Grassmann variables (a_i, \bar{a}_i) , (b_i, \bar{b}_i) , and (c_i, \bar{c}_i) such that

$$\begin{aligned} Z_0 &= \int \prod_{i=1}^L d\bar{\eta}_i d\eta_i d\bar{a}_i da_i d\bar{b}_i db_i d\bar{c}_i dc_i e^{a_i \bar{a}_i + b_i \bar{b}_i + c_i \bar{c}_i} \times \\ &\quad \left[(1 + a_i \eta_i)(1 + u \bar{a}_i \eta_{i+1}) \right] \left[(1 + b_i \bar{\eta}_i)(1 + u \bar{b}_i \bar{\eta}_{i+1}) \right] \left[(1 + c_i \eta_i)(1 + v \bar{c}_i \bar{\eta}_i) \right] \\ &= \int \prod_{i=1}^L d\bar{\eta}_i d\eta_i d\bar{a}_i da_i d\bar{b}_i db_i d\bar{c}_i dc_i e^{a_i \bar{a}_i + b_i \bar{b}_i + c_i \bar{c}_i} \left[A_i \bar{A}_{i+1} \right] \left[B_i \bar{B}_{i+1} \right] \left[C_i \bar{C}_i \right]. \end{aligned} \quad (2)$$

Two pairs correspond to the horizontal bonds for each branch and one pair to the transverse or vertical bonds. Each term in brackets $\left[\dots \right]$ can commute with all the other terms of the product

under the integral symbol. Therefore, we can reorder the product in order to isolate and integrate over the nilpotent variables η_i and $\bar{\eta}_i$, such that

$$\left[A_{i-1}\bar{A}_i\right]\left[B_{i-1}\bar{B}_i\right]\left[A_i\bar{A}_{i+1}\right]\left[B_i\bar{B}_{i+1}\right] \rightarrow \left[B_{i-1}\underbrace{\left[A_{i-1}\bar{A}_i\right]\left[B_i\bar{B}_i\right]}_{\eta_i, \bar{\eta}_i}\right]\left[B_i\left[A_i\bar{A}_{i+1}\right]\bar{B}_{i+1}\right]. \quad (3)$$

We chose to integrate over the $\bar{\eta}_i$ variables first, and this is the reason why we isolate the terms B_i and \bar{B}_i at the center of the product. Another choice would have been to integrate over η_i first, and isolate the quantities A_i and \bar{A}_i at the center of the same product. The other quantity $\left[C_i\bar{C}_i\right]$ is inserted adequately. The two orderings are nevertheless equivalent. The partition function can be rewritten with this reordering as

$$Z_0 = \int \prod_{i=1}^L d\bar{\eta}_i d\eta_i d\bar{a}_i da_i d\bar{b}_i db_i d\bar{c}_i dc_i e^{a_i\bar{a}_i + b_i\bar{b}_i + c_i\bar{c}_i} \\ \times B_1 A_1 \left(\prod_{i=2}^L \bar{A}_i \left[C_i \bar{C}_i \right] \bar{B}_i B_i A_i \right) \bar{A}_1 \left[C_1 \bar{C}_1 \right] \bar{B}_1. \quad (4)$$

After integrating over $(\eta_i, \bar{\eta}_i)$ we obtain, except for the boundary term $i = 1$

$$\int d\eta_i \bar{A}_i C_i \left(\int d\eta_i \bar{C}_i \bar{B}_i B_i \right) A_i = (-a_i + u\bar{a}_{i-1} + c_i)(b_i + u\bar{b}_{i-1} + v\bar{c}_i). \quad (5)$$

The product of these linear Grassmann variables can be exponentiated using a new set of variables (d_i, \bar{d}_i) and the integration over $(a_i, \bar{a}_i, b_i, \bar{b}_i, c_i, \bar{c}_i)$ leads to a quadratic action in (d_i, \bar{d}_i)

$$Z_0 = \int \prod_{i=1}^L d\bar{a}_i da_i d\bar{b}_i db_i d\bar{c}_i dc_i d\bar{d}_i dd_i \exp \left(a_i \bar{a}_i + b_i \bar{b}_i + c_i \bar{c}_i \right. \\ \left. + d_i [-a_i + u\bar{a}_{i-1} + c_i] + [b_i + u\bar{b}_{i-1} + v\bar{c}_i] \bar{d}_i \right) \\ = \int \prod_{i=1}^L d\bar{d}_i dd_i \exp (u d_i d_{i+1} - u \bar{d}_i \bar{d}_{i+1} + v d_i \bar{d}_i) = \int \prod_{i=1}^L d\bar{d}_i dd_i \exp (S_0). \quad (6)$$

The periodic conditions at the boundary imply antiperiodic conditions for the Grassmann variables $\bar{a}_0 = -\bar{a}_L$, $\bar{b}_0 = -\bar{b}_L$, and consequently $d_0 = -d_L$ and $\bar{d}_0 = -\bar{d}_L$. This comes from the fact that in equation (4) the boundary term is $B_1 A_1 \bar{A}_1 [C_1 \bar{C}_1] \bar{B}_1$ instead of $\bar{A}_1 [C_1 \bar{C}_1] \bar{B}_1 B_1 A_1$. We can then use Fourier transforms $d_j = L^{-1/2} \sum_k e^{2i\pi(k+1/2)\frac{j}{L}} d_{k+\frac{1}{2}}$ and $\bar{d}_j = L^{-1/2} \sum_k e^{-2i\pi(k+1/2)\frac{j}{L}} \bar{d}_{k+\frac{1}{2}}$ to rewrite the action S_0 as

$$S_0 = \sum_k' \left[2iu \sin \left(\frac{2\pi}{L} (k + \frac{1}{2}) \right) d_{-k-\frac{1}{2}} d_{k+\frac{1}{2}} + 2iu \sin \left(\frac{2\pi}{N} (k + \frac{1}{2}) \right) \bar{d}_{-k-\frac{1}{2}} \bar{d}_{k+\frac{1}{2}} \right. \\ \left. + v d_{k+\frac{1}{2}} \bar{d}_{k+\frac{1}{2}} + v d_{-k-\frac{1}{2}} \bar{d}_{-k-\frac{1}{2}} \right]. \quad (7)$$

For L even or odd, the prime symbol in the sum corresponds to the set of momenta k located in half of the Brillouin zone with the opposite momenta $-k - \frac{1}{2}$, or equivalently $L - k - \frac{1}{2}$, completing

the zone. The partition function is therefore equal to the product over the momenta k of only four integrals with the measures $\int dd_{k+\frac{1}{2}} d\bar{d}_{k+\frac{1}{2}} dd_{-k-\frac{1}{2}} d\bar{d}_{-k-\frac{1}{2}}$ in half the Brillouin zone, which yields

$$Z_0 = \prod_{\omega} [v^2 + 4u^2 \sin^2(\omega)]^{1/2}, \quad \omega = \frac{2\pi}{L}(k + \frac{1}{2}), \quad k = 1, \dots, L. \quad (8)$$

If $u = v = 1$, Z_0 corresponds to the number of all possible dimer tilings; for example $(L, Z_0) = (1, 1), (2, 5), (3, 4), (4, 9), (5, 11)$ for the first terms. The previous formula has an explicit expression, since the product of trigonometric functions can be simplified using the product formula $\prod_{\omega} (x - e^{i\omega}) = x^L + 1$, which leads to the factorization

$$Z_0 = u^L \sqrt{\left(x_-^{L/2} + 1\right) \left(x_-^{L/2}(-1)^L + 1\right) \left(x_+^{L/2} + 1\right) \left(x_+^{L/2} + (-1)^L\right)},$$

$$x_{\pm} = \frac{v^2 + 2u^2 \pm v\sqrt{v^2 + 4u^2}}{2u^2}. \quad (9)$$

From this result, we can define a configurational entropy per site, $S_e = (2L)^{-1} \log Z_0$, in the asymptotic limit $L \rightarrow \infty$. Since $x_+ x_- = 1$, only the root x_+ is larger than unity, and therefore

$$\lim_{L \rightarrow \infty} S_e = \frac{1}{2} \log(u) + \frac{1}{4} \log(x_+). \quad (10)$$

In particular, when $u = v = 1$, we have $S_e = 4^{-1} \log((3 + \sqrt{5})/2) \simeq 0.2406$, which is slightly lower than the value for the square lattice, or $S_e \simeq 0.2916$ [4]. We already obtain a good approximation for $L = 5$, since in this case $Z_0 = 11$, and $S_e \simeq \log(11)/10 \simeq 0.2398$. If the parameters u and v depend on the temperature T , the previous quantity S_e is not exactly the physical entropy. Normally the real entropy is defined as minus the derivative of the free energy with respect to the temperature, or $F = -k_B T \log Z_0$. In the case where $u = e^{-\beta \epsilon_u}$ and $v = e^{-\beta \epsilon_v}$, then we should consider additional terms that are temperature-dependent. It is clear that if both energies are strictly different, or $u \neq v$, then when $\beta \rightarrow \infty$ only configurations with dimers all horizontal or all vertical are the ground states, depending on the relative values of ϵ_u and ϵ_v . The entropy per site is therefore zero in the thermodynamical limit. It is non-zero at zero temperature when $\epsilon_u = \epsilon_v$.

Let us now consider the case where a set of monomers is inserted at arbitrary locations. The insertion of monomers or defects at given abscissa $\{i_0\}$ (upper branch) or $\{i_1\}$ (lower branch) [20] is performed by introducing additional nilpotent variables η_{i_0} and $\bar{\eta}_{i_1}$ directly in the partition function, which prevents any dimer attachment

$$Z[\{i_0\}, \{i_1\}] = \int \prod_{i=1}^L d\eta_i d\bar{\eta}_i (1 + u\eta_i \eta_{i+1})(1 + w\bar{\eta}_i \bar{\eta}_{i+1})(1 + v\eta_i \bar{\eta}_i) \prod_{i_0, i_1} \eta_{i_0} \bar{\eta}_{i_1}. \quad (11)$$

These insertions can be furthermore exponentiated by introducing Grassmann fields h_{i_0} and \bar{h}_{i_1} such that $\eta_{i_0} = \int dh_{i_0} e^{h_{i_0} \eta_{i_0}}$ and $\bar{\eta}_{i_1} = \int d\bar{h}_{i_1} e^{\bar{h}_{i_1} \bar{\eta}_{i_1}}$. For example, one insertion at the position i on the lower branch will modify the integrals (5), yielding

$$\int d\eta_i \bar{A}_i C_i \left[\int d\bar{\eta}_i \int d\bar{h}_i e^{\bar{h}_i \bar{\eta}_i} \bar{C}_i \bar{B}_i B_i \right] A_i$$

$$\begin{aligned}
 &= \int d\eta_i \bar{A}_i C_i \left[\int d\bar{h}_i (\bar{h}_i + b_i + u\bar{b}_{i-1} + v\bar{c}_i) \right] A_i \\
 &= (a_i + u\bar{a}_{i-1} + c_i) \int d\bar{h}_i (\bar{h}_i + b_i + u\bar{b}_{i-1} + v\bar{c}_i).
 \end{aligned} \tag{12}$$

The difference with expression (5) is the addition of Grassmann field \bar{h}_i in the second linear term as well as a change of sign of the term a_i . This is due to the fact that the integral over \bar{h}_i commutes with the last quantity A_i instead of anticommuting as previously with the term a_i . On the contrary, the insertion of the other field h_i does not change any sign in the expressions, it only adds this field into the first linear term. The presence of a sign induces naturally a "line" of disorder, which starts from the insertion sites and reaches the upper branch by changing the sign of some coupling terms on its path [20, 31]. The line materializes by adding correcting terms in the action. The orientation of this line depends on the choice in ordering the operators $(A_i, \bar{A}_i, B_i, \bar{B}_i)$ in equation (3). Since the insertions corresponding to h_i are already on the upper boundary, there is no line of defects associated with them. We finally obtain, up to a global sign

$$\begin{aligned}
 Z[\{i_0\}, \{i_1\}] &= \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \prod_{i_1} d\bar{h}_{i_1} \prod_{i_0} d h_{i_0} \right) \exp \left(\sum_{i=1}^N [u d_i d_{i+1} - u \bar{d}_i \bar{d}_{i+1} + v d_i \bar{d}_i] \right. \\
 &\quad \left. + \sum_{i_0} d_{i_0} h_{i_0} + \sum_{i_1} [\bar{h}_{i_1} \bar{d}_{i_1} - 2u d_{i_1} d_{i_1+1}] \right).
 \end{aligned} \tag{13}$$

In this expression, the disorder term associated to \bar{h}_{i_1} is $-2u d_{i_1} d_{i_1+1}$, which corrects the sign of $u d_{i_1} d_{i_1+1}$ in the original action. We can introduce local occupation numbers $n_i = 0, 1$ and $\bar{n}_i = 0, 1$ for the upper and lower branches, respectively, so that

$$\begin{aligned}
 Z[\{n_i, \bar{n}_i\}] &= \int \left(\prod_{i=1}^L d\bar{d}_i d d_i d\bar{h}_i e^{(1-\bar{n}_i)\bar{h}_i} d h_i e^{(1-n_i)h_i} \right) \\
 &\quad \times \exp \left(\sum_i [u d_i d_{i+1} - u \bar{d}_i \bar{d}_{i+1} + v d_i \bar{d}_i] \right. \\
 &\quad \left. + \sum_i d_i h_i n_i + \sum_i [\bar{d}_i \bar{h}_i - 2u d_i d_{i+1}] \bar{n}_i \right),
 \end{aligned} \tag{14}$$

where we have performed in (13) the transformation $\bar{h}_i \rightarrow -\bar{h}_i$ for later convenience. We introduce the vector $\mathbf{d} = (d_1, \bar{d}_1, \dots, d_L, \bar{d}_L)^\top$ to rewrite the action S_0 as $S_0 = \frac{1}{2} \mathbf{d}^\top A_0 \mathbf{d}$, where the $2L \times 2L$

skew matrix A_0 is block tridiagonal

$$A_0 = \begin{pmatrix} V_0 & U_0 & 0 & 0 & 0 & \cdots & U_0 \\ -U_0 & V_0 & U_0 & 0 & 0 & \cdots & 0 \\ 0 & -U_0 & V_0 & U_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -U_0 & V_0 & U_0 & 0 \\ 0 & 0 & \cdots & \cdots & -U_0 & V_0 & U_0 \\ -U_0 & 0 & \cdots & \cdots & \cdots & -U_0 & V_0 \end{pmatrix}, \quad (15)$$

with the 2×2 block matrices U_0 and V_0 given by

$$U_0 = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, V_0 = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}. \quad (16)$$

Therefore, the partition function Z_0 is formally equal to the Pfaffian of A_0 , or $Z_0 = \text{Pf}(A_0) = \sqrt{\det A_0}$. However, the partition function (14) contains the additional terms $-2ud_id_{i+1}\bar{n}_i$, which leads to a modification of matrix A_0 :

$$A = A[\{\bar{n}_i\}] = \begin{pmatrix} V_0 & U_1 & 0 & 0 & 0 & \cdots & U_L \\ -U_1 & V_0 & U_2 & 0 & 0 & \cdots & 0 \\ 0 & -U_2 & V_0 & U_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -U_{L-3} & V_0 & U_{L-2} & 0 \\ 0 & 0 & \cdots & \cdots & -U_{L-2} & V_0 & U_{L-1} \\ -U_L & 0 & \cdots & \cdots & \cdots & -U_{L-1} & V_0 \end{pmatrix} \quad (17)$$

where

$$U_i = \begin{pmatrix} u - 2u\bar{n}_i & 0 \\ 0 & -u \end{pmatrix}, \quad \bar{n}_i = 0, 1. \quad (18)$$

If we define the projection or occupation matrix $P = \text{diag}(n_1, \bar{n}_1, \dots, n_L, \bar{n}_L)$ onto the sites occupied by monomers, and $\mathbf{h} = (h_1, \bar{h}_1, \dots, h_L, \bar{h}_L)$, we can integrate directly over the vector \mathbf{d} , which leads to

$$Z[\{n_i, \bar{n}_i\}] = \sqrt{\det A[\{\bar{n}_i\}]} \int \prod_{i=1}^L d\bar{h}_i e^{(1-\bar{n}_i)\bar{h}_i} dh_i e^{(1-n_i)h_i} \exp\left(\frac{1}{2}\mathbf{h}^\top P A^{-1} P \mathbf{h}\right). \quad (19)$$

We have used the following general formula for the Grassmannian Gaussian integrals ($P^\top = P = P^2$)

$$\begin{aligned} \int \prod_{i=1}^L d\bar{d}_i dd_i \exp\left(\frac{1}{2}\mathbf{d}^\top A \mathbf{d} + \mathbf{d}^\top P \mathbf{h}\right) &= \sqrt{\det A} \exp\left(\frac{1}{2}\mathbf{h}^\top P A^{-1} P \mathbf{h}\right) \\ &= \text{Pf}(A) \exp\left(\frac{1}{2}\mathbf{h}^\top P A^{-1} P \mathbf{h}\right). \end{aligned} \quad (20)$$

When all the densities $n_i = \bar{n}_i = 0$, it is clear that $Z = Z_0$. In (19), we can reorder the product

$$\prod_{i=1}^L d\bar{h}_i e^{(1-\bar{n}_i)\bar{h}_i} dh_i e^{(1-n_i)h_i} = \prod_{i=1}^L d\bar{h}_i dh_i e^{-(1-\bar{n}_i)\bar{h}_i} e^{(1-n_i)h_i}, \quad (21)$$

and use the following identity for a set of Grassmann variables $\{\mathcal{L}_i, \bar{\mathcal{L}}_i\}_{i=1,\dots,L}$, with each \mathcal{L}_i or $\bar{\mathcal{L}}_i$ being a linear combination of noncommuting terms

$$\begin{aligned} \prod_{i=1}^L \exp(\bar{\mathcal{L}}_i) \exp(\mathcal{L}_i) &= \exp \left(\sum_{i < j} (\mathcal{L}_i + \bar{\mathcal{L}}_i)(\mathcal{L}_j + \bar{\mathcal{L}}_j) + \sum_i \bar{\mathcal{L}}_i \mathcal{L}_i \right) \\ &\times \exp \left(\sum_i \mathcal{L}_i + \bar{\mathcal{L}}_i \right). \end{aligned} \quad (22)$$

The last sum in the exponential is linear and thus does not contribute to the overall integral. Therefore, we can rewrite the partition function (19) as a Gaussian integral

$$\begin{aligned} Z[\{n_i, \bar{n}_i\}] &= \sqrt{\det A[\{\bar{n}_i\}]} \int \prod_{i=1}^L d\bar{h}_i dh_i \\ &\times \exp \left(\sum_{i < j} [(1-n_i)h_i - (1-\bar{n}_i)\bar{h}_i][(1-n_j)h_j - (1-\bar{n}_j)\bar{h}_j] \right) \\ &\times \exp \left(\sum_i [1-n_i][1-\bar{n}_i]h_i\bar{h}_i + \frac{1}{2}\mathbf{h}^\top P A^{-1} P \mathbf{h} \right), \end{aligned} \quad (23)$$

where the linear Grassmannian terms in the exponential have been discarded. We can then integrate over the field variables \mathbf{h} and more simply express Z as a product of two Pfaffians [32], which can be generalized in the two-dimensional case [20] or as a single Pfaffian of the $4L \times 4L$ partitioned skew matrix B defined below [33, 34]

$$\begin{aligned} Z[\{n_i, \bar{n}_i\}] &= \sqrt{\det A[\{\bar{n}_i\}_i]} \int \prod_{i=1}^L d\bar{h}_i dh_i \exp \left(\frac{1}{2}\mathbf{h}^\top (1-P)R(1-P)\mathbf{h} + \frac{1}{2}\mathbf{h}^\top P A^{-1} P \mathbf{h} \right) \\ &= \sqrt{\det A} \sqrt{\det[(1-P)R(1-P) + P A^{-1} P]} \\ &= \det \begin{pmatrix} A & P \\ -P & Q R Q \end{pmatrix}^{1/2} = \sqrt{\det B}, \end{aligned} \quad (24)$$

where $Q = 1 - P$ and R the $2L \times 2L$ skew matrix defined by the following elements: $R_{i,j} = (-1)^{i+j}$ for $j > i$, except when $j = i + 1$ and j even, in that case $R_{i,j} = 1$. The matrix $\tilde{A} = Q R Q + P A^{-1} P$ is by definition the Schur complement of A in B [35]. We can notice that, in absence of any monomer, $P = 0$, $A = A_0$, and therefore $Z = \sqrt{\det A_0} \sqrt{\det R} = \sqrt{\det A_0} = Z_0$ since $\det R = 1$. On the contrary, if $P = 1$, there is only one configuration with no dimer, and therefore $Z = \sqrt{\det A} \sqrt{\det A^{-1}} = 1$.

In figure 2(a) we have computed numerically, using an inverse matrix algorithm, the partition function $Z_{(\alpha)}(d)$ for different monomer configurations (α) , with $u = v = 1$: For example, $Z_{(a)}(d)$

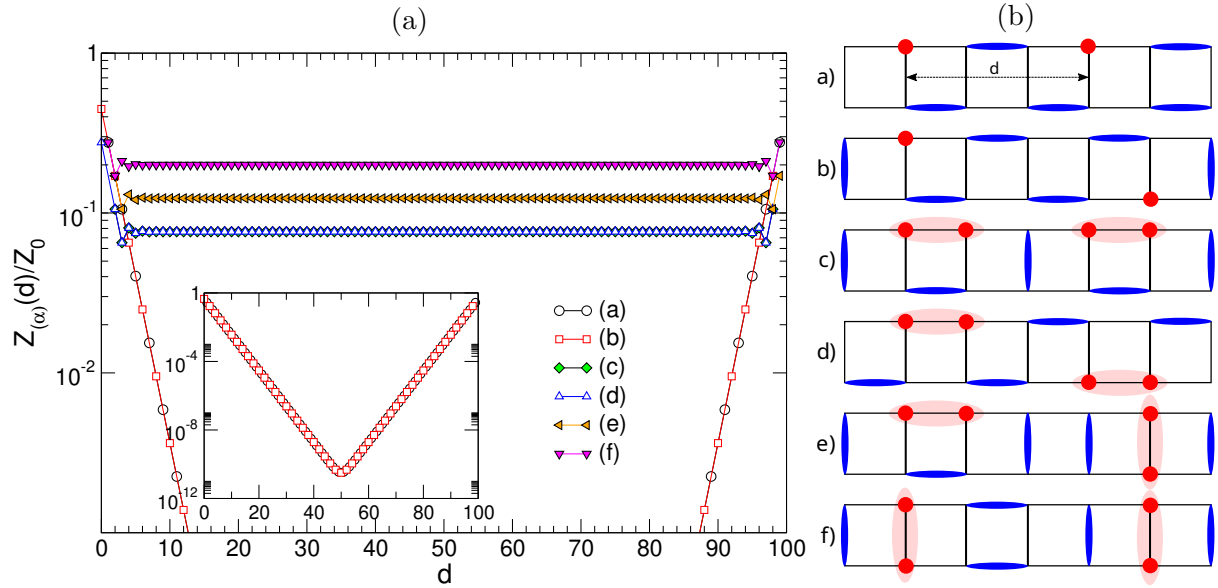


Figure 2: (a) Partition function ratio for different configurations of two or four monomers, as function of their distance d and defined on the right hand chart (b). The partition function is normalized with respect to the value at Z_0 , in absence of monomers. The system size is $L = 100$.

represents the partition function for two monomers on the same branch, see figure 2(b), and separated by an odd distance d . Even distances are indeed not compatible with any valid dimer configuration in this case. $Z_{(b)}(d)$ represents the number of configurations with two monomers, each in different branches, separated longitudinally by an even distance d , as there is no possible configuration with odd distances. In the inset of figure 2(a) we have plotted an exponential fit (dashed line) for the decreasing ratios $Z_{(a)}(d)/Z_0$ and $Z_{(b)}(d)/Z_0$ in the first half of the ladder size, and which agrees well with the data. The corresponding function is found to be accurately close to

$$Z_{(a)}(d)/Z_0 \simeq \varphi^{-d}, \quad Z_{(b)}(d)/Z_0 \simeq \varphi^{-d}, \quad (25)$$

with $\varphi = (1 + \sqrt{5})/2 \simeq 1.6103$ the golden ratio, and d odd and even, respectively. Cases (c) to (f) correspond to configurations with two pairs of monomers separated by a distance d . Contrary to cases (a) and (b), the ratios $Z_{(\alpha)}(d)/Z_0$ tend to a constant at long distance after some initial oscillations. The value of these limits depends on the relative position of the two pairs. It is maximum for the case (f). Moreover, we find that the constants are accurately determined by the following numbers

$$\begin{aligned} Z_{(c)}(d)/Z_0 \simeq Z_{(d)}(d)/Z_0 &\simeq 0.076\,393 \simeq \frac{(\sqrt{5}-1)^2}{20}, \\ Z_{(e)}(d)/Z_0 \simeq 0.123\,607 &\simeq \frac{\sqrt{5}-1}{10}, \quad Z_{(f)}(d)/Z_0 \simeq 0.2 \simeq \frac{1}{5}. \end{aligned} \quad (26)$$

We should notice that all configurations are symmetric by exchange of the variables $n_i \leftrightarrow \bar{n}_i$.

3. Thermodynamics of the model

We now consider a collection of monomers with a fixed density and evaluate the free energy for such ensemble, in order to calculate the different thermodynamical quantities. We first work in the grand canonical ensemble where it is easier to compute the partition function and the grand potential. Then we will use a Legendre transformation to obtain the free energy, entropy and specific heat. We therefore introduce monomers in the grand ensemble for a given temperature T , with chemical potentials μ and $\bar{\mu}$ on the upper and lower branches of the ladder, respectively. We can construct from (14) the partition function in this ensemble by summing over all possible monomer configurations

$$Z(\mu, \bar{\mu}) = \sum_{\{n_i=0,1\}} \sum_{\{\bar{n}_i=0,1\}} Z[\{n_i, \bar{n}_i\}] e^{\beta\mu \sum_i n_i + \beta\bar{\mu} \sum_i \bar{n}_i}, \quad (27)$$

and the grand potential is given by

$$\Omega(\mu, \bar{\mu}) = -k_B T \log Z(\mu, \bar{\mu}). \quad (28)$$

The sum over the n_i, \bar{n}_i can be performed in (14) by considering the factorized partial integrals over the field variables h_i and \bar{h}_i . Contrary to the previous method to obtain the general partition function (19), we first integrate over the field variables, and not over the d_i and \bar{d}_i variables. We can isolate each term corresponding to (n_i, \bar{n}_i) after moving the commuting terms that are quadratic in Grassmann variables

$$\begin{aligned} & \sum_{\{n_i=0,1\}} \sum_{\{\bar{n}_i=0,1\}} \int \left(\prod_{i=1}^L d\bar{h}_i e^{(1-\bar{n}_i)\bar{h}_i} dh_i e^{(1-n_i)h_i} \right) \\ & \times \exp \left(\sum_i d_i h_i n_i + \sum_i [\bar{d}_i \bar{h}_i - 2ud_i d_{i+1}] \bar{n}_i + \beta\mu \sum_i n_i + \beta\bar{\mu} \sum_i \bar{n}_i \right) \\ & = \int \left(\prod_{i=1}^L \sum_{\bar{n}_i=0,1} d\bar{h}_i e^{(1-\bar{n}_i)\bar{h}_i} \exp([\beta\bar{\mu} + \bar{d}_i \bar{h}_i - 2ud_i d_{i+1}] \bar{n}_i) \right. \\ & \left. \times \sum_{n_i=0,1} dh_i e^{(1-n_i)h_i} \exp([\beta\mu + d_i h_i] n_i) \right). \end{aligned} \quad (29)$$

After summation over the occupation states, and integration over h_i and \bar{h}_i , we obtain, for the term depending on n_i

$$\begin{aligned} \int \sum_{n_i=0,1} dh_i e^{(1-n_i)h_i} \exp([\beta\mu + d_i h_i] n_i) &= \int dh_i (e^{h_i} + \exp(\beta\mu + d_i h_i)) \\ &= \exp(-zd_i), \end{aligned} \quad (30)$$

where $z = e^{\beta\mu}$ is the particle activity or fugacity on the upper branch. Similarly, we obtain, for the term depending on \bar{n}_i

$$\int \sum_{\bar{n}_i=0,1} d\bar{h}_i e^{(1-\bar{n}_i)\bar{h}_i} \exp([\beta\bar{\mu} + \bar{d}_i\bar{h}_i - 2ud_id_{i+1}] \bar{n}_i) = \exp(-\bar{z}\bar{d}_i[1 - 2ud_id_{i+1}]), \quad (31)$$

with $\bar{z} = e^{\beta\bar{\mu}}$. Finally, the partition function can be rewritten as the following, after performing an overall change of sign $(d_i, \bar{d}_i) \rightarrow (-d_i, -\bar{d}_i)$,

$$\begin{aligned} Z(\mu, \bar{\mu}) = & \int \left(\prod_{i=1}^L d\bar{d}_i dd_i \right) \left(\prod_{i=1}^L \exp(zd_i) \exp(\bar{z}\bar{d}_i[1 - 2ud_id_{i+1}]) \right) \\ & \times \exp \left(\sum_i [ud_id_{i+1} - u\bar{d}_i\bar{d}_{i+1} + vd_i\bar{d}_i] \right). \end{aligned} \quad (32)$$

We can notice that the effective action is not quadratic for an arbitrary number of monomers in the two branches. Indeed, the presence of the cubic terms $\bar{d}_i d_i d_{i+1}$ generally hinders the search for exact solutions. In the following, we consider the case where the number of monomers is fixed, and evaluate the free energy

$$F = \Omega + L(\mu n + \bar{\mu} \bar{n}), \quad (33)$$

where n and \bar{n} are the average density of monomers on each branch, defined by the derivatives

$$n = \frac{z}{L} \partial_z \log Z(\mu, \bar{\mu}), \quad \bar{n} = \frac{\bar{z}}{L} \partial_{\bar{z}} \log Z(\mu, \bar{\mu}). \quad (34)$$

4. Integrable case, $\bar{z} = 0$

In this section, we focus on the case $\bar{z} = 0$, where the monomers occupy only on the upper ladder, see figure 3 for an example. We can use the formula (22) in order to write $Z = Z(\mu)$ as

$$\begin{aligned} Z(\mu) = & \int \left(\prod_{i=1}^L d\bar{d}_i dd_i \right) \exp \left(z^2 \sum_{i<j} d_i d_j + \sum_i [ud_id_{i+1} - u\bar{d}_i\bar{d}_{i+1} + vd_i\bar{d}_i] \right) \\ = & \int \left(\prod_{i=1}^L d\bar{d}_i dd_i \right) \exp(S). \end{aligned} \quad (35)$$

To compute these Gaussian integrals, we can solve for the eigenvalues, by considering the matrix

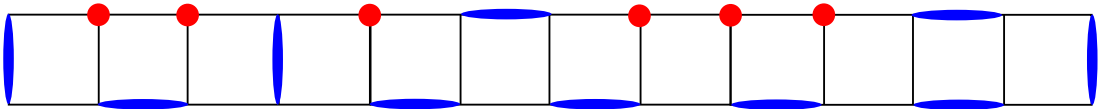


Figure 3: Example of a typical configuration for $L = 12$ and $\bar{z} = 0$, with the upper branch filled partially with particles at half filling $n = 1/2$.

representation of the Grassmannian action. As before, we introduce the following $2L \times 2L$ array

$$A(z) = \begin{pmatrix} V_0 & U_z + W_z & W_z & W_z & W_z & \cdots & U_z \\ -U_z - W_z & V_0 & U_z + W_z & W_z & W_z & \cdots & W_z \\ -W_z & -U_z - W_z & V_0 & U_z + W_z & W_z & \cdots & W_z \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -W_z & -W_z & \cdots & -U_z - W_z & V_0 & U_z + W_z & W_z \\ -W_z & -W_z & \cdots & \cdots & -U_z - W_z & V_0 & U_z + W_z \\ -U_z - W_z & -W_z & \cdots & \cdots & \cdots & -U_z - W_z & V_0 \end{pmatrix}, \quad (36)$$

where the 2×2 block matrices U_0 , V_0 , and W_z are given by

$$U_0 = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, V_0 = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}, W_z = \begin{pmatrix} z^2 & 0 \\ 0 & 0 \end{pmatrix}. \quad (37)$$

Then the action in equation (35) can be formally written as

$$z^2 \sum_{i < j} d_i d_j + \sum_i (u d_i d_{i+1} - u \bar{d}_i \bar{d}_{i+1} + v d_i \bar{d}_i) = \frac{1}{2} \mathbf{d}^\top A(z) \mathbf{d}, \quad (38)$$

with the same notations as before. The eigenvectors $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_L)$ and eigenvalues λ of $A(z)$ satisfy the following set of equations

$$U_0(\mathbf{v}_{l+1} - \mathbf{v}_{l-1}) + V_0 \mathbf{v}_l - W_z(\mathbf{v}_1 + \cdots + \mathbf{v}_{l-1}) + W_z(\mathbf{v}_{l+1} + \cdots + \mathbf{v}_L) = \lambda \mathbf{v}_l, \quad (39)$$

where each component \mathbf{v}_l is itself a two-component vector with antiperiodic conditions $\mathbf{v}_0 = -\mathbf{v}_L$, or $\mathbf{v}_{l \pm L} = -\mathbf{v}_l$. We consider the following eigenvectors $\mathbf{v}_l = \mathbf{v}_0 e^{i\omega l}$, with $\omega = \frac{2\pi}{L}(k + \frac{1}{2})$, $0 \leq k \leq L-1$, and obtain the condition

$$[2i \sin(\omega) U_0 + (V_0 - \lambda \mathbf{1}) + i \cot(\omega/2) W_z] \mathbf{v}_0 = 0. \quad (40)$$

Indeed we have

$$\begin{aligned} & -W_z(\mathbf{v}_1 + \cdots + \mathbf{v}_{l-1}) + W_z(\mathbf{v}_{l+1} + \cdots + \mathbf{v}_L) \\ & = W_z(\mathbf{v}_{l+1} + \cdots + \mathbf{v}_L - \mathbf{v}_1 - \cdots - \mathbf{v}_{l-1}) \\ & = W_z(\mathbf{v}_{l+1} + \cdots + \mathbf{v}_L + \mathbf{v}_{L+1} + \cdots + \mathbf{v}_{L+l-1}) = W_z e^{i\omega l} (\mathbf{v}_1 + \cdots + \mathbf{v}_{L-1}). \end{aligned} \quad (41)$$

The sum in the last line of the previous equation is independent of l , and is proportional to $(1 + e^{i\omega})/(1 - e^{i\omega}) \mathbf{v}_0$. We can therefore factorize $e^{i\omega l}$ in equation (39), and obtain the system of two equations (40) independent of l . This system possesses non-zero solutions $\mathbf{v}_0 \neq 0$ if the determinant of the corresponding 2×2 matrix is zero. This gives two eigenvalues $\lambda_{\pm}(\omega)$, solutions of a quadratic polynomial, and whose product is given by

$$\lambda_+(\omega) \lambda_-(\omega) = v^2 + 4u^2 \sin^2(\omega) + 4uz^2 \cos^2(\omega/2). \quad (42)$$

We can then write the Pfaffian (35) as the square root of the product over all the $2L$ eigenvalues $\lambda_{\pm}(\omega)$

$$\begin{aligned} Z(\mu) &= \prod_{\omega} [v^2 + 4u^2 \sin^2(\omega) + 4uz^2 \cos^2(\omega/2)]^{1/2}, \\ \omega &= \frac{2\pi}{L}(k + \frac{1}{2}), \quad k = 1, \dots, L. \end{aligned} \quad (43)$$

When $z = 0$, we recover the expression (8) in the absence of monomers. This result can also be obtained using a simpler method, by considering the Fourier transform of the action (38). After some algebra consisting in performing geometric series, and taking into account the antisymmetry of the Grassmann variables in the Fourier space, we obtain the factorization

$$\begin{aligned} S &= z^2 \sum_{i < j} d_i d_j + \sum_i (u d_i d_{i+1} - u \bar{d}_i \bar{d}_{i+1} + v d_i \bar{d}_i) \\ &= \sum_k' \left[(i z^2 \cot(\omega/2) + 2iu \sin(\omega)) d_{-k-\frac{1}{2}} d_{k+\frac{1}{2}} \right. \\ &\quad \left. + 2iu \sin(\omega) \bar{d}_{-k-\frac{1}{2}} \bar{d}_{k+\frac{1}{2}} + v d_{k+\frac{1}{2}} \bar{d}_{k+\frac{1}{2}} + v d_{-k-\frac{1}{2}} \bar{d}_{-k-\frac{1}{2}} \right]. \end{aligned} \quad (44)$$

The Fourier transform of the sum over $d_i d_j$ with $i < j$ will be non-zero only for the terms corresponding to $d_{k+\frac{1}{2}} d_{L-k-\frac{1}{2}}$ or equivalently $d_{k+\frac{1}{2}} d_{-k-\frac{1}{2}}$. It is then clear that the form (44) implies the decoupling of the Fourier modes, and the integration $\int d\bar{d}_{k+\frac{1}{2}} d d_{k+\frac{1}{2}} d\bar{d}_{-k-\frac{1}{2}} d d_{-k-\frac{1}{2}}$ over the four variables in each block $(k + \frac{1}{2}, -k - \frac{1}{2})$ leads to the correct result (43). The partition function has the exact expression

$$\begin{aligned} Z(\mu) &= u^L \sqrt{\left(\left(\left(\sqrt{\zeta_-^2 - 1} - \zeta_- \right)^L + 1 \right) \left(\left(\zeta_- + \sqrt{\zeta_-^2 - 1} \right)^L + (-1)^L \right) \right)} \\ &\quad \times \sqrt{\left(\left(\left(\zeta_+ - \sqrt{\zeta_+^2 - 1} \right)^L + 1 \right) \left(\left(\zeta_+ + \sqrt{\zeta_+^2 - 1} \right)^L + 1 \right) \right)}, \end{aligned} \quad (45)$$

with coefficients

$$\zeta_{\pm} = \frac{\sqrt{4v^2 + (z^2 + 4u)^2} \pm z^2}{4u}. \quad (46)$$

We recover expression (9) when $z = 0$ by noticing that $\zeta_+ = \zeta_-$, and therefore $x_{\pm}^{1/2}$ is equal to $\zeta_{\pm} \pm \sqrt{\zeta_{\pm}^2 - 1}$. In the thermodynamical limit $L \rightarrow \infty$, $Z(\mu)$ can be simplified by using the following inequalities, which are valid for any value of z

$$0 < \zeta_+ - \sqrt{\zeta_+^2 - 1} < \zeta_- - \sqrt{\zeta_-^2 - 1} < 1 < \zeta_- + \sqrt{\zeta_-^2 - 1} < \zeta_+ + \sqrt{\zeta_+^2 - 1}. \quad (47)$$

In this case, we obtain

$$\lim_{L \rightarrow \infty} \frac{1}{L} \log Z(\mu) = \log(u) + \frac{1}{2} \log \left(\zeta_- + \sqrt{\zeta_-^2 - 1} \right) + \frac{1}{2} \log \left(\zeta_+ + \sqrt{\zeta_+^2 - 1} \right). \quad (48)$$

From this expression, we can easily obtain the free energy after evaluating the density of states n which is proportional to the derivative of (48) with respect to z

$$\begin{aligned} n &= \frac{4uz^2}{L} \sum_{\omega} \frac{\cos^2(\omega/2)}{v^2 + 4u^2 \sin^2(\omega) + 4uz^2 \cos^2(\omega/2)} \\ &= 2z^2 (\phi(0) + \phi(1)), \quad \phi(d) = \frac{u}{L} \sum_{\omega} \frac{\cos(\omega d)}{z(\omega)}, \end{aligned} \quad (49)$$

where the function ϕ , which is similar to a correlation function, is computed in Appendix B.1 in the limit $L \rightarrow \infty$, see identity (B.2). When z is large, it is clear from the previous sum that $n \rightarrow 1$. An alternative method for evaluating n , but less straightforward, is to consider the average value $\langle n_{i_0} \rangle$. The symbol $\langle \dots \rangle$ is the average over all configurations n_i for a given chemical potential μ , after integrating over all the Grassmann variables, see equations (14) and (27). For this, we have to evaluate the integrals over the following products ordered correctly with the operator $z d_{i_0}$ representing the local variable n_{i_0} , see further below (65)

$$\langle n_{i_0} \rangle = \frac{1}{Z} \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) \left(\prod_{i=1}^{i_0-1} \exp(z d_i) \right) z d_{i_0} \left(\prod_{i=i_0+1}^L \exp(z d_i) \right) \exp(S_0). \quad (50)$$

We can reorder the product inside the integral by moving d_{i_0} through the exponentials on the left, which implies a negative sign for each $e^{z d_i} \rightarrow e^{-z d_i}$, with $i = 1, \dots, i_0 - 1$, and use the relation (22) to obtain the expression

$$\langle n_{i_0} \rangle = \frac{z^2}{Z} \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) d_{i_0} \left(\sum_{i=i_0+1}^L d_i - \sum_{i=1}^{i_0-1} d_i \right) \exp(S). \quad (51)$$

It is clear that the extra linear term in formula (22) combines with the operator d_{i_0} in order to give a non-vanishing result. The final integration, after Fourier transform, is independent of the site location i_0 and is equivalent to (49). The independence of the local density on i_0 is actually incorporated into the sum difference of the d_i s, next to the operator d_{i_0} . In Appendix A, we have computed the sum (49) in the limit $L \rightarrow \infty$. The result can be expressed using the coefficients ζ_{\pm} defined previously

$$n = \frac{z^2}{2u(\zeta_- + \zeta_+)} \left(\sqrt{\frac{\zeta_+ + 1}{\zeta_+ - 1}} - \sqrt{\frac{\zeta_- - 1}{\zeta_- + 1}} \right).$$

For z small, we have $\zeta_- \simeq \zeta_+ \simeq \sqrt{v^2 + 4u^2}/u$, and the dependence of n with z is simply quadratic

$$n \simeq \frac{2uz^2}{v\sqrt{v^2 + 4u^2}}. \quad (52)$$

4.1. Entropy and specific heat

The free energy per site can be written explicitly, using the previous results (45) and (49), in the thermodynamical limit as

$$\lim_{L \rightarrow \infty} \frac{F}{2L} = \frac{\epsilon_u}{2} - k_B T \left[\frac{1}{4} \log \left(\zeta_- + \sqrt{\zeta_-^2 - 1} \right) + \frac{1}{4} \log \left(\zeta_+ + \sqrt{\zeta_+^2 - 1} \right) \right] + \frac{1}{2} n \mu. \quad (53)$$

There are three different cases, depending on the relative values of ϵ_u and ϵ_v . The isotropic case $\epsilon_u = \epsilon_v$, or $u = v$, leads to a simplification of the previous parameters. If we set $x = z^2/u$, we have the parametrization

$$\zeta_{\pm} = \frac{\sqrt{4 + (4 + x)^2} \pm x}{4}, \quad (54)$$

and the density n can also be uniquely parametrized by x

$$\begin{aligned} n &= \frac{x}{\sqrt{y}\sqrt{1+4x}} \left(\sqrt{9+2x+2\sqrt{y}} - \sqrt{9+2x-2\sqrt{y}} \right), \\ y &= x^2 + 8x + 20. \end{aligned} \quad (55)$$

Since n is fixed, x is also constant and independent of the temperature, as it is a solution of the previous implicit equation. This means that the chemical potential is always linear in temperature

$$\mu = -\frac{\epsilon_u}{2} + \frac{k_B T}{2} \log(x). \quad (56)$$

The parameters ζ_{\pm} depend also only on x and not on the temperature. Therefore, F is linear in temperature, implying that the specific heat $C_v = -T\partial^2 F/\partial^2 T$ is zero for any temperature, and the entropy per site S_e is constant. The latter is given by the formula

$$S_e = \frac{1}{4} \log \left(\zeta_- + \sqrt{\zeta_-^2 - 1} \right) + \frac{1}{4} \log \left(\zeta_+ + \sqrt{\zeta_+^2 - 1} \right) - \frac{1}{4} n \log(x). \quad (57)$$

The entropy is plotted in figure 4, it is equal to the configurational entropy (10) for $n = 0$, and

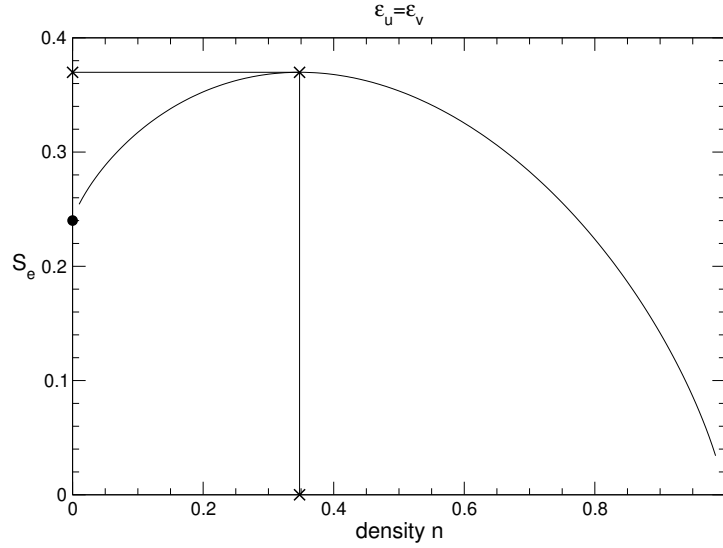


Figure 4: Entropy as function of the density for any temperature, in the case $\epsilon_u = \epsilon_v$. The maximum (cross symbol) entropy $S_e \simeq 0.3698$ is achieved when $n \simeq 0.3477$. At zero density, the entropy is equal to the configurational entropy, or $S_e \simeq 0.2406$.

zero when $n = 1$. The maximal entropy is achieved when $x = 1$, corresponding to $\mu = -\epsilon_u/2$ which remains constant at any temperature. We find $S_e \simeq 0.3698$ and $n \simeq 0.3477$. This value has to be compared with the concentration $n \simeq 0.3935$ for which the number of arrangements of dimers is maximum on a ladder [17], without the branch restriction imposed here on the monomer locations. The authors used in this case an exact recursive technique, and found that the dimers cover a fraction equal to 0.6064927 of the lattice.

In the general case, $\epsilon_u \neq \epsilon_v$, the entropy and specific heat depend on temperature. As low temperature we expect the entropy per site to be zero, since all the dimers will align either vertically (one ground state) or horizontally (four possible states) in the absence of monomers, or $\rho = 0$. In the high temperature limit and for the same concentration $\rho = 0$, we have $u \simeq v \simeq 1$, and we expect to recover the configurational entropy (10). If $u > v$, or $\epsilon_u < \epsilon_v$, then at low temperature $v/u \rightarrow 0$ exponentially, and the previous expressions for the density can be approximated by

$$n \simeq \sqrt{\frac{x}{4+x}}, \quad x \simeq \frac{4n^2}{1-n^2}, \quad (58)$$

where we have dropped the terms v/u which are exponentially small inside the coefficients ζ_{\pm} . The chemical potential is the same as in equation (56). We obtain therefore the entropy per site at zero temperature and for a given density value n

$$S_e = \frac{1}{4} \log \left(1 + \frac{x}{2} + \sqrt{x + \frac{x^2}{4}} \right) - \frac{1}{4} n \log(x). \quad (59)$$

As previously, this entropy is found to be maximal for $x = 1$, which corresponds to $n = 1/\sqrt{5} \simeq 0.4472$, and we have $S_e = 4^{-1} \log((3 + \sqrt{5})/2)$, which is exactly the configurational entropy for the pure dimer system. It is clear that when $n = 0$ or $n = 1$, $S_e = 0$.

In the opposite case, $u < v$, or $\epsilon_u > \epsilon_v$, the terms v/u in the coefficients ζ_{\pm} are exponentially large at low temperature, and we cannot use x as a scaling parameter since it can be comparable with v/u . We write instead $x = \bar{x}(v/u)^{\alpha}$, with \bar{x} slowly varying with temperature, and look for an adequate exponent α for which n depends only on \bar{x} in a non-trivial way. After inspection of all the possible values, we find that only $\alpha = 2$ leads to a correct result. This means that at low temperature, the chemical potential varies linearly as

$$\mu \simeq \frac{\epsilon_u}{2} - \epsilon_v + \frac{k_B T}{2} \log(\bar{x}), \quad (60)$$

with exponential corrections proportional to u/v . The density is simply given by

$$n \simeq 1 - \frac{1}{\sqrt{1+4\bar{x}}}, \quad \bar{x} \simeq \frac{1}{4} \left[\frac{1}{(1-n)^2} - 1 \right]. \quad (61)$$

We can finally write the entropy at zero temperature as

$$S_e = \frac{1}{4} \log \left(1 + \frac{1}{2\bar{x}} + \sqrt{\frac{1}{\bar{x}} + \frac{1}{4\bar{x}}} \right) + \frac{1}{4} (1-n) \log(\bar{x}). \quad (62)$$

As before it is maximal for $\bar{x} = 1$, for which $n = 1 - 1/\sqrt{5} \simeq 0.5528$ and $S_e = 4^{-1} \log((3 + \sqrt{5})/2)$. At this value, μ is independent of the temperature, except for exponentially small corrections. In figure 5, we have plotted the entropy for the two following cases: $\epsilon_u = 1, \epsilon_v = 2$ (left), and $\epsilon_u = 2, \epsilon_v = 1$ (right). As expected, the entropy vanishes at zero temperature when there is no monomer or when $n = 1$, and tends to the entropy of figure 4 at high temperature since $u \simeq v \simeq 1$ in this limit. At low temperature, the maximal entropy value is located at $x = 1$ or $\bar{x} = 1$, where the chemical potential is almost temperature-independent, up to exponentially small terms, see figure

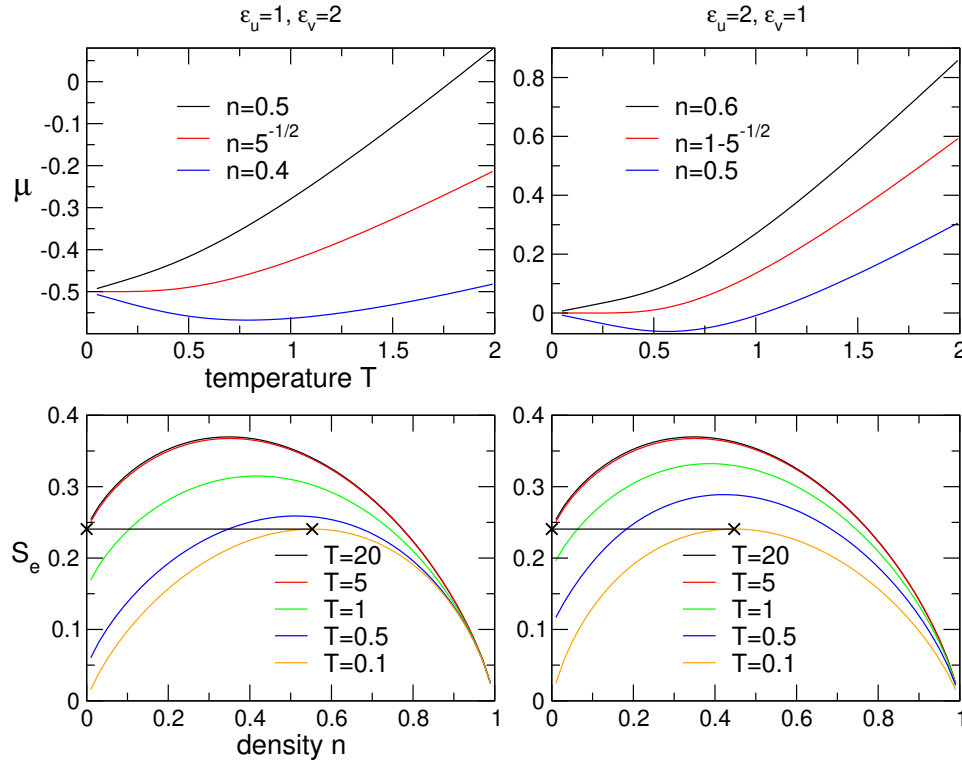


Figure 5: Chemical potential and entropy as function of the temperature and density respectively, in two cases: $\epsilon_u = 1, \epsilon_v = 2$ (left) and $\epsilon_u = 2, \epsilon_v = 1$ (right). The crosses represent the location of the maximal entropy at zero temperature. At high temperature, the entropy is identical to the isotropic case $\epsilon_u = \epsilon_v$, see (57).

5. The corresponding value is the same as the configurational entropy of the full dimer system. To understand the reason why, we first consider the case $\epsilon_u < \epsilon_v$, where all the dimers are supposed to be aligned along the horizontal direction. For the value $x = 1$, the chemical potential is exactly equal to $\mu = -\epsilon_u/2$, and we can pair two monomers to make a pseudo horizontal dimer with the higher weight $e^{-\beta\epsilon_u} = u = z^2$, see figure 6(a). We can also pair one monomer with half of an horizontal dimer on the opposite branch to make a pseudo vertical dimer with the same weight. Since each monomer has half of the higher Boltzmann weight, we can expect to construct all the possible dimer configurations of the full dimer system, with the maximal entropy. When $\epsilon_u > \epsilon_v$, all the dimers tend to be aligned vertically with v the highest weight. The chemical potential is equal to $\mu = \epsilon_u/2 - \epsilon_v$ at the density for which $\bar{x} = 1$, and the pairing of two monomers leads to a Boltzmann weight equal to $e^{\beta(\epsilon_u - 2\epsilon_v)} = u^{-1}v^2$, see figure 6(b). However, if we consider the first configuration on this figure, the pairing of two monomers associated with one horizontal dimer on the lower branch has a total weight equal to $(u^{-1}v^2) \times u = v^2$, which is the weight for two vertical dimers. There must exist on the lower branch horizontal dimers since we can not always form vertical dimers in the presence of monomers on the upper branch of the ladder. Also, we can

form pseudo-vertical dimers by pairing one monomer with half of a horizontal dimer on the lower branch, with a total weight equal to $(u^{-1/2}v \times u^{1/2})^2 = v^2$. We therefore expect to form as well all possible dimer configurations of the full dimer system. This formal construction may at least partially explain the value of the entropy at the specific density for which x or $\bar{x} = 1$.

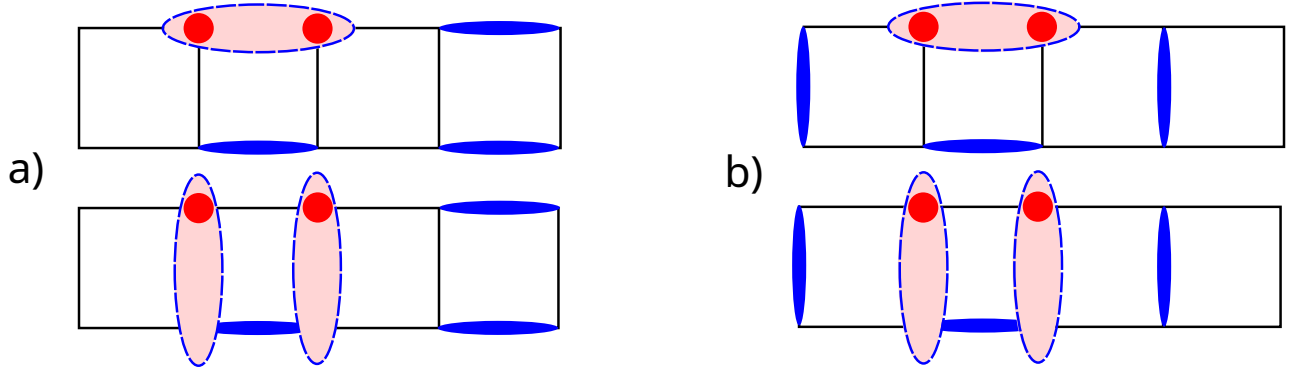


Figure 6: Equivalent configurations between pairings of two monomers and dimers when (a) $\epsilon_u < \epsilon_v$, and (b) $\epsilon_u > \epsilon_v$. In the first case, each monomer has a chemical potential equal to $\mu = -\frac{1}{2}\epsilon_u$, whereas $\mu = \frac{1}{2}\epsilon_u - \epsilon_v$ in the second case.

In figure 7, we have plotted the specific heat for the two cases, $\epsilon_u < \epsilon_v$ on the left, and $\epsilon_u > \epsilon_v$ on the right. All specific heat curves present a Schottky anomaly with an amplitude decreasing with the density of monomers. The Schottky peak usually represents the presence of a gap in the system. We examined the best possible fits with a least-squares minimization method. We first consider a simple two-level system, such as $Z = 1 + ge^{\beta\Delta}$, which contains two parameters g and a gap Δ . This usually gives a specific heat with a broad peak, which provides information about the presence of a gap in the system at low temperature. From the specific heat curves of figure 7, we found however that the best fit for all temperatures and densities is given by a slightly modified two-level system

$$C_v = \frac{C_0 g \beta^2 \Delta^2 e^{\beta\Delta}}{(1 + ge^{\beta\Delta})^2}, \quad (63)$$

where C_0 is a global fitting parameter that comes as a factor of the free energy or C_v . If $C_0 = 1$, we do not find that the formula (63) is accurate for all densities in the cases considered in figure 7, even when there is more than one gap. We indeed assume that for $n = 1$, the specific heat is zero, and therefore C_0 should also tend to zero, since all the sites are occupied by monomers on the upper branch, leaving the dimer system with only two possible horizontal states on the lower branch at all temperatures. The fact that $C_0 \rightarrow 0$ is clearly seen on the representative curves at the bottom of figure 7, when the density approaches unity. The gap Δ is different in the two cases. It is globally equal to $\Delta \simeq 1$ when $u > v$, and close to $\Delta \simeq 2$ in the other case $u < v$. We attribute this to the fact that there are twice as many horizontal bonds as vertical bonds. The fitting curves are plotted in figure 8 as dashed red lines and compared to the numerical results C_v ,

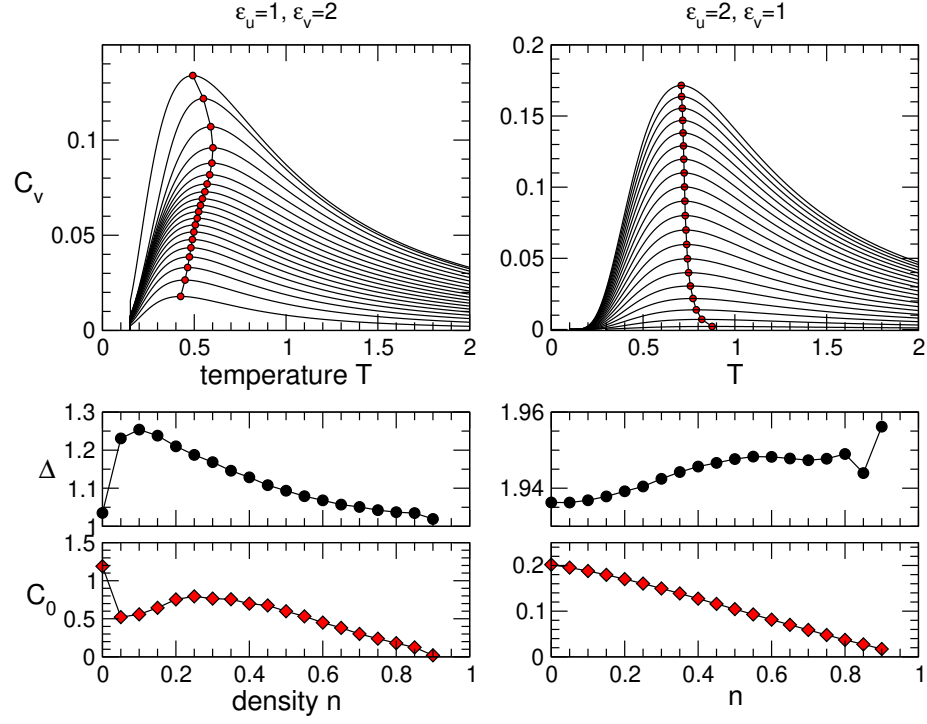


Figure 7: Specific heat C_v as function of the temperature and density, from $n = 0$ to $n = 0.95$ by increment of 0.05. The top curve corresponds to the full dimer system $n = 0$. Two cases are considered: $\epsilon_u = 1, \epsilon_v = 2$ (left) and $\epsilon_u = 2, \epsilon_v = 1$ (right). The red circles represent the location of the specific heat maximum, for each case. The four bottom figures represent the dependence of the fitting parameters as function of the density, see equation (63).

black lines. The fitting analysis gives a better result for the second case, $\epsilon_u > \epsilon_v$, for all densities and temperatures.

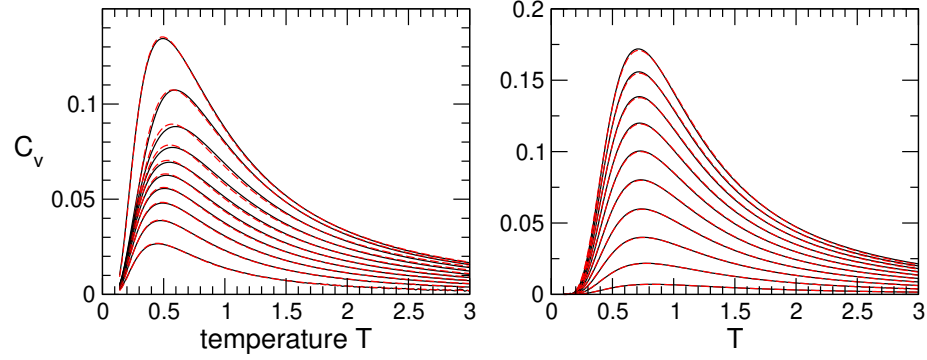


Figure 8: Specific heat C_v , black lines, as function of the temperature and density, from $n = 0$ to $n = 0.9$ by increment of 0.1, from top to bottom. Two cases are considered: $\epsilon_u = 1, \epsilon_v = 2$ (left) and $\epsilon_u = 2, \epsilon_v = 1$ (right). The dashed red lines represent the fitting function, see equation (63).

The macroscopic entropy at zero temperature and the characteristics of the specific heat are similar to the physics of dilute spin-ice [36], in the presence in particular of disorder or vacant sites. Spin ice systems are based on spin orientation conservation rules. For example, dilute pyrochlore lattices $\text{Dy}_{2-x}\text{Y}_x\text{Ti}_2\text{O}_7$ [37] are representative of real materials with disorder, which can be also modeled in two-dimensional geometries [38]. The magnetic ions Dy^{3+} are replaced by non magnetic yttrium Y^{3+} . In the ground state, spin-ice rules impose two spins to point in and two spins to point out at the summits of each tetrahedron, forming the elementary cells of the three-dimensional pyrochlore lattice, and the macroscopic configurational entropy that follows is called the Pauling entropy. In absence of phase transition, the magnetic specific heat also possesses a Schottky anomaly, with the location of the peak varying nonuniformly with the density of disorder, which is equivalent to the density of monomers in the present model. In particular, the fit expression (63) has been proposed in previous references [39, 38] for dilute spin-ices to explain the specific heat data. To be more specific, this analogy should work only in the anisotropic case $\epsilon_u \neq \epsilon_v$, and in the presence of monomers when the zero-temperature entropy per site is nonzero.

4.2. Two-point correlation function

We consider in this section the two-point connected correlation function $G_1(i_0, i_1) = \langle n_{i_0} n_{i_1} \rangle - n^2$, which is not directly related to a relative partition function. Due to the periodicity and symmetry of the lattice, this function only depends on the distance $d = i_1 - i_0 > 0$, or more precisely, $G_1(i_0, i_1) = G_1(d)$. We can evaluate the average by considering the quantity (30), after taking into account the previous change of sign in the variables d_i

$$\int \sum_{n_i=0,1} dh_i e^{(1-n_i)h_i} n_i \exp([\beta\mu - d_i h_i] n_i) = z d_i.$$

For the variable \bar{n}_i , we have the corresponding value, from equation (31)

$$\int \sum_{\bar{n}_i=0,1} d\bar{h}_i e^{(1-\bar{n}_i)\bar{h}_i} \bar{n}_i \exp([\beta\bar{\mu} - \bar{d}_i \bar{h}_i - 2ud_i d_{i+1}] \bar{n}_i) = \bar{z} \bar{d}_i \exp(-2ud_i d_{i+1}). \quad (64)$$

Therefore, the operators n_i and \bar{n}_i are directly associated to Grassmannian insertions with the following rules

$$n_i \rightarrow z d_i, \quad \bar{n}_i \rightarrow \bar{z} \bar{d}_i \exp(-2ud_i d_{i+1}). \quad (65)$$

These operators have to be inserted at the correct position inside the integral (32) when we deal with correlation functions. For the two-point correlation function, we indeed have to evaluate the ordered product

$$\begin{aligned} & \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) \left(\prod_{i=1}^{i_0-1} \exp(z d_i) \right) z d_{i_0} \left(\prod_{i=i_0+1}^{i_1-1} \exp(z d_i) \right) z d_{i_1} \\ & \times \left(\prod_{i=i_1+1}^L \exp(z d_i) \right) \exp \left(\sum_i [u d_i d_{i+1} - u \bar{d}_i \bar{d}_{i+1} + v d_i \bar{d}_i] \right) \end{aligned} \quad (66)$$

$$\begin{aligned}
&= z^2 \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) \left(\prod_{i=1}^{i_0-1} \exp(z d_i) \right) d_{i_0} d_{i_1} \left(\prod_{i=i_0+1}^{i_1-1} \exp(-z d_i) \right) \\
&\quad \times \left(\prod_{i=i_1+1}^L \exp(z d_i) \right) \exp \left(\sum_i [u d_i d_{i+1} - u \bar{d}_i \bar{d}_{i+1} + v d_i \bar{d}_i] \right). \tag{67}
\end{aligned}$$

When moving d_{i_1} through the different exponential terms $e^{z d_i}$, we insert a minus sign inside the exponential arguments $e^{z d_i} \rightarrow e^{-z d_i}$. Since the product $d_{i_0} d_{i_1}$ is a commuting variable, we can take it out of the products, and order the remaining exponentials using the formula (22), which yields

$$\begin{aligned}
&\left(\prod_{i=1}^{i_0-1} \exp(z d_i) \right) \left(\prod_{i=i_0+1}^{i_1-1} \exp(-z d_i) \right) \left(\prod_{i=i_1+1}^L \exp(z d_i) \right) \\
&= \exp \left[z^2 \sum_{i < j} d_i d_j + 2 \left(\sum_{i=i_0+1}^{i_1-1} z d_i \right) \left(\sum_{j=1}^{i_0-1} z d_j - \sum_{j=i_1+1}^L z d_j \right) \right] \\
&\quad \times \exp \left(\sum_{i=1, i \neq i_0, i_1}^L z d_i - 2 \sum_{i=i_0+1}^{i_1-1} z d_i \right). \tag{68}
\end{aligned}$$

The indices i_0 and i_1 can eventually be included in the summation without modifying the result of the integral since the overall product is already multiplied by $d_{i_0} d_{i_1}$ which cancels any other dependence on d_{i_0} and d_{i_1} . Also the linear term in the argument does not contribute to the integral. Therefore, the average density value between the two sites i_0 and i_1 is equal to

$$\begin{aligned}
\langle n_{i_0} n_{i_1} \rangle &= z^2 \left\langle d_{i_0} d_{i_1} \left[1 + 2z^2 \left(\sum_{i=i_0+1}^{i_1-1} d_i \right) \left(\sum_{j=1}^{i_0-1} d_j - \sum_{j=i_1+1}^L d_j \right) \right] \right\rangle_S \\
&= z^2 \langle d_{i_0} d_{i_1} (1 + 2z^2 Q_{i_0 i_1}) \rangle_S, \tag{69}
\end{aligned}$$

where $\langle \dots \rangle_S$ is the integral average over the d_i and \bar{d}_i variables using the action S , see expressions (35) and (44) in the real and Fourier spaces. $Q_{i_0 i_1}$ is a sum of quadratic Grassmannian variables, and only depends on the distance between the two sites. We evaluate this average in the Fourier space, since the momenta can be factorized independently. As S is quadratic, we only need the following average values in the Fourier space

$$\begin{aligned}
\langle d_{k+\frac{1}{2}} d_{-k-\frac{1}{2}} \rangle_S &= \frac{2iu \sin(\omega)}{z(\omega)}, \quad \langle \bar{d}_{k+\frac{1}{2}} \bar{d}_{-k-\frac{1}{2}} \rangle_S = \frac{2iu \sin(\omega) + iz^2 \cot(\omega/2)}{z(\omega)}, \\
\langle d_{k+\frac{1}{2}} \bar{d}_{k+\frac{1}{2}} \rangle_S &= \frac{v}{z(\omega)}, \quad \langle d_{-k-\frac{1}{2}} \bar{d}_{-k-\frac{1}{2}} \rangle_S = \frac{v}{z(\omega)}, \tag{70}
\end{aligned}$$

where $z(\omega) = v^2 + 4u^2 \sin^2(\omega) + 4uz^2 \cos^2(\omega/2)$. There are three different terms in (69) to consider. The first one is the average of the product $\langle d_{i_0} d_{i_1} \rangle_S$, which can be written as a sum over the momenta

$$z^2 \langle d_{i_0} d_{i_1} \rangle_S = \frac{z^2}{L} \sum_{\omega, \omega'} e^{i\omega i_0 + i\omega' i_1} \left\langle d_{k+\frac{1}{2}} d_{k'+\frac{1}{2}} \right\rangle_S$$

$$= -\frac{2iz^2}{L} \sum_{\omega}' \sin(\omega d) \left\langle d_{k+\frac{1}{2}} d_{-k-\frac{1}{2}} \right\rangle_S = \frac{2uz^2}{L} \sum_{\omega} \frac{\sin(\omega) \sin(\omega d)}{z(\omega)}. \quad (71)$$

This quantity can therefore be decomposed into the difference of two functions $\phi(d)$

$$\langle d_{i_0} d_{i_1} \rangle_S = \phi(d-1) - \phi(d+1). \quad (72)$$

The computation of this difference in the thermodynamical limit is done in Appendix B.1. The result is a sign oscillating and decreasing function with two correlation lengths. The other two terms in (69) corresponding to $\langle d_{i_0} d_{i_1} Q_{i_0 i_1} \rangle_S$ involve the product of four operators in the momentum space $\langle d_{k+\frac{1}{2}} d_{k'+\frac{1}{2}} d_{k''+\frac{1}{2}} d_{k'''+\frac{1}{2}} \rangle_S$, which can be decomposed into products of three possible contractions between operator pairs

$$\begin{aligned} \left\langle d_{k+\frac{1}{2}} d_{k'+\frac{1}{2}} d_{k''+\frac{1}{2}} d_{k'''+\frac{1}{2}} \right\rangle_S &= \left\langle d_{k+\frac{1}{2}} d_{-k-\frac{1}{2}} \right\rangle_S \left\langle d_{k''+\frac{1}{2}} d_{-k''-\frac{1}{2}} \right\rangle_S \delta_{k+k'+1} \delta_{k''+k'''+1} \\ &\quad \left\langle d_{k+\frac{1}{2}} d_{-k-\frac{1}{2}} \right\rangle_S \left\langle d_{k'+\frac{1}{2}} d_{-k'-\frac{1}{2}} \right\rangle_S (-\delta_{k+k''+1} \delta_{k'+k'''+1} + \delta_{k+k'''+1} \delta_{k'+k''+1}), \end{aligned} \quad (73)$$

where δ_k is the Kronecker delta function that selects the momenta $k = 0 \pmod{L}$. The explicit expression for each of these terms is given in Appendix B.2 in the thermodynamical limit. We also obtain a sign oscillating exponential function with three different decay rates and with amplitudes depending on parameters ζ_{\pm} and α_{\pm} defined by

$$\alpha_{\pm} = \zeta_{\pm} + \sqrt{\zeta_{\pm}^2 - 1}. \quad (74)$$

In summary, we find that the correlation function is given explicitly by the following expression

$$\begin{aligned} G_1(d) &= \frac{z^2(1-\tilde{n})}{2u(\zeta_- + \zeta_+)} (\alpha_+^{-d} - (-1)^d \alpha_-^{-d}) \\ &\quad - \frac{z^4(\alpha_- + \alpha_+)^2}{u^2(\zeta_- + \zeta_+)^2(\alpha_- + 1)^2(\alpha_+ - 1)^2} (-\alpha_- \alpha_+)^{-d} \\ &= A_1 (\alpha_+^{-d} - (-1)^d \alpha_-^{-d}) - A_2 (-\alpha_- \alpha_+)^{-d}, \end{aligned} \quad (75)$$

where the coefficient \tilde{n} is given by (B.10), or

$$\tilde{n} = \frac{2z^2}{u(\zeta_- + \zeta_+)} \left[\frac{\alpha_+}{(\alpha_+ - 1)^2} + \frac{\alpha_-}{(\alpha_- + 1)^2} \right]. \quad (76)$$

The expression (75) contains two decreasing exponential functions with inverse correlation lengths $\log \alpha_+ > \log \alpha_- > 0$, in addition to a corrective exponential proportional to z^4 , and decreasing with the mixed inverse correlation length $\log(\alpha_- \alpha_+)$. The amplitude of the dominant term is proportional to $(1 - \tilde{n})$, which decreases to zero as z , or the number of monomers, becomes large $1 - \tilde{n} \simeq v^2/(4uz^2)$. At the same time, the main amplitudes A_1 and A_2 of $G_1(d)$ tend to unity

$$A_1 \simeq 1 - \frac{4u}{z^2}, \quad A_2 \simeq 1 - \frac{v}{\sqrt{uz}}. \quad (77)$$

4.3. Correlation function between pairs of monomers

We consider in this section the four-point correlation function defined by

$$G_2(i_0, i_1) = \langle n_{i_0} n_{i_0+1} n_{i_1} n_{i_1+1} \rangle - n_p^2, \quad (78)$$

with $i_1 \geq i_0 + 2$. As for the two-point correlation function, $G_2(d)$ depends only on the difference $d = i_1 - i_0$, or $G_2(i_0, i_1) = G_2(d)$. n_p is the probability to find a pair of monomers horizontally. It is defined by the average $n_p = z^2 \langle d_{i_0} d_{i_0+1} \rangle_S$, for any arbitrary site i_0 . Since d_{i_0} and d_{i_0+1} are two consecutive variables, the product is commuting through the exponentials in (67). A Fourier transformation leads directly to the result

$$n_p = \frac{2uz^2}{L} \sum_{\omega} \frac{\sin^2(\omega)}{z(\omega)}. \quad (79)$$

The computation in the thermodynamical limit of this sum yields the relation $n_p = z^2(\phi(0) - \phi(2))$, or, using the formula (B.2)

$$n_p = \frac{z^2}{2u(\zeta_- + \zeta_+)} \left(\frac{1}{\alpha_+} + \frac{1}{\alpha_-} \right). \quad (80)$$

We can notice this is not equal to the square of the monomer density n which is written as $n = 2z^2(\phi(0) + \phi(1))$ as seen in equation (49). The correlation function (78) can be evaluated in the Fourier space, which consists in computing an integral over a quartic product of Grassmannian variables, using the three possible contractions seen in equation (73). We can therefore express $G_2(d)$ in terms of function ϕ , after subtracting the contribution n_p^2 of individual pairs

$$G_2(d) = z^4 \left(\phi(d) - \phi(d+2) \right)^2 - z^4 \left(\phi(d-1) - \phi(d+1) \right)^2. \quad (81)$$

We obtain, after replacing function ϕ by its explicit expression (B.2) in the thermodynamical limit

$$G_2(d) = \frac{z^4}{4u^2(\zeta_- + \zeta_+)^2} \left[(1 - \alpha_+^2) \alpha_+^{-2d-2} + (1 - \alpha_-^2) \alpha_-^{-2d-2} - 2(1 + \alpha_- \alpha_+) (-\alpha_- \alpha_+)^{-d-1} \right]. \quad (82)$$

This correlation function is oscillating around zero, and takes positive values for even values of d , and negative otherwise. Indeed, $\phi(d)$ is usually negative and close to zero when d is odd, and with a larger positive value when d is even. Therefore, we find that $G_2(d)$ defined by (81) is positive when d is even since the second term of the right-hand side equation does not contribute as much as the first term. We can also define a pair correlation function, which is defined by subtracting n^2 from n_p

$$\Delta_p = n_p - n^2 = z^2(\phi(0) - \phi(2)) - 4z^4(\phi(0) + \phi(1))^2. \quad (83)$$

This quantity is always positive for any density value. In figure 9, we find that it is maximal for $n \simeq 0.4905$, after which the residual correlations tend to decrease as the presence of more monomers destroys the effective interactions through the dimers.

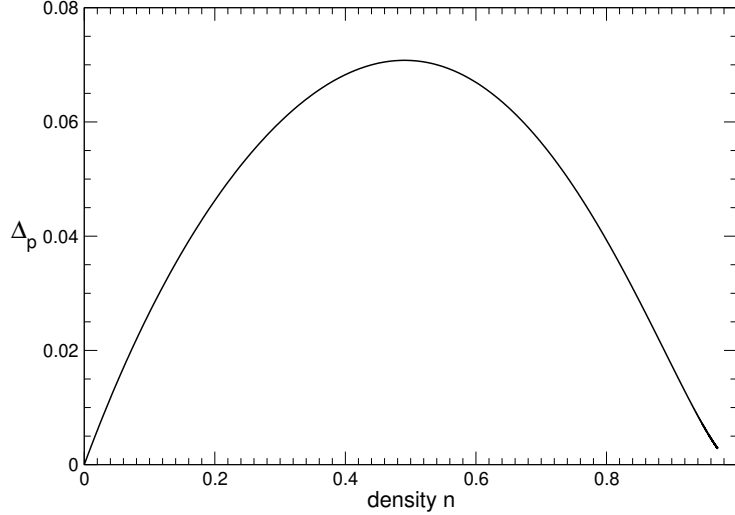


Figure 9: Pair correlation function Δ_p as function of the density. It is maximal at $n \simeq 0.4906$.

5. Relative partition functions for simple monomer configurations, $z = 0$ and $\bar{z} = 0$

In this section, we evaluate the partition function (24) for the configurations proposed previously in figure 2 using Grassmann integrals instead of algebraic methods. We first consider the case (a) where two monomers are separated by a distance $d = i_1 - i_0 \geq 1$ on the same branch of the ladder, which is the easiest case, with the Grassmann integral representation

$$Z_{(a)}(d) = Z[\{n_{i_0} = 1, n_{i_1} = 1\}] = \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) (d_{i_0} d_{i_1}) \exp(S_0), \quad (84)$$

where $S_0 = \lim_{z \rightarrow 0} S$, see definition (7). In the Fourier space, it is easy to see that

$$\langle d_{i_0} d_{i_1} \rangle_{S_0} = \frac{2u}{L} \sum_{\omega} \frac{\sin(\omega) \sin(\omega d)}{z_0(\omega)}, \quad (85)$$

where $z_0(\omega) = \lim_{z \rightarrow 0} z(\omega) = v^2 + 4u^2 \sin^2(\omega)$. We will also define the following function

$$\phi_0(d) = \frac{u}{L} \sum_{\omega} \frac{\cos(\omega d)}{z_0(\omega)} = \lim_{z \rightarrow 0} \phi(d) = \frac{u(1 + (-1)^d)}{2v\sqrt{v^2 + 4u^2}} \alpha_0^{-d}, \quad (86)$$

where $\alpha_0 = \lim_{z \rightarrow 0} \alpha_{\pm}$, or

$$\alpha_0 = \frac{v + \sqrt{v^2 + 4u^2}}{2u}. \quad (87)$$

The function ϕ_0 is zero for d odd. We obtain therefore the relative ratio

$$Z_{(a)}(d)/Z_0 = \langle d_{i_0} d_{i_1} \rangle_{S_0} = \phi_0(d-1) - \phi_0(d+1) = \frac{1 - (-1)^d}{2\sqrt{v^2 + 4u^2}} \alpha_0^{-d}. \quad (88)$$

It is zero when d is even, and decays exponentially with the inverse correlation length $\log \alpha_0$. In particular $\alpha_0 = \varphi$ when $u = v = 1$. This corresponds to the numerical result of figure 2 for the corresponding case $(\alpha) = (a)$.

In the next case (b) with two monomers, each of them in a different branch of the ladder, we have the following partition function

$$Z_{(b)}(d) = Z[\{n_{i_0} = 1, \bar{n}_{i_1} = 1\}] = \int \left(\prod_{i=1}^L dd_i \bar{d}_i \right) (d_{i_0} \bar{d}_{i_1}) \exp(S_0 - 2ud_{i_1} d_{i_1+1}), \quad (89)$$

where $d = i_1 - i_0 \geq 0$. The extra term $d_{i_1} d_{i_1+1}$ takes into account the correlation between monomers on different branches. If we expand the integral for this term, the overall integral over the action S_0 includes not only the average value of $\langle d_{i_0} \bar{d}_{i_1} \rangle_{S_0}$, but also the average value of the quartic term $\langle d_{i_0} \bar{d}_{i_1} d_{i_1} d_{i_1+1} \rangle_{S_0}$, which can be performed with a Fourier transform after considering the three possible contractions (73). The quadratic term involves the averages $\langle d_{k+\frac{1}{2}} \bar{d}_{k+\frac{1}{2}} \rangle_{S_0}$ as well as $\langle d_{-k-\frac{1}{2}} \bar{d}_{-k-\frac{1}{2}} \rangle_{S_0}$, which gives

$$\langle d_{i_0} \bar{d}_{i_1} \rangle_{S_0} = \frac{v}{L} \sum_{\omega} \frac{\cos(\omega d)}{z_0(\omega)}. \quad (90)$$

We then obtain $\langle d_{i_0} \bar{d}_{i_1} \rangle_{S_0} = (v/u)\phi_0(d)$. The quartic term involves products of two functions ϕ_0 . After some algebra, we find

$$\begin{aligned} \frac{Z_{(b)}(d)}{Z_0} &= \frac{v}{u} \phi_0(d) + 2v \left[\phi_0(1)\phi_0(d+1) + \phi_0(2)\phi_0(d) \right. \\ &\quad \left. - \phi_0(0)\phi_0(d+2) - \phi_0(1)\phi_0(d-1) \right] = \frac{1 + (-1)^d}{2\sqrt{v^2 + 4u^2}} \alpha_0^{-d}. \end{aligned} \quad (91)$$

It is clear that $Z_{(b)}$ decreases exponentially with the inverse correlation length $\log \alpha_0 = \log \varphi$, as seen in figure 2. $Z_{(a)}$ and $Z_{(b)}$ are therefore located on the same curve.

Let us consider now the partition function for the configuration of two pairs of monomers on the same upper branch and separated by distance d , corresponding to $Z_{(c)}(d)$. Since both pairs are on the same branch, there are no extra terms in the action, and the value of $Z_{(c)}(d)$ is given by the average of the quartic product

$$\begin{aligned} Z_{(c)}(d) &= Z[\{n_{i_0} = 1, n_{i_0+1} = 1, n_{i_1} = 1, n_{i_1+1} = 1\}] \\ &= \int \left(\prod_{i=1}^L dd_i \bar{d}_i \right) (d_{i_0} d_{i_0+1} d_{i_1} d_{i_1+1}) \exp(S_0). \end{aligned} \quad (92)$$

As before, we can evaluate this average by considering three possible contractions in the Fourier space

$$\begin{aligned} \frac{Z_{(c)}(d)}{Z_0} &= \left(\phi_0(0) - \phi_0(2) \right)^2 \\ &\quad + \left(\phi_0(d) - \phi_0(d+2) \right) \left(\phi_0(d-2) - \phi_0(d) \right) - \left(\phi_0(d-1) - \phi_0(d+1) \right)^2. \end{aligned} \quad (93)$$

The terms dependent of d are found to be oscillating around zero, with positive values for d even, and negative values otherwise. This explains the oscillating behavior in figure 2(a) near the end points. Indeed, in the second line, the last term is clearly non-zero and negative for d odd, whereas the other positive term is only non-zero for even distances, inducing an oscillating behavior. In the limit of large d , the ratio is exponentially close to the asymptotic limit, which is related to the square of the pair probability, or

$$\lim_{d \rightarrow \infty} \frac{Z_{(c)}(d)}{Z_0} = \left(\phi_0(0) - \phi_0(2) \right)^2 = \lim_{z \rightarrow 0} \left(\frac{n_p}{z^2} \right)^2 = \frac{4u^2}{(v^2 + 4u^2)(v + \sqrt{v^2 + 4u^2})^2}. \quad (94)$$

We thus recover the numerical result (26) when $u = v = 1$. The next case (d) represented in figure 2 corresponds to two pairs of monomers located on two different branches and also parallel to each other

$$\begin{aligned} Z_{(d)}(d) &= Z[\{n_{i_0} = 1, n_{i_0+1} = 1, \bar{n}_{i_1} = 1, \bar{n}_{i_1+1} = 1\}] \\ &= \int \left(\prod_{i=1}^L d\bar{d}_i dd_i \right) (d_{i_0} d_{i_0+1} \bar{d}_{i_1} \bar{d}_{i_1+1}) \exp(-2ud_{i_1} d_{i_1+1} - 2ud_{i_1+1} d_{i_1+2} + S_0). \end{aligned} \quad (95)$$

As for case (c), there are terms that are oscillating with the distance d corresponding to pair contractions between operators at i_0 or $i_0 + 1$, and operators at i_1 , $i_1 + 1$, or $i_1 + 2$, as well as a constant term corresponding to contractions between operators at neighboring sites. The former terms are decreasing exponentially with d , whereas the constant term is the asymptotic value of the partition function ratio that is equal to

$$\begin{aligned} \frac{Z_{(d)}(d)}{Z_0} &\simeq \left(\phi_0(0) - \phi_0(2) \right)^2 \left(-1 + 4u(\phi_0(0) - \phi_0(2)) + 2\frac{v^2}{u}\phi_0(0) \right) \\ &\simeq \left(\phi_0(0) - \phi_0(2) \right)^2, \end{aligned} \quad (96)$$

after noticing that the second term on the right-hand side of the first equality is actually equal to unity. We should also notice that the average value $\langle \bar{d}_{i_1} \bar{d}_{i_1+1} \rangle_{S_0}$ is equal to $-\langle d_{i_0} d_{i_0+1} \rangle_{S_0}$, so that $\bar{d}_{i_1} \bar{d}_{i_1+1}$ does not exactly represent a pair of monomers on the lower branch, but only in combination with the extra terms $d_{i_1} d_{i_1+1}$ and $d_{i_1+1} d_{i_1+2}$ in the exponential argument that compensate for the minus sign. The asymptotic constant for $Z_{(d)}$ is therefore the same as for $Z_{(c)}$, whether the pairs of monomers are on the same or opposite branches.

In the case (e), where the two pairs of monomers are perpendicular to each other on the ladder, we have the integral representation

$$\begin{aligned} Z_{(e)}(d) &= Z[\{n_{i_0} = 1, n_{i_0+1} = 1, n_{i_1} = 1, \bar{n}_{i_1} = 1\}] \\ &= \int \left(\prod_{i=1}^L d\bar{d}_i dd_i \right) (d_{i_0} d_{i_0+1} d_{i_1} \bar{d}_{i_1}) \exp(-2ud_{i_1} d_{i_1+1} + S_0). \end{aligned} \quad (97)$$

As before, we can deduce the long distance behavior of $Z_{(e)}$ by considering the pairs $d_{i_0} d_{i_0+1}$ and $d_{i_1} \bar{d}_{i_1+1} \exp(-2ud_{i_1} d_{i_1+1})$ separately. For the first term, we obtain $\phi_0(0) - \phi_0(2)$, whereas the

second term leads to $(v/u)\phi_0(0)$ after computation of all the possible contractions. We therefore obtain

$$\frac{Z_{(e)}(d)}{Z_0} \simeq (\phi_0(0) - \phi_0(2)) \frac{v}{u} \phi_0(0). \quad (98)$$

It is simply the product of the probability to have a pair parallel to the ladder with the probability to have a pair perpendicular. We find in particular for $u = v = 1$ that it is equal to $\lim_{d \rightarrow \infty} Z_{(e)}(d)/Z_0 = (\sqrt{5} - 1)/10$, which corresponds to the value found numerically in figure 2. The last configuration (f) is given by

$$\begin{aligned} Z_{(f)}(d) &= Z[\{n_{i_0} = 1, \bar{n}_{i_0} = 1, n_{i_1} = 1, \bar{n}_{i_1} = 1\}] \\ &= \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) (d_{i_0} \bar{d}_{i_0} d_{i_1} \bar{d}_{i_1}) \exp(-2ud_{i_0} d_{i_0+1} - 2ud_{i_1} d_{i_1+1} + S_0), \end{aligned} \quad (99)$$

and is proportional at long distance to the square of the probability of a pair perpendicular to the ladder, or

$$\frac{Z_{(f)}(d)}{Z_0} \simeq \left(\frac{v}{u} \phi_0(0) \right)^2. \quad (100)$$

The value for $u = v = 1$ is equal to $(\phi_0(0))^2 = 1/5$, which also corresponds to the value found previously. In general, it is easy to show that

$$\frac{v}{u} \phi_0(0) > \phi_0(0) - \phi_0(2). \quad (101)$$

This implies that configurations with pairs of monomers perpendicular to the ladder have higher weight than those parallel to the ladder, for any value of u and v .

6. Insertion of additional monomers for $z \neq 0$

In this section, we would like to analyze the addition of one monomer at site location i_1 on the lower branch, or on the upper branch at site i_0 , in an infinite system with $z \neq 1$, and compare the two cases. We first consider the partition function after inserting one monomer on the lower branch

$$Z_{\bar{d}} = \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) \left(\prod_{i=1}^{i_1} \exp(z d_i) \right) \bar{z} \bar{d}_{i_1} \exp(-2ud_{i_1} d_{i_1+1}) \left(\prod_{i=i_1+1}^L \exp(z d_i) \right) \exp(S). \quad (102)$$

This expression is similar to (50), except we impose the addition of one monomer at the location i_1 on the lower branch, or $\bar{n}_{i_1} = 1$. We can take $\bar{z} = 1$ since there is no reservoir on the corresponding branch. On the other hand, we can insert one monomer on the upper branch, and consider in this case the following integral representation of the partition function, which is the same as (50)

$$Z_d = \int \left(\prod_{i=1}^L d\bar{d}_i d d_i \right) \left(\prod_{i=1}^{i_0-1} \exp(z d_i) \right) z d_{i_0} \left(\prod_{i=i_0+1}^L \exp(z d_i) \right) \exp(S). \quad (103)$$

We can compare $Z_{\bar{d}}$ with Z_d by evaluating their ratio. The partition functions $Z_{\bar{d}}$ and Z_d should be independent of i_1 and i_0 due to the periodic conditions. It is quite straightforward to show that the expression for $Z_{\bar{d}}$ is equivalent to compute the following integral, after reordering the exponentials and gathering all the relevant terms

$$Z_{\bar{d}} = z \int \left(\prod_{i=1}^L d\bar{d}_i dd_i \right) \bar{d}_{i_1} \exp(-2ud_{i_1} d_{i_1+1}) \left(\sum_{i=i_1+1}^L d_i - \sum_{i=1}^{i_1} d_i \right) \exp(S). \quad (104)$$

The integral includes quadratic as well as quartic terms. In the Fourier space, this can be decomposed as

$$\begin{aligned} \frac{Z_{\bar{d}}}{Z} &= z \left\langle \bar{d}_{i_1} (1 - 2ud_{i_1} d_{i_1+1}) \left(\sum_{i=i_1+1}^L d_i - \sum_{i=1}^{i_1} d_i \right) \right\rangle_S \\ &= \frac{2z}{L} \sum_{k,k'} \langle \bar{d}_{k+\frac{1}{2}} d_{k'+\frac{1}{2}} \rangle_S \frac{e^{-i\omega i_1 + i\omega' (i_1+1)}}{1 - e^{i\omega'}} \\ &\quad - \frac{4uz}{L^2} \sum_{k,k',k'',k'''} \langle \bar{d}_{k+\frac{1}{2}} d_{k'+\frac{1}{2}} d_{k''+\frac{1}{2}} d_{k'''+\frac{1}{2}} \rangle_S \frac{e^{-i\omega i_1 + i\omega' i_1 + i\omega'' (i_1+1) + i\omega''' (i_1+1)}}{1 - e^{i\omega'''}}. \end{aligned} \quad (105)$$

where Z is the non-perturbed partition function. The computation of all the contractions for the quartic terms with the adequate selection of momenta, see equations (70), leads to the following result

$$\frac{Z_{\bar{d}}}{Z} = \frac{zv}{u} [\phi(0) + 2u (\phi(0)^2 - 2\phi(1)^2 + \phi(0)\phi(2))]. \quad (106)$$

As expected, this quantity is independent of the site location i_1 . Similarly, we can evaluate Z_d which is equal to $Z_d = 2z^2 (\phi(0) + \phi(1)) Z = nZ$, independent of the site i_0 . The energy difference ΔE between the two configurations, when adding a single monomer on the lower and upper branch respectively, can be evaluated using the free energy difference

$$\begin{aligned} \Delta F &= -T \log(z Z_{\bar{d}} / Z_d) = \Delta E - T \Delta S = \Delta E + T \frac{\partial \Delta F}{\partial T} \\ &= \epsilon_v - \epsilon_u - T \log(\tilde{z}), \quad \tilde{z} = \frac{\phi(0) + 2u (\phi(0)^2 - 2\phi(1)^2 + \phi(0)\phi(2))}{2(\phi(0) + \phi(1))}, \end{aligned} \quad (107)$$

where the fugacity z in the logarithm comes from the Legendre transformation relating the grand potential to the free energy. We find therefore that

$$\Delta E = \epsilon_v - \epsilon_u + T^2 \frac{\partial \log(\tilde{z})}{\partial T}. \quad (108)$$

If $\Delta E < 0$, the addition of an extra monomer on the lower branch is favorable in comparison with the cost of an additional monomer on the upper branch, whereas the energy cost is unfavorable in the opposite case, $\Delta E > 0$. In figure 10, we have plotted ΔE as a function of temperature, for a constant density of monomers $n = 0.2$. For each curve, the relative binding energy $\epsilon_v - \epsilon_u$ varies from -1 to 1 by increment of 0.1. When $\epsilon_v < \epsilon_u$, the zero temperature value of ΔE is simply equal to $\epsilon_v - \epsilon_u$, which corresponds to the constant term in (108), whereas it is equal to zero for

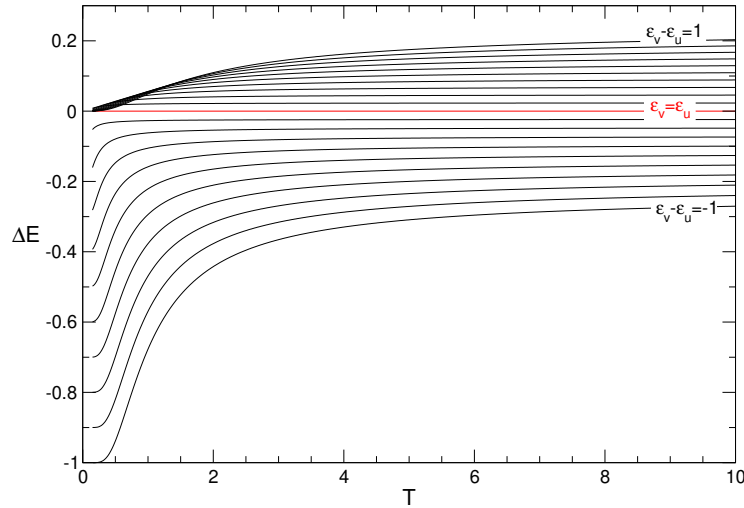


Figure 10: Energy difference ΔE as function of the temperature and relative binding energy $\epsilon_v - \epsilon_u$, at constant density $n = 0.2$.

$\epsilon_v \geq \epsilon_u$ when the dimers tend to align parallel to the ladder in the ground state. This reflects the difference of the ground state structures in both cases.

7. Conclusion

In this manuscript, we have considered the thermodynamics of a mixture of monomers and dimers on a ladder geometry. The restriction to a ladder geometry instead of a square lattice is due to the observation that their configurational entropies per site are very similar, suggesting the possibility of relatively complex physical phenomena in the former case. Also, in this particular geometry, we can evaluate exactly the different correlation functions between monomers, which present an oscillating behavior due to the presence of two correlation lengths. As in the case of monomers on a square lattice, the analytical resolution of the thermodynamical quantities is hampered by the presence of non-local or disorder operators in the fermionic action that results from the Grassmannian formalism that we have applied to this model.

The entropy possesses a maximal value for a specific monomer density, depending on the temperature and relative value between the vertical and horizontal binding energies of the dimers, in the general anisotropic situation. In the isotropic case where the two energies are equal, the system has a zero specific heat and constant entropy for all temperatures, but this entropy increases with the monomer density until a maximal value equal to $S_e \simeq 0.3698$ at $n \simeq 0.3477$.

In the anisotropic case, the zero temperature entropy presents a maximum value equivalent to the configurational entropy of the pure dimer system for a specific monomer density that depends on the relative value of the binding energies. This can be partially explained by a formal pairing of individual monomers which are close neighbors and for which the chemical energy is adjusted

in such a way that the pairing has the same Boltzmann weight as one dimer. This occurs only for a specific density of monomers, and we have shown that the chemical potential is locally almost independent of the temperature.

The specific heat is characterized by a broad peak or Schottky anomaly, which can be accurately represented by a modified two-level system at all densities. The variation of the peak is non-monotonic as the density of monomers varies in the domain $\epsilon_u < \epsilon_v$. We have seen that the macroscopic entropy at zero temperature and the characteristics of the specific heat are typical of dilute spin-ice magnetic material in the presence of disorder where some sites are replaced by non-magnetic atoms. The physics of spin-ice is based on specific spin local orientation conservation rules. There is no phase transition, and the specific heat also possesses a Schottky anomaly, with the location of the peak varying non-uniformly with the density of disorder. The disorder is similar to the presence of monomers. The analogies found between the two models can be explained by the presence of frustration in both cases, disorder, macroscopic degeneracy and residual entropy. Here the spin-ice rules are replaced by the constraint that each site can share only one dimer with one of its neighbors in absence of defects.

Appendix A. Asymptotic limit of the Fourier sum for the density

In this appendix, we consider the thermodynamical limit of the Fourier sum representing the density expression (49). More specifically, we would like to consider the discrete expression

$$n = \frac{4uz^2}{L} \sum_{\omega} \frac{\cos^2(\omega/2)}{v^2 + 4u^2 \sin^2(\omega) + 4uz^2 \cos^2(\omega/2)}, \quad (\text{A.1})$$

and take the limit $L \rightarrow \infty$. We first decompose the denominator $z(\omega) = v^2 + 4u^2 \sin^2(\omega) + 4uz^2 \cos^2(\omega/2)$ into a factorized polynomial in $\cos(\omega)$

$$z(\omega) = 4u^2(\zeta_+ - \cos(\omega))(\zeta_- + \cos(\omega)), \quad (\text{A.2})$$

with coefficients $\zeta_+ > \zeta_- > 1$ defined previously by formula (46), so that we can decompose the trigonometric integrand as

$$\begin{aligned} n &= \frac{z^2}{4\pi u} \int_0^{2\pi} d\omega \frac{1 + \cos(\omega)}{(\zeta_+ - \cos(\omega))(\zeta_- + \cos(\omega))} \\ &= \frac{z^2}{4\pi u(\zeta_- + \zeta_+)} \int_0^{2\pi} d\omega (1 + \cos(\omega)) \left(\frac{1}{\zeta_+ - \cos(\omega)} + \frac{1}{\zeta_- + \cos(\omega)} \right). \end{aligned} \quad (\text{A.3})$$

We then exponentiate the fractions by introducing a secondary integration

$$\begin{aligned} n &= \frac{z^2}{4\pi u(\zeta_- + \zeta_+)} \int_0^{\infty} d\lambda \int_0^{2\pi} d\omega (1 + \cos(\omega)) \left[\exp(-\lambda\zeta_+ + \lambda \cos(\omega)) \right. \\ &\quad \left. + \exp(-\lambda\zeta_- - \lambda \cos(\omega)) \right], \end{aligned} \quad (\text{A.4})$$

in order to integrate over the variable ω , which leads to an expression involving modified Bessel functions

$$n = \frac{z^2}{2u(\zeta_- + \zeta_+)} \int_0^\infty d\lambda \left[\exp(-\lambda\zeta_+) (I_0(\lambda) + I_1(\lambda)) + \exp(-\lambda\zeta_-) (I_0(\lambda) - I_1(\lambda)) \right]. \quad (\text{A.5})$$

The final integration over λ is performed after using the known formula, valid for $\zeta > 1$ [40]

$$\int_0^\infty d\lambda \exp(-\zeta\lambda) I_d(\lambda) = \frac{1}{\sqrt{\zeta^2 - 1}} (\zeta + \sqrt{\zeta^2 - 1})^{-d}. \quad (\text{A.6})$$

We finally obtain the explicit result

$$\begin{aligned} n &= \frac{z^2}{2u(\zeta_- + \zeta_+)} \left[\frac{1}{\sqrt{\zeta_+^2 - 1}} \left(1 + \frac{1}{\zeta_+ + \sqrt{\zeta_+^2 - 1}} \right) \right. \\ &\quad \left. + \frac{1}{\sqrt{\zeta_-^2 - 1}} \left(1 - \frac{1}{\zeta_- + \sqrt{\zeta_-^2 - 1}} \right) \right] \\ &= \frac{z^2}{2u(\zeta_- + \zeta_+)} \left(\sqrt{\frac{\zeta_+ + 1}{\zeta_+ - 1}} - \sqrt{\frac{\zeta_- - 1}{\zeta_- + 1}} \right). \end{aligned} \quad (\text{A.7})$$

Appendix B. Asymptotic limit of Fourier sums for the correlation functions

Appendix B.1. Correlation functions, quadratic contribution

We consider the following Fourier sum, contributing to the average $\langle d_{i_0} d_{i_1} \rangle_S$ between Grassmann variables with the action S

$$\phi(d) = \frac{u}{L} \sum_{\omega} \frac{\cos(\omega d)}{z(\omega)}. \quad (\text{B.1})$$

From the previous techniques, we can take the thermodynamical limit and decompose the denominator as

$$\phi(d) = \frac{1}{8\pi u(\zeta_- + \zeta_+)} \int_0^{2\pi} d\omega \cos(\omega d) \left(\frac{1}{\zeta_+ - \cos(\omega)} + \frac{1}{\zeta_- + \cos(\omega)} \right).$$

We then introduce as before a secondary integral over λ in order to perform the integration over ω , and then use the symmetry relation $I_d(-\lambda) = (-1)^d I_d(\lambda)$ to obtain the following expression

$$\phi(d) = \frac{1}{4u(\zeta_- + \zeta_+)} \int_0^\infty d\lambda \left[\exp(-\zeta_+ \lambda) + (-1)^d \exp(-\zeta_- \lambda) \right] I_d(\lambda).$$

We then apply the formula (A.6), which yields

$$\phi(d) = \frac{1}{4u(\zeta_- + \zeta_+)} \left(\frac{\alpha_+^{-d}}{\sqrt{\zeta_+^2 - 1}} + \frac{(-\alpha_-)^{-d}}{\sqrt{\zeta_-^2 - 1}} \right), \quad (\text{B.2})$$

where we introduced the notation $\alpha_{\pm} = \zeta_{\pm} + \sqrt{\zeta_{\pm}^2 - 1}$. In particular, the difference $\phi(d-1) - \phi(d+1)$ can be written as

$$\phi(d-1) - \phi(d+1) = \frac{1}{2u(\zeta_- + \zeta_+)} (\alpha_+^{-d} - (-1)^d \alpha_-^{-d}). \quad (\text{B.3})$$

Since $\alpha_+ > \alpha_-$, this is a sign oscillating function which is negative when d is even, and positive when d is odd.

Appendix B.2. Correlation functions, quartic contribution

The other terms contributing to the correlation function between monomers and coming from the product $\langle d_{i_0} d_{i_1} Q_{i_0 i_1} \rangle_S$ can be decomposed in the Fourier space, using (73), as

$$\begin{aligned} z^4 \langle d_{i_0} d_{i_1} Q_{i_0 i_1} \rangle_S &= \frac{z^4}{L^2} \sum_{k, k'} \sum_{k'', k'''} \left\langle d_{k+\frac{1}{2}} d_{k'+\frac{1}{2}} d_{k''+\frac{1}{2}} d_{k'''+\frac{1}{2}} \right\rangle_S e^{i\omega i_0 + i\omega' i_1} \\ &\times \left(\frac{e^{i\omega'' i_1} - e^{i\omega''(i_0+1)}}{1 - e^{i\omega''}} \right) \left(\frac{e^{i\omega''' i_0} + e^{i\omega'''(i_1+1)}}{1 - e^{i\omega'''}} \right). \end{aligned} \quad (\text{B.4})$$

There are three possible pair contractions for the quartic product inside the bracket, see (73). We will label each contraction R_a , R_b and R_c . The first term R_a corresponds to the constraint $\delta_{k+k'+1} \delta_{k''+k'''+1}$, or

$$\begin{aligned} R_a &= \frac{z^2}{L} \sum_k' \left\langle d_{k+\frac{1}{2}} d_{-k-\frac{1}{2}} \right\rangle_S (e^{-i\omega d} - e^{i\omega d}) \\ &\times \frac{z^2}{L} \sum_{k''}' \left\langle d_{k''+\frac{1}{2}} d_{-k''-\frac{1}{2}} \right\rangle_S \left(\frac{e^{i\omega'' i_1} - e^{i\omega''(i_0+1)}}{1 - e^{i\omega''}} \frac{e^{-i\omega'' i_0} + e^{-i\omega''(i_1+1)}}{1 - e^{-i\omega''}} - (\omega'' \rightarrow -\omega'') \right) \\ &= \frac{4uz^2}{L} \sum_{\omega}' \frac{\sin(\omega)}{z(\omega)} \sin(\omega d) \times \frac{4uz^2}{L} \sum_{\omega''}' \frac{\sin(\omega'')}{z(\omega'')} \frac{\sin(\omega'' d) - \sin(\omega'' d)}{1 - \cos(\omega'')}. \end{aligned} \quad (\text{B.5})$$

The first sum over ω on the right-hand side of the previous expression is equal to $z^2(\phi(d-1) - \phi(d+1))$ after combination of the trigonometric functions. The second sum over ω' can be decomposed as

$$\begin{aligned} \frac{4uz^2}{L} \sum_{\omega''}' \frac{\sin(\omega'')}{z(\omega'')} \frac{\sin(\omega'' d) - \sin(\omega'' d)}{1 - \cos(\omega'')} &= \frac{8uz^2}{L} \sum_{\omega''}' \frac{\cos^2(\omega''/2)}{z(\omega'')} \\ &- \frac{2uz^2}{L} \sum_{\omega''}' \frac{\cos(\omega''(d-1)) - \cos(\omega''(d+1))}{z(\omega'')(1 - \cos(\omega''))}. \end{aligned} \quad (\text{B.6})$$

The first sum on the right hand side corresponds to n , whereas the rational cosine function of the second sum can be decomposed into a sum of linear cosine functions

$$\frac{\cos(\omega''(d-1)) - \cos(\omega''(d+1))}{1 - \cos(\omega'')} = 2 + 4 \sum_{l=1}^{d-1} \cos(\omega'' l) + 2 \cos(\omega'' d). \quad (\text{B.7})$$

We can therefore express R_a uniquely in terms of functions $\phi(d)$

$$R_a = z^2 \left(\phi(d-1) - \phi(d+1) \right) \left[n - 2z^2 \left(\phi(0) + 2 \sum_{l=1}^{d-1} \phi(l) + \phi(d) \right) \right]. \quad (\text{B.8})$$

Taking into account the expression (B.2) for $\phi(d)$ to perform the geometric sum over l , we obtain

$$\begin{aligned} \phi(0) + 2 \sum_{l=1}^{d-1} \phi(l) + \phi(d) &= \frac{1}{4u(\zeta_- + \zeta_+)} \left[\frac{1}{\sqrt{\zeta_+^2 - 1}} \frac{\alpha_+ + 1}{\alpha_+ - 1} (1 - \alpha_+^{-d}) \right. \\ &\quad \left. + \frac{1}{\sqrt{\zeta_-^2 - 1}} \frac{\alpha_- - 1}{\alpha_- + 1} (1 - (-\alpha_-)^{-d}) \right] \\ &= \frac{z^2}{2u(\zeta_- + \zeta_+)} \left[\frac{\alpha_+}{(\alpha_+ - 1)^2} (1 - \alpha_+^{-d}) + \frac{\alpha_-}{(\alpha_- + 1)^2} (1 - (-\alpha_-)^{-d}) \right] \\ &= \frac{\tilde{n}}{4} - z^2 \tilde{\phi}(d), \end{aligned} \quad (\text{B.9})$$

where $\tilde{\phi}(d)$ corresponds to the exponential decay and \tilde{n} to a constant defined by

$$\tilde{n} = \frac{2z^2}{u(\zeta_- + \zeta_+)} \left[\frac{\alpha_+}{(\alpha_+ - 1)^2} + \frac{\alpha_-}{(\alpha_- + 1)^2} \right]. \quad (\text{B.10})$$

The factor $1/4$ in the last line of (B.9) ensures that $\tilde{n} \rightarrow 1$ when $z \rightarrow \infty$. In particular, we have $\tilde{n} \simeq 1 - v^2/(4uz^2)$ asymptotically. We can finally write R_a as

$$R_a = z^2 \left(\phi(d-1) - \phi(d+1) \right) \left(n - \frac{\tilde{n}}{2} + 2z^2 \tilde{\phi}(d) \right). \quad (\text{B.11})$$

The second quantity R_b corresponds to the contraction $-\delta_{k+k''+1} \delta_{k'+k'''+1}$ with the minus sign coming from pairing $d_{k+\frac{1}{2}}$ with $d_{k''+\frac{1}{2}}$. From equation (B.4), R_b can be factorized into a product of two sums

$$\begin{aligned} R_b &= \frac{2uz^2}{L} \sum_{\omega}' \frac{\sin(\omega)}{z(\omega)} \left[\frac{\sin(\omega) + \sin(\omega(d-1)) - \sin(\omega d)}{1 - \cos(\omega)} \right] \\ &\quad \times \frac{2uz^2}{L} \sum_{\omega'}' \frac{\sin(\omega')}{z(\omega')} \left[\frac{\sin(\omega') + \sin(\omega'(d+1)) - \sin(\omega' d)}{1 - \cos(\omega')} \right]. \end{aligned} \quad (\text{B.12})$$

After simplifying the trigonometric expressions, we obtain

$$R_b = \left(\frac{n}{2} - z^2 \phi(d-1) - z^2 \phi(d) \right) \left(\frac{n}{2} + z^2 \phi(d+1) + z^2 \phi(d) \right). \quad (\text{B.13})$$

We can also show that the last term R_c is equal to R_b , after pairing $d_{k+\frac{1}{2}}$ with $d_{k'''+\frac{1}{2}}$. Therefore, we have the overall result

$$\begin{aligned} z^4 \langle d_{i_0} d_{i_1} Q_{i_0 i_1} \rangle_S &= R_a + R_b + R_c = R_a + 2R_b \\ &= z^2 \left(\phi(d-1) - \phi(d+1) \right) \left(-\frac{\tilde{n}}{2} + 2z^2 \tilde{\phi}(d) \right) \\ &\quad - 2z^4 \left(\phi(d-1) + \phi(d) \right) \left(\phi(d+1) + \phi(d) \right) + \frac{n^2}{2}, \end{aligned}$$

$$\begin{aligned}
G_1(d) &= z^2 \langle d_{i_0} d_{i_1} (1 + 2z^2 Q_{i_0 i_1}) \rangle_S - n^2 \\
&= z^2 (\phi(d-1) - \phi(d+1)) (1 - \tilde{n}) \\
&\quad + 4z^4 \left[\tilde{\phi}(d) (\phi(d-1) - \phi(d+1)) - (\phi(d-1) + \phi(d)) (\phi(d+1) + \phi(d)) \right].
\end{aligned} \tag{B.14}$$

The first term in the right-hand side of the last relation proportional to $(1 - \tilde{n})$ is a sign oscillating and exponentially decreasing function. The last term inside square brackets can be evaluated using the formula for $\phi(d)$ and $\tilde{\phi}(d)$, see equations (B.2) and (B.9). It leads to small oscillating corrections with inverse correlation length $\log(\alpha_- \alpha_+)$

$$\begin{aligned}
&4z^4 \left[\tilde{\phi}(d) (\phi(d-1) - \phi(d+1)) - (\phi(d-1) + \phi(d)) (\phi(d+1) + \phi(d)) \right] \\
&= - \frac{z^4 (\alpha_- + \alpha_+)^2}{u^2 (\zeta_- + \zeta_+)^2 (\alpha_- + 1)^2 (\alpha_+ - 1)^2} (-\alpha_- \alpha_+)^{-d}.
\end{aligned} \tag{B.15}$$

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