



**HAL**  
open science

# An Operational Quantum Information Framework for Experimental Studies on Color Perception

Roberto Leporini, Edoardo Provenzi, Michel Berthier

► **To cite this version:**

Roberto Leporini, Edoardo Provenzi, Michel Berthier. An Operational Quantum Information Framework for Experimental Studies on Color Perception. 2024. hal-04763826

**HAL Id: hal-04763826**

**<https://hal.science/hal-04763826v1>**

Preprint submitted on 2 Nov 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An Operational Quantum Information Framework for Experimental Studies on Color Perception

Roberto Leporini<sup>\*1</sup>, Edoardo Provenzi<sup>†2</sup> and Michel Berthier<sup>‡3</sup>

<sup>1</sup>Department of Economics, University of Bergamo, via dei Caniana 2, 24127,  
Bergamo, Italy

<sup>2</sup>Université de Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, F-33400, 351  
Cours de la Libération, Talence, France

<sup>3</sup>Laboratoire MIA, Batiment Pascal, Pôle Sciences et Technologie, Université de  
La Rochelle, 23, Avenue A. Einstein, BP 33060, 17031 La Rochelle cedex, France

October 31, 2024

## Abstract

Starting from the foundational axiomatization of the perceptual color space initiated by Schrödinger in 1920 and eventually refined by Resnikoff in 1974, Berthier, Provenzi and their collaborators have recently proposed a reformulation of perceptual color attributes within the framework of quantum information. Their work is based on the Jordan algebra formalism of quantum theories and, more specifically, on a quantum system described by a spin factor over the field of real numbers. This theoretical framework is not that of ordinary quantum mechanics, mainly because it requires dealing with rebits, whereas the latter uses qubits. The aim of this paper is to show that this difference in no way hinders the implementation of experimental protocols for testing the validity of the predictions of the color perception model. In particular, we show how to compute the quantum information based perceptual attributes of perceived colors in terms of qubit density matrices.

**Keywords:** Color perception, density matrices, color attributes

## 1 Introduction

In the paper [5], a mathematically rigorous vocabulary for color attributes has been built based on a prior analysis of color perception. This analysis, presented in a series of papers, provided theoretical evidence supporting the development of a quantum-relativistic theory of color perception free from internal contradictions. For more details, see e.g. [3, 6, 4, 7].

A distinctive feature of this theory is that, unlike the majority of quantum models, the state space is defined over the real field rather than the complex one. In fact, the quantum system of this theory is a *rebit*, the real analogue of a qubit. In this model, the two states correspond to the degrees of chromatic opposition as described in Hering's theory of color vision [10]. To the best of our knowledge, this color perception model is the first factual example of a real quantum theory, which had previously been explored only theoretically.

It is a challenging problem to validate this new theory experimentally, for example by testing some of its predictions. In tests designed to characterize a color perceived by a human observer,

---

\*roberto.leporini@unibg.it

†edoardo.provenzi@math.u-bordeaux.fr

‡michel.berthier@univ-lr.fr

photons of visible light are absorbed by the retinal photoreceptors, changing their electrical potential and triggering a very complicated chain of events that culminate in the brain's creation of a color sensation. The perceptual attributes associated with this sensation are defined in the proposed theory from the generalized state of the rebit given by the result of a Lüders measurement operation. Therefore, in order to perform experiments, we must take into account the inherent fact that photons are described in quantum mechanics using qubits, while color sensations are described in the quantum model of color perception using rebits.

Replacing the field of complex numbers of ordinary quantum mechanics by the field of real numbers is not a trivial matter. For instance, it is underlined in [12] that in real-vector-space quantum mechanics, one cannot in general determine the density matrix of a composite system using only local measurements whereas in complex-vector-space quantum mechanics any sets of measurements which are just sufficient for determining the states of the subsystems are, when performed jointly, just sufficient for determining the state of the complete system.

Another example is highlighted in [1], where it is shown that Choi's theorem cannot be applied in the real-vector-space framework to obtain the classification of rebit channels and another strategy must be used.

Nevertheless, if we limit the scope of the experiments to be performed, it is possible to express some of the colorimetric concepts defined in the rebit framework by means of density matrices associated with qubit states. It is precisely the aim of this paper to elaborate on this last point.

The paper is structured as follows: in section 2, we recall the basic facts about the quantum model of color perception together with the quantum information-based definitions of the perceptual attributes that characterize perceived colors.

In section 3, we explain how to compute these perceptual attributes using qubit states and effects, rather than using rebit generalized states and effects. In doing so, we propose an operational framework for conducting color perception experiments.

Finally, in the conclusions we discuss some perspectives for designing such experiments that we deem able to test the quantum color perception model.

## 2 Hering's rebit of color perception

The historical, physiological and psychophysical motivations as well as the main stages of the mathematical reasoning that led to the proposal of the model of color perception described below have already been described in detail, e.g. in the articles [5] and [7].

### 2.1 Perceived colors and color measurements

The relevant algebraic structure of this model is the spin factor  $\mathbb{R} \oplus \mathbb{R}^2$  whose Jordan product is given by  $(\alpha, \mathbf{v}) \circ (\beta, \mathbf{w}) = (\alpha\beta + \mathbf{v} \cdot \mathbf{w}, \alpha\mathbf{w} + \beta\mathbf{v})$ , with  $\alpha, \beta \in \mathbb{R}$  and  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ , and where  $\mathbf{v} \cdot \mathbf{w}$  denotes the Euclidean inner product of  $\mathbb{R}^2$ . It is a non-associative commutative formally real Jordan algebra of rank 2 and of real dimension 3 which, as a Jordan algebra, is isomorphic to the algebra  $\mathcal{H}(2, \mathbb{R})$  of real symmetric  $2 \times 2$  matrices endowed with the Jordan matrix product, i.e.  $A \circ B = (AB + BA)/2$ ,  $A, B \in \mathcal{H}(2, \mathbb{R})$ . An explicit isomorphism is given by

$$\chi : \begin{array}{ccc} \mathcal{H}(2, \mathbb{R}) & \xrightarrow{\sim} & \mathbb{R} \oplus \mathbb{R}^2 \\ A = \begin{pmatrix} \alpha + v_1 & v_2 \\ v_2 & \alpha - v_1 \end{pmatrix} & \longmapsto & \chi(A) = \begin{pmatrix} \alpha \\ \mathbf{v} \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \end{array} \quad (1)$$

For more information on Jordan algebras, see e.g. [2]. The trace of an element  $(\alpha, \mathbf{v})$  of  $\mathbb{R} \oplus \mathbb{R}^2$  is equal to  $2\alpha$ , so the unit trace elements of the spin factor are in one-to-one correspondence via the isomorphism  $\chi$  with the density matrices of the rebit, i.e. the unit trace matrices belonging to the domain of positivity  $\overline{\mathcal{H}^+}(2, \mathbb{R})$  of the algebra  $\mathcal{H}(2, \mathbb{R})$ . We denote

$$\mathcal{S} = \left\{ \rho(s_1, s_2) \equiv \frac{1}{2} \begin{pmatrix} 1 + s_1 & s_2 \\ s_2 & 1 - s_1 \end{pmatrix}, s_1, s_2 \in \mathbb{R}, s_1^2 + s_2^2 \leq 1 \right\} \cong \mathcal{D}, \quad (2)$$

the state space of the rebit, where  $\mathcal{D}$  is the unit disk in  $\mathbb{R}^2$ , referred to as the Bloch disk.

Vectors of the Bloch disk represent chromatic states, and Hering's color opponency involving the two pairs of unique hues is encoded by means of the two Pauli matrices with real entries

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3)$$

In the sequel, we will denote with  $\sigma_0$  the  $2 \times 2$  identity matrix  $I_2$ .

As in the case of qubits, we can decompose any density matrix of the rebit using the basis  $(\sigma_0, \sigma_1, \sigma_2)$ :

$$\rho(s_1, s_2) = \rho_0 + \frac{s_1}{2}\sigma_1 + \frac{s_2}{2}\sigma_2 \equiv \rho_0 + \frac{1}{2}\mathbf{v}_\rho \cdot \vec{\sigma}, \quad (4)$$

where  $\rho_0 := \sigma_0/2$  and the components of the Bloch vector  $\mathbf{v}_\rho = (s_1, s_2) = (\text{Tr}(\rho\sigma_1), \text{Tr}(\rho\sigma_2)) \equiv (\langle\sigma_1\rangle_\rho, \langle\sigma_2\rangle_\rho)$  are the  $\rho$ -expectation values of the real Pauli matrices. Using polar coordinates, i.e.  $(s_1, s_2) = (r \cos \vartheta, r \sin \vartheta)$ ,  $r \in [0, 1]$ ,  $\vartheta \in [0, 2\pi)$ , we can write

$$\rho(r, \vartheta) = \rho_0 + \frac{\langle\sigma_1\rangle_\rho}{2} [\rho(1, 0) - \rho(1, \pi)] + \frac{\langle\sigma_2\rangle_\rho}{2} [\rho(1, \pi/2) - \rho(1, 3\pi/2)]. \quad (5)$$

For all  $\vartheta$ ,  $\rho(1, \vartheta)$  is a rank-1 projector, i.e. a pure state, and  $\rho(1, \vartheta_1)$ ,  $\rho(1, \vartheta_2)$  project on orthogonal directions precisely when  $\vartheta_1$  and  $\vartheta_2$  correspond to antipodal points on the unit circle. Since orthogonality in quantum theories represents incompatible states, eq. (5) codifies a *generic chromatic state* as the *superposition of two chromatic opponencies between incompatible states*, red-green and yellow-blue in Hering's theory, see [10], weighted by the expectation values of the real density matrices, *plus an offset state* represented by  $\rho_0$ .

The density matrix  $\rho_0$ , associated with the center of the Bloch disk  $\mathcal{D}$ , is the maximally mixed state, characterized by the fact of having maximal von Neumann entropy, defined as  $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ , and not carrying any chromatic information. Therefore, the density matrix  $\rho_0$  represents the *achromatic state*, and eq. (5) is exactly the quantum description of the chromatic information that can be gathered from an isolated color stimulus in Hering's theory, see e.g. [11].

The new paradigm at the heart of the model of color perception is based on the interpretation of perceived colors as the results of measurements performed by observers from chromatic states. Perceived colors are no longer simple triplets of coordinates in color spaces derived from XYZ space, but are produced by the duality between states and effects.

The domain of positivity of the spin factor  $\mathbb{R} \oplus \mathbb{R}^2$  is the closure  $\overline{\mathcal{L}^+}$  of the future lightcone in the 3-dimensional Minkowski space. Constraints such as the visibility threshold and the glare limit impose to restrict the set of positive observables to a convex solid of finite volume of  $\overline{\mathcal{L}^+}$ , called color solid in colorimetry.

Effects encode the probabilistic nature of quantum measurements and permit to construct in a natural way a color solid. To see how, we recall that an effect is defined to be an element  $\eta_{\mathbf{e}}$  of  $\overline{\mathcal{H}^+}(2, \mathbb{R})$  bounded between the null and the identity matrix  $\sigma_0$  (with respect to the Loewner ordering of positive semi-definite matrices, i.e.  $B \leq A \iff A - B \in \overline{\mathcal{H}^+}(2, \mathbb{R})$ ). The matrix  $\eta_{\mathbf{e}}$  can be written explicitly as follows

$$\eta_{\mathbf{e}} = \begin{pmatrix} e_0 + e_1 & e_2 \\ e_2 & e_0 - e_1 \end{pmatrix}, \quad (6)$$

with  $e_0, e_1, e_2 \in \mathbb{R}$  belonging to the following effect space:

$$\mathcal{E} = \{(e_0, e_1, e_2) \in \mathbb{R}^3, e_0 \in [0, 1], e_1^2 + e_2^2 \leq \min\{(1 - e_0)^2, e_0^2\}\}. \quad (7)$$

The effect vector in the Bloch disk associated to  $\eta_{\mathbf{e}}$  is given by

$$\mathbf{v}_{\mathbf{e}} := \begin{pmatrix} \frac{e_1}{e_0} & \frac{e_2}{e_0} \end{pmatrix}, \quad e_0 \neq 0. \quad (8)$$

As it can be seen in Figure 1, the effect space  $\mathcal{E}$  is a closed convex double cone with a circular basis of radius  $1/2$  located at height  $e_0 = 1/2$  and vertices in  $(0, 0, 0)$  and  $(1, 0, 0)$ , associated to the null and the unit effect, respectively [7].  $\mathcal{E}$  represents the space of colors that can actually be sensed thanks to a perceptual measurement performed by an observer and its geometry happens to be in perfect agreement with that of the color solid advocated by Ostwald and de Valois, see e.g. [9].

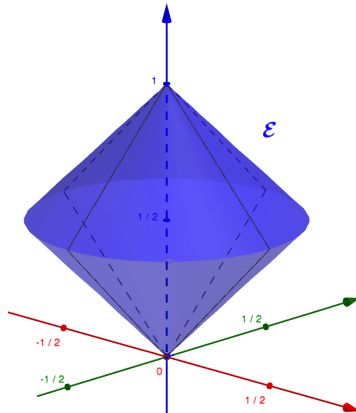


Figure 1: The double cone representing the effect space can be interpreted as the color solid of actually perceived colors inside the infinite cone  $\overline{\mathcal{L}^+}$ .

Effects parameterize a fundamental class of state transformations called *Lüders operations*, which are *convex-linear positive functions*  $\psi_{\mathbf{e}}$  defined on the state space  $\mathcal{S}$  and satisfying the constraint:

$$0 \leq \text{Tr}(\psi_{\mathbf{e}}(\rho)) \leq 1, \quad \text{for all } \rho \in \mathcal{S}. \quad (9)$$

After a Lüders operation,  $\rho$  becomes a *generalized density matrix*, whose set will be denoted with  $\tilde{\mathcal{S}}$ , representing a *post-measurement generalized state* which does not belong to the Bloch disk anymore, but to the cone generated by  $\mathcal{S}$ . The analytical expression of the post-measurement generalized state  $\psi_{\mathbf{e}}(\rho)$ , see [8], is:

$$\psi_{\mathbf{e}}(\rho) = \eta_{\mathbf{e}}^{1/2} \rho \eta_{\mathbf{e}}^{1/2}, \quad (10)$$

where  $\eta_{\mathbf{e}}^{1/2}$  is the Kraus operator associated to  $\mathbf{e}$ , the only symmetric and positive semi-definite matrix such that  $\eta_{\mathbf{e}}^{1/2} \eta_{\mathbf{e}}^{1/2} = \eta_{\mathbf{e}}$ .

Given a chromatic state  $\mathbf{s} = (s_1, s_2)$  and the corresponding density matrix  $\rho_{\mathbf{s}} := \rho(s_1, s_2)$  of  $\mathcal{S}$ , we denote  $\psi_{\mathbf{e}}(\mathbf{s}) := \psi_{\mathbf{e}}(\rho_{\mathbf{s}})$ . Thanks to the cyclic property of the trace,

$$\text{Tr}(\psi_{\mathbf{e}}(\mathbf{s})) = \text{Tr}(\rho_{\mathbf{s}} \eta_{\mathbf{e}}) = \langle \mathbf{e} \rangle_{\mathbf{s}} = e_0(1 + \mathbf{v}_{\mathbf{e}} \cdot \mathbf{v}_{\mathbf{s}}), \quad (11)$$

and

$$\varphi_{\mathbf{e}}(\mathbf{s}) := \frac{\psi_{\mathbf{e}}(\mathbf{s})}{\langle \mathbf{e} \rangle_{\mathbf{s}}} \quad (12)$$

is a density matrix representing to a genuine chromatic state belonging to  $\mathcal{S}$ .

It is shown in [7] that the state change  $\mathbf{s} \mapsto \psi_{\mathbf{e}}(\mathbf{s})$  can be computed with a 3-dimensional normalized Lorentz boost in the direction of  $\mathbf{v}_{\mathbf{e}}$ , which implies

$$\chi(\psi_{\mathbf{e}}(\mathbf{s})) = e_0(1 + \mathbf{v}_{\mathbf{e}} \cdot \mathbf{v}_{\mathbf{s}}) \frac{1}{2} \begin{pmatrix} 1 \\ \mathbf{v}_{\mathbf{e}} \oplus \mathbf{v}_{\mathbf{s}} \end{pmatrix} \in \tilde{\mathcal{S}}. \quad (13)$$

As a consequence the Bloch vector associated to the state  $\varphi_{\mathbf{e}}(\mathbf{s})$  is the vector

$$\mathbf{v}_{\varphi_{\mathbf{e}}(\mathbf{s})} = \mathbf{v}_{\mathbf{e}} \oplus \mathbf{v}_{\mathbf{s}}, \quad (14)$$

where  $\oplus$  denotes the Einstein-Poincaré relativistic sum. When  $\|\mathbf{v}_e\| < 1$ ,  $\oplus$  is defined by

$$\mathbf{v}_e \oplus \mathbf{v}_s := \frac{1}{1 + \mathbf{v}_e \cdot \mathbf{v}_s} \left\{ \mathbf{v}_e + \frac{1}{\gamma_{\mathbf{v}_e}} \mathbf{v}_s + \frac{\gamma_{\mathbf{v}_e}}{1 + \gamma_{\mathbf{v}_e}} (\mathbf{v}_e \cdot \mathbf{v}_s) \mathbf{v}_e \right\}, \quad (15)$$

where  $\gamma_{\mathbf{v}_e}$  is the  $\mathbf{e}$ -Lorentz factor

$$\gamma_{\mathbf{v}_e} := \frac{1}{\sqrt{1 - \|\mathbf{v}_e\|^2}}, \quad (16)$$

and, when  $\|\mathbf{v}_e\| = 1$ ,  $\oplus$  is defined as follows

$$\mathbf{v}_e \oplus \mathbf{v}_s := \mathbf{v}_e. \quad (17)$$

By convex-linearity, Lüders operations can be extended to generalized states as follows:

$$\psi_e(s_0 \mathbf{s}) = s_0 \psi_e(\mathbf{s}), \quad \forall s_0 \in [0, 1], \quad (18)$$

this implies

$$\langle \mathbf{e} \rangle_{s_0 \mathbf{s}} = \text{Tr}(s_0 \rho_s \eta_e) = s_0 \text{Tr}(\rho_s \eta_e) = s_0 \langle \mathbf{e} \rangle_{\mathbf{s}} = e_0 s_0 (1 + \mathbf{v}_e \cdot \mathbf{v}_s), \quad (19)$$

so

$$\varphi_e(s_0 \mathbf{s}) = \frac{\psi_e(s_0 \mathbf{s})}{\langle \mathbf{e} \rangle_{s_0 \mathbf{s}}} = \frac{s_0 \psi_e(\mathbf{s})}{s_0 \langle \mathbf{e} \rangle_{\mathbf{s}}} = \varphi_e(\mathbf{s}), \quad (20)$$

thus the post-measurement chromatic state depends solely on  $\mathbf{s}$  and not on  $s_0$ . This implies the formula

$$\psi_e(s_0 \mathbf{s}) = e_0 s_0 (1 + \mathbf{v}_e \cdot \mathbf{v}_s) \varphi_e(\mathbf{s}), \quad (21)$$

which shows explicitly how the chromatic information about the state  $\mathbf{s}$  and the expectation value of the effect  $\mathbf{e}$  on  $\mathbf{s}$  are *fused together* in the post-measurement generalized state  $\psi_e(s_0 \mathbf{s})$ .

In the case of an achromatic effect  $\mathbf{e}_a$ , for which  $\mathbf{v}_{\mathbf{e}_a} = \mathbf{0}$ , the previous formula gives

$$\psi_{\mathbf{e}_a}(s_0 \mathbf{s}) = e_0 s_0 \varphi_{\mathbf{e}_a}(\mathbf{s}), \quad (22)$$

but  $\eta_{\mathbf{e}_a}^{1/2} = \sqrt{e_0} \sigma_0$  so, by eq. (10),

$$\psi_{\mathbf{e}_a}(s_0 \mathbf{s}) = e_0 s_0 \rho_s, \quad (23)$$

hence  $\varphi_{\mathbf{e}_a}(\mathbf{s}) = \rho_s$ , or, by identifying  $\rho_s$  with the chromatic state  $\mathbf{s}$ ,

$$\varphi_{\mathbf{e}_a}(\mathbf{s}) = \mathbf{s}, \quad (24)$$

this means that the post-measurement state induced by the action of an achromatic effect coincides with the original state.

## 2.2 The quantum information of perceived colors

According to the modeling described in the previous subsection, a formal framework for describing colorimetric perceptual attributes is provided by the following rules [5]:

- any quantity whose chromatic features manifest themselves fused together with a normalized scalar factor will be described through a *generalized state*;
- any act of (physical or perceptual) color measurement and the (physical or perceptual) medium used to perform it will be associated to an *effect*;
- the measurement outcome will be identified with the *post-measurement generalized state* induced by the action of the effect.

The color stimulus hitting the eyes of the (human) observer can be either a light *emitted* by a source of radiation or a light *reflected* from the patch of a surface lit by an illuminant. In the paper [5], both situations were analyzed, however here we will concentrate only on emitted light because the experiments that we aim at performing will be conducted in the scenario of photons emitted by a source and sent to the eyes of the observer.

The first set of definitions concerns the emitted light, the observer, and the color perceived by the observer from the emitted light.

**Def. 2.1 (Emitted light stimulus)** *An emitted light stimulus  $\ell$  is identified with the generalized state  $\ell_0\ell$ ,  $\ell_0 \in [0, 1]$  and  $\ell \in \mathcal{S}$ . The real quantity  $\ell_0$  is the normalized light intensity and  $\ell$  carries the intrinsic physical chromatic features.*

**Def. 2.2 (Achromatic and white light)** *An achromatic light is an emitted light stimulus with  $\ell_0 \in [0, 1]$  and  $\ell = \mathbf{s}_a$ . If, in particular,  $\ell_0 = 1$ , then we call it a white light and we write  $\ell^W = \mathbf{s}_a$ .*

Since a human observer is the medium through which a perceptual color measurement takes place, it is modeled as an effect.

**Def. 2.3 (Observer)** *An observer  $o$  measuring a color stimulus is identified with an effect  $\mathbf{o} = (o_0, \mathbf{v}_o) \in \mathcal{E}$ ,  $o_0 \in [0, 1]$  and  $\mathbf{v}_o \in \mathcal{D}$ .*

The act of measuring an emitted light stimulus  $\ell$  by an observer  $o$  produces a perceived color through the Lüders operation associated to the effect  $\mathbf{o}$ .

**Def. 2.4 (Perceived color from a light stimulus)** *Given an observer  $o$  and an emitted light stimulus  $\ell$ , i.e. the couple  $(\mathbf{o}, \ell_0\ell)$ , the color perceived by  $o$  from  $\ell$  is the post-measurement generalized state  $\psi_{\mathbf{o}}(\ell_0\ell) \in \tilde{\mathcal{S}}$ .*

This definition is coherent with the three-dimensional nature of perceived colors, in fact eq. (12) implies:

$$\psi_{\mathbf{o}}(\ell_0\ell) = o_0\ell_0(1 + \mathbf{v}_o \cdot \mathbf{v}_\ell) \varphi_{\mathbf{o}}(\ell) = \langle \mathbf{o} \rangle_{\ell_0\ell} \varphi_{\mathbf{o}}(\ell), \quad (25)$$

with  $\langle \mathbf{o} \rangle_{\ell_0\ell} \in [0, 1]$  and  $\varphi_{\mathbf{o}}(\ell) \in \mathcal{S}$ .

Thanks to eq. (24), we know that if an observer  $o_a$  is associated to an achromatic effect  $\mathbf{o}_a$ , then  $\varphi_{\mathbf{o}_a}(\ell) = \ell$ , which means that the chromatic state of the color perceived by  $o_a$  from the light source  $\ell = \ell_0\ell$  is exactly its intrinsic physical chromatic state  $\ell$ .

The second set of definitions concerns the achromatic perceptual attributes of perceived colors.

**Def. 2.5 (Brightness of a perceived color from an emitted light)** *Given an observer  $o$ ,  $\mathbf{o} = (o_0, \mathbf{v}_o)$ , the brightness of the color  $\psi_{\mathbf{o}}(\ell_0\ell)$  perceived by  $o$  from an emitted light stimulus  $\ell_0\ell$  is given by*

$$\mathcal{B}(\psi_{\mathbf{o}}(\ell_0\ell)) := \text{Tr}(\psi_{\mathbf{o}}(\ell_0\ell)) = o_0\ell_0(1 + \mathbf{v}_o \cdot \mathbf{v}_\ell). \quad (26)$$

Given any observer  $\mathbf{o} = (o_0, \mathbf{v}_o)$ , the brightness perceived by  $o$  from the white light  $\ell^W$  is:

$$\mathcal{B}(\psi_{\mathbf{o}}(\ell^W)) = o_0, \quad (27)$$

so the brightness of the white light does not depend on the effect vector of  $o$ .

**Def. 2.6 (Lightness of a perceived color from an emitted light)** *Given an observer  $o$ ,  $\mathbf{o} = (o_0, \mathbf{v}_o)$ , the lightness of the color  $\psi_{\mathbf{o}}(\ell_0\ell)$  perceived by  $o$  from an emitted light stimulus  $\ell_0\ell$  is given by the ratio between its brightness, eq. (26), and the brightness of the white light, eq. (27), i.e.*

$$\mathcal{L}(\psi_{\mathbf{o}}(\ell_0\ell)) := \frac{\mathcal{B}(\psi_{\mathbf{o}}(\ell_0\ell))}{\mathcal{B}(\psi_{\mathbf{o}}(\ell^W))} = \ell_0(1 + \mathbf{v}_o \cdot \mathbf{v}_\ell). \quad (28)$$

The lightness perceived from an achromatic emitted light coincides with its normalized intensity  $\ell_0$  independently of the observer:

$$\mathcal{L}(\psi_{\mathbf{o}}(\ell_0 \mathbf{s}_{\mathbf{a}})) = \ell_0, \quad \forall \mathbf{o}. \quad (29)$$

In particular, the lightness of the white light is normalized to 1.

When  $\mathbf{v}_{\mathbf{o}} = \mathbf{0}$ , the lightness of any color perceived from an emitted light coincides with the light intensity independently of the chromatic state of the emitted light:

$$\mathcal{L}(\psi_{\mathbf{o}_{\mathbf{a}}}(\ell_0 \boldsymbol{\ell})) = \ell_0, \quad \forall \boldsymbol{\ell}. \quad (30)$$

Finally, the last set of definitions concerns the chromatic perceptual attributes of perceived colors. Given the perceived color  $\psi_{\mathbf{o}}(\mathbf{s})$ , its saturation is

$$\begin{aligned} \text{Sat}(\psi_{\mathbf{o}}(\rho)) &= R(\rho_{\varphi_{\mathbf{o}}(\rho)} \parallel \rho_{\mathbf{0}}) \\ &= \frac{1}{2} \log_2(1 - r_{\varphi_{\mathbf{o}}(\rho)}^2) + \frac{r_{\varphi_{\mathbf{o}}(\rho)}}{2} \log_2 \left( \frac{1 + r_{\varphi_{\mathbf{o}}(\rho)}}{1 - r_{\varphi_{\mathbf{o}}(\rho)}} \right), \end{aligned} \quad (31)$$

where  $R$  is the *relative entropy*<sup>1</sup> between the states appearing as its arguments and  $r_{\varphi_{\mathbf{o}}(\rho)} = \|\mathbf{v}_{\varphi_{\mathbf{o}}(\rho)}\|$ , while its hue is the pure chromatic state  $\varphi_{\mathbf{o}}^*(\rho)$  defined by

$$\varphi_{\mathbf{o}}^*(\rho) := \arg \min_{\rho \in \mathcal{PS}} R(\rho \parallel \rho_{\varphi_{\mathbf{o}}(\rho)}), \quad (32)$$

where  $\mathcal{PS}$  are the pure chromatic states, parameterized by the points of the border  $\partial \mathcal{D}$  of the Bloch disk. The explicit expression of the density matrix associated to the pure chromatic state  $\varphi_{\mathbf{o}}^*(\rho)$  is

$$\rho_{\varphi_{\mathbf{o}}^*(\rho)} = \frac{1}{2} \begin{pmatrix} 1 + \cos \vartheta_{\varphi_{\mathbf{o}}(\rho)} & \sin \vartheta_{\varphi_{\mathbf{o}}(\rho)} \\ \sin \vartheta_{\varphi_{\mathbf{o}}(\rho)} & 1 - \cos \vartheta_{\varphi_{\mathbf{o}}(\rho)} \end{pmatrix}, \quad (33)$$

where

$$\cos \vartheta_{\rho, \varphi_{\mathbf{o}}(\rho)} = \frac{\mathbf{v}_{\rho} \cdot \mathbf{v}_{\varphi_{\mathbf{o}}(\rho)}}{r_{\varphi_{\mathbf{o}}(\rho)}}. \quad (34)$$

### 3 Perceived colors from the qubit point of view

The aim of this section is to show how to compute the perceptual attributes of perceived colors in the more usual physical context where the Hilbert space of the quantum system is the complex vector space  $\mathbb{C}^2$  rather than the real vector space  $\mathbb{R}^2$  of the rebit system.

#### 3.1 From rebit generalized states and effects to qubit states

The set  $\mathcal{H}(2, \mathbb{C})$  of  $2 \times 2$  hermitian matrices is a  $\mathbb{R}$ -vector space of dimension 4 with an orthogonal basis with respect to the Hilbert-Schmidt inner product  $\langle A, B \rangle_{\text{HS}} := \text{Tr}(A^\dagger B)$  given by the Bloch basis  $(\sigma_0, \sigma_x, \sigma_y, \sigma_z)$ , where  $\sigma_0$  is again  $I_2$  and

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (35)$$

constitute the full set of Pauli matrices. The squared Hilbert-Schmidt norm of all the elements of the Bloch basis is 2, so a density matrix belonging to  $\mathcal{H}(2, \mathbb{C})$  can be written as follows:

$$\rho(x, y, z) = \frac{1}{2} (I_2 + x\sigma_x + y\sigma_y + z\sigma_z) = \frac{1}{2} \begin{pmatrix} 1 + z & x - iy \\ x + iy & 1 - z \end{pmatrix}, \quad (36)$$

<sup>1</sup>we recall that  $R(\rho_{\mathbf{s}} \parallel \rho_{\mathbf{t}}) := \text{Tr}[\rho_{\mathbf{s}} \log_2 \rho_{\mathbf{s}} - \rho_{\mathbf{s}} \log_2 \rho_{\mathbf{t}}]$ .



where  $x^2 + y^2 + z^2 \leq 1$ , a condition that defines the *Bloch sphere*. As is well-known, a quantum system whose states can be described by Bloch vectors belonging to this sphere is a *qubit*. The expression of a Bloch vector in spherical coordinates is

$$\mathbf{v} = r (\sin \vartheta \cos \phi, \sin \vartheta \sin \phi, \cos \vartheta), \quad (37)$$

with  $r \in [0, 1]$ ,  $\theta \in [0, \pi]$  and  $\phi \in [0, 2\pi)$ , and the associated density matrix is

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + r \cos \theta & e^{-i\phi} r \sin \theta \\ e^{i\phi} r \sin \theta & 1 - r \cos \theta \end{pmatrix}. \quad (38)$$

For the sake of a more concise writing, from now on we will indicate with ‘r-’ and ‘q-’ the quantities relative to the rebit and qubit formulations, respectively.

First, we discuss how to define an r-emitted light as a q-state. In Def. 2.1, a r-emitted light stimulus is written as the r-generalized state  $\ell_0 \boldsymbol{\ell}$ , with  $\ell_0$  representing the normalized intensity and  $\boldsymbol{\ell} \in \mathcal{S}$  the intrinsic physical chromatic state. In the r-framework, we need a generalized state in order to take into account the information of both  $\ell_0$  and  $\boldsymbol{\ell}$ , however we are going to show that in the q-formalism a complex density matrix can carry the entire information. The advantage is that this density matrix can be more easily associated to the actual preparation of a photon to be sent in the eyes of an observer.

We are thus led to define a map from the space  $\tilde{\mathcal{S}}$  of r-generalized states to the Bloch sphere of q-states. As mentioned earlier,  $\ell_0 = 0$  represents the *threshold of photopic visibility*, i.e. the minimal light intensity that can induce the response of a cone in the retina, while  $\ell_0 = 1$  represents the *glare limit*, when all light stimuli are perceived as the brightest possible white and above that intensity vision is no longer possible because the cones saturate. This means that the only *meaningful light stimuli* are those belonging to the subset of  $\tilde{\mathcal{S}}$  given by all  $\psi_{\mathbf{e}}(\mathbf{s}_{\mathbf{a}})$ , as  $\mathbf{e}$  varies in  $\mathcal{E}$ , i.e.

$$\mathcal{E}_1 := \{(\ell_0, \ell_1, \ell_2) \in \mathbb{R}^3, \ell_0 \in [0, 1] \text{ and } (\ell_0 \ell_1)^2 + (\ell_0 \ell_2)^2 \leq \min\{\ell_0^2, (1 - \ell_0)^2\}\}, \quad (39)$$

see Figure 2.

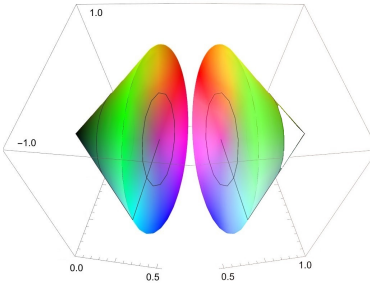


Figure 2: Geometric representation of the set  $\mathcal{E}_1$  opened in half at its basis.

The image  $\mathcal{E}_2$  of the map

$$\begin{aligned} \beta: \quad \mathcal{E}_1 \subset \tilde{\mathcal{S}} &\longrightarrow \mathcal{E}_2 \\ (\ell_0, \ell_1, \ell_2) &\longmapsto \beta(\ell_0, \ell_1, \ell_2) := (\ell_0 \ell_1, \ell_0 \ell_2, 2\ell_0 - 1) \end{aligned} \quad (40)$$

is the subset of the Bloch sphere defined by

$$\mathcal{E}_2 := \{(x, y, z) \in \mathbb{R}^3, z \in [-1, 1] \text{ and } x^2 + y^2 \leq (1 - |z|)^2\}, \quad (41)$$

which is a solid convex double cone, with vertices in  $(0, 0, 1)$  and  $(0, 0, -1)$ . Its circular basis is now of radius 1, located at height  $z = 0$ , and it can be interpreted as the Bloch disk of the r-framework, as depicted in Figure 3.

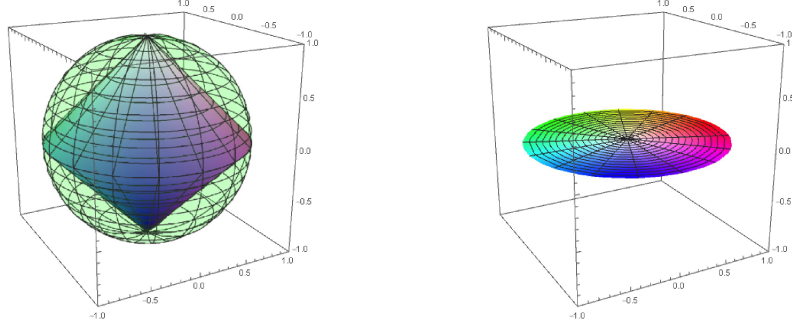


Figure 3: *Left*: geometric representation of the set  $\mathcal{E}_2$  inside the Bloch sphere. *Right*: the equatorial basis of the double cone corresponds to the Bloch disk with Hering's opponency.

The q-Bloch vector of an r-emitted light  $\ell_0\ell$  belonging to  $\mathcal{E}_1$  is given by

$$\mathbf{v}_\ell^q = \begin{pmatrix} \ell_0\ell_1 \\ \ell_0\ell_2 \\ 2\ell_0 - 1 \end{pmatrix}, \quad (42)$$

and the corresponding q-density matrix by:

$$\rho_\ell = \frac{1}{2} (I_2 + \ell_0\ell_1\sigma_x + \ell_0\ell_2\sigma_y + (2\ell_0 - 1)\sigma_z) = \frac{1}{2} \begin{pmatrix} 2\ell_0 & \ell_0\ell_1 - i\ell_0\ell_2 \\ \ell_0\ell_1 + i\ell_0\ell_2 & 2(1 - \ell_0) \end{pmatrix}. \quad (43)$$

It is also possible to write this last matrix using the following decomposition

$$\rho_\ell = p_w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + p_b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{p_c}{2} \begin{pmatrix} 1 & se^{-i\phi} \\ se^{i\phi} & 1 \end{pmatrix} \equiv p_w\rho_w + p_b\rho_b + p_c\rho_c, \quad (44)$$

with

$$\begin{cases} 2\ell_0 = 2p_w + p_c \\ 2(1 - \ell_0) = 2p_b + p_c \\ \ell_0\ell_1 + i\ell_0\ell_2 = p_c se^{i\phi}, \end{cases} \quad (45)$$

and where  $s \in [0, 1]$ ,  $\phi \in [0, 2\pi)$ , and  $p_w, p_b, p_c \in [0, 1]$ . The three matrices  $\rho_w$ ,  $\rho_b$ , and  $\rho_c$  represent respectively the white and black r-emitted lights, and the chromatic component of the r-emitted light  $\ell_0\ell$ .

The q-density matrix of an r-effect  $\mathbf{e}$ , i.e. an element of  $\mathcal{E}$ , can be written

$$\rho_{\mathbf{e}} := \frac{1}{2} \begin{pmatrix} 1 + e_z & e_x - ie_y \\ e_x + ie_y & 1 - e_z \end{pmatrix}, \quad (46)$$

with  $e_z = 2e_0 - 1 \in [-1, 1]$ ,  $e_x = e_0e_1$ , and  $e_y = e_0e_2$ . The condition  $\mathbf{0} \leq \rho_{\mathbf{e}} \leq I_2$  is equivalent to the condition  $0 \leq e_x^2 + e_y^2 + e_z^2 \leq 1$ , i.e. the vector  $\mathbf{v}_{\mathbf{e}}^q := (e_x, e_y, e_z)$  belongs to the Bloch sphere.

It can be checked that the resolution of the identity can be written as  $I_2 = \rho_{\mathbf{e}} + \rho_{-\mathbf{e}}$ , so we can limit the variability of the parameters defining  $\rho_{\mathbf{e}}$  to the upper half Bloch sphere, i.e.  $e_z \in [0, 1]$  and  $e_x^2 + e_y^2 + e_z^2 \leq 1$ , and reconstruct the information from the lower half Bloch sphere via the antipodal q-vectors  $-\mathbf{v}_{\mathbf{e}}^q = -(e_x, e_y, e_z)$ . We adopt the following definition.

**Def. 3.1 (q-effect)** A q-effect is a q-state

$$\rho_{\mathbf{e}} = \frac{1}{2} \begin{pmatrix} 1 + e_z & e_x - ie_y \\ e_x + ie_y & 1 - e_z \end{pmatrix}, \quad (47)$$

with  $e_z \in [0, 1]$  and  $e_x^2 + e_y^2 + e_z^2 \leq 1$ .

As it is well-known, extremal effects are orthogonal projectors and they are labeled by unit vectors in the Bloch sphere. As an example, the Bloch-effect vector  $\mathbf{v}_e^q = (0, 0, 1)$  labels the orthogonal projector on the ‘white point’

$$P_w := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (48)$$

while its antipodal vector  $-\mathbf{v}_e^q = (0, 0, -1)$  labels the orthogonal projector on the ‘black point’

$$P_b := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (49)$$

and together they form a resolution of the identity.

### 3.2 Color measurements with qubit states and effects

Here we introduce the definitions that permit to describe perceived colors as results of measurements using qubit states and effects, rather than rebit states and effects.

**Def. 3.2 (q-emitted light stimulus)** *A q-emitted light stimulus  $\ell$  is identified with the following q-density matrix:*

$$\rho_\ell = \frac{1}{2} \begin{pmatrix} 2p_w + p_c & p_c s e^{-i\phi} \\ p_c s e^{i\phi} & 2p_b + p_c \end{pmatrix}, \quad (50)$$

where  $p_b, p_w, p_c, s \in [0, 1]$  and  $p_b + p_w + p_c = 1$ . In particular:

- a q-achromatic light is an emitted light stimulus with  $p_c = 0$ ;
- a q-white light is a q-achromatic light with  $p_w = 1$ .

Hence, the generic q-achromatic emitted light can be written as

$$\rho_\ell^a = \frac{1}{2} \begin{pmatrix} 1 + z & 0 \\ 0 & 1 - z \end{pmatrix}, \quad z \in [-1, 1].$$

**Def. 3.3 (q-observer)** *A q-observer  $o$  measuring a color stimulus is identified with a q-state*

$$\rho_o = \frac{1}{2} \begin{pmatrix} 1 + o_z & o_x - io_y \\ o_x + io_y & 1 - o_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + r_o \cos \theta_o & e^{-i\phi_o} r_o \sin \theta_o \\ e^{i\phi_o} r_o \sin \theta_o & 1 - r_o \cos \theta_o \end{pmatrix}, \quad (51)$$

with  $o_z, r_o \in [0, 1]$ ,  $o_x^2 + o_y^2 + o_z^2 \leq 1$ , and<sup>2</sup>  $\theta_o \in [0, \frac{\pi}{2}]$ ,  $\phi_o \in [0, 2\pi)$ .

In the r-framework, we obtain a perceived color from an emitted light stimulus  $\ell_0 \ell$  by applying on it the Lüders operation  $\psi_o$  related to the effect that represents the observer  $o$ . The result is the r-generalized state  $\psi_o(\ell_0 \ell)$ . As we have seen in eq. (43), the r-generalized state  $\ell_0 \ell$  is represented in the q-framework by the q-density matrix  $\rho_\ell$ . So, also the perceived color from an emitted light stimulus, i.e. the r-generalized state  $\psi_o(\ell_0 \ell)$ , will be represented in the q-framework as a q-density matrix that we indicate with  $\rho_\ell^o := \varphi_o(\rho_\ell)$ .

If we denote with  $c_0, c_1, c_2$  the matrix element of  $\rho_\ell^o$ , then by eq. (43) we have

$$\rho_\ell^o = \frac{1}{2} \begin{pmatrix} 2c_0 & c_0(c_1 - ic_2) \\ c_0(c_1 + ic_2) & 2(1 - c_0) \end{pmatrix}, \quad (52)$$

and, by eq. (42), the relative Bloch vector is

$$\mathbf{v}_\ell^o := \begin{pmatrix} c_0 c_1 \\ c_0 c_2 \\ 2c_0 - 1 \end{pmatrix}. \quad (53)$$

<sup>2</sup>taking  $\theta_o \in [0, \frac{\pi}{2}]$  is equivalent to considering only the upper half cap of the Bloch sphere.

In order to find the explicit expression of the entries  $c_0, c_1, c_2$ , let us introduce the following ‘chromatic q-Bloch vectors’:

$$\bar{\mathbf{v}}_\ell^q := \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}, \quad \bar{\mathbf{v}}_o^q := \begin{pmatrix} o_x/o_z \\ o_y/o_z \end{pmatrix}, \quad \bar{\mathbf{v}}_\ell^o := \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad (54)$$

where the presence of  $o_z$  in  $\bar{\mathbf{v}}_o^q$  is reminiscent of the presence of  $e_0$  in eq. (8). Using equations from (13) to (17) and eq. (25), with straightforward computations we find

- if  $\|\bar{\mathbf{v}}_o^q\| < 1$ , then

$$\begin{cases} c_0 = o_z \ell_0 (1 + \bar{\mathbf{v}}_o^q \cdot \bar{\mathbf{v}}_\ell^q) \\ \bar{\mathbf{v}}_\ell^o = \frac{1}{1 + \bar{\mathbf{v}}_o^q \cdot \bar{\mathbf{v}}_\ell^q} \left\{ \bar{\mathbf{v}}_o^q + \frac{1}{\gamma_o} \bar{\mathbf{v}}_\ell^q + \frac{\gamma_o}{1 + \gamma_o} (\bar{\mathbf{v}}_o^q \cdot \bar{\mathbf{v}}_\ell^q) \bar{\mathbf{v}}_o^q \right\} \end{cases}, \quad (55)$$

where

$$\gamma_o := 1/\sqrt{1 - \|\bar{\mathbf{v}}_o^q\|^2}; \quad (56)$$

- if  $\|\bar{\mathbf{v}}_o^q\| = 1$ , then

$$\begin{cases} c_0 = o_z \ell_0 (1 + \bar{\mathbf{v}}_o^q \cdot \bar{\mathbf{v}}_\ell^q) \\ \bar{\mathbf{v}}_\ell^o \stackrel{\text{eq. (17)}}{=} \bar{\mathbf{v}}_o^q \end{cases}. \quad (57)$$

It follows that the definition of perceived color from an emitted light stimulus can be formalized as follows.

**Def. 3.4 (q-perceived color from an emitted light stimulus)** *Given the q-effect  $\rho_o$  as in Def. 3.3 associated to the observer  $o$  and an the emitted light stimulus  $\ell$  described by the q-state  $\rho_\ell$  as in eq. (43), the color perceived by  $o$  from  $\ell$  is the post-measurement q-state*

$$\rho_\ell^o = \frac{1}{2} \begin{pmatrix} 2c_0 & c_0(c_1 - ic_2) \\ c_0(c_1 + ic_2) & 2(1 - c_0) \end{pmatrix} \quad (58)$$

where

- $c_0 = o_z \ell_0 (1 + \bar{\mathbf{v}}_o^q \cdot \bar{\mathbf{v}}_\ell^q)$ , with  $\bar{\mathbf{v}}_o^q$  and  $\bar{\mathbf{v}}_\ell^q$  as in eq. (54);
- and

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} := \begin{cases} \bar{\mathbf{v}}_o^q & \text{if } \|\bar{\mathbf{v}}_o^q\| = 1 \\ \frac{1}{1 + \bar{\mathbf{v}}_o^q \cdot \bar{\mathbf{v}}_\ell^q} \left\{ \bar{\mathbf{v}}_o^q + \frac{1}{\gamma_o} \bar{\mathbf{v}}_\ell^q + \frac{\gamma_o}{1 + \gamma_o} (\bar{\mathbf{v}}_o^q \cdot \bar{\mathbf{v}}_\ell^q) \bar{\mathbf{v}}_o^q \right\} & \text{otherwise} \end{cases}, \quad (59)$$

with  $\gamma_o$  as in eq. (56).

It can be seen that the q-chromatic vector  $\mathbf{v}_\ell^o = (c_1, c_2)$  coincides with the vector  $\mathbf{v}_{\varphi_e(\mathbf{s})}$  in eq. (14) defining the post-measurement chromatic state in the r-framework.

### 3.3 Computing perceptual attributes with qubit states and effects

It can be seen that  $c_0$  has the same expression of the brightness of a light stimulus defined in eq. (26) which may seem odd because the brightness was obtained in the r-framework by taking the trace of the post-measurement generalized states  $\psi_o(\ell_0 \ell)$ . However, in the q-formalism we cannot hope to obtain the brightness through the trace of  $\rho_\ell^o$ , because this quantity is always equal to 1.

For this reason, we have to come up with a novel and meaningful way to compute the brightness in the q-formalism which has to be coherent with the result obtained in the r-framework. To this aim, we observe that the  $z$ -axis of the Bloch sphere depicted in Figure 3 contains *non-normalized* achromatic values. Moreover, the product of  $\rho_\ell^o$  with the Pauli matrix  $\sigma_z$  produces the  $z$ -component

of the Bloch vector relative to  $\rho_{\ell}^o$ . So, it makes sense to compute the brightness of  $\rho_{\ell}^o$  as the expectation value of  $\sigma_z$  on the state defined by  $\rho_{\ell}^o$ , that is

$$\text{Tr}(\rho_{\ell}^o \sigma_z) = 2c_0 - 1. \quad (60)$$

However, as previously underlined, the  $z$ -component in the Bloch sphere where  $\mathcal{E}_2$  is embedded is non-normalized and ranges from  $-1$  to  $+1$ , thus, to obtain the brightness of  $\rho_{\ell}^o$  we need to add 1 and divide by 2 in order to arrive to an achromatic value between 0 and 1.

This explains why in the  $q$ -formalism the definition of brightness coherent with the  $r$ -framework is the following.

**Def. 3.5 (q-brightness)** *Given a perceived color  $\rho_{\ell}^o$  from a light stimulus  $\rho_{\ell}$ , its brightness is defined as follows*

$$\mathcal{B}(\rho_{\ell}^o) := \frac{1 + \text{Tr}(\rho_{\ell}^o \sigma_z)}{2} = c_0 = o_z \ell_0 + o_x \ell_0 \ell_1 + o_y \ell_0 \ell_2. \quad (61)$$

Finally, let us define the concept of lightness of a perceived color from a light stimulus. In order to be coherent with the  $r$ -framework, we notice that we just have to divide the brightness formula just given by  $o_z$ . However,  $o_z$  is the expectation value of  $\sigma_z$  on the state  $\rho_{\mathbf{o}}$ , thanks to the already underlined duality state-effect in the  $q$ -formalism.

These remarks lead naturally to the following definition.

**Def. 3.6 (q-lightness)** *Given a perceived color  $\rho_{\ell}^o$  from a light stimulus  $\rho_{\ell}$ , its lightness is defined as follows*

$$\mathcal{L}(\rho_{\ell}^o) := \frac{1 + \text{Tr}(\rho_{\ell}^o \sigma_z)}{2\text{Tr}(\rho_{\mathbf{o}} \sigma_z)} = \ell_0 + \frac{o_x}{o_z} \ell_0 \ell_1 + \frac{o_y}{o_z} \ell_0 \ell_2,$$

where  $\rho_{\mathbf{o}} = \rho(o_x, o_y, o_z)$  is the  $q$ -effect associated with the observer  $o$ .

In the  $r$ -framework, the saturation of a perceived color defined as in eq. (31) provides a measure of how the post-measurement chromatic state can be discerned from the achromatic one using as measure of distinguishability the relative quantum entropy.

To translate eq. (31) in the  $q$ -framework we need the Euclidean norm of the Bloch vector  $\mathbf{v}_{\rho_{\ell}^o} = (c_1, c_2)^t$ , which is

$$r_{\rho_{\ell}^o} = \sqrt{c_1^2 + c_2^2}, \quad (62)$$

where  $c_1$  and  $c_2$  can be obtained via  $\rho_{\ell}^o$  in an analogous way as  $c_0$ , namely

$$c_1 = \frac{\text{Tr}(\rho_{\ell}^o \sigma_x)}{c_0} = \frac{2\text{Tr}(\rho_{\ell}^o \sigma_x)}{1 + \text{Tr}(\rho_{\ell}^o \sigma_z)}, \quad c_2 = \frac{\text{Tr}(\rho_{\ell}^o \sigma_y)}{c_0} = \frac{2\text{Tr}(\rho_{\ell}^o \sigma_y)}{1 + \text{Tr}(\rho_{\ell}^o \sigma_z)}, \quad (63)$$

so that

$$r_{\rho_{\ell}^o} = \frac{2}{1 + \text{Tr}(\rho_{\ell}^o \sigma_z)} \sqrt{\text{Tr}(\rho_{\ell}^o \sigma_x)^2 + \text{Tr}(\rho_{\ell}^o \sigma_y)^2}, \quad (64)$$

with  $\text{Tr}(\rho_{\ell}^o \sigma_z) > -1$ .

**Def. 3.7** *Given a perceived color  $\rho_{\ell}^o$  from a light stimulus  $\rho_{\ell}$ , its saturation is defined as follows*

$$\text{Sat}(\rho_{\ell}^o) := \frac{1}{2} \log_2(1 - r_{\rho_{\ell}^o}^2) + \frac{r_{\rho_{\ell}^o}}{2} \log_2 \left( \frac{1 + r_{\rho_{\ell}^o}}{1 - r_{\rho_{\ell}^o}} \right), \quad (65)$$

with  $r_{\rho_{\ell}^o}$  as in eq. (64).

Finally, let us discuss the hue. In the  $r$ -framework, the hue of a perceived color, see eq. (32), is the pure state that minimizes the relative entropy with the post-measurement state. As shown by eq. (33), in its definition there appear the polar angle of the Bloch vector associated to the post-measurement state.

In the q-framework, the analogous of that angle is given by

$$\phi = 2 \arctan \frac{c_2}{\sqrt{c_1^2 + c_2^2} + c_1} = 2 \arctan \frac{\text{Tr}(\rho_{\ell}^o \sigma_y)}{\sqrt{\text{Tr}(\rho_{\ell}^o \sigma_x)^2 + \text{Tr}(\rho_{\ell}^o \sigma_y)^2} + \text{Tr}(\rho_{\ell}^o \sigma_x)}, \quad (66)$$

with at least one between the values  $\text{Tr}(\rho_{\ell}^o \sigma_x), \text{Tr}(\rho_{\ell}^o \sigma_y)$  strictly larger than 0.

**Def. 3.8** Given a perceived color  $\rho_{\ell}^o$  from a light stimulus  $\rho_{\ell}$ , its hue is defined as the following pure chromatic state

$$\rho_{\ell, \phi}^o := \frac{1}{2} \begin{pmatrix} 1 & e^{-i\phi} \\ e^{i\phi} & 1 \end{pmatrix},$$

with  $\phi$  as in eq. (66).

### 3.4 The operational framework

The operational framework that we propose to perform experiments regarding the quantum model of color perception is defined by the following data:

$$(\Lambda, \xi, \sigma_x, \sigma_y, \sigma_z, \phi_{\text{white}}, \phi_{\text{red}}, \phi_{\text{green}}, \phi_{\text{yellow}}, \phi_{\text{blue}}),$$

where

- $\Lambda$  is the set of all q-emitted light stimuli

$$\Lambda = \left\{ \rho = \begin{pmatrix} p_w + \frac{p_c}{2} & \frac{p_c s e^{-i\phi}}{2} \\ \frac{p_c s e^{i\phi}}{2} & p_b + \frac{p_c}{2} \end{pmatrix} : p_b, p_w, p_c, s \in [0, 1] \text{ and } p_b + p_w + p_c = 1 \right\}; \quad (67)$$

- $\xi$  is the set of q-effects describing q-observers who perform perceptual measurements on  $\Lambda$  giving rise to post-measurement q-states corresponding to perceived colors

$$\xi = \left\{ \frac{1}{2} \begin{pmatrix} 1 + r \cos \theta & e^{-i\phi} r \sin \theta \\ e^{i\phi} r \sin \theta & 1 - r \cos \theta \end{pmatrix} : r \in [0, 1], \theta \in [0, \pi/2], \phi \in [0, 2\pi) \right\}; \quad (68)$$

- the Pauli matrices  $\sigma_x, \sigma_y, \sigma_z$  can be interpreted dually as extremal effects or pure states (rank-1 projectors) given by

$$\sigma_x = \xi_{1, \frac{\pi}{2}, 0}, \quad \sigma_y = \xi_{1, 0, \frac{\pi}{2}}, \quad \sigma_z = \xi_{1, 0, 0}; \quad (69)$$

- the constants  $\phi_{\text{white}}, \phi_{\text{red}}, \phi_{\text{green}}, \phi_{\text{yellow}}, \phi_{\text{blue}}$  are pure states associated to the pure white and the antipodal hues:

$$\phi_{\text{white}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (70)$$

$$\phi_{\text{red}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \phi_{\text{green}} = \sigma_z \phi_{\text{red}} \sigma_z, \quad (71)$$

$$\phi_{\text{yellow}} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \phi_{\text{blue}} = \sigma_z \phi_{\text{yellow}} \sigma_z. \quad (72)$$

We remark that the choice of the names red, green, yellow and blue is just a convention, and, in this abstract setting, any name would be as valid as those used above. Instead, what is really important to underline the role of  $\sigma_z$  in this structure.  $\sigma_z$  acts as a ‘*chromatic antipodal operator*’ in the sense that, given any pure hue state, its antipodal state is obtained by conjugation with  $\sigma_z$  (given that  $\sigma_z = \sigma_z^\dagger = \sigma_z^{-1}$ ).

## 4 Conclusions and future perspectives

In this paper we have shown how to compute the perceptual attributes of perceived colors, initially defined in the model based on Jordan algebras and Hering's rebit, by means of the usual qubit density matrices and effects.

We hope that this will allow scientists familiar with the conventional algebraic language of quantum theories to have an easier access to that model which, analogously to quantum mechanics, underscores the pivotal role of the observer and measurement apparatus in comprehending color perception. This motivates the need of this work, which is a step towards enabling an experimental verification of the theory.

In future works, we plan to perform a series of experiments which will necessarily involve chromatic opposition. As an example, we can test entanglement effects associated to center-surround light stimuli in the following way: from the center stimulus, a minimal number of photons are sent within a specific time interval to a cone area, activating the observer's action potentials; simultaneously, other photons are sent from the surround. The physical properties of the center and its surround can be encoded in the entangled system of color perception.

Specifically, we envision a density operator representing, say, the blue-yellow and yellow-blue Bell state, where the first subsystem corresponds to the center and the second to its surround. By measuring the composite system, we expect a blue center in a yellow surround 50% of the time and a yellow center in a blue surround the other 50%. If the surround stimulus is measured before the center stimulus arrives, then we expect an achromatic sensation for the center.

While the results of such experiments could be explained by the quantum model, they pose a significant challenge to the classical theory of color perception.

## References

- [1] M. Aldé, M. Berthier, and E. Provenzi. The classification of rebit quantum channels. *Journal of Physics A: Mathematical and Theoretical*, 56(495301):1–17, 2023.
- [2] John C Baez. Division algebras and quantum theory. *Foundations of Physics*, 42(7):819–855, 2012.
- [3] M. Berthier. Geometry of color perception. Part 2: perceived colors from real quantum states and Hering's rebit. *The Journal of Mathematical Neuroscience*, 10(1):1–25, 2020.
- [4] M. Berthier, V. Garcin, N. Prencipe, and E. Provenzi. The relativity of color perception. *Journal of Mathematical Psychology*, 103:102562, 2021.
- [5] Michel Berthier, Nicoletta Prencipe, and Edoardo Provenzi. A quantum information-based refoundation of color perception concepts. *SIAM Journal on Imaging Sciences*, 15(4):1944–1976, 2022.
- [6] Michel Berthier and Edoardo Provenzi. The quantum nature of color perception: Uncertainty relations for chromatic opposition. *Journal of Imaging*, 7(40), 2021.
- [7] Michel Berthier and Edoardo Provenzi. Quantum measurement and colour perception: theory and applications. *Proceedings of the Royal Society A*, 478(2258):20210508, 2022.
- [8] Paul Busch, Marian Grabowski, and Pekka J Lahti. *Operational quantum physics*, volume 31. Springer Science & Business Media, 1997.
- [9] Karen K De Valois. *Seeing*. Academic Press, 2000.
- [10] Ewald Hering. *Zur Lehre vom Lichtsinne: sechs Mittheilungen an die Kaiserl. Akademie der Wissenschaften in Wien*. C. Gerold's Sohn, 1878.
- [11] D.H. Hubel. *Eye, Brain, and Vision*. Scientific American Library, 1995.

- [12] William K Wootters. Local accessibility of quantum states. *Complexity, entropy and the physics of information*, 8:39–46, 1990.