



Exact solutions for radiative transfer with partial frequency redistribution

Hélène Frisch

► To cite this version:

Hélène Frisch. Exact solutions for radiative transfer with partial frequency redistribution. Journal of Computational and Theoretical Transport, 2024, pp.1 - 46. <10.1080/23324309.2024.2393598>. <hal-04763602>

HAL Id: hal-04763602

<https://hal.science/hal-04763602v1>

Submitted on 2 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Exact solutions for radiative transfer with partial frequency redistribution

H. Frisch^a

^aUniversité Côte d’Azur, Observatoire de la Côte d’Azur, CNRS, Laboratoire Lagrange, Bd de l’Observatoire, CS34229, 06304 Nice Cedex 4, France

ARTICLE HISTORY

Compiled October 23, 2024

ABSTRACT

The construction of exact solutions for radiative transfer in a plane-parallel medium has been addressed by Hemsch and Ferziger in 1972 for a partial frequency redistribution model of the formation of spectral lines consisting in a linear combination of frequency coherent and fully incoherent scattering. The method of solution is based on an eigenfunction expansion of the radiation field, leading to two singular integral equations with Cauchy or Cauchy-type kernels, that have to be solved one after the other. We reconsider this problem, using as starting point the integral formulation of the radiative transfer equation, where the terms involving the coupling between the two scattering mechanisms are clearly displayed, as well as the primary source of photons.

With an inverse Laplace transform, we recover the singular integral equations previously established and, with Hilbert transforms as in the previous work, recast them as boundary value problems in the complex plane. Their solutions are presented in detail for an infinite and a semi-infinite medium. The coupling terms are carefully analyzed and consistency with either the coherent or the incoherent limit is systematically checked.

We recover the important result of the previous work that an exact solution exists for an infinite medium, whereas for a semi-infinite medium, which requires the introduction of half-space auxiliary functions, the solution is given by a Fredholm integral equation to be solved numerically. The solutions of the singular integral equations are used to construct explicit expressions providing the radiation field for an arbitrary primary source and for the Green’s function. An explicit expression is given for the radiation field emerging from a semi-infinite medium.

KEYWORDS

radiative transfer; scattering; analytical methods; stellar atmospheres

1. Introduction

It has been known since the work of the mathematicians N. Wiener and E. Hopf in 1931 (Wiener and Hopf 1931) that monochromatic scattering in a semi-infinite medium has an exact solution. Monochromatic means that scattering remains frequency coherent, photons suffering at each scattering only changes in direction. The so-called Wiener–Hopf method of solution is based on the possibility to express the radiative transfer

CONTACT H. Frisch. Email: frisch@oca.eu To be published in the Journal of Computational and Theoretical Transport, DOI 10.1080/23324309.2024.2393598

equation as a convolution integral equation on a half-line for a function depending only on the space variable.

For incoherent scattering, that is when the frequencies of photons before and after a scattering are totally uncorrelated, it was shown, thanks to the work done in the late forties and early fifties by the group of V.V. Sobolev in Leningrad (for references see e.g. Sobolev 1963; Ivanov 1973), that the radiative transfer problem can also be recast as a convolution integral equation for a function which depends only on the space variable. The Wiener–Hopf method remains applicable, as rigorously proved by Krein (1962) (see also Levinson 1949). In Astronomy, this type of scattering, which applies to most of the spectral lines observed in the spectra of stellar atmospheres, is usually referred to as *complete frequency redistribution*, a terminology that is followed here. In contrast, monochromatic scattering applies to the formation of continuous spectra emitted by stellar atmospheres.

For spectral lines, the assumption of complete frequency redistribution breaks down, when the frequencies of the absorbed and scattered photons are correlated. *Partial frequency redistribution*, as it is called, is typical of strong spectral lines, known as resonance lines (see Appendix A for details). In principle it is not possible to find exact solutions when there is partial frequency redistribution because the radiative transfer equation cannot be recast into an equation for a function depending only on the space variable. However, for a very simple partial frequency redistribution model, consisting in a linear combination of monochromatic scattering and complete frequency redistribution, it was shown by M. Hemsch in his PhD thesis with a method based on the singular eigenfunction expansion method introduced in Case (1960), that it is possible to construct an exact solution for a infinite medium and to reduce the problem to a Fredholm integral equation for a semi-infinite medium (Hemsch 1971; Hemsch and Ferziger 1972). This simple partial frequency redistribution model was introduced for resonance lines (see the discussion in Appendix A), for spectral lines formed between two excited levels (Jefferies 1968; Frisch 1980; Hubeny and Mihalas 2015; Frisch 2022, p. 252) and to study the resonance scattering of gamma-ray quanta (Yengibarian and Khachatryan 1991). The same type of model is found in Corngold and Durgin (1967) to analyze the decay of a neutron pulse inside a moderator. Several examples of applications to neutron and photons scattering problems of the expansion method introduced in Case (1960) may be found in classical textbooks and reviews (e.g. Case and Zweifel 1967; Williams 1971; McCormick and Kuščer 1973; Duderstadt and Martin 1979). A few of them are also mentioned at the end of Section 2.1.

Because of the interplay between monochromatic scattering and complete frequency redistribution, the solution is not a simple linear combination of a monochromatic scattering and a complete frequency redistribution solution. As shown in Hemsch (1971) and Hemsch and Ferziger (1972), a solution can be constructed by proceeding in two successive steps. The first one consists in solving a monochromatic scattering problem. The integration over frequency of its solution provides a generalized complete frequency redistribution problem, involving only frequency integrated functions. This two-step procedure is also applied in Yengibarian and Khachatryan (1991), where the method of solution is otherwise radically different from the method employed in Hemsch (1971) and the results not as explicit.

In this article, we also consider a linear combination of monochromatic scattering and complete frequency redistribution, reach the same conclusions as in Hemsch (1971) and Hemsch and Ferziger (1972), but with an approach that has differences and similarities with the previous work. In the latter a singular eigenfunction expansion method is used to transform the integro-differential equation for the radiation

field into a Cauchy-type singular integral equation, which has the particularity of having two singular integrals, one coming from the coherent scattering contribution and the other one from the incoherent one. Assuming that the incoherent contribution is a known term, the singular integral reduces to a singular integral equation for monochromatic scattering. An integration over frequency of its solution provides a new singular integral equation for a frequency independent function, the properties of which are akin to those of a singular integral equation for complete frequency redistribution. The solutions of both equations are obtained by transforming them into a boundary value problem in the complex plane. This method of solution, proposed by Carleman (1922), relies on the analyticity properties of Hilbert transforms (e.g., Muskhelishvili 1953; Gakhov 1966; Pogorzelski 1966; Ablowitz and Fokas 1997).

The hard core of our approach is also the solution of a singular integral equation containing two singular integrals, coming from the coherent and incoherent parts respectively, and its solution is constructed in two steps, as described above. The route to the singular integral is however different. The radiative transfer equation is first transformed into a convolution integral equation for the direction averaged radiation field. The latter is then transformed into a singular integral equation by an inverse Laplace transform, more or less equivalent to the singular eigenfunction expansion. Since the convolution equation takes into account the inhomogeneous term in the radiative transfer equation and the boundary conditions on the radiation field, the solution of the singular integral equation directly provides the full solution of the problem. In contrast, the singular integral equation derived with the singular eigenfunction expansion provides only a general solution of the homogeneous radiative transfer equation, to which must be added a particular solution of the inhomogeneous equation. The full solution has then to be constrained by the boundary conditions.

The organization of this work is as follows. The radiative transfer and its two associated integral equations are constructed in Section 2. Section 3 is devoted to the *monochromatic dispersion function*, which plays a critical role in the solution of monochromatic scattering problems. In Section 4 we consider an infinite medium and give an explicit solution of the infinite medium Green's function. In Section 5, we consider a semi-infinite medium and show how to construct the half-space auxiliary functions, corresponding respectively to the monochromatic and the generalized complete frequency redistribution problem. We also show how to calculate the emergent radiation field. Some remarks and technical material are presented in Appendices. In Appendix A we discuss the modelling of resonance lines by a linear combination of monochromatic scattering and complete frequency redistribution. In Appendix B we explain how a convolution integral equation can be transformed into a singular integral equation. Appendix C is devoted to the properties of the dispersion function associated to the generalized complete frequency redistribution problem. Finally, Appendix D presents some details regarding the construction of the half-space auxiliary functions.

2. The radiative transfer equation and associated integral equations

We first present the radiative transfer equation for a combination of coherent and incoherent scattering. We then show how to transform it into a convolution integral equation and then how the latter can then be transformed into a singular integral equation.

2.1. The radiative transfer equation

We consider here an isolated spectral line. Following a notation employed in Chandrasekhar (1960) and a series of articles by D. Hummer and collaborators (see e.g. Avrett and Hummer 1965), we write the one-dimensional radiative transfer equation as

$$\mu \frac{\partial I}{\partial \tau}(\tau, x, \mu) = \varphi(x)[I(\tau, x, \mu) - S(\tau, x)], \quad (1)$$

where $I(\tau, x, \mu)$ is the specific intensity of the radiation field and $S(\tau, x)$ the so-called source function. For the problem considered here, it is defined in Eq. (2). Here τ is the frequency-averaged line optical depth. Its domain of definition is $\tau \in]-\infty, +\infty[$ for an infinite medium and $\tau \in [0, +\infty[$ ¹ for a semi-infinite one. The variable $\mu \in [-1, +1]$ is the direction cosine of the photon-propagation vector, as measured from the negative τ axis. With this choice, usual in Astronomy, μ is positive for rays emerging from the surface of a semi-infinite medium, negative for incoming rays, and the absorption term appears with a positive sign in the right-hand side. The variable $x \in]-\infty, +\infty[$ measures frequencies from the line center in a convenient unit, usually the Doppler width of the line, assumed to be independent of τ . The line absorption profile $\varphi(x)$ is normalized to unity in $-\infty \leq x \leq +\infty$. For spectral lines, $\varphi(x)$ is usually assumed to be a Gaussian or a Voigt function (convolution of a Gaussian with a Lorentzian). Here we assume that the source function $S(\tau, x)$, which describes the emission term, has the form

$$S(\tau, x) = (1 - \epsilon)[bJ(\tau, x) + (1 - b)\bar{J}(\tau)] + Q^*(\tau), \quad (2)$$

where $Q^*(\tau)$ is a given primary source of photons,

$$J(\tau, x) = \frac{1}{2} \int_{-1}^{+1} I(\tau, x, \mu) d\mu, \quad \text{and} \quad \bar{J}(\tau) = \int_{-\infty}^{+\infty} J(\tau, x) \varphi(x) dx. \quad (3)$$

The parameter $\epsilon \in [0, 1]$ is the probability that upon scattering a photon is lost from the line. We shall also make use of $\omega = 1 - \epsilon$ the albedo for single scattering. The coefficient $b \in [0, 1]$ is the branching ratio between monochromatic scattering and complete frequency redistribution, the limit $b = 0$ corresponding to complete frequency redistribution and the limit $b = 1$ to monochromatic scattering. In this limit, the scattering of photons occurring without changes in frequencies, the dependence of $I(\tau, x, \mu)$ on the frequency x can be ignored and $\varphi(x)$ set to one. In both limits the source function depends only on τ . To make a contact with particle transport, we remark here that $I(\tau, \mu, x)$ corresponds to a one-particle distribution function $f(\mathbf{r}, \mathbf{v})$, and that the radiative transfer equation is a time-independent linear Boltzmann equation.

In Hemsch and Ferziger (1972), the branching ratio is assumed to be a function $b(x)$ of the frequency. Here we keep it independent of x , for a fundamental reason, namely that a branching ratio depending on x does not preserve conservation of energy nor symmetry in x and x' (see Appendix A for a more detailed argumentation). The assumption that b is a constant does not influence the main conclusions of the work of Hemsch and Ferziger (1972).

¹The notation $[]$ stands for a closed interval, $[[]$ or $]]$ for semi-open intervals and $]][]$ for an open interval such as $] - \infty, +\infty[$.

In Hensch and Ferziger (1972), it is assumed that the solution of the radiative transfer equation, restricted to its homogeneous part, may be written as

$$I(\tau, x, \mu) = \int_k \Phi(k, x, \mu) e^{-\tau/k} dk. \quad (4)$$

This expansion, once inserted into the homogeneous part of Eq. (1) provides a relation between $\Phi(k, x, \mu)$ and the two averages $\bar{\Phi}(k, x)$ and $\bar{\bar{\Phi}}(k)$, defined as in Eq. (3). The condition that this expansion, say at $\tau = 0$, can represent an arbitrary function $\psi(x, \mu)$, provides a integral equation with two Cauchy-type singular integrals, one for $\bar{\Phi}(k, x)$ and one for $\bar{\bar{\Phi}}(k)$. Its solution is carried out in two steps. The first one provides $\bar{\Phi}(k, x)$ in terms of $\bar{\bar{\Phi}}(k)$. An integration over frequency of $\bar{\Phi}(k, x)$ provides a new equation, now of the complete frequency redistribution type, which can be solved for $\bar{\bar{\Phi}}(k)$. Once $\Phi(k, x, \mu)$ has been determined, it can be used to construct a solution of the inhomogeneous radiative transfer equation, which takes into account the boundary conditions.

The proof given in Hensch (1971) and Hensch and Ferziger (1972) that exact solutions can be constructed for a partial frequency redistribution problem is definitely a great success of the singular eigenfunction expansion method. Another example is the proof by Siewert and Fraley (1967) that conservative Rayleigh scattering in a semi-infinite medium has an exact solution. The first proof of this result was given by Chandrasekhar (1946) (see also Chandrasekhar 1960) with a method in which the most important point is somewhat hidden. With a singular eigenfunction expansion, Siewert and Fraley (1967) show how the vectorial transfer equation for the linearly polarized radiation field can be transformed into a matrix boundary value problem in the complex plane (see also Siewert and Burniston 1972). The latter has an exact solution because the associated matrix dispersion function can be diagonalized. This property breaks down when the scattering loses the monochromatic and/or its conservative characteristics. More generally, the singular eigenfunction expansion method brings a rigorous and deep mathematical understanding to transport problems involving the scattering of neutrons or photons and as such offers an excellent starting point for the construction of accurate numerical solutions of neutron and photons transfer equations (e.g. Siewert and Benoist 1979; Barichello and Siewert 1998, 1999a,b).

Now, as explained in Section 1 we shall construct a convolution integral equation with an assumption somewhat similar to Eq. (4) and then transform it into a singular integral equation with two singular integrals. This method is applied in Section 4 for an infinite medium and in Section 5 for a semi-infinite one.

2.2. Convolution integral equations

We show here how a radiative transfer equation can be transformed into a convolution integral equation for the source function and mention some of the advantages of this formulation. For monochromatic scattering, the systematic introduction of convolution integral equations for handling radiative transfer equations dates back from Milne (1921) and for complete frequency redistribution from the late forties and early fifties, thanks to the work done by the group of V.V. Sobolev in Leningrad (for references see e.g. Sobolev 1963; Ivanov 1973). The integral equation for the source function is obtained by inserting a formal solution of the radiative transfer equation into the definition of the source function.

Let us consider for example a semi-infinite medium with no incident radiation on the surface and complete frequency redistribution. The source function is given by $S(\tau) = (1 - \epsilon)\bar{J}(\tau) + Q^*(\tau)$, where $\bar{J}(\tau)$ is the direction and frequency average of the radiation field (see Eq. (3)). For $\mu > 0$, the formal solution of the radiative transfer may be written as

$$I(\tau, x, \mu) = \int_{\tau}^{\infty} S(\tau') e^{-[\tau' - \tau]\varphi(x)/\mu} \frac{\varphi(x)}{\mu} d\tau'. \quad (5)$$

There is for $\mu < 0$ a similar expression, in which the integration runs from 0 to τ . Integrating Eq. (5) over μ and over x (weighted by $\varphi(x)$), we readily obtain the integral equations

$$S(\tau) = (1 - \epsilon) \int_0^{\infty} K(|\tau - \tau'|) S(\tau') d\tau' + Q^*(\tau), \quad (6)$$

and

$$\bar{J}(\tau) = (1 - \epsilon) \int_0^{\infty} K(|\tau - \tau'|) \bar{J}(\tau') d\tau' + \int_0^{\infty} K(|\tau - \tau'|) Q^*(\tau') d\tau'. \quad (7)$$

The kernel $K(\tau)$ is given by

$$K(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \varphi^2(x) e^{-|\tau|\varphi(x)/\mu} \frac{d\mu}{\mu} dx. \quad (8)$$

For monochromatic scattering, we can ignore the dependence on x and set $\varphi(x) = 1$. The source function may then be written as $S(\tau) = (1 - \epsilon)J(\tau) + Q^*(\tau)$, where $J(\tau)$ is the radiation field averaged over μ . The functions $S(\tau)$ and $J(\tau)$ also satisfy Eqs. (6) and (7). The kernel is given by

$$K(\tau) = \frac{1}{2} \int_1^{\infty} e^{-|\tau|/\mu} \frac{d\mu}{\mu}. \quad (9)$$

For monochromatic scattering, the kernel is one-half of the first integro-exponential function. It is easy to observe that it can be written as a Laplace transform, namely as

$$K(\tau) = \int_0^{\infty} k(\nu) e^{-\nu|\tau|} d\nu, \quad (10)$$

where $k_M(\nu)$ (M for monochromatic) can be defined as

$$k_M(\nu) \equiv 1/2\nu, \quad \nu > 1, \quad k_M(\nu) \equiv 0, \quad 0 < \nu < 1. \quad (11)$$

As stressed by V.V. Sobolev (e.g. Sobolev 1963; Ivanov 1973), for complete frequency redistribution, when $\varphi(x)/\mu$ is chosen as integration variable, $K(\tau)$ can also be written as in Eq. (10). The definition of $k_R(\nu)$ (R for complete frequency redistribution) is given in Eq. (C.15).

Because they are constructed with a solution, albeit formal, of the radiative transfer equation, integral equations can take into account any boundary conditions on the

radiation field. For example, an incident field $I_0(0, x, \mu)$, $\mu < 0$, will appear as an additional primary source describing the directly transmitted incident radiation, provided the diffuse field is separated from the directly transmitted one. Here we always assumed for simplicity that the incident field on the boundary is zero and that there is an internal source.

The integral equations in Eqs. (6) and (7) are the starting points for two of the most famous methods providing exact solutions for radiative transfer in a semi-infinite medium. For monochromatic scattering, when $\epsilon = 0$ and $Q^*(\tau) = 0$, Eq. (6) becomes the Milne integral equation, for which Wiener and Hopf (1931) constructed an exact solution. Actually, the Milne integral was first proposed by Schwarzschild (1914) to solve the problem stated in Schwarzschild (1906) of the run of temperature in a stellar atmosphere in radiative equilibrium. The Wiener–Hopf method of solution is based on the transformation of the semi-infinite convolution integral equation into a convolution equation on a full line. The use of the analyticity properties of complex Fourier transforms leads to a factorization problem in the complex plane. The Wiener–Hopf method, introduced for monochromatic transfer, can also be applied to complete frequency redistribution, as was rigorously shown by Krein (1962). This generalization makes clear the strict analogy between a Wiener–Hopf factorization and a homogeneous Riemann–Hilbert problem. Because it requires only a factorization in the complex plane, the Wiener–Hopf method has been applied to convolution integral equations with a large variety of kernels (see for example Noble 1958; Krein 1962; Pincus 1981; Ablowitz and Fokas 1997; Lawrie and Abrahams 2007). For mono-kinetic neutron transport, it was very popular in the early forties, but the publications came out only after 1945 (e.g. Mark 1947; Placzek 1947; Placzek and Seidel 1947).

The fundamental work of V. A. Ambarzumian for monochromatic scattering in the early forties is also based on the integral equation for the source function (Ambarzumian 1942, 1943; Kourganoff 1963). When $Q^*(\tau)$ is replaced by the Dirac distribution $(1 - \epsilon)\delta(\tau)$, the inhomogeneous term in Eq. (7) becomes $(1 - \epsilon)K(\tau)$ and the solution of Eq. (7) provides the so-called *resolvent function*, which is the regular part of the surface Greens function. Making use of the property that the kernel and the inhomogeneous term are Laplace transforms, Ambarzumian could establish that the resolvent function can be expressed in terms a half-space auxiliary function, widely known now as the H-function, a name attributed to it by S. Chandrasekhar in the late forties, and show that the H-function satisfies a nonlinear integral equation. This approach, often referred to as the resolvent method, has been generalized to complete frequency redistribution (e.g. Sobolev 1963; Ivanov 1973, 1994). It does not invoke analyticity properties of functions of a complex variable. The resolvent method is applied to the partial frequency redistribution problem investigated by Yengibarian and Khachatryan (1991).

For partial frequency redistribution, it is also possible to write an integral equation for the function $S(\tau, x)$, but in general, it can be solved numerically only. In the special case of a linear combination of monochromatic scattering and complete frequency redistribution, it is possible, as we now show, to construct for $J(\tau, x)$ an integral equation somewhat similar to Eq. (7), which will be the starting point for the method of solution presented in this article.

The radiative transfer equation in Eq. (1) shows that the dependence on the frequency x is actually through the absorption profile $\varphi(x)$. The frequency variable is from now on

$$\xi = \varphi(x). \quad (12)$$

The variable ξ is positive and its range of variation is $[0, \varphi_0]$, with $\varphi_0 = \varphi(0)$. For the radiation field and the source function, we use the notation $I(\tau, \xi, \mu)$ and $S(\tau, \xi)$ and for the direction and frequency averaged intensity we introduce $\mathcal{J}(\tau, \xi) = J(\tau, x)$ and $\bar{\mathcal{J}}(\tau)$. The latter is defined by

$$\bar{\mathcal{J}}(\tau) = \bar{J}(\tau) = \int_{-\infty}^{+\infty} \varphi(x) J(\tau, x) dx = \int_0^{\varphi_0} W(\xi) \mathcal{J}(\tau, \xi) d\xi, \quad (13)$$

where

$$W(\xi) = 2\xi \left| \frac{dx}{d\xi} \right|. \quad (14)$$

The normalization of $\varphi(x)$ to one implies

$$\int_0^{\varphi_0} W(\xi) d\xi = 1. \quad (15)$$

The radiative transfer equation becomes

$$\mu \frac{\partial I}{\partial \tau}(\tau, \xi, \mu) = \xi [I(\tau, \xi, \mu) - S(\tau, \xi)]. \quad (16)$$

The source function $S(\tau, \xi)$ is defined in Eq. (2), with $\mathcal{J}(\tau, \xi)$ and $\bar{\mathcal{J}}(\tau)$ instead of $J(\tau, x)$ and $\bar{J}(\tau)$. Integrating over μ the formal solution given in Eq. (5), in which $S(\tau')$ is replaced by $S(\tau', \xi)$, we obtain for $\mathcal{J}(\tau, \xi)$ the integral equation

$$\begin{aligned} \mathcal{J}(\tau, \xi) = & (1 - \epsilon) b \xi \int_{-\infty}^{+\infty} K_M(|\tau' - \tau| \xi) \mathcal{J}(\tau', \xi) d\tau' \\ & + \xi \int_{-\infty}^{+\infty} K_M(|\tau' - \tau| \xi) [Q^*(\tau') + (1 - \epsilon)(1 - b) \bar{\mathcal{J}}(\tau')] d\tau'. \end{aligned} \quad (17)$$

It is written here for an infinite medium, but has exactly the same form for a semi-infinite medium with no incident radiation.

The subscript M indicates that the kernel is the monochromatic one defined in Eq. (9). For $b = 1$, the term depending on $\bar{\mathcal{J}}(\tau)$, which describes the coupling between the two scattering mechanisms, disappears and when in addition ξ is set to one, we recover the integral equation for $\bar{J}(\tau)$ written in Eq. (7). The main interest of Eq. (17) is that $\bar{\mathcal{J}}(\tau)$ can be treated exactly as a primary source similar to $Q^*(\tau)$. This property is fully exploited for the solution of the singular integral equation constructed in Section 2.3. We note that an integration of Eq. (17) over ξ , weighted by $W(\xi)$, does not provide an integral equation for $\bar{\mathcal{J}}(\tau)$, because of the structure of the first integral in the right-hand side.

2.3. Construction of a singular integral equation

There are different methods to transform a radiative transfer equation or a convolution integral equation into a singular integral equation. Some of them are mentioned in the Appendix B. Here we apply to Eq. (17) an inverse Laplace transform method, which

amounts to assume that $\mathcal{J}(\tau, \xi)$ can be written as a Laplace transform. For an infinite medium, we write

$$\mathcal{J}(\tau, \xi) = \begin{cases} \int_0^\infty j(\nu, \xi) e^{-\nu\tau} d\nu & \tau \geq 0, \\ -\int_{-\infty}^0 j(\nu, \xi) e^{-\nu\tau} d\nu & \tau \leq 0. \end{cases} \quad (18)$$

For a semi-infinite medium, it suffices to keep the positive values of τ . In Eq. (18) ν is real and $j(\nu, \xi)$, henceforth referred to as the inverse Laplace transform of $\mathcal{J}(\tau, \xi)$, is a regular function or a distribution. We also introduce the inverse Laplace transforms of $\bar{\mathcal{J}}(\tau)$ and $Q^*(\tau)$, denoted $\bar{j}(\nu)$ and $q^*(\nu)$ and their linear combination

$$p^*(\nu) \equiv \omega(1 - b)\bar{j}(\nu) + q^*(\nu), \quad (19)$$

where ω , the single scattering albedo, is defined by

$$\omega \equiv 1 - \epsilon, \quad (20)$$

and $\bar{j}(\nu)$ is related to $j(\nu, \xi)$ by

$$\bar{j}(\nu) = \int_0^{\varphi_0} W(\xi) j(\nu, \xi) d\xi. \quad (21)$$

A detailed description of the transformation of Eq. (17) into Eq. (22) is given in the Appendix B. The main idea is the separation of the integrations over τ' in two ranges: $\tau' \in [-\infty, \tau]$ and $\tau' \in [\tau, +\infty[$. Expressing then all the functions in Eq. (17) in terms of their inverse Laplace transforms, simple algebra shows that $j(\nu, \xi)$ satisfies the singular integral equation :

$$\begin{aligned} \lambda(\nu, \xi) j(\nu, \xi) &= \omega b \xi k(\nu, \xi) \oint_{-\infty}^{+\infty} \frac{j(\nu', \xi)}{\nu' - \nu} d\nu' \\ &+ \xi \left[p^*(\nu) t(\nu, \xi) + k(\nu, \xi) \oint_{-\infty}^{+\infty} \frac{p^*(\nu')}{\nu' - \nu} d\nu' \right], \end{aligned} \quad (22)$$

where

$$k(\nu, \xi) \equiv 1/2\nu, \quad \nu > \xi, \quad k(\nu, \xi) \equiv 0, \quad 0 < \nu < \xi, \quad (23)$$

$$t(\nu, \xi) \equiv \oint_{\xi}^{\infty} k(\nu', \xi) \left[\frac{1}{\nu' + \nu} + \frac{1}{\nu' - \nu} \right] d\nu', \quad (24)$$

and

$$\lambda(\nu, \xi) \equiv 1 - \omega b \xi t(\nu, \xi). \quad (25)$$

The symbol \oint means that the integral has to be taken in principal part. Equation (22) corresponds to Eq. (18) in Hemsch and Ferziger (1972). A sketch of $\lambda(\nu, \xi)$ for $\nu \in$

$] - \infty, +\infty[$ is shown in Fig. 1. For a semi-infinite medium, $j(\nu, \xi)$ satisfies the same equation, but the range of integration over ν' is limited to $[0, \infty[$.

Equation (22) has two singular integrals, one with $j(\nu, \xi)$ and the other one with the frequency integrated function $p^*(\nu)$. Its solution will be constructed in two steps. First it is assumed that $p^*(\nu)$ is a known function. Equation (22) represents then a singular integral equation of the monochromatic type with a Cauchy kernel for $j(\nu, \xi)$. Its solution, when integrated over ξ , provides a new singular integral equation for $\bar{j}(\nu)$, which is solved in Section 4.

The method of solution, which we apply to Eq. (22) and all the other singular integral equations encountered in this article, was introduced by Carleman (1922). It amounts to transform the singular integral equation into a boundary value problem in the complex plane for the Hilbert transform of the unknown function. Singular integral equations, Hilbert transforms, and their properties are treated in detail in many books, such as Muskhelishvili (1953), Gakhov (1966), Carrier, Krook, and Pearson (1966), Pogorzelski (1966), Williams (1971), Ablowitz and Fokas (1997).

We recall that the Hilbert transform of a function $f(t)$ is the integral

$$\mathcal{H}(z) \equiv \frac{1}{2i\pi} \int_C \frac{f(t)}{t-z} dt, \quad (26)$$

where z belongs to the complex plane and C is a given curve in the complex plane. In the following we use the symbolic notation

$$\mathcal{H}(z) = H_C[f(t)], \quad (27)$$

where the subscript will indicate the integration curve. The function $\mathcal{H}(z)$ has very interesting properties. It is analytic in the complex plane cut along the curve C , goes to zero at infinity in the complex plane, is free of singularities, and satisfies the Plemelj formulae. The latter state that for a point z_0 on the cut, the limiting values $\mathcal{H}^+(z_0)$ and $\mathcal{H}^-(z_0)$ on each side of it obey the relations

$$\mathcal{H}^+(z_0) - \mathcal{H}^-(z_0) = f(z_0), \quad \mathcal{H}^+(z_0) + \mathcal{H}^-(z_0) = \frac{1}{i\pi} \oint_C \frac{f(t')}{t' - z_0} dt'. \quad (28)$$

In this article, the integration curve C is always on the real line and will be defined by its end-points. The Plemelj formulae will allow us to transform Eq. (22) into a boundary value problem in the complex plane for the Hilbert transform of $j(\nu, \xi)$. The latter is then retrieved with the first of the Plemelj formulae.

A function playing a critical role in the solution of Eq. (22) is the *dispersion function* constructed with $\lambda(\nu, \xi)$. We now discuss its properties.

3. The frequency-dependent dispersion function

The *dispersion function* is the analytic continuation of $\lambda(\nu, \xi)$ in the complex plane. It is obtained by replacing in Eq. (25) the real variable ν by a complex variable z . The dispersion function is thus defined by

$$\mathcal{L}(z, \xi) \equiv 1 - \omega b \xi T(z, \xi), \quad (29)$$

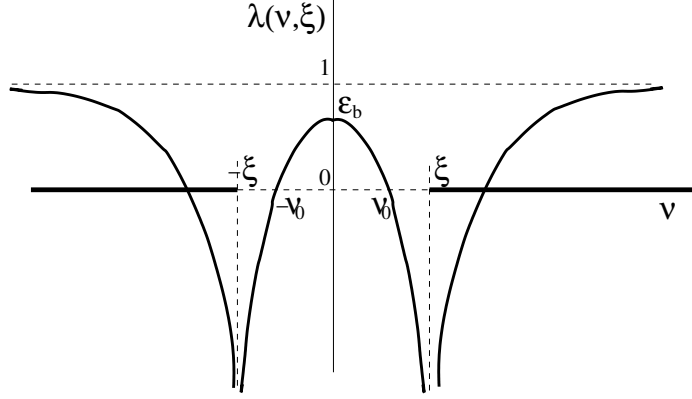


Figure 1. The function $\lambda(\nu, \xi)$, real part of $\mathcal{L}^\pm(\nu, \xi)$, as a function of ν , for a given value of the frequency variable $\xi = \varphi(x)$. At the origin $\lambda(0, \xi) = \epsilon_b = 1 - \omega b$ and at infinity $\lambda(\pm\infty) = 1$. The zeroes at $\pm\nu_o(\xi)$ are also the zeroes of the dispersion function $\mathcal{L}(z, \xi)$. The latter is analytic in the complex plane cut along the semi-infinite intervals $\Re(z) = \nu \in]-\infty, -\xi] \cup [+ \xi, +\infty[$ indicated here by two thick lines.

where z belongs to the complex plane and

$$T(z, \xi) = \int_{\mathcal{C}(\xi)} k(\nu, \xi) \frac{1}{\nu - z} d\nu. \quad (30)$$

$T(z, \xi)$ is actually the Fourier transform of the kernel $K_M(\tau\xi)$, The notation $\mathcal{C}(\xi)$ stands for $] -\infty, -\xi] \cup [+ \xi, +\infty[$, that is the union of the two intervals $] -\infty, -\xi]$ and $[+ \xi, +\infty[$. The properties of $\mathcal{L}(z, \xi)$ are essentially the same as those of the well studied monochromatic dispersion function corresponding to $b = \xi = 1$, henceforth denoted $\mathcal{L}_M(z)$. We summarized here the properties of $\mathcal{L}(z, \xi)$.

As shown by Eqs. (29) and (30) $T(z, \xi)$, which is obviously a Hilbert transform, and $\mathcal{L}(z, \xi)$ are analytic in the complex plane cut along $\mathcal{C}(\xi)$. At infinity, $T(z, \xi) \rightarrow 0$ and $\mathcal{L}(z, \xi) \rightarrow 1$. For $\nu \in \mathcal{C}(\xi)$, the Plemelj formulae lead to

$$T^+(\nu, \xi) - T^-(\nu, \xi) = 2i\pi k(\nu, \xi), \quad (31)$$

$$T^+(\nu, \xi) + T^-(\nu, \xi) = 2t(\nu, \xi), \quad (32)$$

$$\mathcal{L}^+(\nu, \xi) - \mathcal{L}^-(\nu, \xi) = -2i\pi\omega b \xi k(\nu, \xi), \quad (33)$$

$$\mathcal{L}^+(\nu, \xi) + \mathcal{L}^-(\nu, \xi) = 2\lambda(\nu, \xi). \quad (34)$$

For $z \rightarrow 0$,

$$\mathcal{L}(z, \xi) \simeq \epsilon_b - (1 - \epsilon_b) \frac{z^2}{3\xi^2}, \quad (35)$$

hence

$$\mathcal{L}(0, \xi) = \epsilon_b. \quad (36)$$

Here,

$$\epsilon_b = 1 - \omega b = 1 - (1 - \epsilon) b. \quad (37)$$

The parameter ϵ_b is positive and smaller than unity since $b \in [0, 1]$.

Another important property of $\mathcal{L}(z, \xi)$ is that it has a pair of real zeroes in the interval $[-\xi, +\xi]$ coinciding with the zeroes of $\lambda(\nu, \xi)$ at $\pm\nu_0(\xi)$ (see Fig. 1). The dispersion function has no other zero. The two zeroes and the cut along $\mathcal{C}(\xi)$ are known as the *spectrum* of ν or as the *discrete* and *continuous eigenvalues* of the eigenfunctions $\Phi(k, x, \mu)$ (McCormick and Kušcer 1973). We refer here to the interval $[-\xi, +\xi]$ containing the discrete eigenvalues as the *discrete spectrum* of $\mathcal{L}(\nu, \xi)$ and to $\mathcal{C}(\xi) =]-\infty, -\xi] \cup [+ \xi, +\infty[$ as the *continuous spectrum*.

The integration over ν in Eq. (30) is easily carried out and leads to the explicit expression

$$\mathcal{L}(z, \xi) = 1 - \omega b \xi \frac{1}{2z} \ln \frac{\xi + z}{\xi - z}, \quad (38)$$

which shows that $\mathcal{L}(z, \xi)$ is actually a function of the complex variable

$$\tilde{z} = z/\xi. \quad (39)$$

and that we have $\mathcal{L}(z, \xi) = \tilde{\mathcal{L}}(\tilde{z})$ with

$$\tilde{\mathcal{L}}(\tilde{z}) = 1 - (1 - \epsilon_b) \frac{1}{2\tilde{z}} \ln \frac{1 + \tilde{z}}{1 - \tilde{z}}. \quad (40)$$

The zeroes of $\tilde{\mathcal{L}}(z)$ are located at $\pm u_0$, with u_0 solution of $\tilde{\mathcal{L}}(u_0) = 0$ and those of $\mathcal{L}(z, \xi)$ at $\nu_0(\xi) = \xi u_0$. The inverse relation giving the value of ξ for a given position ν of the positive zero is $\xi_0(\nu) = \nu/u_0$.

The partial derivatives of the dispersion function at $\pm\nu_0(\xi)$ play an important role in the calculation of $j(\nu, \xi)$ and $\bar{j}(\nu)$. At these points $\mathcal{L}(z, \xi) = \lambda(\nu, \xi)$, with $\nu = \Re(z)$. Using Eq. (40), one readily obtains

$$\frac{\partial \lambda(\nu, \xi)}{\partial \nu} \Big|_{\pm \nu_0} = \pm \frac{1}{\xi u_0} \left[\frac{\epsilon_b - u_0^2}{1 - u_0^2} \right], \quad (41)$$

and

$$\frac{\partial \lambda(\nu, \xi)}{\partial \xi} \Big|_{\pm \nu_0} = \mp \frac{u_0}{\nu_0} \left[\frac{\epsilon_b - u_0^2}{1 - u_0^2} \right]. \quad (42)$$

The derivatives with respect to ν and ξ have opposite signs, the derivative with respect to ν being negative and the other one positive. Their ratio is $-1/u_0$.

4. The infinite medium

We now apply the Carleman method to solve the singular integral equation for $j(\nu, \xi)$ written in Eq. (22). In this equation, the coefficient $\lambda(\nu, \xi)$, which appears in the left-hand side, has zeroes in the interval $[-\xi, +\xi]$ at $\pm\nu_0(\xi)$. For $\nu \neq \pm\nu_0(\xi)$, $j(\nu, \xi)$ is given by the right-hand side divided by $\lambda(\nu, \xi)$. For $\nu = \pm\nu_0(\xi)$, the right-hand side of Eq. (22) has no reason to be zero. Equation (22) can be satisfied for $\nu = \pm\nu_0(\xi)$ by assuming that $j(\nu, \xi) = f(\nu, \xi)[\delta(\nu - \nu_0(\xi)) + \delta(\nu + \nu_0(\xi))]$, where $f(\nu, \xi)$ is at this

stage an arbitrary function. It is indeed a property of the Dirac delta function that $\int x\delta(x)dx = 0$. The solution of Eq. (22) valid for all values of ν is given in Eq. (51). The same argument is found in the construction of the singular eigenfunctions of the Case (1960) expansion method.

Hilbert transforms are usually introduced for Hölder continuous functions (e.g., Muskhelishvili 1953). This means that one should treat separately the intervals $] - \xi, +\xi[$ and $\mathcal{C}(\xi)$. This is done in Section 5 for the semi-infinite medium. Hilbert transforms can however still be defined for a Dirac distribution (Pandey 1996) and the Plemelj formulae remain applicable. Following Hemsch and Ferziger (1972), the Hilbert transforms are defined in this Section over the full real axis. Using the notation defined in Eq. (27) we introduce:

$$N_\infty(z, \xi) \equiv H_{-\infty}^{+\infty}[j(\nu, \xi)], \quad \bar{N}_\infty(z) \equiv H_{-\infty}^{+\infty}[\bar{j}(\nu)], \quad (43)$$

and

$$N_Q^\infty(z) \equiv H_{-\infty}^{+\infty}[q^*(\nu)], \quad P_\infty(z) \equiv H_{-\infty}^{+\infty}[p^*(\nu)]. \quad (44)$$

The Hilbert transform of $q^*(\nu)$ has all the required properties when the integral of $Q^*(\tau)$ over τ has a finite value. By construction, these four Hilbert transforms are analytic in the complex plane cut along the real axis and go to zero as z goes to infinity.

In Section 4.1 we determine $N_\infty(z, \xi)$ in terms of $\bar{N}_\infty(z)$ and in Section 4.2 $\bar{N}_\infty(z)$ by an integration over ξ of $N_\infty(z, \xi)$. The functions $j(\nu, \xi)$ and $\bar{j}(\nu)$ are then given by the jumps of $N_\infty(z, \xi)$ and $\bar{N}_\infty(z)$ across the real axis (see Eq. (28)). Explicit expressions of $j(\nu, \xi)$ and $\bar{j}(\nu)$ are given in Section 4.3 for the Green's function.

4.1. The frequency dependent Hilbert transform $N_\infty(z, \xi)$

Making use of the expression of $\mathcal{L}(z, \xi)$ in Eq. (29), of the Plemelj formulae for $T(z, \xi)$ and $\mathcal{L}(z, \xi)$, and also of the Hilbert transforms in Eqs. (43) and (44), we can rewrite Eq. (22) as

$$\begin{aligned} \mathcal{L}^+(\nu, \xi)N_\infty^+(\nu, \xi) - \mathcal{L}^-(\nu, \xi)N_\infty^-(\nu, \xi) = \\ \frac{1}{\omega b} \left\{ [1 - \mathcal{L}^+(\nu, \xi)]P_\infty^+(\nu) - [1 - \mathcal{L}^-(\nu, \xi)]P_\infty^-(\nu) \right\}. \end{aligned} \quad (45)$$

We stress that in this Section devoted to frequency dependent functions, we necessarily have $b \neq 0$. To solve Eq. (45) for $N_\infty(z, \xi)$, we use a standard method based on the Liouville theorem. Equation (45) is equivalent to

$$F^+(\nu, \xi) - F^-(\nu, \xi) = 0, \quad \nu \in] - \infty, +\infty[, \quad (46)$$

where

$$F(z, \xi) \equiv \mathcal{L}(z, \xi)N_\infty(z, \xi) - \frac{1}{\omega b} [1 - \mathcal{L}(z, \xi)]P_\infty(z). \quad (47)$$

By construction, $F(z, \xi)$ is analytic in the complex plane cut along the full real axis and, according to Eq. (46), the jump of $F(z, \xi)$ across this axis is zero. Hence $F(z, \xi)$ is

an entire function, which can be determined by its behavior at infinity, by application of the Liouville theorem (e.g. Carrier et al. 1966). For $z \rightarrow \infty$, $N_\infty(z, \xi)$ and $P_\infty(z)$ tend to 0 and $\mathcal{L}(z, \xi) \rightarrow 1$. Hence, $F(z, \xi) \rightarrow 0$ as $z \rightarrow \infty$ and by the Liouville theorem $F(z, \xi)$ is zero for all z . We thus obtain

$$\begin{aligned} & \mathcal{L}(z, \xi) N_\infty(z, \xi) \\ &= \frac{1}{\omega b} [1 - \mathcal{L}(z, \xi)] P_\infty(z) = \frac{1}{\omega b} [1 - \mathcal{L}(z, \xi)] [\omega(1 - b) \bar{N}_\infty(z) + N_Q^\infty(z)]. \end{aligned} \quad (48)$$

Since $\mathcal{L}(z, \xi)$ has zeroes at $\pm \nu_0(\xi) \in [-\xi, +\xi]$, Eq. (48) cannot be straightforwardly divided by $\mathcal{L}(z, \xi)$. We must distinguish between the regions $z \in [-\xi, +\xi]$ and z outside this interval.

For $z \notin [-\xi, +\xi]$, we can divide Eq. (48) by $\mathcal{L}(z, \xi)$. Using the expression of $\mathcal{L}(z, \xi) = 1 - \omega b \xi T(\nu, \xi)$ (see Eq. (29)), we obtain

$$N_\infty(z, \xi) = \xi \frac{T(z, \xi)}{\mathcal{L}(z, \xi)} [\omega(1 - b) \bar{N}_\infty(z) + N_Q^\infty(z)]. \quad (49)$$

In the interval $[-\xi, +\xi]$, according to Eq. (23), $k(\nu, \xi) = 0$, hence Eq. (22) reduces to

$$\lambda(\nu, \xi) j(\nu, \xi) = \xi p^*(\nu) t(\nu, \xi). \quad (50)$$

As explained at the beginning of Section (4), because of the zeroes of $\lambda(\nu, \xi)$ at $\pm \nu_0(\xi)$, the solution of Eq. (50) has the general form

$$j(\nu, \xi) = f(\nu, \xi) [\delta(\nu - \nu_0(\xi)) + \delta(\nu + \nu_0(\xi))] + p^*(\nu) P \frac{\xi t(\nu, \xi)}{\lambda(\nu, \xi)}, \quad (51)$$

where $\delta(\nu \pm \nu_0(\xi))$ are Dirac distributions at $\pm \nu_0(\xi)$ and P stands for Cauchy principal part. Using $\int x \delta(x) dx = 0$, we can check that the solution in Eq. (51) satisfies Eq. (50) for all values of ν .

The function $f(\nu, \xi)$ can be determined by considering the limit $z \rightarrow \pm \nu_0(\xi)$ in Eq. (48). In the right-hand side we can set $\mathcal{L}(z, \xi) = 0$. In the left-hand side, the only terms giving a non-zero contribution come from the Dirac distributions. We then Taylor expand $\mathcal{L}(z, \xi)$ around its zeroes at $\pm \nu_0(\xi)$, that is assume $\mathcal{L}(z, \xi) \simeq (z \pm \nu_0) \partial_\nu \mathcal{L}(\pm \nu_0, \xi)$, where $\partial_\nu \mathcal{L}(\pm \nu_0, \xi) = \partial_\nu \lambda(\pm \nu_0, \xi)$ is the partial derivative of $\lambda(\nu, \xi)$ with respect to ν taken at $\pm \nu_0(\xi)$. Equation (48) thus leads to

$$f(\pm \nu_0, \xi) \partial_\nu \lambda(\pm \nu_0, \xi) = -\frac{2i\pi}{\omega b} \left\langle P_\infty(\pm \nu_0) \right\rangle, \quad (52)$$

where

$$\left\langle P_\infty(\pm \nu_0) \right\rangle \equiv \frac{1}{2} [P_\infty^+(\pm \nu_0) + P_\infty^-(\pm \nu_0)] = \frac{1}{2i\pi} \oint_{-\infty}^{+\infty} \frac{p^*(\nu')}{\nu' \mp \nu_0} d\nu'. \quad (53)$$

We recall that $P_\infty(z)$, the Hilbert transform of $p^*(\nu)$ is analytic in the complex plane cut along the real axis. The derivative $\partial_\nu \lambda(\pm \nu_0, \xi)$ is given in Eq. (41). One can check that $f(\pm \nu_0, \xi)$ is positive.

The expressions that we have obtained for $N_\infty(z, \xi)$ and $f(\pm\nu_0, \xi)$ are still containing the unknown function $\bar{N}_\infty(z)$. We now proceed to the determination of the latter.

4.2. The frequency integrated Hilbert transform $\bar{N}_\infty(z)$

Knowing how to properly define $N_\infty(z, \xi)$ for all z , we can use Eq. (49) to determine $\bar{N}_\infty(z)$. Integrating over ξ , we obtain

$$\Omega(z)\bar{N}_\infty(z) = N_Q^\infty(z) \int_0^{\varphi_0} W(\xi) \xi \frac{T(z, \xi)}{\mathcal{L}(z, \xi)} d\xi, \quad (54)$$

where $W(\xi)$ is the weight function defined in Eq. (14), and

$$\Omega(z) \equiv 1 - \omega(1 - b) \int_0^{\varphi_0} W(\xi) \xi \frac{T(z, \xi)}{\mathcal{L}(z, \xi)} d\xi. \quad (55)$$

The function $\Omega(z)$ plays for $\bar{N}_\infty(z)$ the role played by the dispersion function $\mathcal{L}(z, \xi)$ for $N_\infty(z, \xi)$. It will be referred to as the *generalized dispersion function*. Equation (55) shows that $\Omega(z)$ is analytic in \mathbb{C}/\mathbb{R} .² It is shown in Appendix C.1 that $\Omega(z)$ has no zero and in Appendix C.2 that it goes to the complete frequency dispersion function for $b = 0$. The latter is denoted $\mathcal{L}_R(z)$, the subscript R standing for complete frequency redistribution. For $b = 1$, that is in the monochromatic limit, one simply has $\Omega(z) = 1$. Other properties of $\Omega(z)$ are described in Appendix C.1.

Since $\Omega(z)$ has no zero, we can divide Eq. (54) by $\Omega(z)$. Making use of Eq. (55), we obtain for $\bar{N}_\infty(z)$ an explicit expression, analytic in \mathbb{C}/\mathbb{R} , which may be written as

$$\bar{N}_\infty(z) = \frac{1}{\omega(1 - b)} N_Q^\infty(z) \left[\frac{1}{\Omega(z)} - 1 \right]. \quad (56)$$

We can now obtain closed form expressions for $N_\infty(z, \xi)$ and $f(\pm\nu_0, \xi)$. Inserting Eq. (56) into Eqs. (49) and (52), we obtain

$$N_\infty(z, \xi) = \frac{1}{\omega b} N_Q^\infty(z) \left[\frac{1}{\mathcal{L}(z, \xi)} - 1 \right] \frac{1}{\Omega(z)}, \quad (57)$$

and

$$f(\pm\nu_0, \xi) \partial_\nu \lambda(\pm\nu_0, \xi) = -\frac{2i\pi}{\omega b} \left\langle \bar{N}_Q(\pm\nu_0) \frac{1}{\Omega(\pm\nu_0)} \right\rangle. \quad (58)$$

The expression given for $\bar{N}_\infty(z)$ holds only when $b \neq 1$. For $b = 0$ it provides the complete frequency redistribution limit. The expressions of $N_\infty(z, \xi)$ and $f(\pm\nu_0, \xi)$ hold only for $b \neq 0$. For $b = 1$, they provide the monochromatic limit.

²The notation \mathbb{C}/\mathbb{R} stands for the complex plane cut along the full real axis

4.3. Explicit expressions of $j(\nu, \xi)$ and $\bar{j}(\nu)$ for the Green's function

We give here explicit expressions of $j(\nu, \xi)$ and $\bar{j}(\nu)$ obtained by setting

$$Q^*(\tau) = \delta(\tau). \quad (59)$$

For this special primary source, $S(\tau, \xi)$ is the infinite medium Green's function and $(1 - \epsilon)\mathcal{J}(\tau, \xi)$, its regular part, often referred to as the resolvent function. Replacing $Q^*(\tau)$ by $\delta(\tau)$ in the integral equation for $\mathcal{J}(\tau, \xi)$ (see Eq. (17)), we observe that the contribution from the primary source term is now given by $\xi K_M(|\tau|\xi)$, the inverse Laplace transform of which is

$$\xi k(\nu, \xi) = -\frac{1}{2i\pi\omega b}[\mathcal{L}^+(\nu, \xi) - \mathcal{L}^-(\nu, \xi)]. \quad (60)$$

Equation (22) still provides the integral equation for $j(\nu, \xi)$. It suffices to add $\xi k(\nu, \xi)$ to the right-hand side, while simultaneously setting $q^*(\nu) = 0$.

Proceeding as described in Section 4.1, we obtain the relation

$$\mathcal{L}(z, \xi)N_\infty(z, \xi) = [1 - \mathcal{L}(z, \xi)] \left[\frac{1-b}{b}\bar{N}_\infty(z) + \frac{1}{\omega b 2i\pi} \right]. \quad (61)$$

Comparing it with Eq. (48), we see that $N_Q^\infty(z)$ has been replaced by $1/(2i\pi)$. It suffices to carry out this change in Eqs. (56), (57) and (58) to obtain $\bar{N}_\infty(z)$, $N_\infty(z, \xi)$ and $f(\pm\nu_0, \xi)$.

The function $\bar{j}(\nu)$ is given by the jump of $\bar{N}_\infty(z)$ across the real axis. Taking advantage of the fact that $\Omega^+(\nu)$ and $\Omega^-(\nu)$ are complex conjugate, we can write

$$\bar{j}(\nu) = \frac{1}{\omega\pi(1-b)} \frac{\Im[-\Omega^+(\nu)]}{[\Re[\Omega^+(\nu)]]^2 + [\Im[\Omega^+(\nu)]]^2}. \quad (62)$$

When $\Im[-\Omega^+(\nu)]$ is replaced by the explicit expressions given by Eqs. (C.8), (C.9) (C.12), the factor $\omega(1-b)$ disappears, but there remains a dependence on b , which appears in the functions defined in Eqs. (C.5) and (C.7). The complete frequency redistribution limit $b = 0$ is recovered when $\Omega(z)$ is replaced by the complete frequency redistribution dispersion function $\mathcal{L}_R(z)$. In this case $\bar{j}(\nu)$ becomes

$$\bar{j}_R(\nu) = \frac{k_R(\nu)}{\mathcal{L}_R^+(\nu)\mathcal{L}_R^-(\nu)}, \quad (63)$$

where $k_R(\nu)$ is the inverse Laplace transform of the kernel $K_R(\tau)$ (see Eqs. (C.14)).

The case of $j(\nu, \xi)$ is slightly more complex since we have to distinguish between the discrete part of the spectrum corresponding to $\nu \in [-\xi, +\xi]$ and the continuous part for $\nu \in \mathcal{C}(\xi)$. As shown by Eq. (51), the discrete part contains the two Dirac distributions at $\pm\nu_0(\xi)$, corresponding to the zeroes of $\lambda(\nu, \xi)$, plus another term taken in principal value because it has $1/\lambda(\nu, \xi)$ in the denominator. It is convenient to write the discrete part, denoted $j_d(\nu, \xi)$, with d standing for discrete, as

$$j_d(\nu, \xi) = j_d^{(0)}(\nu, \xi) + j_d^{(1)}(\nu, \xi), \quad (64)$$

where the superscript 0 correspond to the Dirac distributions. The term $j_d^{(1)}(\nu, \xi)$ is given by the jump of $N_\infty(z, \xi)$ across the real axis for $\nu \in [-\xi, +\xi]$. Equations (57) and (58) in which $N_Q^\infty(z)$ is replaced by $1/(2i\pi)$ lead to

$$j_d^{(1)}(\nu, \xi) = \frac{\xi}{\pi} \frac{\Im[-\Omega^+(\nu)]}{\Omega^+(\nu)\Omega^-(\nu)} P \frac{t(\nu, \xi)}{\lambda(\nu, \xi)}, \quad (65)$$

$$j_d^{(0)}(\nu, \xi) = f(\nu, \xi)[\delta(\nu - \nu_0(\xi)) + \delta(\nu + \nu_0(\xi))], \quad (66)$$

with

$$f(\nu, \xi) = -\frac{1}{\omega b} \frac{1}{\partial \lambda_\nu(\nu, \xi)} \left\langle \frac{1}{\Omega^\pm(\nu_0)} \right\rangle. \quad (67)$$

The angle bracket is defined in Eq. (53). One can check that $j_d(\nu, \xi)$ is positive. In the monochromatic limit $j_d(\nu, \xi)$ is reduced to $j_d^{(0)}(\nu, \xi)$ since $\Omega(z) = 1$ when $b = 1$ (see Eq. (55)).

In the continuous part of the spectrum, $j(\nu, \xi)$ is given by the jump of $N_\infty(z, \xi)$ across the real axis for $\nu \in \mathcal{C}(\xi)$. Combining Eq. (57) and the Plemelj formulae for $\mathcal{L}(z, \xi)$, we can write $j(\nu, \xi)$ as

$$j_c(\nu, \xi) = \xi \frac{k(\nu, \xi)}{\mathcal{L}^+(\nu, \xi)\mathcal{L}^-(\nu, \xi)} + \frac{1}{2i\pi\omega b} \left[\frac{(1 - \Omega(z))(1 - \mathcal{L}(z, \xi))}{\Omega(z)\mathcal{L}(z, \xi)} \right]_{\text{jmp}}, \quad (68)$$

where the subscript jmp stands for jump across the real axis and the subscript c for continuous. In the monochromatic limit the right-side is reduced to its first term since $\Omega(z) = 1$ for $b = 1$. There is no point in considering the complete frequency limit $b = 0$ since we are dealing here with a frequency dependent function.

We want to stress here the roles of the zero of $\mathcal{L}(z, \xi)$ at $\nu_0(\xi)$. First it gives a Dirac distribution in $j_d(\nu, \xi)$, hence an exponential term $e^{-\nu_0(\xi)\tau}$ in the source function $S(\tau, \xi)$ as shown in Eq. (72). Additional effects from the zeroes of $\mathcal{L}(z, \xi)$ appear in the functions $f(\nu_0, \xi)$, $j_d^{(1)}(\nu, \xi)$, and also in $\bar{j}(\nu)$ for $\nu \in [0, \xi]$, via the derivatives of $\mathcal{L}(z, \xi)$ with respect to ν or ξ taken at $\nu_0(\xi)$. A derivative with respect to ν appears explicitly in $f(\nu_0, \xi)$ (see Eq. (67)). The derivative with respect to ξ sneaks in for $\nu \in [0, \xi]$, through the imaginary part of $\Omega^\pm(\nu)$, which, as shown in Eq. (C.12), has a term of the form

$$-\pi\omega(1-b)W(\xi_0(\nu))\xi_0(\nu)\frac{t(\nu, \xi_0(\nu))}{\partial_\xi \mathcal{L}(\nu, \xi_0(\nu))}. \quad (69)$$

The zeroes of $\mathcal{L}(z, \xi)$ have thus a very important role for all values of b , except for $b = 0$. In this limit, the dispersion function $\mathcal{L}_R(z)$ is free of zeroes.

4.4. The source function in the physical space

In the physical space the source function is given by

$$S(\tau, \xi) = \omega[b\mathcal{J}(\tau, \xi) + (1-b)\bar{\mathcal{J}}(\tau)] + Q^*(\tau). \quad (70)$$

We limit the discussion to positive values of τ , for which we have

$$\mathcal{J}(\tau, \xi) = \int_0^\infty j(\nu, \xi) e^{-\nu\tau} d\nu, \quad \bar{\mathcal{J}}(\tau) = \int_0^\infty \bar{j}(\nu) e^{-\nu\tau} d\nu. \quad (71)$$

We recall that $\xi = \varphi(x)$, with $\varphi(x)$ the absorption line profile. The analysis of $j(\nu, \xi)$ and $\bar{j}(\nu)$ carried out in the preceding sections shows that we can write the source function as

$$S(\tau, \xi) = \omega \left[b f(\nu_0(\xi), \xi) e^{-\nu_0(\xi)\tau} + \int_0^\xi [b j_d^{(1)}(\nu, \xi) + (1-b)\bar{j}(\nu)] e^{-\nu\tau} d\nu \right. \\ \left. + \int_\xi^\infty [b j_c(\nu, \xi) + (1-b)\bar{j}(\nu)] e^{-\nu\tau} d\nu \right] + Q^*(\tau), \quad (72)$$

where the subscripts d and c stand for discrete and continuous.

In the monochromatic limit corresponding to $b = 1$, Eq. (72) becomes

$$S_M(\tau, \xi) = \omega \left[f(\nu_0(\xi), \xi) e^{-\nu_0(\xi)\tau} + \int_\xi^\infty j_c(\nu, \xi) e^{-\nu\tau} d\nu \right] + Q^*(\tau). \quad (73)$$

There is no contribution from the term $j_d^{(1)}(\nu, \xi)$, which is zero as shown by Eq. (65) because $\Omega(z) = 1$. For the same reason, $j_c(\nu, \xi)$ is reduced to the first term in the right-hand side of Eq. (68). When ξ is set to one, one recovers the usual monochromatic source function. For $\tau \rightarrow \infty$, $S_M(\tau, \xi)$ decreases exponentially as $e^{-\nu_0(\xi)\tau}$. The integral term decreases faster since its lower limit ξ is larger than $\nu_0(\xi)$.

In the complete frequency redistribution limit, that is for $b = 0$, we readily obtain

$$S_R(\tau, \xi) = \omega \int_0^\infty \bar{j}(\nu) e^{-\nu\tau} d\nu + Q^*(\tau). \quad (74)$$

It is quite obvious that $S_R(\tau, \xi)$ is independent of ξ . For $\tau \rightarrow \infty$, $S_R(\tau, \xi)$ decreases algebraically as a power law $\tau^{-\alpha}$ with $\alpha > 0$ depending on the behavior of $\varphi(x)$ for x large. For a Doppler (Gaussian) profile, $\alpha = 1$ (there is also a logarithmic correction). For a Voigt profile $\alpha = 1/2$ (see e.g., Ivanov 1973, p. 159).

Equation (72) shows that as soon as b becomes smaller than one, there will be a competition between the exponential term and the first integral in which ν , belonging to the interval $[0, \xi]$, can take values smaller than $\nu_0(\xi)$. In this interval $\Omega^\pm(\nu)$ depends on a derivative at $\nu_0(\xi)$ of the monochromatic dispersion function (see Eq. (C.12)), hence the functions $\bar{j}(\nu)$ and $j_d^{(1)}(\nu)$ depend also on this derivative (see Eqs. (62) and (65)). The zero of the dispersion function at $\nu_0(\xi)$ has thus several roles. It appears directly in the pure exponential term but also indirectly through derivatives of the dispersion function.

A very precise asymptotic expansion for $\nu \rightarrow 0$ or numerical calculations are needed to describe in some detail the transition between the monochromatic and the complete frequency redistribution limits, that is between an exponential and an algebraic large τ asymptotic behavior. This point is further discussed in Section 6.

5. The semi-infinite medium

We now consider a semi-infinite plane-parallel medium, where the space variable τ varies from 0 at the surface to ∞ in the interior. We also assume that there is no incident radiation field at $\tau = 0$. An incident field can always be transformed into an interior source by separating the contribution of the incident field from that of the diffuse field created by the scattering. The functions $\mathcal{J}(\tau, \xi)$ and $\bar{\mathcal{J}}(\tau)$ for $\tau \geq 0$ are defined as shown in Eqs. (18) to (21). Now ν has only positive values.

We first determine $j(\nu, \xi)$ (Section 5.1) and then $\bar{j}(\nu)$ by integration over ξ (Sections 5.2 and 5.3). Exact solutions of half-space problems always depend on a half-space auxiliary function derived from the dispersion function. We will introduce two half-space auxiliary functions, namely $X(z, \xi)$ constructed with $\mathcal{L}(z, \xi)$ and $\bar{X}(z)$, constructed with $\Omega(z)$. We use a standard construction method based on the solution of a homogeneous Riemann–Hilbert problem (e.g., Ablowitz and Fokas 1997). Finally we show in Section 5.4 how to calculate the emergent intensity.

5.1. The frequency dependent function $j(\nu, \xi)$

The singular integral equation for $j(\nu, \xi)$ is given by Eq. (22), with the domain of definition of ν restricted to $[0, \infty[$. This means that we now have to solve

$$\begin{aligned} \lambda(\nu, \xi) j(\nu, \xi) = & \omega b \xi k(\nu, \xi) \oint_0^\infty \frac{j(\nu', \xi)}{\nu' - \nu} d\nu' \\ & + \xi \left[p^*(\nu) t(\nu, \xi) + k(\nu, \xi) \oint_0^\infty \frac{p^*(\nu')}{\nu' - \nu} d\nu' \right], \end{aligned} \quad (75)$$

where $p^*(\nu)$ has been defined in Eq. (19). For the full-space case, we have introduced $N_\infty(z, \xi)$, the Hilbert transform of $j(\nu, \xi)$ over the full real axis. Here, we consider separately the interval $\nu \in [0, \xi]$, where $j(\nu, \xi)$ has a distributional solution, and the interval $\nu \in [\xi, \infty[$, where an auxiliary function $X(z, \xi)$ is needed to determine $j(\nu, \xi)$.

5.1.1. Interval $[0, \xi]$

As $k(\nu, \xi) = 0$ (see Eq. (23)) and $\lambda(\nu, \xi) = 0$ at $\nu = \nu_0(\xi)$, the solution of Eq. (75) has thus the same form as for the infinite medium, namely

$$j(\nu, \xi) = \xi p^*(\nu) P \frac{t(\nu, \xi)}{\lambda(\nu, \xi)} + f(\nu, \xi) \delta(\nu - \nu_0(\xi)), \quad (76)$$

where $f(\nu, \xi)$ is still undetermined. An expression of $f(\nu_0(\xi), \xi)$ is established when we consider the interval $[\xi, \infty[$.

5.1.2. Interval $[\xi, \infty[$

We now combine Eq. (76) with Eq. (75). Replacing $\nu_0(\xi)$ by ν_0 to simplify the notation, we obtain

$$\begin{aligned} \lambda(\nu, \xi) j(\nu, \xi) &= \omega b \xi k(\nu, \xi) \int_{\xi}^{\infty} \frac{j(\nu', \xi)}{\nu' - \nu} d\nu' \\ &+ \omega b \xi k(\nu, \xi) \frac{f(\nu_0, \xi)}{\nu_0 - \nu} + G(p^*, \nu, \xi). \end{aligned} \quad (77)$$

In $G(p^*, \nu, \xi)$ we have regrouped the terms depending on $p^*(\nu)$. After some simple algebra, which makes use in particular of the definition of $\lambda(\nu, \xi)$ in Eq. (25), we can write

$$\begin{aligned} G(p^*, \nu, \xi) &= \xi k(\nu, \xi) \int_0^{\xi} \frac{p^*(\nu')}{\lambda(\nu', \xi)} \frac{d\nu'}{\nu' - \nu} \\ &+ \xi \left[p^*(\nu) t(\nu, \xi) + k(\nu, \xi) \int_{\xi}^{\infty} \frac{p^*(\nu')}{\nu' - \nu} d\nu' \right]. \end{aligned} \quad (78)$$

The principal part in the integration over $[0, \xi]$ is required by the zero of $\lambda(\nu, \xi)$ at $\nu = \nu_0(\xi)$. In the monochromatic limit, $G(p^*, \nu, \xi)$ depends only on the primary source term $q^*(\nu)$, since $p^*(\nu) = q^*(\nu)$ when $b = 1$ (see Eq. (19)).

To solve Eq. (77) for $j(\nu, \xi)$, we first transform it into a boundary value problem in the complex plane. We introduce the Hilbert transforms

$$N_c(z, \xi) \equiv H_{\xi}^{\infty}[j(\nu, \xi)], \quad P_c(z, \xi) \equiv H_{\xi}^{\infty}[p^*(\nu)]. \quad (79)$$

The subscript c stands for continuous. In $P_c(z, \xi)$ the dependence on ξ comes from the integration domain. The two functions in Eq. (79) are analytic in the complex plane cut along $[\xi, \infty[$ and go to zero at infinity.

Introducing into Eq. (77) the Plemelj formulas for $\mathcal{L}(z, \xi)$, $N_c(z, \xi)$, and $P_c(z, \xi)$, we obtain after some elementary algebra

$$\mathcal{L}^+(\nu, \xi) M^+(\nu, \xi) - \mathcal{L}^-(\nu, \xi) M^-(\nu, \xi) = \frac{1}{\omega b} m(\nu, \xi), \quad (80)$$

with

$$m(\nu, \xi) = p^*(\nu) - [\mathcal{L}^+(\nu, \xi) - \mathcal{L}^-(\nu, \xi)] \frac{1}{2i\pi} \int_0^{\xi} \frac{p^*(\nu')}{\lambda(\nu', \xi)} \frac{d\nu'}{\nu' - \nu}. \quad (81)$$

Here $M^{\pm}(\nu, \xi)$ are the limiting values above and below the real axis of the function $M(z, \xi)$ defined by

$$M(z, \xi) \equiv N_c(z, \xi) + \frac{1}{2i\pi} \frac{f(\nu_0, \xi)}{\nu_0 - z} + \frac{1}{\omega b} P_c(z, \xi). \quad (82)$$

The functions $M(z, \xi)$ and $\mathcal{L}(z, \xi)$ have different analyticity properties: $M(z, \xi)$ is analytic in the complex plane cut along $[\xi, \infty[$, while the singular cut of $\mathcal{L}(z, \xi)$ is along $]-\infty, -\xi] \cup [+ \xi, +\infty[$. A standard method for solving such equations is to introduce an auxiliary function $X(z, \xi)$, with the same analyticity domain as the unknown function,

here $M(z, \xi)$ (see e.g., Ablowitz and Fokas 1997). This means that we have to construct a function $X(z, \xi)$ analytic in the complex plane cut along $[\xi, \infty[$. In addition it should be free of singularities and solution to the homogeneous Riemann–Hilbert problem

$$\frac{X^+(\nu, \xi)}{X^-(\nu, \xi)} = \frac{\mathcal{L}^+(\nu, \xi)}{\mathcal{L}^-(\nu, \xi)}, \quad \nu \in [\xi, \infty[. \quad (83)$$

Taking the logarithm of this equation, making use of appropriate Plemelj formulae and imposing the condition that $X(z, \xi)$ is free of singularity, one finds the exact expression

$$X(z, \xi) = \frac{1}{1-z} \exp \left[\frac{1}{\pi} \int_{\xi}^{\infty} \theta(\nu, \xi) \frac{d\nu}{\nu - z} \right], \quad (84)$$

where $\theta(\nu, \xi)$ is the argument of $\mathcal{L}^+(\nu, \xi)$ (e.g., Muskhelishvili 1953; Gakhov 1966; Ablowitz and Fokas 1997; Frisch 2022, p. 82). In the following, we shall make use of the function

$$X^*(z, \xi) \equiv (\nu_0 - z)X(z, \xi). \quad (85)$$

It satisfies the factorization relation

$$X^*(z, \xi)X^*(-z, \xi) = \mathcal{L}(z, \xi). \quad (86)$$

We see here that the Wiener–Hopf factorization and the Riemann–Hilbert problem stated in Eq. (83) are two sides of the same coin. Combined with $\mathcal{L}(0, \xi) = \epsilon_b = 1 - \omega b$, Eq. (86) leads to $X^*(0, \xi) = \sqrt{\epsilon_b}$. In the monochromatic limit, i.e., when $b = \xi = 1$, $X^*(z, \xi)$ is simply related to the monochromatic Chandrasekhar’s H-function (see Section 5.4).

To pursue the determination $M(z, \xi)$, we multiply Eq. (80) by $(\nu_0 - \nu)$. Making use of Eq. (83), we obtain

$$[X^*]^+(\nu, \xi)M^+(\nu, \xi) - [X^*]^-(\nu, \xi)M^-(\nu, \xi) = \frac{[X^*]^+(\nu, \xi)}{\mathcal{L}^+(\nu, \xi)} \frac{m(\nu, \xi)}{\omega b}, \quad (87)$$

where $m(\nu, \xi)$ is defined in Eq. (81) and $M(z, \xi)$ in Eq. (82).

The left-hand side of this equation is the jump across $[\xi, \infty[$ of the function $(\nu_0 - z)X(z, \xi)M(z, \xi)$, analytic in the complex plane cut along $[\xi, \infty[$. The right-hand side is the jump across $[\xi, \infty[$ of the function

$$\mathcal{M}(z, \xi) \equiv H_{\xi}^{\infty} \left[\frac{[X^*]^+(\nu, \xi)}{\mathcal{L}^+(\nu, \xi)} m(\nu, \xi) \right]. \quad (88)$$

The function $\mathcal{M}(z, \xi)$ has the standard properties of a Hilbert transform, namely, it goes to zero at infinity and is free of singularities.

We can now use the Liouville theorem to solve Eq. (87). We introduce

$$F(z, \xi) \equiv (\nu_0 - z)X(z, \xi)M(z, \xi) - \frac{1}{\omega b} \mathcal{M}(z, \xi). \quad (89)$$

By construction $F(z, \xi)$ is analytic in the complex plane cut along the real semi-infinite line $[\xi, \infty[$. According to Eq. (87) its jump across $[\xi, \infty[$ is zero. It is also free of singularity. The factor $(\nu_0 - z)$ cancels the singularity in $M(z, \xi)$ due to the term $f(\nu_0, \xi)/(\nu_0 - z)$ (see Eq. (82)). One may thus conclude that $F(z, \xi)$ is an entire function, which can be determined by its behavior at infinity. Remembering that $X(z, \xi) \simeq -1/z$ as $z \rightarrow \infty$, we see that all the terms in Eq. (89) go to zero at infinity. Hence

$$F(z, \xi) = 0, \quad (90)$$

for all values of z and ξ . We thus obtain

$$N_c(z, \xi) = \frac{1}{\omega b} \frac{\mathcal{M}(z, \xi)}{(\nu_0 - z)X(z, \xi)} - \frac{1}{2i\pi} \frac{f(\nu_0, \xi)}{\nu_0 - z} - \frac{1}{\omega b} P_c(z, \xi). \quad (91)$$

Since $N_c(z, \xi)$ has been defined as a Hilbert transform, it should be free of singularity in its analyticity domain. Equation (91) shows that $N_c(z, \xi)$ has a singularity at $z = \nu_0$. It is readily eliminated by choosing

$$f(\nu_0, \xi) = \frac{2i\pi \mathcal{M}(\nu_0, \xi)}{\omega b X(\nu_0, \xi)}. \quad (92)$$

The expressions of $N_c(z, \xi)$ and $f(\nu_0, \xi)$ given above still depend on $\bar{j}(\nu)$.

For the infinite medium, we have seen in Section 4.2 that for the determination of $\bar{j}(\nu)$ we need the integral of the Hilbert transform of $j(\nu, \xi)$ over the full real axis. Here we have to calculate $N(z, \xi) = H_0^\infty[j(\nu, \xi)]$, the Hilbert transform of $j(\nu, \xi)$ over the full interval $[0, \infty[$. Summing the contributions from Eqs. (76) and (91), we obtain

$$N(z, \xi) = \frac{1}{2i\pi} \int_0^\infty \frac{j(\nu, \xi)}{\nu - z} d\nu = \frac{1}{\omega b} \frac{\mathcal{M}(z, \xi)}{X^*(z, \xi)} - \frac{1}{2i\pi} \left\{ \frac{1}{\omega b} \int_\xi^\infty p^*(\nu) \frac{d\nu}{\nu - z} - \xi \int_0^\xi p^*(\nu) \frac{t(\nu, \xi)}{\lambda(\nu, \xi)} \frac{d\nu}{\nu - z} \right\}. \quad (93)$$

It is shown in Appendix D that

$$\begin{aligned} \mathcal{M}(z, \xi) &= -X^*(z, \xi) \frac{1}{2i\pi} \int_0^\xi \frac{p^*(\nu)}{\lambda(\nu, \xi)} \frac{d\nu}{\nu - z} \\ &+ \frac{1}{2i\pi} \int_0^\infty \frac{p^*(\nu)}{X^*(-\nu, \xi)} \frac{d\nu}{\nu - z}. \end{aligned} \quad (94)$$

Introducing $\mathcal{M}(z, \xi)$ into Eq. (93), using $\lambda(\nu, \xi) = 1 - \omega b \xi t(\nu, \xi)$ and $p^*(\nu) = \omega(1 - b)\bar{j}(\nu) + q^*(\nu)$, we obtain the relation

$$\omega[bN(z, \xi) + (1 - b)\bar{N}(z)] + N_Q(z) = \frac{1}{X^*(z, \xi)} \frac{1}{2i\pi} \int_0^\infty \frac{p^*(\nu)}{X^*(-\nu, \xi)} \frac{d\nu}{\nu - z}, \quad (95)$$

where $\bar{N}(z)$ and $N_Q(z)$ are the Hilbert transforms of $\bar{j}(\nu)$ and $q^*(\nu)$ defined by

$$\bar{N}(z) = H_0^\infty[\bar{j}(\nu)], \quad \text{and} \quad N_Q(z) = H_0^\infty[q^*(\nu)]. \quad (96)$$

For the infinite medium, the equation corresponding to Eq. (95) is the much simpler Eq. (49). We remark also that Eq. (95) can be obtained without the separation in two intervals $[0, \xi]$ and $[\xi, \infty[$, but we find that the need of a half-space auxiliary function appears more clearly when the intervals are treated separately. In the monochromatic limit $b = 1$, Eq. (95) provides an exact expression for $N(z, \xi)$ from which $j(\nu, \xi)$ can be derived by application of the Plemelj formulae. Equation (95) becomes also simpler for the Green's function. When $Q^*(\tau) = \delta(\tau)$, the term $N_Q(z)$ is replaced by $[[X^*(z, \xi)]^{-1} - 1]/(2i\pi\omega)$ and $p^*(\nu)$ by $\omega(1-b)\bar{j}(\nu)$.

5.2. The frequency integrated function $\bar{j}(\nu)$

Proceeding as for the infinite medium, we integrate Eq. (95) over ξ . Making us of the factorization relation in Eq. (86), we obtain

$$\omega\bar{N}(z) + N_Q(z) = \frac{1}{2i\pi} \int_0^{\varphi_0} \frac{W(\xi)}{\mathcal{L}(z, \xi)} \oint_0^\infty \frac{X^*(-z, \xi)}{X^*(-\nu, \xi)} p^*(\nu) \frac{d\nu}{\nu - z} d\xi. \quad (97)$$

The singular integral in the right-hand side of Eq. (97) cannot be expressed in terms of the Hilbert transform of the unknown function $p^*(\nu)$ because of the factor $1/X^*(-\nu, \xi)$. A method for solving singular integral equations with Cauchy-type kernels is described in Muskhelishvili (1953, Chapter 6). Following this method, also applied in Hemsch and Ferziger (1972), the kernel is separated into a “dominant” part, containing the Cauchy singularity $1/(\nu - z)$, and a “Fredholm” part. This amounts to introduce a function $r(z, \nu, \xi)$ defined by

$$\frac{X^*(-z, \xi)}{X^*(-\nu, \xi)} \frac{1}{\nu - z} = \frac{1}{\nu - z} + r(z, \nu, \xi). \quad (98)$$

Regrouping the integrals with Cauchy kernels and writing now $\Omega(z)$ as

$$\Omega(z) \equiv 1 - \frac{1-b}{b} \int_0^{\varphi_0} W(\xi) \left[\frac{1}{\mathcal{L}(z, \xi)} - 1 \right] d\xi, \quad (99)$$

we can rewrite Eq. (97) as

$$\Omega(z) \left[\bar{N}(z) + \frac{1}{\omega(1-b)} N_Q(z) \right] = G(z) + R(z), \quad (100)$$

where

$$R(z) \equiv \frac{1-b}{b} \int_0^\infty \bar{j}(\nu) V(z, \nu) d\nu, \quad (101)$$

$$G(z) \equiv \frac{1}{\omega(1-b)} N_Q(z) + \frac{1}{\omega b} \int_0^\infty q^*(\nu) V(z, \nu) d\nu, \quad (102)$$

and

$$V(z, \nu) \equiv \frac{1}{2i\pi} \int_0^{\varphi_0} \frac{W(\xi)}{\mathcal{L}(z, \xi)} r(z, \nu, \xi) d\xi. \quad (103)$$

The term $R(z)$, which involves $\bar{j}(\nu)$ and the Fredholm part of the kernel, will prevent us from constructing with Eq. (100) an exact expression for $\bar{j}(\nu)$. It will however let us construct for $\bar{j}(\nu)$ a Fredholm integral equation.

5.3. A Fredholm integral equation for $\bar{j}(\nu)$

In Eq. (100) $\bar{N}(z)$ is analytic in the complex plane cut along $[0, \infty[$, while $\Omega(z)$, $R(z)$ and $G(z)$ are analytic in \mathbb{C}/\mathbb{R} . Thus $\bar{N}(z)$ cannot be derived directly from this equation. However, it can provide for $\nu \in [0, \infty[$ a boundary value problem, namely

$$\begin{aligned} \Omega^+(\nu) [\bar{N}^+(\nu) + \frac{1}{\omega(1-b)} N_Q^+(\nu)] \\ - \Omega^-(\nu) [\bar{N}^-(\nu) + \frac{1}{\omega(1-b)} N_Q^-(\nu)] = \rho(\nu) + g(\nu), \end{aligned} \quad (104)$$

with

$$\rho(\nu) \equiv R^+(\nu) - R^-(\nu), \quad g(\nu) \equiv G^+(\nu) - G^-(\nu). \quad (105)$$

Equation (101) shows that $\rho(\nu)$ may be written as

$$\rho(\nu) = \frac{1-b}{b} \oint_0^\infty \bar{j}(\nu') \mathcal{V}(\nu, \nu') d\nu', \quad (106)$$

where $\mathcal{V}(\nu, \nu')$ is the jump of the function $V(z, \nu)$ defined in Eq. (103). This jump is calculated in Appendix D (see Eqs. (D.11), (D.12), and (D.13)). Equation (102) shows that the jump $g(\nu)$ may be written as

$$g(\nu) = \frac{1}{\omega(1-b)} q^*(\nu) + \frac{1}{\omega b} \oint_0^\infty q^*(\nu') \mathcal{V}(\nu, \nu') d\nu'. \quad (107)$$

For complete frequency redistribution $r(z, \nu, \xi)$ is zero, hence $\rho(\nu) = 0$ and $g(\nu)$ is reduced to its first term.

The problem is now to solve Eq. (104) for $\bar{N}(z)$. Because the analyticity domain of $\Omega(z)$ is cut along the real axis, the solution of this equation requires the introduction of an auxiliary function, denoted $\bar{X}(z)$, analytic in the complex plane cut along $[0, \infty[$, free of zero and singularity, and such that

$$\frac{\bar{X}^+(\nu)}{\bar{X}^-(\nu)} = \frac{\Omega^+(\nu)}{\Omega^-(\nu)}, \quad \nu \in [0, \infty[. \quad (108)$$

As shown in Section C.1, $\Omega(z)$ is free of zeroes. Hence $\bar{X}(z)$ is given by

$$\bar{X}(z) \equiv \exp \left[\frac{1}{\pi} \int_0^\infty \bar{\theta}(\nu) \frac{d\nu}{\nu - z} \right], \quad (109)$$

where $\bar{\theta}(\nu)$ is the argument of $\Omega^+(\nu)$. The definition of $\bar{X}(z)$ shows that $\bar{X}(z) \rightarrow 1$ as $z \rightarrow \infty$. The function $\bar{X}(z)$ satisfies the factorization relation

$$\bar{X}(-z)\bar{X}(z) = \Omega(z), \quad (110)$$

for z outside the cut $[0, \infty[$. In the limit $b = 0$, $\Omega(z)$ becomes identical to the dispersion function for complete frequency redistribution $\mathcal{L}_R(z)$ (see Section C.2) and the function $\bar{X}(z)$ identical to the X -function for complete frequency redistribution, henceforth denoted $X_R(z)$.

Knowing that $\Omega(z)$ is free of zeroes, we can divide Eq. (104) by $\Omega^+(\nu)$ and make use of Eq. (108). We also introduce the two Hilbert transforms

$$\mathcal{G}(z) \equiv H_0^\infty \left[\frac{\bar{X}^+(\nu)}{\Omega^+(\nu)} g(\nu) \right] = H_0^\infty \left[\frac{g(\nu)}{\bar{X}(-\nu)} \right], \quad (111)$$

$$\mathcal{R}(z) \equiv H_0^\infty \left[\frac{\bar{X}^+(\nu)}{\Omega^+(\nu)} \rho(\nu) \right] = H_0^\infty \left[\frac{\rho(\nu)}{\bar{X}(-\nu)} \right]. \quad (112)$$

They are analytic in the complex plane cut along $[0, \infty[$. We can now write Eq. (104) as

$$\begin{aligned} \bar{X}^+(\nu) \left[\bar{N}^+(\nu) + \frac{N_Q^+(\nu)}{\omega(1-b)} \right] - \bar{X}^-(\nu) \left[\bar{N}^-(\nu) + \frac{N_Q^-(\nu)}{\omega(1-b)} \right] \\ = [\mathcal{G}^+(\nu) - \mathcal{G}^-(\nu)] + [\mathcal{R}^+(\nu) - \mathcal{R}^-(\nu)]. \end{aligned} \quad (113)$$

Following the method already used several times, we introduce

$$\bar{F}(z) \equiv \bar{X}(z) \left[\bar{N}(z) + \frac{N_Q(z)}{\omega(1-b)} \right] - \mathcal{G}(z) - \mathcal{R}(z). \quad (114)$$

By construction $\bar{F}(z)$ is analytic in the complex plane cut along $[0, \infty[$ and according to Eq. (113) its jump along this cut is zero. Hence it is an entire function. It is also easy to see that it goes to zero at infinity. We can thus conclude that $\bar{F}(z)$ is zero for all z . We thus obtain

$$\bar{N}(z) + \frac{1}{\omega(1-b)} N_Q(z) = \frac{1}{\bar{X}(z)} [\mathcal{G}(z) + \mathcal{R}(z)]. \quad (115)$$

Applying the Plemelj formulae to all the terms in this equation and using the expression of $\rho(\nu)$ given in Eq. (106), we obtain for $\bar{J}(\nu)$ a Fredholm integral equation, which may be written as

$$\bar{J}(\nu) - \int_0^\infty \bar{J}(\nu') \kappa(\nu, \nu') d\nu' = -\frac{1}{\omega(1-b)} q^*(\nu) + \eta(\nu). \quad (116)$$

The kernel $\kappa(\nu, \nu')$ is given by

$$\begin{aligned} \kappa(\nu, \nu') &= \frac{1}{2} \left[\frac{1}{\Omega^+(\nu)} + \frac{1}{\Omega^-(\nu)} \right] \mathcal{V}(\nu, \nu') \\ &+ \left[\frac{1}{\bar{X}^+(\nu)} - \frac{1}{\bar{X}^-(\nu)} \right] \frac{1}{2i\pi} \int_0^\infty \frac{\mathcal{V}(\nu'', \nu')}{\bar{X}(-\nu'')} \frac{d\nu''}{\nu'' - \nu}, \end{aligned} \quad (117)$$

and $\eta(\nu) = [\mathcal{G}(z)/\bar{X}(z)]_{\text{jmp}}$ by

$$\begin{aligned} \eta(\nu) &= \frac{1}{2} \left[\frac{1}{\Omega^+(\nu)} + \frac{1}{\Omega^-(\nu)} \right] g(\nu) \\ &+ \left[\frac{1}{\bar{X}^+(\nu)} - \frac{1}{\bar{X}^-(\nu)} \right] \frac{1}{2i\pi} \int_0^\infty \frac{g(\nu')}{\bar{X}(-\nu')} \frac{d\nu'}{\nu' - \nu}. \end{aligned} \quad (118)$$

An explicit expression of $\mathcal{V}(\nu, \nu')$ is given in Appendix D and $g(\nu)$ is defined in Eq. (107). Using the definition of $\bar{X}^\pm(\nu)$ in Eq. (108) and the factorization relation in Eq. (110), it is possible to express all the square brackets in terms of $\bar{X}(-\nu)$ and of the imaginary and real parts of $\Omega^+(\nu)$.

For the Green's function, that is for $Q^*(\tau) = \delta(\tau)$, Eq. (116) becomes

$$\bar{j}(\nu) - \int_0^\infty \bar{j}(\nu') \kappa(\nu, \nu') d\nu' = \eta_g(\nu), \quad (119)$$

with

$$\eta_g(\nu) = \frac{1}{2i\pi(1-b)} \left[\frac{1}{\bar{X}^+(\nu)} - \frac{1}{\bar{X}^-(\nu)} \right] + \left[\frac{\mathcal{G}_g(z)}{\bar{X}(z)} \right]_{\text{jmp}}. \quad (120)$$

The function $\mathcal{G}_g(z)$ is a Hilbert transform defined as in Eq. (111), the function $g(\nu)$ being replaced by

$$g_g(\nu) \equiv \frac{1}{2i\pi b} \int_0^{\varphi_0} \left[\frac{W(\xi)}{\mathcal{L}(z, \xi)} \right]_{\text{jmp}} [(\nu_0(\xi) + \nu)X(-\nu, \xi) - 1] d\xi. \quad (121)$$

The equation for $\bar{j}(\nu)$ becomes of course much simpler for complete frequency redistribution and provides an explicit exact solution. First, we have $\kappa(\nu, \nu') = 0$. As for $\eta(\nu)$ it is given by Eq. (118) in which $g(\nu)$ is replaced by $q^*(\nu)/(\omega(1-b))$. A further simplification appears when we consider the Green's function. Indeed, when $b = 0$, we have $\mathcal{L}(z, \xi) = 1$. Hence the jump of $\mathcal{L}(z, \xi)$ is zero, $g_g(\nu)$ is also zero and $\eta_g(\nu)$ is reduced to its first term. We thus obtain

$$\bar{j}_R(\nu) = \frac{1}{\omega 2i\pi} \left[\frac{1}{X_R^+(\nu)} - \frac{1}{X_R^-(\nu)} \right] = \frac{k_R(\nu)}{\mathcal{L}_R^+(\nu)\mathcal{L}_R^-(\nu)} X_R(-\nu). \quad (122)$$

The Laplace transform of $\bar{j}_R(\nu)$ yields the resolvent function for complete frequency redistribution in a semi-infinite medium (see e.g., Ivanov 1973; Frisch 2022, p. 104).

5.4. The emergent intensity

The emergent intensity for a uniform primary source is studied in detail, analytically and numerically in Hemsch and Ferziger (1972), for a frequency dependent branching ratio b . This article shows in particular how the line profile is affected by a monochromatic component acting at large frequencies. Their results are discussed in some detail in Section 6. Here we present some exact expressions for an arbitrary primary source $Q^*(\tau)$ and a constant branching ratio b .

In terms of the source function $S(\tau, \xi)$, the emergent intensity is given by

$$I(0, \xi, \mu) = \int_0^\infty S(\tau, \xi) e^{-\tau\xi/\mu} \frac{\xi}{\mu} d\tau. \quad (123)$$

We recall that $\xi = \varphi(x)$ and that μ is the cosine of the radiation field direction. The source function $S(\tau, \xi)$, defined in Eq. (2), may be written as

$$S(\tau, \xi) = \int_0^\infty s(\nu, \xi) e^{-\nu\tau} d\nu, \quad (124)$$

where $s(\nu, \xi)$ is given by

$$s(\nu, \xi) = \omega[bj(\nu, \xi) + (1-b)\bar{j}(\nu)] + q^*(\nu). \quad (125)$$

Combining Eqs. (123) and (124) and integrating over τ , we obtain

$$I(0, \xi, \mu) = \int_0^\infty \frac{\xi}{\xi + \mu\nu} s(\nu, \xi) d\nu. \quad (126)$$

To calculate this integral we introduce $N_S(z, \xi)$ the Hilbert transform of $s(\nu, \xi)$, denoted $N_S(z, \xi)$. According to Eq. (125), we have

$$N_S(z, \xi) \equiv \omega[bN(z, \xi) + (1-b)\bar{N}(z)] + N_Q(z), \quad (127)$$

where $N(z, \xi)$, $\bar{N}(z)$ and $N_Q(z)$ are the Hilbert transform over $[0, \infty[$ of $j(\nu, \xi)$, $\bar{j}(\nu)$, and $q^*(\nu)$. These functions are analytic in the complex plane cut along the positive real axis, free of singularities, and go to zero at infinity. The same properties hold true for $N_S(z, \xi)$. Applying the Plemleij formulae to $N_S(z, \xi)$, we can write the emergent intensity as

$$I(0, \xi, \mu) = \int_0^\infty \frac{\xi}{\xi + \mu\nu} [N_S^+(\nu, \xi) - N_S^-(\nu, \xi)] d\nu, \quad (128)$$

where $N_S^\pm(\nu, \xi)$ are the limiting values of $N(z, \xi)$ above and below the positive real axis. Integrating over the contour shown in Figure 2, which circulates around the cut of $N_S(z, \xi)$ and around the pole at $z = -\xi/\mu$, and remembering that $N_S(z, \xi)$ goes to zero at infinity, we find

$$I(0, \xi, \mu) = 2i\pi \frac{\xi}{\mu} N_S(-\frac{\xi}{\mu}, \xi). \quad (129)$$

The expression of $N(z, \xi)$ given in Eq. (95) leads to

$$N_S(z, \xi) = \frac{1}{X^*(z, \xi)} \frac{1}{2i\pi} \int_0^\infty \frac{\omega(1-b)\bar{j}(\nu) + q^*(\nu)}{X^*(-\nu, \xi)} \frac{d\nu}{\nu - z}, \quad (130)$$

where $\bar{j}(\nu)$ is the solution of the Fredholm integral equation in Eq. (116). There is thus no explicit expression for $N_S(z, \xi)$ nor for the emergent intensity. In the limiting cases of monochromatic scattering and complete frequency redistribution, Eq. (130) provides well known results.

For **monochromatic scattering**, $b = 1$, and ξ has a constant value, which we take equal to one, following a usual practice. Equation (130) straightforwardly leads to

$$N_S(z, \xi) = \frac{1}{X_M^*(z)} \frac{1}{2i\pi} \int_0^\infty \frac{1}{X_M^*(-\nu)} q^*(\nu) \frac{d\nu}{\nu - z}, \quad (131)$$

where $X_M^*(z) = X^*(z, 1) = (\nu_0(1) - z)X(z, 1)$.

For **complete frequency redistribution**, $b = 0$. Equation (127) leads to

$$N_S(z, \xi) = \omega \bar{N}(z) + N_Q(z). \quad (132)$$

The expression of $\bar{N}(z)$ for complete frequency redistribution can be derived from Eq. (115). Setting $\mathcal{R}(z) = 0$ and $g(\nu) = q^*(\nu)/(\omega(1-b))$ in the definition of $\mathcal{G}(z)$ (see Eq. (111)), we readily obtain

$$N_S(z, \xi) = \frac{1}{X_R(z)} \frac{1}{2i\pi} \int_0^\infty \frac{1}{X_R(-\nu)} q^*(\nu) \frac{d\nu}{\nu - z}. \quad (133)$$

We have the same expression as for monochromatic scattering, with $X_M^*(z)$, replaced by $X_R(z)$. For monochromatic scattering and complete frequency redistribution, $N_S(z, \xi)$ depends only on z , a result which is consistent with the fact that the source function depends only on τ . In these two limits, the emergent intensity may thus be written as

$$I(0, \xi, \mu) = H\left(\frac{\mu}{\xi}\right) \int_0^\infty \frac{\xi}{\xi + \mu\nu} q^*(\nu) H\left(\frac{1}{\nu}\right) d\nu, \quad \mu \geq 0, \quad (134)$$

with $\xi = \varphi(x)$ for complete frequency redistribution and $\xi = 1$ for monochromatic scattering. The H -function is defined by

$$H(z) = \begin{cases} 1/X_R(-1/z), & \text{complete redistribution,} \\ 1/X_M^*(-1/z), & \text{monochromatic scattering.} \end{cases} \quad (135)$$

For monochromatic scattering, $H(z)$ is the Chandrasekhar H -function for isotropic scattering (Chandrasekhar 1960). For complete frequency redistribution, the properties of the H -function are discussed in detail in, e.g., Ivanov (1973) (see also Frisch 2022). They are similar to those of the monochromatic H -function. For example, they both increase monotonically from $H(0) = 1$ to $1/\sqrt{\epsilon}$ at infinity. The value at infinity is easily derived from the factorization relations for $X_R(z)$ and $X_M^*(z)$ and from $\mathcal{L}_M(0) = \mathcal{L}_R(0) = \epsilon$.

Equation (134) is valid for all positive values of μ , but only those belonging to the interval $[0, 1]$ are needed for the calculation of the emergent intensity. This equation

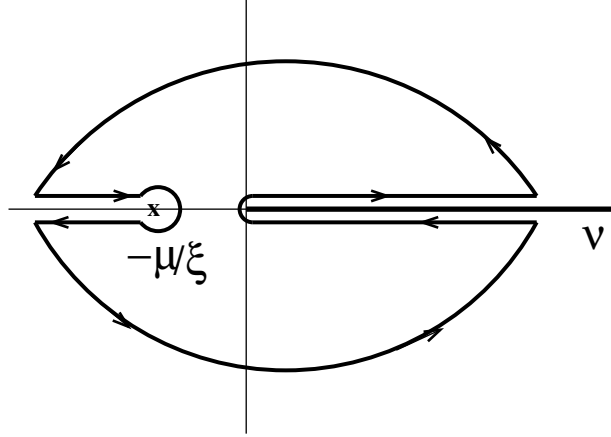


Figure 2. Contour in the complex plane for the calculation of the emergent intensity. The function $N_S(z, \xi)$ is analytic in the complex plane cut along the semi-infinite real line $[0, \infty[$.

provides a very convenient expression for the emergent intensity, especially when $q^*(\nu)$ has a simple form. For example, when the primary source has a uniform value Q^* , its inverse Laplace transform is $q^*(\nu) = Q^*\delta(\nu)$. In the monochromatic and complete frequency redistribution limits, the emergent intensity is thus simply given by

$$I(0, x, \mu) = \frac{Q^*}{\sqrt{\epsilon}} H\left(\frac{\mu}{\varphi(x)}\right), \quad (136)$$

with $\varphi(x) = 1$ in the monochromatic case.

Everywhere in this article it has been assumed implicitly that $\epsilon \neq 0$. The results that have been obtained hold also for $\epsilon = 0$. Indeed, provided $b \neq 1$, the parameter ϵ_b will remain strictly positive, smaller than one and the dispersion function $\mathcal{L}(z, \xi)$ will have two zeroes in the interval $[-\xi, +\xi]$. Only in the monochromatic limit $b = 1$ will there be a difference, since $\mathcal{L}(z, \xi)$ will have a double zero at the origin.

6. Summary of the solution method and concluding remarks

Although the partial frequency redistribution model considered here is fairly simple, namely a linear combination of monochromatic scattering and complete frequency redistribution, the task of constructing exact solutions is not straightforward, due to the coupling between the two different scattering mechanisms. Two successive steps are needed, whatever the geometry. The first one requires the solution of a monochromatic scattering problem and the second one of a complete frequency redistribution one. For an infinite medium, the two steps can be solved explicitly, making it possible to construct an exact solution for the full partial frequency redistribution problem. For a semi-infinite medium, the second step has no explicit solution but can be recast as a Fredholm integral equation, which will require a numerical solution. These results, first established in Hemsch (1971) and Hemsch and Ferziger (1972) are confirmed in this article. In Hemsch and Ferziger (1972), the singular eigenfunction expansion method introduced by Case (1960) is applied to the homogeneous radiative transfer equation. The boundary conditions and known inhomogeneous terms, coming from primary source of photons, are then taken into account to construct the final solution.

The method described here does not require the introduction of singular eigenfunctions and directly provides the final solution. The radiative transfer equation is first transformed into an convolution integral equation, which takes into account primary source terms and boundary conditions. The convolution equation is then transformed into a singular integral equation with an inverse Laplace transform, more or less equivalent to the singular eigenfunction expansion. Both methods lead to a singular integral equation, one of them taking into account internal primary source terms, and boundary conditions, the other not. They are both solved by transforming them into a boundary value problem in the complex plane.

The solution of the partial frequency redistribution problem depends critically on the branching ratio b between complete frequency redistribution for $b = 0$ and monochromatic scattering for $b = 1$. Monochromatic scattering and complete frequency redistribution have well understood but quite different large scale asymptotic properties. For monochromatic scattering, the kernel $K_M(\tau)$ has a finite second order moment and the dispersion function has isolated zeroes. This implies that the large scale behavior of the radiative transfer equation is a diffusion equation and also that the monochromatic Green's function in an infinite medium decreases at infinity as $\exp(-\nu_0|\tau|)$, where ν_0 is the positive root of the dispersion function $\mathcal{L}_M(z)$. In contrast, for complete frequency redistribution, the kernel $K_R(\tau)$ defined in Eq. (8) has an infinite second order moment, the dispersion function $\mathcal{L}_R(z)$ has no zero, and the large scale behavior of the radiation field has the structure of a Lévy walk. More specifically the large scale transport equation contains a Laplacian with a fractional power, equal to $1/2$ or $1/4$ depending on whether $\varphi(x)$ is a Doppler or a Voigt profile. The infinite medium Green's function decreases then algebraically at infinity as $1/\tau^2$ for a Doppler profile and as $1/\tau^{3/2}$ for a Voigt profile (see e.g., Ivanov 1973; Frisch 2022).

Whether the transition between the complete frequency redistribution and the monochromatic regime is rapid or slow and for which values of the branching ratio b it occurs is an interesting question. The exact expression for $S(\tau, \xi)$ given in Eq. (72) suggests that the overall asymptotic behavior at large τ will be dominated by complete frequency redistribution contributions, unless b is close to one. Indeed, as soon as $b \neq 1$, Eq. (72) shows that there will be contributions from $\bar{j}(\nu)$ and $j_d^{(1)}(\nu, \xi)$ for $\nu \in [0, \xi]$. That the large τ behavior is dominated by complete frequency redistribution when there is a combination of monochromatic scattering and complete frequency redistribution is also suggested by a large τ asymptotic analysis of the source function $S(\tau, \xi)$ (see e.g., Frisch 2022, p. 569). This conjecture is also in agreement with the numerical investigation of the emergent intensity reported in Hensch (1971) and Hensch and Ferziger (1972). Line profiles of the emergent intensity for a semi-infinite medium with a uniform primary source are calculated for a frequency dependent parameter b equal to 0 in the line center and equal to some constant value b_W in the wings. Numerical results for $b_W = 0.2, 0.5, 0.9$ show that b_W has to be close to 1 for departures from the complete frequency redistribution limit to become significant. Having been obtained with a model which does preserve energy conservation, these results may not be fully trustworthy.

A numerical solution of this partial frequency redistribution problem for different values of the parameter b appears to be the best way to analyze the transition between the two asymptotic regimes. Although Hensch (1971) has been able to use the exact expressions for the calculation of line profiles, a direct numerical solution of the radiative transfer equation appears easier. Some precautions should however be taken. To be able to observe an asymptotic regime, the number of scatterings undergone by the photons should be large enough, which means ϵ small enough, say of order 10^{-4} .

The total optical depth of the medium, say τ_{\max} , should be roughly ten times the thermalization length τ_{th} . This length scales as $1/\sqrt{\epsilon}$ for monochromatic scattering. For complete frequency redistribution it scales as $1/\epsilon$ (neglecting a logarithmic factor) for a Doppler profile and as a/ϵ^2 for a Voigt profile. Here a , the Voigt parameter of the line, has typical values around 10^{-3} . For complete frequency redistribution, the frequency range should be wide enough to ensure that the last frequency point, say x_{\max} , satisfies $\varphi(x_{\max})\tau_{\text{th}} \ll 1$, otherwise there is a spurious accumulation of photons (Avrett and Hummer 1965). With these constraints it seems better to avoid Monte-Carlo simulations. There is no lack of standard numerical schemes to solve partial frequency redistribution problem. A possibility is an Accelerated Λ -Iteration (ALI) method developed by Paletou and Auer (1995).

7. Acknowledgements

The author is very grateful to Dr. M. Hemsch for having introduced her to this radiative transfer problem with partial frequency redistribution and for having provided a copy of his PhD thesis. She is also very thankful for many stimulating discussions. The author is also very grateful to the referees for asking several clarification questions and in particular to Professor Gary Ganapol for showing how to solve the infinite medium problem with a Fourier transform applied directly to the radiative transfer equation.

References

- Ablowitz, Mark J., and Fokas, Athanassios S. 1997. *Complex Variables, Introduction and Applications*. Cambridge, UK : Cambridge University Press
- Ambarzumian Viktor A. 1942. "On the scattering of light by planetary atmospheres". *Astrophysical Journal of the Soviet Union* 19 : 30–41
- Ambarzumian Viktor A. 1943 "Diffuse reflection of light by a foggy medium". *Dokl. Akad. Nauk. SSSR*. 38 : 257–261
- Avrett, Eugene H., and Hummer, David G. 1965. "Non-coherent scattering II; Line Formation with a Frequency Independent Source Function". *Monthly Notices Roy. Astronomical Soc.* 130 : 295–331
- Barichello, Liliane, and Siewert, Charles. 1998. "The F_N Method for spectral-line formation by completely noncoherent scattering". *J. Quant. Spectrosc. Radiat. Transfer*. 60 : 261–276
- Barichello, Liliane, and Siewert, Charles. 1999. "A discrete-ordinate solution for non-grey model with complete frequency redistribution". *Journal of Quantitative Spectroscopy and Radiative Transfer* 62 : 665–675
- Barichello, Liliane, and Siewert, Charles. 1999. "A discrete-ordinate solution for a polarization model with complete frequency redistribution". *The Astrophysical Journal* 513 : 370–382
- Carleman, Torsten. 1922. "Sur la résolution de certaines équations intégrales." *Arkiv för Matematik Astronomi och Fysik* 16 : 1–19
- Carrier, George F., Krook, Max, and Pearson, Carl E. 1966. *Functions of a Complex Variable*. New York : McGraw-Hill
- Case, Kenneth M. : 1960, "Elementary solutions of the transport equation and their applications." *Annals of Physics (New York)* 9 : 1–23
- Case, Kenneth M., and Zweifel, Paul. 1967. *Linear Transport Theory*. Reading, MA : Addison-Wesley
- Chandrasekhar, Subrahmanyan. 1946. "On the radiative equilibrium of a stellar atmosphere. X. *The Astrophysical Journal* 103 : 351–370

- Chandrasekhar, Subrahmanyan. 1960. *Radiative Transfer*. New York: Dover Publications. 1950 1st ed. Oxford, UK : Oxford University Press
- Corngold, Noel, and Durgin, Kanat. 1967. "Analysis of pulsed-neutron experiments in moderators via a simple model." *Nuclear Science and Engineering* 29 : 354–366
- Duderstadt, James J., and Martin, Williams R. 1979. *Transport Theory*. New York: John Wiley and Sons
- Faurobert, Marianne. 1988. "Linear polarization of resonance lines in the absence of magnetic fields II Semi-infinite atmospheres." *Astronomy and Astrophysics* 194 : 268–278
- Frisch, Hélène. 1980. "Non-LTE transfer V. The asymptotics of partial redistribution." *Astronomy and Astrophysics* 83 : 166–183
- Frisch, Hélène. 1988. "A Cauchy integral equation method for analytic solutions of half-space convolution equations." *Journal of Quantitative Spectroscopy and Radiative Transfer* 39 : 149–162
- Frisch, Hélène. 2022. *Radiative Transfer, Introduction to exact and asymptotic methods*. Switzerland : Springer Nature
- Frisch, Hélène, and Frisch, Uriel. 1982. "A method of Cauchy integral equations for non-coherent transfer in half-space." *Journal of Quantitative Spectroscopy and Radiative Transfer* 28 : 361–375
- Gakhov, Fyodor D. 1966. *Boundary Value Problems*. Trans. I.N. Sneddon. London : Pergamon Press. 1963. *Boundary problems*. 2nd Russian edition. Moscow : Fizmatgiz
- Halpern, Otto, Lueneburg, Rudolf, and Clark, O.. 1938. "On Multiple scattering of Neutrons. I. Theory of the Albedo of a Plane Boundary." *Physical Review* 53 : 173–183
- Hemsch, Michael. J. 1971. "Linear transport theory with partly incoherent scattering." PhD diss., Stanford University
- Hemsch, Michael J., and Ferziger, Joel H. 1972. "Radiative transfer with partially coherent scattering." *Journal of Quantitative Spectroscopy and Radiative Transfer* 12 : 1029–1046
- Hubeny, Ivan, and Mihalas, Dimitri. 2015. *Theory of Stellar Atmospheres*. Princeton, N. J. : University Press
- Hummer, David G. 1962. "Non-coherent scattering I. The redistribution functions with Doppler broadening." *Monthly Notices of the Royal astronomical Society* 125 : 21–37
- Hummer, David G. 1969. "Non-coherent scattering VI. Solutions of the Transfer Problem with a Frequency-dependent Source Function". *Monthly Notices of the Royal astronomical Society* 145 : 95–120
- Ivanov, Vsevolod V. 1973. *Transfer of Radiation in Spectral Lines*. Trans. G. Hummer and E. Weppner. NBS Spec. Publ. 385. Washington DC : Washington Government Printing Office. 1969. *Radiative Transfer and the Spectra of Celestial Bodies*. Moscow : Nauka
- Ivanov, Vsevolod V. 1994. "Resolvent method : exact solutions of half-space transport problems by elementary means." *Astronomy and Astrophysics* 286 : 328–337
- Jefferies, John T. 1968. *Spectral Line Formation*. Waltham, MA : Blaisdell Publishing Company
- Jefferies, John T, and White, Oran R. 1960. "Source function in a non-equilibrium Atmosphere. VI. The Frequency Dependence of the Source Function for Resonance Lines". *The Astrophysical Journal* 132 : 767–774
- Kneer, Franz. 1975. "Comments on the redistribution Function of Jefferies and White". *The Astrophysical Journal* 200 : 367–368
- Kourganoff, Vladimir. 1963. *Basic Methods in Transfer Problems. Radiative Equilibrium and Neutron Diffusion*. New York : Dover Publications; 1952. 1st edition *Basic Methods in Transfer Problems*. Oxford, UK : Oxford University Press
- Krein, Mark G. 1962. "Integral Equations on a half-line with kernels depending upon the difference of the arguments." *American Mathematical Society Translation Series. 2* 22 : 163–288. 1958. Russian original *Uspekhi. Mat. Nauk.* 13 : 3–120
- Lawrie, Jane B. and Abrahams, David, I. 2007. "A brief historical perspective of the Wiener–Hopf technique". *Journal of Engineering Mathematics. The Wiener–Hopf technique : Modern applications and developments* 59 : 1–16
- Levinson, Norman. 1949. Appendix C in N. Wiener *Extrapolation, Interpolation, and Smooth-*

- ing of Time series*. Cambridge, USA : The MIT Press
- Mark, Charles. 1947. "The neutron density near a plane surface". *Physical Review* 72 : 558–564
- McCormick, Norman J., and Kuščer, Ivan. 1973. "Singular eigenfunction expansions in neutron transport theory." in *Advances in Nuclear Science and Technology* 7 : 181–282, eds. E.J. Henley, and J. Lewins. New York : Academic Press
- McCormick, Norman J., and Siewert, Charles E.. 1970. "Spectral Line Formation by noncoherent Scattering". *Astrophys. J.* 162 : 633–647
- Milne, Edward A.. 1921. "Radiative Equilibrium in the Outer Layers of a Star: the temperature distribution and the Law of Darkening". *Monthly Notices of the Royal astronomical Society* 81 : 361–375
- Muskhlishvili, Nicolai I. 1953. *Singular Integral Equations*. Trans. J.M.R. Radok. Groningen, Holland : Noordhoff. 1992. New York : Dover Publications
- Noble, Benjamin. 1958. *Methods based on the Wiener–Hopf Technique for the solution of partial differential equations* London : Pergamon Press
- Paletou, Frederic and Auer, Laurence H. 1995. "A new approximate Operator Method for Partial Frequency Redistribution Problems". *Astronomy and Astrophysics* 297 : 771–778
- Pandey, J. N. 1996. *The Hilbert transform of Schwartz distributions and applications*. New York : John Wiley and Sons
- Pincus, Joel. 1966. "The spectral Theory of self-adjoint Wiener–Hopf Operators." *Bulletin of the American Mathematical Society* 72: 882–887
- Pincus, Joel. 1981. "Commentary on [31a]" in *Norbert Wiener Collected Works with Commentaries, Volume III* 44–53 Ed. P. MASani. Cambridge, Mass. : The MIT Press
- Placzek, George .1947. "The angular Distribution of Neutrons emerging from a plane Surface". *Physical Review* 72 : 556–558
- Placzek, George and Seidel W. 1947. "Milne's Problem in Transport Theory". *Physical Review* 72 : 550–555
- Pogorzelski, Witold. 1966. *Integral Equations and their Applications*. New York : Pergamon Press
- Schwarzschild, Karl. 1906. "Über das Gleichgewicht der Sonnenatmosphäre." *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse* 195 : 41–53
- Schwarzschild, Karl. 1914. "Über Diffusion und absorption in der Sonnenatmosphäre". *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin* 47 : 1183–1200
- Siewert, Charles E. 1984. "On radiative-transfer problems with reflective boundary conditions and internal emission". *Zeitschrift für angewandte Mathematik und Physik* 35 : 144–155
- Siewert, Charles E. and Benoist, Pierre. 1979. "The F_N method in Neutron-Transport Theory. Part I: Theory and Applications". *Nuclear science and Engineering* 69 : 156–160
- Siewert, Charles E. and Burniston, Ernest E. 1972. "On Existence and Uniqueness Theorems Concerning the H -matrix of Radiative Transfer". *The Astrophysical Journal* 174 : 629–641
- Siewert, Charles E. and Fraley, Stanley K. 1967. "Radiative transfer in a free-electron atmosphere". *Annals of Physics (New York)* 43 : 338–359
- Sobolev, Viktor V. 1963. *A Treatise on Radiative Transfer*. Trans. S.I. Gaposchkin. Princeton, N.J. : Von Nostrand Company. 1956. *Transport of Radiant Energy in Stellar and Planetary Atmospheres*. Moscow : Gostechizdat
- Wiener, Norbert, and Hopf, Eberhard. 1931. "Über eine Klasse singulärer Integralgleichungen." *Sitzungsberichte der Preussischen Akademie, Mathematisch-Physikalische Klasse* 3. Dezember 1931 31 : 696–706. 1934. In Paley, Raymond C., and Wiener, Norbert. *Fourier transforms in the Complex Domain. American Mathematical Society Colloquium Publications* XIX : 49–58
- Williams, Michael M. R. 1971. *Mathematical Methods in Particle Transport Theory*. Hoboken, N. J. : John Wiley and Sons
- Yengibarjan, Norair B., and Khachatryan, Agharvard Kh. 1991. "On a problem of multiple resonance scattering of gamma-ray quanta." *Journal of Quantitative Spectroscopy and Radiative Transfer* 46 : 565–575

Appendix A. Remarks on the partial frequency redistribution model

A linear combination of complete frequency redistribution and monochromatic scattering was introduced in Jefferies and White (1960) to model the formation of spectral lines, known as resonance lines. These lines are formed by radiative transitions between an infinitely thin fundamental energy level and a naturally broadened excited one. In the rest frame of the atom, the scattering is coherent in frequency and the absorption profile is a Lorentzian. In the laboratory frame, when the Doppler broadening is taken into account, the absorption profile becomes a Voigt function and partial frequency redistribution sets in. Explicit expressions for this partial frequency redistribution mechanism known as R_{II} have been established by Hummer (1962) (see also Hubeny and Mihalas 2015; Frisch 2022). As can be observed, for example in Figure 1 of Jefferies and White (1960), R_{II} has a complete frequency redistribution behavior in the line core and a monochromatic one in the line wings. In Jefferies and White (1960), this behavior was estimated too complex for application and it was suggested to have $b = 0$ in the line core, $b = 1$ in the lines wings, with the jump from 0 to 1 being positioned at a few Doppler widths. It is shown in Kneer (1975) that this approximation does not meet essential properties of a scattering process. Indeed a redistribution function of the form

$$r(x, x') = b(x)\varphi(x)\delta(x - x') + [1 - b(x)]\varphi(x)\varphi(x'), \quad (\text{A.1})$$

as used in Hemsch and Ferziger (1972) does not satisfy the conservation of the photons. In other terms, when $\epsilon = 0$ and $Q^*(\tau) = 0$ the radiative flux should be a constant, since the medium is conservative and free of primary sources. Integrating the left-hand side of the radiative transfer equation over directions and frequencies, we see that the derivative of the radiative flux is zero, if and only if b is a constant. Moreover $r(x, x')$ is not symmetric in x and x' , unless b is a constant. A simple cure to the problem is proposed in Kneer (1975), but it requires that the parameter b be a function of x and x' . Nowadays the explicit expressions given in Hummer (1962) for the R_{II} redistribution function are commonly used in numerical work and efficient numerical methods have been developed to handle partial frequency redistribution (see e.g. Hummer 1969; Hubeny and Mihalas 2015). Comparisons with the predictions of the Kneer's approximation show that it fails in the line wings at reproducing the intensity and the linear polarization created by the scattering process. The reason is that it ignores the Doppler broadening of the monochromatic component (Faurobert 1988).

Appendix B. From a convolution integral equation to a singular integral equation

We show here how to transform the convolution equation for $\mathcal{J}(\nu, \xi)$ given in Eq. (17) into a singular integral equation for $j(\nu, \xi)$, its inverse Laplace transform defined in Eq. (18). The method described below was proposed for complete frequency redistribution in Frisch and Frisch (1982) and for monochromatic scattering in Frisch (1988) (see also Frisch 2022). Its goal is to write each of the integrals over τ' in the right-hand side of Eq. (17 as a Laplace transform. To fix the idea we assume $\tau > 0$. Let us consider

the integral

$$\int_{-\infty}^{+\infty} K_M(|\tau - \tau'| \xi) \mathcal{J}(\tau', \xi) d\tau'. \quad (\text{B.1})$$

The integration over τ' is divided into three integrals :

$$\int_{-\infty}^{+\infty} d\tau' = \int_{-\infty}^0 d\tau' + \int_0^\tau d\tau' + \int_\tau^{+\infty} d\tau' = [1] + [2] + [3]. \quad (\text{B.2})$$

Using Eqs. (18) and (9), we find after some simple algebra

$$[1] = - \int_{-\infty}^0 d\nu j(\nu, \xi) \frac{1}{2} \int_1^\infty \frac{d\eta}{\eta} \frac{e^{-\eta\xi\tau}}{\eta\xi - \nu}, \quad (\text{B.3})$$

$$[2] = \int_0^\infty d\nu j(\nu, \xi) \frac{1}{2} \int_1^\infty \frac{d\eta}{\eta} \frac{e^{-\eta\xi\tau} - e^{-\nu\tau}}{\nu - \eta\xi}, \quad (\text{B.4})$$

$$[3] = \int_0^\infty d\nu j(\nu, \xi) \frac{1}{2} \int_1^\infty \frac{d\eta}{\eta} \frac{e^{-\nu\tau}}{\nu + \eta\xi}. \quad (\text{B.5})$$

In [2], we separate the two terms. This requires taking the two new integrals in principal part. Regrouping the first one with [1] and the second one with [3], we obtain

$$\begin{aligned} [1] + [2] + [3] &= \int_0^\infty d\nu e^{-\nu\tau} j(\nu, \xi) \frac{1}{2} \oint_1^\infty \frac{d\eta}{\eta} \left[\frac{1}{\eta\xi + \nu} + \frac{1}{\eta\xi - \nu} \right] \\ &+ \int_{-\infty}^{+\infty} d\nu j(\nu, \xi) \frac{1}{2} \oint_1^\infty \frac{d\eta}{\eta} \frac{e^{-\eta\xi\tau}}{\nu - \eta\xi}. \end{aligned} \quad (\text{B.6})$$

The symbol \oint indicates that the integral is taken in principal part. The first integral in the right-hand side has obviously the structure of a Laplace transform. Making in the second integral the change of variable $\nu = \nu'$ and $\eta\xi = \nu$ and interchanging the order of the integration, we can write

$$\int_{-\infty}^{+\infty} d\nu j(\nu, \xi) \frac{1}{2} \oint_1^\infty \frac{d\eta}{\eta} \frac{e^{-\eta\xi\tau}}{\nu - \eta\xi} = \int_\xi^\infty \frac{d\nu}{2\nu} e^{-\nu\tau} \oint_{-\infty}^{+\infty} d\nu' \frac{j(\nu', \xi)}{\nu' - \nu}. \quad (\text{B.7})$$

The second integral has thus also the structure of a Laplace transform. We thus have

$$\begin{aligned} [1] + [2] + [3] &= La \left\{ j(\nu, \xi) \oint_0^\infty k(\nu', \xi) \left[\frac{1}{\nu' + \nu} + \frac{1}{\nu' - \nu} \right] d\nu' \right. \\ &+ \left. k(\nu, \xi) \oint_{-\infty}^{+\infty} \frac{j(\nu', \xi)}{\nu' - \nu} d\nu' \right\}, \end{aligned} \quad (\text{B.8})$$

where La stands for Laplace transform and

$$k(\nu, \xi) \equiv 1/(2\nu), \quad \nu > \xi, \quad k(\nu, \xi) \equiv 0, \quad 0 < \nu < \xi. \quad (\text{B.9})$$

Proceeding in the same way with the second integral in the right-hand side of Eq. (17) and replacing in the left-hand side $\mathcal{J}(\tau, \xi)$ by $La\{j(\nu, \xi)\}$, we obtain for $j(\nu, \xi)$ the singular integral equation in Eq. (22).

There is a large choice of methods to transform a radiative transfer equation or a convolution integral equation into a singular integral equation. Instead of assuming that the radiation field can be represented by a sum of exponentials, which the hypothesis behind the inverse Laplace transform method and the Case expansion method, one can apply a direct Laplace transform to the radiative transfer equation or to the associated convolution integral equation. For example a direct Laplace transform applied to Eq. (17) leads to a singular integral equation adjoint to Eq. (22), in which the coefficient $k(\nu, \xi)$ is inside the singular integral and the inhomogeneous term is represented by its direct Laplace transform. The required algebra is very similar to the algebra described above (e.g. Kourganoff 1963; Frisch 2022). One of the first application of a Laplace transform to construct an exact solution for a semi-infinite medium can be found in Halpern, Lueneburg, and Clark (1938). The resulting singular integral equation is solved by a factorization method à la Wiener–Hopf. Direct and inverse Laplace transforms have also been employed to develop accurate numerical methods such as the F_N method or the discrete-ordinate method (e.g. Barichello and Siewert 1998, 1999b). The direct and inverse Laplace transform methods lead to singular integral equations for infinite as well as semi-infinite media. For an infinite medium, a Fourier transform can provide an algebraic equation. For a semi-infinite medium, it can also lead to a singular integral equation. A very elegant proof, making use of the analyticity properties of the complex Fourier transform of the Wiener–Hopf integral operator, is described in Pincus (1966). We recall that the traditional Wiener–Hopf method leads to a factorization problem in the complex plane.

Appendix C. The generalized dispersion function $\Omega(z)$

The generalized dispersion function $\Omega(z)$ is defined in the text by

$$\Omega(z) \equiv 1 - \omega(1 - b) \int_0^{\varphi_0} W(\xi) \xi \frac{T(z, \xi)}{\mathcal{L}(z, \xi)} d\xi. \quad (\text{C.1})$$

This definition is used in Section C.1 to analyze the properties of $\Omega(z)$ and in Section C.2 to show that $\Omega(z)$ reduces to the complete frequency redistribution dispersion function $\mathcal{L}_R(z)$ when $b = 0$. We recall that $\xi = \varphi(x)$, with $\varphi(x)$ the absorption profile and $\varphi_0 = \varphi(0)$.

C.1. Properties of $\Omega(z)$

We first remark that $\Omega(z)$ is analytic in the complex plane cut along the real axis, the point zero being excluded. Indeed, it has the same singular line as $\mathcal{L}(z, \xi)$, and in the integration over ξ , the lower limit of ξ is zero. For $z = 0$, we have $\mathcal{L}(0, \xi) = \epsilon_b$ (see Eq. (36)). Using $\mathcal{L}(0, \xi) = 1 - \omega b \xi T(0, \xi)$ and the normalization of $W(\xi)$ to unity (see Eq. (15)), we readily infer from Eq. (C.1) that

$$\Omega(0) = \frac{\epsilon}{\epsilon_b}. \quad (\text{C.2})$$

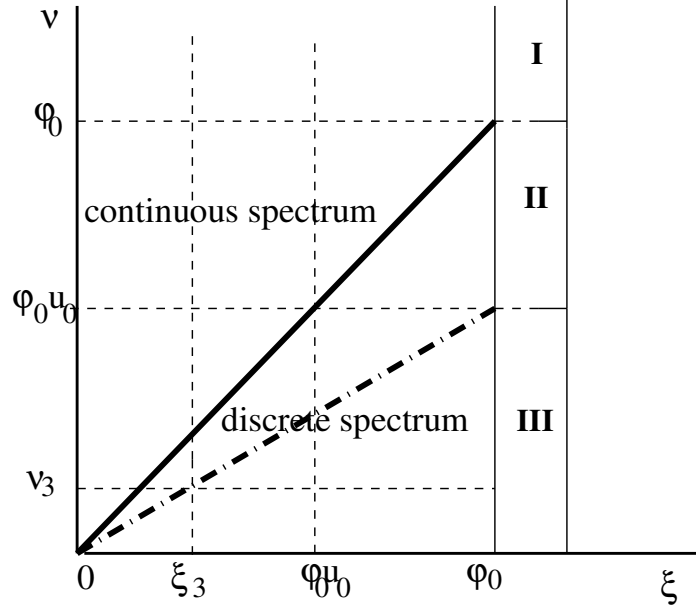


Figure C1. Properties of the spectrum of $\mathcal{L}(\nu, \xi)$ in the variables ν and ξ . The line $\nu = \xi$ shows the separation between the continuous and the discrete spectrum. The dotted-dashed line $\nu = u_0 \xi$ shows the location of the zeroes of $\mathcal{L}(z, \xi)$, with u_0 the positive zero of $\tilde{\mathcal{L}}(\tilde{z})$. The domains **I**, **II**, and **III** corresponds respectively to $\nu > \varphi_0$, $\varphi_0 u_0 < \nu < \varphi_0$ and $0 < \nu < \varphi_0 u_0$. ν_3 is a typical values of ν in the domain **III** and ξ_3 is the value of ξ such that the positive zero of $\mathcal{L}(z, \xi)$ is located at ν_3 . Integrations are along lines parallel to the horizontal ξ -axis for the calculation of $\Omega^\pm(\nu)$ and along lines parallel to the vertical ν -axis for the calculations of $\mathcal{J}(\tau, \xi)$ and $\tilde{\mathcal{J}}(\tau)$.

Hence $\Omega(0) = \epsilon$ for $b = 0$. For $b = 1$, $\Omega(z)$ is equal to 1 for all z . At infinity, $\mathcal{L}(z, \xi) \rightarrow 1$ and $T(z, \xi) \rightarrow 0$, hence $\Omega(z)$ tends to 1.

We now determine for all $\nu \in]-\infty, +\infty[$ ($\nu = \Re(z)$) the limiting values $\Omega^+(\nu)$ and $\Omega^-(\nu)$ of $\Omega(z)$, as z tends to the real axis, from above and from below. With the explicit expressions of $\Omega^\pm(\nu)$, it will be possible to determine whether or not $\Omega(z)$ is free of zeroes. We know that the dispersion function $\mathcal{L}(\nu, \xi)$ has a continuous spectrum for $|\nu| > |\xi|$ and a discrete one for $\nu \in [-\xi, +\xi]$, with zeroes at $\pm\nu_0(\xi)$. The existence of these two different spectra and of these zeroes brings some complexity to the calculation of $\Omega^\pm(\nu)$ and the expressions that are obtained depend on the value of ν .

Three different domains shown in Fig. C1 have to be considered. They are referred to as **I**, for ν larger than φ_0 , **II**, for $\varphi_0 u_0 < \nu < \varphi_0$, and **III**, for $0 < \nu < \varphi_0 u_0$. Because $\Omega(z)$ is an even function of z , we consider only positive values of ν . The straight line $\nu = \xi$ separates the continuous from the discrete spectrum. The line $\nu = u_0 \xi$ shows the location of the zeroes of $\mathcal{L}(\nu, \xi)$. We recall that u_0 is the positive zero of $\tilde{\mathcal{L}}(\tilde{z})$ (see Eq. (40)) and that u_0 and φ_0 are smaller than one.

Domain I $\nu \geq \varphi_0$

For all values of ν in this interval and all values of $\xi \in [0, \varphi_0]$, ν belongs to $\mathcal{C}(\xi)$, that is to the continuous spectrum. The Plemelj formulae can be applied to Eq. (C.1). Using the expressions of $\mathcal{L}^\pm(\nu)$ and $T^\pm(\nu)$ given in Eq.(34) to calculate the jump of

$T(z, \xi)/\mathcal{L}(z, \xi)$, we obtain

$$\Omega_I^+(\nu) - \Omega_I^-(\nu) = -\omega(1-b)2i\pi \int_0^{\varphi_0} W(\xi) \xi \frac{k(\nu, \xi)}{D(\nu, \xi)} d\xi, \quad (\text{C.3})$$

and

$$\Omega_I^+(\nu) + \Omega_I^-(\nu) = 2 - \omega(1-b)2 \int_0^{\varphi_0} W(\xi) \xi \frac{C(\nu, \xi)}{D(\nu, \xi)} d\xi, \quad (\text{C.4})$$

where

$$C(\nu, \xi) = t(\nu, \xi) - \omega b \xi E(\nu, \xi), \quad (\text{C.5})$$

$$E(\nu, \xi) = T^+(\nu, \xi)T^-(\nu, \xi) = t^2(\nu, \xi) + \pi^2 k^2(\nu, \xi), \quad (\text{C.6})$$

and

$$D(\nu, \xi) = \mathcal{L}^+(\nu, \xi)\mathcal{L}^-(\nu, \xi) = \lambda^2(\nu, \xi) + \pi^2 \omega^2 b^2 \xi^2 k^2(\nu, \xi). \quad (\text{C.7})$$

We thus obtain

$$\begin{aligned} \Re[\Omega_I^\pm(\nu)] &= 1 - \omega(1-b) \int_0^{\varphi_0} W(\xi) \xi \frac{C(\nu, \xi)}{D(\nu, \xi)} d\xi, \\ \Im[\Omega_I^\pm(\nu)] &= \mp \pi \omega(1-b) \int_0^{\varphi_0} W(\xi) \xi \frac{k(\nu, \xi)}{D(\nu, \xi)} d\xi. \end{aligned} \quad (\text{C.8})$$

Domain II $u_0\varphi_0 < \nu \leq \varphi_0$

For all $\xi < \nu$, ν belongs to the continuum spectrum and for all $\xi > \nu$, to the discrete spectrum. In the latter region, $\mathcal{L}(z, \xi)$ being analytic, its jump across the real axis is zero. In domain **II** there is no contribution from the zero at $\nu_0(\xi)$ since $\nu > u_0\varphi_0$ (see Fig. C1). We thus obtain

$$\begin{aligned} \Re[\Omega_{II}^\pm(\nu)] &= 1 - \omega(1-b) \left[\int_0^\nu W(\xi) \xi \frac{C(\nu, \xi)}{D(\nu, \xi)} d\xi + \int_\nu^{\varphi_0} W(\xi) \xi \frac{t(\nu, \xi)}{\lambda(\nu, \xi)} d\xi \right], \\ \Im[\Omega_{II}^\pm(\nu)] &= \mp \pi \omega(1-b) \int_0^\nu W(\xi) \xi \frac{k(\nu, \xi)}{D(\nu, \xi)} d\xi. \end{aligned} \quad (\text{C.9})$$

In the imaginary part, which comes from the continuous spectrum, the upper limit of the integral is now ν .

Domain III $0 < \nu < \varphi_0 u_0$

The situation similar to that of the domain **II**, but for a given value of $\nu = \Re(z)$, $\mathcal{L}(z, \xi)$ will have a zero when ξ has the value $\xi_0(\nu) = \nu/u_0$ (see Section 3). There is

thus in Eq. (C.9) an additional contribution coming from the integral

$$\int_{\nu}^{\varphi_0} W(\xi) \xi \frac{T(\nu, \xi)}{\mathcal{L}(\nu, \xi)} d\xi. \quad (\text{C.10})$$

The analyticity of $T(z, \xi)$ and $\mathcal{L}(z, \xi)$ for $\Re(z) \in [-\xi, +\xi]$, allows us to deform the integration line in such a way that it will pass around the pole. The contribution of the pole is found, in the standard way, by introducing a semi-circle around it, placed below the integration line for $\Omega^+(\nu)$ and above it for $\Omega^-(\nu)$. Taylor expanding $\lambda(\nu, \xi)$ around $\xi_0(\nu)$, we find that the contribution of the pole to the integral in Eq. (C.10) may be written as

$$\pm i \pi W(\xi_0(\nu)) \xi_0(\nu) \frac{t(\nu, \xi_0(\nu))}{\partial_{\xi} \lambda(\nu, \xi_0(\nu))}, \quad (\text{C.11})$$

where $\partial_{\xi} \lambda(\nu, \xi_0)$ is the partial derivative of $\lambda(\nu, \xi)$ with respect to ξ taken at $\xi_0(\nu)$. This partial derivative has been calculated in Section 3 (see Eq. (42)). We thus find

$$\Omega_{\text{III}}^{\pm}(\nu) = \Omega_{\text{II}}^{\pm}(\nu) \mp i \pi \omega(1 - b) W(\xi_0(\nu)) \xi_0(\nu) \frac{t(\nu, \xi_0(\nu))}{\partial_{\xi} \lambda(\nu, \xi_0(\nu))}. \quad (\text{C.12})$$

A general theory for the calculation of integrals with a pole in the denominator as in Eq. (C.10) can be found in Gakhov (1966, p. 66-68) and in Hemsch and Ferziger (1972).

We now turn to the determination of the number of zeroes of $\Omega(z)$. A standard method for an analytic function is to examine the variation of its argument along a close contour inside the analyticity region (e.g., Carrier, Krook, and Pearson 1966; Case and Zweifel 1967). For $\Omega(z)$, which is an even function of z , it suffices to calculate the variation of the argument of $\Omega^+(\nu)$ as ν varies along the positive real axis. A phase diagram of $\Omega^+(\nu)$, showing the variation of $\Im[\Omega^+(\nu)]$ versus $\Re[\Omega^+(\nu)]$ for $\nu \in [0, \infty[$ is sketched in Figure C2. For $\nu = 0$, we know that $\Omega(0) = \epsilon/\epsilon_b$ (see Eq. (C.2)) hence $\Im[\Omega^+(\nu)] = 0$ and $\Re[\Omega^+(\nu)]$ is smaller or equal to 1. At infinity $\Omega(z) \rightarrow 1$, hence, $\Im[\Omega^+(\infty)] = 0$ and $\Re[\Omega^+(\infty)] = 1$. Equations (C.8) and (C.9) show that $\Im[\Omega^+(\nu)]$ is negative in domains **I** and **II**. In domain **III**, the imaginary part in Eq. (C.12) contains an additional term coming from the pole. Since the derivative of $\mathcal{L}(\nu, \xi)$ is positive, as pointed in Section 3, this additional term is also negative. The real part $\Re[\Omega^+(\nu)]$ is positive for $\nu = 0$ and at infinity. It can be checked that it remains positive for all ν positive. From these properties, illustrated in the lower panel of Fig. C2, one infers that $(\theta(\infty) - \theta(0))$, the variation of the argument of $\Omega^+(\nu)$ is zero, hence that $\Omega(z)$ is free of zeroes. The upper panel shows for comparison a sketch of the phase diagram of $\mathcal{L}(\nu, \xi)$.

C.2. The complete frequency redistribution limit

We verify in this appendix that $\Omega(z) = \mathcal{L}_R(z)$ for $b = 0$, with $\mathcal{L}_R(z)$ the dispersion function for complete frequency redistribution. In this limit the source function depends only on τ and satisfies a convolution integral equation, given in Eq. (6) for

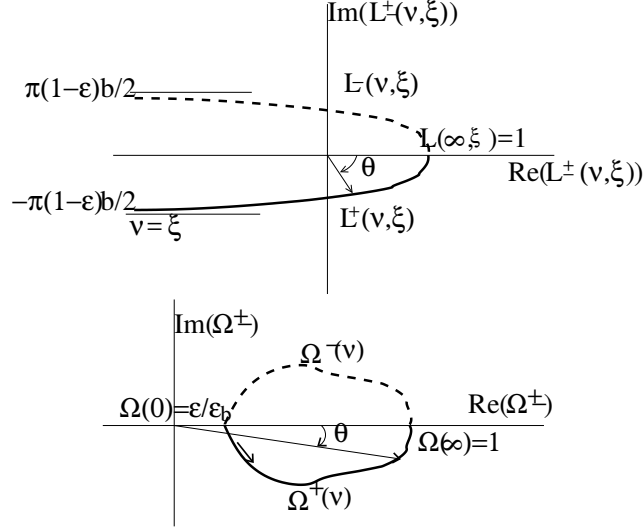


Figure C2. Phase diagrams of the frequency dependent dispersion function $\mathcal{L}(z, \xi)$ and of the generalized dispersion function $\Omega(z)$. The upper panel shows $\Im[\mathcal{L}^\pm(\nu, \xi)]$ versus $\Re[\mathcal{L}^\pm(\nu, \xi)]$ for $\nu \in [\xi, \infty[$ and the lower panel $\Im[\Omega^\pm(\nu)]$ versus $\Re[\Omega^\pm(\nu)]$ for $\nu \in [0, \infty[$.

$\tau \in [0, \infty[$. The kernel, defined by

$$K(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \varphi^2(x) e^{-|\tau|\varphi(x)/\mu} \frac{d\mu}{\mu} dx, \quad (\text{C.13})$$

may be written as

$$K(\tau) = \int_0^{\infty} k_R(\nu) e^{-\nu|\tau|} d\nu. \quad (\text{C.14})$$

Its inverse Laplace transform $k_R(\nu)$ (R for complete frequency redistribution) may be written as

$$k_R(\nu) = \frac{1}{\nu} \int_{y(\nu)}^{\infty} \varphi^2(x) dx, \quad (\text{C.15})$$

where

$$y(\nu) = \begin{cases} \varphi^{-1}(\nu) & \nu < \varphi_0 \\ 0 & \nu > \varphi_0. \end{cases} \quad (\text{C.16})$$

The notation φ^{-1} stands for the inverse function of φ . The dispersion function is then given by

$$\mathcal{L}_R(z) \equiv 1 - \omega \int_0^{\infty} k_R(\nu) \left[\frac{1}{\nu - z} + \frac{1}{\nu + z} \right] d\nu. \quad (\text{C.17})$$

The integral in the right-side is the complex Fourier transform of the kernel.

The frequency dependent dispersion function $\mathcal{L}(z, \xi)$, reduces to $\mathcal{L}(z, \xi) = 1$ when

$b = 0$ (see Eq. (29)). Hence $\Omega(z)$ reduces to

$$\Omega(z) = 1 - \omega \int_0^{\varphi_0} W(\xi) \xi T(z, \xi) d\xi, \quad (\text{C.18})$$

with

$$T(z, \xi) = \int_{\xi}^{\infty} k(\nu, \xi) \left[\frac{1}{\nu - z} + \frac{1}{\nu + z} \right] d\nu, \quad (\text{C.19})$$

and

$$k(\nu, \xi) = \frac{1}{2\nu} Y(\nu - \xi). \quad (\text{C.20})$$

The function $Y(\nu)$ is the Heaviside function. Replacing ξ by $\varphi(x)$ and using

$$W(\xi) = 2\xi \left| \frac{dx}{d\xi} \right|, \quad (\text{C.21})$$

we can rewrite Eq. (C.18) as

$$\Omega(z) = 1 - \omega \int_0^{\infty} \varphi^2(x) \int_{\varphi(\nu)}^{\infty} \frac{1}{\nu} Y(\nu - \varphi(x)) \frac{2\nu}{\nu^2 - z^2} d\nu dx. \quad (\text{C.22})$$

Interchanging the order of the integrations between ν and x and using $Y(\nu - \varphi(x)) = 1$, we obtain

$$\Omega(z) = 1 - \omega \int_0^{\infty} d\nu \left[\frac{1}{\nu - z} + \frac{1}{\nu + z} \right] \frac{1}{\nu} \int_{y(\nu)}^{\infty} \varphi^2(\lambda) d\lambda. \quad (\text{C.23})$$

Referring to the definition of $k_R(\nu)$ given in Eq. (C.15), we can immediately verify that $\Omega(z) = \mathcal{L}_R(z)$.

A similar calculation, applied to the expressions given in Section C.1 for $\Omega^+(\nu) - \Omega^-(\nu)$ show that in the limit $b = 0$,

$$\Omega^+(\nu) - \Omega^-(\nu) = -\omega 2i \pi k_R(\nu), \quad \nu = \Re(z) \in]-\infty, +\infty[. \quad (\text{C.24})$$

This is precisely the jump of $\mathcal{L}_R(z)$, as can be derived from Eq. (C.23).

Appendix D. The functions $\mathcal{M}(z, \xi)$ and $\mathcal{V}(\nu, \nu')$

The function $\mathcal{M}(z, \xi)$ is defined by

$$\mathcal{M}(z, \xi) \equiv H_{\xi}^{\infty} \left[\frac{[X^*]^+(\nu, \xi)}{\mathcal{L}^+(\nu, \xi)} m(\nu, \xi) \right], \quad (\text{D.1})$$

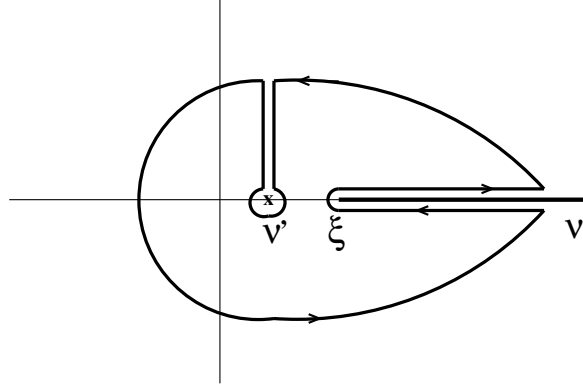


Figure C3. Contour in the complex plane for the calculation of the integral in Eq. (D.6). The function $X(z, \xi)$ is analytic in the complex plane cut along the semi-infinite interval $[\xi, \infty[$. The singularity of the integrand is at ν' , with $\nu' \in [0, \xi]$.

where H_ξ^∞ stands for the Hilbert transform over $[\xi, \infty[$ and

$$m(\nu, \xi) = p^*(\nu) - [\mathcal{L}^+(\nu, \xi) - \mathcal{L}^-(\nu, \xi)] \frac{1}{2i\pi} \int_0^\xi \frac{p^*(\nu')}{\lambda(\nu', \xi)} \frac{d\nu'}{\nu' - \nu}. \quad (\text{D.2})$$

For convenience we write $\mathcal{M}(z, \xi) = \mathcal{M}_1(z, \xi) + \mathcal{M}_2(z, \xi)$ with

$$\mathcal{M}_2(z, \xi) = -\frac{1}{2i\pi} \int_\xi^\infty \left\{ [X^*]^+(\nu, \xi) - [X^*]^-(\nu, \xi) \right\} \frac{1}{2i\pi} \int_0^\xi \frac{p^*(\nu')}{\lambda(\nu', \xi)} \frac{d\nu'}{\nu' - \nu} \frac{d\nu}{\nu - z}. \quad (\text{D.3})$$

This double integral can be transformed into the sum of two simple integrals. Using

$$\frac{1}{\nu' - \nu} \frac{1}{\nu - z} = \left[\frac{1}{\nu' - \nu} + \frac{1}{\nu - z} \right] \frac{1}{\nu' - z}, \quad (\text{D.4})$$

and changing the order of integration, we find

$$\begin{aligned} \mathcal{M}_2(z, \xi) &= -\frac{1}{2i\pi} \int_0^\xi \frac{p^*(\nu')}{\lambda(\nu', \xi)} \frac{d\nu'}{\nu' - z} \\ &\times \frac{1}{2i\pi} \int_\xi^\infty \left\{ [X^*]^+(\nu, \xi) - [X^*]^-(\nu, \xi) \right\} \left[\frac{1}{\nu' - \nu} + \frac{1}{\nu - z} \right] d\nu, \end{aligned} \quad (\text{D.5})$$

where $X^*(z, \xi) = (\nu_0(\xi) - z)X(z, \xi)$. The integration contour shown in Fig. C3 leads to

$$\frac{1}{2i\pi} \int_\xi^\infty \left\{ [X^*]^+(\nu, \xi) - [X^*]^-(\nu, \xi) \right\} \frac{d\nu}{\nu - \nu'} = X^*(\nu', \xi) - 1. \quad (\text{D.6})$$

A similar equation holds when ν' is replaced by a point z in the complex plane. Summing the two equations, we obtain

$$\mathcal{M}_2(z, \xi) = -X^*(z, \xi) \frac{1}{2i\pi} \int_0^\xi \frac{p^*(\nu')}{\lambda(\nu', \xi)} \frac{d\nu'}{\nu' - z} + \frac{1}{2i\pi} \int_0^\xi \frac{p^*(\nu')}{\lambda(\nu', \xi)} X^*(\nu', \xi) \frac{d\nu'}{\nu' - z}. \quad (\text{D.7})$$

We now use the factorization $X^*(\nu, \xi)X^*(-\nu, \xi) = \lambda(\nu, \xi)$, which holds for $\nu \in [0, \xi]$, to transform the second integral in the right-hand side of Eq. (D.7). To calculate $\mathcal{M}_1(z, \xi)$, we use the factorization $[X^*]^+(\nu, \xi)X^*(-\nu, \xi) = \mathcal{L}^+(\nu, \xi)$, which holds for $\nu \in [\xi, \infty]$. We thus finally obtain

$$\mathcal{M}(z, \xi) = -\frac{1}{2i\pi} X^*(z, \xi) \int_0^\xi \frac{p^*(\nu')}{\lambda(\nu', \xi)} \frac{d\nu'}{\nu' - z} + \frac{1}{2i\pi} \int_0^\infty \frac{p^*(\nu')}{X^*(-\nu', \xi)} \frac{d\nu'}{\nu' - z}. \quad (\text{D.8})$$

In Section 5.3 we show that $\rho(\nu)$, the jump of $R(z)$, can be written as

$$\rho(\nu) = \frac{1-b}{b} \int_0^\infty \bar{J}(\nu') \mathcal{V}(\nu, \nu') d\nu', \quad (\text{D.9})$$

where $\mathcal{V}(\nu, \nu')$ is the jump across the positive real axis of the function

$$V(z, \nu) \equiv \frac{1}{2i\pi} \int_0^{\varphi_0} \frac{W(\xi)}{\mathcal{L}(z, \xi)} r(z, \nu, \xi) d\xi. \quad (\text{D.10})$$

To calculate $\mathcal{V}(\nu, \nu')$ we can proceed exactly as for the calculation the jump of $\Omega(z)$ in Appendix C.1. In both cases, the jumps are controlled by the jump of $1/\mathcal{L}(z, \nu)$. We thus obtain

$$\mathcal{V}(\nu, \nu') = V(\nu, \nu'), \quad \nu \in [u_0\varphi_0, \infty[, \quad (\text{D.11})$$

$$\mathcal{V}(\nu, \nu') = V(\nu, \nu') + r(\nu, \nu', \xi_0) \frac{W(\xi_0)}{\partial_\xi \mathcal{L}(\nu, \xi_0)}, \quad \nu \in [0, u_0\varphi_0], \quad (\text{D.12})$$

where

$$V(\nu, \nu') \equiv \frac{1}{2i\pi} \int_0^{\varphi_0} W(\xi) \left[\frac{1}{\mathcal{L}(z, \xi)} \right]_{\text{jmp}} r(\nu, \nu', \xi) d\xi, \quad (\text{D.13})$$

and

$$r(\nu, \nu', \xi_0) = \left[\frac{2\nu X(-\nu, \xi_0)}{(\nu + \nu')X(-\nu', \xi_0)} - 1 \right] \frac{P}{\nu' - \nu}. \quad (\text{D.14})$$

Here, $\xi_0 = \nu/u_0$, where u_0 is the positive zero of the function $\tilde{\mathcal{L}}(\tilde{z})$ introduced in Eq. (40). In the complete frequency redistribution limit $\mathcal{V}(\nu, \nu')$ and $\rho(\nu)$ are zero.