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Regular Grammars for Graph Sets of Tree-Width ≤ 2

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Abstract

Regular and context-free languages form a central pillar of formal language theory. This is because a variety of formalisms are known that define these classes of languages. For example, we have that finite automata, monoids, algebraic recognizability, regular expressions, regular grammars, monadic-second order logic, etc., can be used to represent regular word languages. However, the situation is less clear for formal languages over graphs, and open problems persist. This is because generalizing notions from words to graphs has been more successful for some of the cited formalisms than for the other ones. Bruno Courcelle has introduced hyper-edge replacement (HR) algebras for generalizing the notion of context-free languages from words to graphs. At the same time, HR-algebras support the generalization of algebraic recognizability from words to graphs, a notion that has been proven to be equivalent to definability in (counting) monadic-second order logic (CMSO) over graphs of bounded tree-width. In this paper, we deal with generalizing regular word grammars to graphs. We propose regular grammars for (unordered and unranked) trees, series-parallel graphs, and graphs of tree-width ≤ 2 , where the qualifier regular is justified because these grammars define exactly the recognizable resp. CMSO-definable subsets of the respective graph classes.

1 Introduction

Regular and context-free languages form a central pillar of formal language theory. This foundational role is underpinned by the variety of well-established formalisms that define these classes of languages. For instance, regular word languages can be represented through finite automata, monoids, algebraic recognizability, regular expressions, regular grammars, and monadic-second order logic, among others. These diverse formalisms provide a robust and interconnected framework for understanding the structure and properties of regular languages in the context of words. While the landscape of formal languages of words and (to a lesser extent also) of trees has been mapped extensively, the landscape of formal languages of graphs is less known and open problems persist. Significant advancements in this area have been made by the works of Bruno Courcelle, who introduced hyper-edge replacement (HR) algebras [8, 10]. HR-algebras are a pivotal development as they allow to generalize the notions of context-free languages and algebraic recognizability from words to graphs. Notably, algebraic HR-recognizability has been shown to be equivalent to definability in (counting) monadic-second order logic (CMSO) for graphs of bounded tree-width [3].

While extending the notions of logical definability and algebraic recognizability from words to graph has been successful, the story is less clear for other formalisms such as regular expressions or regular grammars, for which no general results are known to capture precisely the recognizable (resp. CMSO-definable) graph languages. Courcelle has proposed so-called “regular graph grammars” in [9], and proven that all languages resulting from such grammars are CMSO-definable and of bounded tree-width; however, these “regular” grammars fall short of

capturing all CMSO-definable sets of graphs of bounded tree-width. Recently, progress has been made on defining regular expressions that capture exactly the CMSO-definable sets of graphs of tree-width at most two [12], and on defining grammars that capture exactly the CMSO-definable sets of graphs of bounded embeddable tree-width [6] (i.e., an over-approximation of tree-width that considers only tree decompositions whose tree backbones are spanning trees of the input graph).

Contributions. In this paper, we propose regular grammars for (unordered and unranked) trees, oriented and disoriented series-parallel graphs, and for graphs of tree-width ≤ 2 . Calling these grammars *regular* is justified because our grammars are syntactic restrictions of the context-free grammars (as in the word case) and because we prove that our grammars define exactly the recognizable (resp. CMSO-definable) subsets of the respective graph classes. Moreover, we show that the theory developed in this paper has applications for the inclusion problem of languages given by regular grammars in any of the considered graph classes and we provide a 2EXPTIME upper bound on the inclusion problem by a uniform argument.

Related Work. The rise of XML (eXtensible Markup Language) has sparked a systematic study of automata and logics for XML trees, which are unranked trees with a total order on siblings. We refer to [18] for a survey on automata and logics for unranked (ordered and unordered) trees, including some complexity results relative to the model checking problem. Unranked trees with total sibling order and data values are studied in [2]. Deterministic automata on unranked trees with total sibling order are investigated in [11]. The complexity for decision problems of automata on unordered trees (i.e., without a total sibling order) is examined in [1]. Algebraic recognizers for unranked (ordered and unordered) trees are discussed in [5].

The foundational work by Courcelle established that all CMSO-definable sets of graphs are recognizable [8], but there are recognizable sets of graphs that are not CMSO-definable. The picture is completed by the seminal papers by Bojanczyk and Pilipczuk [3, 4], that established the equivalence of recognizability and CMSO-definability for bounded tree-width classes of graphs.

The regular expressions for graphs of tree-width ≤ 2 introduced in [12] capture exactly the CMSO-definable sets, just as the regular grammars we propose in this paper. However, comparing the resulting formalisms and technical developments is challenging and requires further study. For example, [12] defines and employs different graph operations for graphs of tree-width ≤ 2 compared to ours, the syntactic restrictions put on the regular expressions are quite different from the restrictions put on our grammars, with their technical development primarily based on CMSO, while ours is mainly based on algebraic recognizability. Furthermore, we obtain explicit bounds on the size of the algebraic recognizer, whereas this is stated as an open problem in [12].

The paper [6] proposes tree-verifiable graph grammars, a class that strictly generalizes the regular graph grammars introduced by Courcelle in [9] and captures exactly the CMSO-definable sets of graphs of bounded embeddable tree-width — a notion that denotes tree-decompositions whose backbone is a spanning tree of the considered graph. The authors of [6] already proved the CMSO-definability of the regular tree-grammars that we consider in this paper; we reprove their result here, however, based on a different argument, that is, by an explicit construction of a recognizer, which has applications to the language inclusion problem. Our results on series-parallel graphs and graphs of tree-width ≤ 2 , in contrast, are orthogonal.

The paper [13] proposes a finite axiomatisation for term equivalence of two HR-terms for graphs of tree-width ≤ 3 . This is an interesting direction to investigate whether ideas from this paper can be used for defining grammars for graphs of tree-width ≤ 3 .

2 Definitions

The set of natural numbers is denoted by \mathbb{N} . Given numbers $i, j \in \mathbb{N}$, we write $[i, j] \stackrel{\text{def}}{=} \{i, i+1, \dots, j\}$, assumed to be empty if $i > j$. The cardinality of a finite set A is denoted by $\text{card}(A)$. By writing $A \subseteq_{\text{fin}} B$ we mean that A is a finite subset of B . The disjoint union $A \uplus B$ is defined as the union of A and B if $A \cap B = \emptyset$ and undefined, otherwise. For a set A , we denote by $\text{pow}(A)$ its powerset.

We denote by $\text{mpow}(A)$ the multi-powerset of A , i.e., the set of powersets $m : A \rightarrow \mathbb{N}$. By $m_1 \uplus m_2$ we denote the union of the multisets $m_1, m_2 : A \rightarrow \mathbb{N}$, i.e., $(m_1 \uplus m_2)(a) = m_1(a) + m_2(a)$, for all $a \in A$.

For a relation $R \subseteq A \times B$, we denote by $\text{dom}(R)$ and $\text{img}(R)$ the sets consisting of the first and second components of the pairs in R , respectively. We write R^{-1} for the inverse relation and $R(S)$ for the image of a set S via R . Sometimes we write $R(a)$ instead of $R(\{a\})$, for an element $a \in A$.

We denote by $\text{poly}(x)$ the set of functions $p : \mathbb{N} \rightarrow \mathbb{N}$ for which there exist $c, k \in \mathbb{N}$ such that $p(n) \leq c \cdot n^k$, for all $n \in \mathbb{N}$ (the argument x is necessary to specify the argument of these functions). This notation is used in exponentiation, e.g., $f \in 2^{2^{\text{poly}(x)}}$, with the obvious meaning, i.e., $f(n) \leq 2^{2^{p(n)}}$, for some $p(x) \in \text{poly}(x)$, for all $n \in \mathbb{N}$.

2.1 Algebras, Recognizability and Grammars

Let $\Sigma = \{\sigma_1, \sigma_2, \dots\}$ be a set of *sorts*, $\mathcal{F} = \{f_1, f_2, \dots\}$ be a set of *function symbols*, called *signature*, $\mathcal{V} = \{x, y, \dots\}$ and $\mathcal{X} = \{X, Y, \dots\}$ be sets of *individual* and *set variables*, respectively. Each function symbol f has an associated tuple of *argument sorts* $\rho(f) = \langle \sigma_1, \dots, \sigma_n \rangle$ and a *value sort* $\sigma(f)$. We denote by $\#f \stackrel{\text{def}}{=} n$ the arity of f . A function symbol of zero arity is a *constant*. Moreover, each variable x (resp. X) has an associated sort $\sigma(x)$ (resp. $\sigma(X)$).

A *term* of sort σ is either a constant of value sort σ , a variable of sort σ , or $f(t_1, \dots, t_n)$, where $\rho(f) = \langle \sigma_1, \dots, \sigma_n \rangle$, each t_i is a term of sort σ_i and $\sigma(f) = \sigma$. A term is *ground* if it has no variables. We denote by $t[x_1, \dots, x_n]$ a term in which x_1, \dots, x_n occur at least once, and no other variable occurs in t .

A (\mathcal{F}, Σ) -*algebra* is a tuple $\mathcal{A} = (\{A_\sigma\}_{\sigma \in \Sigma}, \{f^A\}_{f \in \mathcal{F}})$, where each set A_σ is the *domain of sort* σ and each function $f^A : A_{\sigma_1} \times \dots \times A_{\sigma_n} \rightarrow A_\sigma$ is the *interpretation* of the function symbol f , having $\rho(f) = \langle \sigma_1, \dots, \sigma_n \rangle$ and $\sigma(f) = \sigma$. The *domain* of \mathcal{A} is $A \stackrel{\text{def}}{=} \bigcup_{\sigma \in \Sigma} A_\sigma$. We denote by $\sigma(a)$ the sort of each element $a \in A$. An algebra is *locally finite* if A_σ is finite, for each $\sigma \in \Sigma$ and *finite* if A is finite. Each finite algebra is locally finite, but there are locally finite algebras that are not finite, i.e., having infinitely many sorts.

A term $t[x_1, \dots, x_n]$ defines a function $t^A : A_{\sigma(x_1)} \times \dots \times A_{\sigma(x_n)} \rightarrow A_{\sigma(t)}$. We also denote the sort of t by $\sigma(t)$ and the argument sorts of t by $\rho(t) = \langle \sigma(x_1), \dots, \sigma(x_n) \rangle$. An algebra is *representable* if its domain is the set of values of ground terms (of any sort). We implicitly assume each algebra considered in this paper to be representable.

We say a $(\mathcal{F}_B, \Sigma_B)$ -algebra \mathcal{B} is *derived* from some $(\mathcal{F}_A, \Sigma_A)$ -algebra \mathcal{A} , if $\Sigma_B \subseteq \Sigma_A$ and for every $f \in \mathcal{F}_B$ there is some first-order \mathcal{F}_A -term t_f such that $f^B = t_f^A$. We note that any *subalgebra*, i.e., an algebra obtained by restricting the signature and sorts of \mathcal{A} , is a derived algebra of \mathcal{A} (this case needs no special treatment).

Given (\mathcal{F}, Σ) -algebras \mathcal{A} and \mathcal{B} , an \mathcal{A} - \mathcal{B} *homomorphism* is a function $h : A \rightarrow B$ such that $h(f^A(a_1, \dots, a_n)) = f^B(h(a_1), \dots, h(a_n))$, for all $f \in \mathcal{F}$, such that $\rho(f) = \langle \sigma_1, \dots, \sigma_n \rangle$ and all $a_i \in A_{\sigma_i}$, $i \in [1, n]$. Because we assumed \mathcal{A} and \mathcal{B} to be representable algebras, the homomorphism h is uniquely determined, hence we denote it by $h_{\mathcal{A}, \mathcal{B}}$.

A (\mathcal{F}, Σ) -recognizer is a pair (\mathcal{B}, C) , where \mathcal{B} is a locally finite (\mathcal{F}, Σ) -algebra and $C \subseteq \mathcal{B}$ is a set. Let \mathcal{A} be a (\mathcal{F}, Σ) -algebra in the following.

Definition 1. A set $\mathcal{L} \subseteq \mathcal{A}$ is recognized by (\mathcal{B}, C) if $\mathcal{L} = h_{\mathcal{A}, \mathcal{B}}^{-1}(C)$. A set is (finitely) recognizable if it is recognized by some (finite) recognizer.

We recall the following relation between an algebra \mathcal{B} derived of some algebra \mathcal{A} :

Lemma 1. $\mathcal{L} \subseteq \mathcal{B}$ is recognizable in \mathcal{B} , if it is recognizable in \mathcal{A} .

Proof By assumption we have $\Sigma_{\mathcal{B}} \subseteq \Sigma_{\mathcal{A}}$ and that for every $f \in \mathcal{F}_{\mathcal{B}}$ there is some first-order $\mathcal{F}_{\mathcal{A}}$ -term t_f such that $f^{\mathcal{B}} = t_f^{\mathcal{A}}$. Let (\mathcal{D}, E) be a (\mathcal{F}, Σ) -recognizer, such that $\mathcal{L} = h_{\mathcal{A}, \mathcal{D}}^{-1}(E)$. We define the $(\mathcal{F}_{\mathcal{B}}, \Sigma_{\mathcal{B}})$ -recognizer (\mathcal{I}, J) , where:

$$\begin{aligned} \mathcal{I} &\stackrel{\text{def}}{=} (\{l_{\sigma}\}_{\sigma \in \Sigma_{\mathcal{B}}}, \{f^{\mathcal{I}}\}_{f \in \mathcal{F}_{\mathcal{B}}}) \text{ is a locally finite algebra, with} \\ \text{domains } l_{\sigma} &\stackrel{\text{def}}{=} \{t^{\mathcal{D}} \mid t \text{ ground } \mathcal{F}_{\mathcal{B}}\text{-term, } \sigma(t) = \sigma\}, \text{ for all } \sigma \in \Sigma_{\mathcal{B}}, \text{ and} \\ \text{functions } f^{\mathcal{I}} &\stackrel{\text{def}}{=} t_f^{\mathcal{D}}, \\ J &\stackrel{\text{def}}{=} l \cap E \end{aligned}$$

We prove that $\mathcal{L} = h_{\mathcal{B}, \mathcal{I}}^{-1}(J)$. We consider some $x \in \mathcal{B}$. By assumption that we consider only representable algebras, there exists a ground $\mathcal{F}_{\mathcal{B}}$ -term t such that $x = t^{\mathcal{B}}$. Let u be the $\mathcal{F}_{\mathcal{A}}$ -term obtained by expanding the terms from $\mathcal{F}_{\mathcal{B}}$, i.e., $x = u^{\mathcal{A}}$. Then, $h_{\mathcal{B}, \mathcal{I}}(x) = h_{\mathcal{B}, \mathcal{I}}(t^{\mathcal{B}}) = t^{\mathcal{I}} = u^{\mathcal{D}} = h_{\mathcal{A}, \mathcal{D}}(u^{\mathcal{A}}) = h_{\mathcal{A}, \mathcal{D}}(x)$. Hence, $x \in h_{\mathcal{B}, \mathcal{I}}^{-1}(J)$ iff $x \in h_{\mathcal{A}, \mathcal{D}}^{-1}(E)$. With $\mathcal{L} \subseteq \mathcal{B}$, we obtain that $\mathcal{L} = h_{\mathcal{A}, \mathcal{D}}^{-1}(E) = h_{\mathcal{B}, \mathcal{I}}^{-1}(J)$. \square

An (\mathcal{F}, Σ) -grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ consists of a finite set \mathcal{N} of set variables, called *nonterminals* and a finite set \mathcal{R} of *rules* of the form, either:

- $X \rightarrow t[X_1, \dots, X_n]$, where $X, X_1, \dots, X_n \in \mathcal{N}$, t is a \mathcal{F} -term, with $\sigma(t) = \sigma(X)$ and $\rho(t) = \langle \sigma(X_1), \dots, \sigma(X_n) \rangle$,
- $\rightarrow X$, for some $X \in \mathcal{N}$; these rules are called *axioms*.

We define $\text{size}(\Gamma)$ to be the number of all symbols appearing in Γ . Given an (\mathcal{F}, Σ) -algebra \mathcal{A} , an \mathcal{A} -solution of Γ is a mapping $\mathcal{S} : \mathcal{N} \rightarrow \text{pow}(\mathcal{A})$, such that

1. $\mathcal{S}(X) \subseteq \mathcal{A}_{\sigma(X)}$, for all $X \in \mathcal{N}$, and
2. $t^{\mathcal{A}}(\mathcal{S}(X_1), \dots, \mathcal{S}(X_n)) \subseteq \mathcal{S}(X)$, for each rule $X \rightarrow t[X_1, \dots, X_n] \in \mathcal{R}$,

where we lift point-wise the interpretation of terms from elements to sets, i.e., $t^{\mathcal{A}}(S_1, \dots, S_n) \stackrel{\text{def}}{=} \{t^{\mathcal{A}}(e_1, \dots, e_n) \mid e_i \in S_i\}$ for $S_i \subseteq \mathcal{A}$ and $i \in [1, n]$. We write $\mathcal{L}_{\mathcal{X}}^{\mathcal{A}}(\Gamma) \stackrel{\text{def}}{=} \mathcal{S}(X)$ for the least \mathcal{A} -solution \mathcal{S} of Γ . Note that, since the evaluation of terms with set variables is monotonic with regard to set containment, a least solution exists and is unique. The *language* of Γ in \mathcal{A} is the set $\mathcal{L}^{\mathcal{A}}(\Gamma) \stackrel{\text{def}}{=} \bigcup_{X \in \mathcal{R}} \mathcal{L}_{\mathcal{X}}^{\mathcal{A}}(\Gamma)$. A language \mathcal{L} is *context-free* or *equational* in \mathcal{A} if $\mathcal{L} = \mathcal{L}^{\mathcal{A}}(\Gamma)$, for some (\mathcal{F}, Σ) -grammar Γ .

A *refinement* of a (\mathcal{F}, Σ) -grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ by a (\mathcal{F}, Σ) -recognizer (\mathcal{B}, C) is a (\mathcal{F}, Σ) -grammar $\Gamma^{(\mathcal{B}, C)}$, having nonterminals $\{X^b \mid X \in \mathcal{N}, b \in \mathcal{B}\}$ and rules either:

1. $X^b \rightarrow t[X_1^{b_1}, \dots, X_n^{b_n}]$, for each rule $X \rightarrow t[X_1, \dots, X_n] \in \mathcal{R}$ and each sequence of elements $b, b_1, \dots, b_n \in \mathcal{B}$, such that $b = f^{\mathcal{B}}(b_1, \dots, b_n)$, or
2. $\rightarrow X^c$, for each axiom $\rightarrow X \in \mathcal{R}$ and each element $c \in C$.

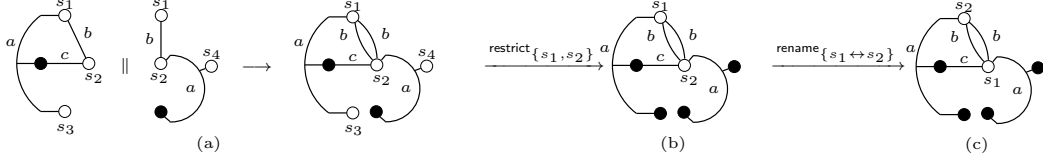


Figure 1: Composition (a), Restriction (b) and Renaming (c) of Graphs. Sources are denoted by hollow and internal vertices by solid circles.

We next state the fundamental property of refinement grammars:

Theorem 1 (Theorem 3.88 in [10]). *Let \mathcal{A} be a (\mathcal{F}, Σ) -algebra. For each (\mathcal{F}, Σ) -grammar Γ and (\mathcal{F}, Σ) -recognizer $(\mathcal{B}, \mathcal{C})$, we have $\mathcal{L}_{X^b}^{\mathcal{A}}(\Gamma^{(\mathcal{B}, \mathcal{C})}) = \mathcal{L}_X^{\mathcal{A}}(\Gamma) \cap h_{\mathcal{A}, \mathcal{B}}^{-1}(b)$, for all $X \in \mathcal{N}$ and $b \in \mathcal{B}$. Consequently, $\mathcal{L}^{\mathcal{A}}(\Gamma^{(\mathcal{B}, \mathcal{C})}) = \mathcal{L}^{\mathcal{A}}(\Gamma) \cap h_{\mathcal{A}, \mathcal{B}}^{-1}(\mathcal{C})$.*

Finally, we state that algebras over finite signature and sorts have a *universal* grammar:

Lemma 2. *Let \mathcal{A} be a (\mathcal{F}, Σ) -algebra, with Σ and \mathcal{F} finite. Let $\Gamma_{\mathcal{A}} = (\{X_{\sigma} \mid \sigma \in \Sigma\}, \mathcal{R}_{\mathcal{A}})$ be the grammar with rules $X_{\sigma} \rightarrow f[X_{\sigma_1}, \dots, X_{\sigma_n}]$, for each $f \in \mathcal{F}_{\mathcal{A}}$ with $\sigma(f) = \sigma$ and $\rho(f) = \langle \sigma_1, \dots, \sigma_n \rangle$, and axioms $\rightarrow X_{\sigma}$, for every $\sigma \in \Sigma$. Then, $\mathcal{A} = \mathcal{L}^{\mathcal{A}}(\Gamma_{\mathcal{A}})$.*

Proof “ \subseteq ” Let $c \in \mathcal{A}$ be an element and u be a ground \mathcal{F} -term such that $c = u^{\mathcal{A}}$. By induction of the structure of u we build a derivation of X in $\Gamma_{\mathcal{A}}$ that produces c . This is possible because there is a rule $X \rightarrow f[X, \dots, X]$, for each $f \in \mathcal{F}$. “ \supseteq ” Let $c \in \mathcal{L}_X^{\mathcal{A}}(\Gamma_{\mathcal{A}})$ be an element. By induction on the derivation that produces c in $\Gamma_{\mathcal{A}}$, we build a ground \mathcal{F} -term t , such that $t^{\mathcal{A}} = c$. \square

3 Graphs

Let \mathbb{S} be a countably infinite set of *source labels* and \mathbb{A} be a finite alphabet of *edge labels*, disjoint from \mathbb{S} . Each edge label $a \in \mathbb{A}$ has an associated *arity* $\#a \geq 1$, i.e., we do not consider edge labels of arity zero. The sets \mathbb{S} and \mathbb{A} are fixed in the rest of this paper.

Definition 2. *Let $\tau \subseteq_{\text{fin}} \mathbb{S}$ be a finite set of source labels. A graph of sort τ is a tuple $G = \langle V_G, E_G, \lambda_G, \nu_G, \xi_G \rangle$, where:*

- V_G is a finite set of vertices,
- E_G is a finite set of edges, disjoint from V_G ,
- $\lambda_G : E_G \rightarrow \mathbb{A}$ is a mapping that defines the labels of the edges,
- $\nu_G : E_G \rightarrow V_G^+$ is a mapping that associates each edge a nonempty sequence of vertices attached to the edge, such that $\#(\lambda_G(e)) = \text{len}(\nu_G(e))$, for each $e \in E_G$,
- $\xi_G : \tau \rightarrow V_G$ is a one-to-one mapping that designates the sources of G . The vertex $\xi_G(s)$ is called the s -source of G . Because ξ_G is injective, a vertex cannot be both an s - and s' -source, for $s \neq s'$. Vertices that are not sources are called internal.

We denote by \mathbf{G} the set of graphs.

For example, the leftmost graph in Fig. 1 (a) has four vertices of which tree sources labeled s_1, s_2 and s_3 and three edges labeled a, b and c . The a -labeled edge is attached to three vertices, whereas the b - and c -labeled edges are binary. The middle graph is of sort $\{s_1, s_2, s_4\}$ and the rightmost one of sort $\{s_1, s_2, s_3, s_4\}$.

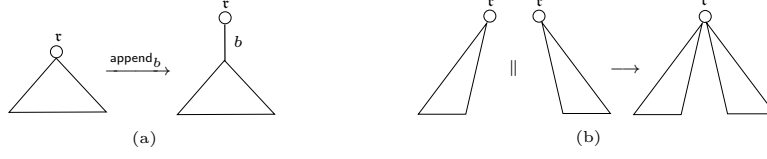


Figure 2: Append (a) and Composition (b) of Trees. The hollow circles are the τ -sources.

Next, we introduce the *hyperedge replacement* (HR) algebra of operations on graphs (for a more detailed treatment, see [10]). We fix Σ_{HR} to be the set of finite subsets of \mathbb{S} . \mathcal{F}_{HR} consists of the constants $\mathbf{0}_\tau$, for all $\tau \subseteq_{\text{fin}} \mathbb{S}$ and $\mathbf{a}_{(s_1, \dots, s_{\#a})}$, for all $a \in \mathbb{A}$ and $s_1, \dots, s_{\#a} \in \mathbb{S}$, the unary function symbols restrict_τ , for all $\tau \subseteq_{\text{fin}} \mathbb{S}$, and rename_α , for all finite permutations $\alpha : \mathbb{S} \rightarrow \mathbb{S}$ and the binary function symbols $\|_{\tau_1, \tau_2}$, for all $\tau_1, \tau_2 \subseteq_{\text{fin}} \mathbb{S}$. Note that \mathcal{F}_{HR} is infinite, because the set \mathbb{S} of source labels is infinite. The *graph algebra* \mathcal{G} interprets the symbols in \mathcal{F}_{HR} as:

1. **sources only:** the graph $\mathbf{0}_\tau^{\mathcal{G}}$ consists of one s -source, for each $s \in \tau$, and no edges.
2. **single edge:** the graph $\mathbf{a}_{(s_1, \dots, s_{\#a})}^{\mathcal{G}}$ consists of an s_i -source, for each $i \in [1, \#a]$, and a single edge labeled with a attached to the $s_1, \dots, s_{\#a}$ -sources, in this order.
3. **restriction:** the unary function $\text{restrict}_\tau^{\mathcal{G}}$ takes as input any graph of sort τ' and returns the graph of sort $\tau \cap \tau'$ obtained by removing the source labels in $\tau' \setminus \tau$ from G .
4. **rename:** the unary function $\text{rename}_\alpha^{\mathcal{G}}$ takes as input a graph of sort τ and returns the graph of sort $\alpha^{-1}(\tau)$ obtained by renaming its sources according to α .
5. **composition:** the binary function $\|_{\tau_1, \tau_2}^{\mathcal{G}}$ takes the disjoint union of two graphs of sorts τ_1 and τ_2 , respectively, and fuses the vertices labeled by the same source label in both. For simplicity, we omit the sorts and write $\|$ (resp. $\|^{\mathcal{G}}$) whenever the argument sorts can be inferred from the context.

For example, Fig. 1 (a) shows the result of the composition of two graphs, whereas (b) and (c) show the result of applying restriction and renaming to this composition, respectively.

In the following, we will define several derived graph algebras over finite signatures, which include the parallel composition operator. For this, we introduce the following notion:

Definition 3. A graph class (or simply class) \mathcal{C} is a (\mathcal{F}, Σ) -algebra derived from the graph algebra \mathcal{G} with \mathcal{F} and Σ finite, with $\| \in \mathcal{F}$ and $\|^{\mathcal{C}} \stackrel{\text{def}}{=} \|^{\mathcal{G}}$.

3.1 Trees

Let $\tau, l_1, l_2, \dots \in \mathbb{S}$ be source labels. The τ -source of a graph is called its *root*. We denote by \mathcal{T} the class of trees, with universe \mathbb{T} , having the signature $\mathcal{F}_{\mathcal{T}}$ defined below:

$$\mathcal{F}_{\mathcal{T}} \stackrel{\text{def}}{=} \{\text{append}_b[x_1, \dots, x_{n-1}] \mid b \in \mathbb{A}, \#b = n\} \cup \{\|, \mathbf{0}_{\{\tau\}}\}, \text{ where}$$

$$\text{append}_b(x_1, \dots, x_{n-1}) \stackrel{\text{def}}{=} \text{restrict}_{\{\tau\}}(\mathbf{b}_{(\tau, l_1, \dots, l_{n-1})} \parallel \text{rename}_{\tau \leftrightarrow l_1}(x_1) \parallel \dots \parallel \text{rename}_{\tau \leftrightarrow l_{n-1}}(x_{n-1}))$$

Here $x \leftrightarrow y$ is the permutation that switches x with y and does not change anything else. Figure 2 shows the interpretation of the function symbols from $\mathcal{F}_{\mathcal{T}}$. The set of sorts used by the class of trees is $\Sigma_{\mathcal{T}} \stackrel{\text{def}}{=} \{\tau\}$. Note that $\mathcal{F}_{\mathcal{T}}$ is finite, because the alphabet \mathbb{A} of edge labels is finite.

The standard terminology is immediately retrieved from the above definition. The vertices V_T of a tree T are called *nodes*. For an edge $e \in E_T$, labeled with a symbol of arity n , we say that $v_T(e)_1$ is the *parent* of $v_T(e)_2, \dots, v_T(e)_{n-1}$ and the latter are the *children* of $v_T(e)_1$. A

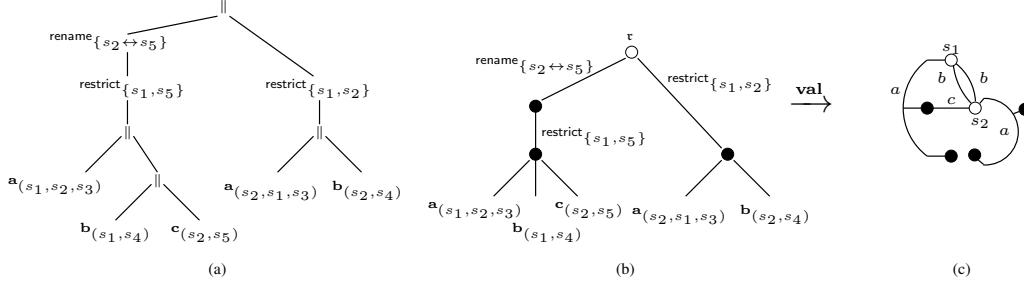


Figure 3: An \mathcal{F}_{HR} -term (a), its corresponding parse tree (b) and the canonical evaluation (c). Sources are denoted by hollow and internal vertices by solid circles.

node with no children is called a *leaf*. We denote by $T|_n$ the subtree of T rooted at some node $n \in V_T$. The *rank* of a tree is the maximum number of children of a node. A set of trees is *ranked* if the corresponding set of ranks is finite and *unranked*, otherwise. In general, the set of trees labels is unranked (over any alphabet), because each node can be attached on the first position to any number of edges. Note that the trees in this paper are *unordered* because an arbitrary number of children can be attached to a node using the \parallel operator and their order is not important.

Let \mathcal{C} be some (\mathcal{F}, Σ) -graph class \mathcal{C} . The *parse trees* of \mathcal{C} are compact representations of \mathcal{F} -terms, obtained by grouping adjacent \parallel -nodes together. Hence, we consider the following alphabet:

$$\mathbb{A}_{\mathcal{C}} \stackrel{\text{def}}{=} \{\bar{f} \mid f \in \mathcal{F} \setminus \{\parallel\}\}, \text{ where } \#\bar{f} = \#f + 1, \text{ for all } f \in \mathcal{F} \setminus \{\parallel\}$$

The class of parse trees $\mathcal{P}_{\mathcal{C}}$ is the (\mathcal{F}, Σ) -algebra derived from the tree algebra \mathcal{T} over the alphabet $\mathbb{A}_{\mathcal{C}}$, where each function symbol $f \in \mathcal{F} \setminus \{\parallel\}$ is interpreted as $\mathbf{append}_{\bar{f}}$. We denote by $\mathbf{val}_{\mathcal{C}}$ the canonical homomorphism $h_{\mathcal{P}_{\mathcal{C}}, \mathcal{C}}$ that evaluates each ground \mathcal{F} -term to the graph obtained by interpreting the function symbols in the algebra \mathcal{C} . For instance, Fig. 3 shows a \mathcal{F}_{HR} -term (a), the corresponding parse tree (b) and the result of the canonical homomorphism applied to it.

3.2 Series-Parallel Graphs

Let $\mathbb{B} \subseteq \mathbb{A}$ be an alphabet of binary edge labels and $1, 2, 3 \in \mathbb{S}$ be source labels. Note that \mathbb{B} is finite, because \mathbb{A} was assumed to be finite. We denote by \mathcal{SP} (resp. \mathcal{DSP}) the class of *oriented* (resp. *disoriented*) *series-parallel graphs*, with universe SP (resp. DSP), signature \mathcal{F}_{SP} (resp. \mathcal{F}_{DSP}) given below and sorts $\Sigma_{\text{SP}} \stackrel{\text{def}}{=} \{\{1, 2\}\}$ (resp. $\Sigma_{\text{DSP}} \stackrel{\text{def}}{=} \{\{1, 2\}\}$):

$$\mathcal{F}_{\text{SP}} \stackrel{\text{def}}{=} \{\mathbf{b}_{(1,2)} \mid b \in \mathbb{B}\} \cup \{\circ, \parallel\} \quad \mathcal{F}_{\text{DSP}} \stackrel{\text{def}}{=} \mathcal{F}_{\text{SP}} \cup \{\mathbf{b}_{(2,1)} \mid b \in \mathbb{B}\}, \text{ where}$$

$$x \circ y \stackrel{\text{def}}{=} \mathbf{rename}_{2 \leftrightarrow 3}(\mathbf{restrict}_{\{1,3\}}(x \parallel \mathbf{rename}_{[1,2,3]}(y)))$$

Here $[1, 2, 3]$ is the permutation mapping $1 \rightarrow 2$, $2 \rightarrow 3$ and $3 \rightarrow 1$, which acts as the identity everywhere else. We use the more standard infix notation $x \circ y$ instead of $\circ(x, y)$. Figure 4 (a) shows the interpretation of the function symbols from \mathcal{F}_{SP} .

Definition 4. A graph $G \in \text{DSP}$ (resp. SP) is said to be \circ -atomic (resp. \parallel -atomic) if there are no graphs $G_1, G_2 \in \text{DSP}$ (resp. SP), such that $G = G_1 \circ^{SP} G_2$ (resp. $G = G_1 \parallel^{SP} G_2$).

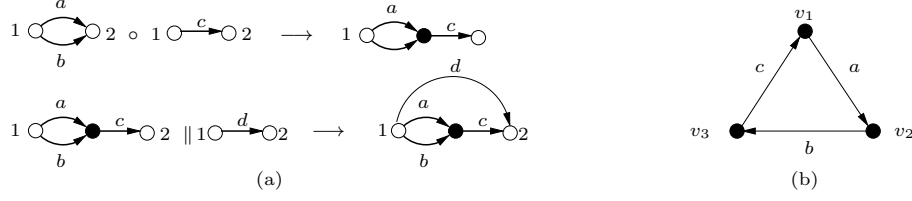


Figure 4: Construction of series-parallel graphs (a). A disoriented series-parallel graph (b). Sources are denoted by hollow and internal vertices by solid circles.

For example, the graph in the top-right of Figure 4 is \parallel -atomic and the graph in the bottom-right is \circ -atomic. Clearly, each graph $G \in \text{DSP}$ (resp. SP) is either \circ -atomic or \parallel -atomic. Moreover, each series-parallel graph has a unique decomposition:

Lemma 3 (Lemma 6.3 in [9]). *Let $G \in \text{SP}$. Then, the following hold:*

1. *If G is not \circ -atomic then there exists a unique sequence G_1, \dots, G_k , for $k \geq 2$, of \circ -atomic subgraphs of G , such that $G = G_1 \circ^{\text{SP}} \dots \circ^{\text{SP}} G_k$,*
2. *If G is \circ -atomic then either G is a single edge, or there exists a unique nonempty set $\{G_1, \dots, G_k\}$, for $k \geq 2$, of \parallel -atomic subgraphs of G , such that $G = G_1 \parallel^{\text{SP}} \dots \parallel^{\text{SP}} G_k$.*

Note that the above does not apply to disoriented series-parallel graphs, e.g., the triangle in Figure 4 (b) can be equally decomposed as $(\mathbf{a}_{(1,2)} \circ \mathbf{b}_{(1,2)}) \parallel \mathbf{c}_{(2,1)}$, $(\mathbf{b}_{(1,2)} \circ \mathbf{c}_{(1,2)}) \parallel \mathbf{a}_{(2,1)}$ or $(\mathbf{c}_{(1,2)} \circ \mathbf{a}_{(1,2)}) \parallel \mathbf{b}_{(2,1)}$.

3.3 Graphs of Bounded Tree-Width

Let parent be a binary edge label and T be a tree whose edges are all labeled by parent . A set of nodes $C \subseteq V_T$ is *connected* in T iff between any two nodes in C there exists an undirected path of edges from E_T that traverses only nodes from C . Tree decompositions are used to formalize the notion of *tree-width*:

Definition 5. *A tree decomposition of a graph G is a pair (T, β) , where T is a tree with edges labeled by parent and $\beta : V_T \rightarrow \text{pow}(V_G)$ is a mapping, such that:*

1. *there exists $n \in V_T$ such that all sources of G belong to $\beta(n)$,*
2. *for each $e \in E_G$ there exists $n \in V_T$ such that $v_G(e)_i \in \beta(n)$, for all $1 \leq i \leq \#\lambda_T(e)$,*
3. *for each $v \in V_G$, the set $B_T(v) \stackrel{\text{def}}{=} \{n \in V_T \mid v \in \beta(n)\}$ is nonempty and connected in T .*

The width of the tree decomposition is $\text{wd}(T, \beta) \stackrel{\text{def}}{=} \max\{\text{card}(\beta(n)) \mid n \in V_T\} - 1$ and the tree-width of G is $\text{twd}(G) \stackrel{\text{def}}{=} \min\{\text{wd}(T, \beta) \mid (T, \beta) \text{ is a tree decomposition of } G\}$. The sets $\beta(n)$, $n \in V_T$ are called bags. If m is the parent of n in T , the set $\text{adh}_{T, \beta}(n) \stackrel{\text{def}}{=} \beta(m) \cap \beta(n)$ is the adhesion of n in (T, β) (by convention, $\text{adh}_{T, \beta}(n) = \emptyset$ if n is the root of T).

Note that point (1) is not needed if the graph has no sources. We define $\mathbf{G}^{\leq k} \stackrel{\text{def}}{=} \{G \in \mathbf{G} \mid \text{twd}(G) \leq k\}$.

It has been proved [10, Theorem 2.83] that $\mathbf{G}^{\leq k}$ is the domain of a class $\mathcal{G}^{\leq k}$ whose sorts are subsets of $[1, k+1]$ and whose signature is the set:

$$\begin{aligned} \mathcal{F}_{\mathcal{G}^{\leq k}} \stackrel{\text{def}}{=} & \{\mathbf{0}_\tau, \text{restrict}_\tau \mid \tau \subseteq [1, k+1]\} \cup \{\text{rename}_\alpha \mid \alpha \text{ is a } [1, k+1] \text{ - permutation}\} \\ & \cup \{\mathbf{a}_{(i_1, \dots, i_n)} \mid a \in \mathbb{A}, \#a = n, i_1, \dots, i_n \in [1, k+1]\} \end{aligned}$$

Note that $\mathcal{F}_{\mathcal{G}^{\leq k}}$ is finite, because \mathbb{A} is finite. This class has sorts $\Sigma_{\mathcal{G}^{\leq k}} \stackrel{\text{def}}{=} \text{pow}(\{1, \dots, k+1\})$.

As a remark, series-parallel graphs (either oriented or disoriented) can easily be seen to have tree-width (at most) two, because the algebra \mathcal{SP} is derived from the algebra $\mathcal{G}^{\leq 2}$.

4 Counting Monadic Second Order Logic

The *Counting Monadic Second Order Logic* (CMSO) is the set of formulæ written using the individual $\mathcal{V} = \{x, y, \dots\}$ and set variables $\mathcal{X} = \{X, Y, \dots\}$ according to the following syntax:

$$\psi := x = y \mid a(x, y_1, \dots, y_{\#a}), a \in \mathbb{A} \mid X(x) \mid \text{card}_{q,p}(X) \mid \psi_1 \wedge \psi_2 \mid \neg\psi \mid \exists x . \psi \mid \exists X . \psi$$

where $p, q \in \mathbb{N}$ are constants, such that $p \in [0, q-1]$. By MSO we denote the subset of CMSO consisting of formulæ that do not contain atomic propositions of the form $\text{card}_{q,p}(X)$, also called *cardinality constraints*. A variable is *free* in a formula ϕ if it does not occur in the scope of a quantifier. A *sentence* is a formula with no free variables.

The semantics of CMSO is defined in terms of a satisfaction relation $G \models^{\mathfrak{s}} \psi$, where G is a graph and $\mathfrak{s} : \mathcal{V} \cup \mathcal{X} \rightarrow V_G \cup E_G \cup \text{pow}(V_G \cup E_G)$ is a variable assignment that maps each individual variable into a vertex or an edge, respectively each set variable into a set of vertices and edges from G . This relation is defined inductively on the structure of formulæ (we omit the semantics for the boolean operators as they are standard):

$$\begin{array}{lll} G \models^{\mathfrak{s}} x = y & \iff & \mathfrak{s}(x) = \mathfrak{s}(y) \\ G \models^{\mathfrak{s}} a(x, y_1, \dots, y_n) & \iff & \mathfrak{s}(x) \in E_G, \lambda_G(\mathfrak{s}(x)) = a \text{ and } v_G(\mathfrak{s}(x)) = \langle \mathfrak{s}(y_1), \dots, \mathfrak{s}(y_n) \rangle \\ G \models^{\mathfrak{s}} X(x) & \iff & \mathfrak{s}(x) \in \mathfrak{s}(X) \\ G \models^{\mathfrak{s}} \text{card}_{q,p}(X) & \iff & \text{card}(\mathfrak{s}(X)) = p + kq, \text{ for some } k \in \mathbb{N} \\ G \models^{\mathfrak{s}} \exists x . \psi & \iff & G \models^{\mathfrak{s}[x \leftarrow u]} \psi, \text{ for some } u \in V_G \cup E_G \\ G \models^{\mathfrak{s}} \exists X . \psi & \iff & G \models^{\mathfrak{s}[X \leftarrow U]} \psi, \text{ for some } U \subseteq V_G \cup E_G \end{array}$$

If ϕ is a sentence, the satisfaction relation does not depend on \mathfrak{s} and we write $G \models \phi$ instead of $G \models^{\mathfrak{s}} \phi$. A set of graphs \mathcal{L} is (C)MSO-definable if there exists a (C)MSO sentence ϕ such that $\mathcal{L} = \{G \mid G \models \phi\}$.

Theorem 2 (Theorem 4.4 in [8]). *Each CMSO-definable set of graphs is recognizable in \mathcal{G} .*

A *transduction* is a relation $\delta \subseteq \mathbf{G} \times \mathbf{G}$. We consider a class of transductions that first initialize given set variables X_1, \dots, X_n called *parameters*, then produce k disjoint copies of the input graph, for some constant $k \geq 1$, and finally define the output graph from the k copies, based on the valuation of the parameters, using MSO formulæ. Formally, a *k-copying transduction scheme* (resp. *copyless*, if $k = 1$) is a tuple of MSO formulæ:

$$\Theta = \langle \varphi, \{\psi_i^{\text{vert}}\}_{i \in [1, k]}, \{\psi_i^{\text{edge}}\}_{i \in [1, k]}, \{\theta_{(a, i_1, \dots, i_{\#a})}\}_{a \in \mathbb{A}, i_1, \dots, i_{\#a} \in [1, k]} \rangle, \text{ where:}$$

- $\varphi(X_1, \dots, X_n)$ defines for which input graphs the transduction has an output, and constrains the set parameters X_1, \dots, X_n ,
- $\psi_i^{\text{vert}}(x, X_1, \dots, X_n)$ (resp. $\psi_i^{\text{edge}}(x, X_1, \dots, X_n)$) defines the set of vertices (resp. edges) of the output graph taken from the i -th copy, and
- $\theta_{(a, i, j_1, \dots, j_{\#a})}(x, y_1, \dots, y_{\#a}, X_1, \dots, X_n)$ defines the incidence relation for the edges labeled by $a \in \mathbb{A}$, taken from the i -th copy and vertices from the $j_1, \dots, j_{\#a}$ -th copies of the input graph, respectively.

Formally, given a graph G and an assignment \mathfrak{s} as above, the output of the transduction associated with G and \mathfrak{s} is the graph $\text{def}_{\Theta}^{\mathfrak{s}}(G)$, defined iff $G \models^{\mathfrak{s}} \varphi$, where:

$$\begin{aligned} V_{\text{def}_{\Theta}^{\mathfrak{s}}(G)} &\stackrel{\text{def}}{=} \{u \in V_G \cup E_G \mid G \models^{\mathfrak{s}[x \leftarrow u]} \psi_i^{\text{vert}}, i \in [1, k]\} \\ E_{\text{def}_{\Theta}^{\mathfrak{s}}(G)} &\stackrel{\text{def}}{=} \{u \in V_G \cup E_G \mid G \models^{\mathfrak{s}[x \leftarrow u]} \psi_i^{\text{edge}}, i \in [1, k]\} \\ v_{\text{def}_{\Theta}^{\mathfrak{s}}(G)}(e) &\stackrel{\text{def}}{=} \langle u_1, \dots, u_{\#a} \rangle \iff \\ &G \models^{\mathfrak{s}[x \leftarrow e, y_1 \leftarrow u_1, \dots, y_{\#a} \leftarrow u_{\#a}]} \bigvee_{i, j_1, \dots, j_{\#a} \in [1, k]} \theta_{(a, i, j_1, \dots, j_{\#a})}, \text{ for all } a \in \mathbb{A} \end{aligned}$$

We require that $v_{\text{def}_{\Theta}^{\mathfrak{s}}(G)}$ be a function from edges to tuples of vertices, for each assignment \mathfrak{s} , and let $\text{def}_{\Theta}(G) \stackrel{\text{def}}{=} \{\text{def}_{\Theta}^{\mathfrak{s}} \mid \text{variable assignment } \mathfrak{s}\}$. A transduction δ is (C)MSO-*definable* if there exists a (C)MSO transduction scheme Θ such that $\delta = \text{def}_{\Theta}$. The main property of definable transductions is the *Backwards Translation Theorem*:

Theorem 3 (Theorem 1.40 in [10]). *If \mathcal{L} is an (C)MSO-definable set of graphs and δ is an MSO-definable transduction then the set $\delta^{-1}(\mathcal{L})$ is (C)MSO-definable.*

The following property is a direct consequence of the above theorem:

Proposition 1. *The composition of definable transductions is definable.*

In the following, we consider some graph class \mathcal{C} . We recall that $\mathbf{val}_{\mathcal{C}}$ denotes the canonical homomorphism between the algebra of parse trees $\mathcal{P}_{\mathcal{C}}$ and the algebra \mathcal{C} . We say that \mathcal{C} is *parsable* iff there exists a definable transduction $\pi \subseteq \mathbf{val}_{\mathcal{C}}^{-1}$ with $\text{dom}(\pi) = \mathcal{C}$. Parsable classes of graphs enjoy the equivalence between recognizability and CMSO-definability:

Theorem 4 (Theorem 4.8(2) in [9]). *Let \mathcal{C} be some parsable graph class and $\mathcal{L} \subseteq \mathcal{C}$ be a set. Then, \mathcal{L} is CMSO-definable iff \mathcal{L} is recognizable in \mathcal{C} .*

5 Regular Grammars for Graphs of Tree-Width ≤ 2

Our main goal in this paper is the definition of grammars whose languages precisely characterize the recognizable (resp. CMSO-definable) graph classes. The following theorem states that such grammars always exist for parsable graph classes (item 2):

Theorem 1. *Let \mathcal{C} be a parsable graph class. For each set $\mathcal{L} \subseteq \mathcal{C}$, the following conditions are equivalent:*

1. \mathcal{L} is CMSO-definable,
2. $\mathcal{L} = \mathcal{L}^c((\Gamma_{\mathcal{C}})^{(\mathcal{D}, E)})$, for some $(\mathcal{F}_{\mathcal{C}}, \Sigma_{\mathcal{C}})$ -recognizer (\mathcal{D}, E) , where $\Gamma_{\mathcal{C}}$ is the universal grammar for \mathcal{C} (see Lemma 2),
3. \mathcal{L} is recognizable in \mathcal{C} .

Proof (1) \Rightarrow (3) Since \mathcal{L} is CMSO-definable, \mathcal{L} is recognizable in \mathcal{G} , by Theorem 2. By Lemma 1, \mathcal{L} is recognizable in \mathcal{C} , because \mathcal{C} is a derived algebra of \mathcal{G} .

(3) \Rightarrow (2) Since \mathcal{L} is recognizable in \mathcal{C} , there exists a $(\mathcal{F}_{\mathcal{C}}, \Sigma_{\mathcal{C}})$ -recognizer (\mathcal{D}, E) such that $\mathcal{L} = h_{\mathcal{C}, \mathcal{D}}^{-1}(E)$. Since $\mathcal{C} = \mathcal{L}^c(\Gamma_{\mathcal{C}})$, by Lemma 2, we obtain $\mathcal{L} = \mathcal{C} \cap \mathcal{L} = \mathcal{L}^c(\Gamma_{\mathcal{C}}) \cap h_{\mathcal{C}, \mathcal{D}}^{-1}(E) = \mathcal{L}^c(\Gamma_{\mathcal{C}}^{(\mathcal{D}, E)})$, by Theorem 1.

(2) \Rightarrow (1) Let (\mathcal{D}, E) be a $(\mathcal{F}_{\mathcal{C}}, \Sigma_{\mathcal{C}})$ -recognizer, such that $\mathcal{L} = \mathcal{L}^g(\Gamma_{\mathcal{C}}^{(\mathcal{D}, E)})$. Since $\mathcal{C} = \mathcal{L}^c(\Gamma_{\mathcal{C}})$, by Lemma 2, we obtain $h_{\mathcal{C}, \mathcal{D}}^{-1}(E) = \mathcal{C} \cap h_{\mathcal{C}, \mathcal{D}}^{-1}(E) = \mathcal{L}^c(\Gamma_{\mathcal{C}}) \cap h_{\mathcal{C}, \mathcal{D}}^{-1}(E) = \mathcal{L}^c(\Gamma_{\mathcal{C}}^{(\mathcal{D}, E)}) = \mathcal{L}$, by Theorem 1. Hence, \mathcal{L} is recognizable. Since \mathcal{C} is parsable, we obtain that \mathcal{L} is CMSO-definable, by Theorem 4. \square

We would like to take item 2 of Theorem 1 as definition of regular grammars. However, the statement of item 2 is not constructive, and there is an inherent limitation here: given an arbitrary grammar Γ , the problem of whether $\mathcal{L}^c(\Gamma)$ is recognizable is undecidable. In fact, it is already undecidable if a given context-free word grammar defines a recognizable language [16]. In order to work around this limitation, we consider the following view on regular word grammars: the right-recursive (resp. left-recursive) word grammars can be seen as a *syntactic restriction* of the context-free grammars. We follow this idea of introducing syntactic restrictions and propose *regular* grammars for trees, series-parallel graphs, and graphs of tree-width ≤ 2 . The qualifier regular is justified because our grammars define exactly the recognizable (resp. CMSO-definable) subsets of the respective graph classes.

The rest of this section is organized as follows: We first introduce the Periodic Filtering Theorem (in subsection 5.1), which is used for establishing the completeness of the proposed regular grammars. We then give regular grammars for trees (subsection 5.2), series-parallel graphs (subsection 5.3) and graphs of tree-width at most two (subsection 5.4). Proving that our regular grammars capture the recognizable (resp. CMSO-definable) subsets follows the same principle for these three graph classes: (1) we establish that the class is parsable, (2) we provide a universal regular grammar for the class, which in conjunction with the Periodic Filtering Theorem, establishes that every recognizable (resp. CMSO-definable) can be represented by a regular grammar and (3) we construct a recognizer from the rules of the given regular grammar.

5.1 Periodic Grammars and the Periodic Filtering Theorem

Let us fix some graph class \mathcal{C} with signature \mathcal{F} and set of sorts Σ . All regular grammars defined later on share the following *periodicity* property:

Definition 6. A k -partitioned periodic grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ has non-terminals $\mathcal{N} = \mathcal{X}_1 \uplus \mathcal{Y}_1 \uplus \dots \uplus \mathcal{X}_k \uplus \mathcal{Y}_k$, for some $k \geq 1$, and rules of one of the following forms:

A. $X \rightarrow X \parallel Y^{\sharp q}$, for some $i \in [1, k]$, non-terminals $X \in \mathcal{X}_i$, $Y \in \mathcal{Y}_i$ and $q \geq 0$, where $Y^{\sharp q} \stackrel{\text{def}}{=} \underbrace{Y \parallel \dots \parallel Y}_{q \text{ times}}$; by convention, we consider that $Y^{\sharp 0} = \mathbf{0}_{\sigma(Y)}$,

B. $X \rightarrow Y_1^{\sharp q_1} \parallel \dots \parallel Y_k^{\sharp q_k}$, $X \in \mathcal{X}_i$ and $Y_1, \dots, Y_k \in \mathcal{Y}_i$,

C. $Y \rightarrow t[Z_1, \dots, Z_m]$, for some $i \in [1, k]$, $Y \in \mathcal{Y}_i$, $Z_1, \dots, Z_m \in \mathcal{N}$, where t is an \mathcal{F} -term,

D. $\rightarrow X$, for some $i \in [1, k]$, $X \in \mathcal{X}_i$.

We denote by $\text{Terms}(\Gamma)$ the terms over variables $\{x_1, \dots, x_k, y_1, \dots, y_k\}$ such that $t[v_1, \dots, v_m] \in \text{Terms}(\Gamma)$ if $t[Z_1, \dots, Z_m]$ occurs in a rule of form (C), $v_i = x_j$ if $Z_i \in \mathcal{X}_j$ and $v_i = y_j$ if $Z_i \in \mathcal{Y}_j$.

The following lemma is helpful in dealing with periodic grammars (see Lemma 1 of [6]):

Lemma 4. Let $\Gamma = (\mathcal{N}, \mathcal{R})$ be a k -partitioned periodic grammar. There is a k -partitioned periodic grammar $\Gamma' = (\mathcal{N}, \mathcal{R}')$ having $\mathcal{L}^c(\Gamma') = \mathcal{L}^c(\Gamma)$, $\text{Terms}(\Gamma') = \text{Terms}(\Gamma)$ and $\text{size}(\Gamma') \in \text{poly}(\text{size}(\Gamma))$, such that for each $i \in [1, k]$, $X \in \mathcal{X}_i$, $Y \in \mathcal{Y}_i$, there is exactly one rule $X \rightarrow X \parallel Y^{\sharp q}$ of form (A) in \mathcal{R}' .

Proof Let us fix $i \in [1, k]$, $X \in \mathcal{X}_i$ and $Y \in \mathcal{Y}_i$. Let $X \rightarrow X \parallel Y^{\#q_1}, \dots, X \rightarrow X \parallel Y^{\#q_h}$ be the rules of the form (A) of \mathcal{R} , for the given nonterminals. We assume w.l.o.g. that $q_1 < \dots < q_h$. In case of $h \geq 1$, we let $d \geq 1$ be the gcd of the integers $q_1, \dots, q_h \geq 1$. By Schur's theorem, there exists a computable integer $n \geq 1$ such that for each integer $x > n$ there exist $x_1, \dots, x_h \in \mathbb{N}$ such that $dx = q_1x_1 + \dots + q_hx_h$. We further observe that the set $M \stackrel{\text{def}}{=} \{q_1x_1 + \dots + q_hx_h \mid x_1, \dots, x_h \in \mathbb{N}\} \cap [1, dn]$ is computable. In case of $h = 0$, we set $d = 0$ and $M = \emptyset$. Otherwise, n is the Frobenius number of the co-prime numbers $\frac{q_1}{d}, \dots, \frac{q_h}{d}$. We now obtain the set of rules \mathcal{R}' from \mathcal{R} as follows:

- (a) we keep the rules $Y \rightarrow t[Z_1, \dots, Z_m]$,
- (b) we remove the rules $X \rightarrow X \parallel Y^{\#q_1}, \dots, X \rightarrow X \parallel Y^{\#q_h}$ and add the rule $X \rightarrow X \parallel Y^{\#d}$,
- (c) for every rule $X \rightarrow Y_1^{\#q_1} \parallel \dots \parallel Y_k^{\#q_k}$ such that $Y = Y_\ell$, for some $\ell \in [1, k]$, we add a rule:

$$X \rightarrow Y_1^{\#q_1} \parallel \dots \parallel Y_{\ell-1}^{\#q_{\ell-1}} \parallel Y^{\#q_\ell+m} \parallel Y_{\ell+1}^{\#q_{\ell+1}} \parallel \dots \parallel Y_k^{\#q_k}$$

for each $m \in M$.

It is immediate to show that each derivation of Γ can be simulated by a derivation of Γ' with the same outcome, and vice-versa. As we can iteratively perform the above transformation for every $i \in [1, k]$, $X \in \mathcal{X}_i$ and $Y \in \mathcal{Y}_i$, we obtain the grammar Γ' such that $\mathcal{L}^c(\Gamma') = \mathcal{L}^c(\Gamma)$ and $\text{Terms}(\Gamma') = \text{Terms}(\Gamma)$, having the stated property regarding the rules of the form (A).

It remains to prove that $\text{size}(\Gamma') \in \text{poly}(\text{size}(\Gamma))$. Consider the changes described in the previous. Clearly, step (a) does not increase the size of the grammar. Because $d \leq q_1$, step (b) does not increase the size of the grammar. In contrast, step (c) adds, for each $m \in M$ and each rule ρ of form (B), m rules of size at most $\text{size}(\rho) + m$. Since $m \leq dn \leq q_1n$, it is sufficient to prove that $n \in \text{poly}(\text{size}(\Gamma))$. However, this follows from the upper bound on the Frobenius number $n \leq (\frac{q_1}{d} - 1)(\frac{q_h}{d} - 1) - 1$, see, e.g., [14]. Then, the polynomial bound on n follows from $q_1 < q_h \leq \text{size}(\Gamma)$. \square

The next result strengthens the Filtering Theorem of Courcelle [10] (Theorem 1). Our refined version states that the intersection of the language defined by a periodic grammar and a recognizable set results into a language that is defined by a periodic grammar such that, moreover, no new terms are introduced in the right-hand sides of the rules of form (C):

Theorem 2 (Periodic Filtering Theorem). *For each k -partitioned periodic grammar Γ and recognizer (\mathcal{A}, B) there exists a k -partitioned periodic grammar Γ' such that $\text{Terms}(\Gamma') \subseteq \text{Terms}(\Gamma)$ and $\mathcal{L}^c(\Gamma') = \mathcal{L}^c(\Gamma^{(\mathcal{A}, B)})$.*

Proof Let $n \stackrel{\text{def}}{=} \text{card}(\mathcal{A})$. We consider the k -partitioned periodic grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ with non-terminals $\mathcal{N} = \mathcal{X}_1 \uplus \mathcal{Y}_1 \uplus \dots \uplus \mathcal{X}_k \uplus \mathcal{Y}_k$. By Lemma 4, we can assume that for each $i \in [1, k]$, $X \in \mathcal{X}_i$ and $Y \in \mathcal{Y}_i$, there is exactly one rule $X \rightarrow X \parallel Y^{\#q}$ of the form (A) in \mathcal{R} .

We define $\Gamma' \stackrel{\text{def}}{=} (\mathcal{N}', \mathcal{R}')$ by taking $\mathcal{N}' \stackrel{\text{def}}{=} \mathcal{X}'_1 \uplus \mathcal{Y}'_1 \uplus \dots \uplus \mathcal{X}'_k \uplus \mathcal{Y}'_k$ where for all $i \in [1, k]$ we set $\mathcal{X}'_i \stackrel{\text{def}}{=} \{X_a \mid X \in \mathcal{X}_i, a \in \mathcal{A}\}$, $\mathcal{Y}'_i \stackrel{\text{def}}{=} \{Y_a \mid Y \in \mathcal{Y}_i, a \in \mathcal{A}\}$, such that moreover $\sigma(X_a) = \sigma(X)$, $\sigma(Y_a) = \sigma(Y)$ for any $X \in \mathcal{X}_i, Y \in \mathcal{Y}_i, a \in \mathcal{A}$. We define \mathcal{R}' to contain the following rules:

- E. $X_b \rightarrow X_b \parallel Y_a^{\#qr}$, for every rule $X \rightarrow X \parallel Y^{\#q}$ in \mathcal{R} , $a, b \in \mathcal{A}$ and $r \in [1, n]$ such that $b = \underbrace{a \parallel^{\mathcal{A}} a^{\#A} \parallel^{\mathcal{A}} \dots \parallel^{\mathcal{A}} a}_{n \text{ times}}$.

F. $X_b \rightarrow \parallel_{Y \in \mathcal{Y}_i} \parallel_{a \in \mathbb{A}} Y_a^{\#q(Y,a)+r(Y,a)}$, for every $i \in [1, k]$, $X \in \mathcal{X}_i$, $b \in \mathbb{A}$, where $q(Y, a) \in \mathbb{N}$ and $0 \leq r(Y, a) < n$ are integers such that:

- there exist rules $X \rightarrow \parallel_{Y \in \mathcal{Y}_i} Y^{\#q(Y)}$ and $X \rightarrow X \parallel Y^q$ in \mathcal{R} , with $\sum_{a \in \mathbb{A}} q(Y, a) = q(Y)$ and $\sum_{a \in \mathbb{A}} r(Y, a) = kq$ for some $k \in \mathbb{N}$, and
- $b = \parallel_{Y \in \mathcal{Y}_i} \parallel_{a \in \mathbb{A}} a^{\#q(Y,a)+r(Y,a)}$.

G. $Y_b \rightarrow t[(Z_1)_{a_1}, \dots, (Z_m)_{a_m}]$ for every rule $Y \rightarrow t[Z_1, \dots, Z_m]$ in \mathcal{R} and $b, a_1, \dots, a_m \in \mathbb{A}$ such that $b = t^{\mathbb{A}}[a_1, \dots, a_m]$,

H. $\rightarrow X_b$ for any rule $\rightarrow X$ in \mathcal{R} and $b \in B$.

First, it is an easy check that Γ' is indeed a k -partitioned periodic grammar, for the partitioning $\mathcal{X}'_1 \uplus \mathcal{Y}'_1 \uplus \dots \uplus \mathcal{X}'_k \uplus \mathcal{Y}'_k$ of \mathcal{N}' , and moreover satisfies $\text{Terms}(\Gamma') \subseteq \text{Terms}(\Gamma)$ by construction.

Second, the set of \parallel -terms that can be derived from a non-terminal X_b in $\Gamma^{(\mathbb{A}, B)}$ using the refined rules of form (A) and (B) of Γ is equal to:

$$\left\{ \begin{array}{l} \parallel_{Y \in \mathcal{Y}_i} \parallel_{a \in \mathbb{A}} Y_a^{\#q(Y,a)+r(Y,a)+f(Y,a) \cdot e(Y)} \mid \\ q(Y, a) \in \mathbb{N}, 0 \leq r(Y, a) < n, f(Y, a) \in \mathbb{N}, \\ X \rightarrow \parallel_{Y \in \mathcal{Y}_i} Y^{\#q(Y)} \in \mathcal{R} \text{ rule of form (B) and } \sum_{a \in \mathbb{A}} q(Y, a) = q(Y) \text{ for all } Y \in \mathcal{Y}_i, \\ X \rightarrow X \parallel Y^{\#e(Y)} \in \mathcal{R} \text{ rule of form (A) and } \sum_{a \in \mathbb{A}} r(Y, a) = k \cdot e(Y) \text{ for some } k \in \mathbb{N} \text{ for all } Y \in \mathcal{Y}_i, \\ b = \parallel_{Y \in \mathcal{Y}_i} \parallel_{a \in \mathbb{A}} a^{\#q(Y,a)+r(Y,a)+f(Y,a) \cdot e(Y)} \end{array} \right\}$$

Any term in the set above is necessarily obtained by first applying a refined (terminal) rule of form (B) followed by an arbitrary number of refined (pumping) rules of the form (A) from Γ . With the notations above, the refined terminal rule generates the sub-term $\parallel_{Y \in \mathcal{Y}_i} \parallel_{a \in \mathbb{A}} Y_a^{\#q(Y,a)+r(Y,a)}$ given the specific constraints on the existence of the rule, the choice of refinement, etc. The additional refined pumping rules add the sub-terms $Y_a^{\#f(Y,a) \cdot e(Y)}$ with the additional constraints on the existence and the refinement of such rules. The rules of form (E) and (F) of Γ' are defined such that they produce exactly the same set of \parallel -terms as above, for the non-terminal X_a . Actually, all *small* terms, that is, where $q(Y, a) + r(Y, a) < q(Y) + n$ are included by construction in terminal rules of form (F). Any other *big* terms, that is, where $q(Y, a) + r(Y, a) \geq q(Y) + n$ can be constructed then using the pumping rules of form (E) and some terminal rule.

Third, note that the rules of form (G) are exactly the rules in $\Gamma^{(\mathbb{A}, B)}$ obtained by refinement of the rules of form (C) of Γ . Therefore, by using the fact above, the two sets of \mathcal{F} -terms generated from the same non-terminal Z_a in $\Gamma^{(\mathbb{A}, B)}$ and Γ' respectively, lead to the same sets of parse trees for Z_a in $\Gamma^{(\mathbb{A}, B)}$ and Γ' . Moreover, $\Gamma^{(\mathbb{A}, B)}$ and Γ' have the same set of axioms of form (H), and that concludes the proof. \square

5.2 Trees

We recall that the class \mathcal{T} has signature $\mathcal{F}_{\mathcal{T}} = \{\text{append}_b \mid b \in \mathbb{A}\} \cup \{\parallel, \mathbf{0}_{\{\mathbf{r}\}}\}$, for some finite alphabet \mathbb{A} of edge labels, and set of sorts $\Sigma_{\mathcal{T}} = \{\mathbf{r}\}$, where the \mathbf{r} -source of a tree is its root, and the universe of \mathcal{T} consists of all unordered and unranked trees with edges labels in \mathbb{A} . The domain of trees is denoted \mathbb{T} (subsection 3.1).

Proposition 1. *The class of tress \mathcal{T} is a parsable graph class.*

Proof We consider a copyless transduction scheme that maps any edge $e \in E_T$ of a tree T , having label $\lambda_T(e) = b$ of arity $\#b = n$, into an edge labeled with \mathbf{append}_b , with the same vertices attached in the same order. Note that the first vertex $v_T(e)_1$ attached to e is the parent of $v_T(e)_2, \dots, v_T(e)_n$, for each edge $e \in E_T$. The output is a parse tree t such that $\mathbf{val}_{\mathcal{T}}(t) = T$. \square

Definition 7. *A regular tree grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ has nonterminals partitioned as $\mathcal{N} = \mathcal{U} \uplus \mathcal{W}$ and rules of one of the forms:*

- (A) $W \rightarrow \mathbf{append}_b[U_1, \dots, U_{\#b-1}]$, where $b \in \mathbb{A}$, $W \in \mathcal{W}$, $U_1, \dots, U_{n-1} \in \mathcal{U}$,
- (B) $U \rightarrow U \parallel W^{\#q}$, where $U \in \mathcal{W}$, $W \in \mathcal{W}$ and $q \in \mathbb{N}$,
- (C) $U \rightarrow W_1^{\#q_1} \parallel \dots \parallel W_n^{\#q_n}$, where $U \in \mathcal{U}$, $W_1, \dots, W_n \in \mathcal{W}$ are distinct and $q_1, \dots, q_n \geq 1$,
- (D) $\rightarrow U$, where $U \in \mathcal{U}$.

Note that any regular tree grammar is a 1-partitioned periodic grammar (Definition 6). Below, we show that \mathbb{T} is the language of a regular tree grammar, i.e, we establish the existence of a universal regular grammar for the class of trees:

Lemma 5. *Let $\Gamma_{\mathcal{T}} \stackrel{\text{def}}{=} (\{U, W\}, \mathcal{R})$ be the regular tree grammar having the following rules:*

$$\begin{array}{ll} \rightarrow U & U \rightarrow U \parallel W \\ W \rightarrow \mathbf{append}_b(U, \dots, U), b \in \mathbb{A} & U \rightarrow \mathbf{0}_{\{\tau\}} \end{array}$$

Then, we have $\mathcal{L}^{\mathcal{T}}(\Gamma_{\mathcal{T}}) = \mathbb{T}$.

Proof “ \subseteq ” Obvious, as $\Gamma_{\mathcal{T}}$ uses only operations from $\mathcal{F}_{\mathcal{T}}$. “ \supseteq ”. Let $T \in \mathbb{T}$ be a tree. Let t be an $\mathcal{F}_{\mathcal{T}}$ -term with $t^{\mathcal{T}} = T$. By a straight-forward induction on the structure of t we can then establish that $T \in \mathcal{L}^{\mathcal{T}}(\Gamma_{\mathcal{T}})$. \square

Let $\Gamma = (\mathcal{U} \uplus \mathcal{W}, \mathcal{R})$ be an arbitrary regular tree grammar. We build a recognizer (\mathcal{A}, B) such that $\mathcal{L}^{\mathcal{T}}(\Gamma) = h_{\mathcal{T}, \mathcal{A}}^{-1}(B)$. To define the finite algebra \mathcal{A} , we require a few auxiliary notions. First, we assume w.l.o.g. that, for each pair $(U, W) \in \mathcal{U} \times \mathcal{W}$, there exists at most one rule $U \rightarrow U \parallel W^q \in \mathcal{R}$, see Lemma 4. For each $W \in \mathcal{W}$, we define:

$$\begin{aligned} b(W) &\stackrel{\text{def}}{=} 1 + \max\{q_i \mid U \rightarrow W_1^{\#q_1} \parallel \dots \parallel W_n^{\#q_n} \in \mathcal{R}, W_i = W\} \\ p(W) &\stackrel{\text{def}}{=} \max\{1, \text{lcm}\{q \mid U \rightarrow U \parallel W^{\#q} \in \mathcal{R}\}\} & q(W) &\stackrel{\text{def}}{=} b(W) + p(W) \end{aligned}$$

For a multiset $m : \mathcal{W} \rightarrow \mathbb{N}$, we define the operation:

$$\lceil m \rceil(W) \stackrel{\text{def}}{=} \begin{cases} m(W), & \text{if } m(W) < b(W) \\ b(W) + (m(W) - b(W)) \bmod p(W), & \text{otherwise} \end{cases}$$

We call a multiset m *reduced*, if $m = \lceil m \rceil$. We prove the following properties of the $\lceil \cdot \rceil$ operation:

Lemma 6. *For all multi-sets m_1 and m_2 , we have that $\lceil m_1 \uplus m_2 \rceil = \lceil \lceil m_1 \rceil \uplus \lceil m_2 \rceil \rceil$.*

Proof Immediate from the definitions. \square

We denote by $U \rightsquigarrow_{\Gamma} m$ the fact that the multiset m can be derived in Γ from U only by applying rules of types (B) and (C). The grammar Γ is clear from the context and will be omitted in the following:

Lemma 7. *For each $U \in \mathcal{U}$ and multiset $m : \mathcal{W} \rightarrow \mathbb{N}$, we have $U \rightsquigarrow m \iff U \rightsquigarrow \lceil m \rceil$.*

Proof “ \Rightarrow ” Let W be a nonterminal. If $m(W) = \lceil m \rceil(W)$ there is nothing to prove, hence we assume that $m(W) > \lceil m \rceil(W)$, i.e., $\lceil m \rceil(W) = b(W) + (m(W) - b(W)) \bmod p(W)$. Then, there exists $n \in \mathbb{N}$ such that $m(W) = n \cdot q(W) + \lceil m \rceil(W)$. We cut the derivation $U \rightsquigarrow m$ in Γ by removing nk rules of the form $U \rightarrow U \parallel W^{\sharp q}$, where $p(W) = kq$. Since $m(W) > \lceil m \rceil(W)$, at least one such rule must have been used in the derivation $U \rightsquigarrow m$ and, by the definition of $p(W)$, we know that such a $k \in \mathbb{N}$ exist. This step is repeated for each $W \in \mathcal{W}$, until we obtain a derivation of $U \rightsquigarrow \lceil m \rceil$.

“ \Leftarrow ” Let W be a nonterminal. If $m(W) = \lceil m \rceil(W)$ there is nothing to prove, hence we assume that $m(W) > \lceil m \rceil(W)$, i.e., $\lceil m \rceil(W) = b(W) + (m(W) - b(W)) \bmod p(W)$. Suppose, for a contradiction, that no rule $U \rightarrow U \parallel W^{\sharp q}$ exists in Γ . In this case, $m(W) = \lceil m \rceil(W)$, contradiction. Then, let $U \rightarrow U \parallel W^{\sharp q}$ be such a rule and $k \in \mathbb{N}$ be such that $p(W) = kq$; by the definition of $p(W)$, such a $k \in \mathbb{N}$ exists. We extend the derivation $U \rightsquigarrow \lceil m \rceil$ by adding nk times this rule to it, where $n \in \mathbb{N}$ is such that $m(W) = n \cdot q(W) + \lceil m \rceil(W)$. By repeating this step for each $W \in \text{supp}(m)$, we obtain a derivation of $U \rightsquigarrow m$. \square

A multiset $m : \mathcal{W} \rightarrow \mathbb{N}$ is a *view* of a graph G if, for each $W \in \mathcal{W}$ there exist graphs $G_1^W, \dots, G_{m(W)}^W \in \mathcal{L}_W^\tau(\Gamma)$ such that $G = \parallel_{W \in \mathcal{W}} \left(G_1^W \parallel^\tau \dots \parallel^\tau G_{m(W)}^W \right)$. The *profile abstraction* is the mapping $h : \mathbf{G} \rightarrow \text{mpow}(\mathcal{W})$ defined as:

$$h(G) \stackrel{\text{def}}{=} \{ \lceil m \rceil \mid m \text{ is a view of } G \}$$

We define the $(\mathcal{F}_\tau, \Sigma_\tau)$ -algebra \mathcal{A} as follows:

- The domain \mathbf{A} consists of the reduced multisets. In particular, the domain of \mathbf{A} is finite. In more detail, there are at most $Q^{\text{card}(\mathcal{W})}$ such multisets, where $Q \stackrel{\text{def}}{=} \max\{q(W) \mid W \in \mathcal{W}\}$, hence $\text{card}(\mathbf{A}) \leq 2^{Q^{\text{card}(\mathcal{W})}}$, leading to $\text{card}(\mathbf{A}) \in 2^{\text{poly}(\text{size}(\Gamma))}$.
- The function symbols from \mathcal{F}_τ are interpreted as follows:

$$\begin{aligned} \mathbf{0}_{\{\tau\}}^{\mathbf{A}} &\stackrel{\text{def}}{=} \{\emptyset_m\} \text{ (the set containing the empty multiset)} \\ \text{append}_b^{\mathbf{A}}(a_1, \dots, a_{\#b-1}) &\stackrel{\text{def}}{=} \{ \llbracket W \rrbracket \mid W \rightarrow \text{append}_b(U_1, \dots, U_{\#b-1}) \in \mathcal{R}, \\ &\quad \forall i \in [1, \#b-1] \exists m_i \in a_i . U_i \rightsquigarrow m_i \} \\ a_1 \parallel^{\mathbf{A}} a_2 &\stackrel{\text{def}}{=} \{ \lceil m_1 \uplus m_2 \rceil \mid m_1 \in a_1, m_2 \in a_2 \} \end{aligned}$$

First, the profile abstraction refines membership in $\mathcal{L}^\tau(\Gamma)$:

Lemma 8. *Let G_1, G_2 be graphs with $h(G_1) = h(G_2)$. Then, $G_1 \in \mathcal{L}^\tau(\Gamma) \Leftrightarrow G_2 \in \mathcal{L}^\tau(\Gamma)$.*

Proof We assume $G_1 \in \mathcal{L}^\tau(\Gamma)$. Then, there is a derivation of G_1 starting with non-terminal U for some axiom $\rightarrow U$ of Γ . In particular, this derivation starts with several rules of type (B) followed by a single rule of type (C). Let m be the view that results from these rule applications. By definition of m we have $U \rightsquigarrow m$. By Lemma 7 we then have $U \rightsquigarrow \lceil m \rceil$ (*). Then, $\lceil m \rceil \in h(G_1)$. Hence, $\lceil m \rceil \in h(G_2)$. In particular, there is a view m' of G_2 with $\lceil m \rceil = \lceil m' \rceil$. By (*) and $\lceil m \rceil = \lceil m' \rceil$, we clearly also have $U \rightsquigarrow \lceil m' \rceil$. Then, by Lemma 7, we have $U \rightsquigarrow m'$. Because $\rightarrow U$ is an axiom of Γ , we get that $G_2 \in \mathcal{L}^\tau(\Gamma)$. The other direction is symmetric. \square

Second, the profile abstraction is a homomorphism between \mathcal{T} and \mathcal{A} . Because both algebras, i.e., \mathcal{T} and \mathcal{A} , are representable, this means that h and $h_{\mathcal{T}, \mathcal{A}}$ are the same homomorphism:

Lemma 9. *The profile abstraction is the homomorphism between the algebras \mathcal{T} and \mathcal{A} .*

Proof It is sufficient that h is a homomorphism between \mathcal{T} and \mathcal{A} , because this homomorphism is unique. To this end, we prove the following points:

$h(\mathbf{0}_{\{\tau\}}^{\mathcal{T}}) = \mathbf{0}_{\{\tau\}}^{\mathcal{A}}$: the graph having one vertex and no edges cannot be split into graphs that can be derived starting from some $W \in \mathcal{W}$.

$h(G_1) \parallel^{\mathcal{A}} h(G_2) = h(G_1 \parallel^{\mathcal{T}} G_2)$, for all $G_1, G_2 \in \mathbb{G}$: “ \subseteq ” Let $[m_1] \in h(G_1)$ and $[m_2] \in h(G_2)$, for some views m_1 and m_2 of G_1 and G_2 , respectively. Then, $m_1 \uplus m_2$ is a view of $G_1 \parallel^{\mathcal{T}} G_2$. Hence, $[m_1 \uplus m_2] \in h(G_1 \parallel^{\mathcal{T}} G_2)$. By Lemma 6, we have that $[m_1 \uplus m_2] = [[m_1] \uplus [m_2]] = h(G_1) \parallel^{\mathcal{A}} h(G_2)$. Thus, $h(G_1) \parallel^{\mathcal{A}} h(G_2) \in h(G_1 \parallel^{\mathcal{T}} G_2)$. “ \supseteq ” Let $[m] \in h(G_1 \parallel^{\mathcal{T}} G_2)$, for some view m of $G_1 \parallel^{\mathcal{T}} G_2$. Then, there are multisets m_1, m_2 with $m_1 \uplus m_2 = m$ such that m_1 (resp. m_2) is a view of G_1 (resp. G_2). Hence, $[m_1] \in h(G_1)$ (resp. $[m_2] \in h(G_2)$). By definition of the operation $\parallel^{\mathcal{A}}$ we get that $[[m_1] \uplus [m_2]] \in h(G_1) \parallel^{\mathcal{A}} h(G_2)$. By Lemma 6, we have that $[m_1 \uplus m_2] = [[m_1] \uplus [m_2]]$. Thus, $[m] = [m_1 \uplus m_2] \in h(G_1) \parallel^{\mathcal{A}} h(G_2)$.

$h(\text{append}_b^{\mathcal{T}}(G_1, \dots, G_{\#b-1})) = \text{append}_b^{\mathcal{A}}(h(G_1), \dots, h(G_{\#b-1}))$, for all $b \in \mathbb{A}$ and $G_1, \dots, G_{\#b-1} \in \mathbb{G}$: “ \subseteq ” Let $m \in h(\text{append}_b^{\mathcal{T}}(G_1, \dots, G_{\#b-1}))$ be a multiset. We note that we must have $m = \{\!\!\{W\}\!\!\}$ for some $W \in \mathcal{W}$, because $G \stackrel{\text{def}}{=} \text{append}_b^{\mathcal{T}}(G_1, \dots, G_{\#b-1})$ cannot be written as a $\parallel^{\mathcal{T}}$ -composition in this case. By definition of the profile abstraction, there must be a derivation of G starting with non-terminal W . This derivation must start with some rule $W \rightarrow \text{append}_b^{\mathcal{T}}(U_1, \dots, U_{\#b-1})$ of type (A), for some non-terminals $U_1, \dots, U_{\#b-1}$. Then, the derivation continues for each U_i with several rules of type (B) followed by a single rule of type (C). Let m_i be the multi-sets resulting from these rule applications. Clearly, we have $U_i \rightsquigarrow m_i$. Further, we get that each m_i is a view of graph G_i . Hence, $[m_i] \in h(G_i)$. By Lemma 7, we then have $U_i \rightsquigarrow [m_i]$. Then, we get by the definition of $\text{append}_b^{\mathcal{A}}$ that $m = \{\!\!\{W\}\!\!\} \in \text{append}_b^{\mathcal{A}}(h(G_1), \dots, h(G_{\#b-1}))$. “ \supseteq ” Let $m \in \text{append}_b^{\mathcal{A}}(h(G_1), \dots, h(G_{\#b-1}))$ be a multiset. By the definition of $\text{append}_b^{\mathcal{A}}$, we must have $m = \{\!\!\{W\}\!\!\}$ for some $W \in \mathcal{W}$. Hence, there are some multisets $[m_i] \in h(G_i)$ for some views m_i of G_i such that $U_i \rightsquigarrow [m_i]$, for some non-terminals U_i , and a rule $W \rightarrow \text{append}_b^{\mathcal{T}}(U_1, \dots, U_{\#b-1})$ of type (A). By Lemma 7, we then have $U_i \rightsquigarrow m_i$. In particular, we can construct a derivation of G that starts with W by using rule $W \rightarrow \text{append}_b^{\mathcal{T}}(U_1, \dots, U_{\#b-1})$ combined with the derivations $U_i \rightsquigarrow m_i$. Hence, $m = \{\!\!\{W\}\!\!\} \in h(\text{append}_b^{\mathcal{T}}(G_1, \dots, G_{\#b-1}))$. \square

The main result of this subsection is that regular tree grammars define precisely the recognizable, and by Theorem 1, also the CMSO-definable, sets of trees:

Theorem 3. *A set \mathcal{L} is recognizable for the class \mathcal{T} if and only if there exists a regular tree grammar Γ , such that $\mathcal{L} = \mathcal{L}^{\mathcal{T}}(\Gamma)$.*

Proof “ \Rightarrow ” We consider some $(\mathcal{F}_{\mathcal{T}}, \Sigma_{\mathcal{T}})$ -algebra (\mathcal{A}, B) that recognizes $\mathcal{L} \subseteq \mathbb{T}$. By Lemma 5, there is a regular tree grammar $\Gamma_{\mathcal{T}}$ such that $\mathcal{L}^{\mathcal{T}}(\Gamma_{\mathcal{T}}) = \mathbb{T}$. Because regular tree grammars are 1-partitioned periodic grammars, we get by the Periodic Filtering Theorem (Theorem 2) that there is a 1-partitioned periodic grammar Γ with $\mathcal{L}^{\mathcal{T}}(\Gamma) = \mathcal{L}^{\mathcal{T}}(\Gamma_{\mathcal{T}}) \cap h_{\mathcal{T}, \mathcal{A}}^{-1}(B) = \mathbb{T} \cap \mathcal{L} = \mathcal{L}$. Moreover, because of $\text{Terms}(\Gamma) \subseteq \text{Terms}(\Gamma_{\mathcal{T}})$ we obtain that Γ is a regular tree grammar.

“ \Leftarrow ” We consider the profile abstraction h and the algebra \mathcal{A} defined above. By Lemma 9, h is the unique homomorphism between \mathcal{T} and \mathcal{A} . By Lemma 8, we get that $\mathcal{L}^{\mathcal{T}}(\Gamma) = h^{-1}(B) =$

$h_{\mathcal{T}, \mathcal{A}}^{-1}(B)$, for $B \stackrel{\text{def}}{=} \{h(G) \mid G \in \mathcal{L}^{\tau}(\Gamma)\}$. Hence, (\mathcal{A}, B) is a recognizer for $\mathcal{L}^{\tau}(\Gamma) = \mathcal{L}$. \square

5.3 Series-Parallel Graphs

We recall that the class of oriented series-parallel graphs \mathcal{SP} (resp. disoriented series-parallel graphs \mathcal{DSP}) has signature $\mathcal{F}_{\mathcal{SP}} = \{\mathbf{b}_{(1,2)} \mid b \in \mathbb{B}\} \cup \{\circ, \parallel\}$ (resp. $\mathcal{F}_{\mathcal{DSP}} = \mathcal{F}_{\mathcal{SP}} \cup \{\mathbf{b}_{(2,1)} \mid b \in \mathbb{B}\}$) and set of sorts $\Sigma_{\mathcal{T}} = \{1, 2\}$. The domain of series-parallel graphs (resp. disoriented series-parallel graphs) is denoted by \mathcal{SP} (resp. \mathcal{DSP}) ([subsection 3.2](#)).

Proposition 2. *The classes of series-parallel \mathcal{SP} and disoriented series-parallel \mathcal{DSP} graphs are parsable graph classes.*

Proof The class \mathcal{SP} is parsable, by [9, Theorem 6.10] and let $\delta_{\mathcal{SP}}$ be the MSO-definable transduction from the proof of this theorem. To establish the same for \mathcal{DSP} , let $\delta_{\mathcal{D}}$ be the copy-less MSO-transduction having one parameter X_1 , that contains zero or more (binary) edges from the input graph and produces an output graph in which these edges are reversed, i.e., for each $e \in E_G \cap \mathfrak{s}(X_1)$, where G is the input graph, the output graph G' will have $v_{G'}(e) = \langle v_G(e)_2, v_G(e)_1 \rangle$. Then $\delta_{\mathcal{SP}} \circ \delta_{\mathcal{D}}$ is a parsing function for \mathcal{DSP} . By Proposition 1, this function is MSO-definable. \square

Definition 8. *A regular series-parallel grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ has nonterminals, all of sort $\{1, 2\}$, partitioned as $\mathcal{N} = \mathcal{S} \uplus \mathcal{P}$ and rules of one of the forms:*

- (A) $S_1 \rightarrow P \circ S_2$, where $S_1, S_2 \in \mathcal{S}$ and $P \in \mathcal{P}$,
- (B) $S \rightarrow P_1 \circ P_2$, where $S \in \mathcal{S}$ and $P_1, P_2 \in \mathcal{P}$,
- (C) $S_1 \rightarrow \mathbf{b}_{(1,2)} \circ S_2$, where $S_1, S_2 \in \mathcal{S}$, $P \in \mathcal{P}$, and $b \in \mathbb{B}$,
- (D) $S \rightarrow \mathbf{b}_{(1,2)} \circ P$, where $S \in \mathcal{S}$, $P \in \mathcal{P}$, and $b \in \mathbb{B}$,
- (E) $S \rightarrow \mathbf{b}_{(1,2)}$, for some $b \in \mathbb{B}$, where $S \in \mathcal{S}$,
- (F) $P \rightarrow P \parallel S^{\sharp q}$, for some $q \in \mathbb{N}$, where $P \in \mathcal{P}$ and $S \in \mathcal{S}$,
- (G) $P \rightarrow S_1^{\sharp q_1} \parallel \dots \parallel S_k^{\sharp q_k}$, where $P \in \mathcal{P}$ and $S_1, \dots, S_k \in \mathcal{S}$ are pairwise distinct, respectively, and $q_1, \dots, q_k \geq 1$ are integers such that $\sum_{i=1}^k q_i \geq 2$,
- (H) $\rightarrow X$, where $X \in \mathcal{S} \cup \mathcal{P}$.

Note that any regular series-parallel grammar is a 1-partitioned periodic grammar (Definition 6). We prove that \mathcal{SP} is the language of a regular series-parallel grammar, i.e., we establish the existence of a universal regular series-parallel grammar:

Lemma 10. *Let $\Gamma_{\mathcal{SP}} \stackrel{\text{def}}{=} (\{\mathcal{S}, \mathcal{P}\}, \mathcal{R})$ be the regular series-parallel grammar having the following rules:*

$$\begin{array}{lll}
 \rightarrow P & S \rightarrow \mathbf{b}_{(1,2)} \circ P & S \rightarrow \mathbf{b}_{(1,2)}, b \in \mathbb{B} \\
 \rightarrow S & S \rightarrow P \circ S & P \rightarrow P \parallel S \\
 S \rightarrow \mathbf{b}_{(1,2)} \circ S & S \rightarrow P \circ P & P \rightarrow S \parallel S
 \end{array}$$

Then, we have $\mathcal{L}^{\mathcal{SP}}(\Gamma_{\mathcal{SP}}) = \mathcal{SP}$.

Proof “ \subseteq ” Each graph $G \in \mathcal{L}^{\mathcal{SP}}(\Gamma)$ is oriented series-parallel, because the definition of $\Gamma_{\mathcal{SP}}$ uses only operations from $\mathcal{F}_{\mathcal{SP}}$.

“ \supseteq ” Let $G \in \mathcal{SP}$ be an oriented series-parallel graph. By S -derivation (resp. P -derivation) of G we mean a derivation of $\Gamma_{\mathcal{SP}}$ starting in S (resp. P) and ending in a ground $\mathcal{F}_{\mathcal{SP}}$ -term t such

that $t^{S\mathcal{P}} = G$. We prove the following facts simultaneously, by induction on the decomposition of G according to Lemma 3:

- If G is a single edge or not \circ -atomic then there exists a S -derivation of G ,
- If G is not a single edge and \circ -atomic then there exists a P -derivation of G .

Consider the following cases for the induction:

$G = \mathbf{b}_{(1,2)}$ for some $b \in \mathbb{B}$: The S -derivation consists of applying the rule $S \rightarrow \mathbf{b}_{(1,2)}$ of $\Gamma_{S\mathcal{P}}$.

G is not a single edge and not \circ -atomic: in this case, we have:

$$G = \mathbf{b}^1_{(1,2)} \circ^{S\mathcal{P}} \dots \circ^{S\mathcal{P}} \mathbf{b}^\ell_{(1,2)} \circ^{S\mathcal{P}} G_{\ell+1} \circ^{S\mathcal{P}} \dots \circ^{S\mathcal{P}} G_k$$

for some $k \geq 2$, where $G_{\ell+1}, \dots, G_k$ are not single-edge and \circ -atomic. By the inductive hypothesis, there is a P -derivation of each G_i in $\Gamma_{S\mathcal{P}}$, for $i \in [\ell + 1, k]$. We distinguish the following subcases:

- $\ell = 0$: in this case, we obtain a S -derivation of G using the rule $S \rightarrow P \circ P$ once and the rule $S \rightarrow P \circ S$, $k - 2$ times (note that $k \geq 2$).
- $\ell = k$: in this case, we obtain a S -derivation of G using the rules $S \rightarrow \mathbf{b}^\ell_{(1,2)}$, $S \rightarrow \mathbf{b}^{\ell-1}_{(1,2)} \circ S$, \dots , $S \rightarrow \mathbf{b}^1_{(1,2)} \circ S$, in this order.
- $1 \leq \ell < k$: in this case, we obtain a S -derivation of G using the rules $S \rightarrow \mathbf{b}^\ell_{(1,2)}$, $S \rightarrow \mathbf{b}^{\ell-1}_{(1,2)} \circ S$, \dots , $S \rightarrow \mathbf{b}^1_{(1,2)} \circ S$, followed by $S \rightarrow P \circ P$ once and $S \rightarrow P \circ S$, $k - \ell - 1$ times.

G is \circ -atomic and not a single edge: in this case, we have:

$$G = G_1 \parallel^{S\mathcal{P}} \dots \parallel^{S\mathcal{P}} G_k$$

for some $k \geq 2$, where G_1, \dots, G_k are \parallel -atomic. Then each G_i is either a single edge or not \circ -atomic hence, by the inductive hypothesis, there is a S -derivation of each G_i in $\Gamma_{S\mathcal{P}}$, for $i \in [1, k]$. We obtain a P -derivation of G by applying the rule $P \rightarrow S \parallel S$ once and $P \rightarrow P \parallel S$, $k - 2$ times (note that $k \geq 2$). \square

Let $\Gamma = (\mathcal{S} \uplus \mathcal{P}, \mathcal{R})$ be an arbitrary regular series-parallel grammar. We build a $(\mathcal{F}_{S\mathcal{P}}, \Sigma_{S\mathcal{P}})$ -recognizer (\mathcal{A}, B) such that $\mathcal{L}^{S\mathcal{P}}(\Gamma) = h_{\mathcal{T}, \mathcal{A}}^{-1}(B)$. As in the case of trees, we can assume w.l.o.g. that, for each pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ there is at most one rule of type (F), see Lemma 4. Moreover, we assume w.l.o.g. that, for each edge label $b \in \mathbb{B}$ there exists a rule $S_b \rightarrow \mathbf{b}_{(1,2)} \in \mathcal{R}$, where $S_b \in \mathcal{S}$ is a nonterminal that does not occur in any other rule from \mathcal{R} and S_b is unique, for each $b \in \mathbb{B}$. Note that any regular series-parallel grammar can be transformed into a grammar having this property by adding at most $\text{card}(\mathbb{B})$ rules to it. Since \mathbb{B} is not considered to be part of the input to any algorithm (resp. decision problem) in the following, this increase in the size of the grammar is linear.

For each $S \in \mathcal{S}$, we define $b(S)$, $p(S)$, $q(S)$ as before, see subsection 5.2. For a multiset $m : \mathcal{S} \rightarrow \mathbb{N}$, the operation $\lceil m \rceil$ is defined as in subsection 5.2, using the definitions of $b(S)$, $p(S)$ and $q(S)$. For any $P \in \mathcal{P}$, by $P \rightsquigarrow_\Gamma m$, we denote the fact that m can be derived in Γ from P

using only rules of types (F) and (G), as given in Definition 8. As before, we omit to specify the grammar when it is clear from the context.

A multiset $m : \mathcal{S} \rightarrow \mathbb{N}$ is a *view* of a graph G if, for each $S \in \mathcal{S}$ there exist graphs $G_1^S, \dots, G_{m(S)}^S$ such that $G_1^S, \dots, G_{m(S)}^S \in \mathcal{L}_S^{\text{SP}}(\Gamma)$ and $G = \parallel_{S \in \mathcal{S}} (G_1^S \parallel^{\text{SP}} \dots \parallel^{\text{SP}} G_{m(S)}^S)$.

The *profile abstraction* $h : \text{SP} \rightarrow \text{mpow}(\mathcal{S})$ is defined as $h(G) \stackrel{\text{def}}{=} \{\lceil m \rceil \mid m \text{ is a view of } G\}$ (i.e., in the same way as in subsection 5.2). We define the algebra \mathcal{A} below:

- The domain \mathbf{A} consists reduced multisets $m : \mathcal{S} \rightarrow \mathbb{N}$. In particular, the domain of \mathbf{A} is finite. In more detail, there are at most $Q_S^{\text{card}(\mathcal{S})}$ reduced multisets $m : \mathcal{S} \rightarrow \mathbb{N}$, where $Q_S \stackrel{\text{def}}{=} \max\{q(S) \mid S \in \mathcal{S}\}$. Hence, $\text{card}(\mathbf{A}) \leq 2^{Q_S^{\text{card}(\mathcal{S})}}$, leading to $\text{card}(\mathbf{A}) \in 2^{\text{poly}(\text{size}(\Gamma))}$.
- The function symbols from \mathcal{F}_{SP} are interpreted as follows:

$$\begin{aligned} \mathbf{b}_{(1,2)}^{\mathbf{A}} &\stackrel{\text{def}}{=} \{\{\{S\}\} \mid S \rightarrow \mathbf{b}_{(1,2)} \in \mathcal{R}\}, \text{ for all } b \in \mathbb{B} \\ a_1 \circ^{\mathbf{A}} a_2 &\stackrel{\text{def}}{=} \{\{\{S\}\} \mid S \rightarrow \mathbf{b}_{(1,2)} \circ S_2 \in \mathcal{R}, \mathbf{b}_{(1,2)}^{\mathbf{A}} = a_1, \{\{S_2\}\} \in a_2\} \cup \\ &\quad \{\{\{S\}\} \mid S \rightarrow \mathbf{b}_{(1,2)} \circ P_2 \in \mathcal{R}, \mathbf{b}_{(1,2)}^{\mathbf{A}} = a_1, P_2 \rightsquigarrow m_2 \in a_2\} \cup \\ &\quad \{\{\{S\}\} \mid S \rightarrow P_1 \circ S_2 \in \mathcal{R}, P_1 \rightsquigarrow m_1 \in a_1, \{\{S_2\}\} \in a_2\} \cup \\ &\quad \{\{\{S\}\} \mid S \rightarrow P_1 \circ P_2 \in \mathcal{R}, P_1 \rightsquigarrow m_1 \in a_1, P_2 \rightsquigarrow m_2 \in a_2\} \\ a_1 \parallel^{\mathbf{A}} a_2 &\stackrel{\text{def}}{=} \{\lceil m_1 \uplus m_2 \rceil \mid m_1 \in a_1, m_2 \in a_2\} \end{aligned}$$

First, we note that Lemma 7, and hence Lemma 8, continues to hold because the definitions of \rightsquigarrow and $\lceil \cdot \rceil$ carry over to regular series-parallel grammars. Next we show that the profile abstraction is the (unique) homomorphism between the algebras SP and \mathcal{A} :

Lemma 11. *The profile abstraction is the homomorphism between the algebras SP and \mathcal{A} .*

Proof It is sufficient that h is a homomorphism between SP and \mathcal{A} , because this homomorphism is unique. We first prove the following fact, for all graphs G and edge labels $b \in \mathbb{B}$:

Fact 1. $h(G) = \mathbf{b}_{(1,2)}^{\mathbf{A}} \iff G = \mathbf{b}_{(1,2)}^{\text{SP}}$

Proof. A direct consequence of the definition of $\mathbf{b}_{(1,2)}^{\mathbf{A}}$ and the assumption that there is a rule $S_b \rightarrow \mathbf{b}_{(1,2)}$ in Γ , where S_b uniquely identifies the edge label b . \square

The proof of the statement requires showing the following points:

$h(\mathbf{b}_{(1,2)}^{\text{SP}}) = \mathbf{b}_{(1,2)}^{\mathbf{A}}$, for all $b \in \mathbb{B}$: by Fact 1.

$h(G_1) \parallel^{\mathbf{A}} h(G_2) = h(G_1 \parallel^{\text{SP}} G_2)$, for all $G_1, G_2 \in \text{SP}$: this case is identical to the case of \parallel -composition from Lemma 9. In particular, the definition of $\parallel^{\mathbf{A}}$ is identical and Lemma 6 carries over from trees to series-parallel graphs.

$h(G_1) \circ^{\mathbf{A}} h(G_2) = h(G_1 \circ^{\text{SP}} G_2)$, for all $G_1, G_2 \in \text{SP}$ of sort $\{1, 2\}$:

“ \subseteq ” We consider some $\{\{S\}\} \in h(G_1) \circ^{\mathbf{A}} h(G_2)$ (we note that all elements of $h(G_1) \circ^{\mathbf{A}} h(G_2)$ are of this shape). By the definition of $\circ^{\mathbf{A}}$, we distinguish four cases:

1. $h(G_1) = \mathbf{b}_{(1,2)}^{\mathbf{A}}$ and $S \rightarrow \mathbf{b}_{(1,2)} \circ S_2 \in \mathcal{R}$, for some $b \in \mathbb{B}$. By Fact 1, we have $G_1 = \mathbf{b}_{(1,2)}^{\text{SP}}$.

2. $h(G_1) = \mathbf{b}_{(1,2)}^A$, $P_2 \rightsquigarrow [m_2]$ and $S \rightarrow \mathbf{b}_{(1,2)} \circ P_2 \in \mathcal{R}$, for some $b \in \mathbb{B}$. By Fact 1, we have $G_1 = \mathbf{b}_{(1,2)}^{\mathcal{SP}}$. By Lemma 7, we have $P_2 \rightsquigarrow m_2$.
3. $P_1 \rightsquigarrow [m_1]$ and $S \rightarrow P_1 \circ S_2 \in \mathcal{R}$. By Lemma 7, we have $P_1 \rightsquigarrow m_1$.
4. $P_i \rightsquigarrow [m_i]$, for $i = 1, 2$ and $S \rightarrow P_1 \circ P_2 \in \mathcal{R}$. By Lemma 7, we have $P_i \rightsquigarrow m_i$, for $i = 1, 2$.

In all cases above, there is a S -derivation of Γ that witnesses the fact that $\{\{S\}\}$ is a view of $G_1 \circ^{\mathcal{SP}} G_2$, leading to $\{\{S\}\} \in h(G_1 \circ^{\mathcal{SP}} G_2)$.

“ \supseteq ” We consider some $\{\{S\}\} \in h(G_1 \circ^{\mathcal{G}} G_2)$ (we note that all elements of $h(G_1 \circ^{\mathcal{G}} G_2)$ are of this shape). We now consider some S -derivation that witnesses the fact that $\{\{S\}\}$ is a view of $G_1 \circ^{\mathcal{G}} G_2$. There are four cases to consider, based on the first rule applied on this derivation (assuming w.l.o.g. that the first rule introduces the \circ -decomposition of $G_1 \circ^{\mathcal{SP}} G_2$):

- $S \rightarrow \mathbf{b}_{(1,2)} \circ S_2$: in this case, $G_1 = \mathbf{b}_{(1,2)}^{\mathcal{SP}}$ and there is a S_2 -derivation of G_2 in Γ . Then, $\{\{S_2\}\} \in h(G_2)$. By Fact 1, we have $h(G_1) = \mathbf{b}_{(1,2)}^A$. By the first case of the definition of \circ^A , we obtain $\{\{S\}\} \in h(G_1) \circ^A h(G_2)$.
- $S \rightarrow \mathbf{b}_{(1,2)} \circ P_2$: in this case, $G_1 = \mathbf{b}_{(1,2)}^{\mathcal{SP}}$ and there is a view m_2 of G_2 , such that $P_2 \rightsquigarrow m_2$. By Lemma 7, we obtain $P_2 \rightsquigarrow [m_2]$. By Fact 1, we have $h(G_1) = \mathbf{b}_{(1,2)}^A$. By the second case of the definition of \circ^A , we obtain $\{\{S\}\} \in h(G_1) \circ^A h(G_2)$.
- $S \rightarrow P_1 \circ S_2$: in this case, there is a view m_1 of G_1 , such that $P_1 \rightsquigarrow m_1$ and $\{\{S_2\}\} \in h(G_2)$ is a view of G_2 . By Lemma 7, we obtain $P_1 \rightsquigarrow [m_1]$. Hence, $[m_1] \in h(G_1)$. Thus $\{\{S\}\} \in h(G_1) \circ^A h(G_2)$, by the third case of the definition of \circ^A .
- $S \rightarrow P_1 \circ P_2$: in this case, there are views m_i of G_i , such that $P_i \rightsquigarrow m_i$, for $i = 1, 2$. By Lemma 7, we obtain $P_i \rightsquigarrow [m_i]$, hence $[m_i] \in h(G_i)$, for $i = 1, 2$. Thus $\{\{S\}\} \in h(G_1) \circ^A h(G_2)$, by the fourth case of the definition of \circ^A .

□

The main result of this subsection is that regular series-parallel grammars define precisely the recognizable, and by Theorem 1, also the CMSO-definable, sets of series-parallel graphs:

Theorem 4. *A set \mathcal{L} is recognizable for the class \mathcal{SP} if and only if there exists a regular series-parallel grammar Γ , such that $\mathcal{L} = \mathcal{L}^{\mathcal{SP}}(\Gamma)$.*

Proof “ \Rightarrow ” We consider some $(\mathcal{F}_{\mathcal{SP}}, \Sigma_{\mathcal{SP}})$ -algebra (\mathcal{A}, B) that recognizes $\mathcal{L} \subseteq \mathcal{SP}$. By Lemma 10 there is a regular series-parallel grammar $\Gamma_{\mathcal{SP}}$ such that $\mathcal{L}^{\mathcal{SP}}(\Gamma_{\mathcal{SP}}) = \mathcal{SP}$. Moreover, $\Gamma_{\mathcal{SP}}$ is a 1-partitioned periodic grammar. By Theorem 2, there is a 1-partitioned periodic grammar Γ with language $\mathcal{L}^{\mathcal{SP}}(\Gamma) = \mathcal{L}^{\mathcal{SP}}(\Gamma_{\mathcal{SP}}) \cap h_{\mathcal{SP}, \mathcal{A}}^{-1}(B) = \mathcal{SP} \cap \mathcal{L} = \mathcal{L}$. Moreover, we obtain that $\text{Terms}(\Gamma) \subseteq \text{Terms}(\Gamma_{\mathcal{SP}})$, thus Γ is a regular tree grammar.

“ \Leftarrow ” We consider the profile abstraction h and the algebra \mathcal{A} defined above. By Lemma 11, h is the unique homomorphism between \mathcal{SP} and \mathcal{A} . By (the analogue of) Lemma 8, we get that $\mathcal{L}^{\mathcal{SP}}(\Gamma) = h_{\mathcal{SP}, \mathcal{A}}^{-1}(B)$, for $B \stackrel{\text{def}}{=} \{h(G) \mid G \in \mathcal{L}^{\mathcal{SP}}(\Gamma)\}$. Hence, (\mathcal{A}, B) is a recognizer for $\mathcal{L}^{\mathcal{SP}}(\Gamma)$. □

A (universal) regular disoriented series-parallel grammar can be defined by adding reversed edges as additional cases to the grammars in Definition 8 and Lemma 10. We state the definition

of regular disoriented series-parallel grammars in the appendix (Definition 9), and note that a $(\mathcal{F}_{\mathcal{DSP}}, \Sigma_{\mathcal{DSP}})$ -recognizer for any regular disoriented series-parallel grammar can be defined in the same way as for regular oriented series-parallel grammars. Hence, we obtain an analogous result as Theorem 4 for DSP.

Definition 9. A regular disoriented series-parallel grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ has nonterminals partitioned as $\mathcal{N} = \mathcal{S} \uplus \mathcal{P}$ and rules of one of the forms:

- (A) $S_1 \rightarrow P \circ S_2$, where $S_1, S_2 \in \mathcal{S}$ and $P \in \mathcal{P}$,
- (B) $S \rightarrow P_1 \circ P_2$, where $S \in \mathcal{S}$ and $P_1, P_2 \in \mathcal{P}$,
- (C) $S_1 \rightarrow \mathbf{b}_{(1,2)} \circ S_2$, where $S_1, S_2 \in \mathcal{S}$, $P \in \mathcal{P}$, and $b \in \mathbb{B}$,
- (D) $S \rightarrow \mathbf{b}_{(1,2)} \circ P$, where $S \in \mathcal{S}$, $P \in \mathcal{P}$, and $b \in \mathbb{B}$,
- (E) $S_1 \rightarrow \mathbf{b}_{(2,1)} \circ S_2$, where $S_1, S_2 \in \mathcal{S}$, $P \in \mathcal{P}$, and $b \in \mathbb{B}$,
- (F) $S \rightarrow \mathbf{b}_{(2,1)} \circ P$, where $S \in \mathcal{S}$, $P \in \mathcal{P}$, and $b \in \mathbb{B}$,
- (G) $S \rightarrow \mathbf{b}_{(1,2)}$, for some $b \in \mathbb{B}$, where $S \in \mathcal{S}$,
- (H) $S \rightarrow \mathbf{b}_{(2,1)}$, for some $b \in \mathbb{B}$, where $S \in \mathcal{S}$,
- (I) $P \rightarrow P \parallel S^{\sharp q}$, for some $q \in \mathbb{N}$, where $P \in \mathcal{P}$ and $S \in \mathcal{S}$,
- (J) $P \rightarrow S_1^{\sharp q_1} \parallel \dots \parallel S_k^{\sharp q_k}$, where $P \in \mathcal{P}$ and $S_1, \dots, S_k \in \mathcal{S}$ are pairwise distinct, respectively, and $q_1, \dots, q_k \geq 1$ are integers such that $\sum_{i=1}^k q_i \geq 2$,
- (K) $\rightarrow X$, where $X \in \mathcal{S} \cup \mathcal{P}$.

5.4 Graphs of Tree-Width ≤ 2

Any class of graphs defined by a bound on the tree-width of its elements is parsable. This occurs as a consequence of the seminal result of Bojanczyk and Pilipczuk [3, 4], which states the existence of a definable transduction that outputs an optimal tree decomposition of the input graph. This transduction can be composed with a definable transduction for translating any tree decomposition of a graph G into a parse tree P of an HR-grammar such that $\mathbf{val}_G(P) = G$. This leads to the following result:

Proposition 3. For each integer $k \geq 1$, the class of bounded tree-width graphs $\mathcal{G}^{\leq k}$ is a parsable class of \mathcal{G} .

Proof The existence of an MSO-definable parsing function was proved as [17, Theorem 6.9]. \square

However, the problem of defining regular grammars that capture precisely the classes $\mathcal{G}^{\leq k}$, for $k \geq 2$, is more complex. In this paper, we propose a regular grammar for graphs of tree-width ≤ 2 , and leave open the problem whether such grammars exist for $k \geq 3$. In order to define regular grammars for graphs of tree-width ≤ 2 , we do not use directly the algebra $\mathcal{G}^{\leq 2}$ (subsection 3.3). Instead, we work with a derived algebra that is based on both the tree and series-parallel algebras considered earlier. We make this choice in order to exploit the graph-theoretic properties of trees and series-parallel graphs, in particular, the fact that every graph of tree-width ≤ 2 can be decomposed as a tree of blocks, where the blocks are \circ -atomic disoriented series parallel graph. This decomposition is formalized and detailed below.

We make the following simplifying assumptions, without loss of generality:

1. We work with an alphabet \mathbb{B} of binary edge labels only (as in the case of series-parallel graphs). Note that edges of arity 3 can be encoded as cliques of size 3 (since we assume graphs of tree-width ≤ 2 , no edges of arity more than 3 exist), whereas a unary edge e attached to a vertex v can be encoded by a binary edge attached \bar{e} to v on the first position and to a fresh vertex \bar{v} on the second position, such that \bar{v} is attached only to \bar{e} .

2. We do not consider graphs with self-loops. Each edge attached to the same vertex on both first and second positions can be encoded in a way similar to the encoding of unary edges, described above.
3. We only consider connected graphs. This is in line with our earlier treatment of trees and series-parallel graphs. However, the generalization of regular grammars and the main result of this section (Theorem 5) to disconnected graphs of tree-width at most 2 is immediate and is considered for an extended version of this paper.

We now introduce the $(\mathcal{F}_2, \Sigma_2)$ -algebra \mathcal{G}_2 (derived from the graph algebra \mathcal{G}) with sorts $\Sigma_2 \stackrel{\text{def}}{=} \{\{1\}, \{1, 2\}\}$ and function symbols:

$$\mathcal{F}_2 \stackrel{\text{def}}{=} \{\|, \circ, \triangleright, \text{restrict}_{\{1\}}, \mathbf{0}_{\{1\}}\} \cup \{\mathbf{b}_{(1,2)}, \mathbf{b}_{(2,1)} \mid b \in \mathbb{B}\}, \text{ where}$$

$$x \triangleright y \stackrel{\text{def}}{=} x \parallel \text{rename}_{1 \leftrightarrow 2}(y),$$

where we require x (resp. y) to be a graph of sort $\{1, 2\}$ (resp. $\{1\}$) in the above. We further require that the arguments of $\|$ are either both of sort $\{1\}$ or both of sort $\{1, 2\}$, that both arguments of \circ are of sort $\{1, 2\}$ and that the argument of $\text{restrict}_{\{1\}}$ is of sort $\{1, 2\}$. Moreover, $\mathbf{0}_{\{1\}}$ is a constant of sort $\{1\}$, and $\mathbf{b}_{(1,2)}, \mathbf{b}_{(2,1)}$ are constants of sort $\{1, 2\}$. As before, the domain \mathbb{G}_2 of the algebra \mathcal{G}_2 is the set of graphs that are values of ground \mathcal{F}_2 -terms.

We introduce several graph-theoretic notions necessary in the upcoming developments. A path π from u to v in a graph G , where $u, v \in V_G$ is an ordered set of edges $e_1, \dots, e_n \in E_G$, such that $u = v_G(e_1)_1$, $v_G(e_i)_2 = v_G(e_{i+1})_1$, for all $i \in [1, n-1]$ and $v = v_G(e_n)_2$. A graph is *connected* if there is a path between any two of its vertices. We denote the set of connected graphs of tree-width ≤ 2 by $\mathbb{G}_c^{\leq 2}$.

A *cutvertex* $w \in V_G$ is a vertex for which there exist vertices $u \neq v \in V_G \setminus \{w\}$ such that there is a path between u and v in G and no path between u and v in the graph obtained from G by removing the vertex w and all edges incident to it. A *block* of G is a maximal subgraph of G without a cutvertex (i.e., maximal w.r.t. the subgraph partial order). A block is *nontrivial* if it has more than one vertex.

Lemma 12. *Let G be a connected graph having at least two vertices. Then, G has only nontrivial blocks.*

Proof Suppose, for a contradiction, that G has a trivial block B consisting of one vertex $u \in V_G$. Then, G has another vertex $v \in V_G \setminus \{u\}$. Because G is connected, there is a path between u and v , hence there is an edge e attached to u and some other vertex w (possibly $w = v$). The graph with vertices $\{u, v\}$ and edges $\{e\}$ has no cutvertex and it subsumes B , which contradicts the fact that B is a maximal subgraph of G without a cutvertex. \square

A *block tree* of G is a tree whose vertices are the blocks and the cutvertices of G and there is a binary edge between a block B and a cutvertex c if and only if $c \in V_B$ (note that this means that block vertices and cutvertices alternate in a block tree). It is easy to see that each block tree of a graph can be obtained from any other block tree of the same graph by choosing a node as root and reorienting some of the edges.

The following lemma establishes a connection between (connected) graphs of tree-width ≤ 2 and series-parallel graphs:

Lemma 13 (Lemma 6.15 in [9]). *Let G be a graph of sort \emptyset , i.e., G has no sources. Then $G \in \mathbb{G}_c^{\leq 2}$ if and only if its blocks are either trivial or \circ -atomic disoriented series-parallel graphs.*

We are going to need the following stronger statement about the nontrivial blocks of a graph of tree-width 2 at most:

Lemma 14. *Let $G \in \mathcal{G}_c^{\leq 2}$, let B be a nontrivial block of G of sort \emptyset and let x be a vertex of B . Then, there exists another vertex y of B such that the graph $B_{(x,y)}$, i.e., B with x (resp. y) taken as first (resp. second) source, is a \circ -atomic disoriented series-parallel graph.*

Proof Let (T, β) be a tree decomposition of B such that $\text{wd}(T, \beta) = \text{tw}(B)$. Then, there is some node whose bag contains x and at least some other node y . Because B is connected and has at least two vertices, there must be an edge from x to some other vertex y and there must be some node whose bag contains both x and y . We can w.l.o.g. assume that this node is the root of T (otherwise we can reorient the edges of T such that this node becomes the root). We can further assume w.l.o.g. that the bag of the root contains exactly $\{x, y\}$ (otherwise we add a new root node to the tree, connected to the old root, whose bag contains exactly $\{x, y\}$). Further, w.l.o.g. we can choose (T, β) such that $\beta(n) \setminus \beta(m) \neq \emptyset$, for each pair (m, n) of parent and child nodes of T (if $\beta(n) \subseteq \beta(m)$ then the edge of T between m and n can be contracted). Since B has no cutvertex, it follows that $\text{card}(\text{adh}_{T,\beta}(n)) \geq 2$, for all non-root nodes $n \in V_T$. This is a consequence of the fact that the adhesion of each non-root node of T is a separator of B , see e.g. [15, Lemma 11.3]. Since $\text{card}(\beta(n)) \leq 3$, the bag of each non-root node can be uniquely decomposed into its adhesion $\{u_1, u_2\}$ and a third element $v \notin \{u_1, u_2\}$. Where no confusion arises, we shall use the names u_1, u_2 (for the adhesion) and v (for the remaining vertex) throughout the proof. Moreover, w.l.o.g. we can choose (T, β) such that $\text{adh}_{T,\beta}(n) \neq \text{adh}_{T,\beta}(m)$ for each pair (m, n) of parent and child of T ; if $\text{adh}_{T,\beta}(n) = \text{adh}_{T,\beta}(m)$, then m can be attached as a child to the parent of n . For every pair of vertices u_1, u_2 of B we define the graph $B^{\{u_1, u_2\}}$ of sort $\{1, 2\}$ as the subgraph of B consisting only of the vertices u_1 and u_2 and all the edges of B between these two vertices. For every non-root node $n \in V_T$, with $\text{adh}_{T,\beta}(n) = \{u_1, u_2\}$, we denote by $B_n^{\{u_1, u_2\}}$ the subgraph of B of sort $\{1, 2\}$, with u_1 resp. u_2 as the first resp. second source, induced by $\bigcup \{\beta(p) \mid p \text{ is a descendant of } n\}$ minus the edges of $B^{\{u_1, u_2\}}$.

Fact 2. *For every non-root node $n \in V_T$, with $\text{adh}_{T,\beta}(n) = \{u_1, u_2\}$, we have $B_n^{\{u_1, u_2\}} \in \text{DSP}$.*

Proof. By induction on the structure of the subtree $T|_n$ of T rooted at n , where $\{u_1, u_2, v\} \stackrel{\text{def}}{=} \beta(n)$ and $\{u_1, u_2\} \stackrel{\text{def}}{=} \text{adh}_{T,\beta}(n)$. Let p_1, \dots, p_k be the children of n in T . Then, for each $i \in [1, k]$, $\text{adh}_{T,\beta}(p_i)$ is either $\{u_1, v\}$ or $\{u_2, v\}$. Let $P_{u_1, v} \uplus P_{v, u_2} = \{p_1, \dots, p_k\}$ be the sets such that $\text{adh}_{T,\beta}(p) = \{u_1, v\}$ resp. $\text{adh}_{T,\beta}(p) = \{u_2, v\}$, for all $p \in P_{u_1, v}$ resp. $p \in P_{v, u_2}$. We now observe that

$$B_n^{\{u_1, u_2\}} = \left(\left(\parallel_{p \in P_{u_1, v}}^{\mathcal{G}_2} B|_p^{\{u_1, v\}} \parallel^{\mathcal{G}_2} B^{\{u_1, v\}} \right) \circ^{\mathcal{G}_2} \left(\parallel_{p \in P_{v, u_2}}^{\mathcal{G}_2} B|_p^{\{v, u_2\}} \parallel^{\mathcal{G}_2} B^{\{v, u_2\}} \right) \right).$$

□

Back to the proof, let n be the root of T , and let p_1, \dots, p_k be the children of n in T . We recall that $\beta(n) = \{x, y\}$. We now observe that:

$$B_{(x,y)} = B^{\{x,y\}} \parallel^{\mathcal{G}_2} \left(\parallel_{i \in [1, k]}^{\mathcal{G}_2} B|_{p_i}^{\{x,y\}} \right).$$

This establishes that $B_{(x,y)} \in \text{DSP}$. Now, we use the fact that B is a block (i.e., has no cutvertex) to obtain that $B_{(x,y)}$ is \circ -atomic: Assume that $B_{(x,y)}$ is not \circ -atomic. Then, by Lemma 3, there are some graphs G_1, G_2 of sort $\{1, 2\}$ with $B_{(x,y)} = G_1 \circ G_2$. In particular, the

2-source of G_1 (which is equal to the 1-source of G_2) is a cutvertex of $B_{(x,y)}$. However, this contradicts the definition of a block, which cannot contain a cutvertex. \square

We are now ready to state the regular grammars of tree-width at most 2:

Definition 10. A regular tree-width ≤ 2 grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ has nonterminals partitioned as $\mathcal{N} = \mathcal{U} \uplus \mathcal{W} \uplus \mathcal{P} \uplus \mathcal{S}$, with $\sigma(U) = \sigma(W) = \{1\}$, for all $U \in \mathcal{U}$, $W \in \mathcal{W}$ and $\sigma(P) = \sigma(S) = \{1, 2\}$, for all $P \in \mathcal{P}$, $S \in \mathcal{S}$. The rules of Γ have one of the following forms:

- (A) $W \rightarrow \text{restrict}_{\{1\}}(P \triangleright U)$, for some $W \in \mathcal{W}$, $P \in \mathcal{P}$ and $U \in \mathcal{U}$,
- (B) $U \rightarrow U \parallel W^{\sharp q}$, for some $U \in \mathcal{U}$, $W \in \mathcal{W}$ and $q \in \mathbb{N}$,
- (C) $U \rightarrow W_1^{\sharp q_1} \parallel \dots \parallel W_k^{\sharp q_k}$, for some $U \in \mathcal{U}$ and $W_1, \dots, W_k \in \mathcal{W}$ pairwise distinct and $q_1, \dots, q_k \geq 1$ integers,
- (D) $S_1 \rightarrow (P \triangleright U) \circ S_2$, for some $S_1, S_2 \in \mathcal{S}$, $P \in \mathcal{P}$ and $U \in \mathcal{U}$,
- (E) $S \rightarrow (P_1 \triangleright U) \circ P_2$, for some $S \in \mathcal{S}$, $P_1, P_2 \in \mathcal{P}$ and $U \in \mathcal{U}$,
- (F) $S_1 \rightarrow (\mathbf{b}_{(1,2)} \triangleright U) \circ S_2$, for some $S_1, S_2 \in \mathcal{S}$, $b \in \mathbb{B}$ and $U \in \mathcal{U}$,
- (G) $S \rightarrow (\mathbf{b}_{(1,2)} \triangleright U) \circ P$, for some $S \in \mathcal{S}$, $b \in \mathbb{B}$, $P \in \mathcal{P}$ and $U \in \mathcal{U}$,
- (H) $S_1 \rightarrow (\mathbf{b}_{(2,1)} \triangleright U) \circ S_2$, for some $S_1, S_2 \in \mathcal{S}$, $b \in \mathbb{B}$ and $U \in \mathcal{U}$,
- (I) $S \rightarrow (\mathbf{b}_{(2,1)} \triangleright U) \circ P$, for some $S \in \mathcal{S}$, $b \in \mathbb{B}$, $P \in \mathcal{P}$ and $U \in \mathcal{U}$,
- (J) $S \rightarrow \mathbf{b}_{(1,2)}$, for some $b \in \mathbb{B}$,
- (K) $S \rightarrow \mathbf{b}_{(2,1)}$, for some $b \in \mathbb{B}$,
- (L) $P \rightarrow P \parallel S^{\sharp q}$, for some $S \in \mathcal{S}$, $P \in \mathcal{P}$ and $q \in \mathbb{N}$,
- (M) $P \rightarrow S_1^{\sharp q_1} \parallel \dots \parallel S_k^{\sharp q_k}$, for some $P \in \mathcal{P}$ and $S_1, \dots, S_k \in \mathcal{S}$ pairwise distinct, respectively, $q_1, \dots, q_k \geq 1$ integers such that $\sum_{i=1}^k q_i \geq 2$,
- (N) $\rightarrow U$, for some $U \in \mathcal{U}$.

Note that tree-width ≤ 2 regular grammars are 2-partitioned periodic grammars (Definition 6).

In the rest of this section, we prove that the regular tree-width ≤ 2 grammars define exactly the recognizable (and, by Theorem 1, the CMSO-definable) subsets of $\mathcal{G}_c^{\leq 2}$ of sort $\{1\}$. As the first step, we establish the existence of a universal regular tree-width ≤ 2 grammar, i.e., a grammar Γ_2 with $\mathcal{L}^{\mathcal{G}_2}(\Gamma_2) = \text{restrict}_{\{1\}}^{\mathcal{G}_2}(\mathcal{G}_c^{\leq 2})$. This proof uses the decomposition of a graph into its block-tree. We will use the fact that the universal regular tree-width ≤ 2 grammar can mimic, for every nontrivial block B of the input graph, a derivation of this block by the universal DSP grammar (Lemma 13). Further, by Lemma 14, we can assign the sources of a block $B \in \text{DSP}$ such that the 1-source of B is the cutvertex that is the parent of B in the block-tree:

Lemma 15. Let $\Gamma_2 \stackrel{\text{def}}{=} (\{U, W, S, P\}, \mathcal{R})$ be the regular tree-width ≤ 2 grammar having the following rules:

$$\begin{array}{lll}
 \rightarrow U & S \rightarrow (P \triangleright U) \circ S & S \rightarrow (P \triangleright U) \circ P \\
 U \rightarrow U \parallel W & S \rightarrow (\mathbf{b}_{(1,2)} \triangleright U) \circ S, b \in \mathbb{B} & S \rightarrow (\mathbf{b}_{(1,2)} \triangleright U) \circ P \\
 U \rightarrow \mathbf{0}_{\{1\}} & S \rightarrow (\mathbf{b}_{(2,1)} \triangleright U) \circ S, b \in \mathbb{B} & S \rightarrow (\mathbf{b}_{(2,1)} \triangleright U) \circ P \\
 W \rightarrow \text{restrict}_{\{1\}}(P \triangleright U) & &
 \end{array}$$

$$\begin{array}{l}
 S \rightarrow \mathbf{b}_{(1,2)}, b \in \mathbb{B} \\
 S \rightarrow \mathbf{b}_{(2,1)}, b \in \mathbb{B} \\
 P \rightarrow P \parallel S \\
 P \rightarrow S \parallel S
 \end{array}$$

Then, we have $\mathcal{L}^{\mathcal{G}_2}(\Gamma_2) = \text{restrict}_{\{1\}}^{\mathcal{G}_2}(\mathcal{G}_c^{\leq 2})$.

Proof “ \subseteq ” Each graph $G \in \mathcal{L}^{\mathcal{G}_2}(\Gamma_2)$ has tree-width at most 2, because the grammar Γ_2 only uses operations from the algebra \mathcal{G}_2 , and \mathcal{G}_2 is derived from the algebra $\mathcal{G}^{\leq 2}$, whose universe consists exactly of all graphs of tree-width at most 2 (see [subsection 3.3](#)).

“ \supseteq ” Let $G \in \text{restrict}_{\{1\}}^{\mathcal{G}_2}(\mathcal{G}_c^{\leq 2})$ be a connected graph of tree-width at most 2 of sort $\{1\}$, and let T be its block tree. Without loss of generality (by rotating the block tree), we can assume that the block that contains the 1-source is the root of the block tree. We extend this block tree by adding the 1-source of G as the new root. We prove that $G \in \mathcal{L}_U^{\mathcal{G}_2}(\Gamma_2)$. The construction of the derivation follows the structure of the (extended) block-tree T inductively.

If G has one vertex and no edges, we obtain $G \in \mathcal{L}_U^{\mathcal{G}_2}(\Gamma_2)$ by applying the rule $U \rightarrow \mathbf{0}_{\{1\}}$ from Γ_2 . Else, by [Lemma 12](#), each block of G is nontrivial. We consider some node of the block tree consisting of a single vertex $v \in V_G$ and denote by G_v the subgraph of G induced by v and its descendants in the block tree, considered as graph of sort $\{1\}$ with 1-source v . It is sufficient to prove that $G_v \in \mathcal{L}_U^{\mathcal{G}_2}(\Gamma_2)$: the claim $G \in \mathcal{L}_U^{\mathcal{G}_2}(\Gamma_2)$ is established by choosing v as the root of the block tree, i.e., the 1-source of G . Let B_1, \dots, B_ℓ be the blocks that are the children of v in the block tree. Let $c_1^j, \dots, c_{n_j}^j$ be the vertices that are the children of B_j in T , for $j \in [1, \ell]$. By the inductive hypothesis, there are U -derivations of the subgraphs $G_{c_i^j}$ in Γ_2 , for every $1 \leq j \leq \ell$ and $1 \leq i \leq n_j$, induced by c_i^j and its descendants, considered as graphs of sort $\{1\}$ with 1-source v . By [Lemma 14](#), there is some vertex w_j of B_j , such that $(B_j)_{(v, w_j)}$, the graph B_j considered as graph of sort $\{1, 2\}$, with v (resp. w_j) as its first (resp. second) source, is a \circ -atomic disoriented series-parallel graph. By [Lemma 10](#) (analogously extended to the case disoriented series-parallel graphs) there is a derivation of the universal series-parallel grammar Γ that witnesses $(B_j)_{(v, w_j)} \in \mathcal{L}^{\mathcal{S}\mathcal{P}}(\Gamma)$. Moreover, we have $(B_j)_{(v, w_j)} \in \mathcal{L}_P^{\mathcal{S}\mathcal{P}}(\Gamma)$ because $(B_j)_{(v, w_j)}$ is \circ -atomic. From this derivation we obtain a derivation that witnesses $(B_j)_{(v, w_j)} \in \mathcal{L}_P^{\mathcal{G}_2}(\Gamma_2)$ as follows: every application of a rule $S \rightarrow P \circ S$ is replaced by the two rules $S \rightarrow (P \triangleright U) \circ S$ and $U \rightarrow \mathbf{0}_{\{1\}}$, every application of a rule $S \rightarrow P \circ P$ is replaced the two rules $S \rightarrow (P \triangleright U) \circ P$ and $U \rightarrow \mathbf{0}_{\{1\}}$, and all the other rule applications are kept the same. We now obtain a U -derivation of G_v by concatenating the rule applications $U \rightarrow U \parallel W$, $U \rightarrow \mathbf{0}_{\{1\}}$, $W \rightarrow \text{restrict}_{\{1\}}(P \triangleright U)$ and $U \rightarrow \mathbf{0}_{\{1\}}$ with the P -derivations of $(B_j)_{(v, w_j)}$, and by replacing the rules $U \rightarrow \mathbf{0}_{\{1\}}$ with the U -derivation of $G_{c_i^j}$, whenever a cutpoint c_i^j is encountered in the constructed derivation, for every $1 \leq j \leq \ell$ and $1 \leq i \leq n_j$. \square

Let $\Gamma = (\mathcal{U} \uplus \mathcal{W} \uplus \mathcal{S} \uplus \mathcal{P}, \mathcal{R})$ be a regular tree-width ≤ 2 grammar ([Definition 10](#)). As in [subsection 5.3](#), we assume w.l.o.g. that, for each pair $(P, S) \in \mathcal{P} \times \mathcal{S}$ there is at most one rule of type [\(F\)](#), see [Lemma 4](#). Moreover, we assume w.l.o.g. that, for each edge label $b \in \mathbb{B}$, there exists two rules $S_{1,2}^b \rightarrow \mathbf{b}_{(1,2)}$ and $S_{2,1}^b \rightarrow \mathbf{b}_{(2,1)}$, where $S_{1,2}^b, S_{2,1}^b \in \mathcal{S}$ are distinguished nonterminals that uniquely identify the constant symbols $\mathbf{b}_{(1,2)}$ and $\mathbf{b}_{(2,1)}$, respectively, and do not occur in any other rule from \mathcal{R} .

We define the *profile abstraction* for graphs of tree-width ≤ 2 . A *view* of a graph G of sort $\{1\}$ is a multiset $m : \mathcal{W} \rightarrow \mathbb{N}$ such that for each $W \in \mathcal{W}$ there exist graphs $G_1^W, \dots, G_{m(W)}^W \in \mathcal{L}_W^{\mathcal{G}_2}(\Gamma)$ such that $G = \parallel_{W \in \mathcal{W}} (G_1^W \parallel^{\mathcal{G}_2} \dots \parallel^{\mathcal{G}_2} G_{m(W)}^W)$. A *view* of a graph G of sort $\{1, 2\}$ is a pair of multisets (k, m) , with $k : \mathcal{S} \rightarrow \mathbb{N}$ and $m : \mathcal{W} \rightarrow \mathbb{N}$, such that for each $S \in \mathcal{S}$ there exist graphs $G_1^S, \dots, G_{k(S)}^S \in \mathcal{L}_S^{\mathcal{G}_2}(\Gamma)$ and for each $W \in \mathcal{W}$ there exist graphs $G_1^W, \dots, G_{m(W)}^W \in \mathcal{L}_W^{\mathcal{G}_2}(\Gamma)$ such that:

$$G = \left(\parallel_{S \in \mathcal{S}} \left(G_1^S \parallel^{\mathcal{G}_2} \dots \parallel^{\mathcal{G}_2} G_{k(S)}^S \right) \right) \triangleright \left(\parallel_{W \in \mathcal{W}} \left(G_1^W \parallel^{\mathcal{G}_2} \dots \parallel^{\mathcal{G}_2} G_{m(W)}^W \right) \right).$$

Then, the *profile abstraction* $h : \mathbf{G} \rightarrow (\text{mpow}(\mathcal{W}) \cup (\text{mpow}(\mathcal{S}) \times \text{mpow}(\mathcal{W})))$ is defined as:

$$h(G) \stackrel{\text{def}}{=} \begin{cases} \{\lceil m \rceil \mid m \text{ is a view of } G\}, & G \text{ is of sort } \{1\} \\ \{\lceil k \rceil, \lceil m \rceil \mid (k, m) \text{ is a view of } G\}, & G \text{ is of sort } \{1, 2\} \end{cases}$$

We define the $(\mathcal{F}_2, \Sigma_2)$ -algebra \mathcal{A} as follows:

- The domain \mathbf{A} consists of reduced multisets $m : \mathcal{W} \rightarrow \mathbb{N}$ and pairs (k, m) of reduced multisets $k : \mathcal{S} \rightarrow \mathbb{N}$, $m : \mathcal{W} \rightarrow \mathbb{N}$. In particular, the domain of \mathbf{A} is finite. In more detail, there are at most $Q_{\mathcal{W}}^{\text{card}(\mathcal{W})}$ reduced multisets $m : \mathcal{W} \rightarrow \mathbb{N}$, where $Q_{\mathcal{W}} \stackrel{\text{def}}{=} \max\{q(W) \mid W \in \mathcal{W}\}$, and at most $Q_{\mathcal{S}}^{\text{card}(\mathcal{S})}$ reduced multisets $k : \mathcal{S} \rightarrow \mathbb{N}$, where $Q_{\mathcal{S}} \stackrel{\text{def}}{=} \max\{q(S) \mid S \in \mathcal{S}\}$. Hence, $\text{card}(\mathbf{A}) \leq 2^{Q_{\mathcal{W}}^{\text{card}(\mathcal{W})}} + 2^{Q_{\mathcal{S}}^{\text{card}(\mathcal{S})}} \cdot 2^{Q_{\mathcal{W}}^{\text{card}(\mathcal{W})}}$, leading to $\text{card}(\mathbf{A}) \in 2^{2^{\text{poly}(\text{size}(\Gamma))}}$.
- The function symbols from \mathcal{F}_2 are interpreted as follows:

$$\begin{aligned} \mathbf{0}_{\{1\}}^{\mathcal{A}} &\stackrel{\text{def}}{=} \{\emptyset_m\} & a_1 \parallel^{\mathcal{A}} a_2 &\stackrel{\text{def}}{=} \{\lceil m_1 \uplus m_2 \rceil \mid m_1 \in a_1, m_2 \in a_2\}, \text{ if } \sigma(a_1) = \sigma(a_2) = \{1\} \\ \mathbf{b}_{(i,j)}^{\mathcal{A}} &\stackrel{\text{def}}{=} \{(\{\!\!\{S\}\!\!\}, \emptyset_m) \mid S \rightarrow \mathbf{b}_{(i,j)} \in \mathcal{R}\}, \text{ for all } b \in \mathbb{B} \text{ and } (i, j) \in \{(1, 2), (2, 1)\} \\ a_1 \triangleright^{\mathcal{A}} a_2 &\stackrel{\text{def}}{=} \{(k_1, \lceil m_1 \uplus m_2 \rceil) \mid (k_1, m_1) \in a_1, m_2 \in a_2\} \\ \text{restrict}_{\{1\}}^{\mathcal{A}}(a) &\stackrel{\text{def}}{=} \{\{\!\!\{W\}\!\!\} \mid W \rightarrow \text{restrict}_{\{1\}}(P \triangleright U) \in \mathcal{R}, P \rightsquigarrow k, U \rightsquigarrow m, (k, m) \in a\} \\ a_1 \circ^{\mathcal{A}} a_2 &\stackrel{\text{def}}{=} \{(\{\!\!\{S_1\}\!\!\}, m_2) \mid (i, j) \in \{(1, 2), (2, 1)\}, S_1 \rightarrow (\mathbf{b}_{(i,j)} \triangleright U) \circ S_2 \in \mathcal{R}, \\ &\quad (k_1, \emptyset_m) \in \mathbf{b}_{(i,j)}^{\mathcal{A}}, U \rightsquigarrow m_1, (k_1, m_1) \in a_1, (\{\!\!\{S_2\}\!\!\}, m_2) \in a_2\} \\ &\quad \cup \{(\{\!\!\{S_1\}\!\!\}, m_2) \mid (i, j) \in \{(1, 2), (2, 1)\}, S_1 \rightarrow (\mathbf{b}_{(i,j)} \triangleright U) \circ P_2 \in \mathcal{R}, \\ &\quad (k_1, \emptyset_m) \in \mathbf{b}_{(i,j)}^{\mathcal{A}}, U \rightsquigarrow m_1, (k_1, m_1) \in a_1, P_2 \rightsquigarrow k_2, (k_2, m_2) \in a_2\} \\ &\quad \cup \{(\{\!\!\{S_1\}\!\!\}, m_2) \mid S_1 \rightarrow (P_1 \triangleright U) \circ S_2 \in \mathcal{R}, P_1 \rightsquigarrow k_1, \\ &\quad U \rightsquigarrow m_1, (k_1, m_1) \in a_1, (\{\!\!\{S_2\}\!\!\}, m_2) \in a_2\} \\ &\quad \cup \{(\{\!\!\{S_1\}\!\!\}, m_2) \mid S_1 \rightarrow (P_1 \triangleright U) \circ P_2 \in \mathcal{R}, P_1 \rightsquigarrow k_1, \\ &\quad U \rightsquigarrow m_1, (k_1, m_1) \in a_1, P_2 \rightsquigarrow k_2, (k_2, m_2) \in a_2\} \\ a_1 \parallel^{\mathcal{A}} a_2 &\stackrel{\text{def}}{=} \{(\lceil k_1 \uplus k_2 \rceil, \lceil m_1 \uplus m_2 \rceil) \mid (k_1, m_1) \in a_1, (k_2, m_2) \in a_2\}, \text{ if } \sigma(a_1) = \sigma(a_2) \in \{\{1\}, \{1, 2\}\} \end{aligned}$$

First, we note that Lemma 7, and hence Lemma 8, continues to hold because the definitions of \rightsquigarrow and $\lceil \cdot \rceil$ carry over to tree-width ≤ 2 grammars. Next we show that the profile abstraction is the (unique) homomorphism between the algebras \mathcal{G}_2 and \mathcal{A} :

Lemma 16. *The profile abstraction is the homomorphism between the algebras \mathcal{G}_2 and \mathcal{A} .*

Proof It is sufficient that h is a homomorphism between \mathcal{G}_2 and \mathcal{A} , because this homomorphism is unique. The following fact is proved in a similar way to Fact 1:

Fact 3. $h(G) = \mathbf{b}_{(i,j)}^{\mathcal{A}} \iff G = \mathbf{b}_{(i,j)}^{\mathcal{G}}$, for $(i, j) \in \{(1, 2), (2, 1)\}$.

We prove the following points:

$h(\mathbf{0}_{\{1\}}^{\mathcal{G}_2}) = \mathbf{0}_{\{1\}}^{\mathcal{A}}$: the graph having one vertex and no edges cannot be split into graphs that can

be derived starting from some $W \in \mathcal{W}$.

$\underline{h(\mathbf{b}_{(i,j)}^{\mathcal{G}_2}) = \mathbf{b}_{(2,1)}^A}$, for all $b \in \mathbb{B}$ and $(i,j) \in \{(1,2), (2,1)\}$: by Fact 3.

$\underline{h(G_1) \triangleright^A h(G_2) = h(G_1 \triangleright^{\mathcal{G}_2} G_2)}$, for every graph $G_1 \in \mathbf{G}_2$ of sort $\{1,2\}$ and every graph $G_2 \in \mathbf{G}_2$ of sort $\{1\}$: “ \subseteq ” Let $([k], [m_1]) \in h(G_1)$ and $[m_2] \in h(G_2)$, for some view (k, m_1) of G_1 and some view m_2 of graph G_2 . Then, $(k, m_1 \uplus m_2)$ is a view of $G_1 \triangleright^{\mathcal{G}_2} G_2$. Hence, $([k], [m_1 \uplus m_2]) \in h(G_1 \triangleright^{\mathcal{G}_2} G_2)$. By Lemma 6, we have that $[m_1 \uplus m_2] = [[m_1] \uplus [m_2]]$. Hence, $([k], [[m_1] \uplus [m_2]]) \in h(G_1 \triangleright^{\mathcal{G}_2} G_2)$. “ \supseteq ” Let $([k], [m]) \in h(G_1 \triangleright^{\mathcal{G}_2} G_2)$, for some view (k, m) of $G_1 \triangleright^{\mathcal{G}_2} G_2$. Then, there are multisets m_1, m_2 with $m_1 \uplus m_2 = m$ such that (k, m_1) (resp. m_2) is a view of G_1 (resp. G_2). Hence, $([k], [m_1]) \in h(G_1)$ (resp. $[m_2] \in h(G_2)$). By definition of the operation \triangleright^A we get that $([k], [[m_1] \uplus [m_2]]) \in h(G_1) \parallel^A h(G_2)$. By Lemma 6, we have that $[m_1 \uplus m_2] = [[m_1] \uplus [m_2]]$. Thus, $([k], [m]) = ([k], [m_1 \uplus m_2]) \in h(G_1) \parallel^A h(G_2)$.

$\underline{\text{restrict}_{\{1\}}^A(h(G)) = h(\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G))}$, for every graph $G \in \mathbf{G}_2$ of sort $\{1,2\}$: “ \subseteq ” We consider some $\{\{W\}\} \in \text{restrict}_{\{1\}}^A(h(G))$ (we note that all elements of $\text{restrict}_{\{1\}}^A(h(G))$ are of this shape). Then, there is a rule $W \rightarrow \text{restrict}_{\{1\}}(P \triangleright U) \in \mathcal{R}$ and a view (k, m) of G with $P \rightsquigarrow [k]$ and $U \rightsquigarrow [m]$. By Lemma 7 we then have $P \rightsquigarrow k$ and $U \rightsquigarrow m$. In particular, we get that $\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G) \in \mathcal{L}_W^{\mathcal{G}_2}(\Gamma)$. Hence, $\{\{W\}\} \in h(\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G))$. “ \supseteq ” We consider some $\{\{W\}\} \in h(\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G))$ (we note that all elements of $h(\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G))$ are of this shape because for every graph $G \in \mathbf{G}_2$ of sort $\{1,2\}$ we have that $\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G)$ is \parallel -atomic). Then, because $\{\{W\}\}$ is a view of $\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G)$, we have $\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G) \in \mathcal{L}_W^{\mathcal{G}_2}(\Gamma)$. In particular, there is a derivation of $\text{restrict}_{\{1\}}^{\mathcal{G}_2}(G)$ starting with a rule $W \rightarrow \text{restrict}_{\{1\}}(P \triangleright U) \in \mathcal{R}$. Hence, there is a view (k, m) of G with $P \rightsquigarrow k$ and $U \rightsquigarrow m$. By Lemma 7 we then have $P \rightsquigarrow [k]$ and $U \rightsquigarrow [m]$. Further, $([k], [m]) \in h(G)$. Hence, we get $\{\{W\}\} \in \text{restrict}_{\{1\}}^A(h(G))$.

$\underline{h(G_1) \circ^A h(G_2) = h(G_1 \circ^{\mathcal{G}_2} G_2)}$, for all $G_1, G_2 \in \mathbf{G}_2$ of sort $\{1,2\}$: “ \subseteq ” We consider some $(\{\{S\}\}, m) \in h(G_1) \circ^A h(G_2)$ (we note that all elements of $h(G_1) \circ^A h(G_2)$ are of this shape). By the definition of \circ^A , there exist $([k_i], [m_i]) \in h(G_i)$ for views (k_i, m_i) of G_i , $i = 1, 2$ such that either:

1. $S \rightarrow (\mathbf{b}_{(i,j)} \triangleright U) \circ S_2 \in \mathcal{R}$, $([k_1], \emptyset_m) \in \mathbf{b}_{(i,j)}^A$, $U \rightsquigarrow [m_1]$ and $[k_2] = \{\{S_2\}\}$, for some $(i,j) \in \{(1,2), (2,1)\}$. By the definition of $\mathbf{b}_{(i,j)}^A$, we obtain $k_1 = [k_1]$ and, by Lemma 7, we have $U \rightsquigarrow m_1$. Moreover, we have $k_2 = \{\{S_2\}\}$.
2. $S \rightarrow (\mathbf{b}_{(i,j)} \triangleright U) \circ P_2 \in \mathcal{R}$, $([k_1], \emptyset_m) \in \mathbf{b}_{(i,j)}^A$, $U \rightsquigarrow [m_1]$ and $P_2 \rightsquigarrow [k_2]$, for some $(i,j) \in \{(1,2), (2,1)\}$. By the definition of $\mathbf{b}_{(i,j)}^A$, we obtain $k_1 = [k_1]$ and, by Lemma 7, we have $U \rightsquigarrow m_1$ and $P_2 \rightsquigarrow k_2$.
3. $S \rightarrow (P_1 \triangleright U) \circ S_2 \in \mathcal{R}$, $P_1 \rightsquigarrow [k_1]$, $U \rightsquigarrow [m_1]$ and $[k_2] = \{\{S_2\}\}$. By Lemma 7, we obtain $P_1 \rightsquigarrow k_1$ and $U \rightsquigarrow m_1$. Moreover, we have $k_2 = \{\{S_2\}\}$.
4. $S \rightarrow (P_1 \triangleright U) \circ P_2 \in \mathcal{R}$, $P_1 \rightsquigarrow [k_1]$, $U \rightsquigarrow [m_1]$ and $P_2 \rightsquigarrow [k_2]$. By Lemma 7, we obtain $P_1 \rightsquigarrow k_1$, $U \rightsquigarrow m_1$ and $P_2 \rightsquigarrow k_2$.

In all cases, we can build a S -derivation of $G_1 \circ^{\mathcal{G}_2} G_2$ in Γ , thus $\{\{S\}\}$ is a view of $G_1 \circ^{\mathcal{G}_2} G_2$ and

$\{\!\{S}\!\} \in h(G_1 \circ^{\mathcal{G}_2} G_2)$ follows easily.

“ \supseteq ” We consider some $(\{\!\{S}\!\}, [m]) \in h(G_1 \circ^{\mathcal{G}_2} G_2)$ for some view $(\{\!\{S}\!\}, m)$ of $G_1 \circ^{\mathcal{G}_2} G_2$ (we note that all elements of $h(G_1 \circ^{\mathcal{G}_2} G_2)$ are of this shape). We now consider some derivation, starting with S , which witnesses the view $(\{\!\{S}\!\}, m)$ of $G_1 \circ^{\mathcal{G}_2} G_2$. There are four cases to consider, based on the first rule applied on this derivation (we assume w.l.o.g. that the first rule introduces the \circ -decomposition of $G_1 \circ^{\mathcal{G}_2} G_2$):

1. $S \rightarrow (\mathbf{b}_{(i,j)} \triangleright U) \circ S_2$, for some $(i, j) \in \{(1, 2), (2, 1)\}$: in this case, there is a view (k_1, m_1) of G_1 such that (k_1, \emptyset_m) is a view of $\mathbf{b}_{(i,j)}$ and $U \rightsquigarrow m_1$, and a view $(\{\!\{S_2}\!\}, m)$ of G_2 . By Lemma 7, we have $U \rightsquigarrow [m_1]$ and, since $[\{\!\{S_2}\!\}] = \{\!\{S_2}\!\}$, we obtain $([k_1], [m_1]) \in h(G_1)$. Moreover, $(\{\!\{S_2}\!\}, [m_2]) \in h(G_2)$ and $(\{\!\{S}\!\}, [m]) \in h(G_1) \circ^{\mathcal{A}} h(G_2)$ follows, by the first case of the definition of $\circ^{\mathcal{A}}$.
2. $S \rightarrow (\mathbf{b}_{(i,j)} \triangleright U) \circ P_2$, for some $(i, j) \in \{(1, 2), (2, 1)\}$: in this case, there is a view (k_1, m_1) of G_1 such that (k_1, \emptyset_m) is a view of $\mathbf{b}_{(i,j)}$ and $U \rightsquigarrow m_1$, and a view (k_2, m) of G_2 , such that $P_2 \rightsquigarrow k_2$. By Lemma 7, we have $U \rightsquigarrow [m_1]$ and $P_2 \rightsquigarrow [k_2]$, hence $([k_i], [m_i]) \in h(G_i)$, for $i = 1, 2$. Thus, $(\{\!\{S}\!\}, [m]) \in h(G_1) \circ^{\mathcal{A}} h(G_2)$ follows, by the second case of the definition of $\circ^{\mathcal{A}}$.
3. $S \rightarrow (P_1 \triangleright U) \circ S_2$: in this case, there is a view (k_i, m_i) of G_i , for $i = 1, 2$ such that $P_1 \rightsquigarrow k_1$, $U \rightsquigarrow m_1$, $k_2 = \{\!\{S_2}\!\}$ and $m_2 = m$. By Lemma 7, we obtain $P_1 \rightsquigarrow [k_1]$ and $U \rightsquigarrow [m_1]$. Hence, $([k_1], [m_1]) \in h(G_1)$. Moreover, $(\{\!\{S_2}\!\}, [m]) \in h(G_2)$. Thus $(\{\!\{S}\!\}, [m]) \in h(G_1) \circ^{\mathcal{A}} h(G_2)$, by the third case of the definition of $\circ^{\mathcal{A}}$.
4. $S \rightarrow (P_1 \triangleright U) \circ P_2$: in this case, there is a view (k_i, m_i) of G_i , for $i = 1, 2$ such that $P_1 \rightsquigarrow k_1$, $U \rightsquigarrow m_1$, $P_2 \rightsquigarrow k_2$ and $m_2 = m$. By Lemma 7, we obtain $P_1 \rightsquigarrow [k_1]$, $U \rightsquigarrow [m_1]$ and $P_2 \rightsquigarrow [k_2]$. Hence, $([k_1], [m_1]) \in h(G_1)$ and $([k_2], [m]) \in h(G_2)$. Thus $(\{\!\{S}\!\}, [m]) \in h(G_1) \circ^{\mathcal{A}} h(G_2)$, by the fourth case of the definition of $\circ^{\mathcal{A}}$.

$h(G_1) \|\!^{\mathcal{A}} h(G_2) = h(G_1 \|\!^{\mathcal{G}_2} G_2)$, for all $G_1, G_2 \in \mathcal{G}_2$, both of sort either $\{1\}$ or $\{1, 2\}$: this case is identical to the case of $\|\!$ -composition from Lemma 9. \square

The main result of this section is stated below:

Theorem 5. *A set $\mathcal{L} \subseteq \text{restrict}_{\{1\}}^{\mathcal{G}_2}(\mathcal{G}_c^{\leq 2})$ is recognizable for the class \mathcal{G}_2 if and only if there exists a regular tree-width ≤ 2 grammar Γ , such that $\mathcal{L} = \mathcal{L}^{\mathcal{G}_2}(\Gamma)$.*

Proof “ \Rightarrow ” We consider some $(\mathcal{F}_2, \Sigma_2)$ -algebra (\mathcal{A}, B) that recognizes $\mathcal{L} \subseteq \mathcal{G}_2$. By Lemma 15 there is a regular tree-width ≤ 2 grammar Γ_2 such that $\mathcal{L}^{\mathcal{G}_2}(\Gamma_2) = \text{restrict}_{\{1\}}^{\mathcal{G}_2}(\mathcal{G}_c^{\leq 2})$. Moreover, this is also a 2-partitioned periodic grammar, for the partitioning $\{U\} \uplus \{W\} \uplus \{P\} \uplus \{S, E\}$. By Theorem 2, we get that there is a 2-partitioned periodic grammar Γ with $\mathcal{L}^{\mathcal{G}_2}(\Gamma) = \mathcal{L}^{\mathcal{G}_2}(\Gamma_2) \cap h_{\mathcal{G}_2, \mathcal{A}}^{-1}(B) = \text{restrict}_{\{1\}}^{\mathcal{G}_2}(\mathcal{G}_c^{\leq 2}) \cap \mathcal{L} = \mathcal{L}$. Moreover, we obtain that $\text{Terms}(\Gamma) \subseteq \text{Terms}(\Gamma_2)$, hence Γ is a regular tree-width ≤ 2 grammar.

“ \Leftarrow ” We consider the profile abstraction h and the algebra \mathcal{A} defined above. By Lemma 16, h is the unique homomorphism between \mathcal{G}_2 and \mathcal{A} . By (the analogue of) Lemma 8, we get that $\mathcal{L}^{\mathcal{G}_2}(\Gamma) = h_{\mathcal{G}_2, \mathcal{A}}^{-1}(B)$, for $B \stackrel{\text{def}}{=} \{h(G) \mid G \in \mathcal{L}^{\mathcal{G}_2}(\Gamma)\}$. Hence, (\mathcal{A}, B) is a recognizer for $\mathcal{L}^{\mathcal{G}_2}(\Gamma)$. \square

6 The Inclusion Problem for Regular Graph Grammars

In this section we apply the previously obtained results on regular grammars (section 5) to the problem of inclusion of the sets of graphs defined by such grammars. In the following, we fix a graph class \mathcal{C} , having signature $\mathcal{F}_{\mathcal{C}}$ and set of sorts $\Sigma_{\mathcal{C}}$, where \mathcal{C} is either the algebra of trees (subsection 5.2), series-parallel graphs (subsection 5.3) or connected graphs of tree-width ≤ 2 (subsection 5.4). The main result of this section is a generic algorithm for the following decision problem:

Definition 11. *The regular inclusion problem asks if $\mathcal{L}^c(\Gamma_1) \subseteq \mathcal{L}^c(\Gamma_2)$, where Γ_1 and Γ_2 are (\mathcal{F}, Σ) -grammars and Γ_2 is regular for the class \mathcal{C} .*

We propose a uniform algorithm for deciding the above problem based on the construction of a recognizer for $\mathcal{L}^c(\Gamma_2)$. We recall that for each of the classes of trees, series-parallel graphs and graphs of tree-width ≤ 2 , we can construct a recognizer (\mathcal{A}, B) for $\mathcal{L}^c(\Gamma_2)$ with $\text{card}(\mathcal{A}) \leq 2^{2^{\text{poly}(\text{size}(\Gamma_2))}}$, which is also an upper bound on the time needed for building the recognizer. We then apply the algorithm stated in Figure 5, which we will discuss below, obtaining the following:

Theorem 6. *The regular inclusion problem belongs to 2EXPTIME, for the classes of trees, (disoriented) series-parallel and tree-width ≤ 2 graphs.*

Proof Let $\Gamma_i \stackrel{\text{def}}{=} (\mathcal{N}_i, \mathcal{R}_i)$, for $i = 1, 2$. By Lemma 17, the algorithm stated in Figure 5, runs in $O(\text{card}(\mathcal{R}_1) \cdot \text{card}(\mathcal{A})^{\text{card}(\mathcal{N}_1)} \cdot \Omega)$, where (\mathcal{A}, B) is the recognizer built for $\mathcal{L}^c(\Gamma_2)$ and Ω is an upper bound on the time needed to evaluate $t^{\mathcal{A}}(a_1, \dots, a_n)$, for each rule $X_0 \rightarrow t[X_1, \dots, X_n] \in \mathcal{R}_2$ and $a_1, \dots, a_n \in \mathcal{A}$. Since Γ_2 is a regular (tree, series-parallel or tree-width ≤ 2) grammar, each such term is a derived operation (resp. taken from $\mathcal{F}_{\mathcal{T}}$, $\mathcal{F}_{\mathcal{SP}}$ or \mathcal{F}_2) of arity at most 2. Hence, each such operation can be evaluated using a pre-computed table of size at most $\text{card}(\mathcal{A})^2$, i.e., $\Omega \leq \text{card}(\mathcal{A})^2$, in this case. We then verify:

$$\text{card}(\mathcal{R}) \cdot \text{card}(\mathcal{A})^{\text{card}(\mathcal{N})} \cdot \Omega \in \text{size}(\Gamma_1) \cdot \left(2^{2^{\text{poly}(\text{size}(\Gamma_2))}}\right)^{\text{size}(\Gamma_1)} \cdot 2^{2^{\text{poly}(\text{size}(\Gamma_2))}} \in 2\text{EXPTIME}$$

□

Currently, it is an open problem whether the 2EXPTIME upper bound is optimal or can be further improved. In particular, the inclusion problem between recognizable ranked sets of trees is EXPTIME-hard [7, Corollary 1.7.9], which sets a lower bound for the regular inclusion problem in the class of graphs of tree-width 2 at most.

We now discuss the generic algorithm, stated in Figure 5, for deciding inclusion between the language of a (\mathcal{F}, Σ) -grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ and that of a (\mathcal{F}, Σ) -recognizer (\mathcal{A}, B) . The algorithm maintains a workset $WorkSet \subseteq \mathcal{N} \times \mathcal{A}$, initialized as the empty set. Then the algorithm enumerates the rules $X_0 \rightarrow t[X_1, \dots, X_n]$ of Γ and adds a new pair (X_0, a_0) to the workset whenever (X_0, a_0) is not already a member of $WorkSet$ and there exist pairs $(X_1, a_1), \dots, (X_n, a_n)$ such that a_0 is the value $t^{\mathcal{A}}(a_1, \dots, a_n)$ (line 12). We note that in the first iteration of the algorithm the workset will be populated with pairs $(X, c^{\mathcal{A}}) \in \mathcal{N} \times \mathcal{A}$ for which there exist a rule $X \rightarrow c \in \mathcal{R}$ such that $c \in \mathcal{F}$ is a constant symbol. If X_0 is the right-hand side of an axiom of Γ and $a_0 \notin B$, we found a counterexample to the inclusion, i.e., a graph G that can be derived from X_0 in Γ such that $h_{\mathcal{C}, \mathcal{A}}(G) \notin B$ (line 8). The iteration stops when no rules can be fired, either because no matching rule is found or no new elements can be added to $WorkSet$. By the finiteness of \mathcal{A} and \mathcal{N} , the iteration will eventually stop, for any input. If the iteration

```

1:  $WorkSet := \emptyset \subseteq \mathcal{N} \times \mathbf{A}$ 
2:  $changed := \text{true}$ 
3: while  $changed$  do
4:    $changed := \text{false}$ 
5:   for each rule  $X_0 \rightarrow t[X_1, \dots, X_n] \in \mathcal{R}$  and  $(X_1, a_1), \dots, (X_n, a_n) \in WorkSet$  do
6:      $a_0 := t^{\mathbf{A}}(a_1, \dots, a_n)$ 
7:     if  $\rightarrow X_0 \in \mathcal{R}$  and  $a_0 \notin B$  then
8:       report no
9:     end if
10:    if  $(X_0, a_0) \notin WorkSet$  then
11:       $changed := \text{true}$ 
12:       $WorkSet := WorkSet \cup \{(X_0, a_0)\}$ 
13:    end if
14:  end for
15: end while
16: report yes

```

Figure 5: Algorithm for checking $\mathcal{L}^c(\Gamma) \subseteq h_{\mathcal{C}, \mathbf{A}}^{-1}(B)$ for a given (\mathcal{F}, Σ) -grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ and (\mathcal{F}, Σ) -recognizer (\mathcal{A}, B)

stops and no counterexample has been found, the inclusion holds (line 16). We now state the correctness and time complexity of this algorithm:

Lemma 17. *Given a (\mathcal{F}, Σ) -grammar $\Gamma = (\mathcal{N}, \mathcal{R})$ and a (\mathcal{F}, Σ) -recognizer (\mathcal{A}, B) the algorithm in Figure 5 decides whether $\mathcal{L}^c(\Gamma) \subseteq h_{\mathcal{C}, \mathbf{A}}^{-1}(B)$ in time $\mathcal{O}(\text{card}(\mathcal{R}) \cdot \text{card}(\mathbf{A})^{\text{card}(\mathcal{N})} \cdot \Omega)$, where Ω is the maximum time required to compute $t^{\mathbf{A}}(a_1, \dots, a_n)$, for all $X_0 \rightarrow t[X_1, \dots, X_n] \in \mathcal{R}$ and $a_1, \dots, a_n \in \mathbf{A}$.*

Proof The correctness argument is based the following invariant:

Fact 4. *For each pair $(X, a) \in WorkSet$ there is a graph G for which there is an X -derivation in Γ such that $h_{\mathcal{C}, \mathbf{A}}(G) = a$.*

Proof. By induction on the number of loop iterations. We consider some rule $X \rightarrow t[X_1, \dots, X_n]$ and $(X_1, a_1), \dots, (X_n, a_n) \in WorkSet$. Since each (X_i, a_i) has been previously, there exist a graph G_i for which there exists a X_i -derivation of Γ and $h_{\mathcal{C}, \mathbf{A}}(G_i) = a_i$, for all $i \in [1, n]$, by induction assumption. We take $G \stackrel{\text{def}}{=} t^c(G_1, \dots, G_n)$ and obtain an X -derivation of G in Γ by appending the rule $X \rightarrow t[X_1, \dots, X_n]$ to the X_i -derivation of G_i , for each $i \in [1, n]$. Then, we have $h_{\mathcal{C}, \mathbf{A}}(G) = h_{\mathcal{C}, \mathbf{A}}(t^c(G_1, \dots, G_n)) = t^{\mathbf{A}}(h_{\mathcal{C}, \mathbf{A}}(G_1), \dots, h_{\mathcal{C}, \mathbf{A}}(G_n)) = t^{\mathbf{A}}(a_1, \dots, a_n) = a$. \square

Assume that the algorithm reaches line 8 with a pair (X, a) . Then, by the above fact, there exists a graph $G \in \mathcal{L}_X^c(\Gamma)$ such that $h_{\mathcal{C}, \mathbf{A}}(G) = a$. Moreover, this line is reached only if there exists an axiom $\rightarrow X$ in Γ and $a \notin B$, hence $G \in \mathcal{L}^c(\Gamma)$ and $h_{\mathcal{C}, \mathbf{A}}(G) \notin B$, i.e., $G \notin h_{\mathcal{C}, \mathbf{A}}^{-1}(B)$. Dually, if there exists a graph $G \in \mathcal{L}^c(\Gamma)$ such that $h_{\mathcal{C}, \mathbf{A}}(G) = a \notin B$, there exists an axiom $\rightarrow X$ in Γ and an X -derivation of G in Γ . Then, the pair (X, a) is eventually generated at lines 5–6 and the algorithm reaches line 8 subsequently.

The loop between lines 3 and 15 is iterated at most $\text{card}(\mathcal{N}) \cdot \text{card}(\mathbf{A})$ times, because then all elements of $\mathcal{N} \times \mathbf{A}$ have been added to $WorkSet$. Line 5 needs to be evaluated at most $\text{card}(\mathcal{R}) \cdot \text{card}(\mathbf{A})^{\text{card}(\mathcal{N})}$ times over all iterations of the outer loop, because we do not need to repeat the evaluation for the same arguments, i.e., for each rule $X_0 \rightarrow t[X_1, \dots, X_n] \in \mathcal{R}$, we

need to consider at most once every combination of the non-terminals $\{(X_i, a) \mid a \in \mathbf{A}\}$, with $0 \leq i \leq n \leq \text{card}(\mathcal{N})$. We finally account Ω time for each inner step at line 6. \square

7 Conclusion

We introduce regular grammars for the classes of unranked and unordered trees, series-parallel graphs and, more generally, for graphs of tree-width at most 2. These grammars provide finite and compact representations for the recognizable and, at the same time, the CMSO-definable sets in each class. As a by-product, the inclusion of a context-free language of graphs into a language defined by a regular grammar can be decided in doubly-exponential time. It is currently an open problem whether such grammars exist for the classes of graphs of tree-width 3 or more.

References

- [1] A. Boiret, V. Hugot, J. Niehren, and R. Treinen. Automata for unordered trees. *Inf. Comput.*, 253:304–335, 2017.
- [2] M. Bojanczyk, A. Muscholl, T. Schwentick, and L. Segoufin. Two-variable logic on data trees and XML reasoning. *J. ACM*, 56(3):13:1–13:48, 2009.
- [3] M. Bojańczyk and M. Pilipczuk. Definability equals recognizability for graphs of bounded treewidth. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16*, page 407–416, New York, NY, USA, 2016. Association for Computing Machinery.
- [4] M. Bojanczyk and M. Pilipczuk. Optimizing tree decompositions in MSO. *Log. Methods Comput. Sci.*, 18(1), 2022.
- [5] M. Bojanczyk and I. Walukiewicz. Forest algebras. In J. Flum, E. Grädel, and T. Wilke, editors, *Logic and Automata: History and Perspectives [in Honor of Wolfgang Thomas]*, volume 2 of *Texts in Logic and Games*, pages 107–132. Amsterdam University Press, 2008.
- [6] M. Chimes, R. Iosif, and F. Zuleger. Tree-verifiable graph grammars. In N. S. Bjørner, M. Heule, and A. Voronkov, editors, *LPAR 2024: Proceedings of 25th Conference on Logic for Programming, Artificial Intelligence and Reasoning, Port Louis, Mauritius, May 26-31, 2024*, volume 100 of *EPiC Series in Computing*, pages 165–180. EasyChair, 2024.
- [7] H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, D. Lugiez, C. Löding, S. Tison, and M. Tommasi. *Tree Automata Techniques and Applications*. 2008.
- [8] B. Courcelle. The monadic second-order logic of graphs. i. recognizable sets of finite graphs. *Information and Computation*, 85(1):12–75, 1990.
- [9] B. Courcelle. The monadic second-order logic of graphs v: on closing the gap between definability and recognizability. *Theoretical Computer Science*, 80(2):153–202, 1991.
- [10] B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2012.
- [11] J. Cristau, C. Löding, and W. Thomas. Deterministic automata on unranked trees. In M. Liśkiewicz and R. Reischuk, editors, *Fundamentals of Computation Theory*, pages 68–79, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.
- [12] A. Doumane. Regular expressions for tree-width 2 graphs. In M. Bojanczyk, E. Merelli, and D. P. Woodruff, editors, *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France*, volume 229 of *LIPICs*, pages 121:1–121:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.

- [13] A. Doumane, S. Humeau, and D. Pous. A finite presentation of graphs of treewidth at most three. In K. Bringmann, M. Grohe, G. Puppis, and O. Svensson, editors, *51st International Colloquium on Automata, Languages, and Programming, ICALP 2024, July 8-12, 2024, Tallinn, Estonia*, volume 297 of *LIPICs*, pages 135:1–135:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024.
- [14] P. Erdős and R. Graham. On a linear diophantine problem of frobenius. *Acta Arithmetica*, 21(1):399–408, 1972.
- [15] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006.
- [16] S. Greibach. A note on undecidable properties of formal languages. *Math. Systems Theory*, 2:1–6, 1968.
- [17] R. Iosif and F. Zuleger. Characterizations of monadic second order definable context-free sets of graphs. *CoRR*, abs/2310.04764, 2023.
- [18] L. Libkin. Logics for unranked trees: An overview. In L. Caires, G. F. Italiano, L. Monteiro, C. Palamidessi, and M. Yung, editors, *Automata, Languages and Programming*, pages 35–50, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.