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Neighborhood complexity of planar graphs

Gwenaël Joret*
 Département d'Informatique
 Université libre de Bruxelles
 Brussels, Belgium
 gwenael.joret@ulb.be

Clément Rambaud
 Département d'Informatique
 ENS Paris, PSL University
 Paris, France
 Université Côte d'Azur
 CNRS, Inria, I3S
 Sophia Antipolis, France
 clement.rambaud@inria.fr

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Abstract

Reidl, Sánchez Villaamil, and Stravopoulos (2019) characterized graph classes of bounded expansion as follows: A class \mathcal{C} closed under subgraphs has bounded expansion if and only if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph $G \in \mathcal{C}$, every nonempty subset A of vertices in G and every nonnegative integer r , the number of distinct intersections between A and a ball of radius r in G is at most $f(r)|A|$. When \mathcal{C} has bounded expansion, the function $f(r)$ coming from existing proofs is typically exponential. In the special case of planar graphs, it was conjectured by Sokolowski (2021) that $f(r)$ could be taken to be a polynomial.

In this paper, we prove this conjecture: For every nonempty subset A of vertices in a planar graph G and every nonnegative integer r , the number of distinct intersections between A and a ball of radius r in G is $\mathcal{O}(r^4|A|)$. We also show that a polynomial bound holds more generally for every proper minor-closed class of graphs.

1 Introduction

Nešetřil and Ossona de Mendez [22] introduced the notion of graph classes with bounded expansion as a way to capture graphs that are sparse in a robust way: Very informally, the graphs should not only have linearly many edges but also all their minors with ‘bounded depth’ should have linearly many edges (where the bound depends on the depth). Nowadays, there are several characterizations of graph classes with bounded expansion. These classes have been characterized using generalized coloring numbers [30], low treedepth colorings [22], p -centered colorings [22], and quasi-wideness [11], to name a few. See the textbook by Nešetřil and Ossona de Mendez [20] and their more recent survey [21] for an overview of this rich area.

In this paper, we are interested in the following characterization of graph classes with bounded expansion, due to Reidl, Sánchez Villaamil and Stravopoulos [25]: A class \mathcal{C} of graphs closed under taking subgraphs has *bounded expansion* if and only if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph $G \in \mathcal{C}$, for every nonempty subset $A \subseteq V(G)$ of vertices in G and every nonnegative integer r ,

$$|\{N^r[v] \cap A \mid v \in V(G)\}| \leq f(r)|A|.$$

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Here, $N^r[v]$ denotes the *ball of radius r around v* , namely, the set of vertices at distance at most r from v in G . When the function f exists, we say that the class \mathcal{C} has *bounded neighborhood complexity*, and the minimum such function f is called the *neighborhood complexity* of \mathcal{C} . (We remark that in some papers a different terminology is used: ‘bounded neighborhood complexity’ is called ‘linear neighborhood complexity’ instead, emphasizing the linear dependency on $|A|$. In this paper, ‘linear neighborhood complexity’ has thus a different meaning, namely that f can be taken in $\mathcal{O}(r)$.)

Neighborhood complexity has algorithmic applications. In particular, it has been used in [15] to build a linear kernel for DISTANCE- r -DOMINATING-SET in graph classes with bounded expansion.

However, the general bound given by Reidl, Sánchez Villaamil and Stravopoulos in [25] is exponential, and no effective bound for neighborhood complexity of some particular classes, such as planar graphs, was known before this work. Using the fact that the family of all balls in a planar graph has VC-dimension at most 4 [10] in combination with the Sauer-Shelah Lemma (Lemma 4), it is not difficult to show that $|\{N^r[v] \cap A \mid v \in V(G)\}| = \mathcal{O}(|A|^4)$ holds for every planar graph G , every nonempty subset A of vertices, and every nonnegative integer r ; see Corollary 7. For planar graphs, the best known upper bound is due to Sokołowski [28], who showed that for every planar graph G , every nonempty subset A of vertices, and every nonnegative integer r ,

$$|\{N^r[v] \cap A \mid v \in V(G)\}| = \mathcal{O}(r^7 |A|^3).$$

Thus, a bound polynomial in r can be achieved at the price of a cubic dependence on $|A|$, instead of a linear one. Sokołowski [28] conjectured that the neighborhood complexity of planar graphs is polynomial, that is, there exists a constant c such that $|\{N^r[v] \cap A \mid v \in V(G)\}| = \mathcal{O}(r^c |A|)$.

Our contributions

Our main contribution is a proof of the above conjecture, that planar graphs have polynomial neighborhood complexity.

Theorem 1. *The neighborhood complexity of planar graphs is in $\mathcal{O}(r^4)$.*

Actually this upper bound holds for the slightly more general notion of ‘profile complexity’:

Theorem 2. *The profile complexity of planar graphs is in $\mathcal{O}(r^4)$.*

Profile complexity is a refined version of neighborhood complexity, which we now introduce. The *profile* of a vertex u on a set A of vertices at distance r is the $|A|$ -tuple of all the distances $\text{dist}(u, a)$ for $a \in A$, where values larger than r are replaced by $+\infty$. Since the profile of u on A at distance r determines the intersection $N^r[u] \cap A$, the number of distinct profiles at distance r on A is an upper bound on $|\{N^r[u] \cap A \mid u \in V(G)\}|$. In fact, most upper bounds on $|\{N^r[u] \cap A \mid u \in V(G)\}|$ in the literature are actually obtained by bounding the number of distinct profiles at distance r on A . This motivates the introduction of the notion of *profile complexity* of a class \mathcal{C} of graphs, defined as the minimum function f (if it exists) such that for every $G \in \mathcal{C}$, for every nonempty set $A \subseteq V(G)$ and for every nonnegative integer r , the number of distinct profiles on A at distance r is at most $f(r)|A|$. Furthermore, under mild conditions on \mathcal{C} , a class \mathcal{C} of graphs with bounded neighborhood complexity also has bounded profile complexity (c.f. Lemma 8).

As it turns out, the upper bound in Theorem 2 is almost tight. In Section 8, we present an $\Omega(r^3)$ lower bound.

A second contribution of this paper is a proof that every proper minor-closed class of graphs has polynomial profile complexity, and thus polynomial neighborhood complexity.

Theorem 3. *For every integer $t \geq 1$ the profile complexity of K_t -minor-free graphs is in $\mathcal{O}_t(r^{t^2-1})$.*

(The subscript in the notation $\mathcal{O}_t(\cdot)$ indicates that the hidden constant factor depends on t , and likewise for the other asymptotic notations in the paper.)

We also consider graphs of bounded treewidth and graphs on surfaces, and give upper bounds on their profile complexities that are nearly tight, see Sections 4, 7, and 8. Table 1 gives a summary of the main upper and lower bounds proved in this paper.

Graph class	Upper bound	Lower bound	Reference
PLANAR	$\mathcal{O}(r^4)$	$\Omega(r^3)$	Thm. 30 and Thm. 36
EULER GENUS g	$\mathcal{O}(r^5)$	$\Omega(r^3)$	Cor. 35 and Thm. 36
TREewidth t	$\mathcal{O}_t(r^{2t})$	$\Omega_t(r^{t+1})$	Cor. 18 and Thm. 36
K_t -MINOR-FREE	$\mathcal{O}_t(r^{t^2-1})$	$\Omega_t(r^{t-1})$	Thm. 15 and Thm. 36

Table 1: Summary of our upper and lower bounds for the profile complexities of various graph classes.

While planar graphs, and more generally proper minor-closed classes of graphs, have polynomial profile complexity (and so polynomial neighborhood complexity), we show in Section 8 that 1-planar graphs, which are graphs that can be drawn in the plane in such a way that every edge crosses at most one other edge, have super-polynomial profile and neighborhood complexities. 1-planar graphs are relatively close to planar graphs—in particular, they have $\mathcal{O}(\sqrt{n})$ -size separators, where n denotes the number of vertices—and are among the “simplest” non-minor closed graph classes within the realm of classes with bounded expansion. Thus, the latter result suggests that, besides proper minor closed graph classes, there are probably not many other bounded expansion classes of interest that have polynomial neighborhood complexity.

Related works

Very recently, the relation between neighborhood complexity and twin-width has been studied in [5] and independently in [24]. In particular, it is shown in these two papers that graphs with bounded twin-width have bounded neighborhood complexity. (We refer the reader to [5, 24] for the definition of twin-width.) These bounds have been recently improved in [4].

We remark that every proper minor-closed class of graphs has both bounded expansion (as is well known) and bounded twin-width (as proved in [6]). Thus, bounded neighborhood complexity for such a class also follows from the above mentioned results on twin-

width. It is worth pointing out though that bounded expansion and bounded twin-width are two incomparable properties for graph classes, as neither implies the other. In particular, there are graph classes that do not have bounded expansion but have bounded twin-width (e.g. cographs). Also, let us emphasize that the characterization of bounded expansion classes in terms of bounded neighborhood complexity mentioned at the beginning of the introduction only applies to classes closed under subgraphs, which is not the case of bounded twin-width graphs.

We also note that neighborhood complexity at distance 1 and 2 has also been used recently in [7] for problems related to twin-width.

We close this introduction by mentioning another concept that is closely related to neighborhood complexity. The *metric dimension* of a connected graph G is the minimum size k of a set R of vertices in G such that for every vertex v , the k -tuple $(\text{dist}_G(v, r))_{r \in R}$ uniquely determines v . Beaudou, Foucaud, Dankelmann, Henning, Mary, and Parreau [3] gave upper bounds on the number of vertices of graphs with given diameter and metric dimension. In particular, they showed that for every connected n -vertex graph G with diameter d and metric dimension k , if G has treewidth at most t then

$$n \in \mathcal{O}(d^{3t+3}k)$$

and if G is K_t -minor free then

$$n \leq (dk + 1)^{t-1} + 1.$$

As it turns out, upper bounds on profile complexities can be translated to upper bounds in the latter setting, and as a result we obtain improved bounds on the number of vertices in the latter setting: if G has treewidth at most t then

$$n \in \mathcal{O}_t(d^{2t}k)$$

and if G is K_t -minor free then

$$n \in \mathcal{O}_t(d^{t^2-1}k),$$

see Theorem 40 in Section 9 (The latter bound is of course an improvement only when $k \gg d$.) As far as we are aware, this connection between neighborhood complexity and metric dimension has not been made before in the literature.

Paper organization

The paper is organized as follows. First, in Section 2 we introduce all necessary definitions, as well as some preliminary lemmas. Then in Section 3 we prove the aforementioned bound of $\mathcal{O}_t(r^{t^2-1})$ on the profile complexity of K_t -minor-free graphs (Theorem 3). For the special case of graphs of treewidth t (which are K_{t+2} -minor free), we give an improved bound of $\mathcal{O}_t(r^{2t})$ in Section 4, see Corollary 18. In Section 5, we consider graphs admitting a so-called ‘product structure’, which includes planar graphs, and show the existence of certain types of ‘guarding sets’ in these graphs (to be used in Section 6), see Section 5 for the definitions. The next section, Section 6, contains our main result, namely a bound of $\mathcal{O}(r^4)$ on the profile complexity of planar graphs (Theorem 2). Since planar graphs are K_5 -minor free, a bound of $\mathcal{O}(r^{24})$ already follows Theorem 3. As a warm-up to the proof of the $\mathcal{O}(r^4)$ bound, we first give two easier proofs in Section 6, giving bounds of respectively $\mathcal{O}(r^{11})$ and $\mathcal{O}(r^6)$. Next, in Section 7, we lift the result for planar graphs to graphs on surfaces and obtain a $\mathcal{O}_g(r^5)$ bound on the profile complexity of graphs of Euler genus g (Corollary 35). In Section 8, we prove the lower bounds mentioned in Table 1.

In Section 9, we study the connection between profile complexity and metric dimension, and obtain a number of new bounds on metric dimension using our results. Finally, we conclude the paper by emphasizing two open problems in Section 10.

2 Preliminaries

All graphs considered in this paper are finite, simple, and undirected. We will use the following notations. Let G be a graph and let u be a vertex of G .

- $[n] = \{1, \dots, n\}$ for every integer $n \geq 1$.
- $V(G)$ is the set of vertices of G , $E(G)$ its set of edges.
- For any set X and integer $d \geq 0$, $\binom{X}{d} = \{Y \subseteq X \mid |Y| = d\}$ and $\binom{X}{\leq d} = \{Y \subseteq X \mid |Y| \leq d\}$.
- For any set X , $\mathcal{P}(X)$ is the set of all subsets of X .
- $N_G(u) = \{v \in V(G) \mid uv \in E(G)\}$, and more generally for every $r \geq 0$ we define $N_G^r(u) = \{v \in V(G) \mid \text{dist}_G(u, v) = r\}$. Similarly $N_G^r[u] = \{v \in V(G) \mid \text{dist}_G(u, v) \leq r\}$. Here, $\text{dist}_G(u, v)$ denotes the distance between u and v in G .
- If P is a path, we denote by $\ell(P)$ its length, which is defined as its number of edges.

A *tree decomposition* of a graph G is a pair $(T, (X_z)_{z \in V(T)})$, where T is a tree and $X_z \subseteq V(G)$ for every $z \in V(T)$, with the following properties:

- (i) for every vertex $u \in V(G)$, the subgraph T_u of T induced by $\{z \in V(T) \mid u \in X_z\}$ is a non empty subtree, and
- (ii) for every edge $uv \in E(G)$, $V(T_u) \cap V(T_v) \neq \emptyset$.

We call the sets X_z the *bags* of the tree decomposition. The *width* of a tree decomposition is the maximum size of a bag minus one. The *treewidth* of a graph G is defined to be the minimum width of a tree decomposition of G . For standard results on treewidth, see for example [12, Section 12.3]. We will in particular use the following facts:

- if C is a clique of G , then C is included in a bag,
- for every edge $yz \in E(T)$, if T_1 (resp. T_2) denotes the connected component of y (resp. z) in $T - yz$, then $X_y \cap X_z$ intersects every $(\bigcup_{x \in V(T_1)} X_x, \bigcup_{x \in V(T_2)} X_x)$ -path in G .

The tree decompositions considered in this paper will be arbitrarily (and implicitly) rooted at a node $s \in V(T)$. This induces a root t_u of the subtree T_u for every vertex $u \in V(G)$ (namely, the node of T_u closest to s in T). Moreover, we write $x <_T y$ for two distinct nodes $x, y \in V(T)$ if x is in the (unique) (y, s) -path in T . This defines a partial order on $V(T)$.

Given a subset $A \subseteq V(G)$ of vertices, a vertex $u \in V(G)$ and an integer $r \geq 0$, the *profile* of u on A at distance r is the function $\pi_{r,G}[u \rightarrow A]$ that maps a vertex $a \in A$ to $\text{dist}_G(u, a)$ if $\text{dist}_G(u, a) \leq r$, to $+\infty$ otherwise. Formally, $\pi_{r,G}[u \rightarrow A] : A \rightarrow \{0, \dots, r, +\infty\}$, $a \mapsto \text{Cap}_r(\text{dist}_G(u, a))$, where Cap_r is the function that maps an integer x to itself if $x \leq r$, to $+\infty$ otherwise. For every $U \subseteq V(G)$, we denote by $\Pi_{r,G}[U \rightarrow A]$

the set of all profiles $\pi_{r,G}[u \rightarrow A]$ for vertices $u \in U$ except the ‘all $+\infty$ ’ profile; that is, $\Pi_{r,G}[U \rightarrow A] = \{\pi_{r,G}[u \rightarrow A] \mid u \in U\} \setminus \{\lambda a. +\infty\}$. (We exclude the ‘all $+\infty$ ’ profile because it is convenient in the proofs.) Clearly, $|\{N^r[u] \cap A \mid u \in V(G)\}| \leq |\Pi_{r,G}[V(G) \rightarrow A]| + 1$, so it is enough to bound $|\Pi_{r,G}[V(G) \rightarrow A]|$ to get directly a bound on the neighborhood complexity. The advantage of considering profiles is the following “transitivity” property: Given $U \subseteq V(G)$, if $S \subseteq V(G)$ meets every (U, A) -path of length at most r in G , then the profile of a vertex $u \in U$ on A is completely determined by its profile on S . This observation is also used in [25] to prove graph classes with bounded expansion have bounded neighborhood complexity. This is a key idea, which justifies the introduction of our main tool, the guarding sets, defined in Subsection 2.3.

2.1 VC-dimension

A set system \mathcal{F} over a universe Ω is a collection of subsets of Ω . The *Vapnik–Chervonenkis dimension* (VC-dimension) [29] of a set system \mathcal{F} is the largest integer d such that there exists $X \subseteq \Omega$ of size d which is *shattered* by \mathcal{F} : for every $X' \subseteq X$, there exists $F \in \mathcal{F}$ such that $X \cap F = X'$. Our first tool is a direct consequence of the well-known Sauer–Shelah Lemma [26, 27].

Lemma 4 (Corollary of Sauer–Shelah Lemma [26, 27]). *Let \mathcal{F} be a set system over a universe Ω of VC-dimension at most d with $d \geq 2$. If $A \subseteq \Omega$ is non-empty then*

$$|\{A \cap F \mid F \in \mathcal{F}\}| \leq |A|^d.$$

Proof. Let $n = |A|$. The Sauer–Shelah Lemma states that $|\{A \cap F \mid F \in \mathcal{F}\}| \leq \sum_{i=0}^d \binom{n}{i}$, so we only need to show the inequality

$$\sum_{i=0}^d \binom{n}{i} \leq n^d.$$

Consider $f : [n]^d \rightarrow \binom{[n]}{\leq d} \setminus \{\emptyset\}, (x_1, \dots, x_d) \mapsto \{x_i\}_{i=1, \dots, d}$. Then f is surjective but not injective (because $d \geq 2$) so $|\binom{[n]}{\leq d} \setminus \{\emptyset\}| < n^d$ and thus $\sum_{i=0}^d \binom{n}{i} \leq n^d$. \square

We now prove that some particular set systems related to balls in planar graphs have bounded VC-dimension. Let us emphasize that, in the following lemma, A is not an arbitrary vertex subset but is restricted to be a vertex subset of the outerface. This allows for a better bound on the VC-dimension (and this special case is sufficient for our purposes).

Lemma 5. *Let G be a plane graph and let A be a subset of vertices of the outerface. Then $\{N_G^r[v] \cap A \mid r \geq 0, v \in V(G)\}$ has VC-dimension at most 3.*

Proof. Suppose for contradiction that there exist $a, b, c, d \in A$ such that $\{a, b, c, d\}$ is shattered by some balls in G . Permuting a, b, c, d if necessary, we may assume that there is a walk on the outerface such that a, b, c, d appear for the first time in the walk in this order. In particular, there exist $x_{ac}, x_{bd} \in V(G)$ and $r_{ac}, r_{bd} \geq 0$ such that $N_G^{r_{ac}}[x_{ac}] \cap A = \{a, c\}$ and $N_G^{r_{bd}}[x_{bd}] \cap A = \{b, d\}$.

Let P_{ac} (resp. P_{bd}) be the union of a shortest (a, x_{ac}) -path (resp. (b, x_{bd}) -path) and a shortest (x_{ac}, c) -path (resp. (x_{bd}, d) -path). By planarity, P_{ac} and P_{bd} have at least one common vertex x' . Permuting again a, b, c, d if necessary (and possibly changing the orientation of the facial walk), we may assume that x' is in a shortest (a, x_{ac}) -path P'_{ac} and in a shortest (b, x_{bd}) -path P'_{bd} . Moreover, we may also suppose that $\text{dist}(x', a) \leq \text{dist}(x', b)$.

As $a \notin N^{r_{bd}}[x_{bd}]$ we have $\text{dist}(x_{bd}, a) > r_{bd}$. But $\text{dist}(x_{bd}, a) \leq \text{dist}(x_{bd}, x') + \text{dist}(x', a) \leq \text{dist}(x_{bd}, x') + \text{dist}(x', b) = \text{dist}(x_{bd}, b) \leq r_{bd}$. This contradiction completes the proof of the lemma. \square

The above lemma will be used later on in our proofs. We conclude this section with the following general theorem due to Bousquet and Thomassé [8].

Theorem 6 (Bousquet and Thomassé [8]). *Let G be a K_t -minor-free graph, and let \mathcal{H} be the set system on $V(G)$ consisting of all the balls in G (of all possible radii). Then \mathcal{H} has VC-dimension at most $t - 1$.*

Together with Lemma 4, this gives directly the following corollary.

Corollary 7. *Let t be an integer with $t \geq 3$. If G is K_t -minor-free and $A \subseteq V(G)$ is nonempty, then*

$$|\{N^r[v] \cap A \mid r \geq 0, v \in V(G)\}| \leq |A|^{t-1}.$$

2.2 From balls to profiles

Recall that the profile at distance r of a vertex u on a set A of vertices in a graph G is the function $\pi_{r,G}[u \rightarrow A] : A \rightarrow \{0, \dots, r, +\infty\}$ that maps a vertex $a \in A$ to $\text{dist}_G(u, A)$ if this distance is at most r , to $+\infty$ otherwise. Also, for $U \subseteq A$, $\Pi_{r,G}[U \rightarrow A]$ denotes the set of all profiles $\pi_{r,G}[u \rightarrow A]$ for vertices $u \in U$ except the ‘all $+\infty$ ’ profile. Thus,

$$|\{N^r[u] \cap A \mid u \in V(G)\}| \leq |\Pi_{r,G}[V(G) \rightarrow A]| + 1.$$

In this subsection, we show that the reverse inequality also holds in some approximate sense.

Lemma 8. *Let \mathcal{C} be a class of graphs such that*

1. *if $G \in \mathcal{C}$, then every graph obtained from G by adding a new vertex adjacent to exactly one vertex of G is in \mathcal{C} as well, and*
2. *there exists a function $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for every integer $r \geq 0$, for every graph $G \in \mathcal{C}$ and every $A \subseteq V(G)$,*

$$|\{N_G^r[v] \cap A \mid v \in V(G)\}| \leq f(r, |A|).$$

Then, for every graph $G \in \mathcal{C}$ and every $A \subseteq V(G)$, we have

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq f(r, (r+1)|A|).$$

Proof. We build G' from G by gluing on every vertex $a \in A$ a path $a_0 a_1 \dots a_r$ of length r with $a_r = a$, and we let $A' = \{a_i \mid a \in A, 0 \leq i \leq r\}$ denote the union of the vertex set of these paths (including A). By hypothesis, G' is in \mathcal{C} .

We claim that for every $u, v \in V(G)$, if $N_{G'}^r[u] \cap A' = N_{G'}^r[v] \cap A'$ then $\pi_{r,G}[u \rightarrow A] = \pi_{r,G}[v \rightarrow A]$. Indeed, for every $a \in A$, $\pi_{r,G}[u \rightarrow A](a) = \min\{i \mid a_i \in N^r[u]\} = \min\{i \mid a_i \in N^r[v]\} = \pi_{r,G}[v \rightarrow A](a)$ with the convention $\min \emptyset = +\infty$.

Hence, the result follows from the fact that $|A'| = (r+1)|A|$. \square

Combining Lemma 8 with Corollary 7 gives the following corollary.

Corollary 9. *Let t be an integer with $t \geq 3$ and let G be a K_t -minor-free graph. Then, for every nonempty set $A \subseteq V(G)$ and every integer $r \geq 0$*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq (r+1)^{t-1}|A|^{t-1}$$

In particular, if G is planar, then $|\Pi_{r,G}[V(G) \rightarrow A]| \leq (r+1)^4|A|^4$.

We remark that a similar bound for metric dimension is proved in [3] using similar methods.

Using Lemma 5, we also deduce the following bound.

Corollary 10. *If G is a plane graph and $A \subseteq V(G)$ is non empty and is included in the outerface of G , then for every integer $r \geq 0$*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq (r+1)^3|A|^3.$$

Proof. Consider G' as in the proof of Lemma 8. Observe that G' is planar and can be drawn in the plane in such a way that A' is included in its outerface. Hence, by Lemma 5 and Lemma 4,

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq |\{A' \cap N_{G'}^r[u] \mid u \in V(G)\}| \leq |A'|^3$$

and the result follows as $|A'| = (r+1)|A|$. \square

Corollaries 9 and 10 are our base blocks to derive polynomial bounds on the profile and neighborhood complexities. Most of the work will consist in reducing to the case where A has size polynomial in r .

2.3 Guarding sets

Definition 11. *Let G be a graph, let $A \subseteq V(G)$, and let $r, p \geq 0$ be integers. We say that a family $\mathcal{S} \subseteq \mathcal{P}(V(G))$ is an (r, p) -guarding set for A in G if*

- (i) $|S| \leq p$ for every $S \in \mathcal{S}$, and
- (ii) for every vertex $v \in V(G)$, there exists $S \in \mathcal{S}$ such that S intersects every (v, A) -path of length at most r in G (if any).

Typically we will be looking for an $(r, \text{poly}(r))$ -guarding set of size $\text{poly}(r)|A|$. This is a useful tool to bound the profile complexity thanks to the following property.

Lemma 12. *Let G be a graph, let $A \subseteq V(G)$, let $r, p \geq 0$ be integers, and let \mathcal{S} be an (r, p) -guarding set for A . Suppose that, for some non-decreasing function f ,*

$$|\Pi_{r,G}[V(G) \rightarrow A']| \leq f(r, |A'|)$$

for every $A' \subseteq V(G)$. Then

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq f(r, p)|\mathcal{S}|.$$

Proof. For every $S \in \mathcal{S}$, let V_S be the set of vertices $v \in V(G)$ such that S intersects every shortest (v, a) -path in G for all $a \in A \cap N^r[v]$. The family $\{V_S\}_{S \in \mathcal{S}}$ covers $V(G)$ by (ii), hence $|\Pi_{r,G}[V(G) \rightarrow A]| \leq \sum_{S \in \mathcal{S}} |\Pi_{r,G}[V_S \rightarrow A]|$.

Now fix some $S \in \mathcal{S}$, and let us bound $|\Pi_{r,G}[V_S \rightarrow A]|$. Let $v \in V_S$. For every $a \in A$ we have

$$\begin{aligned}\pi_{r,G}[v \rightarrow a] &= \text{Cap}_r \circ \min_{s \in S} (\text{dist}_G(v, s) + \text{dist}_G(s, a)) \\ &= \text{Cap}_r \circ \min_{s \in S} (\pi_{r,G}[v \rightarrow s] + \text{dist}_G(s, a)).\end{aligned}$$

Therefore, $\pi_{r,G}[v \rightarrow A]$ is fully determined by $\pi_{r,G}[v \rightarrow S] \in \Pi_{r,G}[V_S \rightarrow S]$. But by assumption, $|\Pi_{r,G}[V_S \rightarrow S]| \leq f(r, |S|) \leq f(r, p)$. Hence $|\Pi_{r,G}[V(G) \rightarrow A]| \leq |\mathcal{S}|f(r, p)$ as claimed. \square

We remark that Lemma 12 still holds if we relax the condition (ii) in the definition of guarding sets by:

(ii') for every $v \in V(G)$, there exists $S \in \mathcal{S}$ such that for every $a \in A$, either $\text{dist}_G(v, a) > r$, or S intersects at least one shortest (v, a) -path in G .

However, all our constructions of guarding sets will satisfy (ii) so we prefer to keep this stronger definition.

2.4 Weak coloring numbers

Given a graph G , a linear ordering $<$ on $V(G)$, an integer $r \geq 0$, and two vertices $u, v \in V(G)$, we say that v is *r-weakly reachable* from u if there exists a (u, v) -path P of length at most r such that the minimum of P for $<$ is v . We denote by $\text{WReach}_r[G, <, u]$ the set of r -weakly reachable vertices from u in G for $<$. (Observe that this set contains u .) The *r-weak coloring number* of a graph G is the minimum over all linear orderings $<$ of $V(G)$ of the quantity $\max_{u \in V(G)} |\text{WReach}_r[G, <, u]|$.

Weak coloring numbers give another characterization of graph classes with bounded expansion: Zhu [30] proved that a class \mathcal{C} of graphs has bounded expansion if and only if, for every $r \geq 0$, the r -weak coloring number of every graph in \mathcal{C} is bounded by a function of r (which depends only on \mathcal{C}).

Weak coloring numbers will be a useful tool to design guarding sets when there is a suitable tree decomposition, in particular in the proofs of Theorems 17 and 20. We will use the following result on weak coloring numbers. (Let us recall that, given a tree decomposition $(T, (X_z)_{z \in V(T)})$ of a graph G , we consider T to be rooted, and for $v \in V(G)$, we denote by t_v the root of the subtree of T induced by v .)

Theorem 13 (Grohe, Kreutzer, Rabinovich, Siebertz, and Stavropoulos [16]). *Let t, r be nonnegative integers, let G be a graph, and let $(T, (X_z)_{z \in V(T)})$ be a tree decomposition of G of width at most t . Let $<$ be a linear ordering of $V(G)$ such that for all $v, w \in V(G)$, if $t_v <_T t_w$, then $v < w$. Then, for every vertex $v \in V(G)$,*

$$|\text{WReach}_r[G, <, v]| \leq \binom{r+t}{t} = \mathcal{O}_t(r^t).$$

3 Proper minor-closed classes of graphs

In this section we prove Theorem 3, which states that proper minor-closed graph classes have polynomial profile complexity. Our proof is a modification of the proof in [25] that graph classes with bounded expansion have bounded neighborhood complexity. The latter proof uses weak coloring numbers as their main tool to bound neighborhood complexity. While weak coloring numbers can be exponential for graph classes with bounded expansion, it is known that graphs excluding a fixed minor have polynomial weak coloring numbers:

Theorem 14 ([17]). *For every positive integers t and r with $t \geq 4$, and for every K_t -minor-free graph G ,*

$$\text{wcol}_r(G) \leq \binom{r+t-2}{t-2} (t-3)(2r+1) \leq 2(t-3)(r+1)^{t-1}.$$

However, this fact alone is not enough to turn the proof from [25] into a proof that graphs excluding a minor have polynomial neighborhood complexity. This is because in one of the last steps of the proof in [25], one needs to enumerate all subsets of a given set of vertices, which inherently results in a bound on neighborhood complexity that is exponential. In our proof, we avoid this costly step and use instead an argument based on VC-dimension to finish the proof.

Theorem 3 follows from the following result.

Theorem 15. *For every positive integers t and r with $t \geq 4$, for every K_t -minor-free graph G , for every set A of vertices of G ,*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq (2^t(t-3))^t \cdot (r+1)^{t^2-1} |A|.$$

Proof. Using Theorem 14, let $<$ be an ordering of $V(G)$ witnessing the fact that

$$\text{wcol}_{2r}(G) \leq 2(t-3)(2r+1)^{t-1} \leq 2^t(t-3)(r+1)^{t-1}.$$

Let

$$B = \bigcup_{a \in A} \text{WReach}_r[G, <, a].$$

Observe that $|B| \leq 2^t(t-3)(r+1)^{t-1} |A|$. For every vertex $b \in B$, let

$$S_b = \text{WReach}_{2r}[G, <, b].$$

For every vertex $u \in V(G)$ at distance at most r from a vertex in A , let

$$\phi(u) = \max_{<} (B \cap \text{WReach}_r[G, <, u]).$$

Claim 16. *For every vertex $u \in V(G)$ at distance at most r from A , $S_{\phi(u)}$ intersects every (u, A) -path of length at most r in G .*

Proof. Let Q be a path from u to $\phi(u)$ witnessing the fact that $\phi(u) \in \text{WReach}_r[G, <, u]$. Consider a path P of length at most r from u to some vertex $a \in A$. Let $w = \min_{<} V(P)$, let P_1 be the subpath of P from u to w and let P_2 the subpath of P from w to a . Then P_1 witnesses the fact that $w \in \text{WReach}_r[G, <, u]$ and P_2 witnesses the fact that $w \in \text{WReach}_r[G, <, a] \subseteq B$. Hence $w \in B \cap \text{WReach}_r[G, <, u]$. In particular, $w \leq \phi(u)$ by definition of $\phi(u)$. Then the concatenation of Q with P_1 witnesses the fact that $w \in \text{WReach}_{2r}[G, <, \phi(u)] = S_{\phi(u)}$, and so $S_{\phi(u)}$ intersects $V(P)$ as claimed. \square

Hence $\{S_w \mid w \in B\}$ is an $(r, 2^t(t-3)(r+1)^{t-1})$ -guarding set since $|S_w| \leq 2^t(t-3)(r+1)^{t-1}$ for every $w \in B$.

Now, we use the fact that K_t -minor-free graphs have bounded VC-dimension: By Corollary 9,

$$|\Pi_{r,G}[V(G) \rightarrow S]| \leq (r+1)^{t-1} |S|^{t-1} \leq (2^t(t-3))^{t-1} (r+1)^{t^2-t}$$

for every set S of at most $2^t(t-3)(r+1)^{t-1}$ vertices of G . Using Lemma 12, we deduce that

$$\begin{aligned} |\Pi_{r,G}[V(G) \rightarrow A]| &\leq (2^t(t-3))^{t-1}(r+1)^{t^2-t} \cdot |\{S_w \mid w \in B\}| \\ &\leq (2^t(t-3))^{t-1}(r+1)^{t^2-t} \cdot |B| \\ &\leq (2^t(t-3))^{t-1}(r+1)^{t^2-t} \cdot 2^t(t-3)(r+1)^{t-1}|A| \\ &= (2^t(t-3))^t(r+1)^{t^2-1}|A|, \end{aligned}$$

which concludes the proof of the theorem. \square

4 Graphs of bounded treewidth

Since graphs of treewidth at most t are K_{t+2} -minor-free, Theorem 15 implies that these graphs have profile complexity in $\mathcal{O}_t(r^{t^2+2t})$. In this section, we show the more precise upper bound $\mathcal{O}_t(r^{2t})$.

Theorem 17. *Let t, r be nonnegative integers, let G be a graph of treewidth at most t , and let $A \subseteq V(G)$. Then, there is an (r, t) -guarding set for A in G of size at most $(t+1)\binom{r+t}{t}|A|$.*

Proof. Consider a tree decomposition $(T, (X_z)_{z \in V(T)})$ of width at most t rooted at an arbitrarily chosen node $s \in V(T)$ and a linear order $<$ on $V(G)$ as in Theorem 13. In addition, we suppose that for every edge $yz \in E(T)$, $|X_y \cap X_z| \leq t$ (otherwise $X_y = X_z$ and we can contract yz). As before, we denote by t_v the root of the subtree T_v for every vertex $v \in V(G)$.

Let $A' = \bigcup_{a \in A} \text{WReach}_r[G, <, a]$ and $B = \{t_v \mid v \in A'\}$. Finally, we set $\mathcal{S} = \{X_z \mid z \in B, |X_z| \leq t\} \cup \bigcup_{z \in B, |X_z|=t+1} \binom{X_z}{t}$. Clearly $|\mathcal{S}| \leq \binom{t+1}{t}|A| \cdot \max_{a \in A} \text{WReach}_r[G, <, a]$ and so by Theorem 13 $|\mathcal{S}| \leq (t+1)\binom{r+t}{t}|A|$. Moreover, for every $S \in \mathcal{S}$, $|S| \leq t$, so it only remains to show that for every vertex $v \in V(G)$, there exists $S \in \mathcal{S}$ that intersects every (v, A) -path of length at most r in G . Let thus $v \in V(G)$, and assume that v is at distance at most r from A (otherwise, there is nothing to show).

We find the desired set S as follows: Consider the unique (t_v, s) -path P in T , and let z be the first node in P that belongs to B . Such a node z exists, which can be seen as follows: Consider a (v, A) -path Q of length at most r in G . Let w be the minimum of Q . Then $w \in \text{WReach}_r[G, <, a]$ for some $a \in A$, which implies $t_w \in B$. Furthermore, $\{y \in V(T) \mid V(Q) \cap X_y \neq \emptyset\}$ induces a connected subgraph of T , thus a subtree of T , and by our choice of the ordering $<$, t_w is the root of this subtree.

If $z = t_v$, then take for S an arbitrary subset of X_{t_v} of size t that contains v if $|X_{t_v}| = t+1$, or $S = X_{t_v}$ if $|X_{t_v}| \leq t$, and the result is clear. Otherwise, let y be the predecessor of z in P , and take $S \subseteq X_y$ of size at most t such that $X_y \cap X_z \subseteq S$ if $|X_z| = t+1$, and take $S = X_z$ if $|X_z| \leq t$. In both cases, $S \in \mathcal{S}$. Now consider a (v, a) -path Q of length at most r in G for some $a \in A$. Then Q contains a minimal vertex w for $<$, and so t_w is in P . Moreover, $w \in \text{WReach}_r[G, <, a]$ so $w \in A'$ and $t_w \in B$. It follows that the node z is on the (t_v, t_w) -path in T . Given our choice of S , the set S intersects every (v, w) -path, and in particular intersects Q . This concludes the proof that \mathcal{S} is an (r, t) -guarding set for A in G of size at most $(t+1)\binom{r+t}{t}|A|$. \square

Combining Theorem 17 with Lemma 12 and the fact that $|\Pi_{r,G}[V(G) \rightarrow A']| \leq (r+2)^{|A'|}$ for every $A' \subseteq V(G)$, we deduce that graphs of bounded treewidth have polynomial neighborhood complexity.

Corollary 18. *For every graph G of treewidth at most t , every $A \subseteq V(G)$, and every nonnegative integer r ,*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq (t+1)(r+2)^t \binom{r+t}{t} |A| = \mathcal{O}_t(r^{2t}|A|).$$

5 Graphs admitting a product structure

The *strong product* of two graphs A and B is the graph $A \boxtimes B$ with vertex set $V(A) \times V(B)$ where two distinct vertices $(a, b), (a', b')$ are adjacent if

$$(a = a' \text{ or } aa' \in E(A)) \text{ and } (b = b' \text{ or } bb' \in E(B)).$$

For example, the strong product of two paths is a grid with the diagonals in each square. Our interest in strong products of graphs comes from the fact that planar graphs have the following ‘product structure’.

Theorem 19 (Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [13]). *For every planar graph G , there is a graph H of treewidth at most 3 and a path P such that G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_3$.*

In this section, we lift the construction of guarding sets in graphs of bounded treewidth to graphs having such a product structure.

Theorem 17 can be extended to graphs with a product structure as follows.

Theorem 20. *Let t, r be nonnegative integers. Let H be a graph of treewidth at most t , let c be a positive integer and let P be a path. Let $G \subseteq H \boxtimes P \boxtimes K_c$ and let $A \subseteq V(G)$. Then, G has a $(r, 4c(t+1)r)$ -guarding set of size at most $2 \binom{r+t}{t} |A|$.*

Proof. Let $(T, (X_z)_{z \in V(T)})$ be a tree decomposition of H of width at most t rooted at an arbitrary node $s \in V(T)$.

Each vertex of P defines a corresponding *row* of the product $H \boxtimes P$ in a natural way. Ordering the vertices of P from one of its ends to the other, consider the corresponding rows, and let L_0, L_1, \dots, L_p denote their intersections with $V(G)$, which we call a *layering* of G . Observe that, for every edge $uv \in E(G)$, the two vertices u, v are either in the same layer or in consecutive layers. Observe also that every ball of radius r in G is included in at least one of the intervals

$$I_j = \bigcup_{k=0}^{4r-1} L_{2rj+k}$$

for $j \geq 0$. (We use the convention that $L_i = \emptyset$ for $i > p$.)

Fix one of these intervals I_j , and let $A_j = A \cap I_j$. For every $a \in A_j$, we denote by u_a the (unique) vertex in H such that $a \in V(H[\{u_a\}] \boxtimes P \boxtimes K_c)$. Let $A'_j = \bigcup_{a \in A_j} \text{WReach}_r[H, \langle \cdot, u_a \rangle] \subseteq V(H)$ where $\langle \cdot, u_a \rangle$ is as in Theorem 13. By Theorem 13, $|A'_j| \leq \binom{r+t}{t} |A_j|$.

We set $\mathcal{S}_j = \{V(H[X_{t_u}] \boxtimes P \boxtimes K_c) \cap I_j \mid u \in A'_j\}$, and finally $\mathcal{S} = \bigcup_{j \geq 0} \mathcal{S}_j$.

First, we have $|\mathcal{S}| \leq (t+1) \cdot c \cdot 4r$ for every $S \in \mathcal{S}$, by construction. Second, $|\mathcal{S}| \leq \sum_{j \geq 0} |\mathcal{S}_j| \leq \sum_{j \geq 0} \binom{r+t}{t} |A_j| \leq 2 \binom{r+t}{t} |A|$ (since each $a \in A$ is included in A_j for at most two distinct indices j).

It remains to show property (ii). Let $v \in V(G)$ be a vertex such that $\text{dist}_G(u, A) \leq r$. There exists j such that $N^r[v] \subseteq I_j$, and so $A \cap N^r[v] \subseteq A_j$. Let $z \in V(T)$ be such that $v \in X_z$. Consider the (unique) (z, s) -path $P_{z,s}$ in T , and let y be the first node in $P_{z,s}$ that also belongs to $\{t_u \mid u \in A'_j\}$. Note that the node y is well defined because there is

at least one (v, A) -path Q of length at most r in G , and the path Q induces a connected subgraph \tilde{Q} of H , which has a minimum w with respect to $<$. Moreover, $w \in A'_j$ (as follows from the definition of A'_j and the existence of \tilde{Q}), and $t_w \in V(P_{z,s})$ (because of our specific ordering $<$).

We take $S = V(H[X_y] \boxtimes P \boxtimes K_c) \cap I_j$, which is in \mathcal{S}_j . We claim that S intersects every (v, A) -path of length at most r in G . The argument is similar to the one showing the existence of y above: For every $a \in A$ and every (v, a) -path Q of length at most r , we have $V(Q) \subseteq N^r[v] \subseteq I_j$. Moreover, Q induces in H a connected subgraph \tilde{Q} , which has a minimum w with respect to $<$. Then t_w belongs to the unique (z, s) -path in T , and it follows that X_y intersects $V(\tilde{Q})$ by our choice of y , and so S intersects Q . This proves property (ii), and concludes the proof of the theorem. \square

6 Planar graphs

Since planar graphs are K_5 -minor-free, Theorem 15 directly implies that planar graphs have profile complexity (and so neighborhood complexity) in $\mathcal{O}(r^{24})$. When restricting the proof of Theorem 15 to planar graphs, one may use a better bound of $\mathcal{O}(r^3)$ for their weak coloring numbers (as proved in [17]), which results in a better upper bound of $\mathcal{O}(r^{16})$ for the profile complexity of planar graphs. In this section we improve on these bounds.

We give three proofs. The first one is very short, it combines the product structure of planar graphs [13] mentioned in the previous section together with Theorem 20 and gives a $\mathcal{O}(r^{11})$ bound on the profile complexity. The second one starts by applying a lemma of Sokolowski [28] to reduce to the case where the set A under consideration is the vertex set of the outerface, and also relies on the existence of ‘sparse covers’ for planar graphs (see below). It results in a $\mathcal{O}(r^6)$ bound. The third proof is a variant of the second one, giving a $\mathcal{O}(r^4)$ bound. It uses two extra ingredients, a recent theorem of Li and Parter [18] (Theorem 27 below) and some form of decomposition of planar graphs using ‘tripods’ that is directly inspired from the proof of the product structure of planar graphs [13]. We note that this third proof resulted from discussions with Jacob Holme, Erik Jan van Leeuwen, and Marcin Pilipczuk, who upon reading an earlier version of this paper pointed out reference [18] to us, and the fact that it could be used to improve the $\mathcal{O}(r^6)$ bound to $\mathcal{O}(r^4)$. We are grateful to them for their helpful feedback.

6.1 First proof

We start with the $\mathcal{O}(r^{11})$ bound on the profile complexity of planar graphs.

Theorem 21. *If G is a planar graph, r is a nonnegative integer, and A is a nonempty set of vertices of G , then*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq 10^8(r+1)^{11}|A|.$$

Proof. By Theorems 19 and 20, G has an $(r, 48r)$ -guarding set $\mathcal{S} \subseteq \mathcal{P}(V(G))$ of size at most $2\binom{r+3}{3}|A|$. It follows from Lemma 12 and Corollary 9 that

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq (r+1)^4(48r)^4|\mathcal{S}| \leq (48)^4(r+1)^8 2\binom{r+3}{3}|A| \leq 2(48)^4(r+1)^{11}|A|$$

and so $|\Pi_{r,G}[V(G) \rightarrow A]| \leq 10^8(r+1)^{11}|A|$. \square

6.2 Second proof

We begin this second proof with a lemma similar to Lemma 12.

Lemma 22. *Let G be a graph, let A be a nonempty set of vertices of G , and let r be a nonnegative integer. Suppose that V_0, V_1, \dots, V_q is a collection of subsets of $V(G)$ with $\bigcup_{i=0}^q V_i = V(G)$ and S_0, \dots, S_q is a collection of set of vertices such that for every $i \in [0, q]$ $S_i \subseteq V_i$, and for every vertex $v \in V(G)$, there exists $i \in [0, q]$ so that $v \in V_i$ and for every $a \in A \cap N_G^r[v]$, there is a shortest (v, a) -path P in G whose maximal prefix in V_i intersects S_i (informally, P must intersect S_i before leaving V_i). Then*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq \sum_{i=0}^q |\Pi_{r,G[V_i]}[V_i \rightarrow S_i]|.$$

Proof. Let $v \in V(G)$. Let $i \in [0, q]$ be as in the statement, then for every $a \in A$

$$\begin{aligned} \pi_{r,G}[v \rightarrow a] &= \text{Cap}_r \circ \min_{s \in S_i} (\text{dist}_{G[V_i]}(v, s) + \text{dist}_G(s, a)) \\ &= \text{Cap}_r \circ \min_{s \in S_i} (\pi_{r,G[V_i]}[v \rightarrow s] + \text{dist}_G(s, a)). \end{aligned}$$

It follows that $|\Pi_{r,G}[V(G) \rightarrow A]| \leq \sum_{i=0}^q |\Pi_{r,G[V_i]}[V_i \rightarrow S_i]|$. \square

The following lemma due to Sokołowski [28] is a reduction for planar graphs to the case where the vertices in A are exactly the vertices of the outerface, at the cost of a small increase in the size of A . It is a key ingredient of our second and third proofs.

Lemma 23 (Sokołowski [28]). *Let G be a connected planar graph, let A be a nonempty set of vertices of G , and let r be a nonnegative integer. If every vertex of G is at distance at most r from A , then one can construct a plane graph G' and a set $A' \subseteq V(G')$ such that*

- (i) $|A'| \leq 2(2r + 1)|A|$;
- (ii) the outerface of G' is bounded by a cycle;
- (iii) A' is the vertex set of the outerface of G' , and
- (iv) $|\Pi_{r,G}[V(G) \rightarrow A]| \leq |\Pi_{r,G'}[V(G') \rightarrow A']|$.

Since this lemma is not explicitly stated in [28], but rather used implicitly in a proof (see proof of Theorem 26 in [28]), we sketch here its elegant proof.

Sketch of proof. Since G is connected and every vertex in G is at distance at most r from a vertex in A , we deduce that there is a tree T in G such that $A \subseteq V(T)$ and $|E(T)| \leq (2r + 1)(|A| - 1)$. Such a tree can be obtained by considering a tree T with $A \subseteq V(T)$ of minimal size.

The graph G' is now obtained from G by “cutting the plane open” along T : the tree T is replaced by a face A' whose vertices are the occurrences of the vertices in T along an Euler tour of T compatible with the embedding. See Figure 1. Since every vertex u in T corresponds to $d_T(u)$ vertices in A' , we have $|A'| = \sum_{u \in V(T)} d_T(u) = 2|E(T)| \leq 2(2r + 1)|A|$. Moreover, one can check that for every $x, y \in V(G) \setminus A$, if x and y have the

same profile at distance r on A' in G' , then x and y have the same profile at distance r on A in G . We conclude that

$$\begin{aligned} |\Pi_{r,G}[V(G) \rightarrow A]| &\leq |A| + |\Pi_{r,G}[V(G) \setminus A \rightarrow A]| \\ &\leq |A| + |\Pi_{r,G'}[V(G') \setminus A' \rightarrow A']| \\ &\leq |A'| + |\Pi_{r,G'}[V(G') \setminus A' \rightarrow A']| \\ &\leq |\Pi_{r,G'}[V(G') \rightarrow A']|. \end{aligned}$$

□

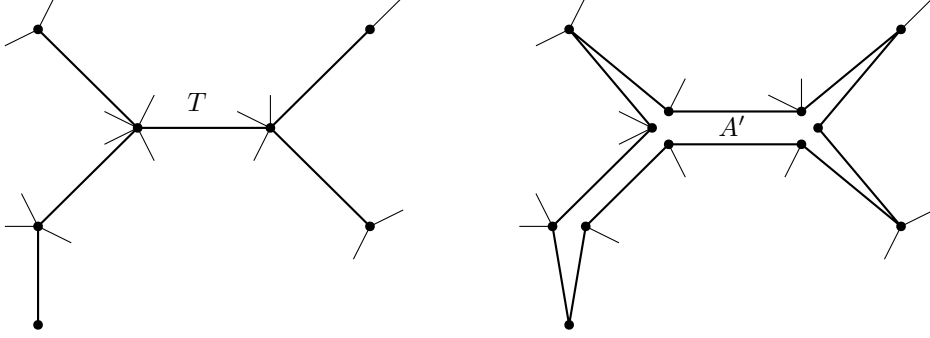


Figure 1: The construction of G' and A' in the proof of Lemma 23

We will also use the following objects called ‘sparse covers’, which will allow us to focus on graphs with radius $\mathcal{O}(r)$ when bounding the profile complexity at distance r . Let G be a graph. Given integers $r, d, k \geq 0$, a family $\mathcal{X} \subseteq \mathcal{P}(V(G))$ is an (r, d, k) -sparse cover of G if

- (i) for every vertex $v \in V(G)$, there exists $X \in \mathcal{X}$ such that $N_G^r[v] \subseteq X$,
- (ii) for every $X \in \mathcal{X}$, the graph $G[X]$ has radius at most d , and
- (iii) for every vertex $v \in V(G)$, we have $|\{X \in \mathcal{X} \mid v \in X\}| \leq k$.

The following two theorems provide sparse covers for planar graphs and graphs excluding a minor.

Theorem 24 (Busch, LaFortune, and Tirthapura [9]). *For every planar graph G and nonnegative integer r , G has an $(r, 24r, 18)$ -sparse cover.*

Theorem 25 (Abraham, Gavaille, Malkhi, and Wieder [1]). *For every K_t -minor-free graph G and nonnegative integer r , G has an $(r, \mathcal{O}(t^2r), 2^{\mathcal{O}(t)}t!)$ -sparse cover.*

We are now ready to prove a $\mathcal{O}(r^6)$ bound on the profile complexity of planar graphs.

Theorem 26. *Let G be a planar graph, let A be a nonempty set of vertices of G , and let r be a nonnegative integer. Then,*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq 10^8(r+1)^6|A|.$$

Proof. If some vertex is at distance more than r from A , then we can remove it without decreasing the size of $\Pi_{r,G}[V(G) \rightarrow A]$ (remember that for convenience, the profile which is $+\infty$ everywhere does not belong to this set). Thus, we may assume that every vertex in G is at distance at most r from at least one vertex of A . Moreover, each connected component of G can be considered separately (with the restriction of A to the component), so we may suppose that G is connected.

By Lemma 23, there exists a plane graph G' and $A' \subseteq V(G)$ such that

- (i) $|A'| \leq 2(2r+1)|A|$;
- (ii) the outerface of G' is bounded by a cycle C' ;
- (iii) A' is the vertex set of the outerface of G' , and
- (iv) $|\Pi_{r,G}[V(G) \rightarrow A]| \leq |\Pi_{r,G'}[V(G') \rightarrow A']|$.

Next, apply Theorem 24 to G' , and let \mathcal{X} be an $(r, 24r, 18)$ -sparse cover of G' . Our goal is to bound $|\Pi_{r,G'}[X \rightarrow A' \cap X]|$ for each $X \in \mathcal{X}$.

Now consider some set $X \in \mathcal{X}$. We will show that $|\Pi_{r,G'}[X \rightarrow A' \cap X]| \leq 2 \times 49^3(r+1)^5|A' \cap X|$.

Case 1: $|A' \cap X| \leq r+1$.

We apply directly Corollary 10 and obtain

$$|\Pi_{r,G'}[X \rightarrow A' \cap X]| \leq (r+1)^3|A' \cap X|^3 \leq (r+1)^5|A' \cap X|.$$

Case 2: $|A' \cap X| > r+1$.

Consider the cycle C' as being oriented clockwise. Let a_1, \dots, a_ℓ denote the vertices in $A' \cap X$ according to their order on C' . Let us mark each vertex a_i with $i \in [\ell]$ such that $i = k(r+1) + 1$ for some integer $k \geq 0$. Note that there are at least two marked vertices, since $\ell = |A' \cap X| > r+1$. We say that two marked vertices are *consecutive* if they are consecutive among marked vertices in the clockwise order around C' . As $G'[X]$ has radius at most $24r$, there exists a vertex $s \in X$ that is at distance at most $24r$ in $G'[X]$ from every vertex in X . By considering the union of a shortest (a, s) -path for every marked vertex a , one can find a tree T in G' rooted at s containing all marked vertices and having height at most $24r$; see Figure 2. (The *height* of a rooted tree is defined as the maximum length of a path from a root to a leaf in the tree.)

Consider a pair a, b of consecutive marked vertices. Then there is a (unique) path $P_{a,b}$ linking a and b in T . Furthermore, $P_{a,b}$ contains at most $48r+1$ vertices. Let $V_{a,b}$ be the set of vertices of G' enclosed by the cycle $B_{a,b}$ defined as the union of $P_{a,b}$ and the path from a to b on the (oriented) cycle C' , including vertices in the cycle $B_{a,b}$ itself. We also denote by $G'_{a,b}$ the subgraph of G' with vertex set $V_{a,b}$ and edge set all the edges drawn in the closed disc bounded by $B_{a,b}$. In other words, $G'_{a,b}$ is obtained from $G'[V_{a,b}]$ by removing the chords of $B_{a,b}$ drawn outside $B_{a,b}$ in G' . Let $A'_{a,b}$ be the union of $V(P_{a,b})$ and all vertices in $A' \cap X$ on the (a, b) -path in C' going from a to b on C' in clockwise order. Observe that $|A'_{a,b}| \leq 49r+1$ and that all vertices in $A'_{a,b}$ lie on the outerface of $G'_{a,b}$. Thus, by Corollary 10, $|\Pi_{r,G'_{a,b}}[V_{a,b} \rightarrow A'_{a,b}]| \leq (r+1)^3(49r+1)^3$.

Now, let Q be the set of pairs (a, b) of consecutive marked vertices. Note that $|Q| = \left\lceil \frac{|A' \cap X|}{r+1} \right\rceil \leq 2 \frac{|A' \cap X|}{r+1}$. Hence,

$$\begin{aligned} |\Pi_{r,G'[X]}[X \rightarrow A' \cap X]| &\leq \sum_{(a,b) \in Q} |\Pi_{r,G'_{a,b}}[V_{a,b} \rightarrow A'_{a,b}]| \\ &\leq |Q|(r+1)^3(49r+1)^3 \\ &\leq 2 \times 49^3(r+1)^5|A' \cap X|. \end{aligned}$$

Thus, in both cases $|\Pi_{r,G'[X]}[X \rightarrow A' \cap X]| \leq 2 \times 49^3(r+1)^5|A' \cap X|$ holds for every $X \in \mathcal{X}$, as claimed.

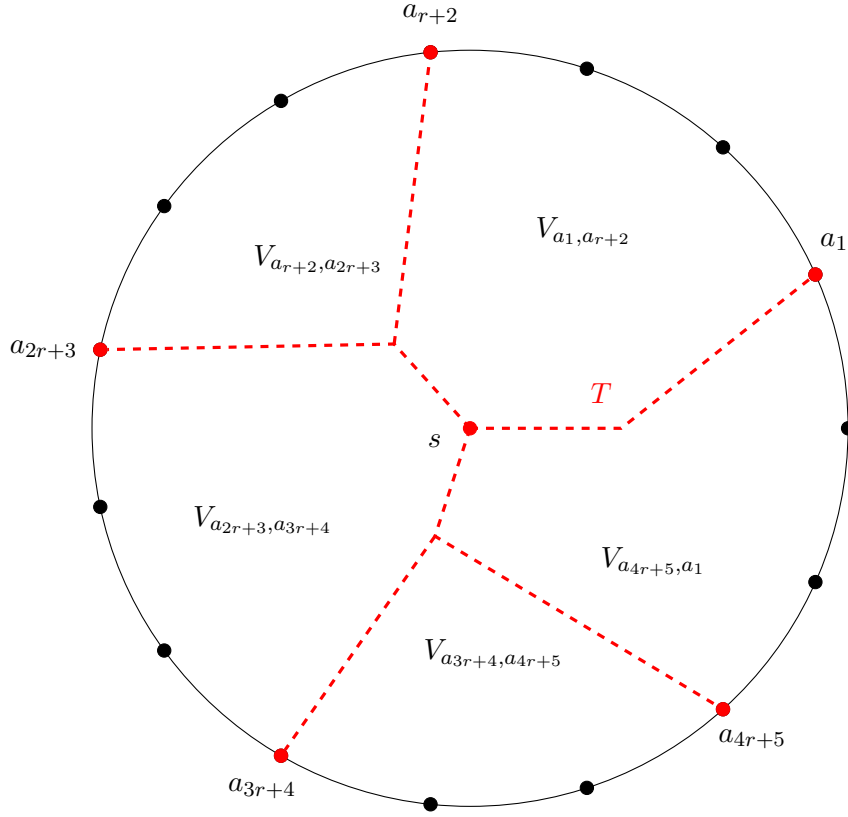


Figure 2: The main step in the proof of Theorem 26.

It follows that

$$\begin{aligned}
|\Pi_{r,G}[V(G) \rightarrow A]| &\leq |\Pi_{r,G'}[V(G') \rightarrow A']| && \text{by (iv)} \\
&\leq \sum_{X \in \mathcal{X}} |\Pi_{r,G'}[X \rightarrow A' \cap X]| && \text{by Lemma 22} \\
&\leq \sum_{X \in \mathcal{X}} 2 \times 49^3 (r+1)^5 |A' \cap X| \\
&\leq 2 \times 49^3 (r+1)^5 18 |A'| \\
&\leq 49^3 \times 36 (r+1)^5 2 (2r+1) |A| && \text{by (i)} \\
&\leq 10^8 (r+1)^6 |A|
\end{aligned}$$

which concludes the proof of the theorem. (Note that we made no attempt to optimize the constant 10^8 in the proof.) \square

6.3 Third proof

We will now improve the bound in Theorem 26 from $\mathcal{O}(r^6|A|)$ to $\mathcal{O}(r^4|A|)$ using the following theorem of Li and Parter [18].

Theorem 27 (Li and Parter [18]). *Let G be a plane graph, let A be a nonempty set of consecutive vertices on the boundary of a face of G , and let r be a nonnegative integer. Then $|\Pi_{G,r}[V(G) \rightarrow A]| \in \mathcal{O}(r|A|^3)$.*

A *near-triangulation* is a plane graph whose outerface is bounded by a cycle of the graph, and all of whose inner faces are bounded by triangles of the graph. We will use the following well-known variation on a lemma of Sperner, see e.g. [2].

Lemma 28 (Sperner’s Lemma). *Let G be a near-triangulation and suppose that the cycle bounding its outerface is partitioned into three non empty paths P_1, P_2, P_3 . Let $\phi : V(G) \rightarrow \{1, 2, 3\}$ be a coloring of the vertices of G such that $\phi(u) = i$ for every $u \in V(P_i), i = 1, 2, 3$. Then there is an inner face bounded by a triangle xyz in G with $\phi(x) = 1, \phi(y) = 2, \phi(z) = 3$.*

Given a vertex subset R of a graph G and a rooted forest F in G , we say that F is *rooted in R* if R coincides with the set of roots of the trees in F . The *height* of a rooted forest F is the maximum length of a path from a root to a leaf in F . The proof strategy for the following lemma is adapted from ideas in [23, 13], in particular it relies on a variant of the so-called ‘tripod decomposition’ from [13].

Lemma 29. *Let r be a positive integer. Let G be a connected plane graph whose outerface is bounded by a cycle C . Let \tilde{G} be a plane near-triangulation with $V(\tilde{G}) = V(G)$ obtained from G by triangulating every inner face of G arbitrarily. Suppose that G has a spanning forest F rooted in $V(C)$ whose height is at most r . Then, there exists a subgraph H of \tilde{G} satisfying:*

1. C is contained in H ;
2. H has at most $3|V(C)|/r + 1$ inner faces, and
3. every inner face of H is bounded by a cycle of H of length at most $6r$ containing at most three edges not in G .

Proof. We start by introducing some terminology used in the proof. A *vertical path* in F is a subpath of a path between a vertex and a root in F . Vertical paths are naturally oriented towards the root; the *source* and *sink* of a vertical path are defined according to this orientation. A *tripod* Y is the union of three pairwise disjoint vertical paths P_1, P_2, P_3 whose sources form a triangle t bounding a face in \tilde{G} . We call P_1, P_2, P_3 the *branches* of Y , t the *center* of Y , and the sinks of P_1, P_2, P_3 the *feet* of Y . The path obtained by taking the union of two branches and the edge of the center linking their sources is called a *side* of the tripod. (Thus, every tripod has three sides.)

We maintain a collection \mathcal{Y} of tripods, a distinguished foot f_Y of Y for every tripod $Y \in \mathcal{Y}$, and a subgraph H of \tilde{G} satisfying the following properties:

1. every face of H is bounded by a cycle of H (i.e. H is 2-connected);
2. C is contained in H , and thus C bounds the outerface of H ;
3. $V(H) = V(C) \cup \bigcup_{Y \in \mathcal{Y}} V(Y)$;
4. for every inner face f of H , the cycle C_f of H bounding f satisfies at least one of the following three properties:
 - (a) C_f has length at most $6r$ and contains at most two edges in $E(\tilde{G}) \setminus E(G)$,
 - (b) the intersection of C_f with C is a path, and moreover C_f is the union of two internally disjoint paths P_f, Q_f where $P_f = C_f \cap C$ and Q_f is a side of some tripod in \mathcal{Y} ,
 - (c) C_f is the center of a tripod Y in \mathcal{Y} ,
5. the number of inner faces of H is at most $3|\mathcal{Y}| + 1$, and

6. for every tripod $Y \in \mathcal{Y}$, f_Y belongs to C , and the elements of $(f_Y)_{Y \in \mathcal{Y}}$ are pairwise at distance at least r on C .

For the reader familiar with the tripod decomposition from [13], let us emphasize that here the tripods in \mathcal{Y} are allowed to intersect, and they always have three branches.

First, we show that objects $\mathcal{Y}, (f_Y)_{Y \in \mathcal{Y}}, H$ satisfying properties 1–6 do exist. This can be seen as follows. Partition C into three non-empty paths P_1, P_2, P_3 , and define a coloring ϕ of $V(\tilde{G})$ as follows:

- for every $i = 1, 2, 3$, for every $v \in V(P_i)$, let $\phi(v) = i$, and
- every vertex of \tilde{G} not in C receives the color of its parent in F .

Observe that the coloring is well defined, since all roots of F are colored in the first step. By Lemma 28, there is a triangle uvw in \tilde{G} such that $\phi(u) = 1, \phi(v) = 2$ and $\phi(w) = 3$. Let Y be the tripod with center uvw and whose branches are the three vertical paths starting at respectively u, v, w and ending on C (including their endpoints on C), and let f_Y be the sink of one (arbitrarily chosen) branch of Y . Then the triple $\mathcal{Y} = \{Y\}, (f_Y), H = C \cup Y$ satisfies properties 1–6. Then the triple $\mathcal{Y} = \{Y\}, (f_Y), H = C \cup Y$ satisfies properties 1–6: 1, 2, 3, 5 and 6 are clear from the definition, and for every inner face f of H , either 4b or 4c holds.

We now consider a choice for $\mathcal{Y}, (f_Y)_{Y \in \mathcal{Y}}, H$ satisfying properties 1–6 and maximizing $|\mathcal{Y}|$. We claim that every inner face f of H satisfies property 3 of the lemma, that is:

- (*) f is bounded by a cycle C_f of H of length at most $6r$ and C_f contains at most three edges not in G .

To see this, suppose for a contradiction that this does not hold for some inner face f of H . Then the cycle C_f does not satisfy properties 4a and 4c, and thus it must satisfy property 4b, that is, C_f is the union of two internally disjoint paths P_f, Q_f where $P_f = C_f \cap C$ and Q_f is a side of some tripod in $Y^* \in \mathcal{Y}$. Since F is a subgraph of G , this implies that C_f contains at most *one* edge not in G , namely the unique edge of the center of Y^* in C_f (if this edge is not in G). Since C_f does not satisfy property 4a, it follows that C_f has length at least $6r + 1$. Observe that Q_f has length at most $2r + 1$, since the rooted forest F has height at most r . Thus, P_f has length at least $6r + 1 - (2r + 1) = 4r$.

Let \tilde{G}_f be the subgraph of \tilde{G} consisting of all the vertices and edges of \tilde{G} that are in the closed disc defined by C_f . Let $Q_{f,1}$ and $Q_{f,2}$ be the two branches of Y^* that are contained in C_f . We orient C_f clockwise. Starting from P_f and going along C_f in this order, we may assume that we meet $Q_{f,1}$ before $Q_{f,2}$. Note that the orientation of C_f induces an orientation of P_f , which we use below.

Let $\phi : V(\tilde{G}_f) \rightarrow \{1, 2, 3\}$ be the coloring of the vertices of \tilde{G}_f defined as follows:

- vertices in $Q_{f,1}$ and the last r vertices of P_f are colored 1;
- vertices in $Q_{f,2}$ and the first r vertices of P_f are colored 2;
- the remaining uncolored vertices of P_f are colored 3, and
- every vertex in the proper interior of f receives the color of its parent in F .

Note that the coloring of a vertex v considered in the last item above is defined inductively, on the distance from v to the root r of F of the corresponding tree in F . It is well defined

since r is not in the proper interior of f and thus the vertical path from v to r in F intersects C_f , which is already colored.

Observe that there are vertices colored 3 on C_f since $|V(P_f)| \geq 4r \geq 2r + 1$. Thus, by Lemma 28, there exists a triangle uvw in \tilde{G}_f with $\phi(u) = 1, \phi(v) = 2, \phi(w) = 3$. Let Y' be the tripod with center uvw and whose three branches are the three vertical paths starting at respectively u, v, w and ending in C_f (including their endpoints in C_f), see Figure 3.

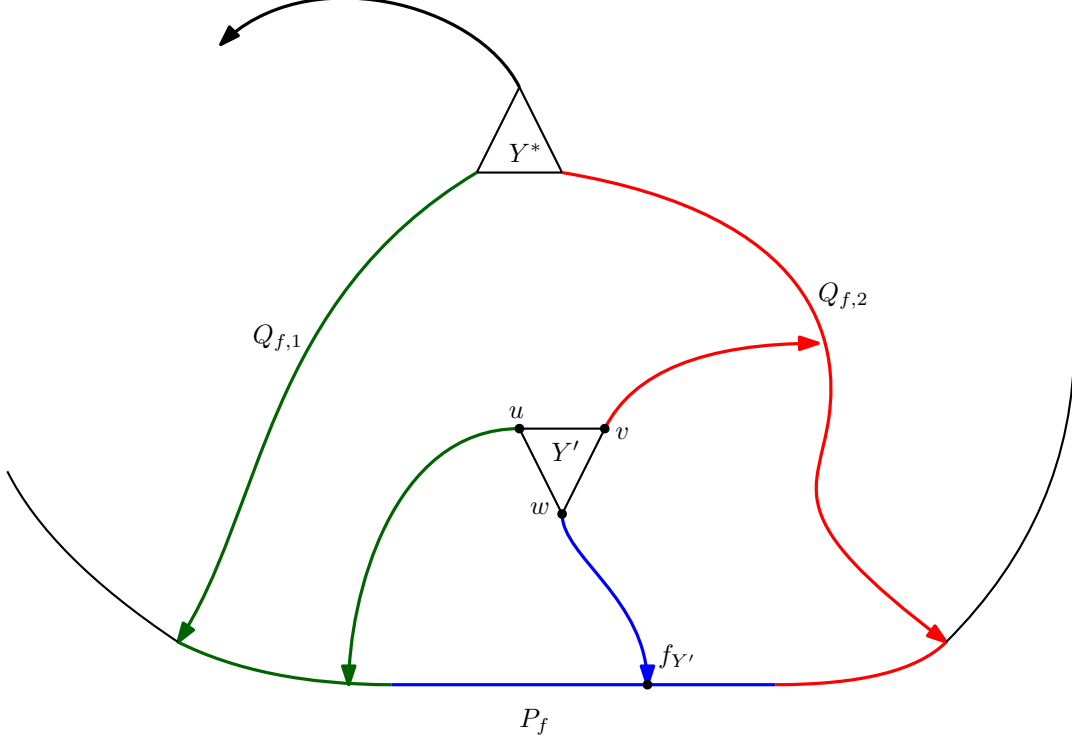


Figure 3: Illustration of the proof of Lemma 29.

We now define $\mathcal{Y}' := \mathcal{Y} \cup \{Y'\}$, we let $f_{Y'}$ be the sink of the branch of Y' starting at w , and let H' be obtained from H by adding the tripod Y' to H (that is, $H' := H \cup Y'$). We now check that \mathcal{Y}' , $(f_Y)_{Y \in \mathcal{Y}'}$, H' satisfy properties 1–6:

1. Every face of H' is bounded by a cycle of H' .
2. C is contained in H' .
3. $V(H') = V(C) \cup \bigcup_{Y \in \mathcal{Y}'} V(Y)$.
4. The three inner faces of H' that were not faces of H were created by adding the tripod Y' inside the face f of H . First the face bounded by the center of Y' satisfies 4c. We now consider the other created faces. Each of these inner faces is bounded by a cycle of H' . Observe that if such a cycle has a non-empty intersection with C then this intersection is a path contained in C . Now, among these three cycles, the cycle that corresponds to colors 1 and 2 contains at most $6r$ vertices, since there are at most $3r$ vertices colored 1 ($2r$ on C_f plus r from Y') and the same for color 2. This cycle contains at most two edges that are possibly not in G , namely one edge from the center of Y^* and the edge uv from the center of Y' . Thus, this cycle satisfies property 4a. By construction, the other two cycles corresponding to the pairs of colors (1, 3) and (2, 3) respectively, satisfy property 4b, since each such cycle

is the union of a path P contained in C_f with a side of Y' , and P intersects P_f in a subpath of P_f .

5. The number of faces of H' is at most $3|\mathcal{Y}| + 1 + 4 - 1 = 3|\mathcal{Y}'| + 1$.
6. In order to verify this property, one only needs to check $f_{Y'}$: It is in C and moreover it is at distance at least r in C from all the vertices in $\{f_Y\}_{Y \in \mathcal{Y}}$ since $f_{Y'}$ was colored with color 3. (Indeed, observe that every path in C from $f_{Y'}$ to some f_Y with $Y \in \mathcal{Y}$ either contains the last r vertices of P_f , colored 1, or the first r vertices of P_f , colored 2.)

However, this shows that the triple \mathcal{Y}' , $(f_Y)_{Y \in \mathcal{Y}'}$, H' is a better choice than \mathcal{Y} , $(f_Y)_{Y \in \mathcal{Y}}$, H , which contradicts the maximality of $|\mathcal{Y}|$.

We deduce that every inner face f of H satisfies property 3 of the lemma. Moreover, it follows from property 6 that $|\mathcal{Y}| \leq |V(C)|/r$, implying that H has at most $3|\mathcal{Y}| + 1 \leq 3|V(C)|/r + 1$ inner faces. Therefore, H satisfies all the required properties from the lemma. \square

We turn to the $\mathcal{O}(r^4)$ bound on the profile complexity of planar graphs, Theorem 1. It follows from the following theorem.

Theorem 30. *Let G be a planar graph, let A be a nonempty set of vertices of G , and let r be a nonnegative integer. Then, $|\Pi_{r,G}[V(G) \rightarrow A]| \in \mathcal{O}(r^4|A|)$.*

Proof. The theorem trivially holds for $r = 0$, so we may assume $r \geq 1$ (which is needed below when using Lemma 29). If there is a vertex at distance at least $r + 1$ from A in G , then we can remove it without decreasing $|\Pi_{r,G}[V(G) \rightarrow A]|$. Thus, we may assume that every vertex of G is at distance at most r from A . By Lemma 23, there is a plane graph G' whose outerface is bounded by a cycle such that $|\Pi_{r,G}[V(G) \rightarrow A]| \leq |\Pi_{r,G'}[V(G') \rightarrow A']|$ and $|A'| \leq 2(2r + 1)|A|$, where A' is the set of vertices on the outerface of G' .

Our goal is to bound $|\Pi_{r,G'}[V(G') \rightarrow A']|$ from above. Again, we may assume that every vertex in G' is at distance at most r from A' , and thus there is a spanning rooted forest F in G' whose set of roots coincide with A' and with height at most r . By Lemma 29, there is a subgraph H of a near triangulation \tilde{G} obtained from G' by triangulating every inner face of G' such that H has at most $3|A'|/r + 1$ inner faces, and every inner face of H is bounded by a cycle of length at most $6r$ and contains at most 3 edges in $E(\tilde{G}) \setminus E(G')$.

Let f_1, f_2, \dots, f_q be the inner faces of H , and for every $i \in [q]$ let C_{f_i} denote the cycle of H bounding f_i . For every $i \in [q]$, let \hat{G}_i be the union of the cycle C_{f_i} with the subgraph of G' drawn inside the face f_i of H . (Note that \hat{G}_i is not necessarily a subgraph of G' , since C_{f_i} could contain edges not in G' ; also, note that in G' the cycle C_{f_i} could have chords drawn outside f_i , which are then not included in \hat{G}_i .) We denote by V_i the vertex set of \hat{G}_i . Also, we let E_i be the set of edges in C_{f_i} that are not in $E(G')$. Thus, $|E_i| \leq 3$.

Let $i \in [q]$. For each edge $uv \in E_i$ (if any), replace in \hat{G}_i the edge uv by a path P_i between u and v of length $r + 1$, where the inner vertices are new vertices, and drawn where the edge uv was drawn in \tilde{G} . This results in a plane graph G_i whose outerface is bounded by a cycle C_i of length at most $9r$. Moreover, one can check that for every $v \in V_i$ and every $c \in V(C_{f_i})$, if $\text{dist}_{G'[V_i]}(v, c) \leq r$, then

$$\text{dist}_{G'[V_i]}(v, c) = \min_{c' \in V(C_{f_i})} (\text{dist}_{G_i}(v, c') + \text{dist}_{G'[V_i]}(c', c)).$$

(We remark that we do not simply have $\text{dist}_{G'[V_i]}(v, c) = \text{dist}_{G_i}(v, c)$ because in $G'[V_i]$ some chords of C_{f_i} could be drawn outside f_i , and those chords are not included in G_i .) Moreover, if $\text{dist}_{G'[V_i]}(v, c) > r$, then $\text{dist}_{G_i}(v, c) > r$. Hence

$$\pi_{r, G'[V_i]}(v, c) = \text{Cap}_r \circ \min_{c' \in V(C_{f_i})} (\pi_{r, G_i}(v, c') + \text{dist}_{G'[V_i]}(c', c)).$$

We deduce that

$$|\Pi_{r, G'[V_i]}[V_i \rightarrow V(C_{f_i})]| \leq |\Pi_{r, G_i}[V(G_i) \rightarrow V(C_{f_i})]| \leq |\Pi_{r, G_i}[V(G_i) \rightarrow V(C_i)]| \in \mathcal{O}(r^4),$$

where the rightmost bound comes from Lemma 27 applied on the pair $G_i, V(C_i)$.

Applying Lemma 22 with $S_i = V(C_{f_i})$ for every $i \in [q]$, we conclude that

$$|\Pi_{r, G'}[V(G') \rightarrow A]| \leq \sum_{i=1}^q |\Pi_{r, G[V_i]}[V_i \rightarrow V(C_{f_i})]| \in \mathcal{O}(r^4 |A'|/r)$$

and we conclude that $|\Pi_{r, G}[V(G) \rightarrow A]| \leq |\Pi_{r, G'}[V(G') \rightarrow A']| \in \mathcal{O}(r^4 |A|)$. This concludes the proof of the theorem. \square

7 Graphs on surfaces

In this section, we bound the profile complexity of graphs on surfaces. We start by recalling some definitions and results about graphs on surfaces. We assume some familiarity of the reader with this topic, see the textbook of Mohar and Thomassen [19] for background on this topic, and for undefined terms. The *Euler genus* of the sphere with g handles added is $2g$, and that of the sphere with g crosscaps added is g . The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G can be embedded into. A graph G is *cellularly embedded* in a surface if every face of the embedding is homeomorphic to an open disk. A cycle C in a graph embedded in a surface Σ is said to be *noncontractible* if C is noncontractible in Σ when seen as a closed curve. The cycle C is *separating* if removing C from Σ separates Σ in two pieces, and *nonseparating* otherwise. The cycle C is *1-sided* if its neighborhood on the surface is homeomorphic to the Möbius band, and *2-sided* otherwise, in which case its neighborhood is homeomorphic to the cylinder.

We need two lemmas about the standard operation of ‘cutting along’ a cycle in a graph embedded in a surface. We refer the reader to [19, Section 4.2] for the definition of this operation. Here, we limit ourselves to recalling informally the three different situations than can occur when cutting along a cycle C : (1) If C is separating, then cutting along C produces two graphs, each having a copy of the cycle C . (2) If C is nonseparating and 2-sided, then cutting along C produces one graph, with two vertex-disjoint cycles C_1 and C_2 that are copies of C . (3) If C is nonseparating and 1-sided, then cutting along C produces one graph, with one cycle C_1 corresponding to C but having twice the length of C .

Lemma 31 ([19, Proposition 4.2.1]). *If G is a graph cellularly embedded in a surface of Euler genus g , and C is a noncontractible separating cycle C , then cutting along C gives two graphs G_1 and G_2 of Euler genera g_1 and g_2 , respectively, such that $g_1 < g$, $g_2 < g$, and $g_1 + g_2 = g$.*

Lemma 32 ([19, Proposition 4.2.4]). *If G is a graph cellularly embedded in a surface of Euler genus g , and C is a noncontractible nonseparating cycle C , then cutting along C gives one graph \tilde{G} with Euler genus $\tilde{g} < g$.*

A *geodesic* in a graph G is a path in G that is shortest among all paths connecting its two endpoints. We will need the following well-known lemma, which follows from Proposition 4.3.1(a) in [19].

Lemma 33 ([19, Proposition 4.3.1(a)]). *If G is a graph cellularly embedded in a surface of Euler genus $g > 0$, then there exists a noncontractible cycle C that is the union of two geodesics in G having the same endpoints.*

We may now bound the number of profiles for graphs embedded in a fixed surface.

Theorem 34. *Let $h : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $|\Pi_{r,G}[V(G) \rightarrow A]| \leq h(r)|A|$ for every planar graph G , every nonempty set A of vertices of G , and every nonnegative integer r . Then, there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every nonnegative integer g , every graph G of Euler genus g , every nonempty set A of vertices of G , and every nonnegative integer r ,*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq f(g)h(r)(r + |A|);$$

in particular,

$$|\Pi_{r,G}[V(G) \rightarrow A]| \in \mathcal{O}_g(r^4(r + |A|))$$

using the bound $h(r) \in \mathcal{O}(r^4)$ from Theorem 30.

The proof works as follows: While there exists a noncontractible cycle of size $\mathcal{O}(r)$, we cut it, which decreases the genus. Finally, when there is no more such short noncontractible cycle, the resulting graph is locally planar, and we can apply Theorem 30.

Proof of Theorem 34. Graphs of Euler genus g forbid K_t as a minor for some $t \in \mathcal{O}(\sqrt{g})$ (see e.g. [19]). By Theorem 25, there are thus constants c_g, c'_g with $c_g \in \mathcal{O}(g)$ such that, for every nonnegative integer r , graphs of Euler genus g have $(r, c_g r, c'_g)$ -sparse covers. These sparse covers will be used in Case 2 of the proof below.

We prove the theorem with the function $f(g)$ defined as follows:

$$\begin{aligned} f(0) &:= 1 \\ f(g) &:= \max\{12(c_g + 1)f(g-1), c'_g\} \quad \forall g \geq 1. \end{aligned}$$

Let G be a graph of Euler genus g and consider an embedding of G in a surface Σ of Euler genus g . We prove the theorem by induction on the Euler genus g . We may assume $g > 0$, since the case $g = 0$ follows from Theorem 30. Note that G is cellularly embedded in Σ (otherwise, one could embed G in a surface of smaller Euler genus).

Case 1 (induction step): There exists a noncontractible cycle C of length at most $4(c_g + 1)r$ in G . First suppose that C is separating. Let G_1 and G_2 be the two graphs resulting from cutting along C , and for $i = 1, 2$ let C_i denote the copy of C in G_i , and let g_i be the Euler genus of G_i . We also denote by A_i the set $A \cap V(G_i)$ for every $i = 1, 2$. Then $g_1 < g$ and $g_2 < g$ with $g_1 + g_2 = g$, by Lemma 31. By Lemma 22 and the induction hypothesis

$$\begin{aligned} |\Pi_{r,G}[V(G) \rightarrow A]| &\leq |\Pi_{r,G_1}[V(G_1) \rightarrow A_1 \cup V(C_1)]| + |\Pi_{r,G_2}[V(G_2) \rightarrow A_2 \cup V(C_2)]| \\ &\leq f(g-1)h(r)(8(c_g + 1)r + |A|) \\ &\leq f(g)h(r)(r + |A|), \end{aligned}$$

as desired.

Next, assume that C is nonseparating. Let \tilde{G} be the graph obtained by cutting along C . If C is 2-sided, let C_1 and C_2 be the two resulting copies of C in \tilde{G} . If C is 1-sided, let C_1 be the cycle in \tilde{G} corresponding to C (which thus has twice the length of C), and define C_2 as being empty (to help uniformize the discussion below). By Lemma 32, \tilde{G} has Euler genus at most $g - 1$.

For every vertex $w \in V(C)$, we denote by w_1, w_2 the two corresponding vertices in \tilde{G} (one in C_1 and another in C_2 if C is 2-sided, or the two in C_1 if C is 1-sided).

Let $A' := (A \setminus V(C)) \cup V(C_1) \cup V(C_2)$. We prove the following:

$$|\Pi_{r,G}[V(G) \rightarrow A]| \leq |\Pi_{r,\tilde{G}}[V(\tilde{G}) \rightarrow A']| + |V(C)| \quad (1)$$

To see this, let $v \in V(G) \setminus V(C)$ and $a \in A$. If $a \notin V(C)$ then

$$\text{dist}_G(v, a) = \min \left(\min_{w \in V(C)} \min_{i=1,2} (\text{dist}_{\tilde{G}}(v, w_i) + \text{dist}_G(w, a)), \text{dist}_{\tilde{G}}(v, a) \right)$$

and as a consequence

$$\pi_{r,G}[v \rightarrow a] = \text{Cap}_r \circ \min \left(\min_{w \in V(C)} \min_{i=1,2} (\pi_{r,\tilde{G}}[v \rightarrow w_i] + \text{dist}_G(w, a)), \pi_{r,\tilde{G}}[v \rightarrow a] \right).$$

Similarly, if $a \in V(C)$ then

$$\pi_{r,G}[v \rightarrow a] = \text{Cap}_r \circ \min_{w \in V(C)} \min_{i=1,2} (\pi_{r,\tilde{G}}[v \rightarrow w_i] + \text{dist}_G(w, a)).$$

Finally, note that we trivially have $|\Pi_{r,G}[V(C) \rightarrow A]| \leq |V(C)|$.

Combining the above observations, we obtain that $|\Pi_{r,G}[V(G) \rightarrow A]| \leq |\Pi_{r,\tilde{G}}[V(\tilde{G}) \rightarrow A']| + |V(C)|$, which proves (1). Using the induction hypothesis on \tilde{G} , we deduce that

$$\begin{aligned} |\Pi_{r,G}[V(G) \rightarrow A]| &\leq |\Pi_{r,\tilde{G}}[V(\tilde{G}) \rightarrow A']| + |V(C)| \\ &\leq f(g-1)h(r)(8(c_g+1)r + |A|) + 4(c_g+1)r \\ &\leq f(g)h(r)(r + |A|). \end{aligned}$$

Case 2 (base case): Every noncontractible cycle in G has length more than $4(c_g+1)r$. Observe that in every induced subgraph H of G with radius at most $(c_g+1)r$, geodesics have length at most $2(c_g+1)r$, thus by Lemma 33, H has no noncontractible cycle, and hence H is planar.

Now apply Theorem 25 to G to obtain a $(r, c_g r, c'_g)$ -sparse cover $\mathcal{X} \subseteq \mathcal{P}(V(G))$ of G .

Let $X \in \mathcal{X}$ and let $v \in V(G)$ be such that every vertex of X is at distance at most $c_g r$ from v in $G[X]$. Observe that every vertex at distance at most r from $A \cap X$ in G is at distance at most $(c_g+1)r$ from v in G . It follows that

$$\Pi_{r,G}[V(G) \rightarrow A \cap X] = \Pi_{r,G[N^{(c_g+1)r}[v]]}[N^{(c_g+1)r}[v] \rightarrow A \cap X].$$

But as $G[N^{(c_g+1)r}[v]]$ is an induced subgraph of G with radius at most $(c_g+1)r$, it is planar. Thus,

$$|\Pi_{r,G}[V(G) \rightarrow A \cap X]| \leq h(r)|A \cap X|.$$

Summing up this inequality for all $X \in \mathcal{X}$, we obtain

$$\begin{aligned} |\Pi_{r,G}[V(G) \rightarrow A]| &\leq \sum_{X \in \mathcal{X}, A \cap X \neq \emptyset} h(r)|A \cap X| \\ &\leq h(r) \sum_{X \in \mathcal{X}, A \cap X \neq \emptyset} |A \cap X| \\ &\leq c'_g h(r)|A| \\ &\leq f(g)h(r)(r + |A|). \end{aligned}$$

This concludes the proof of Theorem 34. \square

Using that $r^4(r + |A|) \leq r^5|A|$, we obtain the following corollary from Theorem 34.

Corollary 35. *For every nonnegative integer g , the profile complexity of graphs of Euler genus g is in $\mathcal{O}_g(r^5)$.*

8 Lower bounds

So far we have seen that classes of graphs excluding a minor have polynomial profile and neighborhood complexity. In this section, we give a lower bound on the degree of this polynomial for graphs of treewidth at most t , and show that 1-planar graphs, which are some of the simplest graphs excluding no minor, have super polynomial profile and neighborhood complexity.

8.1 Bounded treewidth

Let t, r be positive integers with r even. Consider the graph formed by an independent set $A = \{a_1, \dots, a_t\}$, and for every $x = (x_1, \dots, x_t) \in [r/2, r]^t$, add a new vertex v_x and link it to a_i via a path of length x_i , for every $i \in [t]$. One can easily check that G has treewidth at most t and that the profile of v_x on A is x for every $x \in [r/2, r]^t$. This simple construction gives an example of a graph with treewidth t and with a number of profiles that is at least $(r/2)^t = \Omega_t(r^t|A|)$.¹ The following theorem improves this lower bound to $\Omega_t(r^{t+1}|A|)$.

Theorem 36. *For every positive integers t, r such that $2^t(t+1)$ divides r , there exists a graph G of treewidth at most t and a set $A \subseteq V(G)$ such that*

$$|\Pi_{r,G}[V(G) \rightarrow A]| \geq \frac{1}{(2^{t+1})^{t+1}} \cdot r^{t+1}|A| = \Omega_t(r^{t+1}|A|^{t+1}).$$

Graphs of treewidth at most 2 are planar. Hence, there are planar graphs whose numbers of profiles are at least $\Omega(r^3|A|)$. This is illustrated on Figure 4.

Proof of Theorem 36. Let $\ell = \frac{r}{2^{t+1}}$. We define a tree T as follows:

- $V(T) = [0, \ell - 1]^{t+1}$,
- $\mathbf{0} \in V(T)$ is the root of T , and
- the parent of $x_0 \dots x_i 0 \dots 0$ with $x_i \neq 0$ is $x_0 \dots (x_i - 1) 0 \dots 0$.

Set $A = \{a_0, a_1, \dots, a_t\}$ with $a_0 = 0 \in V(T)$, and $a_i \notin V(T)$ for every $i \in [t]$ (that is, a_1, \dots, a_t are new vertices not in T). Finally, for every $i \in [t]$, for every $x = (x_0, \dots, x_{i-1}) \in [0, \ell - 1]^i$, link a_i to $x_0 \dots x_{i-2}(\ell - 1)0 \dots 0$ via a path of length $r/2$. This gives the desired graph G , see Figure 4.

Observe that $G - \{a_2, \dots, a_t\}$ is a tree, so G has treewidth at most $1 + (t - 1) = t$ as wanted.

Let $x = x_0 \dots x_t \in V(T)$, $d_0 = \text{dist}_G(x, a_0)$ and $d_i = \text{dist}_G(x, a_i) - r/2$ for every $i \in \{1, \dots, t\}$. One can easily check the following relations:

$$\begin{cases} d_0 &= \sum_{i=0}^t x_i \leq r/2 \\ d_i &= (\ell - 1 - x_{i-1}) + \sum_{j=i}^t x_j \leq r/2, \quad \forall i \geq 1 \end{cases} \quad (2)$$

¹Note that if one wants to have $|A| \gg t$ then it suffices to take many disjoint copies of this construction.

In particular, $\text{dist}_G(x, a_i) \leq r$ for every $x \in V(T)$ and $i \in \{0, \dots, t\}$.

We claim that this system is injective, that is, it has at most one solution x_0, \dots, x_t when d_0, \dots, d_t is fixed. Indeed, we can recover x_0, \dots, x_t inductively using the following relations:

$$x_{i+1} = \frac{1}{2}(\ell - 1 - d_{i+2} + d_0 - x_0 - \dots - x_i) \quad \text{for } i \in [-1, t-1]$$

As a consequence, $|\Pi_{r,G}[V(G) \rightarrow A]| \geq |V(T)| = \ell^{t+1} = \frac{1}{(2^{t+1})^{t+1}} \cdot r^{t+1}$, and the result follows as $|A| = t + 1$. \square

Observe that this construction is tight for A of constant size. Indeed, Corollary 9 implies that if G has treewidth at most t , then $|\Pi_{r,G}[V(G) \rightarrow A]| = \mathcal{O}(r^{t+1}|A|^{t+1})$ (because G is then K_{t+2} -minor-free).

In terms of neighborhood complexity, we deduce the following corollary from the above theorem using Lemma 8.

Corollary 37. *Let t be a positive integer. Then, for every nonnegative integer r such that $2^t(t+1)$ divides r , there exists a graph G of treewidth at most t and $A \subseteq V(G)$ such that*

$$|\{A \cap N^r[v] \mid v \in V(G)\}| \geq \frac{1}{2^{(2^{t+1})^{t+1}}} \cdot r^t |A| = \Omega_t(r^t |A|).$$

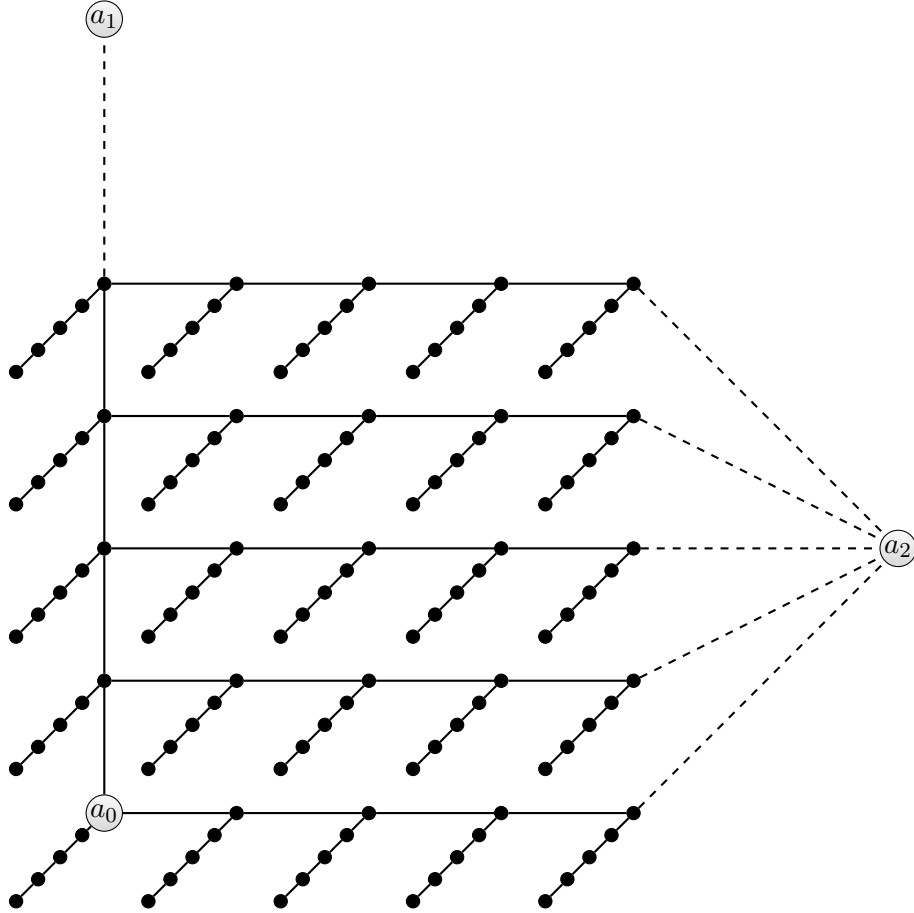


Figure 4: The construction of Theorem 36 for $t = 2$ and $\ell = 5$. The dashed edges represent paths of length $r/2$, and the central tree is a subgraph of an $\ell \times \ell \times \ell$ -grid where $\ell = \frac{r}{2^{t+1}}$.

8.2 1-planar graphs

Given a nonnegative integer k , a graph G is said to be k -planar if it can be drawn in the plane so that every edge of G crosses at most k other edges. Note that 0-planar graphs are exactly planar graphs. However, for $k \geq 1$, the class of k -planar excludes no graph as a minor.

In this subsection, we show that there is no polynomial function f such that the neighborhood complexity of every 1-planar graph on a set A of vertices at distance r is at most $f(r)|A|$. This also shows that the existence of a product structure is not enough to guarantee polynomial neighborhood complexity, despite Theorem 20 which reduces to the case where $|A|$ is bounded by a polynomial in r . Indeed, Dujmović, Morin and Wood [14] proved that every k -planar graph is a subgraph of $H \boxtimes P \boxtimes K_{18k^2+48k+30}$ for some path P , and some graph H of treewidth at most $\binom{k+4}{3} - 1$.

Theorem 38. *For every positive integer r such that $r+1$ is a perfect square, there exists a 1-planar graph G and $A \subseteq V(G)$ such that $|\{N^r[v] \cap A \mid v \in V(G)\}| \geq \frac{2\sqrt{r+1}}{\sqrt{r+1}}|A|$.*

Proof. Let $\ell = \sqrt{r+1}$, and let $A = \{a_1, \dots, a_\ell\}$ be a set of ℓ vertices. For every $i \in [\ell]$, we consider a complete binary tree T_i of depth ℓ , and with leaves $\{a_i^X\}_{X \subseteq [\ell]}$ (this indexing is possible because a complete binary tree of depth h has 2^h leaves). For every $X \subseteq [\ell]$, we add a vertex v_X with neighborhood $N(v_X) = \{a_i^X \mid i \in X\}$. This gives an $(\ell-1)$ -planar graph G_ℓ , see Figure 5. Moreover, $\text{dist}(a_i^X, a_i) = \ell$ for every $i \in [\ell]$ and $X \subseteq [\ell]$, hence $\text{dist}(v_X, a_i) \leq \ell+1$ if and only if $i \in X$. It follows that for every $X \subseteq [\ell]$, $N^{\ell+1}[v_X] \cap A = \{a_i \mid i \in X\}$, and so $|\{N^{\ell+1}[v] \cap A \mid v \in V(G_\ell)\}| \geq 2^\ell$.

To convert this graph G_ℓ into a 1-planar graph, we subdivide every edge $\ell-2$ times, and the new distance is now $(\ell+1)(\ell-1) = r$. Therefore, we obtain a 1-planar graph G such that $|\{N^r[v] \cap A \mid v \in V(G)\}| \geq 2^\ell = \frac{2\sqrt{r+1}}{\sqrt{r+1}}|A|$. \square

Observe that in this construction, A is shattered by the balls of radius r . Hence this shows that there are 1-planar graphs whose hypergraph of r -balls has VC-dimension at least $\sqrt{r+1}$, whereas for K_t -minor-free graphs this hypergraph has VC-dimension at most $t-1$ (which is independent of r) by Theorem 6.

9 Application to metric dimension

In a connected graph G , a *resolving set* S is a set of vertices such that no two vertices have the same profile on S at distance $\text{diam}(G)$, where $\text{diam}(G)$ denotes the diameter of G . The *metric dimension* of G is the smallest size of a resolving set. Clearly, if G has metric dimension k and diameter d , then the number n of vertices of G is bounded as follows:

$$n \leq d^k + k.$$

In [3], the authors improved this upper bound for several particular graph classes.

Theorem 39 (Beaudou, Foucaud, Dankelmann, Henning, Mary, and Parreau [3]). *Let G be an n -vertex graph with diameter d and metric dimension k .*

1. *If G is a tree, then $n \in \mathcal{O}(d^2k)$,*
2. *if G is K_t -minor free, then $n \in \mathcal{O}((dk)^{t-1})$,*
3. *if G has treewidth t , then $n \in \mathcal{O}(d^{3t+3}k)$.*

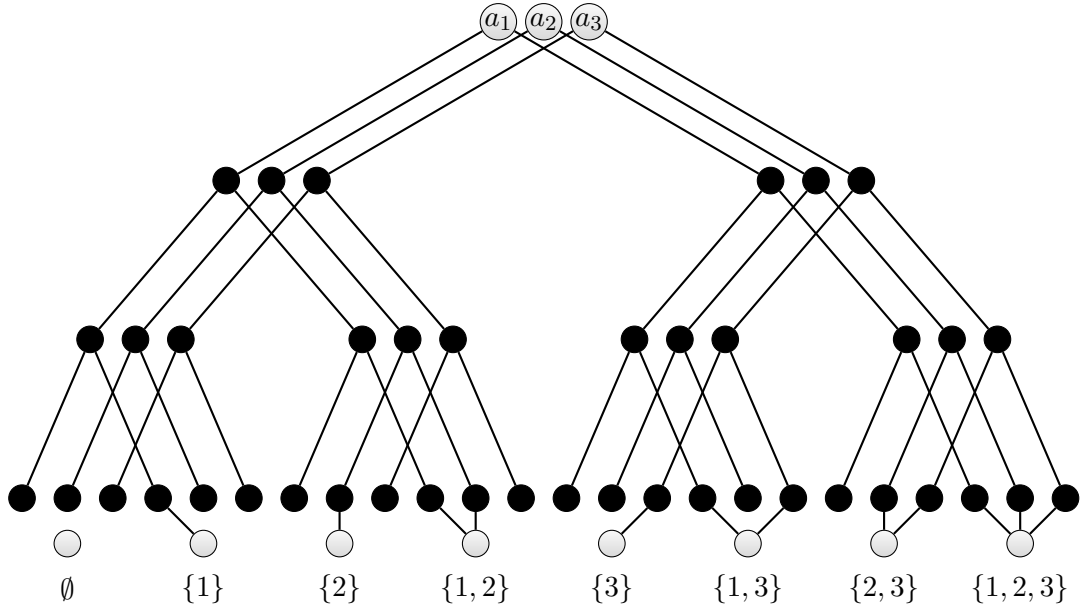


Figure 5: The graph G_ℓ of Theorem 38 for $\ell = 3$. This graph is $(\ell - 1)$ -planar and has neighborhood complexity on $A = \{a_1, a_2, \dots, a_\ell\}$ at distance $\ell + 1$ at least $2^\ell/\ell$.

By observing that $n \leq |\Pi_{d,G}[V(G) \rightarrow S]|$ if S is a resolving set, we obtain the following bounds.

Theorem 40. *Let G be a connected n -vertex graph with diameter d and metric dimension k .*

1. *If G is in a fixed class \mathcal{C} of bounded expansion, then $n \leq f(d)k$ for some function f depending only on \mathcal{C} (by [25] and Lemma 8).*
2. *If G is planar, then $n \in \mathcal{O}(d^4k)$ (by Theorem 30).*
3. *If G has Euler genus g , then $n \in \mathcal{O}_g(d^4(d + k))$ (by Theorem 34).*
4. *If G is K_t -minor-free, then $n \in \mathcal{O}_t(d^{2t-1}k)$ (by Theorem 15).*
5. *If G has treewidth $t \geq 1$, then $n \in \mathcal{O}(d^{2t}k)$ (by Corollary 18).*

10 Conclusion

We conclude the paper with two open problems. The first one is about getting a tight bound on the profile complexity of planar graphs, since the existing lower and upper bounds are tantalizing close to each other.

Problem 41. *Determine the profile and neighborhood complexities of planar graphs up to a constant factor. For profile complexity, it is known to be in $\Omega(r^3)$ (Theorem 36) and in $\mathcal{O}(r^4)$ (Theorem 2). For neighborhood complexity, it is known to be in $\Omega(r^2)$ (Theorem 36 and Lemma 8) and in $\mathcal{O}(r^4)$ (Theorem 2).*

The second open problem is about getting better bounds for K_t -minor-free graphs.

Problem 42. *Are the profile and neighborhood complexities of K_t -minor-free graphs in $r^{\mathcal{O}(t)}$?*

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