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ON THE NON-SPECIAL DIVISORS IN ALGEBRAIC FUNCTION FIELDS DEFINED OVER \mathbb{F}_q

S. BALLEZ AND M. KOUTCHOUKALI

ABSTRACT. In the theory of algebraic function fields and their applications to the information theory, the Riemann-Roch theorem plays a fundamental role. But its use, delicate in general, is efficient and practical for applications especially in the case of non-special divisors. So, in this paper, we give a survey of the known results concerning the non-special divisors in the algebraic function fields defined over finite fields, enriched with some new results about the existence of such divisors in curves of defect k . In particular, we have chosen to be self-contained by giving the full proofs of each result, the original proofs or shorter alternative proofs.

1. INTRODUCTION

1.1. General context. This article mainly is a survey highlighting the current state-of-the-art on the existence and the construction of non-special divisors in algebraic function fields defined over finite fields. The growing importance of this topic has attracted many mathematicians and computer scientists, who developed new ideas and obtained new results. Finite fields constitute an important area of mathematics. They arise in many applications, particularly in areas related to information theory as for example cryptography and error-correcting codes. Moreover, the algebraic geometric codes "à la Goppa" whose performance as error-correcting codes has been proven since the results of Tsfasman-Vladuts-Zink in [14] also plays an important role in cryptography and in algorithmic (computational algebraic geometry) as demonstrated by the work carried out in recent years on sharing secret schemas (see for example [4], [8], [9] and [30]) or on the algebraic complexity of the multiplication in the finite fields (see survey [22]). In all these cases, the use of Riemann-Roch Theorem with non-special divisors of degree d in algebraic function fields of genus g over finite

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fields, in particular in the non-trivial case where $g - 1 \leq d \leq 2g - 2$, is crucial.

1.2. Organisation. In this paper, we give a survey of the known results concerning the non-special divisors in the algebraic function fields defined over finite fields, enriched with some unpublished recent results. In particular, we have chosen to be self-contained by giving the full proofs of each result that relates directly to the non-special divisors, the original proofs or shorter alternative proofs.

In Section 2, we present Notation and the well-known elementary results about non-special divisors in algebraic function fields defined over an arbitrary field. In Section 3, we focus on the results concerning the existence of non-special divisors of degree g and $g - 1$. In particular, in section 3.6, new results are proposed on the existence of these divisors in curves of defect k . Finally, in Section 4, we give known results concerning the explicit construction of non-special divisor of degree g and $g - 1$.

2. PRELIMINARIES

2.1. Notation. Let \mathbf{F}/\mathbb{F}_q be an algebraic function field of one variable defined over a finite field \mathbb{F}_q . We will always suppose that the full constant field of \mathbf{F}/\mathbb{F}_q is \mathbb{F}_q and denote by g the genus of \mathbf{F} . If D is a (rational) divisor we recall that the \mathbb{F}_q -Riemann-Roch vector space associated to D and denoted $\mathcal{L}(D)$ is the following subspace of rational functions

$$(1) \quad \mathcal{L}(D) = \{x \in \mathbf{F} \mid (x) \geq -D\} \cup \{0\}.$$

By Riemann-Roch Theorem we know that the dimension of this vector space, denoted by $\dim(D)$, is related to the genus of \mathbf{F} and the degree of D by

$$(2) \quad \dim(D) = \deg(D) - g + 1 + \dim(\kappa - D),$$

where κ denotes a canonical divisor of \mathbf{F}/\mathbb{F}_q . In this relation, the complementary term $i(D) = \dim(\kappa - D)$ is called the index of speciality and is not easy to compute in general. In particular, a divisor D is non-special when the index of speciality $i(D)$ is zero. Let us recall the usual notation (for the basic notions related to an algebraic function field see [26]). For any integer $k \geq 1$ we denote by $\mathbf{P}_k(\mathbf{F}/\mathbb{F}_q)$ the set of places of degree k , by $B_k(\mathbf{F}/\mathbb{F}_q)$ the cardinality of this set and by $\mathbf{P}(\mathbf{F}/\mathbb{F}_q) = \cup_k \mathbf{P}_k(\mathbf{F}/\mathbb{F}_q)$. The divisor group of \mathbf{F}/\mathbb{F}_q is denoted by

$\mathbf{D}(\mathbf{F}/\mathbb{F}_q)$. If a divisor $D \in \mathbf{D}(\mathbf{F}/\mathbb{F}_q)$ is such that

$$D = \sum_{P \in \mathbf{P}(\mathbf{F}/\mathbb{F}_q)} n_P P,$$

the support of D is the following finite set

$$\text{supp}(D) = \{P \in \mathbf{P}(\mathbf{F}/\mathbb{F}_q) \mid n_P \neq 0\}$$

and its degree is

$$\deg(D) = \sum_{P \in \mathbf{P}(\mathbf{F}/\mathbb{F}_q)} n_P \deg(P).$$

We denote by $\mathbf{D}_n(\mathbf{F}/\mathbb{F}_q)$ the set of divisors of degree n . We say that the divisor D is effective if for each $P \in \text{supp}(D)$ we have $n_P \geq 0$ and we denote by $\mathbf{D}_n^+(\mathbf{F}/\mathbb{F}_q)$ the set of effective divisors of degree n and by $A_n = \#\mathbf{D}_n^+(\mathbf{F}/\mathbb{F}_q)$ the cardinal of the set $\mathbf{D}_n^+(\mathbf{F}/\mathbb{F}_q)$. In general, we will note $\#\mathcal{U}$ the cardinal of the set \mathcal{U} . The dimension of a divisor D , denoted by $\dim(D)$, is the dimension of the vector space $\mathcal{L}(D)$ defined by the formula (1). Let $x \in \mathbf{F}/\mathbb{F}_q$, we denote by (x) the divisor associated to the rational function x , namely

$$(x) = \sum_{P \in \mathbf{P}(\mathbf{F}/\mathbb{F}_q)} v_P(x)P,$$

where v_P is the valuation at the place P . Such a divisor (x) is called a principal divisor, and the set of principal divisors is a subgroup of $\mathbf{D}_0(\mathbf{F}/\mathbb{F}_q)$ denoted by $\text{Princ}(\mathbf{F}/\mathbb{F}_q)$. The factor group

$$\mathcal{C}(\mathbf{F}/\mathbb{F}_q) = \mathbf{D}(\mathbf{F}/\mathbb{F}_q)/\text{Princ}(\mathbf{F}/\mathbb{F}_q)$$

is called the divisor class group. If D_1 and D_2 are in the same class, namely if the divisor $D_1 - D_2$ is principal, we will write $D_1 \sim D_2$. We will denote by $[D]$ the class of the divisor D .

If $D_1 \sim D_2$, the following holds

$$\deg(D_1) = \deg(D_2), \quad \dim(D_1) = \dim(D_2),$$

so that we can define the degree $\deg([D])$ and the dimension $\dim([D])$ of a class. Since the degree of a principal divisor is 0, we can define the subgroup $\mathcal{C}(\mathbf{F}/\mathbb{F}_q)^0$ of classes of degree 0 divisors in $\mathcal{C}(\mathbf{F}/\mathbb{F}_q)$. It is a finite group and we denote by h its order, called the *class number* of \mathbf{F}/\mathbb{F}_q . Moreover if

$$L(t) = \sum_{i=0}^{2g} a_i t^i = \prod_{i=1}^g [(1 - \alpha_i t)(1 - \bar{\alpha}_i t)]$$

with $|\alpha_i| = \sqrt{q}$ is the numerator of the Zeta function of \mathbf{F}/\mathbb{F}_q , we have $h = L(1)$. Finally we will denote $h_{n,k}$ the number of classes of divisors

of degree n and of dimension k .

In the sequel, we may simultaneously use the dual language of (smooth, absolutely irreducible, projective) curves by associating to \mathbf{F}/\mathbb{F}_q a unique (\mathbb{F}_q -isomorphism class of) curve \mathbf{C}/\mathbb{F}_q of genus g and conversely to such a curve its function field $\mathbf{C}(\mathbb{F}_q)$ is the set of \mathbb{F}_q -rational points on \mathbf{C} , and $\mathbb{F}_q(\mathbf{C})$ is the field of rational functions on \mathbf{C} over \mathbb{F}_q . Because, by F.K. Schmidt's theorem (cf. [26, Corollary V.1.11]) there always exists a rational divisor of degree one, the group $\mathcal{C}(\mathbf{F}/\mathbb{F}_q)^0$ is isomorphic to the group of \mathbb{F}_q -rational points on the Jacobian of \mathbf{C} , denoted by $\mathcal{J}\text{ac}(\mathbf{C})$. In particular $h(\mathbf{F}/\mathbb{F}_q) = \#\mathcal{J}\text{ac}(\mathbf{C})(\mathbb{F}_q)$ is the cardinal of the set $\mathcal{J}\text{ac}(\mathbf{C})(\mathbb{F}_q)$.

2.2. Elementary results on non-special divisors. Let \mathbf{F}/\mathbb{F}_q be a function field of genus $g > 0$. First recall some results about non-special divisors (cf. [26]). If $\deg(D) < 0$, then $\dim(D) = 0$ and if $\deg(D) \geq 0$ then $\dim(D) \geq \deg(D) - g + 1$. When $0 \leq \deg(D) \leq 2g - 2$, the computation of $\dim(D)$ is difficult, but we have the following general results.

- Proposition 2.1.** (1) $\mathbb{F}_q \subset \mathcal{L}(D)$ if and only if $D \geq 0$.
(2) If $\deg(D) > 2g - 2$, then D is non-special.
(3) The property of a divisor D being special or non-special depends only on the class of D up to equivalence.
(4) Any canonical divisor κ is special, $\deg(\kappa) = 2g - 2$ and $\dim(\kappa) = g$.
(5) Any divisor D with $\dim(D) > 0$ and $\deg(D) < g$ is special.
(6) If D is non-special and $D' \geq D$ then D' is non-special.
(7) For any divisor D with $0 \leq \deg(D) \leq 2g - 2$, $\dim(D) \leq 1 + \frac{1}{2}\deg(D)$ holds.

For the rational function field $\mathbf{F} = \mathbb{F}_q(x)$ ($g = 0$), there is no non-zero regular differential, thus, all divisors of degree $d \geq 0$ are non-special. From now, we focus on the existence of non-special divisors of degree g or $g - 1$. Note that $g - 1$ is the least possible degree for a divisor D to be non-special. We have the following trivial observations.

Lemma 2.2. Assume $g \geq 1$. Let $D \in \mathbf{D}(\mathbf{F}/\mathbb{F}_q)$ and set $d = \deg(D)$.

- (1) If $d = g$, D is a non-special divisor if and only if $\dim(D) = 1$. Assume that D is a non-special divisor of degree g ; then, $D \sim D_0$, where D_0 is effective. If $D \geq 0$ and $d = g$, D is non-special divisor if and only if $\mathcal{L}(D) = \mathbb{F}_q$.
(2) If $d = g - 1$, D is a non-special divisor if and only if $\dim(D) = 0$. A non-special divisor of degree $g - 1$, if any, is never effective.

- (3) If $g > 1$ and $A_{g-1} = 0$, then any divisor of degree $g - 1$ is non-special.

A consequence of assertion 1 is

Lemma 2.3. *Assume that D is an effective non-special divisor of degree $g \geq 1$. If there exists a degree one place P such that $P \notin \text{supp}(D)$, then $D - P$ is a non-special divisor of degree $g - 1$*

3. EXISTENCE OF NON-SPECIAL DIVISORS OF DEGREE g AND $g - 1$

Unless otherwise specified, the results in this section come from [3] by S. Ballet and D. Le Brigand. They determine the necessary and sufficient conditions for the existence of non-special divisors of degree g and $g - 1$ in the general case and they apply them to the cases of $g = 1$ and $g = 2$.

3.1. General case. Here, we give some general results about non-special divisors of degree g and $g - 1$.

Proposition 3.1. *Let \mathbf{F}/\mathbb{F}_q be an algebraic function field of genus $g \geq 1$.*

- (1) *If $B_1(\mathbf{F}/\mathbb{F}_q) \geq g$, there exists a non-special divisor D such that $D \geq 0$, $\deg(D) = g$ and $\text{supp}(D) \subset \mathbf{P}_1(\mathbf{F}/\mathbb{F}_q)$.*
- (2) *If $B_1(\mathbf{F}/\mathbb{F}_q) \geq g + 1$, there exists a non-special divisor such that $\deg(D) = g - 1$ and $\text{supp}(D) \subset \mathbf{P}_1(\mathbf{F}/\mathbb{F}_q)$.*

Proof. (1) cf. [26, Proposition I.6.10].

- (2) Let $T \subset \mathbf{P}_1(\mathbf{F}/\mathbb{F}_q)$ be such that $\#T = g$ and, using assertion 1, let $D \geq 0$ be a non-special divisor such that $\deg(D) = g$ and $\text{supp}(D) \subset T$. Select $P \in \mathbf{P}_1(\mathbf{F}/\mathbb{F}_q) \setminus \text{supp}(D)$ and apply Lemma 2.3.

□

We denote by \mathcal{E}_g and \mathcal{E}_{g-1} the following holds:

\mathcal{E}_g : \mathbf{F}/\mathbb{F}_q has an effective non-special divisor of degree g , and
 \mathcal{E}_{g-1} : \mathbf{F}/\mathbb{F}_q has a non-special divisor of degree $g - 1$.

If \mathbf{F}/\mathbb{F}_q has enough rational places compared to the genus, then \mathcal{E}_g and \mathcal{E}_{g-1} are true.

Proposition 3.2. *Let \mathbf{F}/\mathbb{F}_q be a function field of genus g . Denote by h the order of the divisor class group of \mathbf{F}/\mathbb{F}_q .*

- (1) *If $A_g < h(q + 1)$, then \mathcal{E}_g is true.*
- (2) *If $A_{g-1} < h$, then \mathcal{E}_{g-1} is true.*

- (3) Assume $g \geq 2$. If $A_{g-2} < h$, then \mathcal{E}_g is true.
(4) If $g = 2$ or 3 , \mathcal{E}_g is untrue if and only if $A_{g-2} = h$.

Proof. Recall that, in any function field, there exists a degree 1 divisor (this result of F.K. Schmidt; see [26, Corollary V.1.11] for instance), so there exist divisors of any degree ≥ 1 . Let $d \geq 1$ and $D_0 \in \mathbf{D}_d^+(\mathbf{F}/\mathbb{F}_q)$, and consider the map ψ_{d,D_0} :

$$\begin{aligned} \psi_{d,D_0} : \mathbf{D}_d^+(\mathbf{F}/\mathbb{F}_q) &\longrightarrow \mathcal{J}ac(\mathbf{F}/\mathbb{F}_q) \\ D &\longmapsto [D - D_0] \end{aligned}$$

- (1) First, it is well known that $1 \leq h \leq A_g$ is true for any function field. Indeed, let D be such that $\deg(D) = g$. By Riemann-Roch, $\dim(D) \geq 1$; thus, there exists an effective divisor of degree g which is equivalent to D . So assume $D_0 \in \mathbf{D}_g^+(\mathbf{F}/\mathbb{F}_q)$ and consider the map ψ_{g,D_0} . For all $[R] \in \mathcal{J}ac(\mathbf{F}/\mathbb{F}_q)$, we have $\deg(R + D_0) = g$; thus, $\dim(R + D_0) \leq 1$ and there exists $u \in \mathbf{F}^*$ such that $D := R + D_0 + \text{div}(u)$ is in $\mathbf{D}_g^+(\mathbf{F}/\mathbb{F}_q)$ and $[R] = [D - D_0] = \psi_{g,D_0}(D)$. This proves that ψ_{g,D_0} is surjective and that $h \leq A_g$. Assume now that \mathbf{F}/\mathbb{F}_q has no non-special divisor D of degree g . Then, $\dim(D) \geq 2$ for all degree g divisors; thus, for all $[R] \in \mathcal{J}ac(\mathbf{F}/\mathbb{F}_q)$, we have

$$\#\{D \in \mathbf{D}_g^+(\mathbf{F}/\mathbb{F}_q), [D - D_0] = [R]\} = \frac{q^{\dim(R+D_0)} - 1}{q - 1} \geq \frac{q^2 - 1}{q - 1} = q + 1$$

and $A_g \geq h(q + 1)$.

- (2) A divisor D of degree $g - 1$ is non-special if and only if $\dim(D) = 0$. If $g = 1$, there exists a non-special divisor of degree $g - 1 = 0$ if and only if $h = B_1 > 1 = A_0$, since two distinct degree one places are not equivalent. Assume now that $g > 1$. Hence, it is sufficient to prove the existence of a divisor of degree $g - 1$ which is not equivalent to any effective divisor. If $A_{g-1} = 0$, the result is proved. otherwise, let D_0 be an effective divisor of degree $g - 1 \geq 1$ and consider the map ψ_{g-1,D_0} . If $A_{g-1} < h$, this map is not surjective. Hence, there exists a zero-degree divisor R such that $[R]$ is not in the image of ψ_{g-1,D_0} . Consequently, $D = R + D_0$ is a divisor of degree $g - 1$ which is not equivalent to an effective divisor. Thus, D is non-special.
- (3) From the functional equation of the zeta function, it can be deduced (see [18, Lemma 3(i)]) that, for $g \geq 1$, one has

$$A_n = q^{n+1-g} A_{2g-2-n} + h \frac{q^{n+1-g} - 1}{q - 1} \text{ for all } 0 \leq n \leq 2g - 2.$$

For $g \geq 2$ and $n = g$ this gives

$$A_g = h + qA_{g-2}.$$

Thus, if $g \geq 2$,

$$A_g < (q+1)h \iff A_{g-2} < h.$$

- (4) Assume that $g = 2$ or 3 . Then if $\deg(D) = g$ and $\dim(D) \geq 2$, one has $\dim(D) = 2$ by assertion 7 of Proposition 2.1, since $\dim(D) \leq \frac{g}{2} + 1$. Thus, \mathcal{E}_g is untrue if and only if $A_g = (q+1)h$, which is equivalent to $A_{g-2} = h$.

□

We quote the following consequence of assertion 2.

Corollary 3.3. *Let \mathbf{F}/\mathbb{F}_q be an algebraic function field of genus $g \geq 2$ such that $A_{g-1} \geq 1$. Denote by h the order of the divisor class group of \mathbf{F}/\mathbb{F}_q . Then \mathcal{E}_{g-1} is untrue if and only if there exists h elements of $\mathcal{D}_{g-1}^+(\mathbf{F}/\mathbb{F}_q)$ pairwise non-equivalent.*

Proof. Let r be the maximum number of pairwise non-equivalent elements of $\mathcal{D}_{g-1}^+(\mathbf{F}/\mathbb{F}_q)$ and let D_1, \dots, D_r be elements of $\mathcal{D}_{g-1}^+(\mathbf{F}/\mathbb{F}_q)$ pairwise non-equivalent. Then:

$$\{[0] = [D_1 - D_1], [D_2 - D_1], \dots, [D_r - D_1]\}$$

is a subset of $\mathcal{J}ac(\mathbf{F}/\mathbb{F}_q)$ of order r . If $r = h$, for any divisor D of degree $d = g - 1$, we have $[D - D_1] = [D_i - D_1]$ for some i , $1 \leq i \leq h$, and then $D \sim D_i$. Thus, $\dim(D) \geq 1$. If $r < h$, ψ_{g-1, D_1} is not surjective and the result follows. □

3.2. Existence of non-special divisors of degree g . Now, we particularly focus on the non-special divisors of degree g . First, we give useful properties and interesting information to study the existence non-special divisors. For example, it is known that this existence is relied on the number of effective divisors of certain degrees A_n .

Lemma 3.4. *If $B_1 \geq m \geq 1$, then for all $n \geq 2$ one has*

$$(3) \quad A_n \geq mA_{n-1} - \frac{m(m-1)}{2} \cdot A_{n-2}.$$

Proof. See [18, Lemma 4]. □

Moreover, it also is linked to the class number h of algebraic function fields.

Proposition 3.5. *Let \mathbf{F}/\mathbb{F}_q be a function field of genus $g \geq 2$. We denote by h its divisor class number.*

- Up to isomorphism, there are 4 function fields \mathbf{F}/\mathbb{F}_q , 2 of them being hyperelliptic, such that $h = 1$ and $g \geq 2$. They are obtained for $\mathbf{F} = \mathbb{F}_2(x, y)$ with:

g	Equation	B_1	B_2	B_3
2	$y^2 + y + (x^5 + x^3 + 1) = 0$	1	2	
2	$y^2 + y + (x^3 + x^2 + 1)/(x^3 + x + 1) = 0$	0	3	
3	$y^4 + xy^3 + (x + 1)y + (x^4 + x^3 + 1) = 0$	0	0	1
3	$y^4 + xy^3 + (x + 1)y + (x^4 + x + 1) = 0$	0	1	1

- Up to isomorphism, there are 15 functions fields \mathbf{F}/\mathbb{F}_q , 7 of them being hyperelliptic, such that $h = 2$ and $g \geq 2$. They are obtained for $\mathbf{F} = \mathbb{F}_2(x, y)$ with:

q	g	Equation	B_1	B_2	B_3	B_4
3	2	$y^2 - 2(x^2 + 1)(x^4 + 2x^3 + x + 1) = 0$	1	5		
2	2	$y^2 + y + (x^3 + x + 1)/(x^2 + x + 1) = 0$	1	3		
		$y^2 + y + (x^4 + x + 1)/x = 0$	2	1		
2	3	$y^2 + y + (x^4 + x^3 + x^2 + x + 1)/(x^3 + x + 1) = 0$	1	2	1	
		$y^2 + y + (x^5 + x^2 + 1)/(x^2 + x + 1) = 0$	1	3	0	
		$y^2 + y + (x^6 + x + 1)/(x^2 + x + 1)^3 = 0$	0	4	2	
		$y^2 + y + (x^4 + x^3 + 1)/(x^4 + x + 1) = 0$	0	3	2	
		$y^4 + xy^3 + (x + 1)y + (x^4 + x^2 + 1) = 0$	0	2	2	
		$y^3 + (x^2 + x + 1)y + (x^4 + x^3 + 1) = 0$	1	0	3	
		$y^3 + y + (x^4 + x^3 + 1) = 0$	1	1	2	
2	4	$y^3 + (x^4 + x^3 + 1)y + (x^6 + x^3 + 1) = 0$	0	0	4	2
		$y^3 + (x^4 + x^2 + 1)y + (x^6 + x^5 + 1) = 0$	0	0	4	2
		$y^3 + (x^4 + x^3 + 1)y + (x^6 + x + 1) = 0$	0	1	3	3
		$y^6 + xy^5 + (x^2 + 1)y^4 + (x^3 + x^2)y^3 + (x^6 + x^5 + x^3 + x + 1) = 0$	0	1	1	3
		$y^6 + xy^5 + x^3y^3 + y^2 + (x^5 + x^2)y + (x^6 + x^2 + 1) = 0$	0	1	2	3

Proof. See [13] and [16] for the solutions of the ($h = 1$) problem and [12, Proposition 3.1 and Theorem 4.1] for the solutions of the ($h = 2$) problem. \square

Proposition 3.6. *An algebraic function field \mathbf{F}/\mathbb{F}_q of genus $g \geq 2$ has an effective non-special divisor of degree g in the following cases:*

- (i) $q \geq 3$.
- (ii) $q = 2$ and $g = 2$, unless $\mathbf{F} = \mathbb{F}_2(x, y)$ with:

$$y^2 + y + (x^5 + x^3 + 1) = 0$$

and

$$y^2 + y + (x^3 + x^2 + 1)/(x^3 + x + 1) = 0$$

- (iii) $q = 2$ and $g = 3$.
- (iv) $q = 2$, $g \geq 4$ and $B_1(\mathbf{F}/\mathbb{F}_q) \geq 3$.

Proof. We set $L(t) := L(\mathbf{F}/\mathbb{F}_q, t)$. for $g \geq 2$, it follows that (see [18, Lemma 3]):

$$\sum_{n=0}^{g-2} A_n t^n + \sum_{n=0}^{g-1} q^{g-1-n} A_n t^{2g-2-n} = \frac{L(t) - ht^g}{(1-t)(1-qt)}.$$

Substituting $t = q^{-1/2}$ into the last identity, we obtain:

$$2 \sum_{n=0}^{g-2} q^{-n/2} A_n + q^{-(g-1)/2} A_{g-1} = \frac{h - q^{g/2} L(q^{-1/2})}{(q^{1/2} - 1)^2 q^{(g-1)/2}}.$$

and since $L(q^{-1/2}) = \prod_{i=1}^g |1 - \pi_i q^{-1/2}|^2 \geq 0$, we have:

$$(4) \quad 2 \sum_{n=0}^{g-2} q^{(g-1-n)/2} A_n + A_{g-1} \leq \frac{h}{(q^{1/2} - 1)^2}.$$

(1) $q \geq 3$. Using (4), $A_{g-2} \geq h$ implies that:

$$2q^{1/2} \leq \frac{1}{(q^{1/2} - 1)^2},$$

which is absurd if $q \geq 3$. Thus, $A_{g-2} < h$ is always satisfied and so \mathcal{E}_g is true.

(2) If $q = 2$ and $g \geq 3$, (4) implies:

$$(5) \quad 4A_{g-3} + 2\sqrt{2}A_{g-2} + A_{g-1} \leq \frac{h}{(\sqrt{2} - 1)^2} = (3 + 2\sqrt{2})h.$$

Assume that $B_1(\mathbf{F}/\mathbb{F}_q) \geq m = 3$; then, by (3) with $n = g - 1$, we have $A_{g-1} + 3A_{g-3} \geq 3A_{g-2}$, and finally using (5):

$$A_{g-3} + (3 + 2\sqrt{2})A_{g-2} \leq (3 + 2\sqrt{2})h.$$

Since $A_{g-3} \geq 1$, because if $g = 3$, $A_{g-3} = A_0 = 1$ and if $g > 3$, $A_{g-3} \geq B_1(\mathbf{F}/\mathbb{F}_q) \geq m = 3$, we deduce that, if $B_1(\mathbf{F}/\mathbb{F}_q) \geq 3$ and $g \geq 3$, then $A_{g-2} < h$ and so \mathcal{E}_g is true.

(3) If $q = 2$ and $g = 2$, using assertion (4) of Proposition 3.2. In fact, \mathcal{E}_g is untrue if and only if $h = A_{g-2} = A_1 = B_1(\mathbf{F}/\mathbb{F}_q)$. Since \mathcal{E}_g is true if $B_1(\mathbf{F}/\mathbb{F}_q) \geq 3$, we are left with $h = B_1(\mathbf{F}/\mathbb{F}_q) = 1$ or 2 and we deduce from Proposition 3.5 that there is no solution.

(4) If $q = 2$ and $g = 2$, using assertion (4) of Proposition 3.2, \mathcal{E}_g is untrue if and only if $h = A_{g-2} = A_0 = 1$. By Proposition 3.5 there are only two function fields \mathbf{F}/\mathbb{F}_q of genus 2 such that $h = 1$. They are such that $q = 2$ and $\mathbf{F} = \mathbb{F}_2(x, y)$, with

- $y^2 + y + (x^5 + x^3 + 1) = 0$ and $B_1(\mathbf{F}/\mathbb{F}_q) = 1$, $B_2(\mathbf{F}/\mathbb{F}_q) = 2$.
- $y^2 + y + (x^3 + x^2 + 1)/(x^3 + x + 1) = 0$ and $B_1(\mathbf{F}/\mathbb{F}_q) = 0$, $B_2(\mathbf{F}/\mathbb{F}_q) = 3$.

Since $h = 1$, all divisors of a given degree $d > 0$ are equivalent. In particular, all the divisors of degree $g = 2$ are equivalent to any divisor of $D_2^+(\mathbf{F}/\mathbb{F}_q)$, and therefor they are special.

□

The following lemma from H. Niederreiter and C. Xing in [18, Lemma 6] is another characterization of the existence of these divisors. This result is less precise than Proposition 3.6 and the proof is based on the same tools.

Lemma 3.7. *There exists an effective divisor D of \mathbf{F}/\mathbb{F}_q with $\deg(D) = g$ and $\dim(D) = 1$ if either $B_1(\mathbf{F}/\mathbb{F}_q) \geq 2$ and $q \geq 3$, or $B_1(\mathbf{F}/\mathbb{F}_q) \geq 4$ and $q = 2$.*

Proof. The lemma is trivial for $g = 0$. if $g = 1$, let D be a rational place of \mathbf{F}/\mathbb{F}_q , then $\dim(D) = 1$. Now let $g \geq 2$. Suppose that $\dim(D) \geq 2$ for any positive divisor D with $\deg(D) = g$. If $g = 2$, then by [18, Lemma 3(i)] we have $A_2 = q + h$ and by [18, Lemma 5] we have $A_2 \geq (q + 1)h$. Thus $h \leq 1$, which contradicts $h \geq A_1 \geq 2$. So we may assume $g \geq 3$. Substituting $t = q^{-1/2}$ in the identity in [18, Lemma 3(ii)], we obtain

$$2 \sum_{n=0}^{g-2} q^{-n/2} A_n + q^{-(g-1)/2} A_{g-1} = \frac{h - q^{g/2} L(q^{-1/2})}{(q^{1/2} - 1)^2 q^{(g-1)/2}}.$$

Since

$$L(q^{-1/2}) = \prod_{j=1}^g |1 - \alpha_j q^{-1/2}|^2 \geq 0,$$

We infer that

$$(6) \quad 2 \sum_{n=0}^{g-2} q^{(g-1-n)/2} A_n + A_{g-1} \leq \frac{h}{(q^{1/2} - 1)^2}.$$

[18, Lemma 3(i)] yields $A_g = h + qA_{g-2}$ and [18, Lemma 5] yields $A_g \geq (q + 1)h$, and thus $A_{g-2} \geq h$. From (6) we then get

$$2q^{1/2} \leq \frac{1}{(q^{1/2} - 1)^2}$$

This inequality is impossible if $q \geq 3$, hence it remains to prove the lemma for $q = 2$.

If $q = 2$ and $B_1(\mathbf{F}/\mathbb{F}_q) \geq 4$, then from (6) we obtain

$$(7) \quad 4A_{g-3} + 2\sqrt{2}A_{g-2} + A_{g-1} \leq \frac{h}{(\sqrt{2} - 1)^2}.$$

Together with [18, Lemma 4] with $m = 3$ and $n = g - 1$ this yields

$$(8) \quad A_{g-3} + (3 + 2\sqrt{2})A_{g-2} \leq \frac{h}{(\sqrt{2} - 1)^2}.$$

if we use [18, Lemma 4] with $m = 4$ and $n = g - 1$ in (7), then we get

$$(9) \quad -2A_{g-3} + (4 + 2\sqrt{2})A_{g-2} \leq \frac{h}{(\sqrt{2} - 1)^2}.$$

By eliminating A_{g-3} from (8) and (9), we arrive at

$$(10 + 6\sqrt{2})A_{g-2} \leq \frac{3h}{(\sqrt{2} - 1)^2},$$

and therefore

$$10 + 6\sqrt{2} \leq \frac{3}{(\sqrt{2} - 1)^2},$$

which is absurd. □

3.3. Existence of non-special divisors of degree $g - 1$. In this section, we are interested by the non-special divisors of degree $g - 1$. We begin with the particular case where $g = 1$. If the genus of F/\mathbb{F}_q is $g = 1$, any divisor of degree $d = g$ is non-special since $d \geq 2g - 1 = 1$ and there exists a non-special divisor of degree $g - 1 = 0$ if and only if the divisor class number h is > 1 , i.e. $B_1(\mathbf{F}/\mathbb{F}_q) \geq 2$. So there are exactly 3 function fields of genus 1 which have no non-special divisor of degree $g - 1$. They are the elliptic solutions to the divisor class number one problem (see [15] and [16]):

$$\begin{aligned} q = 2, & \quad y^2 + y + (x^3 + x + 1) = 0, \\ q = 3, & \quad y^2 - (x^3 + 2x + 2) = 0, \\ q = 4, & \quad y^2 + y + (x^3 + a) = 0, \quad \text{where } \mathbb{F}_4 = \mathbb{F}_2(a). \end{aligned}$$

So, in the rest of this paper, except otherwise stated, we assume that the genus of a function field is ≥ 2 .

Theorem 3.8. *Let \mathbf{F}/\mathbb{F}_q be a function field of genus $g \geq 2$. Then \mathcal{E}_{g-1} is true in the following cases:*

- (i) $q \geq 4$.
- (ii) $g = 2$, unless $\mathbf{F}/\mathbb{F}_q := \mathbb{F}_2(x, y)/\mathbb{F}_2$, with:

$$y^2 + y + (x^5 + x^3 + 1) = 0$$

or

$$y^2 + y + (x^4 + x + 1)/x = 0$$

Proof. Recall that, $A_{g-1} = 0$, the existence is clear.

(1) $q \geq 4$. By (4), for $g \geq 2$ we have:

$$A_{g-1} < 2q^{(g-1)/2}A_0 + A_{g-1} \leq 2 \sum_{n=0}^{g-2} q^{(g-1-n)/2}A_n + A_{g-1} \leq \frac{h}{(q^{1/2} - 1)^2}.$$

Thus, if $q \geq 4$, we have $A_{g-1} < h$ and the result follows from Proposition 3.2.

(2) $g = 2$. If $A_{g-1} = B_1(\mathbf{F}/\mathbb{F}_q) < h$, the result follows from Proposition 3.2. This is the case when $B_1(\mathbf{F}/\mathbb{F}_q) = 0$ and then all divisors of degree $g - 1$ are non-special. If $B_1(\mathbf{F}/\mathbb{F}_q) \geq g + 1 = 3$, the result is true by Proposition 3.1. The remaining cases are $B_1(\mathbf{F}/\mathbb{F}_q) = 1$ or 2 with $h = B_1(\mathbf{F}/\mathbb{F}_q)$. By Proposition 3.5, there are two solutions:

(a) $B_1(\mathbf{F}/\mathbb{F}_q) = 1$ and $h = 1$. There is a unique function field satisfying these conditions. It is $\mathbf{F}/\mathbb{F}_q := \mathbb{F}_2(x, y)/\mathbb{F}_2$, with:

$$y^2 + y + (x^5 + x^3 + 1) = 0$$

Since $h = 1$, all divisors of degree $g - 1 = 1$ are equivalent to the place of degree 1; thus, they are special.

(b) $B_1(\mathbf{F}/\mathbb{F}_q) = 2$ and $h = 2$. There is a unique function field satisfying these conditions. It is $\mathbf{F}/\mathbb{F}_q := \mathbb{F}_2(x, y)/\mathbb{F}_2$, with:

$$y^2 + y + (x^4 + x + 1)/x = 0$$

Since the two degree one places are non-equivalent, it follows from Corollary 3.3 that \mathcal{E}_{g-1} is untrue. □

In the following lemma, the value of A_{g-1} is given in terms of coefficients of the polynomial $L(\mathbf{F}/\mathbb{F}_q, t)$.

Lemma 3.9. *Let \mathbf{F}/\mathbb{F}_q be a function field of genus g and let $L(t) = \sum_{i=0}^{2g} a_i t^i$ be the numerator of its Zeta function. Then*

$$A_{g-1} = \frac{1}{q-1} \left(h - \left(a_g + 2 \sum_{i=0}^{g-1} a_i \right) \right)$$

Proof. This is a well-known result

$$Z(t) = \sum_{m=0}^{+\infty} A_m t^m = \frac{\sum_{i=0}^{2g} a_i t^i}{(1-t)(1-qt)}$$

We deduce that for all $m \geq 0$,

$$A_m = \sum_{i=0}^m \frac{q^{m-i+1} - 1}{q-1} a_i$$

In particular,

$$(q-1)A_{g-1} = \sum_{i=0}^{g-1} (q^{g-i} - 1)a_i$$

Since $a_i = q^{i-g}a_{2g-i}$, for all $i = 0, \dots, g$, we obtain

$$(q-1)A_{g-1} = q^g \sum_{i=0}^{g-1} q^{-i}a_i - \sum_{i=0}^{g-1} a_i = q^g \sum_{i=0}^{g-1} q^{-i}q^{i-g}a_{2g-i} - \sum_{i=0}^{g-1} a_i$$

Hence,

$$(q-1)A_{g-1} = \sum_{i=0}^{g-1} (a_{2g-i} - a_i)$$

Furthermore, we know that $h = L(1) = \sum_{i=0}^{2g} a_i$, therefore,

$$A_{g-1} = \frac{1}{q-1} \left(h - \left(a_g + 2 \sum_{i=0}^{g-1} a_i \right) \right).$$

□

Using the preceding lemma, Corollary 3.3 and Assertion 2 of Proposition 3.2, one has

Corollary 3.10. *Let \mathbf{F}/\mathbb{F}_q be a function field of genus g and $L(t) = \sum_{i=0}^{2g} a_i t^i$ be the L -polynomial of \mathbf{F}/\mathbb{F}_2 . Then*

- For $q \geq 3$, $a_g + 2 \sum_{i=0}^{g-1} a_i \geq 0$ if and only if \mathcal{E}_{g-1} is true.
- For $q = 2$, $a_g + 2 \sum_{i=0}^{g-1} a_i > 0$ if and only if \mathcal{E}_{g-1} is true.

Example 3.11. *The Hermitian function field $\mathbf{F}/\mathbb{F}_{q^2}$ is such that $\mathbf{F} = \mathbb{F}_q(x, y)$ with $y^q + y - x^{q+1} = 0$. It is a maximal function field of genus $g = \frac{q(q-1)}{2}$ and it is the constant field extension of \mathbf{G}/\mathbb{F}_q , where $\mathbf{G} = \mathbb{F}_q(x, y)$, with $y^q + y - x^{q+1} = 0$. We can say that \mathbf{G}/\mathbb{F}_q is a "constant field restriction" of $\mathbf{F}/\mathbb{F}_{q^2}$. All subfields of the Hermitian function field $\mathbf{F}/\mathbb{F}_{q^2}$ are maximal function fields.*

Corollary 3.12. *If the algebraic function field \mathbf{G}/\mathbb{F}_q is a constant field restriction of a maximal function field $\mathbf{F}/\mathbb{F}_{q^2} = \mathbf{G}.\mathbb{F}_{q^2}/\mathbb{F}_{q^2}$, then \mathbf{G}/\mathbb{F}_q contains a non-special divisor of degree $g-1$.*

3.4. Particular cases : ordinary curves over \mathbb{F}_2 and \mathbb{F}_3 . The following results treat the particular case of ordinary curves in [21, Section 4].

Let \mathbf{C}/k be a genus g (smooth projective absolutely irreducible) curve over a finite field $k = \mathbb{F}_{p^n}$. Classically, one defines the p -rank γ of this curve as the integer $0 \leq \gamma \leq g$ such that $\#\mathcal{J}\text{ac}(\mathbf{C})[p](\bar{k}) = p^\gamma$. In

particular \mathbf{C} is said to be ordinary if $\gamma = g$. There is another equivalent characterization in terms of the L -polynomial, namely $\gamma = \deg(L(t) \pmod{p})$ (see [17]). In particular, \mathbf{C} is ordinary if and only if p does not divide a_g .

Proposition 3.13. *Let \mathbf{C} be an ordinary curve of genus $g > 0$ over a finite field k of characteristic 2. There is always a non-special divisor of degree $g - 1$ on \mathbf{C} .*

Proof. Let $f \in k(\mathbf{C})$ such that $df \neq 0$. Developing f in power series at any point of \mathbf{C} , we see that df has only zeros and poles of even multiplicity. Hence there exists a rational divisor of degree $(2g-2)/2 = g - 1$ such that $(df) = 2D_0$. It is easy to show that the class of this divisor does not depend on the choice of f and it is called the canonical theta characteristic divisor. In [27, Prop.3.1], it is shown that there is a bijection between $\mathcal{L}(D_0)$ and the space of exact regular differentials (i.e. the regular differentials ω such that $\omega = df$ for $f \in k(\mathbf{C})$). Now by [24, Prop.8], a regular differential ω is exact if and only if $\mathcal{C}(\omega) = 0$ where \mathcal{C} is the Cartier operator. Moreover by [24, Prop.10], $\mathcal{J}\text{ac}(\mathbf{C})$ is ordinary if and only if \mathcal{C} is bijective. So the only exact regular differential is 0 and $\dim(D_0) = 0$. Hence D_0 is the divisor we were looking for. \square

Note, that the previous proof gives a way to explicitly construct a degree $g - 1$ divisor of dimension zero. We will now generalize Proposition 3.13 (and Lemma 3.9) but without such an explicit construction.

Lemma 3.14. *Let \mathbf{F}/\mathbb{F}_q be a function field of genus g and let $L(t) = \sum_{i=0}^{2g} a_i t^i$ be the numerator of its Zeta function. Then*

$$A_{g-k} = \frac{1}{q-1} \left[q^{-k+1} \left(h - \sum_{i=0}^{g+k-1} a_i \right) - \sum_{i=0}^{g-k} a_i \right].$$

Proof. The proof is similar to Lemma 3.9 (see [21, Lemma 3.6]). \square

Proposition 3.15. *Let \mathbf{C} be a curve of genus $g > 0$ over a finite field \mathbb{F}_q of characteristic p and of p -rank γ . There is always a degree $\gamma - 1$ zero dimension divisor on \mathbf{C} .*

Proof. Recall that $h_{n,k}$ the number of classes of divisors of degree n and of dimension k , for all $g \geq k > 0$ we get

$$h = \sum_{i=0}^{\infty} h_{g-k,i}$$

so

$$h_{g-k,0} = h - \sum_{i=1}^{\infty} h_{g-k,i}.$$

Now

$$A_{g-k} = \sum_{i=1}^{\infty} \frac{q^i - 1}{q - 1} h_{g-k,i}$$

hence we can write

$$\sum_{i=1}^{\infty} h_{g-k,i} = \sum_{i=1}^{\infty} q^i h_{g-k,i} - (q - 1)A_{g-k}.$$

Using the expression of A_{g-k} from Lemma 3.14 and then

$$h = \sum_{i=0}^{g+k-1} a_i + \sum_{i=g+k}^{2g} a_i \equiv \sum_{i=0}^{\gamma} a_i \pmod{p}$$

we get for $k = g - \gamma + 1$

$$\begin{aligned} h_{g-k,0} &= h(1 + q^{-k+1}) - q^{-k+1} \sum_{i=0}^{g+k-1} a_i - \sum_{i=0}^{g-k} a_i - \sum_{i=1}^{\infty} q^i h_{g-k,i} \\ &= \sum_{i=0}^{2g} a_i + q^{-k+1} \sum_{i=g+k}^{2g} a_i - \sum_{i=0}^{g-k} a_i - \sum_{i=1}^{\infty} q^i h_{g-1,i} \\ &= \sum_{i=g-k+1}^{2g} a_i + q^{-k+1} \sum_{i=g+k}^{2g} a_i - \sum_{i=1}^{\infty} q^i h_{g-1,i} \\ &\equiv \sum_{i=g-k+1}^{\gamma} a_i \pmod{p} \\ &\equiv a_{\gamma} \not\equiv 0 \pmod{p}. \end{aligned}$$

Hence $h_{\gamma-1,0}$ is not zero and hence is positive. \square

Remark 3.16. *Note that this proposition is interesting only in the case where $q = 2$ and $\gamma = g - k$ with $k \leq 3$ or $q = 3$ and $\gamma = g$.*

Corollary 3.17. *Let \mathbf{C} be an ordinary curve of genus $g > 0$ over a finite field \mathbb{F}_q . There is always a non-special divisor of degree $g - 1$ on \mathbf{C} .*

3.5. Particular case : Asymptotically exact towers. In this section we adapt the results in [21, Section 5.2] to prove the existence of non-special divisors of degree $g - 1$ in asymptotically exact towers.

First, let us recall the notion of asymptotically exact sequence of algebraic function fields introduced in [28].

Definition 3.18. *Consider a sequence $\mathcal{F}/\mathbb{F}_q = (\mathbf{F}_k/\mathbb{F}_q)_{k \geq 1}$ of algebraic function fields $\mathbf{F}_k/\mathbb{F}_q$ defined over \mathbb{F}_q of genus g_k . We suppose that the sequence of genus g_k is an increasing sequence growing to infinity. The sequence \mathcal{F}/\mathbb{F}_q is called asymptotically exact if for all $m \geq 1$ the following limit exists:*

$$\beta_m(\mathcal{F}/\mathbb{F}_q) = \lim_{g_k \rightarrow \infty} \frac{B_m(\mathbf{F}_k/\mathbb{F}_q)}{g_k}$$

where $B_m(\mathbf{F}_k/\mathbb{F}_q)$ is the number of places of degree m on $\mathbf{F}_k/\mathbb{F}_q$.

Now, let us recall the following two results used by I. Shparlinski, M. Tsfasman and S. Vladut in [10]. These results follow easily from Corollary 2 and Theorem 6 of [28].

Lemma 3.19. *Let $\mathcal{F}/\mathbb{F}_q = (\mathbf{F}_k/\mathbb{F}_q)_{k \geq 1}$ be an asymptotically exact sequence of algebraic function fields defined over \mathbb{F}_q and h_k be the class number of $\mathbf{F}_k/\mathbb{F}_q$. Then*

$$\log_q h_k = g_k \left(1 + \sum_{m=1}^{\infty} \beta_m \cdot \log_q \frac{q^m}{q^m - 1} \right) + o(g_k)$$

Lemma 3.20. *Let A_{a_k} be the number of effective divisors of degree a_k on $\mathbf{F}_k/\mathbb{F}_q$. If*

$$a_k \geq g_k \left(\sum_{m=1}^{\infty} \frac{m\beta_m}{q^m - 1} \right) + o(g_k)$$

then

$$\log_q A_{a_k} = a_k + g_k \cdot \sum_{m=1}^{\infty} \beta_m \cdot \log_q \frac{q^m}{q^m - 1} + o(g_k).$$

These asymptotical properties were established in [28] and [29] in order to estimate the class number h of algebraic function fields of genus g defined over \mathbb{F}_q and also in order to estimate their number of classes of effective divisors of degree $m \leq g - 1$. Namely, I. Shparlinski, M. Tsfasman and S. Vladut used in [10] the inequality $2A_{g_k(1-\epsilon)} < h_k$ where $0 < \epsilon < \frac{1}{2}$ and k big enough, under the hypothesis of Lemma 3.19. In the same spirit, we give here a particular case of their result in the following proposition.

Proposition 3.21. *Let $\mathcal{F}/\mathbb{F}_q = (\mathbf{F}_k/\mathbb{F}_q)_{k \geq 1}$ be an asymptotically exact sequence of algebraic function field defined over \mathbb{F}_q . Then, there exists an integer k_0 such that for any integer $k \geq k_0$, we get:*

$$A_{g_k-1} < h_k$$

and there is a non-special divisor of degree $g - 1$ in $\mathbf{F}_k/\mathbb{F}_q$.

Proof. The total number of linear equivalence classes of an arbitrary degree equals to the divisor class number h_k of $\mathbf{F}_k/\mathbb{F}_q$, which is given by Lemma 3.19. Moreover, for g_k sufficiently large, we have:

$$\sum_{m=1}^{\infty} \frac{m\beta_m}{q^m - 1} \leq \frac{1}{\sqrt{q} + 1} \sum_{m=1}^{\infty} \frac{m\beta_m}{q^{\frac{m}{2}} - 1} < \frac{1}{2}$$

since $q \geq 2$ and $\sum_{m=1}^{\infty} \frac{m\beta_m}{q^{\frac{m}{2}} - 1} \leq 1$ by Corollary 1 of [28]. As $\frac{1}{g_k} < \frac{1}{2}$, one has

$$g_k \left(1 - \frac{1}{g_k}\right) \geq g_k \left(\sum_{m=1}^{\infty} \frac{m\beta_m}{q^m - 1}\right) + o(g_k)$$

for k big enough. Therefore, we can apply Lemma 3.20 and compare $\log_q A_{g_k(1-1/g_k)}$ with $\log_q h_k$ given by Lemma 3.19. Hence, there exists an integer k_0 such that for $k \geq k_0$, $A_{g_k-1} < h_k$. We conclude by Proposition 3.2. \square

3.6. Particular case : curves of defect k over \mathbb{F}_2 and \mathbb{F}_3 . In this section, we will focus on curves over \mathbb{F}_2 and \mathbb{F}_3 of genus $g \geq 3$. The existence of non-special divisors of degree $g - 1$ is assured for $q \geq 4$, moreover, the cases of curves of genus $g = 1, 2$ were studied in the introduction of section 3.3 and Theorem 3.8. The goal is to give some examples of function fields that contain these divisors.

We consider for $r \geq 1$ the number

$$N_r := N(\mathbf{F}_r) = \#\{P \in \mathcal{P}_k(\mathbf{F}/\mathbb{F}_q); \deg(P) = 1\}$$

where $\mathbf{F}_r = \mathbf{F}\mathbb{F}_q^r$ is the constant field extension of \mathbf{F}/\mathbb{F}_q of degree r . Let us remind the equation from [26, Corollary 5.1.16], for all $r \geq 1$,

$$(10) \quad N_r = q^r + 1 - \sum_{i=1}^{2g} \alpha_i^r = q^r + 1 - \sum_{i=1}^g 2q^{r/2} \cos(\pi r \phi_i)$$

where $(\alpha_1, \dots, \alpha_{2g}) = (q^{1/2}e^{i\pi\phi_1}, \dots, q^{1/2}e^{i\pi\phi_g}, q^{1/2}e^{-i\pi\phi_1}, \dots, q^{1/2}e^{-i\pi\phi_g})$ are the reciprocals of the roots of $L(t)$ with $\phi_i \in [0, 1]$. In particular, since $N_1 = N(\mathbf{F})$, we have

$$N(\mathbf{F}) = q + 1 - \sum_{i=1}^{2g} \alpha_i.$$

Proposition 3.22. [26, Corollary 5.1.17] *Let $L(t) = \sum_{i=0}^{2g} a_i t^i$ be the L -polynomial of \mathbf{F}/\mathbb{F}_q , and $S_r := N_r - (q^r + 1)$. Then we have:*

- (a) $L'(t)/L(t) = \sum_{r=1}^{\infty} S_r t^{r-1}$.
 (b) $a_0 = 1$ and

$$(11) \quad i a_i = S_i a_0 + S_{i-1} a_1 + \cdots + S_1 a_{i-1}$$

for $i = 1, \dots, g$.

Given N_1, \dots, N_g and using $a_{2g-i} = q^{g-i} a_i$, we can determine $L(t)$ from (11).

Let \mathbf{F} be an algebraic function field over \mathbb{F}_2 or \mathbb{F}_3 of genus g , and k its defect, *i.e.*

$$|N_1(\mathbf{F}/\mathbb{F}_2) - (q + 1)| = g[2\sqrt{q}] - k$$

where $[x]$ denotes the largest integer $\leq x$. Let

$$L(t) = \sum_{i=0}^{2g} a_i t^i = \prod_{i=1}^g [(1 - \alpha_i t)(1 - \bar{\alpha}_i t)]$$

be the numerator of the Zeta function of \mathbf{F} . Recall that with the previous conditions, one has

$$|N_1(\mathbf{F}/F_q) - (q + 1)| = \left| - \sum_{i=1}^{2g} \alpha_i \right| = g[2\sqrt{q}] - k$$

and then

$$(12) \quad k = g[2\sqrt{q}] - \left| \sum_{i=1}^{2g} \alpha_i \right|.$$

We know that if $N_1 \geq g + 1$, there exists a non-special divisor of degree $g - 1$. Moreover we know all the curves which contain this kind of divisors over \mathbb{F}_2 for $g = 1$ and 2. Since for defect 2 curves over \mathbb{F}_2 with $g \geq 3$ one has

$$|N_1 - 3| = 2g - 2 \Rightarrow N_1 = 2g + 1 \Rightarrow N_1 \geq g + 1$$

the existence of these divisors is obvious and we do not need to use the coefficients a_n for this purpose. Nevertheless, for a defect $k \geq 3$ curves over \mathbb{F}_2 and \mathbb{F}_3 ($q = 2$ or 3, hence $[2\sqrt{q}] = q$), one has

$$|N_1 - (q + 1)| = qg - k \Rightarrow N_1 = -qg + k + q + 1 \text{ or } N_1 = qg - k + q + 1$$

the goal of this section is to prove the existence of non-special divisor of degree $g - 1$ for curves that satisfy

$$(13) \quad 0 \leq N_1 = -qg + k + q + 1 \leq g \text{ or } 0 \leq N_1 = qg - k + q + 1 \leq g.$$

the following results generalize the cases (a) and (d) of [25, Theorem 2.5.1] by J.P Serre and will help us determine the sign of $a_g + 2 \sum_{i=0}^{g-1} a_i$ (in order to apply 3.10).

Lemma 3.23. *Let α be a totally positive algebraic integer and $k(\alpha) = \text{Tr}(\alpha) - d(\alpha)$ (recall that $\text{Tr}(\alpha)$ is the sum of the conjugates of α , it is totally positive if all its conjugates are real > 0 , that $d(\alpha)$ is the degree of its minimal polynomial and $\text{Tr}(\alpha) \geq \text{deg}(\alpha)$ (see [25, Remark 2.2.3])). One has*

- If $k(\alpha) = 0$, then $\alpha = 1$.
- If $d(\alpha) = 1$, then $\alpha = k(\alpha) + 1$.
- If $d(\alpha) = 2$, then $\alpha = \frac{k(\alpha)+2 \pm \sqrt{(k(\alpha)+2)^2 - 4n}}{2}$ with $\frac{(k(\alpha)+2)^2}{4} > n$.

Proof. • For the case $k(\alpha) = 0$ see [25, Corollary 2.2.4].
 • If $d(\alpha) = 1$, then the minimal polynomial of α is $x - \text{Tr}(\alpha)$ and $\alpha = \text{Tr}(\alpha)$, we conclude that $k(\alpha) = \text{Tr}(\alpha) - 1 = \alpha - 1$.
 • If $d(\alpha) = 2$, then the minimal polynomial of α is $x^2 - \text{Tr}(\alpha)x + n = x^2 - (k(\alpha) + 2)x + n$ with real positive roots, namely $\frac{k(\alpha)+2 \pm \sqrt{(k(\alpha)+2)^2 - 4n}}{2}$ only if $\frac{(k(\alpha)+2)^2}{4} > n$. □

Theorem 3.24. *For a curve over \mathbb{F}_q such that $|N_1(\mathbf{F}/\mathbb{F}_q) - (q+1)| = g[2\sqrt{q}] - k$ with $g \geq 2$, one has*

- (a) *For $3 \leq k \leq 2 \cdot [2\sqrt{q}]$, if there exists $i \in \{1, \dots, g\}$ such that $\alpha_i + \bar{\alpha}_i = [2\sqrt{q}] - k$, then we can reorganize the tuple $(\alpha_1 + \bar{\alpha}_1, \dots, \alpha_g + \bar{\alpha}_g)$ to get*

$$(\alpha_1 + \bar{\alpha}_1, \dots, \alpha_g + \bar{\alpha}_g) = \pm([2\sqrt{q}], \dots, [2\sqrt{q}], [2\sqrt{q}] - k)$$

- (b) *If there exists $i, j \in \{1, \dots, g\}$ such that $[2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i$ and $[2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j$ are conjugated with $d([2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i) = 2$ and*

$$\alpha_i + \bar{\alpha}_i + \alpha_j + \bar{\alpha}_j = 2[2\sqrt{q}] - k$$

then we can reorganize the tuple $(\alpha_1 + \bar{\alpha}_1, \dots, \alpha_i + \bar{\alpha}_i, \dots, \alpha_g + \bar{\alpha}_g)$ to get

$$(\alpha_1 + \bar{\alpha}_1, \dots, \alpha_g + \bar{\alpha}_g) = \pm \left([2\sqrt{q}], \dots, [2\sqrt{q}], [2\sqrt{q}] + 1 - \frac{k+2+\Delta}{2}, [2\sqrt{q}] + 1 - \frac{k+2-\Delta}{2} \right)$$

with $\Delta = \sqrt{(k+2)^2 - 4n}$ and $\frac{(k+2)^2}{4} > n$. This holds for $-2(2\sqrt{q} - [2\sqrt{q}]) \leq k \pm \Delta \leq 4\sqrt{q} + 2[2\sqrt{q}]$.

Proof. It is enough to prove the proposition in the case $N_1(\mathbf{F}/\mathbb{F}_q) - (q + 1) = g[2\sqrt{q}] - k$ which means $k = g[2\sqrt{q}] - \sum_{i=1}^{2g} \alpha_i$ (by (12)).

Let $\kappa : \mathbb{Z}[X] \rightarrow \mathbb{Z}$ be the map defined by

$$\kappa(b_0X^n - b_1X^{n-1} + \dots + b_n) = b_1 - n$$

and $P \in \mathbb{Z}[X]$ be the polynomial

$$P(X) = X^g - a_1X^{g-1} + \dots + a_g = \prod_{i=1}^g (X - [2\sqrt{q}] - 1 + \alpha_i + \bar{\alpha}_i)$$

its roots are real positive since $[2\sqrt{q}] + 1 \geq \alpha_i + \bar{\alpha}_i = 2\sqrt{q} \cdot \cos(\phi_i)$ with ϕ_i the argument of α_i . then

$$\kappa(P(X)) = \sum_{i=1}^g ([2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i) - g = g[2\sqrt{q}] + g - \sum_{i=1}^g (\alpha_i + \bar{\alpha}_i) - g = k$$

Now, let $P(X) = \prod_{\lambda=1}^r Q_\lambda(X) = \prod_{\lambda=1}^r (X^{\deg(Q_\lambda)} - a_{1,\lambda}X^{\deg(Q_\lambda)-1} + \dots)$ be the decomposition of P in a product of irreducible polynomials. We have

$$a_1 = \sum_{\lambda=1}^r a_{1,\lambda}$$

and

$$a_1 - g = \sum_{\lambda=1}^r a_{1,\lambda} - \sum_{\lambda=1}^r \deg(Q_\lambda)$$

thus

$$(14) \quad \kappa(P(X)) = \sum_{\lambda=1}^r \kappa(Q_\lambda(X)) = k.$$

- (a) Let $i \in \{1, \dots, g\}$ such that $\alpha_i + \bar{\alpha}_i = [2\sqrt{q}] - k$, then $[2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i \in \mathbb{Z}$ which means that there exists $\lambda' \in \{1, \dots, r\}$ such that $Q_{\lambda'}(X) = X - [2\sqrt{q}] - 1 + \alpha_i + \bar{\alpha}_i$ then $d([2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i) = 1$ and $Tr([2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i) =$

$[2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i$. By (14)

$$\begin{aligned}
\kappa(P(X)) &= \sum_{\lambda=1}^r \kappa(Q_\lambda(X)) \\
&= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + \kappa(Q_{\lambda'}(X)) \\
&= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + [2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i - 1 \\
&= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + [2\sqrt{q}] - [2\sqrt{q}] + k \\
&= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + k = k
\end{aligned}$$

Notice that if $[2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j$ is a root of Q_λ ($\lambda \neq \lambda'$) then with the notation of Lemma 3.23 one has

$$\begin{aligned}
\kappa(Q_\lambda(X)) &= Tr([2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j) - d([2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j) \\
&= k([2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j) \geq 0
\end{aligned}$$

we conclude that $\kappa(Q_\lambda(X)) = k([2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j) = 0$ for $\lambda \neq \lambda'$. By Lemma 3.23, one has $[2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j = 1$ thus $[2\sqrt{q}] = \alpha_j + \bar{\alpha}_j$ for $j \neq i$ and by assumption $\alpha_i + \bar{\alpha}_i = [2\sqrt{q}] - k$. Finally, the argument θ_i with $2\sqrt{q} \cdot \cos(\theta_i) = \alpha_i + \bar{\alpha}_i = [2\sqrt{q}] - k$ exists if $0 \leq k \leq 2 \cdot [2\sqrt{q}]$. We are interested, here, by $3 \leq k \leq 2 \cdot [2\sqrt{q}]$ since the cases 0, 1 et 2 were studied in [25, Theorem 2.5.1].

- (b) Let $\beta_i = [2\sqrt{q}] + 1 - \alpha_i - \bar{\alpha}_i$ and $\beta_j = [2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j$. There exists $\lambda' \in \{1, \dots, r\}$ such that

$$Q_{\lambda'}(X) = X^2 - a_{1,\lambda'}X + a_{2,\lambda'}$$

with

$$a_{1,\lambda'} = Tr(\beta_i) = Tr(\beta_j) = 2[2\sqrt{q}] + 2 - \alpha_i - \bar{\alpha}_i - \alpha_j - \bar{\alpha}_j$$

By (14), one has

$$\begin{aligned}
 \kappa(P(X)) &= \sum_{\lambda=1}^r \kappa(Q_\lambda(X)) \\
 &= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + \kappa(Q_{\lambda'}(X)) \\
 &= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + 2[2\sqrt{q}] + 2 - \alpha_i - \bar{\alpha}_i - \alpha_j - \bar{\alpha}_j - 2 \\
 &= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + 2[2\sqrt{q}] - 2[2\sqrt{q}] + k \\
 &= \sum_{\substack{\lambda=1 \\ \lambda \neq \lambda'}}^r \kappa(Q_\lambda(X)) + k = k
 \end{aligned}$$

As in a), we conclude that $\kappa(Q_\lambda(X)) = \mathbf{k}([2\sqrt{q}] + 1 - \alpha_j - \bar{\alpha}_j) = 0$ for $\lambda \neq \lambda'$ and since $d(\beta_i) = d(\beta_j) = 2$, by Lemma 3.23 one has

$$\beta_i = \frac{\mathbf{k}(\beta_i) + 2 - \sqrt{(\mathbf{k}(\beta_i) + 2)^2 - 4n}}{2} = \frac{k + 2 + \sqrt{(k + 2)^2 - 4n}}{2}$$

and

$$\beta_j = \frac{\mathbf{k}(\beta_j) + 2 - \sqrt{(\mathbf{k}(\beta_j) + 2)^2 - 4n}}{2} = \frac{k + 2 - \sqrt{(k + 2)^2 - 4n}}{2}$$

thus $\alpha_i + \bar{\alpha}_i = [2\sqrt{q}] + 1 - \frac{k+2+\Delta}{2}$ and $\alpha_j + \bar{\alpha}_j = [2\sqrt{q}] + 1 - \frac{k+2-\Delta}{2}$. Finally, the arguments θ_i, θ_j with $2\sqrt{q} \cdot \cos(\theta_i) = \alpha_i + \bar{\alpha}_i$ and $2\sqrt{q} \cdot \cos(\theta_j) = \alpha_j + \bar{\alpha}_j$ exist if

$$-1 \leq \frac{[2\sqrt{q}] + 1 - \frac{k+2\pm\Delta}{2}}{2\sqrt{2}} \leq 1$$

which means

$$-2(2\sqrt{q} - [2\sqrt{q}]) \leq k \pm \Delta \leq 4\sqrt{q} + 2[2\sqrt{q}].$$

□

Remark 3.25. *The conditions of Theorem 3.24 are too constraining but they describe a possibly infinitely many curves if we refer to the following theorem ([25, Theorem 2.4.1]).*

Theorem 3.26. *Let $\{\alpha_1 + \overline{\alpha_1}, \dots, \alpha_g + \overline{\alpha_g}\}$ be a set of algebraic integers. Suppose that the polynomial $\prod_{i=1}^g (X - \alpha_i - \overline{\alpha_i})$ can be factored as $P_1 \cdot P_2$ such that P_1 and P_2 are monic, non constant polynomials in $\mathbb{Z}[X]$ and their resultant is equal to 1 or -1 . Then the $(\alpha_i)_{1 \leq i \leq g}$ can not come from a curve.*

Now, we have enough information to calculate the sum $a_g + 2 \sum_{i=0}^{g-1} a_i$ using (10) and Proposition 3.22, in order to apply Corollary 3.10. Therefor, under the conditions of (13), Theorem 3.24 case a) and Theorem 3.26, we can build the following table:

q	g	k	\mathcal{E}_{g-1}
2	3	3	True
2	3	4	True

q	g	k	\mathcal{E}_{g-1}
3	3	5	True
3	3	6	True

TABLE 1.

and under the conditions of (13), Theorem 3.24 case b) and Theorem 3.26, we can build the following table:

q	g	k	n	\mathcal{E}_{g-1}
2	3	3	1	False
2	3	3	2	True
2	3	3	4	True
2	3	3	6	False
2	3	4	1	True
2	3	4	2	False
2	3	4	3	False
2	3	4	5	True
2	3	4	7	True
2	3	4	8	True
2	3	4	9	True
2	3	5	8	True
2	4	5	8	True
2	3	5	9	True
2	4	5	9	True
2	3	5	10	True
2	4	5	10	True
2	3	5	11	True
2	4	5	11	True
2	3	5	12	True
2	4	5	12	True
2	3	6	13	True
2	4	6	13	True
2	3	6	14	True
2	4	6	14	True
2	3	6	15	True
2	4	6	15	True
2	3	6	16	True
2	4	6	16	True
2	3	7	19	True
2	4	7	19	True
2	5	7	19	True
2	3	7	20	True
2	4	7	20	True
2	5	7	20	True
2	3	8	25	True
2	4	8	25	True

q	g	k	n	\mathcal{E}_{g-1}
2	5	8	25	True
3	3	5	4	True
3	3	5	6	True
3	3	5	8	True
3	3	5	9	False
3	3	5	10	False
3	3	5	11	False
3	3	5	12	False
3	3	6	4	False
3	3	6	5	False
3	3	6	7	True
3	3	6	9	True
3	3	6	10	True
3	3	6	11	True
3	3	6	12	True
3	3	6	13	True
3	3	6	14	True
3	3	6	15	True
3	3	6	16	True
3	3	7	12	False
3	3	7	13	False
3	3	7	14	False
3	3	7	15	False
3	3	7	16	True
3	3	7	17	True
3	3	7	18	True
3	3	7	19	True
3	3	7	20	True
3	3	8	19	True
3	4	8	19	True
3	3	8	20	False
3	4	8	20	False
3	3	8	21	True
3	4	8	21	True
3	3	8	22	True
3	4	8	22	True
3	3	8	23	True

q	g	k	n	\mathcal{E}_{g-1}
3	4	8	23	True
3	3	8	24	True
3	4	8	24	True
3	3	8	25	True
3	4	8	25	True
3	3	9	27	True
3	4	9	27	True
3	3	9	28	False
3	4	9	28	False
3	3	9	29	True
3	4	9	29	True
3	3	9	30	True
3	4	9	30	True

q	g	k	n	\mathcal{E}_{g-1}
3	3	10	34	True
3	4	10	34	True
3	3	10	35	True
3	4	10	35	True
3	3	10	36	True
3	4	10	36	True
3	3	11	42	True
3	4	11	42	True
3	5	11	42	True
3	3	12	49	True
3	4	12	49	True
3	5	12	49	True

TABLE 2.

where g is the genus, k the defect and n the integer defined in Lemma 3.23.

In the table provided, we present the signs of the sum $a_g + 2 \sum_{i=0}^{g-1} a_i$ for curves with defect $k > 2$. These results were obtained through direct computation, without a formal proof. However, it is worth noting that these signs can be theoretically demonstrated using the same method developed in [11] for curves with defect 2. That method relies on an explicit form of the coefficients of the L-polynomial, which provides a rigorous framework for extending these results to higher defect values, should one wish to pursue a theoretical proof.

4. CONSTRUCTION OF NON-SPECIAL DIVISORS OF DEGREE g AND $g - 1$

4.1. Example on Kummer extension and hermitian curves.

This section presents the work of E. Camps Moreno, H. H. Lopez and G. L. Matthews in [5] which is based on the Weierstrass semigroup to find the explicit form of non-special divisors of degree g and $g - 1$ on Kummer extensions and Hermitian curves.

For a curve \mathbf{C} defined by $f(y) = g(x)$ over a finite field \mathbb{F}_q , we denote P_{ab} a point on \mathbf{C} corresponding to $x = a$ and $y = b$. If \mathbf{C} has a unique point at infinity, we note it by P_∞ .

We say that a divisor A is supported by a point $P \in \mathbf{C}(\mathbb{F}_q)$ if and only if $v_p(A) \neq 0$.

Given m distinct rational points P_1, \dots, P_m on a curve \mathbf{C} , the Weierstrass semigroup of the m -tuple is

$$H(P_1, \dots, P_m) := \{\alpha \in \mathbb{N}^m : \exists f \in \mathbf{C}(\mathbb{F}_q) \text{ with } (f)_\infty = \sum_{i=1}^m \alpha_i P_i\}$$

Equivalently, $\alpha \in H(P_1, \dots, P_m)$ if and only if

$$\dim\left(\sum_{i=1}^m \alpha_i P_i\right) = \dim\left(\sum_{i=1}^m \alpha_i P_i - P_j\right) + 1$$

For all j , $1 \leq j \leq m$. The Weierstrass gap of (P_1, \dots, P_m) is

$$G(P_1, \dots, P_m) := \mathbb{N}^m \setminus H(P_1, \dots, P_m)$$

The multiplicity of the semigroup $H(P)$, where $P \in \mathbf{C}(\mathbb{F}_q)$ is

$$\gamma(H(P)) := \min\{a : a \in H(P) \setminus \{0\}\}.$$

We define a partial order \preceq on \mathbb{N}^m by $(n_1, \dots, n_m) \preceq (p_1, \dots, p_m)$ if and only if $n_i \leq p_i$ for all i , $1 \leq i \leq m$.

For i , $1 \leq i \leq m$, let $\Gamma^+(P_i) := H(P_i)$, and for $l \geq 2$, let

(15)

$$\Gamma^+(P_1, \dots, P_m) := \{v \in \mathbb{Z}^{+l} : v \text{ is minimal in } \{w \in H(P_{i_1}, \dots, P_{i_l}) : v_i = w_i\} \\ \text{for some } i, 1 \leq i \leq l\}$$

For each $I \subseteq \{1, \dots, m\}$ let ι_I denote the natural inclusion $\mathbb{N}^l \rightarrow \mathbb{N}^m$ into the coordinate indexed by I . The minimal generating set of $H(P_1, \dots, P_m)$ is

$$\Gamma(P_1, \dots, P_m) := \bigcup_{l=1}^m \bigcup_{\substack{I=\{i_1, \dots, i_l\} \\ i_1 < \dots < i_l}} \iota_I(\Gamma^+(P_{i_1}, \dots, P_{i_l}))$$

The Weierstrass semigroup is completely determined by the minimal generating set $\Gamma(P_1, \dots, P_m)$: if $1 \leq m < \#\mathbb{F}_q$, then

(16)

$$H(P_1, \dots, P_m) = \{\text{lub}\{v_1, \dots, v_m\} : v_1, \dots, v_m \in \Gamma(P_1, \dots, P_m)\}.$$

with:

$$\text{lub}\{v^{(1)}, \dots, v^{(t)}\} := (\max\{v_1^{(1)}, \dots, v_1^{(t)}\}, \dots, \max\{v_n^{(1)}, \dots, v_n^{(t)}\}).$$

The *least upper bound* of $v^{(1)}, \dots, v^{(t)} \in \mathbb{N}^n$.

Proposition 4.1. *Let $A = \sum_{i=1}^m \alpha_i P_i$ be an effective divisor of degree g . If $\gamma \not\leq \alpha$ for all $\gamma \in \Gamma(P_1, \dots, P_m)$, then A is non-special.*

Proof. Let $A = \sum_{i=1}^m \alpha_i P_i \in \mathbf{D}(\mathbf{F}/\mathbb{F}_q)$ be an effective divisor and $\sum_{i=1}^m \alpha_i = g$. Suppose A is special. Then $\dim(A) > \deg(A) + 1 - g = 1$. Since $\dim(A) \geq 2$, there exists $w \in H(P_1, \dots, P_m)$ such that $w \leq \alpha$. According to (16), there exist $v_1, \dots, v_m \in \Gamma(P_1, \dots, P_m)$ satisfying $v_1 \leq \text{lub}\{v_1, \dots, v_m\} = w \leq \alpha$, the contrapositive of the statement completes the proof. \square

Proposition 4.2. *Let $\mathbf{F}/\mathbb{F}_q(y)$ be the Kummer extension defined by*

$$x^m = \prod_{i=1}^r (y - \alpha_i)$$

and let P_i the place associated with $y - \alpha_i$. Then

$$\Gamma^+(P_1) = \mathbb{N} \setminus \left\{ mk + j \mid 1 \leq j \leq m - 1 - \left\lfloor \frac{m}{r} \right\rfloor, 0 \leq k \leq r - 2 - \left\lfloor \frac{rj}{m} \right\rfloor \right\}$$

and for $2 \leq l \leq r - \left\lfloor \frac{r}{m} \right\rfloor$, $\Gamma^+(P_1, \dots, P_l)$ is given by

$$\left\{ (ms_1 + j, \dots, ms_l + j) \mid 1 \leq j \leq m - 1 - \left\lfloor \frac{m}{r} \right\rfloor, s_i \geq 0, \sum_{i=1}^l s_i = r - l - \left\lfloor \frac{rj}{m} \right\rfloor \right\}$$

Proof. See [1, Theorem 3.2] and [7, Theorem 10] \square

Lemma 4.3. *Let $r, m \in \mathbb{Z}^+$ be relatively prime.*

(1) *Let $1 \leq j \leq m - 1$ and set $t = r \bmod m$. then*

$$\left\lfloor \frac{r(j+1)}{m} \right\rfloor - \left\lfloor \frac{rj}{m} \right\rfloor = \begin{cases} \left\lfloor \frac{r}{m} \right\rfloor + 1 & j = \left\lfloor \frac{km}{t} \right\rfloor, 1 \leq k \leq t - 1 \\ \text{otherwise.} & \end{cases}$$

(2) *If $t < m$, then*

$$\sum_{k=1}^{t-1} \left\lfloor \frac{km}{t} \right\rfloor = \frac{(m-1)(t-1)}{2}.$$

Proof. See [5, Lemma 7]. \square

Theorem 4.4. *Let $\mathbf{F}/\mathbb{F}_q(y)$ by the Kummer extension defined by*

$$x^m = \prod_{i=1}^r (y - \alpha_i)$$

where $\alpha_i \in \mathbb{F}_q$ and $(r, m) = 1$. For $1 \leq j \leq m - 1 - \left\lfloor \frac{m}{r} \right\rfloor$, define the following values:

- $l_j = r - \left\lfloor \frac{rj}{m} \right\rfloor$.

- $s_j = l_j - l_{j+1}$ if $j < m-1 - \lfloor \frac{m}{r} \rfloor$ and $s_{m-1 - \lfloor \frac{m}{r} \rfloor} = l_{m-1 - \lfloor \frac{m}{r} \rfloor} - 1 = \max\{1, \lfloor \frac{m}{r} \rfloor\}$.

Then A is an effective non-special divisor of degree g with support contained in the set $\{P_{0b} : \prod_{i=1}^r (b - \alpha_i) = 0\}$ if and only if

$$A = \sum_{j=1}^{m-1 - \lfloor \frac{m}{r} \rfloor} j \sum_{i=1}^{s_j} P_{0b_{j_i}}.$$

In particular, if $r < m$,

$$A = \sum_{j=1}^{r-1} \left\lfloor \frac{jm}{r} \right\rfloor P_{0b_j}.$$

Proof. First, note that

$$A = \sum_{j=1}^{m-1 - \lfloor \frac{m}{r} \rfloor} j D_j = \sum_{j=1}^{m-1 - \lfloor \frac{m}{r} \rfloor} j \sum_{i=1}^{s_j} P_{0b_{j_i}}$$

where

$$D_j = \begin{cases} \sum_{i=1}^{s_j} P_{0b_{j_i}} & \text{if } s_j > 0 \\ 0 & \text{if } s_j = 0 \end{cases}$$

for $1 \leq j \leq m-1 - \lfloor \frac{m}{r} \rfloor$. We will prove that A is of degree g . Let $t = r \bmod m$, we have

$$\begin{aligned} \deg(A) &= \sum_{i=1}^{m-1 - \lfloor \frac{m}{r} \rfloor} j s_j \\ &= \sum_{i=1}^{m-1} j \left\lfloor \frac{r}{m} \right\rfloor + \sum_{k=1}^{t-1} \left\lfloor \frac{km}{t} \right\rfloor \\ &= \frac{(m-1)m}{2} \left\lfloor \frac{r}{m} \right\rfloor + \frac{(m-1)(t-1)}{2} \\ &= \frac{(m-1)}{2} \left(m \left\lfloor \frac{r}{m} \right\rfloor + t - 1 \right) \\ &= \frac{(m-1)(r-1)}{2} = g \end{aligned}$$

where the second equality follows from Lemma 4.3 (1) and the third one from Lemma 4.3 (2). Therefore, A is effective of degree g . Now, we are going to prove that A is non-special using Proposition 4.1.

Take $v \in \mathbb{N}^{l_1-1}$ such that $A = \sum_{i=1}^{l_1-1} v_i P_i$. Since $v_i \leq m - 1 - \lfloor \frac{m}{r} \rfloor$ for all i , then by Proposition 4.2 $v_i < w$ for any $w \in \Gamma^+(P_i)$, and so

$$\iota_{\{i\}}(w) \not\leq v$$

for any $w \in \Gamma^+(P_i)$. Take $w \in \Gamma^+(P_{i_j} | j \in I \subset \{1, \dots, l_1 - 1\})$. If for some i , $w_i > m - \lfloor \frac{m}{r} \rfloor$, then $w \not\leq v$, so assume $w = (k, \dots, k)$ for some $1 \leq k \leq m - 1 - \lfloor \frac{m}{r} \rfloor$. By Proposition 4.2, we know that $\#I = l_k$ and the number of entries of v greater or equal than k are $\sum_{i=k}^{m-1-\lfloor \frac{m}{r} \rfloor} s_i = l_k - 1$, therefore $\iota_I(w) \not\leq v$ for any I of cardinality l_k . This concludes that $w \not\leq v$ for all $w \in \Gamma(\text{Supp}(A))$ and then A is non-special.

Set $\gamma = m - 1 - \lfloor \frac{m}{r} \rfloor$ and choose B an effective non-special divisor of degree g supported on $\text{Supp}(x)$. If $v_P(B) \geq \gamma + 1$ for some P , then $\iota(\gamma + 1) \leq B$, contradicting the non-specialty of B .

Write $B = \sum_{j=1}^{\gamma} j D_j$ where D_j is zero or is the sum of distinct rational places of degree 1 and $\text{Supp}(D_j) \cap \text{Supp}(D_h) = \emptyset$ for $j \neq h$.

Observe that $\#\text{Supp}(B) \leq l_1 - 1 < r = \#\text{Supp}(x)_0$. For D_γ we know

$$\deg(D_\gamma) \leq l_\gamma - 1 = s_\gamma.$$

Similarly, for $1 \leq h \leq \gamma$, we know

$$\sum_{j=h}^{\gamma} \deg(D_j) \leq \deg(D_h) + \sum_{j=h+1}^{\gamma} \deg(D'_j) \leq l_h - 1,$$

so take $D'_h \geq D_h$, $\text{Supp}(D_h) \subseteq \text{Supp}(x)_0 \setminus \text{Supp}(B + \sum_{j=h+1}^{\gamma} D'_j)$ such that

$$\sum_{j=h}^{\gamma} D'_h = \sum_{j=h}^{\gamma} \#\text{Supp}(D'_j) = l_h - 1.$$

From this construction, it is clear

$$\deg(D'_h) = s_h$$

and so

$$g = \deg(B) \leq \sum_{j=1}^{\gamma} j \cdot \deg(D'_j) = \sum_{j=1}^{\gamma} j s_j = g.$$

Then $D'_h = D_h$ for any $1 \leq h \leq \gamma$ and B has the desired form.

In the case $r < m$, By lemma 4.3 we have $s_j = 1$ if $j = \lfloor \frac{km}{r} \rfloor$ and 0 otherwise and then $D_j = P_k$ or $D_j = 0$. \square

Corollary 4.5. *On the norm-trace curve given by $y^{q^{r-1}} + y^{q^{r-2}} + \dots + y = x^{\frac{q^r-1}{q-1}}$ over \mathbb{F}_{q^r} , any effective non-special divisor of degree g supported by points P_{0b} is of the form*

$$\sum_{i=1, q \nmid i}^{\frac{q^r-1}{q-1}-2} iP_{0b_i}.$$

Proof. Take $u = \frac{q^r-1}{q-1}$. Given

$$\left\lfloor \frac{(r-1)u}{q^{r-1}} \right\rfloor = u - 2$$

it is enough to prove that $\left\lfloor \frac{ju}{q^{r-1}} \right\rfloor$ cannot be divisible by q , since

$$u-2-\#\{i \in \{1, \dots, u-2\} \mid q \mid i\} = u-2-\frac{u-1}{q}+1 = \frac{u-1}{q}(q-1) = q^{r-1}-1.$$

and then A should be $\sum_{j=1}^{q^{r-1}-1} \left\lfloor \frac{ju}{q^{r-1}} \right\rfloor P_j$, implying that A is non-special of degree g . By Theorem 4.4, any other divisor with these characteristics should be of this form.

Thus, we will prove $q \nmid \left\lfloor \frac{ju}{q^{r-1}} \right\rfloor$ for any $1 \leq j \leq q^{r-1} - 1$. If j is such that $\left\lfloor \frac{ju}{q^{r-1}} \right\rfloor = qk$ for some $1 \leq k \leq \frac{u-1}{q} - 1$, then

$$ju = q^r k + z = u(q-1)k + k + z.$$

This expression implies that $u \mid k + z$, but

$$k + z < \frac{u-2}{q} + q^{r-1} = \frac{uq-1}{q} < u.$$

Then no such k exists, and the conclusion follows. \square

Corollary 4.6. *On the Hermitian curve $y^q + y - x^{q+1} = 0$ over \mathbf{F}_{q^2} , any effective non-special divisor of degree g with support contained in $\{P_{0b_i} : 1 \leq i \leq q\}$ is of the form*

$$A = \sum_{i=1}^{q-1} iP_{0b_i}.$$

Using 2.3 and 4.4, we have the following explicit construction.

Theorem 4.7. *Let $\mathbf{F}/\mathbf{F}_q(y)$ by the Kummer extension defined by*

$$x^m = \prod_{i=1}^r (y - \alpha_i)$$

where $\alpha_i \in \mathbf{F}_q$ and $(r, m) = 1$. Then

$$A = \sum_{j=1}^{m-1-\lfloor \frac{m}{r} \rfloor} j \sum_{i=1}^{s_j} P_{0b_{j_i}} - P.$$

is a non-special divisor of degree $g-1$ for all $P \in \{P_{ab} | a \neq 0 \text{ or } b \neq b_{j_i}\} \cup \{P_\infty\}$. In particular there exist non-special divisors of degree $g-1$ supported on $\text{supp}((x)_0) \cup \{P_{ab}\}$ for any $a \neq 0$.

Proof. Note that $A + P_{ab}$ is non-special of degree g by Theorem 4.4 and, by Lemma 2.3, we have A is non-special too. Given

$$\# \text{Supp}(A) = r - \left\lfloor \frac{r}{m} \right\rfloor - 1 \leq r - 1,$$

we can take $P \in \text{Supp}(x) \setminus \text{Supp}(A) = \emptyset$. \square

To complete this section, [2, Lemma 4.1] provides the explicit form of the non-special divisors of degree g and $g-1$ in the case of a curve $\mathbf{F}/\mathbb{F}_{q^2}$ defined by

$$(17) \quad y^{q+1} = x^2 + x$$

Lemma 4.8. *Let q be odd, and $\mathbf{F}/\mathbb{F}_{q^2}$ be the function field defined by (17). Let $P \in \{P_{ab} \in \mathbf{P}(\mathbf{F}/\mathbb{F}_{q^2}) \mid 2a+1 \neq 0\}$ be a rational place of \mathbf{F} . Then, gP is a non-special divisor of degree g . In particular, $gP - P'$ is a non-special divisor of degree $g-1$ for all rational places $P' \in \mathbf{P}(\mathbf{F}/\mathbb{F}_{q^2})$ distinct from P .*

Proof. The hyperelliptic involution of $y^{q+1} = x^2 + x$ is given by $\varphi(x, y) = (-x-1, y)$, so the fixed points of φ are those $(a, b) \in \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ such that $2a+1=0$. Let $P \in \{P_{ab} \in \mathbf{P}(\mathbf{F}/\mathbb{F}_{q^2}) \mid 2a+1 \neq 0\}$. We know that if P is a rational place of an hyperelliptic function field which is not fixed by the hyperelliptic involution, then $H(P) = \{0, g+1, g+2, \dots\}$ (see [23, Satz 8]). Therefore $\mathcal{L}(gP) = \{0\}$ and gP is a non-special divisor of degree g . Finally, given a rational place $P' \in \mathbf{P}(\mathbf{F}/\mathbb{F}_{q^2}) \setminus \{P\}$, we have that $gP - P'$ is also a non-special divisor by [3, Lemma 3]. \square

4.2. Example on an asymptotically good tower. A tower $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ of algebraic function fields over a finite field \mathbb{F}_q is said to be asymptotically good if

$$\lim_{m \rightarrow \infty} \frac{B_1(F_m/\mathbb{F}_q)}{g(F_m/\mathbb{F}_q)} > 0.$$

We will consider the following tower $\mathcal{F} = (F_m)_{m \geq 1}$ of function fields \mathbb{F}_m/K when $K = \mathbb{F}_{q^2}$:

$$F_m = K(x_1, \dots, x_m) \text{ with } x_{i+1}^q + x_{i+1} = \frac{x_i^q}{x_i^{q-1} + 1} \text{ for } i = 1, \dots, m-1.$$

This tower was studied in [6]:

Proposition 4.9. i) For all $m \geq 2$, the extension F_m/F_{m-1} is Galois extension of degree q .

ii) The pole of x_1 in F_1 is totally ramified in F_m/T_1 , i.e.,

$$(x_1)_{\infty}^{F_m} = q^{m-1} \cdot P_{\infty}^{(m)}$$

with a place $P_{\infty}^{(m)} \in \mathbf{P}_1(F_m)$ of degree one.

iii) The genus $g(F_m)$ is

$$g(F_m) = \begin{cases} (q^{m/2} - 1)^2 & \text{if } m \equiv 0 \pmod{2} \\ (q^{(m+1)/2} - 1)(q^{(m-1)/2} - 1) & \text{if } m \equiv 1 \pmod{2} \end{cases}$$

Proof. (i), (ii) See [6, Lemma 3.3]. (iii) See [6, Remark 3.8] □

Definition 4.10. For $m \geq 1$, let

$$c_m = \begin{cases} q^m - q^{m/2} & \text{if } m \equiv 0 \pmod{2} \\ q^m - q^{(m+1)/2} & \text{if } m \equiv 1 \pmod{2} \end{cases}$$

Definition 4.11. For $1 \leq j \leq m$ we define

$$\pi_j = \prod_{i=1}^j (x_i^{q-1} + 1)$$

and

$$\mathbf{L}_j^{(m)} = \{P \in \mathbf{P}(F_m) \mid P \text{ is a zero of } x_i^{q-1} + 1, \text{ for some } i \in \{1, \dots, j\}\}.$$

Lemma 4.12. i) Let $1 \leq j \leq m$. Then the principle divisor of π_j is given by

$$(\pi_j)^{F_m} = C_j^{(m)} - (q^m - q^{m-j})P_{\infty}^{(m)},$$

where $C_j^{(m)} \geq 0$ is a divisor of F_m with

$$\text{supp}(C_j^{(m)}) = \mathbf{L}_j^{(m)}.$$

ii) Let $1 \leq j \leq m-1$ and $0 \leq e \leq q-1$. Then the principal divisor of $\pi_j x_{j+1}^e$ in F_m is given by

$$(\pi_j x_{j+1}^e)^{F_m} = D_{j,e}^{(m)} - (q^m - q^{m-j} + eq^{m-j-1})P_{\infty}^{(m)},$$

where $D_{j,e}^{(m)} \geq 0$ is a divisor of F_m with

$$\mathbf{L}_j^{(m)} \subseteq \text{supp}(D_{j,e}^{(m)}).$$

Proof. See [19, Lemma 3.4] □

Definition 4.13. For $1 \leq j \leq m$, let

$$A_j^{(m)} = \sum_{P \in \mathcal{L}_j^{(m)}} P.$$

Remark 4.14. From Lemma 4.12 we have: $\pi_j \in \mathcal{L}((q^m - q^{m-j})P_\infty^{(m)} - A_j^{(m)})$.

Proposition 4.15. For $1 \leq j \leq m$,

$$\mathcal{L}((q^m - q^{m-j})P_\infty^{(m)} - A_j^{(m)}) = \langle \pi_j \rangle;$$

i.e., the space $\mathcal{L}((q^m - q^{m-j})P_\infty^{(m)} - A_j^{(m)})$ is one-dimensional.

Proof. See [19, Proposition 3.6] □

Lemma 4.16. [19, Lemma 3.7] Let $1 \leq j \leq m/2$. Then

$$\deg(A_j^{(m)}) = q^j - 1$$

Proof. Let $\mathcal{A}_i^{(m)} = \{P \in \mathcal{P}(F_m) \mid P \text{ is a zero of } x_i^{q-1} + 1\}$. It follows from [6, Lemma 3.6] that for $1 \leq i \leq m/2$,

$$\deg \left(\sum_{P \in \mathcal{A}_i^{(m)}} P \right) = (q-1)q^{i-1}.$$

Since

$$A_j^{(m)} = \sum_{i=1}^j \sum_{P \in \mathcal{A}_i^{(m)}} P,$$

we obtain

$$\deg(A_j^{(m)}) = \sum_{i=1}^j (q-1)q^{i-1} = q^j - 1. \quad \square$$

Definition 4.17. We define a divisor $A^{(m)}$ of F_m as follows: $A^{(1)} = 0$ and, for $m \geq 2$,

$$A^{(m)} = A_j^{(m)} \quad \text{with } j = \begin{cases} \frac{m}{2} & \text{if } m \equiv 0 \pmod{2} \\ \frac{m-1}{2} & \text{if } m \equiv 1 \pmod{2} \end{cases}$$

Lemma 4.18. [19, Lemma 3.9]

- i) $\deg(A^{(m)}) = c_m - g(F_m)$.
- ii) $\dim(c_m P_\infty^{(m)} - A^{(m)}) = 1$.

i.e., $c_m P_\infty^{(m)} - A^{(m)}$ is non-special of degree $g(F_m)$.

Proof. For $m = 1$, all assertions are obvious since $c_1 = g(F_1) = 0$ and $A^{(1)} = 0$. Let $m \geq 2$.

- If $m \equiv 0 \pmod{2}$. Then

$$c_m = q^m - q^{m/2} \quad \text{and} \quad g_m = (q^{m/2} - 1)^2.$$

Hence

$$c_m - g_m = q^{m/2} - 1 = \deg(A^{(m)}),$$

By Lemma 4.16. On the other hand, we have

$$\mathcal{L}(c_m P_\infty^{(m)} - A^{(m)}) = \mathcal{L}((q^m - q^{m/2})P_\infty^{(m)} - A_{m/2}^{(m)}) = \langle \pi_{m/2} \rangle,$$

By Proposition 4.15.

- If $m \equiv 1 \pmod{2}$. The proof is similar.

□

4.3. Example of construction with a sufficiently large number of points. This section presents the tools proposed by H. Randriam in [20] which allow for the construction of non-special divisors of degree $g - 1$.

Lemma 4.19. *Let \mathbf{C} be a curve of genus g over \mathbb{F}_q , and $\mathcal{S} \subset \mathbf{C}(\mathbb{F}_q)$.*

- (1) *Let A be an \mathbb{F}_q -rational divisor on \mathbf{C} such that*

$$i(A) = \dim(A) - (\deg(A) + 1 - g) \geq 1.$$

*Suppose that for all $P \in \mathcal{S}$ we have $\dim(A + P) > \dim(A)$.
Then*

$$(18) \quad \#\mathcal{S} \leq g - \dim(A).$$

(if $\deg(A) = -1$, then, we have also $\#\mathcal{S} \leq 1$).

- (2) *Let B a \mathbb{F}_q -rational divisor on \mathbf{C} such that $\dim(B) \geq 1$. Suppose that for all $P \in \mathcal{S}$ we have $\dim(B - P) > \dim(B) - 1$.
Then*

$$(19) \quad \#\mathcal{S} \leq \deg(B) + 1 - \dim(B).$$

(if $\deg(B) = 2g - 1$, we have also $\#\mathcal{S} \leq 1$).

Proof. See [20, Lemma 1].

□

The main result of this section is based on the following Lemma.
For all $q > 1$ and all integer $n \geq 2$ we define:

$$\begin{aligned} G_q(n) &= \sum_{k=1}^{n-2} \frac{(q^{n-k} - 1)(q^{n-k-1} - 1)}{(q^n - 1)(q^{n-1} - 1)} \\ &= \frac{1}{q^2 - 1} - \frac{1 - \frac{(q-1)^n}{q^n} - 1}{(q-1)(q^{n-1} - 1)} \end{aligned}$$

Lemma 4.20. *Let \mathbf{C} be a curve of genus g over \mathbb{F}_q , and $\mathcal{S} \subset \mathbf{C}(\mathbb{F}_q)$.*

- (1) *Let A be an \mathbb{F}_q -rational divisor on \mathbf{C} such that $\deg(A) \geq -2$ and*

$$i(A) = \dim(A) - (\deg(A) + 1 - g) \geq 2.$$

*Suppose that for all $P \in \mathcal{S}$ we have $\dim(A + 2P) > \dim(A)$.
Then*

$$(20) \quad \#\mathcal{S} \leq 3g + 3 + \deg(A) - 3\dim(A).$$

and

$$(21) \quad \#\mathcal{S} \leq \left(1 + \frac{q^{i(A)-2} - 1}{q^{i(A)} - 1}\right)^{-1} (6g - 6 - 2\deg(A) - 2G_q(i(A))) \cdot \#\mathbf{C}(\mathbb{F}_q).$$

more generally, for all integers w such that $2 \leq w \leq i(A)$,

$$(22) \quad \begin{aligned} \#\mathcal{S} \leq (i(A) - w) + \left(1 + \frac{q^{w-2} - 1}{q^w - 1}\right)^{-1} (6g - 6 - 2\deg(A) - 4(i(A) - w) \\ - 2G_q(w)) \cdot \#\mathbf{C}(\mathbb{F}_q). \end{aligned}$$

- (2) *Let B an \mathbb{F}_q -rational divisor on \mathbf{C} such as $\deg(B) \leq 2g$ and $\dim(B) \geq 2$. Suppose that for all $P \in \mathcal{S}$ we have $\dim(B - 2P) > \dim(B) - 2$. Then*

$$(23) \quad \#\mathcal{S} \leq 2\deg(B) + 2g + 4 - 3\dim(B).$$

and

$$(24) \quad \#\mathcal{S} \leq \left(1 + \frac{q^{\dim(B)-2} - 1}{q^{\dim(B)} - 1}\right)^{-1} (2\deg(B) + 2g - 2 - 2G_q(\dim(B))) \cdot \#\mathbf{C}(\mathbb{F}_q).$$

more generally, for all integers w such that $2 \leq w \leq \dim(B)$,

$$(25) \quad \#S \leq (l(B) - w) + \left(1 + \frac{q^{w-2} - 1}{q^w - 1}\right)^{-1} (2\deg(B) + 2g - 2 - 4(\dim(B) - w) - 2G_q(w)) \cdot \#\mathbf{C}(\mathbb{F}_q).$$

Proof. See [20, Lemma 2]. \square

For the following results, we introduce two functions $f_{1,\mathbf{C}}$ and $f_{2,\mathbf{C}}$ defined over \mathbb{Z} by:

$$f_{1,\mathbf{C}}(a) = \begin{cases} 1 & \text{if } a = -1 \\ g & \text{if } 0 \leq a \leq g - 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{2,\mathbf{C}}(a) = \begin{cases} g & \text{if } a = g - 2 \\ \min_{2 \leq w \leq g-1-a} \left[(g-1-a-w) + \left(1 + \frac{q^{w-2}-1}{q^w-1}\right)^{-1} (2g-2+2a+4w-2G_q(w)) \#\mathbf{C}(\mathbb{F}_q) \right] & \text{if } -2 \leq a \leq g-3 \\ 0 & \text{otherwise} \end{cases}$$

Definition 4.21. Let $\mathbf{C}(\mathbb{F}_q)$ be a curve of genus $g \geq 1$. A divisor D on \mathbf{C} is called ordinary if

$$\dim(D) = \max(0, \deg(D) + 1 - g).$$

otherwise D is called exceptional.

Lemma 4.22. Let A be a divisor on \mathbf{C} , $\mathcal{S} \subset \mathbf{C}(\mathbb{F}_q)$ a set of points and $s \in \{1, 2\}$. Assume that A is ordinary, and $A + sP$ is exceptional for all $P \in \mathcal{S}$. Then

$$(26) \quad \#S \leq f_{s,\mathbf{C}}(\deg(A)).$$

Proof. Let $a = \deg(A)$.

i) Assume that $s = 1$.

- If $a \leq -2$ or $a \geq 2g - 2$, then $A + P$ is ordinary, thus \mathcal{S} is empty, and we have $f_{1,\mathbf{C}}(a) = 0$.
- If $g - 1 \leq a \leq 2g - 3$, then ordinary A mean that $\dim(A) = a + 1 - g$, thus $A + P$ is ordinary by Riemann-Roch, and we have the same conclusion.
- If $-1 \leq a \leq g - 2$, then ordinary A mean that $\dim(A) = 0$, and exceptional $A + P$ mean that $\dim(A + P) \geq 1$. We conclude by Lemma 4.19-1.

ii) Assume that $s = 2$.

- If $a \leq -3$ or $a \geq 2g - 1$, then $A + 2P$ is ordinary, thus \mathcal{S} is empty.
- If $g - 1 \leq a \leq 2g - 2$, then ordinary A mean that $\dim(A) + a + 1 - g$, thus $A + 2P$ is ordinary by Riemann-Roch, and we have the same conclusion.
- If $a = g - 2$, then ordinary A mean that $\dim(A) = 0$, and exceptional $A + 2P$ mean that $\dim(A + 2P) \geq 1$. By Lemma 4.19-1, we have $\#\mathcal{S} \leq g = f_{2, \mathbf{C}}(a)$.
- If $-2 \leq a \leq g - 3$, then ordinary A mean that $\dim(A) = 0$, and exceptional $A + 2P$ mean that $\dim(A + 2P) \geq 1$. We conclude by Lemma 4.20-1.

□

Proposition 4.23. *Let $\mathbf{C}(\mathbb{F}_q)$ be a curve of genus g , $r \geq 1$ an integer, $s_1, \dots, s_r \in \{1, 2\}$, \mathbb{F}_q -rational divisors T_1, \dots, T_r on \mathbf{C} , and $d \in \mathbb{Z}$ an integer. The degree of T_i is denoted $t_i = \deg(T_i)$. Let D_0 be a \mathbb{F}_q -rational divisor on \mathbf{C} of degree $\deg(D_0) = d_0 \leq d$, such that $s_i D_0 - T_i$ is ordinary. Finelly, let $\mathcal{S} \subset \mathbf{C}(\mathbb{F}_q)$ be a set of points such that*

$$(27) \quad \#\mathcal{S} > \max_{d_0 \leq d' < d} \sum_{i=1}^r f_{s_i, \mathbf{C}}(s_i d' - t_i).$$

Then, there exists an \mathbb{F}_q -rational divisor D on \mathbf{C} of degree $\deg(D) = d$, such that $s_i D - T_i$ is ordinary. Furthermore, D can be chosen such that $D - D_0$ is effective and supported by points in \mathcal{S} .

Proof. We will built D incrementally. Let d' such that $d_0 \leq d' < d$. Assume that a divisor D' of degree d' is already built, with $s_i D' - T_i$ ordinary and $D' - D_0$ effective supported in \mathcal{S} . Lemma 4.22 applied with $s = s_i$ and $A = s_i D' - T_i$ shows that there are no more than $f_{s_i, \mathbf{C}}(s_i d' - t_i)$ points such that $s_i(D' + P) - T_i$ is exceptional. Applying (27), we can find $P \in \mathcal{S}$ such that $s_i(D' + P) - T_i$ is ordinary and $(D' + P) - D_0$ effective (by construction) supported in \mathcal{S} . We conclude by induction on d' . □

Lemma 4.24. *With the previous notations, $f_{s, \mathbf{C}}(a)$ is an increasing function when $a \leq g - 1 - s$, their maximum value on \mathbb{Z} is $f_{s, \mathbf{C}}(g - 1 - s) = s^2 g$.*

Proof. See [20, Lemma 8]. □

Proposition 4.25. *Let $\mathbf{C}(\mathbb{F}_q)$ be a curve of genus g , $r \geq 1$ an integer, $s_1, \dots, s_r \in \{1, 2\}$, \mathbb{F}_q -rational divisors T_1, \dots, T_r on \mathbf{C} . Assume that*

$$\#\mathbf{C}(\mathbb{F}_q) > \sum_{i=1}^r (s_i)^2 g.$$

Then, for all integer d , there exists an \mathbb{F}_q -rational divisor D on \mathbf{C} of degree $\deg(D) = d$ and supported in $\mathbf{C}(\mathbb{F}_q)$, such that $s_i D - T_i$ is ordinary.

Proof. We apply Proposition 4.23 with $\mathcal{S} = \mathbf{C}(\mathbb{F}_q)$, and Lemma 4.24. \square

Corollary 4.26. *Let $\mathbf{C}(\mathbb{F}_q)$ be a curve of genus g , Q and G two \mathbb{F}_q -rational divisors. We denote by $k = \deg(Q)$ and $n = \deg(G)$ their degrees, assume that*

$$\#\mathbf{C}(\mathbb{F}_q) > 5g$$

and

$$n \geq 2k + g - 1.$$

Then, there exists an \mathbb{F}_q -rational divisor D on \mathbf{C} supported in $\mathbf{C}(\mathbb{F}_q)$, such that $D - Q$ is non-special of degree $g - 1$ and $\dim(2D - G) = 0$. In particular, if $n = 2k + g - 1$, then $D - Q$ and $2D - G$ are non-special of degree $g - 1$.

Proof. We apply Proposition 4.25 with $r = 2$, $s_1 = 1$, $T_1 = Q$, $s_2 = 2$, $T_2 = G$ and $d = k + g - 1$. \square

Remark 4.27. *Notice that the divisor D in the previous corollary can be built as seen in the proof of Proposition 4.23. Below is the summary of the steps with the conditions of Corollary 4.26:*

- (1) *Let Q and G be two \mathbb{F}_q -rational divisors with $\deg(Q) = k$ and $\deg(G) = 2k + g - 1$.*
- (2) *Let D_0 be a divisor such that $\deg(D_0) = d_0 \leq k + g - 1$ and $D_0 - Q$, $2D_0 - G$ are ordinary.*
- (3) *Build a divisor D' of degree d' , $d_0 \leq d' < k + g - 1$ such that $D' - Q$, $2D' - G$ are ordinary and $D' - D_0$ effective.*
- (4) *We can find $P \in \mathbf{C}(\mathbb{F}_q)$ such that $D' + P - Q$, $2(D' + P) - G$ are ordinary.*
- (5) *We reapply step 4 until we obtain the the desired divisor D with the desired degree $k + g - 1$.*

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