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1 **MULTI-MODAL USER EQUILIBRIUM AND SYSTEM OPTIMUM IN URBAN**
2 **TRANSPORT SYSTEMS INCLUDING SHARED MOBILITY SERVICES**

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28 **CONFLICT OF INTEREST**

29 The authors declare no potential conflict of interests.

30 **AUTHOR CONTRIBUTION STATEMENT**

31 All the authors have contributed to all aspects of this study, ranging from the conception and
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33 manuscript preparation.

34

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36

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1 ABSTRACT

2 Shared Mobility Services (SMSs) have reshaped urban transportation systems, providing effective
3 and flexible mobility options. As these services can reduce traffic congestion and emissions by
4 reducing the number of cars, they are likely to become an essential part of the urban transportation
5 landscape, which emphasizes the importance of analyzing and understanding their impacts on
6 the system and users' travel choices, especially when integrated with public transport (PT). Most
7 existing works do not consider interactions between SMSs and PT, leading to inaccurate PT and
8 SMS estimates, as in most cases, PT does not cover the first and last mile of journeys. In this study,
9 we first provide a comprehensive literature review on modeling SMSs in multi-modal systems.
10 Then, we present an analytical traffic assignment model formulated as a Mixed-Integer Quadratic
11 Problem, which can be solved with existing solvers. In particular, the proposed model can represent
12 various travel options and any logical combination. We adapt the Beckmann formulation with a
13 link-wise objective function, including time and monetary costs. We perform an in-depth analysis
14 of commuter behavior on two test cases and investigate the difference between user equilibrium
15 and system optimum in such a complex system.

16

17 *Keywords:* Shared Mobility Services, Multi-Modal Transportation Network, Intermodality, User
18 Equilibrium, System Optimum.

1 INTRODUCTION

2 With the rapid growth of demand for urban transportation systems, addressing the population's
3 travel needs has become challenging. Because expanding the physical transportation network is
4 not always feasible and beneficial, new mobility services have emerged, transforming and revolu-
5 tionizing how people move within cities (1). In particular, shared mobility services (SMSs) have
6 gained popularity due to their convenience and cost-effectiveness. By promoting shared mobil-
7 ity and reducing the number of cars in the network, these services have the potential to reduce
8 traffic congestion, minimize emissions, and alleviate parking space demands. Furthermore, SMSs
9 can radically improve mobility, especially in low-demand and low-density regions where public
10 transport (PT) is unable to provide first and last-mile transportation (e.g., from origin to the trans-
11 portation hub or from the hub to destination) (2).

12 Shared mobility describes various mobility options. For instance, carpooling is an SMS
13 where commuters with overlapping itineraries and similar schedules share a privately owned car
14 for the trip. In this context, taxi services can be redefined as an on-demand service, called e-hailing,
15 where one commuter requests a driver to travel from his origin to his destination without sharing
16 the car with other passengers. When the sharing is allowed, the service is called ridesharing (3).

17 Integrating SMSs with PT has shifted traveling behavior and created a more complex multi-
18 modal transportation system (4). On the one hand, commuters face mode and route choices, aiming
19 to reduce travel expenses, while on the other hand, the system owners or traffic engineers strive to
20 optimize mobility services and meet demand with minimized overall costs. This needs a complete
21 understanding of the various transportation services accessible and their associated costs.

22 Numerous studies have focused on modeling multi-modal transportation systems. Re-
23 cently, many works have incorporated SMSs into their models. However, a less explored aspect
24 concerns modeling and evaluating combinations of multiple mobility options to accomplish one
25 trip (intermodality). Particularly, using SMSs for first and last mile transportation is often not ex-
26 plored. This lack of integration results in inaccurate traffic predictions and planning. Consequently,
27 individuals may not be able to fully optimize their travel choices, leading to potential inefficiencies
28 and higher expenses (5).

29 Addressing this issue requires the development of comprehensive transportation models
30 that consider all travel options available to commuters. By integrating SMSs and PT into a unified
31 framework and providing accurate information on costs and travel times, commuters can make
32 more optimal decisions, optimizing individual and system travel expenses.

33 We first conduct a literature review on multi-modal traffic models in this paper. Second, we
34 formulate an analytical model for multi-modal urban transportation networks, considering SMSs
35 and intermodality. For model tractability, we consider a time-independent setting, leading to a
36 non-convex Mixed-Integer Quadratic Program (MIQP) formulation. In this model, commuters can
37 travel with a single mode or use any combination of the travel options. We formulate commuters'
38 travel decisions using two traffic assignment principles: User equilibrium (UE) and system op-
39 timum (SO), to represent the two baseline scenarios for the network. In UE, known as the first
40 Wardrop principle (6), every commuter tries to minimize his own cost, and equilibrium is reached
41 when no one is willing to change his choices to achieve lower costs. The second principle of
42 Wardrop is system-oriented and assumes that the system's total cost is minimized (6). We ad-
43 dress the equilibrium problem with the Beckmann transformation and prove that the solution is
44 equivalent to the UE. Therefore, formulating other types of equilibria, such as stochastic user equi-
45 librium (SUE), in which commuters are assumed to minimize their perceived travel costs, will be

1 straightforward (*1*).

2 The proposed model enables the investigation of different planning scenarios by providing
3 network traffic distribution. In this study, we perform a comparative analysis of the two equilibria
4 to investigate the impact of intermodality on the transportation system performance. In particular,
5 we compare the system, including PT and SMSs, competing with the cooperative scenario wherein
6 intermodality is allowed.

7 **LITERATURE REVIEW**

8 Most of the studies in the literature addressed the modeling of SMSs, including background traffic,
9 but they mostly did not consider PT (*5*). (*7*) proposed a UE framework to analyze congestion
10 effects arising from carpooling and e-hailing services. Similarly, (*8*) analyzed the efficiency of
11 e-hailing and ridesharing services in reducing traffic congestion. They showed that fleet size is an
12 important factor for ridesharing efficiency due to the congestion effects of empty fleet vehicles. (*9*)
13 studied the carpooling demand, with the same origin and destination for riders and drivers, under
14 the SUE principle. They showed that carpooling is preferred for long trips.

15 Some recent works have included both SMSs and PT in their modeling. For example, (*10*)
16 analyzed the e-hailing and ridesharing pricing strategies in a multi-modal network, considering
17 PT. (*11*) proposed a UE framework, under time uncertainties, to analyze the commuters' mode and
18 departure time choices. (*12*) studied the impacts of e-hailing service on the PT demand and usage
19 of personal cars for the city of Amsterdam. They showed that e-hailing could potentially replace
20 private cars and attract PT demand with a larger fleet. (*13*) modeled the carpooling service in a
21 multi-modal single-region network and proposed two pricing schemes to alleviate traffic conges-
22 tion. (*14*) investigated the use of underutilized bus lanes for ridesharing service and its impacts on
23 the solo-pooled rides split. (*15*) investigated the integration of shared autonomous vehicles into
24 the multi-modal network, considering PT and parking space constraints. All these studies have
25 not considered interactions between SMSs and PT, referred to as intermodality in this paper. This
26 means that, with these models, commuters have single-mode trips and cannot switch modes be-
27 tween their origin and destination. This results in the use of PT and SMSs being under-estimated
28 or over-estimated, as in most cases, PT does not cover the first and last mile of the journey.

29 Multiple studies have considered intermodality exclusively in the form of Park-and-Ride
30 mode ((*16*), (*17*), (*18*) and (*19*)). The use of bikes to complete first-mile and last-mile has also
31 been intensively investigated (See e.g., (*20*) and (*21*)). However, only a few recent works have
32 targeted intermodality with SMSs and PT. (*22*) analyzed the competitiveness and complementarity
33 of e-hailing with PT. They showed that fleet size and trip fares impact using e-hailing services as
34 a first-mile/last-mile option. Similarly, (*5*) investigated the impacts of e-hailing and ridesharing
35 services on PT ridership and showed that ridesharing can be a competitor of PT when low fares
36 are applied. In (*1*), the authors proposed an SUE multi-modal framework where commuters access
37 metro stations via bus, car, or e-hailing. The mode and path choices are handled by discrete
38 choice models. In (*4*), commuters can choose inter-modes with solo driving (park-and-ride mode),
39 carpooling, and e-hailing. Travel behavior is modeled via a multi-layer nested logit model. To our
40 knowledge, these two last-mentioned works present the most comprehensive and highly relevant
41 models related to our research. However, the ridesharing service, biking and walking are not
42 modeled. Additionally, to the best of our knowledge, no work has compared and investigated the
43 difference between UE and SO in such a multi-modal complex system.

44 One of the goals of this study is to fill this research gap by proposing a comprehensive

1 analytical model for multi-modal transportation systems, integrating SMSs with PT. We propose a
 2 combined framework for carpooling, e-hailing and ridesharing, as well as intermodality. We aim
 3 to address UE and SO principles in a tractable (analytical) framework. Insights are provided to
 4 understand better the impacts of SMSs on PT and commuters' behavior when these latter act for
 5 their benefit or the good of the system.

6 To help position our work and its contribution, we present a characterization of the existing
 7 models in the literature in Table 1. The traffic assignment column categorizes the works based
 8 on equilibrium principles. Generalized cost presents the terms included in the travel costs: travel
 9 time states for in-vehicle time, waiting time for the time spent before the vehicle arrives at the
 10 passenger's position, the service time is the time to be matched, to get in and out of the vehicle and
 11 the time spent at PT stops, the monetary cost is the cost of fuel and trip fares. Finally, the travel
 12 options column denotes the categories of travel modes and their travel options considered in each
 13 model. There are four categories: Personal means, Public transport, Shared mobility services, and
 14 Intermodality which includes park-and-ride, walking and/or biking combined with PT (W/B), and
 15 using SMSs with PT.

16 We characterized 22 studies in the literature and compared them with our study (the last
 17 row in Table 1) to highlight the research gap we are addressing. Hence, the contributions of this
 18 paper can be summarized as follows:

- 19 • It establishes a comprehensive Mixed-Integer Quadratic Program for multi-modal trans-
 20 portation networks considering personal transportation means (car, walking, and bik-
 21 ing), public transport (bus and metro), as well as shared mobility services (carpooling,
 22 ridesharing, and e-hailing)
- 23 • It explicitly formulates interactions between shared mobility services and public transport
 24 regarding congestion effects and intermodality. It also considers park-and-ride mode and
 25 the use of walking and biking for first-mile and last-mile transportation.
- 26 • It solves the UE and SO problems when the model simultaneously handles mode choice
 27 and path choice, and passenger-driver matching.
- 28 • The proposed model is used to analyze the behavior of commuters within a transportation
 29 network encompassing multiple travel modes.

30 **MODEL FORMULATION**

31 This section presents the analytical model for multi-modal urban transportation networks consid-
 32 ering SMSs integrated with PT. To clearly present the model, we first present a simplified version
 33 without intermodality. Then, we reformulate the model to consider intermodality. The list of
 34 notations used in this paper is presented in Table 2.

35 **Multi-Modal Transportation Network without Intermodality**

36 Let us consider an urban transportation network represented as a directed graph $G(E, A)$, wherein
 37 links (A) represent physical routes, and nodes (E) can represent intersections or zones, depending
 38 on the level of network aggregation. Let $q^{(i,j)}$ be the travel demand between Origin-Destination
 39 (OD) pair (i, j) , $\forall i, j \in E$. This study assumes the demand for each OD pair is given. Commuters
 40 traveling between (i, j) must simultaneously choose the mode and path. For travel modes, as il-
 41 lustrated in Figure 1, commuters can take their car and be solo drivers, be a bus or metro (M)
 42 passenger, carpooling driver (CD), or carpooling passenger (CP) as long as the matching happens,
 43 e-hailing passenger (EH) or ridesharing passenger (RS) and thus, be matched with a service vehi-

TABLE 1: Comparison of the different studies on modeling of Shared Mobility Services

Research	Traffic Assignment			Generalized Cost					Travel Options								
	User Equilibrium	System Optimum		Travel Time	Waiting Time	Service Time	Monetary Cost	Personal Means			PT		SMSs		Intermodality		
		Stochastic UE	Optimum					Car	Walking	Biking	Bus	Metro	Carpooling	E-hailing		Ride Sharing	Park & Ride
Di, Xuan and Ban, 2019 (7)	x			x	x		x						x				
Beojone et al., 2021 (8)				x	x								x				
Yan et al., 2019 (9)				x	x		x						x*				
Zhong et al., 2020 (23)	x		x	x	x								x				
Wang et al., 2021 (24)	x			x		x							x				
Nourinejad and Ramezani, 2020 (25)				x	x												
Zhang and Nie, 2021 (10)				x	x												
Liang et al., 2023 (11)	x			x	x												
Narayan et al., 2019 (12)	x			x	x												
Wei et al., 2020 (13)				x	x												
Fayed et al., 2023 (14)				x	x												
Tang et al., 2021 (15)				x	x												
Ye et al., 2021 (16)		x		x	x												
Zheng et al., 2020 (17)				x	x												
Liu and Geroliminis, 2017 (18)				x	x												
Fan et al., 2014 (19)				x	x												
Wang et al., 2022 (20)				x	x												
Geurs et al., 2016 (21)				x	x												
Du et al., 2022 (1)				x	x												
Pi et al., 2019 (4)	x			x	x												
Ke et al., 2021 (22)	x			x	x												
Zhu et al., 2020 (5)	x			x	x												
This study	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

*: The sharing is for the same Origin-Destination (OD) pair

TABLE 2: List of notations.

Network Structure	
E	Set of nodes.
O	Set of origin nodes.
D	Set of destination nodes.
TR	Set of transfer nodes.
N	Set of networks ; $N = \{Road(RN), Metro(MN), Bike(BN), Walk(WN)\}$.
A_n	Set of links in network n .
A	Set of links: $A = \cup_{n \in N} A_n$.
Ψ	Set of all travel modes ; $\Psi = \{ car, bus, M, CP, CD, EH, RS, W, B, I_{m1,m2} \}$ where $I_{m1,m2}$ is a combination of mode $m1$ and $m2$.
V	Set of fleet vehicles for ridesharing and e-hailing services.
P_{ij}^m	Set of paths for origin-destination (OD) pair (i, j) with mode m .
o_a	Binary coefficient equals to 1 if link a is an origin link ; 0 Otherwise.
d_a	Binary coefficient equals to 1 if link a is a destination link ; 0 Otherwise.
$\delta_{a,p,m}^{(i,j)}$	Binary coefficient equals to 1 if path p of mode m traverse link a between OD pair (i, j) ; 0 Otherwise.
$q^{(i,j)}$	Travel demand for OD pair (i, j) .
Indices	
i, j, r, s	Index of node, $i, j, r, s \in E$.
a	Index of link (edge), $a \in A$.
m	Index of mode, $m \in \Psi$.
p	Index of path, $p \in P_{ij}^m$.
n	Index of network, $n \in N$.
Input Parameters - [Value for the experiments]	
L_a	Length of link a (Km).
Sp_n	Mean speed in network n (Km/h) - [M: 60 , B: 10 , W: 3].
$freq_{m,a}$	Frequency of travel mode m on link a (units/h) - [Bus: 3 , M: 6].
R_m	Meeting rate of travel mode m (service/h) - [CP: {Toy network: 100 , Sioux Falls: 500}, RS: {Toy network: 200 , Sioux Falls: 1000}, EH: {Toy network: 100 , Sioux Falls: 1000}].
S_m	Average service time for mode m (h) - [Bus: 0.04 , M: 0.02 , CP: 0.04 , RS: 0.05, EH: 0.03].
P_m	Average parking time for mode m (h) - [car: 0.17, B: 0.08].
$TF_{m,a}$	Trip fare of travel mode m on link a (\$) - [Bus: 0.3 , M: 0.3 , CP: 0.7 , CD: 0.7 , RS: 0.9 , EH: 1.1].
PF_m	Parking fare for travel mode m (\$) - [car: 1].
CAP_m	Maximum passenger capacity for the travel mode m - [CP: 1 , RS: 2].
Pk_cap_s	Parking capacity for transfer node s - [Toy network: 100 , Sioux Falls: 800].
$ V $	Fleet size for the ridesharing and e-hailing services - [Toy network: 1000 , Sioux Falls: 3500].
α	Monetary cost per unit of time (\$/h) - [Toy network: 5 , Sioux Falls: 20].
β	Monetary cost per unit of distance (\$/Km) - [0.25].
Variables	
$t_{a,n}$	Travel time on link a belonging to network n .
$WT_{m,a}$	Waiting time for travel mode m on link a .
$ST_{m,a}$	Service time for travel mode m on link a .
$C_{m,a}$	Monetary cost of using travel mode m on link a .
x_a	Aggregated traffic flow on link a .
$x_{a,m}$	Aggregated traffic flow on link a with travel mode m .
q_m	Travel demand for mode m .
$q_m^{(i,j)}$	Travel demand for mode m and OD pair (i, j) .
$q_t^{(i,j)}$	Number of empty or occupied service vehicles between OD pair (i, j) ; $t \in \{e, o\}$.
$q_o^{(i,j)r}$	Number of occupied service vehicles between OD pair (i, j) stopping at node r .
$c_{p,m}^{(i,j)}$	Generalized cost of path p with travel mode m between OD pair (i, j) .
$f_{p,m}^{(i,j)}$	Traffic flow of mode m on path p between OD pair (i, j) .
$f_{p,CD}^{(i,j)r,s}$	Traffic flow of carpooling drivers on path p between (i, j) stopping at nodes r and s .
$y_{p,t}^{(i,j)}$	Flow of empty or occupied service vehicles on path p between OD pair (i, j) ; $t \in \{e, EH, RS\}$.
$y_{p,RS}^{i,r,s,j}$	Flow of ridesharing occupied vehicles on path p stopping at nodes i, r, s and j .

1 cle. Additionally, they can go to their destination by walking (W) or biking (B). Accordingly, we
 2 define Ψ as the set of travel modes ($\Psi = \{car, bus, M, W, B, CD, CP, EH, RS\}$).

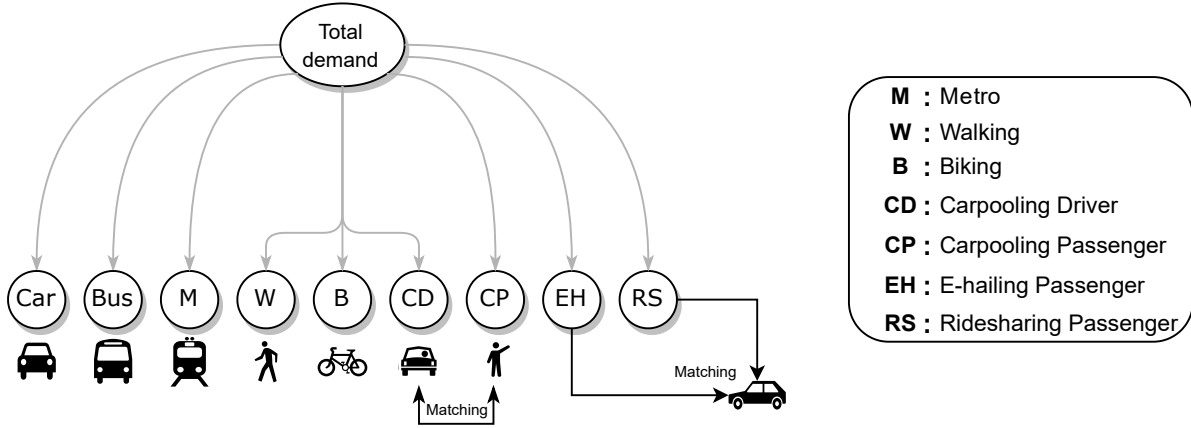


FIGURE 1: Available travel options for multi-modal networks, without intermodality.

3 To consider interactions between the modes in terms of congestion effects, we represent the
 4 physical network by four subnetworks: road (RN), walking (WN), biking (BN), and metro (MN)
 5 network. Buses and cars both use the road network and contribute to traffic congestion. N denotes
 6 the set of networks, $N = \{RN, WN, BN, MN\} \forall RN, WN, BN, MN \subset A$. The model is formulated
 7 for a given time period to represent the state of the network, following two well-known principles:
 8 UE and SO. Thus, the model is presented in a time-independent context, which keeps it tractable
 9 and analytically solvable.

10 *UE and SO Objective Function*

11 This subsection presents the objective functions of UE and SO and how each component of these
 12 functions is calculated. In particular, we present the cost functions for commuters at the link level
 13 for all the modes mentioned in Figure 1.

The generalized cost is the normalized sum of travel, waiting, service, and monetary costs
 (4). Every one of these terms is defined separately for each mode. This leads to a piece-wise
 objective function and, thus, a non-convex MIQP program. To assign traffic based on the UE prin-
 ciple and calculate the equilibrium through Beckmann transformation (26), we define the objective
 function as follows:

$$14 \quad \min Z_{UE} = \sum_{n \in N} \sum_{a \in A_n} \int_0^{x_a} \alpha \cdot t_{a,n}(\omega) d\omega + \sum_{a \in A} \sum_{m \in \Psi} [\alpha(WT_{m,a} + ST_{m,a}) + C_{m,a}] \cdot x_{a,m} \quad (1)$$

15 where, α denotes the value of time (27), $x_{a,m}$ is the flow of mode m on link a and x_a denotes the
 16 total flow of link a . $t_{a,n}$ denotes the travel time on the link a (i.e., the in-vehicle time), which is
 17 calculated by Equation (2) based on the network to which the link belongs. For the road network
 18 (RN), the link travel time increases with the flow. In this study, we use the well-known Bureau of
 19 Public Roads (BPR) function (26). For the other networks, the link travel time depends on the link
 20 length (L_a) and the average speed (Sp_n) of walking, biking, and metro, respectively.

$$t_{a,n}(x) = \begin{cases} BPR(x) & \text{if } n = RN \\ \frac{L_a}{Sp_n} & \text{if } n \in \{MN, WN, BN\} \end{cases} \quad (2)$$

1 Equation (3) defines $WT_{m,a}$ as the waiting time of mode m on link a . A commuter has
 2 a waiting time at the start of the trip, so it is only counted for links attached to an origin node
 3 (ensured by o_a). For PT, the waiting time is assumed to be half of the headway (inspired by (4)
 4 and (20)). For SMSs, based on (25), we assume that the waiting time depends on the number of
 5 passengers in the system (q_m) and the meeting rate (R_m). R_m expresses the rate of pickups in the
 6 network and can be estimated by the cobb-douglas function (13). Here, we set an average constant
 7 rate that should be estimated from real-world SMSs data.

$$WT_{m,a} = \begin{cases} 0 & \text{if } m \in \{car, CD, W, B\} \\ \frac{1}{2freq_{m,a}} \cdot o_a & \text{if } m \in \{Bus, M\} \\ \frac{q_m}{R_m} \cdot o_a & \text{if } m \in \{CP, RS, EH\} \end{cases} \quad (3)$$

$ST_{m,a}$ in Equation (4) is the service time of mode m on link a . For PT, it denotes the average time
 spent at each stop (S_{Bus} , S_M for bus and metro). For car or bike users, it represents the parking
 time (P_{car} and P_B) and is only counted for destination links (represented by d_a). $ST_{m,a}$ for SMSs
 represent the time to be matched with a driver/passenger and the boarding and drop-off times (S_m ;
 $m \in \{CP, RS, EH\}$) on origin and destination links. Additionally, a CD experiences parking time
 at his destination. Walking does not have a service time.

$$ST_{m,a} = \begin{cases} 0 & \text{if } m = W \\ S_m & \text{if } m \in \{Bus, M\} \\ S_m \cdot (o_a + d_a) & \text{if } m \in \{CP, RS, EH\} \\ S_m \cdot (o_a + d_a) + P_1 \cdot d_a & \text{if } m = CD \\ P_m \cdot d_a & \text{if } m \in \{car, B\} \end{cases} \quad (4)$$

8

9 Equation (5) denotes the monetary cost of mode m on link a . Based on (1), for solo drivers, it
 10 includes the cost of using the car (i.e., fuel cost) for the traveled distance and the parking fare
 11 (PF_{car}) at the destination. For all other passengers (PT and SMSs passengers), $C_{m,a}$ represents
 12 the trip fare to be paid ($TF_{m,a}$). For carpooling drivers, the trip fare received from passengers is
 13 retrieved from the cost of using their car. For walking and biking, we assume no monetary cost.

$$C_{m,a} = \begin{cases} \gamma \cdot L_a + PF_m \cdot d_a & \text{if } m = car \\ TF_{m,a} & \text{if } m \in \{Bus, M, CP, EH, RS\} \\ \gamma \cdot L_a + PF_m \cdot d_a - TF_{m,a} & \text{if } m = CD \\ 0 & \text{if } m \in \{W, B\} \end{cases} \quad (5)$$

14 Using these same definitions, we define the SO objective function as:

$$\min Z_{SO} = \sum_{n \in N} \sum_{a \in A_n} \alpha \cdot t_{a,n}(x_a) \cdot x_a + \sum_{a \in A} \sum_{m \in \Psi} [\alpha(WT_{m,a} + ST_{m,a}) + C_{m,a}] \cdot x_{a,m} \quad (6)$$

1 This objective function, Z_{SO} , minimizes the system's total cost. Before introducing inter-
2 modality, we present the model's constraints for a multi-modal network without intermodality.

3 *Model's Linear Constraints for Mode and Path Choice*

4 The objective functions previously described are formulated in terms of link flow and cost. How-
5 ever, commuters are confronted with mode and path choices. Thus, let $f_{p,m}^{(i,j)}$ be the path flow
6 variable. Equation (7) presents the link-path flow conversion, and Equation (8) represents the
7 demand conservation constraint.

$$x_{a,m} = \sum_{i,j \in E} \sum_{p \in P_{ij}^m} f_{p,m}^{(i,j)} \cdot \delta_{a,p,m} \quad \forall a \in A \quad ; \forall m \in \Psi \quad (7)$$

$$q^{(i,j)} = \sum_{m \in \Psi} \sum_{p \in P_{ij}^m} f_{p,m}^{(i,j)} \quad \forall i, j \in E \quad (8)$$

8

9 Here, we present the construction for every considered mobility option. Equations (9)-(11) are
10 for the carpooling service. Based on (1), we define $f_{p,CD}^{(i,j) r,s}$ as the number of carpooling drivers
11 traveling between (i, j) , with path p and stopping at nodes r and s to pick up or drop off their
12 passengers. Note that a CD can have more than one passenger onboard. Thus, $f_{p,CD}^{(i,j) i,j}$ expresses
13 the number of CD having passengers with the same OD pair (i, j) . Constraint (9) ensures that the
14 total number of CD between (i, j) is the sum of the drivers stopping at the different passengers'
15 OD pairs on path p . To formulate the correspondence between the drivers' and passengers' paths,
16 let us define the following sets:

17

- $Q_p^{(r,s)}$: Set of paths l between (r, s) , where the path p is a sub-path of l .

18

- $R_p^{(r,s)}$: Set of paths l (r, s) , where l is a sub-path of p .

Constraint (10) ensures that drivers can pick up all carpooling passengers. Recall that the
drivers can have the same OD pair as the passengers or not. Figure 2 illustrates the correspondence
between the carpooling passenger and driver paths, expressed by these constraints. Constraint (11)
ensures that all CD stopping at nodes r and s have enough passengers between (r, s) . We formu-
lated these constraints regarding path flow variables to ensure the passenger-driver conservation
over the path p .

$$f_{p,CD}^{(i,j)} = \sum_{r \in E} f_{p,CD}^{(i,j) r} \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{CD} \quad (9)$$

$$f_{p,CP}^{(i,j)} \leq CAP_{CP} \cdot \left[\sum_{r,s \in E} \sum_{l \in Q_p^{(r,s)}} f_{l,CD}^{(r,s) i,j} \right] \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{CP} \quad (10)$$

$$f_{l,CD}^{(i,j) r,s} \leq f_{p,CP}^{(r,s)} \quad \forall r, s, i, j \in E \quad ; \forall l \in P_{ij}^{CD} \quad ; \forall p \in R_l^{(r,s)} \quad (11)$$

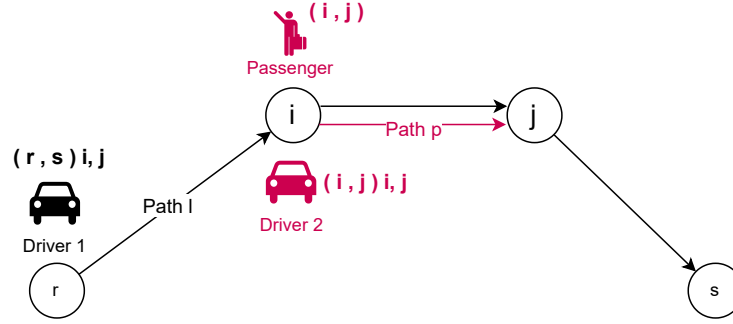


FIGURE 2: Correspondence between the carpooling passenger's and drivers' path.

- 1 For the e-hailing, $y_{p,EH}^{(i,j)}$ denotes the number of fleet vehicles having an e-hailing passenger
 2 between (i, j) with path p . Constraint (12) ensures that this variable equals the corresponding
 3 e-hailing path flow variable.

$$f_{p,EH}^{(i,j)} = y_{p,EH}^{(i,j)} \quad \forall i, j \in E \quad ; \quad \forall p \in P_{ij}^{EH} \quad (12)$$

- 4 Inspired by the formulation of carpooling in (1) and (4), we propose a ride-sharing service
 5 formulation. Let $y_{p,RS}^{i,r,s,j}$ represents the number of fleet vehicles participating in a ridesharing service
 6 and stopping at nodes i, j, r and s to pick up or drop off passengers. The ordering of the nodes in
 7 this variable is whether a First In First Out (FIFO) or Last In First Out (LIFO) service, as illustrated
 8 in Figure 3.

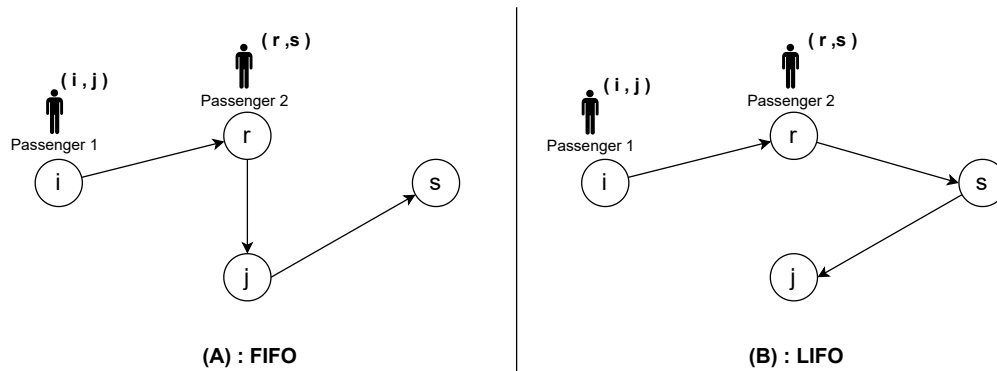


FIGURE 3: Order of ridesharing service. (A): The first picked-up passenger is the first dropped-off; (B): The first picked-up passenger is the last dropped-off.

- 9 Equation (13) defines the number of ridesharing vehicles traveling between (i, j) with path
 10 p . Similar to the carpooling service, Constraint (14) ensures enough seats for the passengers, and
 11 Constraint (15) ensures enough passengers for the ridesharing cars. CAP_{RS} denotes the ridesharing
 12 passenger capacity, which is the same for all RS vehicles.

$$y_{p,RS}^{(i,j)} = \sum_{r,s \in E} y_{p,RS}^{i,r,s,j} \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{RS} \quad (13)$$

$$f_{p,RS}^{(i,j)} \leq CAP_{RS} \cdot \left[\sum_{r,s \in E} \sum_{l \in Q_p^{(r,s)}} (y_{l,RS}^{r,i,j,s} + y_{l,RS}^{r,i,s,j}) \right] \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{RS} \quad (14)$$

$$y_{l,RS}^{i,r,s,j} + y_{l,RS}^{i,r,j,s} \leq f_{p,RS}^{(r,s)} \quad \forall r, s, i, j \in E \quad ; \forall l \in P_{ij}^{RS} \quad ; \forall p \in R_p^{(r,s)} \quad (15)$$

1 Constraint (16) ensures that the ridesharing passengers are sharing the trip with another
 2 passenger (not an e-hailing service). Note that $y_{p,RS}^{i,i,j,j}$ represents the ridesharing vehicles on path
 3 p having passengers with the same OD pair (i, j) . This is different from carpooling, as the driver
 4 providing the service is not a commuter himself.

$$f_{p,RS}^{(i,j)} \geq 2 \cdot y_{p,RS}^{i,i,j,j} + \sum_{r,s \in E} \sum_{l \in Q_p^{(r,s)}} (y_{l,RS}^{r,i,j,s} + y_{l,RS}^{r,i,s,j}) \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{RS} \quad (16)$$

5
 6 Constraints (17) and (18) define the number of occupied fleet cars as the sum of cars participating
 7 in the e-hailing and ridesharing services.

$$q_o^{(i,j)r} = \sum_{p \in P_{ij}^{RS}} y_{p,RS}^{i,r,s,j} + y_{p,RS}^{i,s,r,j} \quad \forall i, j, r \in E \quad (17)$$

$$q_o^{(i,j)} = \sum_{r \in E} q_o^{(i,j)r} + \sum_{p \in P_{ij}^{EH}} y_{p,EH}^{(i,j)} \quad \forall i, j \in E \quad (18)$$

8 Based on the formulation of the SMSs, we need to conserve the passengers and drivers flow
 9 over the paths. $y_{p,e}^{(i,j)}$ denotes the number of empty fleet vehicles traveling between (i, j) with path
 10 p . Constraint (20) ensures, for every destination node, the number of entering vehicles equals the
 11 number of exiting vehicles. Constraint (21) ensures, for every origin node, there are enough empty
 12 cars to pick up the e-hailing and ridesharing passengers. Constraint (22) ensures the fleet size ($|V|$)
 13 conservation. Constraints (23)-(25) are integrality conditions on the decision variables.

$$q_e^{(i,j)} = \sum_{p \in P_{ij}^{EH}} y_{p,e}^{(i,j)} \quad \forall i, j \in E \quad (19)$$

$$\sum_{i,j \in E} q_o^{(i,j)s} + \sum_{j \in E} q_e^{(j,s)} = \sum_{j \in E} q_o^{(s,j)} + \sum_{j \in E} q_e^{(s,j)} \quad \forall s \in D \quad (20)$$

$$\sum_{i \in E} q_e^{(i,j)} \geq \sum_{s \in E} q_o^{(j,s)} \quad \forall j \in O \quad (21)$$

$$\sum_{i,j \in E} q_o^{(i,j)} + q_e^{(i,j)} = |V| \quad (22)$$

$$f_{p,m}^{(i,j)}, f_{p,CD}^{(i,j)r,s} \geq 0 \quad \forall p \in P_{ij}^m \quad ; \forall i, j, r, s \in E \quad ; \forall m \in \Psi \quad (23)$$

$$y_{p,m}^{(i,j)} \geq 0 \quad \forall p \in P_{ij}^m \quad ; \forall i, j \in E \quad ; \forall m \in \{EH, RS, e\} \quad (24)$$

$$y_{p,RS}^{i,r,s,j} \geq 0 \quad \forall p \in P_{ij}^{RS} \quad ; \forall i, j, r, s \in E \quad (25)$$

1 Link Flow and Path Flow Correspondence

2 The UE and SO objective functions, formulated in section 4.1.1, require the calculation of x_a and
 3 $x_{a,m}$. Following Beckmann's formulation, $x_{a,m}$ is defined by Equation (7). However, to calculate
 4 the x_a variable used in links travel time, we consider the travel modes that contribute to traffic
 5 congestion, occupied and empty fleet vehicles, and PT flow. Considering PT flow means that we
 6 need to count PT units independently of the passenger flow. This is consistent with reality since PT
 7 adheres to a well-defined schedule and operates even with no passengers. It is worth mentioning
 8 that multiple studies (e.g., (7), (1) and (28)) formulate only non-empty PT for links, which is not
 9 the case in our study. To do so, we use the frequency of PT on every link to count the units, as
 10 described by Equation (26).

$$x_a = \sum_{m \in \Psi'} x_{a,m} + \sum_{l \in PT} freq_{l,a} + \sum_{i,j \in E} \sum_{p \in P_{ij}^{RS}} y_{p,RS}^{(i,j)} \delta_{a,p,RS}^{(i,j)} + \sum_{i,j \in E} \sum_{p \in P_{ij}^{EH}} (y_{p,EH}^{(i,j)} + y_{p,e}^{(i,j)}) \cdot \delta_{a,p,EH}^{(i,j)} \quad (26)$$

$$\forall a \in A \quad ; \Psi' = \{car, CD, W, B\} \quad ; PT = \{bus, M\}$$

11

12 Regarding the carpooling service, we only count the driver flow. For the ridesharing service, we are
 13 adding the number of cars instead of the passenger flow since a vehicle may carry multiple passen-
 14 gers. With this, we keep the link and path flows as the number of vehicles. At this point, all modes
 15 are defined, and their travel cost functions are formulated. Now we can introduce intermodality.

16 Multi-Modal Transportation Network with Intermodality

17 This section introduces the transformation of the model's constraints and variables to consider
 18 intermodality. In this study, an inter-mode is any combination of the previously enumerated modes
 19 (Figure 1) and is considered as a separate mode. For example, Park-and-Ride as a travel option is
 20 the commuter who uses his car to a transfer node and then takes the metro (3).

21 The proposed model can consider any logical combination of modes as a travel option
 22 without further modifications. However, we are mainly interested in inter-modes, including PT
 23 and SMSs. We will focus on intermodality, as represented in Figure 4. Thus, the set of modes Ψ is
 24 extended to include all modes in the form of $I_{m1,m2}$ where $m1$ and $m2 \in \Psi$.

25 As we are considering every inter-mode as a separate mode, commuters between (i, j) with
 26 an inter-mode $m = I_{m1,m2}$ on path p ($p = p1 \oplus p2$, where \oplus is the concatenation of two paths), use
 27 mode $m1$ with path $p1$, up to a transfer node. Then, they use $m2$ with path $p2$ to their destination.
 28 The number of commuters between (i, j) with an inter-mode m using path p is represented by the
 29 path flow variable $f_{p,m}^{(i,j)}$. Note that the model can consider a combination of more than two modes
 30 $I_{m1,m2}$, where $m1$ is a 'simple' mode and $m2$ is an inter-mode (and vice-versa).

31 Let $PI((i, j), s)$ be the set of all inter-modal paths between (i, j) in which the node s is
 32 the transfer node. Constraint (27) ensures the number of commuters using their car as part of an
 33 inter-mode and transferring at node s does not exceed the parking capacity at node s , represented
 34 by Pk_cap_s .

$$\sum_{i,j \in E} \sum_{p \in PI((i,j),s)} f_{p,I_{car,M}}^{(i,j)} + f_{p,I_{CD,M}}^{(i,j)} \leq Pk_cap_s \quad \forall s \in TR \quad (27)$$

35
 36 To clearly represent intermodality, let $I(m)$ denotes an inter-mode including the simple mode m

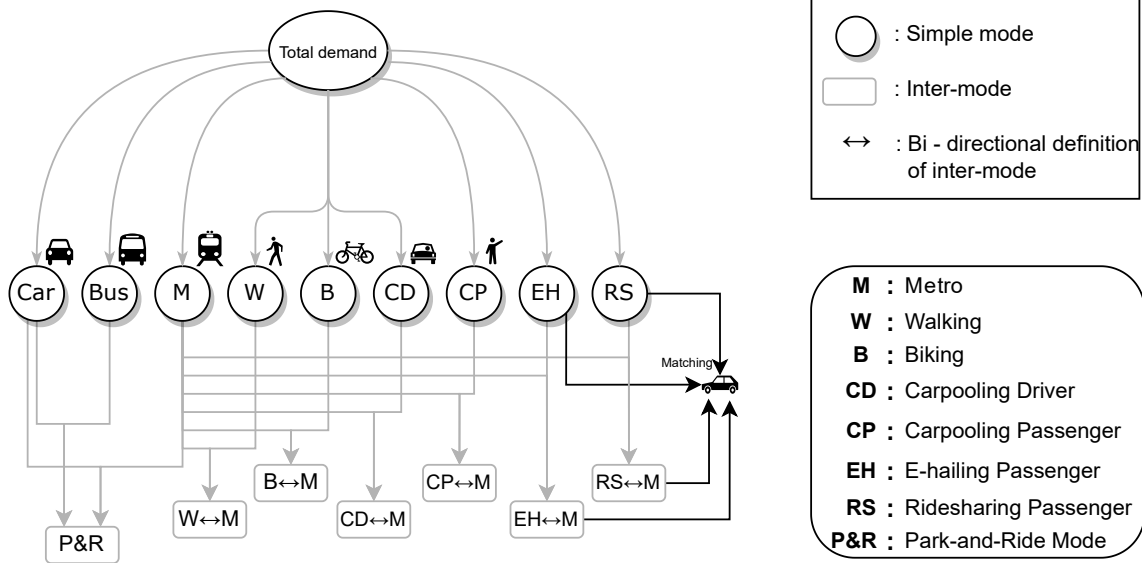


FIGURE 4: Available travel options for multi-modal networks, with intermodality.

1 (either as the first part or the second part of the inter-mode). $m(p)$ denotes the part of path p where
 2 the mode m is used, and $TR(p)$ denotes the transfer node of the inter-modal path p . To consider
 3 intermodality with carpooling, we add Constraints (28) and (29) to the model. Constraint (28)
 4 ensures that drivers can pick up the passengers participating in carpooling as part of an inter-mode.
 5 The right-hand side of the constraint considers drivers participating in carpooling as part of an
 6 inter-modal trip, as well as carpooling drivers between any OD pair (u, v) passing by the transfer
 7 node t of the passengers. This is illustrated by Figure 5. In other words, a passenger can be picked
 8 up by a driver going from his origin to his destination or by a driver going from his origin to the
 9 transfer node t .

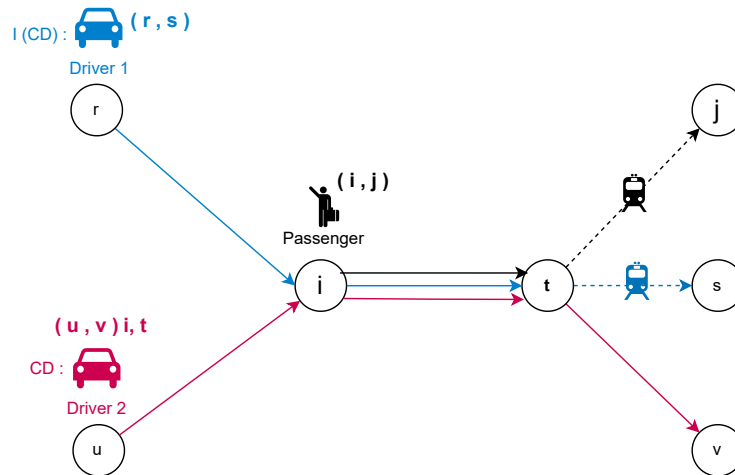


FIGURE 5: Correspondence between a carpooling passenger and drivers paths with intermodality.

$$f_{p,I(CP)}^{(i,j)} \leq CAP_{CP} \cdot \left(\sum_{I(CD) \in \Psi} \sum_{r,s \in E} \sum_{k \in \{x \in P_{rs}^{I(CD)}; CP(x)=CP(p)\}} f_{k,I(CD)}^{(r,s)} + \sum_{u,v \in E} \sum_{l \in P_{uv}^{CP}} f_{l,CD}^{(u,v),i,t} + f_{l,CD}^{(u,v),t,j} \right) \quad (28)$$

$$\forall i, j \in E \quad ; \forall p \in P_{ij}^{I(CP)} \quad ; t = TR(p)$$

1 Constraint (29) ensures all carpooling drivers between (i, j) have enough carpooling pas-
 2 sengers to pick. This means that no CD is driving without a passenger on board, either for the full
 3 trip (i.e., same OD pair as passenger), or a part of the trip.

$$f_{p,I(CD)}^{(i,j)} \leq \sum_{I(CP) \in \Psi} \sum_{r,s \in E} \sum_{p' \in \{x \in P_{rs}^{I(CP)}; CP(x)=CP(p)\}} f_{p',I(CP)}^{(r,s)} + \sum_{u,v \in E} \sum_{k \in \{x \in P_{uv}^{CP}; x=CP(p)\}} f_{k,CP}^{(u,v)} \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{I(CD)} \quad (29)$$

4 Thus, we need to reformulate Constraints (10) and (11) to include intermodality.

$$f_{p,CP}^{(i,j)} \leq CAP_{CP} \cdot \left(\sum_{r,s \in E} \sum_{k \in Q_p^{(r,s)}} f_{k,CD}^{(r,s),i,j} + \sum_{I(CD) \in \Psi} \sum_{u,v \in E} \sum_{k \in \{x \in P_{uv}^{I(CD)}; CP(x)=p\}} f_{k,I(CD)}^{(u,v)} \right) \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{CP} \quad (30)$$

$$f_{p,CD}^{(i,j),r,s} \leq f_{k,CP}^{(r,s)} + \sum_{I(CP) \in \Psi} \sum_{u,v \in E} \sum_{k \in \{x \in P_{uv}^{I(CP)}; CP(x)=p\}} f_{k,I(CP)}^{(u,v)} \quad \forall r, s, i, j \in E \quad ; \forall p \in P_{ij}^{CD} \quad ; \forall k \in R_p^{(r,s)} \quad (31)$$

5 Similarly, for e-hailing and ridesharing, Constraint (32) is added to the model, and Constraints (12)
 6 and (15) are updated accordingly.

$$f_{p,I(RS)}^{(i,j)} \leq CAP_{RS} \cdot \sum_{r,s \in E} \sum_{k \in Q_p^{(r,s)}} (y_{k,RS}^{r,i,TR(p),s} + y_{k,RS}^{r,i,s,TR(p)} + y_{k,RS}^{r,TR(p),j,s} + y_{k,RS}^{r,TR(p),s,j}) \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{I(RS)} \quad (32)$$

$$f_{p,EH}^{(i,j)} + \sum_{I(EH) \in \Psi} \sum_{u,v \in E} \sum_{k \in \{x \in P_{uv}^{I(EH)}; EH(x)=p\}} f_{k,I(EH)}^{(u,v)} = y_{p,EH}^{(i,j)} \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^{EH} \quad (33)$$

$$y_{p,RS}^{i,r,s,j} + y_{p,RS}^{i,r,j,s} \leq f_{k,RS}^{(r,s)} + \sum_{I(RS) \in \Psi} \sum_{u,v \in E} \sum_{k \in \{x \in P_{uv}^{I(RS)}; RS(x)=p\}} f_{k,I(RS)}^{(u,v)} \quad \forall r, s, i, j \in E \quad ; \forall p \in P_{ij}^{RS} \quad ; \forall k \in R_p^{(r,s)} \quad (34)$$

7 To have the link-path flow correspondence, we update the $x_{a,m}$ variable as described by Equation
 8 (35).

$$x_{a,m} = \sum_{i,j \in E} \sum_{p \in P_{ij}^m} f_{p,m}^{(i,j)} \cdot \delta_{a,p,m}^{(i,j)} + \sum_{I(m) \in \Psi} \sum_{u,v \in E} \sum_{k \in P_{uv}^{I(m)}} f_{k,I(m)}^{(u,v)} \cdot \delta_{a,m(k),m}^{(u,TR(k))} \quad \forall a \in A \quad ; \forall m \in \Psi \quad (35)$$

9
 10 Thus, the final comprehensive multi-modal mathematical formulation of UE (UE-CMF) is:

$$\min Z_{UE} = \sum_{n \in N} \sum_{a \in A_n} \int_0^{x_a} \alpha \cdot t_{a,n}(\omega) d\omega + \sum_{a \in A} \sum_{m \in \Psi} [\alpha(WT_{m,a} + ST_{m,a}) + C_{m,a}] \cdot x_{a,m} \quad (\text{UE-CMF})$$

s.t.

$$\left\{ \begin{array}{ll} \text{Cost functions} & (2) - (5) \\ x_{a,m} = \sum_{i,j \in E} \sum_{p \in P_{ij}^m} f_{p,m}^{(i,j)} \cdot \delta_{a,p,m} & \forall a \in A; \forall m \in \Psi \quad (7) \\ q^{(i,j)} = \sum_{m \in \Psi} \sum_{p \in P_{ij}^m} f_{p,m}^{(i,j)} & \forall i, j \in E \quad (8) \\ f_{p,CD}^{(i,j)} = \sum_{r \in E} f_{p,CD}^{(i,j)r} & \forall i, j \in E; \forall p \in P_{ij}^{CD} \quad (9) \\ y_{p,RS}^{(i,j)} = \sum_{r \in E} y_{p,RS}^{(i,j)r} & \forall i, j \in E; \forall p \in P_{ij}^{RS} \quad (13) \\ f_{p,RS}^{(i,j)} \leq CAP_{RS} \cdot [\sum_{r,s \in E} \sum_{l \in Q_p^{(r,s)}} (y_{l,RS}^{r,i,j,s} + y_{l,RS}^{r,i,s,j})] & \forall i, j \in E; \forall p \in P_{ij}^{RS} \quad (14) \\ y_{l,RS}^{i,r,s,j} + y_{l,RS}^{i,r,j,s} \leq f_{p,RS}^{(r,s)} & \forall r, s, i, j \in E; \forall l \in P_{ij}^{RS}; \forall p \in R_p^{(r,s)} \quad (15) \\ f_{p,RS}^{(i,j)} \geq 2 \cdot y_{p,RS}^{i,i,j,j} + \sum_{r,s \in E} \sum_{l \in Q_p^{(r,s)}} (y_{l,RS}^{r,i,j,s} + y_{l,RS}^{r,i,s,j}) & \forall i, j \in E; \forall p \in P_{ij}^{RS} \quad (16) \\ q_o^{(i,j)r} = \sum_{p \in P_{ij}^{RS}} y_{p,RS}^{i,r,s,j} + y_{p,RS}^{i,s,r,j} & \forall i, j, r \in E \quad (17) \\ q_o^{(i,j)} = \sum_{r \in E} q_o^{(i,j)r} + \sum_{p \in P_{ij}^{EH}} y_{p,EH}^{(i,j)} & \forall i, j \in E \quad (18) \\ q_e^{(i,j)} = \sum_{p \in P_{ij}^e} y_{p,e}^{(i,j)} & \forall i, j \in E \quad (19) \\ \sum_{i,j \in E} q_o^{(i,j)s} + \sum_{j \in E} q_e^{(j,s)} = \sum_{j \in E} q_o^{(s,j)} + \sum_{j \in E} q_e^{(s,j)} & \forall s \in D \quad (20) \\ \sum_{i \in E} q_e^{(i,j)} \geq \sum_{s \in E} q_o^{(j,s)} & \forall j \in O \quad (21) \\ \sum_{i,j \in E} q_o^{(i,j)} + q_e^{(i,j)} = |V| & (22) \\ f_{p,m}^{(i,j)}, f_{p,m}^{(i,j)r,s} \geq 0 & \forall p \in P_{ij}^m; \forall i, j, r, s \in E; \forall m \in \Psi \quad (23) \\ y_{p,m}^{(i,j)} \geq 0 & \forall p \in P_{ij}^m; \forall i, j \in E; \forall m \in \{EH, RS, e\} \quad (24) \\ y_{p,RS}^{i,r,s,j} \geq 0 & \forall p \in P_{ij}^{RS}; \forall i, j, r, s \in E \quad (25) \\ \text{Intermodality Constraints} & (26) - (35) \end{array} \right.$$

1 To address the SO principle, only the objective function in UE-CMF needs to be replaced by
2 Equation (6) and the problem can be treated as any minimization problem and solved by common
3 (open-source or commercial) solvers. Our model's formulation allowed the use of these solvers
4 for UE as well. In what follows, we provide the proof that the MIQP UE-CMF follows Beckmann
5 transformation, and thus, the solutions satisfy UE conditions.

6 **Proposition 1.** Solution of optimization problem UE-CMF satisfies user equilibrium conditions.

7 **Proof.** Following (26) and (24) procedures, let μ_{rs} ($r, s \in E$) denote the dual variable
8 associated with the flow conservation constraint in Equation (8). Thus, the Lagrangian can be
9 given by:

$$L(f, \mu) = Z + \sum_{r,s \in E} \mu_{r,s} (q^{(r,s)} - \sum_{m \in \Psi} \sum_{p \in P_{rs}^m} f_{p,m}^{(r,s)}) \quad (36)$$

10 We compute the derivatives of the Lagrangian, w.r.t the path flow and the dual variables, to
11 express the KKT conditions. To simplify the equations, we assume in what is next that $\alpha = 1$.

12 **Derivatives w.r.t path flow variable:**

13 1. If $m \notin \{CP, RS, EH\}$: the terms WT, ST, and C in the objective function are constants
14 w.r.t the path flow. Thus, we obtain the following derivatives:

$$\begin{aligned} \frac{\partial Z}{\partial f_{p,m}^{(i,j)}} &= \sum_{b \in A} \frac{\partial}{\partial x_b} \left\{ \sum_{n \in N} \sum_{a \in A_n} \int_0^{x_a} t_{a,n}(\omega) d\omega \right\} \times \frac{\partial x_b}{\partial f_{p,m}^{(i,j)}} \\ &+ \sum_{b \in A} \sum_{m' \in \Psi} \frac{\partial}{\partial x_{b,m'}} \left\{ \sum_{a \in A} \sum_{m \in \Psi} [WT_{m,a} + ST_{m,a} + C_{m,a}] \cdot x_{a,m} \right\} \times \frac{\partial x_{b,m'}}{\partial f_{p,m}^{(i,j)}} \end{aligned} \quad (37)$$

$$= \sum_{b \in A} t_{b,n}(x_b) \cdot \delta_{b,p,m}^{(i,j)} + \sum_{b \in A} [WT_{m,b} + ST_{m,b} + C_{m,b}] \times \delta_{b,p,m}^{(i,j)} \quad (38)$$

$$= \sum_{b \in A} [t_{b,n}(x_b) + WT_{m,b} + ST_{m,b} + C_{m,b}] \times \delta_{b,p,m}^{(i,j)} \quad (39)$$

$$= c_{p,m}^{(i,j)} \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^m \quad (40)$$

2. If $m \in \{CP, RS, EH\}$: the terms ST and C in the objective function are constants w.r.t the path flow, while WT is not. Let us define G_b ($\forall b \in N$) as bellow:

$$G_{b,m} = \frac{\partial WT_{m,b}}{\partial x_{b,m}} \quad (41)$$

$$= \frac{\partial WT_{m,b}}{\partial f_{p,m}^{(i,j)}} \times \frac{\partial f_{p,m}^{(i,j)}}{\partial x_{b,m}} \quad (42)$$

$$= \frac{1}{R_m} \times \frac{1}{\delta_{b,p,m}^{(i,j)}} \quad (43)$$

1 The transition from Equation (41) to Equation (42) is by applying the chain rule of dif-
2 ferentiation. The transition from Equation (42) to Equation (43) is explained by the fact that the
3 waiting time for a passenger using an SMS depends only on the number of passengers using that
4 same service. Also, the incidence matrix $[(\delta_{a,p,m}^{(i,j)})_{i,j \in E ; a \in A ; m \in \Psi ; p \in P_{ij}^m}]$ is predefined and fixed for
5 every network, and depends on neither the flow variables nor the OD demand. Then, as we are
6 working with the same boundaries for functions f and x , we can use the property of derivatives,
7 which states that: if $\frac{\partial x}{\partial f} = g$, then $\frac{\partial f}{\partial x} = \frac{1}{g}$. With this, we calculate the derivative of the objective
8 function w.r.t the flow variable.

$$\begin{aligned} \frac{\partial Z}{\partial f_{p,m}^{(i,j)}} &= \sum_{b \in A} \frac{\partial}{\partial x_b} \left\{ \sum_{n \in N} \sum_{a \in A_n} \int_0^{x_a} t_{a,n}(\omega) d\omega \right\} \times \frac{\partial x_b}{\partial f_{p,m}^{(i,j)}} \\ &+ \sum_{b \in A} \sum_{m' \in \Psi} \frac{\partial}{\partial x_{b,m'}} \left\{ \sum_{a \in A} \sum_{m \in \Psi} [WT_{m,a} + ST_{m,a} + C_{m,a}] \cdot x_{a,m} \right\} \times \frac{\partial x_{b,m'}}{\partial f_{p,m}^{(i,j)}} \end{aligned} \quad (44)$$

$$= \sum_{b \in A} t_{b,n}(x_b) \cdot \delta_{b,p,m}^{(i,j)} + \sum_{b \in A} [WT_{m,b} + ST_{m,b} + C_{m,b} + \frac{\partial WT_{m,b}}{\partial x_{b,m}} \cdot x_{b,m}] \times \delta_{b,p,m}^{(i,j)} \quad (45)$$

$$= \sum_{b \in A} [t_{b,n}(x_b) + WT_{m,b} + ST_{m,b} + C_{m,b} + G_{b,m} \cdot x_{b,m}] \times \delta_{b,p,m}^{(i,j)} \quad (46)$$

$$= c_{p,m}^{(i,j)} \quad \forall i, j \in E \quad ; \forall p \in P_{ij}^m \quad (47)$$

1 In conclusion, the derivative of the Lagrangian is calculated as follows:

$$\frac{\partial L(f, u)}{\partial f_{p,m}^{(i,j)}} = \frac{\partial Z}{\partial f_{p,m}^{(i,j)}} + \frac{\partial}{\partial f_{p,m}^{(i,j)}} \left\{ \sum_{r,s \in E} \mu_{r,s} (q^{(r,s)} - \sum_{m \in \Psi} \sum_{p \in P_{rs}^m} f_{p,m}^{(r,s)}) \right\} \quad (48)$$

$$= c_{p,m}^{(i,j)} - \mu_{ij} \quad \forall i, j \in E \quad ; \forall m \in \Psi \quad ; \forall p \in P_{ij}^m \quad (49)$$

2 Using these derivatives, we can explicitly formulate the minimization problem in UE-CMF with
3 the following UE conditions, which complete our proof.

$$f_{p,m}^{(i,j)} (c_{p,m}^{(i,j)} - \mu_{ij}) = 0 \quad \forall i, j \in E \quad \forall m \in \Psi \quad \forall p \in P_{ij}^m \quad (50)$$

$$c_{p,m}^{(i,j)} - \mu_{ij} \geq 0 \quad \forall i, j \in E \quad \forall m \in \Psi \quad \forall p \in P_{ij}^m \quad (51)$$

$$\sum_{m \in \Psi} \sum_{p \in P_{ij}^m} f_{p,m}^{(i,j)} = q^{(i,j)} \quad \forall i, j \in E \quad (52)$$

$$f_{p,m}^{(i,j)} \geq 0 \quad \forall i, j \in E \quad \forall m \in \Psi \quad \forall p \in P_{ij}^m \quad (53)$$

4 Since the objective function in program UE-CMF is non-convex, the uniqueness of equilibrium so-
5 lutions can be proven at the link level if all cost functions remain strictly monotone (29). However,
6 the uniqueness w.r.t path flows is not straightforward. In other words, there could exist an infinite
7 number of path flow solutions leading to the same unique link flow solution (30).

8 NUMERICAL RESULTS

9 In this section, we present the numerical results obtained from our study, which aims to analyze the
10 commuter's behavior under different scenarios. We first evaluate the proposed framework for a toy
11 network and perform an analysis to validate the model. Second, we apply the model for a larger
12 multi-modal test case, the Sioux Falls network (31). By conducting this analysis, we also aim to
13 offer valuable insights into the decision-making process of commuters when selecting their mode

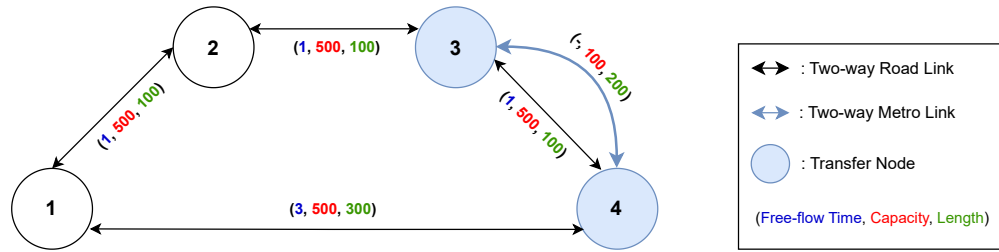


FIGURE 6: A multi-modal network with four nodes and twelve OD pairs.

1 of transportation and the influential factors contributing to their choices. We solve the UE and SO
 2 models using the Gurobi optimizer. Table 2 displays the experiment settings, with explicit mention
 3 of differing values for the two test cases. Experiments were conducted on a 64-bit computer with
 4 Intel i7 CPU 2.90GHz and 16GB RAM.

5 Validation and Analysis on a Toy Network

6 The proposed model is implemented for the network illustrated by Figure 6. The network has four
 7 nodes, serving as both origins and destinations. Nodes 3 and 4 are transfer nodes to the metro
 8 system. We assume a 100 commuter as demand for every OD pair.

9 We consider three scenarios, illustrated by Figure 7, and analyze the commuters' behavior
 10 under UE and SO principles. In Scenario (1), we only consider personal means of transport (car,
 11 walking, and biking) and PT (bus and metro). In Scenario (2), we allow SMSs as door-to-door
 12 services (without intermodality). Scenario (3) includes all possible travel options, i.e., SMSs and
 13 inter-modes are available. For this scenario, the model's solving time is 12 seconds.

14 Figure 8 presents the modes usage: the proportion of commuters using the travel mode.
 15 Additionally, Table 3 presents the traffic assignment regarding mode and path choices for the UE
 16 solution.

17 In Figure 8, Scenario (1), 66.67% of commuters choose PT. The bus mode is more used
 18 (with 50%) due to its accessibility. Since intermodality is not allowed, only commuters between
 19 OD pairs (3,4) and (4,3) can access the metro. This demonstrates that using PT when accessible
 20 is always preferable to personal means under both UE and SO principles. Let us take the bus
 21 service as an example. Bus units operate in the system independently of their passenger flow,
 22 contributing simultaneously to traffic congestion in the road network. Besides, a bus can hold
 23 more passengers than a simple car, and since we consider BPR as our travel time function, having
 24 as many commuters as possible going with PT is the best option. However, it is essential to
 25 acknowledge that PT has inherent limitations, like limited capacity and frequency, restricting its
 26 availability to all commuters. Consequently, some commuters will use private cars when PT is
 27 not viable. Consider the OD pair (2,3) as an example. This pair has no metro links between
 28 the origin and the destination. However, there are bus connections between Node 2 and Node 3.
 29 The commuters cannot use the bus mode because it has reached capacity due to hanging onboard
 30 passengers between OD pair (2,4).

31 Moreover, since the network is simple and uncongested, solutions under UE and SO are
 32 identical, following the observations in (32). None of the commuters in this scenario have opted
 33 for walking or biking due to the higher travel time associated with these modes than other options.

34 In Scenario (2), commuters can choose all modes without intermodality. Similar to the pre-

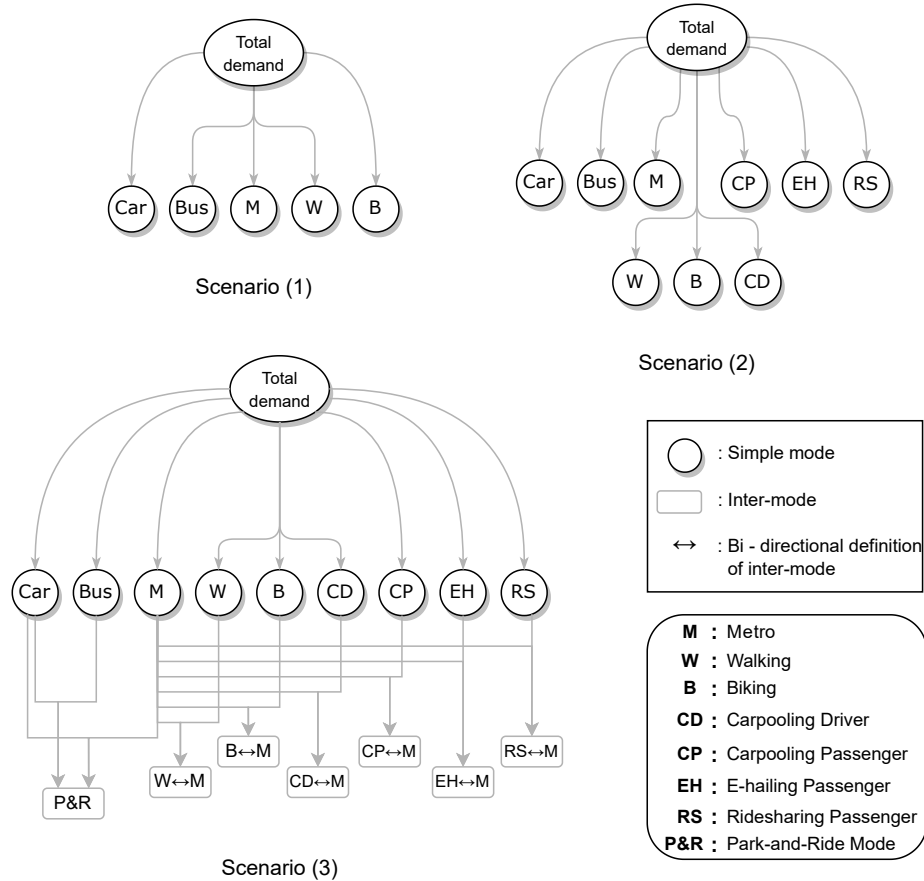


FIGURE 7: Available travel options for three scenarios.

1 vious scenario, bus mode is used to its full capacity. The metro is also preferred when accessible.
 2 For SMSs, 16.66% choose carpooling (both as passengers and drivers), and 16.67% used rideshar-
 3 ing. Table 3 shows that ridesharing is used for short-distance trips (OD pairs (2,3) and (3,2)) while
 4 carpooling is chosen for long-distance trips. This can be explained by the fact that ridesharing
 5 has a high trip fare, increasing with the length of the trip (traversed links). In this scenario, no
 6 commuter is willing to use his car or the e-hailing service because, without considering personal
 7 preferences, it is always beneficial to share the cost of the trip with a passenger through carpooling
 8 or ridesharing.

9 Note that under the UE conditions and for the same OD pair, the cost of carpooling pas-
 10 sengers is not the same as that of drivers. For example, consider the OD pair (1,3). For these
 11 commuters, the bus is the option with minimum cost. Nevertheless, it reaches full capacity be-
 12 cause of commuters between (1,2). The second best option is carpooling as a passenger. However,
 13 this can only be feasible if there are enough drivers, so only half of these commuters can choose
 14 CP mode. The cost for a carpooling driver is high due to the cost of using his car, but since this
 15 cost is less than driving alone or ridesharing, this option is used by the other half of commuters,
 16 described by Table 3.

17 When traffic is assigned under SO, the ridesharing service is not used. Our SO objective
 18 function does not consider the ridesharing service provider's profits. Thus, from a system point of

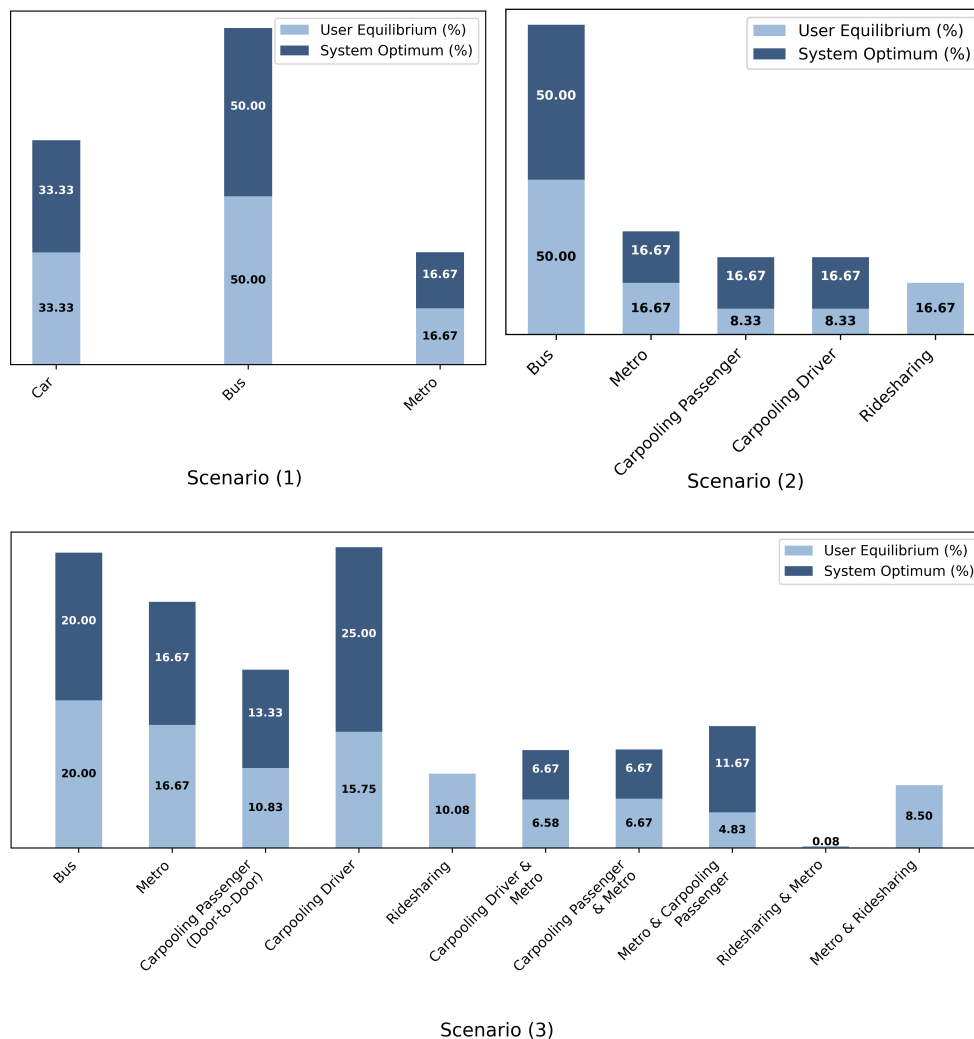


FIGURE 8: Modes usage in the small network, for three scenarios.

1 view, the goal is to reduce both the passengers’ and drivers’ costs. If the number of CP increases,
 2 the number of CD also increases accordingly, which will attract the demand to switch from the
 3 ridesharing service to the carpooling service.

4 In the third scenario, where the intermodality is available, we observed that, with the same
 5 model configuration as the two previous scenarios, more than 80% of commuters will use PT due
 6 to its accessibility, while the rest of the commuters will use carpooling. However, to present a more
 7 realistic and complete analysis, we reduce the bus frequency ($freq_{Bus} = 2$) and provide the traffic
 8 assignment information in Table 3. In this case, when traffic is assigned based on UE, 36.67% of
 9 commuters use PT, 10.83% carpool as passengers for door-to-door trips and not for intermodality,
 10 and 10.08% participate in ridesharing, while 15.75% of commuters use CD mode. These drivers
 11 benefit from the possibility of having a passenger on board (CP or I(CP)), for part or the entire trip
 12 to compensate for the high cost of using their car. Intermodality is used with SMSs, representing
 13 26.66% of the trips (neither P&R mode nor walking and biking for the first/last mile). Similar to
 14 the previous scenario, ridesharing is mainly used for short trips (either as a door-to-door service or

TABLE 3: Traffic distribution for three scenarios.

OD pair	Mode	Path	Flow	Gen. Cost*	OD pair	Mode	Path	Flow	Gen. Cost*
Scenario 1: Baseline Network									
(1,2)	Bus	1-2	100	6.487	(1,3)	car	1-2-3	100	64.158
(1,4)	Bus	1-4	100	16.346	(2,1)	Bus	2-1	100	6.487
(2,3)	car	2-3	100	32.154	(2,4)	Bus	2-3-4	100	12.975
(3,1)	car	3-2-1	100	64.158	(3,2)	car	3-2	100	32.154
(3,4)	M	3-4	100	17.483	(4,1)	Bus	4-1	100	16.346
(4,2)	Bus	4-3-2	100	12.975	(4,3)	M	4-3	100	17.483
Scenario 2: Multi-modal Network with SMSs									
(1,2)	Bus	1-2	100	1.522	(1,3)	CP	1-2-3	50	5.406
(1,3)	CD	1-2-3	50	10.446	(1,4)	Bus	1-4	100	3.509
(2,1)	Bus	2-1	100	1.522	(2,3)	RS	2-3	100	11.06
(2,4)	Bus	2-3-4	100	3.045	(3,1)	CP	3-2-1	50	5.406
(3,1)	CD	3-2-1	50	10.446	(3,2)	RS	3-2	100	11.060
(3,4)	M	3-4	100	3.736	(4,1)	Bus	4-1	100	3.509
(4,2)	Bus	4-3-2	100	3.045	(4,3)	M	4-3	100	3.736
Scenario 3: Multi-modal Network with Intermodality (with $freq_{Bus} = 2$)									
(1,2)	Bus	1-2	40	1.623	(1,2)	RS	1-2	60	11.063
(1,3)	CP	1-2-3	80	7.342	(1,3)	CD	1-2-3	20	10.482
(1,4)	Bus	1-4	40	3.591	(1,4)	CD&M	1-2-3-4	60	14.218
(2,1)	Bus	2-1	40	1.629	(2,1)	RS	2-1	60	11.069
(2,3)	Bus	2-3	40	1.638	(2,3)	CD	2-3	60	5.248
(2,4)	CD&M	2-3-4	20	8.985	(2,4)	CP&M	2-3-4	80	7.415
(3,1)	CD	3-2-1	100	10.494	(3,2)	Bus	3-2	40	1.644
(3,2)	CP	3-2	30	3.684	(3,2)	CD	3-2	30	5.254
(3,4)	M	3-4	100	3.736	(4,1)	Bus	4-1	40	3.591
(4,1)	M&CP	4-1	60	11.090	(4,2)	M&RS	4-3-2	100	14.821
(4,3)	M	4-3	100	3.736					

*: Generalized cost for using the corresponding mode and path between OD pairs.

1 combined with the metro for a short distance). In this case, ridesharing is used instead of carpooling
2 due to the high demand for the carpooling service and, thus, a high waiting time. The trip with
3 intermodality at the UE showed that our model could capture these complex trips. Recall that the
4 model with inter-mode method can represent any trip, including multiple modes.

5 Under the SO principle, the ridesharing demand is directed toward carpooling. Here, many
6 commuters use M&CP because increasing PT usage improves the system's overall objectives. This
7 increase in the M&CP mode attracts more drivers to participate in carpooling (CD mode) to reduce
8 their costs since they can transport passengers from the transfer node to their destination.

9 We show in Figure 9(A) the total cost evolution under both UE and SO, when the total
10 demand increases. The total cost under UE is approximately 1.2 times higher than SO when the
11 demand is increased by a factor of 10. This means when the network is congested, commuters'
12 personal decisions decrease the overall performance level of the transport system. It is expected
13 due to the UE objective function. In such a case, increasing the capacity of PT, or promoting more
14 carpooling through subsidies, for example, may help bring the solution under UE to that under SO.

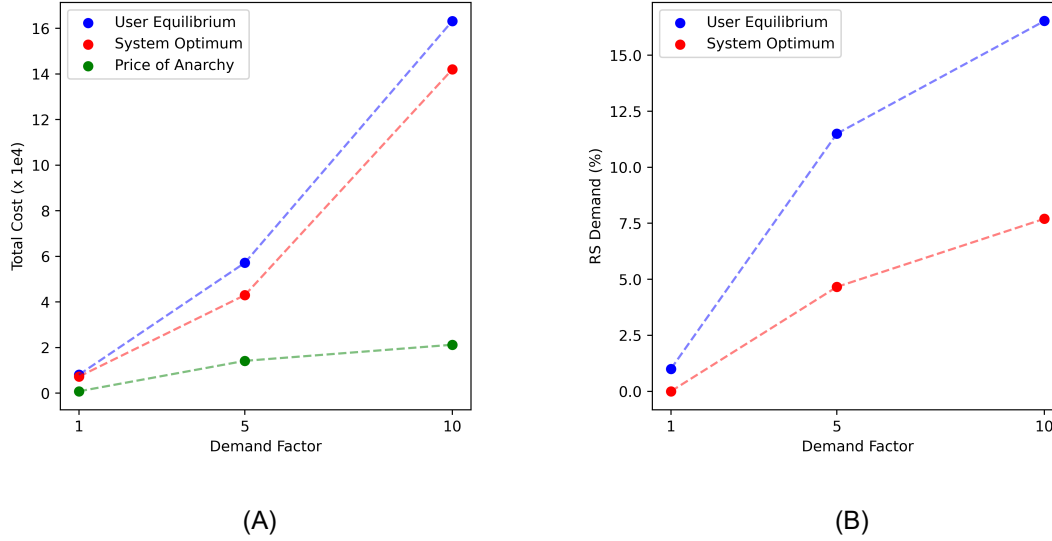


FIGURE 9: (A): Price of Anarchy - Evolution of the total cost in the small network; (B): Evolution of ridesharing usage for the small network, w.r.t the total demand.

1 Moreover, we investigate the variation in ridesharing usage when the total demand in-
 2 creases. Figure 9(B) illustrates, under UE, the use of ridesharing services increases which can
 3 be explained by the fact that, for carpooling drivers, ridesharing becomes more attractive since
 4 the travel time is high and the difference in terms of monetary cost between using their car and
 5 ridesharing becomes comparable. Besides, under SO, since the PT reaches its total capacity, de-
 6 mand is directed to carpooling and ridesharing. However, the ridesharing service is more present
 7 due to the high waiting time for carpooling passengers.

8 **Application: Travel Behavior in Sioux Falls Network**

9 This section analyzes travel behavior in the multi-modal Sioux Falls network, with PT infrastruc-
 10 ture extracted from (31) and illustrated by Figure 10. The network comprises 24 nodes, with nodes
 11 4, 6, 15, and 19 as transfer nodes. We execute our model on 30 randomly selected OD pairs among
 12 552 in total. The solving time is 61 seconds.

13 Figure 11 illustrates the proportion of commuters using each available mode. We can ob-
 14 serve similar travel behavior as described in section 5.1. PT is widely used under both UE and SO
 15 principles. For UE, 26.65% of commuters choose intermodality with SMSs, which is higher than
 16 the SO mode choice distribution (16%). Similar to the previous case study, ridesharing is used
 17 for short-distance trips (OD pair (23,12) as door-to-door service, and (4,6), (6,4), (15,12), (19,12)
 18 combined with metro). For SO, the demand for the door-to-door ridesharing service and some of
 19 the metro and M&RS demand was directed towards the door-to-door carpooling service. As pre-
 20 viously explained, reducing the carpooling drivers' cost (by satisfying more passengers) improve
 21 the system's objectives. In addition, 4.67% of commuters used the M&RS mode because of the
 22 high waiting time for the carpooling service, induced by the high number of passengers.

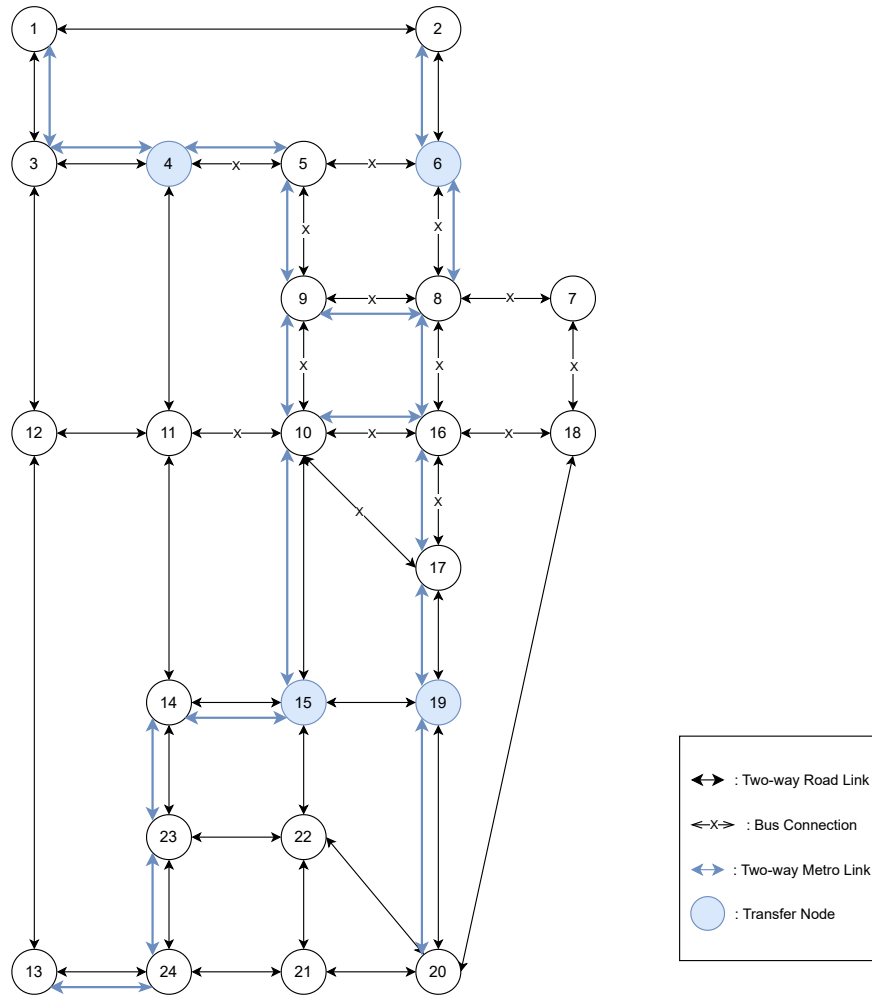


FIGURE 10: Sioux Falls Network with PT infrastructure.

1 CONCLUSION

2 This research proposed a comprehensive mathematical assignment model for multi-modal urban
 3 transportation networks, including shared mobility services (SMSs) and public transport (PT). The
 4 model is able to address the impacts of SMSs on transportation systems and travelers' choices
 5 when integrated with PT. Existing studies have shown shortcomings in providing an integrated
 6 framework for evaluating, combining, and optimizing available travel options effectively, leading
 7 to potential inefficiencies in planning and higher expenses. In response, our proposed model,
 8 where commuters can seamlessly select and combine various travel modes, including SMSs and
 9 PT, allows for optimized travel choices.

10 Two fundamental assignment principles, user equilibrium (UE) and system optimum (SO),
 11 are considered for the model formulation (Program UE-CMF). The model not only includes tran-
 12 sit, personal, and shared transportation modes but also captures the intermodality in which users
 13 can combine more than one travel mode. To the best of our knowledge, this is the most complete
 14 assignment model for the multi-modal network (refer to Table 1). We examined commuters' travel
 15 decisions under UE and SO. Our findings reveal carpooling is preferred over ridesharing when PT
 16 is not directly accessible. However, in situations of high demand and network congestion, rideshar-

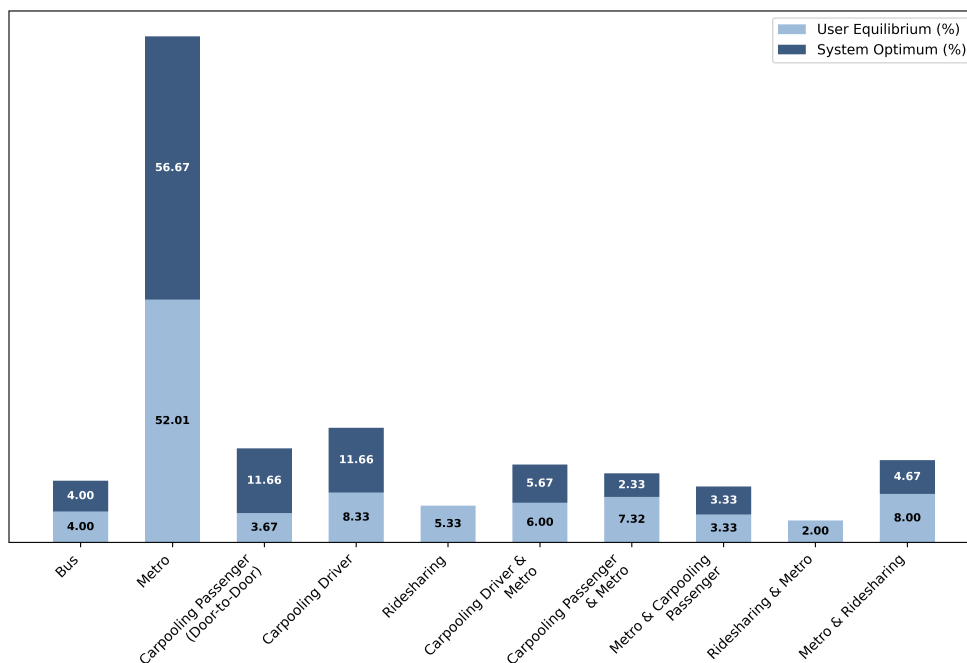


FIGURE 11: Modes usage in the multi-modal Sioux Falls network, with intermodality.

1 ing becomes more attractive due to increased waiting times for carpooling passengers and reduced
 2 costs between cars and ridesharing services. The proposed framework can provide valuable in-
 3 sights into travel behavior under different scenarios, contributing to a better understanding of the
 4 integrated transportation system.

5 As an initial step toward a unified framework incorporating SMSs and PT, our proposed
 6 model offers a baseline for analyzing commuters' behavior within a multi-modal transportation
 7 network. It facilitates the investigation of planning scenarios by providing insightful analyses of
 8 mode usage and traffic assignment in the network.

9 Future research should focus on integrating personal preferences and social criteria into
 10 commuters' decision-making processes, incorporating service providers' objectives, and consid-
 11 ering market equilibrium in the model's formulation. In addition, including traffic dynamics
 12 can enhance the model while increasing the system's complexity and solution calculation. The
 13 simulation-based approach can be applied while they make the system intractable for large-scale
 14 applications.

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