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# TRANSMISSION EIGENVALUE-FREE REGIONS NEAR THE REAL AXIS.

## II

GEORGI VODEV

ABSTRACT. In this paper we extend the results in [16] to more general domains. More precisely, we obtain transmission eigenvalue-free regions for the interior transmission problem with one complex-valued refraction index, that is, with a damping term which does not vanish on the boundary. In particular, we remove the non-trapping condition as well as the strict concavity condition from [16]. Instead, we impose new, more general conditions in terms of the high-frequency behavior of certain cut-off resolvents associated to exterior problems.

Key words: interior transmission problems, transmission eigenvalues.

### 1. INTRODUCTION

Our goal in this paper is to extend the recent results on the location of the transmission eigenvalues obtained in [16] to more general domains. Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a bounded, connected domain with a  $C^\infty$  smooth boundary  $\Gamma = \partial\Omega$ . In [16] the following interior transmission problem has been considered:

$$(1.1) \quad \begin{cases} (\nabla c_1(x)\nabla + \lambda^2 n_1(x) + i\lambda m(x))u_1 = 0 & \text{in } \Omega, \\ (\nabla c_2(x)\nabla + \lambda^2 n_2(x))u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, c_1\partial_\nu u_1 = c_2\partial_\nu u_2 & \text{on } \Gamma, \end{cases}$$

where  $\lambda \in \mathbb{C}$ ,  $\nu$  denotes the Euclidean unit inner normal to  $\Gamma$  and  $c_j, n_j, m \in C^\infty(\overline{\Omega})$ ,  $j = 1, 2$ , are real-valued functions satisfying  $c_j(x) > 0$ ,  $n_j(x) > 0$ . We also suppose that either  $m(x) \geq 0$  for all  $x \in \overline{\Omega}$  or  $m(x) \leq 0$  for all  $x \in \overline{\Omega}$ . In other words, the function  $m$  does not change the sign. If the equation (1.1) has a non-trivial solution  $(u_1, u_2)$  the complex number  $\lambda$  is said to be an interior transmission eigenvalue. When the function  $m$  is not identically zero, it is well-known that there are no real transmission eigenvalues. Moreover, Robbiano showed (see Theorem 8 of [12]) that there are no transmission eigenvalues in the region

$$|\operatorname{Im} \lambda| \leq C_1 e^{-C_2|\lambda|}, \quad C_1, C_2 > 0.$$

When  $m \equiv 0$  it is known (e.g. see [9], [10], [7]) that the transmission eigenvalues form a discrete set in  $\mathbb{C}$  with no finite accumulation points under the conditions

$$(1.2) \quad c_1(x) \neq c_2(x) \quad \text{on } \Gamma$$

and

$$(1.3) \quad c_1(x)n_1(x) \neq c_2(x)n_2(x) \quad \text{on } \Gamma.$$

This is also true in the case  $c_1 \equiv c_2 \equiv 1$  under the condition (1.3) (e.g. see [9], [5], [8], [14]). Note also that under these conditions Weyl asymptotics for the counting function of the transmission eigenvalues are obtained in [5], [10], [11], [13]. We refer to the survey article [3] for more results and references in this case.

In the present paper we will be interested in the case when the function  $m$  satisfies the condition

$$(1.4) \quad m(x) \neq 0 \quad \text{on} \quad \Gamma.$$

The location of the transmission eigenvalues near the real axis has been studied in [16] under (1.2) and (1.4) as well as the condition that all geodesics associated to the Hamiltonians  $g_j = \frac{c_j(x)}{n_j(x)}|\xi|^2$ ,  $j = 1, 2$ , reach the boundary  $\Gamma$  in a finite time. It was also supposed that  $\Gamma$  is  $g_2$ -strictly concave (e.g. see [16] for the precise definition). It is shown in [16] that under these conditions the region

$$(1.5) \quad C_N(|\lambda| + 1)^{-N} \leq |\operatorname{Im} \lambda| \leq C(|\lambda| + 1)^{-\ell}$$

is free of transmission eigenvalues,  $N > 1$  being arbitrary, where  $\ell \in \{0, 1\}$  is defined as follows. We put  $\ell = 0$  if the condition

$$(1.6) \quad \frac{n_1(x)}{c_1(x)} > \frac{n_2(x)}{c_2(x)}, \quad c_1(x) < c_2(x), \quad \text{on} \quad \Gamma$$

is satisfied, and  $\ell = 1$  otherwise.

In the present paper we will extend this result to more general situations. To describe them we first take an open, bounded domain  $\mathcal{O} \subset \mathbb{R}^d$  (which may be empty) with a  $C^\infty$  smooth boundary  $\Gamma_0 = \partial\mathcal{O}$ , such that  $\mathbb{R}^d \setminus \mathcal{O}$  is connected. Let  $\tilde{c}_j, \tilde{n}_j \in C^\infty(\mathbb{R}^d \setminus \mathcal{O})$ ,  $j = 1, 2$ , be real-valued functions satisfying  $\tilde{c}_j(x) \geq b_0$ ,  $\tilde{n}_j(x) \geq b_0$  for some constant  $b_0 > 0$ . We also suppose that there are a constant  $R_0 \gg 1$  and constants  $c_j^0 > 0$ ,  $n_j^0 > 0$  such that  $\tilde{c}_j(x) = c_j^0$ ,  $\tilde{n}_j(x) = n_j^0$  for  $|x| \geq R_0 + 1$ . Denote by  $\tilde{G}_j$  the selfadjoint Dirichlet realization of the operator  $-\tilde{n}_j(x)^{-1} \nabla \tilde{c}_j(x) \nabla$  on the Hilbert space  $\tilde{H}_j = L^2(\mathbb{R}^d \setminus \mathcal{O}; \tilde{n}_j(x) dx)$ . We define the outgoing and incoming resolvents,  $\mathcal{R}_j^+(\lambda)$  and  $\mathcal{R}_j^-(\lambda)$ , so that

$$\mathcal{R}_j^\pm(\lambda) = (\tilde{G}_j - \lambda^2)^{-1} : \tilde{H}_j \rightarrow \tilde{H}_j$$

is bounded for  $\pm \operatorname{Im} \lambda > 0$ . Clearly, we have  $\mathcal{R}_j^+(\lambda)^* = \mathcal{R}_j^-(\bar{\lambda})$ . Let  $\chi \in C^\infty(\mathbb{R}^d)$  be a real-valued function of compact support such that  $\chi(x) = 1$  for  $|x| \leq R_0 + 2$ . It is well-known that the cutoff resolvents  $\chi \mathcal{R}_j^+(\lambda) \chi$  and  $\chi \mathcal{R}_j^-(\lambda) \chi$  extend through the real axis to meromorphic operator-valued functions with no poles on the real axis. Moreover, we have

$$\left( \chi \mathcal{R}_j^+(\lambda) \chi \right)^* = \chi \mathcal{R}_j^-(\lambda) \chi$$

for real  $\lambda$ . We suppose that the cut-off resolvent of  $\tilde{G}_j$  satisfies the following high-frequency bound on the real axis:

$$(1.7) \quad \left\| \partial_x^\alpha \chi \mathcal{R}_j^\pm(\lambda) \chi \partial_x^\beta \right\|_{\tilde{H}_j \rightarrow \tilde{H}_j} \lesssim \lambda^{|\alpha| + |\beta| - 1} \tau_j(\lambda), \quad \lambda \gg 1,$$

for all multi-indices  $\alpha$  and  $\beta$  such that  $|\alpha| + |\beta| \leq 2$ , where the function  $\tau_j$  is non-decreasing and satisfies

$$(1.8) \quad 1 \leq \tau_j(\lambda) \leq \lambda^q,$$

with some constant  $q \geq 0$ . Let  $\Omega_\sharp \subset \mathbb{R}^d$  be a bounded, connected domain with a  $C^\infty$  smooth boundary  $\Gamma = \partial\Omega_\sharp$ , such that

$$\overline{\Omega} \subset \Omega_\sharp \subset \{x \in \mathbb{R}^d : |x| \leq R_0\}.$$

Then  $\Omega = \Omega_\sharp \setminus \overline{\Omega}$  is a bounded, connected domain with a  $C^\infty$  smooth boundary  $\partial\Omega = \Gamma \cup \Gamma_0$  and  $\Gamma \cap \Gamma_0 = \emptyset$ . We denote by  $c_j, n_j$  the restrictions of the functions  $\tilde{c}_j, \tilde{n}_j$  on  $\Omega$ . In the present paper we will consider the following interior transmission problem:

$$(1.9) \quad \begin{cases} (\nabla c_1(x)\nabla + \lambda^2 n_1(x) + i\lambda m(x))u_1 = 0 & \text{in } \Omega, \\ (\nabla c_2(x)\nabla + \lambda^2 n_2(x))u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 & \text{on } \Gamma, \\ u_1 = u_2 = 0 & \text{on } \Gamma_0, \end{cases}$$

where the function  $m$  is as above. We will be interested in obtaining eigenvalue-free regions near the real axis under the above conditions. Our first result is the following

**Theorem 1.1.** *Suppose that the conditions (1.4) and (1.7) are satisfied with  $j = 1$ . Then there exists a constant  $C > 0$  such that there are no transmission eigenvalues in the region*

$$(1.10) \quad -C\tau_1(|\lambda|)^{-2} \leq \text{sign}(m) \text{Im } \lambda \leq 0, \quad |\lambda| > 1.$$

This theorem is relatively easy to prove. It follows from the a priori estimates for the solution of the first equation in (1.9) obtained in Section 3 together with the Green formula applied to the second equation in (1.9). Therefore we do not need to impose any conditions on the coefficients  $c_2$  and  $n_2$ .

In the half-plane  $\{\text{sign}(m) \text{Im } \lambda > 0\}$ , however, the situation is much more complicated. Therefore, our main goal in the present paper is to obtain eigenvalue-free regions also in this half-plane under the conditions (1.2), (1.4), (1.7) and (1.8). Define  $\ell_1, \ell_2 \in \{0, 1\}$  as follows. We put  $\ell_1 = 0$  if the condition (1.6) is satisfied and the function  $\tau_1$  satisfies

$$(1.11) \quad \lambda^{-1/6} \tau_1(\lambda) \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty,$$

and we put  $\ell_1 = 1$  otherwise. We put  $\ell_2 = 0$  if  $\Gamma$  is  $g_2$ - strictly concave and the function  $\tau_2$  satisfies

$$(1.12) \quad \lambda^{-1/2} \tau_2(\lambda) \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty,$$

and we put  $\ell_2 = 1$  otherwise. Given a parameter  $C > 0$  we denote by  $\Lambda(C) \subset \mathbb{C}$  the region

$$|\text{Im } \lambda| \leq C|\lambda|^{-\ell_1 - \ell_2} \tau_1(|\lambda|)^{-10 + 2\ell_1} \tau_2(|\lambda|)^{-2}, \quad |\lambda| > 1.$$

Denote by  $G_2$  the selfadjoint Dirichlet realization of the operator  $-n_2(x)^{-1} \nabla c_2(x) \nabla$  on the Hilbert space  $H_2 = L^2(\Omega; n_2(x) dx)$ . Clearly, the operator  $G_2$  is non-negative. It is also well-known that the spectrum of  $G_2$  is discrete consisting of infinitely many non-negative eigenvalues. Given  $\lambda \in \mathbb{C}^\pm := \{\lambda \in \mathbb{C} : \pm \text{Re } \lambda > 0\}$ , set

$$\theta_\pm(\lambda) = \text{dist}(\lambda, \text{spec } \pm \sqrt{G_2}) = \min_{\mu_k \in \text{spec } \sqrt{G_2}} |\lambda \mp \mu_k| \geq |\text{Im } \lambda|.$$

Given parameters  $C, N > 0$  we denote by  $\mathcal{L}_N^\pm(C) \subset \mathbb{C}^\pm$  the region

$$\theta_\pm(\lambda) \leq C|\lambda|^{-N}, \quad |\lambda| > 1.$$

Let  $\mathcal{L}_N(C) = \mathcal{L}_N^+(C) \cup \mathcal{L}_N^-(C)$ . We are now ready to state our main result.

**Theorem 1.2.** *Suppose that the conditions (1.2), (1.4), (1.7) and (1.8) are satisfied. Then for every  $N > 12q + 2$  there exist constants  $C, C_N > 0$ ,  $C$  being independent of  $N$ , such that there are no transmission eigenvalues in the region  $\Lambda(C) \setminus \mathcal{L}_N(C_N)$ .*

**Remark 1.3.** *Obviously, if  $\lambda$  is a transmission eigenvalue associated to the problem (1.9), then  $-\lambda$  is a transmission eigenvalue associated to the problem (1.9) with the function  $m$  replaced by  $-m$ . Therefore, it suffices to prove the above theorems in  $\mathbb{C}^+$ , only.*

**Remark 1.4.** *It follows from Theorem 1.2 that the transmission eigenvalues in the region  $\Lambda(C) \cap \mathbb{C}^\pm$  are either finitely many or they are infinitely many and asymptotically very close to the eigenvalues of the self-adjoint operator  $\pm\sqrt{G_2}$ .*

**Remark 1.5.** *It is well-known (e.g. see [1]) that the cutoff resolvents always satisfy the bound (1.7) with  $\tau_j = e^{c\lambda}$ ,  $c > 0$  being some constant, and that this bound cannot be improved without imposing extra conditions. Therefore, imposing the polynomial bound (1.8) is important. Indeed, our arguments do not work anymore without this condition.*

**Remark 1.6.** *When  $\mathcal{O} = \emptyset$  and the coefficients  $\tilde{c}_j, \tilde{n}_j$ ,  $j = 1, 2$ , are constants, then  $\mathcal{R}_j^+(\lambda)$  and  $\mathcal{R}_j^-(\lambda)$  are the outgoing and incoming resolvents of the operator  $-\frac{\tilde{c}_j}{\tilde{n}_j}\Delta$ , where  $\Delta$  is the Euclidean Laplacian on  $\mathbb{R}^d$ . In this case it is well-known that the bound (1.7) holds with  $\tau_j = 1$ . Such a bound still holds for non-constant coefficients but one has to assume a non-trapping condition, namely that all geodesics associated to the Hamiltonians  $\frac{\tilde{c}_j(x)}{\tilde{n}_j(x)}|\xi|^2$  escape to the infinity. This conclusion remains valid when  $\mathcal{O} \neq \emptyset$  as well, but in this case one has to impose the non-trapping condition on the broken geodesics. The simplest example of such a non-trapping obstacle  $\mathcal{O}$  is the strictly convex one, provided the coefficients  $\tilde{c}_j, \tilde{n}_j$ ,  $j = 1, 2$ , are supposed constants.*

**Remark 1.7.** *Our result also applies to trapping obstacles  $\mathcal{O}$  which may have periodic broken geodesics, provided these geodesics are of hyperbolic type. A typical example is an obstacle consisting of several strictly convex bodies and the coefficients are supposed constants (in which case the geodesics are just rays). More precisely, let  $\mathcal{O} = \cup_{j=1}^J \mathcal{O}_j$ ,  $J \geq 2$ , where each  $\mathcal{O}_j$  is strictly convex and  $\overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_j} = \emptyset$  if  $i \neq j$ . When  $J = 2$  there is only one periodic ray between  $\mathcal{O}_1$  and  $\mathcal{O}_2$  (which is of hyperbolic type), while when  $J \geq 3$  there are infinitely many. Therefore, in this case one needs to impose some conditions in order to be able to get a nice resolvent bound on the real axis. The first one is the Ikawa no-eclipse condition: if  $i, j$  and  $k$  are all different, then  $\overline{\mathcal{O}_k}$  does not intersect the convex hull of  $\overline{\mathcal{O}_i} \cup \overline{\mathcal{O}_j}$ . The second one is a dynamical assumption involving the topological pressure of the billiard flow (see [6] for the precise definition). Under these conditions Ikawa [6] obtained that the cut-off resolvent extends analytically to a strip  $|\operatorname{Im} \lambda| \leq C$  and its norm is polynomially bounded there. Using this Burq [2] applied the Fragmèn-Lindelöf principle to conclude that on the real axis the norm of the cut-off resolvent is logarithmically bounded. Therefore, under the Ikawa conditions mentioned above, in this case we have the bound (1.7) fulfilled with  $\tau_j = \log \lambda$ .*

The eigenvalue-free region in Theorem 1.2 is more precise than the region (1.5) obtained in [16] in the sense that the region  $\{|\operatorname{Im} \lambda| \leq C_N |\lambda|^{-N}\}$  is replaced by the smaller one  $\mathcal{L}_N(C_N)$ . This is due to the observation that the parametrix of the interior Dirichlet-to-Neumann map (without damping) in the elliptic region is valid in  $\mathbb{C} \setminus \mathcal{L}_N(C_N)$  (see Section 6). Note that this parametrix plays a crucial role in our proof. Therefore, studying the transmission eigenvalues in  $\mathcal{L}_N(C_N)$ , that is, close to the eigenvalues of  $\pm\sqrt{G_2}$ , requires a completely different approach. Most probably the region  $\mathcal{L}_N(C_N)$  is also free of transmission eigenvalues but proving this remains an open problem.

To prove Theorem 1.2 we follow the same strategy as in [16] with some important modifications. The first one is, as mentioned above, that we extend the domain of validity of the parametrix of the interior Dirichlet-to-Neumann map in the elliptic region (see Theorem 6.1). Secondly, we obtain more general a priori estimates for the solutions of the interior Helmholtz equation without damping (see Theorem 2.1). In particular, the boundary  $\Gamma$  is not necessarily supposed strictly concave. This is done by using the so-called jump formula that allows to express the solution by the term in the right-hand side and the restriction on  $\Gamma$  of the solution

and its normal derivative (see Lemma 2.5). The a priori estimates for the solutions of the interior Helmholtz equation with damping are also more general than those obtained in [16] (see Theorem 3.1).

The proof of Theorems 1.1 and 1.2 is carried out in Section 7. Let us sketch our approach to the proof of Theorem 1.2 which is much more complicated. We need to show that if  $\lambda$  belongs to the eigenvalue-free regions, then the solution  $(u_1, u_2)$  to the equation (1.9) is identically zero. In fact, it suffices to show that the function  $f = u_1|_\Gamma = u_2|_\Gamma$  is identically zero, which would imply that  $u_1$  and  $u_2$  are identically zero. Clearly, if  $\lambda$  is a transmission eigenvalue, we have  $T(\lambda)f \equiv 0$  with  $f$  not identically zero, where

$$T(\lambda) = c_1 \mathcal{N}_1(\lambda, m) - c_2 \mathcal{N}_2(\lambda),$$

$\mathcal{N}_1(\lambda, m)$  and  $\mathcal{N}_2(\lambda)$  being the corresponding interior Dirichlet-to-Neumann maps (see Section 5). In other words, we have to show that if  $\lambda$  belongs to the eigenvalue-free region of Theorem 1.2 and  $T(\lambda)f \equiv 0$ , then  $f \equiv 0$ . To this end, we use the parametrix of the interior Dirichlet-to-Neumann maps in the corresponding elliptic regions in the form of  $h$ - $\Psi$ DOs (see Section 6). This allows us to build a parametrix for the operator  $T(\lambda)$  in the intersection of the two elliptic regions and to compute its principal symbol. The assumption (1.2) is crucial since it implies that this principal symbol is an elliptic one in the deep elliptic region belonging to the class  $S^1(\Gamma)$  (see Section 4 for the definition). This fact in turn implies that the parametrix of  $T(\lambda)$  sends the Sobolev space  $H^1(\Gamma)$  into  $L^2(\Gamma)$  and it is invertible in the deep elliptic region. That is what we use in the proof of Lemma 7.2. The assumption (1.6) allows to extend the invertibility of the parametrix to the whole elliptic region associated to the first equation in (1.9). This is used in the proof of Lemma 7.3. Finally, we combine Lemmas 7.2 and 7.3 with the a priori estimates from the previous sections to obtain the desired eigenvalue-free region.

## 2. STUDY OF THE INTERIOR HELMHOLTZ EQUATION WITHOUT DAMPING

Throughout this paper  $\|\cdot\|$ ,  $\|\cdot\|_1$ ,  $\|\cdot\|_0$  and  $\|\cdot\|_{1,0}$  will denote the norms in  $L^2(\Omega)$ ,  $H^1(\Omega)$ ,  $L^2(\Gamma)$  and  $H^1(\Gamma)$ , respectively, where the norms in  $H^1(\Omega)$  and  $H^1(\Gamma)$  are the semiclassical ones, that is,

$$\begin{aligned} \|u\|_1^2 &:= \sum_{0 \leq |\alpha| \leq 1} \|(h\partial_x)^\alpha u\|^2, \\ \|u\|_{1,0}^2 &:= \sum_{0 \leq |\alpha| \leq 1} \|(h\partial_x)^\alpha u\|_0^2, \end{aligned}$$

where  $0 < h \ll 1$  is a semiclassical parameter to be fixed below. Also,  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_0$  will denote the scalar products in  $L^2(\Omega)$  and  $L^2(\Gamma)$ , respectively.

In this section we consider the equation

$$(2.1) \quad \begin{cases} (\nabla c(x)\nabla + \lambda^2 n(x))u = \lambda v & \text{in } \Omega, \\ u = f & \text{on } \Gamma, \\ u = 0 & \text{on } \Gamma_0, \end{cases}$$

where  $\lambda \in \mathbb{C}$ ,  $\text{Re } \lambda \gg 1$ , and  $c, n \in C^\infty(\overline{\Omega})$  are real-valued functions satisfying  $c(x) > 0$ ,  $n(x) > 0$  for all  $x \in \overline{\Omega}$ . Let  $\tilde{c}, \tilde{n} \in C^\infty(\mathbb{R}^d \setminus \mathcal{O})$  be real-valued functions satisfying  $\tilde{c}(x) \geq b_0$ ,  $\tilde{n}(x) \geq b_0$ ,  $b_0 > 0$ , such that  $\tilde{c} = c$ ,  $\tilde{n} = n$  in  $\Omega$ . We also suppose that there are a constant  $R_0 \gg 1$  and constants  $c^0 > 0$ ,  $n^0 > 0$  such that  $\overline{\Omega} \subset \{x \in \mathbb{R}^d : |x| \leq R_0\}$  and  $\tilde{c}(x) = c^0$ ,  $\tilde{n}(x) = n^0$  for  $|x| \geq R_0 + 1$ . Denote by  $\tilde{G}$  the selfadjoint Dirichlet realization of the operator  $-\tilde{n}(x)^{-1}\nabla\tilde{c}(x)\nabla$

on the Hilbert space  $\tilde{H} = L^2(\mathbb{R}^d \setminus \mathcal{O}; \tilde{n}(x)dx)$ . We define the outgoing and incoming resolvents,  $\mathcal{R}^+(\lambda)$  and  $\mathcal{R}^-(\lambda)$ , so that

$$\mathcal{R}^\pm(\lambda) = (\tilde{G} - \lambda^2)^{-1} : \tilde{H} \rightarrow \tilde{H}$$

is bounded for  $\pm \text{Im } \lambda > 0$ . Clearly, we have  $\mathcal{R}^+(\lambda)^* = \mathcal{R}^-(\bar{\lambda})$ . Let  $\chi \in C^\infty(\mathbb{R}^d)$  be a real-valued function of compact support such that  $\chi(x) = 1$  for  $|x| \leq R_0 + 2$ . As mentioned in the introduction, the cutoff resolvents  $\chi\mathcal{R}^+(\lambda)\chi$  and  $\chi\mathcal{R}^-(\lambda)\chi$  extend through the real axis to meromorphic operator-valued functions with no poles on the real axis, and we have  $(\chi\mathcal{R}^+(\lambda)\chi)^* = \chi\mathcal{R}^-(\lambda)\chi$  for real  $\lambda$ . We suppose that the cut-off resolvent of  $\tilde{G}$  satisfies the bound

$$(2.2) \quad \left\| \partial_x^\alpha \chi \mathcal{R}^\pm(\lambda) \chi \partial_x^\beta \right\|_{\tilde{H} \rightarrow \tilde{H}} \lesssim \lambda^{|\alpha|+|\beta|-1} \tau(\lambda), \quad \lambda \gg 1,$$

for all multi-indices  $\alpha$  and  $\beta$  such that  $|\alpha| + |\beta| \leq 2$ , where the function  $\tau \geq 1$  is non-decreasing, that is,  $\tau(\lambda_1) \leq \tau(\lambda_2)$  if  $\lambda_1 \leq \lambda_2$ .

In  $\Omega$  we define the Hamiltonian  $g = \frac{c(x)}{n(x)}|\xi|^2$ . We now define  $\ell \in \{0, 1\}$  as follows. We put  $\ell = 0$  if  $\Gamma$  is  $g$ - strictly concave and  $\tau$  satisfies

$$(2.3) \quad \lambda^{-1/2} \tau(\lambda) \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty,$$

and we put  $\ell = 1$  otherwise. We introduce the semiclassical parameter  $h = (\text{Re } \lambda)^{-1}$ . Given a parameter  $0 < \epsilon \ll 1$ , independent of  $\lambda$ , set  $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \Gamma) < \epsilon\}$ . One of our goals in this section is to prove the following

**Theorem 2.1.** *Suppose that the condition (2.2) is fulfilled. Let  $u \in H^2(\Omega)$  satisfy equation (2.1) and set  $\omega = h\partial_\nu u|_\Gamma$ . Then there are constants  $C, \lambda_0 > 0$  such that for all  $\lambda \in \mathbb{C}$  such that*

$$(2.4) \quad |\text{Im } \lambda| \leq C\tau(|\lambda|)^{-1}, \quad \text{Re } \lambda \geq \lambda_0,$$

*we have the estimates*

$$(2.5) \quad \|u\| \lesssim \tau(|\lambda|) (\|v\| + \|u\|_{L^2(\Omega_\epsilon)}),$$

$$(2.6) \quad \|u\|_1 \lesssim \tau(|\lambda|) \left( \|v\| + |\lambda|^{\ell/2} \|f\|_0 + |\lambda|^{\ell/2} \|\omega\|_0 \right).$$

*Proof.* It is easy to see that it suffices to prove the estimates (2.5) and (2.6) for real  $\lambda \geq \lambda_0$ . Indeed, if  $u$  satisfies equation (2.1) with complex  $\lambda$ , then  $u$  satisfies equation (2.1) with  $\lambda$  replaced by  $\text{Re } \lambda$  and  $v$  replaced by

$$\tilde{v} = \frac{\lambda}{\text{Re } \lambda} v + \frac{\text{Im } \lambda}{\text{Re } \lambda} (\text{Im } \lambda - 2i\text{Re } \lambda)nu.$$

Therefore, by (2.6) applied with  $\lambda$  replaced by  $\text{Re } \lambda$  and  $v$  replaced by  $\tilde{v}$  we get

$$(2.7) \quad \|u\|_1 \lesssim \tau(|\lambda|) \left( \|v\| + |\lambda|^{\ell/2} \|f\|_0 + |\lambda|^{\ell/2} \|\omega\|_0 \right) + |\text{Im } \lambda| \tau(|\lambda|) \|u\|.$$

If  $\lambda$  satisfies (2.4), taking  $C$  properly we can absorb the last term in the right-hand side of (2.7). Therefore we can conclude that (2.6) also holds for  $\lambda$  satisfying (2.4). Clearly, a similar analysis applies to the estimate (2.5). Thus, in what follows we will prove the estimates (2.5) and (2.6) for real  $\lambda \geq \lambda_0$ . Then we have  $h = \lambda^{-1}$ .

Let  $\phi, \phi_1 \in C_0^\infty(\mathbb{R}^d)$  be independent of  $\lambda$  and such that  $\phi = 1$  in  $(\Omega \cup \mathcal{O}) \setminus \Omega_{\epsilon/3}$ ,  $\text{supp } \phi \subset (\Omega \cup \mathcal{O}) \setminus \Omega_{\epsilon/4}$ , and  $\phi_1 = 1$  in  $(\Omega \cup \mathcal{O}) \setminus \Omega_\epsilon$ ,  $\text{supp } \phi_1 \subset (\Omega \cup \mathcal{O}) \setminus \Omega_{\epsilon/2}$ .

**Lemma 2.2.** *The solution  $u$  of the equation (2.1) satisfies the formula*

$$(2.8) \quad \phi u = -\chi \mathcal{R}^\pm(\lambda) \chi n^{-1} (\lambda \phi v + [\nabla c \nabla, \phi](1 - \phi_1)u)$$

*for real  $\lambda$ .*

*Proof.* Clearly, the function  $\phi u$  satisfies the equation

$$(2.9) \quad \begin{cases} (\nabla c(x)\nabla + \lambda^2 n(x))\phi u = U & \text{in } \mathbb{R}^d \setminus \mathcal{O}, \\ \phi u = 0 & \text{on } \Gamma_0, \end{cases}$$

where  $U \in \tilde{H}$  is given by

$$U = \lambda\phi v + [\nabla c(x)\nabla, \phi]u = \lambda\phi v + [\nabla c(x)\nabla, \phi](1 - \phi_1)u.$$

Given a parameter  $0 < k \ll 1$ , set

$$U_k^\pm = (\nabla c(x)\nabla + (\lambda \pm ik)^2 n(x))\phi u.$$

We have

$$\|U - U_k^\pm\|_{\tilde{H}} \lesssim k|\lambda| \|\phi u\|_{\tilde{H}},$$

which implies

$$\lim_{k \rightarrow 0} \|U - U_k^\pm\|_{\tilde{H}} = 0.$$

On the other hand, since  $\chi = 1$  on  $\text{supp } \phi$ , we have

$$\phi u = -\chi \mathcal{R}^\pm(\lambda \pm ik) \chi n^{-1} U_k^\pm.$$

Thus, taking the limit  $k \rightarrow 0$  we arrive at the formula

$$\phi u = -\chi \mathcal{R}^\pm(\lambda) \chi n^{-1} U,$$

as desired.  $\square$

By (2.2) and (2.8), we obtain

$$(2.10) \quad \|\phi u\| \lesssim \tau(\lambda)\|v\| + \tau(\lambda)\|(1 - \phi_1)u\|,$$

which clearly implies (2.5). When  $\ell = 0$  the estimate (2.6) for real  $\lambda \gg 1$  follows from (2.10) and the following

**Proposition 2.3.** *Let  $\Gamma$  be  $g$ -strictly concave. Then, for a suitable choice of the function  $\phi_1$  and the parameter  $\epsilon$ , both independent of  $\lambda$ , we have the estimate*

$$(2.11) \quad \|(1 - \phi_1)u\| \lesssim \|v\| + \|f\|_0 + \|\omega\|_0 + h^{1/2}\|u\|_1.$$

Indeed, combining the estimates (2.10) and (2.11) leads to

$$(2.12) \quad \|u\| \lesssim \tau(\lambda) (\|v\| + \|f\|_0 + \|\omega\|_0) + \lambda^{-1/2} \tau(\lambda) \|u\|_1.$$

On the other hand, by the Green formula we have

$$(2.13) \quad \langle \lambda^2 n u - \lambda v, u \rangle = \langle -\nabla c \nabla u, u \rangle = \int_{\Omega} c |\nabla u|^2 + h^{-1} \langle c \omega, f \rangle_0,$$

which clearly still holds for complex  $\lambda$ . Taking the real part of (2.13) leads to the estimate

$$(2.14) \quad \|u\|_1 \lesssim h\|v\| + \|u\| + h^{1/2}\|f\|_0^{1/2}\|\omega\|_0^{1/2}.$$

Hence

$$(2.15) \quad \|u\|_1 \lesssim \|u\| + \|v\| + \|f\|_0 + \|\omega\|_0.$$

We now combine the estimates (2.12) and (2.15) and use (2.3) to absorb the term  $\lambda^{-1/2} \tau(\lambda) \|u\|_1$  by taking  $\lambda$  big enough. Clearly, this leads to the estimate (2.6) in this case. Note that the above proposition is in fact Proposition 2.2 of [4] and we refer the reader to Section 2 of [4] for the proof.

Consider now the case  $\ell = 1$ . Let  $\gamma : H^s(\Omega) \rightarrow H^{s-1/2}(\Gamma)$ ,  $s > 1/2$ , denote the restriction on  $\Gamma$  and let  $\gamma^* : H^{-s+1/2}(\Gamma) \rightarrow H^{-s}(\Omega)$  be its adjoint. We will need the following

**Lemma 2.4.** *The restriction on the boundary satisfies the estimate*

$$(2.16) \quad \|\gamma u\|_0 \lesssim \lambda^{1/2} (\varepsilon \|u\|_1 + \varepsilon^{-1} \|u\|)$$

for every  $0 < \varepsilon \leq 1$ .

*Proof.* Clearly, it suffices to prove (2.16) locally near  $\Gamma$ . Let  $\mathcal{V} \subset \mathbb{R}^d$  be a small open domain such that  $\mathcal{V}^0 := \mathcal{V} \cap \Gamma \neq \emptyset$ . Let  $(x_1, x') \in \mathcal{V}^+ := \mathcal{V} \cap \Omega$ ,  $0 < x_1 \ll 1$ ,  $x' = (x_2, \dots, x_d) \in \mathcal{V}^0$ , be the local normal geodesic coordinates near the boundary. Let also  $\mathcal{V}_1 \subset \mathcal{V}$  be a small open domain such that  $\mathcal{V}_1^0 := \mathcal{V}_1 \cap \Gamma \neq \emptyset$ . Choose a function  $\psi \in C_0^\infty(\mathcal{V})$ ,  $0 \leq \psi \leq 1$ , such that  $\psi = 1$  on  $\mathcal{V}_1$ . Set  $u^b := \psi(1 - \phi)u$ , the function  $\phi$  being as above, and  $\mathcal{D}_{x_1} = -ih\partial_{x_1}$ . We have

$$\begin{aligned} -\frac{d}{dx_1} \|u^b(x_1, \cdot)\|_0^2 &= -2\operatorname{Re}\langle u^b(x_1, \cdot), \partial_{x_1} u^b(x_1, \cdot) \rangle_0 \\ &\leq h^{-1}\varepsilon^{-2} \|u^b(x_1, \cdot)\|_0^2 + h^{-1}\varepsilon^2 \|\mathcal{D}_{x_1} u^b(x_1, \cdot)\|_0^2. \end{aligned}$$

Hence

$$\begin{aligned} \|\gamma\psi\gamma u\|_0^2 &= \|u^b(0, \cdot)\|_0^2 = -\int_0^\varepsilon \frac{d}{dx_1} \|u^b(x_1, \cdot)\|_0^2 dx_1 \\ &\leq h^{-1}\varepsilon^{-2} \|u^b\|^2 + h^{-1}\varepsilon^2 \|\mathcal{D}_{x_1} u^b\|^2 \lesssim h^{-1}\varepsilon^{-2} \|u\|^2 + h^{-1}\varepsilon^2 \|u\|_1^2. \end{aligned}$$

Since  $\Gamma$  is compact, there exist a finite number of smooth functions  $\psi_i$ ,  $0 \leq \psi_i \leq 1$ ,  $i = 1, \dots, I$ , such that  $1 = \sum_{i=1}^I \psi_i$  and the above estimate holds with  $\gamma\psi$  replaced by each  $\psi_i$ . Therefore, the estimate (2.16) is obtained by summing up all such estimates.  $\square$

To prove (2.6) in this case we will express the solution  $u$  in terms of the resolvent  $\mathcal{R}^\pm(\lambda)$  and the functions  $v$ ,  $f$  and  $\omega$  by using the so-called jump formula. It can be derived from (2.8) by letting  $\varepsilon \rightarrow 0$ . We have the following

**Lemma 2.5.** *There exists a first-order differential operator  $Q$  near  $\Gamma$  so that the solution to the equation (2.1) satisfies the formula*

$$(2.17) \quad \phi_0 u = -\lambda \chi \mathcal{R}^\pm(\lambda) \chi n^{-1} (\phi_0 v + hQ\gamma^* f + \gamma^* c_0 \omega),$$

for real  $\lambda$ , where  $c_0 = \gamma c$  and  $\phi_0$  denotes the characteristic function of  $\Omega \cup \mathcal{O}$ .

*Proof.* We take a family of smooth real-valued functions  $\phi_\varepsilon \rightarrow \phi_0$  as  $\varepsilon \rightarrow 0$  for which (2.8) holds. We will derive from the Green formula that

$$(2.18) \quad \langle [\nabla c \nabla, \phi_\varepsilon] u, w \rangle \rightarrow \langle c_0 \gamma \partial_\nu u, \gamma w \rangle_0 + \langle \gamma u, \gamma \tilde{Q} w \rangle_0$$

for every  $w \in H^1(\Omega)$  such that  $w = 0$  on  $\Gamma_0$ , where  $\tilde{Q}$  is a first-order differential operator. Indeed, by the Green formula we have the identity

$$(2.19) \quad \langle -\nabla c \nabla u, w \rangle = \langle c \nabla u, \nabla w \rangle,$$

provided either  $w = 0$  or  $\partial_\nu u = 0$  on  $\partial\Omega$ . Using (2.19) we obtain

$$(2.20) \quad \langle [\nabla c \nabla, \phi_\varepsilon] u, w \rangle = \langle c \nabla u, [\nabla, \phi_\varepsilon] w \rangle - \langle c [\nabla, \phi_\varepsilon] u, \nabla w \rangle.$$

On the other hand, near  $\Gamma$  we have  $\nabla = \partial_\nu + \mathcal{Q}$ , where  $\mathcal{Q}$  is a tangential first-order differential operator. Therefore,  $\gamma \nabla u = \gamma \partial_\nu u + \tilde{\mathcal{Q}} \gamma u$ , where  $\tilde{\mathcal{Q}}$  is the first-order differential operator obtained by restricting the coefficients of  $\mathcal{Q}$  on  $\Gamma$ . We also have in the sense of distributions

$$[\nabla, \phi_\varepsilon] \rightarrow [\partial_\nu, \phi_\varepsilon] \rightarrow [\partial_\nu, \phi_0] = \delta_\Gamma,$$

where  $\delta_\Gamma$  denotes the Dirac delta function on  $\Gamma$ . Thus, by (2.20) we get

$$\begin{aligned} \langle [\nabla c \nabla, \phi_\epsilon] u, w \rangle &\rightarrow \langle c_0 \gamma \partial_\nu u, \gamma w \rangle_0 - \langle c_0 \gamma u, \gamma \partial_\nu w \rangle_0 \\ &\quad + \langle c_0 \tilde{Q} \gamma u, \gamma w \rangle_0 - \langle c_0 \gamma u, \tilde{Q} \gamma w \rangle_0 \\ &= \langle c_0 \gamma \partial_\nu u, \gamma w \rangle_0 - \langle c_0 \gamma u, \gamma \partial_\nu w \rangle_0 \\ &\quad + \langle \gamma u, (\tilde{Q}^* c_0 - c_0 \tilde{Q}) \gamma w \rangle_0, \end{aligned}$$

which implies (2.18) with

$$\tilde{Q} = -c_0 \partial_\nu + \tilde{Q}^* c_0 - c_0 \tilde{Q}.$$

Given any  $w_1 \in L^2(\Omega)$ , by (2.8) we have

$$-\langle \phi_\epsilon u, w_1 \rangle = \lambda \langle \chi \mathcal{R}^\pm(\lambda) \chi n^{-1} \phi_\epsilon v, w_1 \rangle + \langle [\nabla c \nabla, \phi_\epsilon] u, n^{-1} \chi \mathcal{R}^\mp(\lambda) \chi w_1 \rangle.$$

Letting  $\epsilon \rightarrow 0$  and using (2.18) with  $w = n^{-1} \chi \mathcal{R}^\mp(\lambda) \chi w_1$ , we get

$$\begin{aligned} -\langle \phi_0 u, w_1 \rangle &= \lambda \langle \chi \mathcal{R}^\pm(\lambda) \chi n^{-1} \phi_0 v, w_1 \rangle \\ (2.21) \quad &+ \langle c_0 \gamma \partial_\nu u, \gamma n^{-1} \chi \mathcal{R}^\mp(\lambda) \chi w_1 \rangle_0 + \langle \gamma u, \gamma \tilde{Q} n^{-1} \chi \mathcal{R}^\mp(\lambda) \chi w_1 \rangle_0, \end{aligned}$$

which clearly implies (2.17) with  $Q = \tilde{Q}^*$ .  $\square$

We will now use the form (2) of the formula (2.17) together with Lemma 2.4 applied with  $\varepsilon = 1$  and assumption (2.2). We get

$$\begin{aligned} |\langle \phi_0 u, w_1 \rangle| &\leq \lambda \|\chi \mathcal{R}^\pm(\lambda) \chi n^{-1} \phi_0 v\| \|w_1\| \\ &+ \lambda^{3/2} \|\chi \mathcal{R}^\mp(\lambda) \chi w_1\|_1 \|\omega\|_0 + \lambda^{3/2} \|h Q \chi \mathcal{R}^\mp(\lambda) \chi w_1\|_1 \|f\|_0 \\ &\lesssim \tau(\lambda) \|w_1\| (\|v\| + \lambda^{1/2} \|f\|_0 + \lambda^{1/2} \|\omega\|_0), \end{aligned}$$

which implies

$$(2.22) \quad \|u\| \lesssim \tau(\lambda) \left( \|v\| + \lambda^{1/2} \|f\|_0 + \lambda^{1/2} \|\omega\|_0 \right).$$

Now (2.6) follows from (2.15) and (2.22) in this case.  $\square$

Denote by  $G$  the selfadjoint Dirichlet realization of the operator  $-n(x)^{-1} \nabla c(x) \nabla$  on the Hilbert space  $H = L^2(\Omega; n(x) dx)$ . Then  $G \geq 0$  and the spectrum of  $G$  is discrete consisting of infinitely many non-negative eigenvalues. Given  $\lambda \in \mathbb{C}^+$ , set

$$\theta(\lambda) = \text{dist}(\lambda, \text{spec } \sqrt{G}) = \min_{\mu_k \in \text{spec } \sqrt{G}} |\lambda - \mu_k| \geq |\text{Im } \lambda|.$$

In the next sections we will also need the following

**Theorem 2.6.** *Let  $u \in H^2(\Omega)$  satisfy equation (2.1). Then the function  $\omega = h \partial_\nu u|_\Gamma$  satisfies the estimate*

$$(2.23) \quad \|\omega\|_0 \lesssim (1 + \theta(\lambda)^{-1}) \|v\| + (1 + |\lambda| \theta(\lambda)^{-1}) \|f\|_{1,0}$$

for  $\theta(\lambda) > 0$ ,  $|\text{Im } \lambda| \leq C$ ,  $|\lambda| \gg 1$ ,  $C > 0$  being any constant.

*Proof.* The theorem follows from the next two lemmas.

**Lemma 2.7.** *We have the estimate*

$$(2.24) \quad \|\omega\|_0 \lesssim \|v\| + \|f\|_{1,0} + \|u\|.$$

*Proof.* As in the proof of Lemma 2.4, it suffices to prove (2.24) locally near  $\Gamma$ . We keep the same notations. In the coordinates  $(x_1, x')$  the principal symbol of the Euclidean Laplacian  $-\Delta$  is equal to  $\xi_1^2 + r(x, \xi')$ , where  $(\xi_1, \xi')$  are the dual variables to  $(x_1, x')$ , and  $r$  is a homogeneous polynomial of order two and satisfies  $C_1|\xi'|^2 \leq r \leq C_2|\xi'|^2$  with some constants  $C_1, C_2 > 0$ . Therefore, the principal symbol of the positive Laplace-Beltrami operator on  $\Gamma$  is equal to  $r_0(x', \xi') = r(0, x', \xi')$ . Note that  $\Gamma$  can be considered as a Riemannian manifold without boundary with a Riemannian metric induced by the Euclidean one. Therefore, we can write the operator

$$P = -h^2 n(x)^{-1} \nabla c(x) \nabla - 1$$

in the coordinates  $(x_1, x')$  as follows

$$P = \frac{c(x)}{n(x)} (\mathcal{D}_{x_1}^2 + r(x, \mathcal{D}_{x'})) - 1 + h\mathcal{R}(x, \mathcal{D}_x),$$

where  $\mathcal{D}_{x_1} = -ih\partial_{x_1}$ ,  $\mathcal{D}_{x'} = -ih\partial_{x'}$ ,  $\mathcal{D}_x = -ih\partial_x$  and  $\mathcal{R}$  is a first-order differential operator. Introduce the function

$$F(x_1) = \left\| \mathcal{D}_{x_1} u^b \right\|_0^2 - \left\langle r(x_1, \cdot, \mathcal{D}_{x'}) u^b, u^b \right\rangle_0 + \left\langle n_b(x_1, \cdot) u^b, u^b \right\rangle_0,$$

where  $n_b = c^{-1}n$ . Clearly,

$$(2.25) \quad \operatorname{Re} F(0) \geq \|\gamma\psi\omega\|_0^2 - C\|f\|_{1,0}^2, \quad C > 0.$$

On the other hand,

$$(2.26) \quad F(0) = - \int_0^\epsilon F'(x_1) dx_1$$

where  $F'$  denotes the first derivative with respect to  $x_1$ . We have

$$\begin{aligned} F'(x_1) &= -2\operatorname{Re} \left\langle (\mathcal{D}_{x_1}^2 + r - n_b) u^b, \partial_{x_1} u^b \right\rangle_0 - \left\langle (r' - n'_b) u^b, u^b \right\rangle_0 \\ &= 2h^{-1} \operatorname{Im} \left\langle n_b(P - h\mathcal{R}) u^b, \mathcal{D}_{x_1} u^b \right\rangle_0 - \left\langle (r' - n'_b) u^b, u^b \right\rangle_0. \end{aligned}$$

Hence

$$(2.27) \quad |F'(x_1)| \lesssim h^{-2} \|Pu^b\|_0^2 + \|u^b\|_{1,0}^2 + \|\mathcal{D}_{x_1} u^b\|_0^2.$$

By (2.26) and (2.27) we obtain

$$(2.28) \quad \operatorname{Re} F(0) \leq \int_0^\epsilon |F'(x_1)| dx_1 \lesssim h^{-2} \|Pu^b\|^2 + \|u^b\|_1^2.$$

By (2.25) and (2.28),

$$\begin{aligned} \|\gamma\psi\omega\|_0 &\lesssim h^{-1} \|Pu^b\| + \|u^b\|_1 + \|f\|_{1,0} \\ &\lesssim \lambda \|P((1-\phi)u)\| + \|(1-\phi)u\|_1 + \|f\|_{1,0} \\ &\lesssim \lambda \|Pu\| + \|u\|_1 + \|f\|_{1,0}, \end{aligned}$$

which implies

$$(2.29) \quad \|\omega\|_0 \lesssim \|v\| + \|f\|_{1,0} + \|u\|_1.$$

It is easy to see now that (2.24) follows from (2.14) and (2.29).  $\square$

**Lemma 2.8.** *Under the assumptions of Theorem 2.6 we have the estimate*

$$(2.30) \quad \|u\| \lesssim \theta(\lambda)^{-1} \|v\| + (1 + |\lambda|\theta(\lambda)^{-1}) \|f\|_{1,0}.$$

*Proof.* Let  $\zeta \in C^\infty(\overline{\Omega})$  be supported in a small neighbourhood of  $\Gamma$  and  $\zeta = 1$  on  $\Gamma$ . Clearly, if  $u$  is a solution to the equation (2.1), then  $u - \zeta f$  is a solution to (2.1) with  $f$  replaced by 0 and  $v$  replaced by

$$v - \lambda^{-1}(\nabla c(x)\nabla + \lambda^2 n(x))\zeta f.$$

Hence

$$(2.31) \quad u - \zeta f = -\lambda(G - \lambda^2)^{-1}n^{-1}v + (G - \lambda^2)^{-1}n^{-1}(\nabla c(x)\nabla + \lambda^2 n(x))\zeta f =: u_1 + u_2.$$

We have

$$(2.32) \quad \begin{aligned} \|u_1\| &\lesssim |\lambda| \|(G - \lambda^2)^{-1}\|_{H \rightarrow H} \|v\| \\ &\lesssim |\lambda| \max_{\mu_k \in \text{spec } \sqrt{G}} |\mu_k^2 - \lambda^2|^{-1} \|v\| \\ &\lesssim \max_{\mu_k \in \text{spec } \sqrt{G}} |\mu_k - \lambda|^{-1} \|v\| \lesssim \theta(\lambda)^{-1} \|v\|. \end{aligned}$$

To bound the norm of  $u_2$ , observe that given any  $w \in L^2(\Omega)$ , in view of (2.19), we have

$$\begin{aligned} \langle u_2, w \rangle &= \langle (\nabla c(x)\nabla + \lambda^2 n(x))\zeta f, n^{-1}(G - \overline{\lambda}^2)^{-1}w \rangle \\ &= \langle c(x)\nabla(\zeta f), \nabla(n^{-1}(G - \overline{\lambda}^2)^{-1}w) \rangle + \lambda^2 \langle \zeta f, n^{-1}(G - \overline{\lambda}^2)^{-1}w \rangle. \end{aligned}$$

Hence

$$(2.33) \quad |\langle u_2, w \rangle| \lesssim |\lambda|^2 \|\zeta f\|_1 \|(G - \overline{\lambda}^2)^{-1}w\|_1.$$

On the other hand, since the function  $W = (G - \overline{\lambda}^2)^{-1}w$  satisfies equation (2.1) with  $f = 0$ ,  $\lambda$  replaced by  $\overline{\lambda}$  and  $v$  replaced by  $\lambda^{-1}w$ , the Green formula implies the estimate

$$(2.34) \quad \|W\|_1 \lesssim \|W\| + |\lambda|^{-2} \|w\|.$$

As in (2), we get

$$(2.35) \quad \|W\| \lesssim |\lambda|^{-1} \theta(\lambda)^{-1} \|w\| + |\lambda|^{-2} \|w\|,$$

where we have used the fact that  $\theta(\overline{\lambda}) = \theta(\lambda)$ . Since  $\|\zeta f\|_1 \lesssim \|f\|_{0,1}$ , it follows from (2.33), (2.34) and (2.35) that

$$|\langle u_2, w \rangle| \lesssim (1 + |\lambda| \theta(\lambda)^{-1}) \|f\|_{0,1} \|w\|,$$

which clearly implies

$$(2.36) \quad \|u_2\| \lesssim (1 + |\lambda| \theta(\lambda)^{-1}) \|f\|_{0,1}.$$

Now (2.30) follows from (2.31), (2) and (2.36).  $\square$

### 3. STUDY OF THE INTERIOR HELMHOLTZ EQUATION WITH DAMPING

In this section we will obtain a priori estimates for the solution to the equation

$$(3.1) \quad \begin{cases} (\nabla c(x)\nabla + \lambda^2 n(x) + i\lambda m(x))u = \lambda v & \text{in } \Omega, \\ u = f & \text{on } \Gamma, \\ u = 0 & \text{on } \Gamma_0, \end{cases}$$

where  $\lambda \in \mathbb{C}$ ,  $\text{Re } \lambda \gg 1$ ,  $c, n, m \in C^\infty(\overline{\Omega})$ ,  $c > 0$ ,  $n > 0$ , and the function  $m$  does not change the sign. We keep the same notations as in the previous section. We define in the same way the operator  $\tilde{G}$  and suppose that its resolvent satisfies the condition (2.2). Our goal in this section is to prove the following

**Theorem 3.1.** *Suppose that  $m$  satisfies (1.4). Let  $u \in H^2(\Omega)$  satisfy equation (3.1) and set  $\omega = h\partial_\nu u|_\Gamma$ . Then there are constants  $C, \lambda_0 > 0$  such that for all  $\lambda \in \mathbb{C}$  such that*

$$(3.2) \quad |\operatorname{Im} \lambda| \leq C\tau(|\lambda|)^{-1}, \quad \operatorname{Re} \lambda \geq \lambda_0,$$

*we have the estimate*

$$(3.3) \quad \|u\| \lesssim \tau(|\lambda|) (\|v\| + \|mu\|),$$

*while for*

$$(3.4) \quad |\operatorname{Im} \lambda| \leq C\tau(|\lambda|)^{-2}, \quad \operatorname{Re} \lambda \geq \lambda_0,$$

*we have the estimates*

$$(3.5) \quad \|\omega\|_0 + \|u\|_1 \lesssim \tau(|\lambda|)^2 (\|v\| + \|f\|_0) + \|f\|_{1,0},$$

$$(3.6) \quad \|u\| \lesssim \tau(|\lambda|)^2 \|v\| + \tau(|\lambda|) |\operatorname{Im} \langle c\omega, f \rangle_0|^{1/2},$$

$$(3.7) \quad \|u\|_1 \lesssim \tau(|\lambda|)^2 \|v\| + h^{1/2} \tau(|\lambda|) \|f\|_0 + h^{1/2} \|f\|_{1,0} + \tau(|\lambda|) |\operatorname{Im} \langle c\omega, f \rangle_0|^{1/2}.$$

*Moreover, under the condition*

$$(3.8) \quad \operatorname{sign}(m) \operatorname{Im} \langle c\omega, f \rangle_0 \leq 0$$

*we have the better estimate*

$$(3.9) \quad \|u\| \lesssim \tau(|\lambda|)^2 \|v\|.$$

*Proof.* The estimate (3.3) is a consequence of (2.5). Indeed, in view of assumption (1.4) we can find  $\epsilon > 0$  such that  $|m| \geq C$  on  $\Omega_\epsilon$  with some constant  $C > 0$ . Applying (2.5) with  $v$  replaced by  $v - imu$ , we get

$$\|u\| \lesssim \tau(|\lambda|) (\|v - imu\| + \|u\|_{L^2(\Omega_\epsilon)}) \lesssim \tau(|\lambda|) (\|v\| + \|mu\|).$$

As in the proof of Theorem 2.1, it is easy to see that it suffices to prove the estimates (3.5), (3.6), (3.7) and (3.9) for real  $\lambda \gg 1$ . The Green formula in this case takes the form

$$(3.10) \quad \langle (\lambda^2 n + i\lambda m)u - \lambda v, u \rangle = \langle -\nabla c \nabla u, u \rangle = \int_\Omega c |\nabla u|^2 + \lambda \langle c\omega, f \rangle_0.$$

Taking the imaginary part of this identity we get

$$(3.11) \quad \langle mu, u \rangle = \operatorname{Im} \langle v, u \rangle + \operatorname{Im} \langle c\omega, f \rangle_0,$$

which implies

$$(3.12) \quad \| |m|^{1/2} u \|^2 \leq \|v\| \|u\| + \operatorname{sign}(m) \operatorname{Im} \langle c\omega, f \rangle_0.$$

Since  $|m| \lesssim |m|^{1/2}$ , we obtain from (3.3) and (3.12),

$$\begin{aligned} \|u\|^2 &\lesssim \tau(\lambda)^2 \|v\|^2 + \tau(\lambda)^2 \| |m|^{1/2} u \|^2 \\ &\lesssim \tau(\lambda)^2 \|v\| \|u\| + \tau(\lambda)^2 \|v\|^2 + \tau(\lambda)^2 \operatorname{sign}(m) \operatorname{Im} \langle c\omega, f \rangle_0, \end{aligned}$$

which implies

$$(3.13) \quad \|u\|^2 \lesssim \tau(\lambda)^4 \|v\|^2 + \tau(\lambda)^2 \operatorname{sign}(m) \operatorname{Im} \langle c\omega, f \rangle_0.$$

Clearly, (3.6) and (3.9) follow from (3.13). On the other hand, taking the real part of (3.10) leads to the estimate

$$(3.14) \quad \|u\|_1 \lesssim h \|v\| + \|u\| + h^{1/2} \|f\|_0^{1/2} \|\omega\|_0^{1/2}.$$

By (3.13) and (3.14),

$$(3.15) \quad \|u\|_1^2 \lesssim \tau(\lambda)^4 \|v\|^2 + h\|f\|_0 \|\omega\|_0 + \tau(\lambda)^2 \text{sign}(m) \text{Im} \langle c\omega, f \rangle_0.$$

Applying (2.24) with  $v$  replaced by  $v - imu$ , we get

$$(3.16) \quad \|\omega\|_0 \lesssim \|v\| + \|f\|_{1,0} + \|u\|_1.$$

By (3.15) and (3.16),

$$(3.17) \quad \begin{aligned} \|u\|_1 &\lesssim \tau(\lambda)^2 \|v\| + \tau(\lambda) \|f\|_0^{1/2} \|\omega\|_0^{1/2} \\ &\lesssim \tau(\lambda)^2 \|v\| + \varepsilon^{-1} \tau(\lambda)^2 \|f\|_0 + \varepsilon \|\omega\|_0 \\ &\lesssim \tau(\lambda)^2 \|v\| + \varepsilon^{-1} \tau(\lambda)^2 \|f\|_0 + \varepsilon (\|u\|_1 + \|f\|_{1,0}) \end{aligned}$$

for every  $0 < \varepsilon \leq 1$ . We now get (3.5) from (3.16) and (3) by taking  $\varepsilon$  small enough in order to absorb the term  $\varepsilon \|u\|_1$  in the right-hand side. By (3.15) we also have

$$\begin{aligned} \|u\|_1^2 &\lesssim \tau(\lambda)^4 \|v\|^2 + \tau(\lambda)^2 |\text{Im} \langle c\omega, f \rangle_0| \\ &\quad + h\|f\|_0 (\tau(\lambda)^2 \|v\| + \tau(\lambda)^2 \|f\|_0 + \|f\|_{1,0}) \\ &\lesssim \tau(\lambda)^4 \|v\|^2 + \tau(\lambda)^2 |\text{Im} \langle c\omega, f \rangle_0| \\ &\quad + h\tau(\lambda)^2 \|f\|_0^2 + h\tau(\lambda)^{-2} \|f\|_{1,0}^2, \end{aligned}$$

which clearly implies (3.7).  $\square$

#### 4. A PRIORI ESTIMATES FOR THE BOUNDARY DATA

Given any  $k \in \mathbb{R}$ ,  $S^k(\Gamma)$  will denote the set of all functions  $a \in C^\infty(T^*\Gamma)$  satisfying

$$\left| \partial_x^\alpha \partial_\xi^\beta a \right| \leq C_{\alpha,\beta} (|\xi| + 1)^{k-|\beta|}$$

for all multi-indices  $\alpha$  and  $\beta$ . Given a symbol  $a \in S^k(\Gamma)$ ,  $\text{Op}_h(a)$  will denote the  $h$ - $\Psi$ DO defined by

$$(\text{Op}_h(a)f)(x) = (2\pi h)^{1-d} \int_{T^*\Gamma} e^{i\langle x-y, \xi \rangle/h} a(x, \xi) f(y) d\xi dy.$$

Let  $\chi \in C^\infty(T^*\Gamma)$  be of compact support, independent of  $\lambda$ . We will derive from Theorem 3.1 the following

**Proposition 4.1.** *Suppose that  $m$  satisfies (1.4). Let  $u \in H^2(\Omega)$  satisfy equation (3.1) and set  $\omega = h\partial_\nu u|_\Gamma$ . Then for all  $\lambda \in \mathbb{C}$  satisfying (3.4) we have the estimate*

$$(4.1) \quad \|f\|_{1,0} \lesssim |\lambda|^{1/2} \tau(|\lambda|)^3 \|v\| + |\lambda|^{1/2} \tau(|\lambda|)^2 |\text{Im} \langle c\omega, f \rangle_0|^{1/2} + \|\text{Op}_h(1 - \chi)f\|_{1,0}.$$

*Proof.* By (2.16) used with  $\varepsilon$  replaced by  $\varepsilon\tau(|\lambda|)^{-1}$ , together with (3.6) and (3.7) we get

$$(4.2) \quad \|f\|_0 \lesssim \varepsilon^{-1} |\lambda|^{1/2} \tau(|\lambda|)^3 \|v\| + \varepsilon^{-1} |\lambda|^{1/2} \tau(|\lambda|)^2 |\text{Im} \langle c\omega, f \rangle_0|^{1/2} + \varepsilon \|f\|_{1,0}.$$

On the other hand, we have

$$(4.3) \quad \|f\|_{1,0} \lesssim \|\text{Op}_h(\chi)f\|_{1,0} + \|\text{Op}_h(1 - \chi)f\|_{1,0} \lesssim \|f\|_0 + \|\text{Op}_h(1 - \chi)f\|_{1,0}.$$

We now combine (4.2) and (4.3) and take  $\varepsilon$  small enough in order to absorb the term  $\varepsilon \|f\|_{1,0}$  in the right-hand side. This clearly leads to the estimate (4.1).  $\square$

Let  $\chi \in C^\infty(T^*\Gamma)$  be supported in the hyperbolic region,  $\mathcal{H}$ , of the boundary value problem (3.1), that is,

$$\text{supp } \chi \subset \mathcal{H} := \{(x', \xi') \in T^*\Gamma : r_0(x', \xi') < n_{b,0}(x')\},$$

where  $n_{b,0} = n_b|_\Gamma$ ,  $n_b = n/c$ . With this choice of the function  $\chi$  we will now prove the following improved estimates.

**Proposition 4.2.** *Under the assumptions of Proposition 4.1, we have the estimates*

$$(4.4) \quad \begin{aligned} & \|\mathrm{Op}_h(\chi)f\|_0 + \|\mathrm{Op}_h(\chi)\omega\|_0 \\ & \lesssim \tau(|\lambda|)^2\|v\| + \tau(|\lambda|)|\mathrm{Im}\langle c\omega, f\rangle_0|^{1/2} + h^{1/2}\tau(|\lambda|)\|f\|_0 + h^{1/2}\|f\|_{1,0}, \end{aligned}$$

$$(4.5) \quad \|f\|_{1,0} \lesssim \tau(|\lambda|)^2\|v\| + \tau(|\lambda|)|\mathrm{Im}\langle c\omega, f\rangle_0|^{1/2} + h^{1/2}\tau(|\lambda|)\|f\|_0 + \|\mathrm{Op}_h(1-\chi)f\|_{1,0}.$$

*Proof.* By (4.3) we have

$$(4.6) \quad \|f\|_{1,0} \lesssim \|\mathrm{Op}_h(\chi)f\|_0 + \|\mathrm{Op}_h(1-\chi)f\|_{1,0}.$$

Clearly, the estimate (4.5) follows from (4.2) and (4.6). On the other hand, it is easy to see that (4.2) follows from (3.7) and the following

**Lemma 4.3.** *We have the estimate*

$$(4.7) \quad \|\mathrm{Op}_h(\chi)f\|_0 + \|\mathrm{Op}_h(\chi)\omega\|_0 \lesssim \|v\| + \|u\|_1 + h\|f\|_0.$$

*Proof.* It suffices to prove the estimate (4.7) for real  $\lambda \gg 1$ . Indeed, this would imply (4.7) for complex  $\lambda$  such that  $|\mathrm{Im}\lambda| \leq C$ ,  $C > 0$  being any constant. Let the function  $u^b$  and the operator  $P$  be as in the proof of Lemma 2.7. Set

$$u^\sharp = \mathrm{Op}_h(\chi)u^b = \mathrm{Op}_h(\chi)\psi(1-\phi)u.$$

Clearly,

$$(4.8) \quad \begin{aligned} \|Pu^\sharp\| & \lesssim \|P((1-\phi)u)\| + \|[P, \mathrm{Op}_h(\chi)\psi](1-\phi)u\| \\ & \lesssim \|P((1-\phi)u)\| + h\|(1-\phi)u\|_1 \\ & \lesssim \|Pu\| + h\|u\|_1 \lesssim h\|v\| + h\|u\|_1. \end{aligned}$$

We define the function  $F(x_1)$  as in the proof of Lemma 2.7 replacing  $u^b$  by  $u^\sharp$ . Observe now that the choice of  $\chi$  guarantees that

$$n_b(0, x') - r_0(x', \xi') \geq C > 0$$

on  $\mathrm{supp}\chi$ . Therefore, by Gårding's inequality we have

$$(4.9) \quad \mathrm{Re}\langle (n_b(0, \cdot) - r_0(\cdot, \mathcal{D}_{x'}))\mathrm{Op}_h(\chi)\psi_0f, \mathrm{Op}_h(\chi)\psi_0f \rangle_0 \geq C_1\|\mathrm{Op}_h(\chi)\psi_0f\|_0^2,$$

where  $C_1 > 0$  is some constant and  $\psi_0 = \psi|_{x_1=0}$ . Since

$$\mathcal{D}_{x_1}u^\sharp|_{x_1=0} = -i\mathrm{Op}_h(\chi)\psi_0\omega - ih\mathrm{Op}_h(\chi)\psi_1f,$$

where  $\psi_1 = \partial_{x_1}\psi|_{x_1=0}$ , we deduce from (4.9),

$$(4.10) \quad \begin{aligned} \mathrm{Re}F(0) & \geq C_1\|\mathrm{Op}_h(\chi)\psi_0f\|_0^2 + \|\mathrm{Op}_h(\chi)\psi_0\omega\|_0^2 - \mathcal{O}(h^2)\|f\|_0^2 \\ & \geq C_1\|\psi_0\mathrm{Op}_h(\chi)f\|_0^2 + \|\psi_0\mathrm{Op}_h(\chi)\omega\|_0^2 - \mathcal{O}(h^2)\|f\|_0^2. \end{aligned}$$

On the other hand, the upper bound (2.28) still holds with  $u^b$  replaced by  $u^\sharp$ . This fact together with (4) and (4) imply (4.7).  $\square$

Given a parameter  $0 < \varepsilon \ll 1$ , independent of  $\lambda$ , choose a function  $\chi_\varepsilon \in C_0^\infty(T^*\Gamma)$  such that  $\chi_\varepsilon = 1$  in the region  $\{|r_0/n_{b,0} - 1| \leq \varepsilon\}$  and  $\chi_\varepsilon = 0$  in  $T^*\Gamma \setminus \{|r_0/n_{b,0} - 1| \leq 2\varepsilon\}$ . We have the following

**Proposition 4.4.** *Under the assumptions of Proposition 4.1, we have the estimate*

$$(4.11) \quad \|\mathrm{Op}_h(\chi_\varepsilon)\omega\|_0 \lesssim \tau(|\lambda|)^2 \|v\| + \tau(|\lambda|) |\mathrm{Im} \langle c\omega, f \rangle_0|^{1/2} + (\varepsilon + h^{1/2}\tau(|\lambda|)) \|f\|_{1,0}.$$

*Proof.* It is easy to see that the proposition follows from (3.7) and the following

**Lemma 4.5.** *We have the estimate*

$$(4.12) \quad \|\mathrm{Op}_h(\chi_\varepsilon)\omega\|_0 \lesssim \|v\| + \|u\|_1 + (\varepsilon + h) \|f\|_0.$$

*Proof.* Again, it suffices to prove the lemma for real  $\lambda \gg 1$ . We will proceed in the same way as in the proof of Lemma 4.3 with  $\chi$  replaced by  $\chi_\varepsilon$  making the following modification. Since in this case the function  $\chi_\varepsilon$  is no longer supported in the hyperbolic region, we do not have the Gårding inequality (4.9) fulfilled anymore. Instead, since  $n_b(0, x') - r_0(x', \xi') = \mathcal{O}(\varepsilon)$  on  $\mathrm{supp} \chi_\varepsilon$ , we have the bound

$$\|(n_b(0, \cdot) - r_0(\cdot, \mathcal{D}_{x'}))\mathrm{Op}_h(\chi_\varepsilon)\psi_0 f\|_0 \lesssim (\varepsilon + h) \|\psi_0 f\|_0.$$

Therefore in this case the function  $\mathrm{Re} F(0)$  is lower bounded as follows

$$(4.13) \quad \begin{aligned} \mathrm{Re} F(0) &\geq \|\mathrm{Op}_h(\chi_\varepsilon)\psi_0 \omega\|_0^2 - \mathcal{O}((\varepsilon + h)^2) \|f\|_0^2 \\ &\geq \|\psi_0 \mathrm{Op}_h(\chi_\varepsilon)\omega\|_0^2 - \mathcal{O}((\varepsilon + h)^2) \|f\|_0^2. \end{aligned}$$

The estimate (4.12) follows from (4) and the upper bound (2.28) adapted to this case.  $\square$

## 5. THE DIRICHLET-TO-NEUMANN MAP

Let  $u$  solve equation (3.1) with  $v \equiv 0$  and define the Dirichlet-to-Neumann map

$$\mathcal{N}(\lambda, m) : H^1(\Gamma) \rightarrow L^2(\Gamma)$$

by

$$\mathcal{N}(\lambda, m)f := h\partial_\nu u|_\Gamma.$$

When  $m \equiv 0$  we will denote the Dirichlet-to-Neumann map by  $\mathcal{N}(\lambda)$ . Clearly, under the conditions of Theorem 3.1, by (3.5) we have that the Dirichlet-to-Neumann map in this case satisfies the estimate

$$(5.1) \quad \|\mathcal{N}(\lambda, m)f\|_0 \lesssim \tau(|\lambda|)^2 \|f\|_{1,0}$$

for  $\lambda$  satisfying (3.4). On the other hand, when  $m \equiv 0$ , by (2.23) we have the estimate

$$(5.2) \quad \|\mathcal{N}(\lambda)f\|_0 \lesssim (1 + |\lambda|\theta(\lambda)^{-1}) \|f\|_{1,0},$$

for  $\theta(\lambda) > 0$ ,  $|\mathrm{Im} \lambda| \leq C$ ,  $|\lambda| \gg 1$ ,  $C > 0$  being any constant. Let  $\chi, \eta \in C^\infty(T^*\Gamma)$  be compactly supported functions such that  $\eta = 1$  on  $\mathrm{supp} \chi$ . In Section 7 we will need the following

**Lemma 5.1.** *Under the assumptions of Theorem 3.1, we have the estimates*

$$(5.3) \quad \|[\mathcal{N}(\lambda, m), \mathrm{Op}_h(\chi)]f\|_0 \lesssim h^{1/2}\tau(|\lambda|)^3 \|f\|_{1,0} + \tau(|\lambda|)^3 |\mathrm{Im} \langle c\mathcal{N}(\lambda, m)f, f \rangle_0|^{1/2},$$

$$(5.4) \quad \|\mathrm{Op}_h(1 - \eta)\mathcal{N}(\lambda, m)\mathrm{Op}_h(\chi)f\|_0 \lesssim h^{1/2}\tau(|\lambda|)^3 \|f\|_{1,0} + \tau(|\lambda|)^3 |\mathrm{Im} \langle c\mathcal{N}(\lambda, m)f, f \rangle_0|^{1/2}.$$

*Proof.* Let  $u$  solve equation (3.1) with  $v \equiv 0$ . Then the function  $\tilde{u} = \mathrm{Op}_h(\chi)(1 - \phi)u$  solves equation (3.1) with  $v$  and  $f$  replaced by  $\tilde{v}$  and  $\tilde{f}$ , respectively, where

$$\begin{aligned} \tilde{v} &= \lambda^{-1}[\nabla c(x)\nabla + \lambda^2 n(x) + i\lambda m(x), \mathrm{Op}_h(\chi)(1 - \phi)]u, \\ \tilde{f} &= \mathrm{Op}_h(\chi)f. \end{aligned}$$

Let  $w$  solve equation (3.1) with  $v \equiv 0$  and  $f$  replaced by  $\tilde{f}$ . Then the function  $\tilde{u} - w$  solves equation (3.1) with  $v$  replaced by  $\tilde{v}$  and  $f = 0$ . Moreover, we have

$$h\partial_\nu(\tilde{u} - w)|_\Gamma = -[\mathcal{N}(\lambda, m), \text{Op}_h(\chi)]f.$$

Therefore, the estimates (3.5) and (3.7) lead to

$$\begin{aligned} \|[\mathcal{N}(\lambda, m), \text{Op}_h(\chi)]f\|_0 &\lesssim \tau(|\lambda|)^2 \|\tilde{v}\| \lesssim \tau(|\lambda|)^2 \|u\|_1 \\ &\lesssim h^{1/2} \tau(|\lambda|)^3 \|f\|_{1,0} + \tau(|\lambda|)^3 |\text{Im} \langle c\mathcal{N}(\lambda, m)f, f \rangle_0|^{1/2}. \end{aligned}$$

To prove (5.4) we will use that

$$\text{Op}_h(1 - \eta)\text{Op}_h(\chi) = O(h^\infty) : L^2(\Gamma) \rightarrow L^2(\Gamma).$$

Thus, in view of (5.1), we get

$$\begin{aligned} \|\text{Op}_h(1 - \eta)\mathcal{N}(\lambda, m)\text{Op}_h(\chi)f\|_0 &\lesssim \|\text{Op}_h(1 - \eta)[\mathcal{N}(\lambda, m), \text{Op}_h(\chi)]f\|_0 \\ &\quad + \|\text{Op}_h(1 - \eta)\text{Op}_h(\chi)\mathcal{N}(\lambda, m)f\|_0 \\ &\lesssim \|[\mathcal{N}(\lambda, m), \text{Op}_h(\chi)]f\|_0 + h^\infty \|\mathcal{N}(\lambda, m)f\|_0 \\ &\lesssim \|[\mathcal{N}(\lambda, m), \text{Op}_h(\chi)]f\|_0 + h^\infty \tau(|\lambda|)^2 \|f\|_{1,0}. \end{aligned}$$

We now obtain (5.4) from (5.3).  $\square$

Denote by  $\mathcal{N}(\lambda, m)^*$  the adjoint of  $\mathcal{N}(\lambda, m)$  with respect to the scalar product  $\langle \cdot, \cdot \rangle_0$  in  $L^2(\Gamma)$ . In Section 6 we will need the following

**Lemma 5.2.** *The adjoint operator of the Dirichlet-to-Neumann map satisfies the identity*

$$(5.5) \quad \mathcal{N}(\lambda, m)^* c_0 = c_0 \mathcal{N}(\bar{\lambda}, -m),$$

where  $c_0 = c|_\Gamma$ .

*Proof.* Given any  $f_1, f_2 \in L^2(\Gamma)$ , let  $u_1$  be the solution of equation (3.1) with  $v \equiv 0$  and  $f$  replaced by  $f_1$ , and let  $u_2$  be the solution of equation (3.1) with  $v \equiv 0$ ,  $f$  replaced by  $f_2$  and  $\lambda$  replaced by  $\bar{\lambda}$ . By the Green formula we have

$$\begin{aligned} 0 &= -\langle \nabla c \nabla u_1, u_2 \rangle + \langle u_1, \nabla c \nabla u_2 \rangle = \langle c_0 \partial_\nu u_1|_\Gamma, f_2 \rangle_0 - \langle f_1, c_0 \partial_\nu u_2|_\Gamma \rangle_0 \\ &= h^{-1} \langle c_0 \mathcal{N}(\lambda, m) f_1, f_2 \rangle_0 - h^{-1} \langle f_1, c_0 \mathcal{N}(\bar{\lambda}, -m) f_2 \rangle_0, \end{aligned}$$

which clearly implies (5.5).  $\square$

## 6. PARAMETRIX OF THE DIRICHLET-TO-NEUMANN MAP IN THE ELLIPTIC REGION REVISITED

Let  $\eta \in C^\infty(T^*\Gamma)$  be such that  $1 - \eta$  is supported in the elliptic region,  $\mathcal{E}$ , of the boundary value problem (3.1), that is,

$$\text{supp}(1 - \eta) \subset \mathcal{E} := \{(x', \xi') \in T^*\Gamma : r_0(x', \xi') > n_{b,0}(x')\}.$$

For  $(x', \xi') \in \mathcal{E}$  set

$$\rho(x', \xi', z) = \sqrt{r_0(x', \xi') - zn_{b,0}(x')}, \quad \text{Re } \rho > 0,$$

where  $z = (h\lambda)^2 = 1 - (h\text{Im } \lambda)^2 + 2ih\text{Im } \lambda$ . On  $\text{supp}(1 - \eta)$  we have the lower bound

$$(6.1) \quad \text{Re } \rho \geq C \langle \xi' \rangle, \quad C > 0.$$

The main result in this section is the following

**Theorem 6.1.** *Suppose that the function  $\tau$  satisfies the condition*

$$(6.2) \quad \tau(\lambda) \leq \lambda^q$$

with some constant  $q \geq 0$ . Then, under the conditions of Theorem 3.1 we have the estimate

$$(6.3) \quad \|\mathcal{N}(\lambda, m)\text{Op}_h(1 - \eta)f + \text{Op}_h(\rho(1 - \eta))f\|_0 \lesssim h\|f\|_0$$

for  $\lambda$  satisfying (3.4). When  $m \equiv 0$  the estimate (6.3) still holds without assuming (6.2) for  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$ ,  $|\text{Im } \lambda| \leq C$ ,  $C > 0$  being any constant, while  $C_N > 0$  is a suitable constant depending on  $N$ .

*Proof.* The theorem follows from the parametrix construction carried out in [15]. In what follows we will recall it (see also Section 5 of [16]). In fact, in [15] the case  $m \equiv 0$  is considered, but it is easy to see that the presence of the function  $m$  does not change anything. Indeed, the eikonal equation does not depend on  $m$  and only the transport equations do. Note also that it suffices to build the parametrix locally and then sum up all pieces.

Let  $(x_1, x')$  be the local normal geodesic coordinates near the boundary. Take a function  $\chi \in C^\infty(T^*\Gamma)$ ,  $0 \leq \chi \leq 1$ , such that  $\pi_{x'}(\text{supp } \chi) \subset \mathcal{V}^0$ , where  $\pi_{x'} : T^*\Gamma \rightarrow \Gamma$  denotes the projection  $(x', \xi') \rightarrow x'$ . Moreover, we require that  $\chi \in S^0(\Gamma)$  with  $\text{supp } \chi \subset \text{supp}(1 - \eta)$ . We will be looking for a parametrix of the solution to equation (3.1) (with  $v \equiv 0$ ) in the form

$$\tilde{u} = \phi_0(x_1)(2\pi h)^{-d+1} \iint e^{\frac{i}{h}(\langle y', \xi' \rangle + \varphi(x, \xi', z))} a(x, \xi', z, h) f(y') d\xi' dy',$$

where  $\phi_0 \in C_0^\infty(\mathbb{R})$ ,  $\phi_0(t) = 1$  for  $|t| \leq \delta/2$ ,  $\phi_0(t) = 0$  for  $|t| \geq \delta$ . Here  $0 < \delta \ll 1$  is a small parameter independent of  $\lambda$ . We require that  $\tilde{u}$  satisfies the boundary condition  $\tilde{u} = \text{Op}_h(\chi)f$  on  $x_1 = 0$ . The phase  $\varphi$  and the amplitude  $a$  are chosen in such a way that the function  $\tilde{u}$  satisfies equation (3.1) mod  $\mathcal{O}(h^M)$ , where  $M \geq 1$  is an arbitrary integer. The phase function satisfies

$$\varphi|_{x_1=0} = -\langle x', \xi' \rangle$$

as well as the eikonal equation

$$(6.4) \quad (\partial_{x_1} \varphi)^2 + r(x, \nabla_{x'} \varphi) - z\tilde{n}(x) = x_1^M \Psi_M,$$

where the function  $|\Psi_M|$  is bounded as  $x_1 \rightarrow 0$ . It is shown in Section 4 of [15] that (6.4) has a solution of the form

$$\varphi = \sum_{j=0}^M x_1^j \varphi_j,$$

where the functions  $\varphi_j$  do not depend on  $x_1$ ,  $\varphi_0 = -\langle x', \xi' \rangle$ ,  $\varphi_1 = i\rho$ . It follows from (6.1) that

$$(6.5) \quad \text{Im } \varphi \geq Cx_1 \langle \xi' \rangle / 2,$$

for  $0 \leq x_1 \leq \delta$ , provided  $\delta$  is taken small enough. The amplitude is of the form

$$a = \sum_{j=0}^M h^j a_j,$$

where the functions  $a_j$  do not depend on  $h$ ,  $a_0|_{x_1=0} = \chi$ . Then all functions  $a_j$  can be determined from the transport equations and we have  $a_j \in S^{-j}(\Gamma)$  uniformly in  $x_1$  and  $z$  (see Section 4 of [15]). Clearly, we have

$$(6.6) \quad h\partial_\nu \tilde{u}|_{x_1=0} = \text{Op}_h(b_M)f,$$

where

$$b_M = ia \frac{\partial \varphi}{\partial x_1} \Big|_{x_1=0} + h \frac{\partial a}{\partial x_1} \Big|_{x_1=0} = -\chi \rho + h \sum_{j=0}^M h^j \frac{\partial a_j}{\partial x_1} \Big|_{x_1=0}.$$

Hence  $h^{-1}(b_M + \chi \rho) \in S^0(\Gamma)$  uniformly in  $h$ . This implies

$$(6.7) \quad \text{Op}_h(b_M + \chi \rho) = O(h) : L^2(\Gamma) \rightarrow L^2(\Gamma).$$

On the other hand, the function

$$\tilde{v} = (\nabla c(x) \nabla + \lambda^2 n(x) + i \lambda m(x)) \tilde{u}$$

is of the form

$$\tilde{v} = (2\pi h)^{-d+1} \int \int e^{\frac{i}{h} \langle y', \xi' \rangle + \varphi(x, \xi', z)} V_M(x, \xi', z, h) f(y') d\xi' dy',$$

where  $V_M = V_M^{(1)} + \phi_0(x_1) V_M^{(2)}$ ,

$$V_M^{(1)} = [\nabla c(x) \nabla, \phi_0(x_1)] a,$$

$$V_M^{(2)} = e^{-i\varphi/h} (\nabla c(x) \nabla + \lambda^2 n(x) + i \lambda m(x)) e^{i\varphi/h} a.$$

As shown in Section 4 of [15], the functions  $a_j$  can be chosen in such a way that the function  $V_M^{(2)}$  is of the form

$$(6.8) \quad V_M^{(2)} = x_1^M A_M + h^M B_M,$$

where  $A_M$  and  $B_M$  are smooth functions. More precisely, since  $\chi$  is supported in the elliptic region, we have  $A_M \in S^2(\Gamma)$ ,  $B_M \in S^{1-M}(\Gamma)$  uniformly in  $h$ ,  $z$  and  $0 < x_1 \leq \delta$  (see Proposition 3.4 of [15]). Note that in view of (6.5) we have the bound

$$\left| x_1^M e^{i\varphi/h} \right| \lesssim h^M \langle \xi' \rangle^{-M}.$$

Thus we get that the function  $V_M$  satisfies the bound

$$(6.9) \quad \left| V_M e^{i\varphi/h} \right| \lesssim h^M \langle \xi' \rangle^{-M+1}.$$

By (6.9) we obtain the estimate

$$(6.10) \quad \|\tilde{v}\| \leq \tilde{C}_M h^{M/2} \|f\|_0,$$

provided  $M$  is taken big enough. Let  $u$  solve equation (3.1) with  $v \equiv 0$  and  $f$  replaced by  $\text{Op}_h(\chi) f$ . Then the function  $\tilde{u} - u$  solves equation (3.1) with  $v$  replaced by  $\tilde{v}$  and  $f = 0$ . Therefore, under the conditions of Theorem 3.1, by (3.5) together with (6.2), (6.6) and (6.10) we get

$$(6.11) \quad \|\mathcal{N}(\lambda, m) \text{Op}_h(\chi) f - \text{Op}_h(b_M) f\|_0 = \|h \partial_{x_1} (\tilde{u} - u) \Big|_{x_1=0}\| \lesssim \tau(|\lambda|)^2 \|\tilde{v}\| \lesssim h^{M/2-2q} \|f\|_0.$$

Taking  $M \geq 4q + 2$ , by (6.7) and (6.11), we get

$$(6.12) \quad \|\mathcal{N}(\lambda, m) \text{Op}_h(\chi) f + \text{Op}_h(\chi \rho) f\|_0 \lesssim h \|f\|_0,$$

which implies (6.3) in this case since  $1 - \eta$  can be written as a finite sum of functions  $\chi$  for which (6.12) holds. Consider now the case when  $m \equiv 0$ . We proceed similarly with the difference that we use the estimate (2.23) instead of (3.5). For  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$ ,  $|\text{Im } \lambda| \leq C$ , we obtain

$$(6.13) \quad \|\mathcal{N}(\lambda) \text{Op}_h(\chi) f - \text{Op}_h(b_M) f\|_0 \lesssim (1 + \theta(\lambda)^{-1}) \|\tilde{v}\| \leq \tilde{C}_M^\sharp C_N^{-1} h^{M/2-N} \|f\|_0 \leq h \|f\|_0,$$

provided we take  $M = 2N + 1$  and  $C_N \geq \tilde{C}_M^\sharp$ . Thus we conclude that the estimate (6.12) (and hence (6.3)) still holds in this case as long as  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$ ,  $|\text{Im } \lambda| \leq C$ .  $\square$

Let  $\chi, \eta \in C^\infty(T^*\Gamma)$  be compactly supported functions such that  $\eta = 1$  on  $\text{supp } \chi$  and  $\text{supp}(1 - \eta) \subset \mathcal{E}$ . We will use Theorem 6.1 to prove the following

**Lemma 6.2.** *Under the conditions of Theorem 6.1 we have the estimates*

$$(6.14) \quad \|\text{Op}_h(\chi)\mathcal{N}(\lambda, m)\text{Op}_h(1 - \eta)f\|_0 \lesssim h\|f\|_0,$$

$$(6.15) \quad \|\text{Op}_h(1 - \eta)\mathcal{N}(\lambda, m)\text{Op}_h(\chi)f\|_0 \lesssim h\|f\|_0.$$

When  $m \equiv 0$  the estimates (6.14) and (6.15) still hold for  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$ ,  $|\text{Im } \lambda| \leq C$ .

*Proof.* Since

$$\text{Op}_h(\chi)\text{Op}_h(\rho(1 - \eta)) = O(h^\infty) : L^2(\Gamma) \rightarrow L^2(\Gamma),$$

the estimate (6.14) follows from (6.3). In view of Lemma 5.2 the adjoint of the operator

$$\mathcal{A} := \text{Op}_h(1 - \eta)\mathcal{N}(\lambda, m)\text{Op}_h(\chi)$$

is

$$\mathcal{A}^* = \text{Op}_h(\chi)^*c_0\mathcal{N}(\bar{\lambda}, -m)c_0^{-1}\text{Op}_h(1 - \eta)^*.$$

Choose compactly supported functions  $\chi_1, \eta_1 \in C^\infty(T^*\Gamma)$  such that  $\eta_1 = 1$  on  $\text{supp } \chi_1$ ,  $\text{supp}(1 - \eta_1) \subset \mathcal{E}$ ,  $\chi_1 = 1$  on  $\text{supp } \chi$  and  $\eta = 1$  on  $\text{supp } \eta_1$ . The standard  $h - \Psi$ DO calculus give

$$\text{Op}_h(\chi)^*c_0\text{Op}_h(1 - \chi_1) = O(h^\infty) : H^{-1}(\Gamma) \rightarrow L^2(\Gamma),$$

$$\text{Op}_h(\eta_1)c_0^{-1}\text{Op}_h(1 - \eta)^* = O(h^\infty) : L^2(\Gamma) \rightarrow H^1(\Gamma).$$

We now apply the estimate (6.14) with  $\chi, \eta, \lambda$  replaced by  $\chi_1, \eta_1, \bar{\lambda}$ , respectively. We will also use (6.3) with  $\eta, \lambda, m$  and  $\rho$  replaced by  $\eta_1, \bar{\lambda}, -m$  and  $\bar{\rho}$ , respectively. Note that  $\bar{\rho}(1 - \eta_1) \in S^1(\Gamma)$ . Thus, in view of (5.1) and (6.2), we get

$$\begin{aligned} \|\mathcal{A}^*f\|_0 &\lesssim \|\text{Op}_h(\chi_1)\mathcal{N}(\bar{\lambda}, -m)\text{Op}_h(1 - \eta_1)c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_0 \\ &\quad + \|\mathcal{N}(\bar{\lambda}, -m)\text{Op}_h(\eta_1)c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_0 \\ &\quad + \|\text{Op}_h(\chi)^*c_0\text{Op}_h(1 - \chi_1)\mathcal{N}(\bar{\lambda}, -m)c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_0 \\ &\lesssim h\|c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_0 \\ &\quad + |\lambda|^{2q}\|\text{Op}_h(\eta_1)c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_{1,0} \\ &\quad + h^\infty\|\mathcal{N}(\bar{\lambda}, -m)\text{Op}_h(\eta_1)c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_0 \\ &\quad + h^\infty\|\mathcal{N}(\bar{\lambda}, -m)\text{Op}_h(1 - \eta_1)c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_{-1,0} \\ &\lesssim h\|f\|_0 + h^\infty\|f\|_0 \\ &\quad + h^\infty\|\text{Op}_h(\bar{\rho}(1 - \eta_1))c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_{-1,0} \\ &\quad + h^\infty\|(\mathcal{N}(\bar{\lambda}, -m)\text{Op}_h(1 - \eta_1) + \text{Op}_h(\bar{\rho}(1 - \eta_1)))c_0^{-1}\text{Op}_h(1 - \eta)^*f\|_0 \\ &\lesssim h\|f\|_0 + h^\infty\|f\|_0 \lesssim h\|f\|_0, \end{aligned}$$

where  $\|\cdot\|_{-1,0}$  denotes the semiclassical norm in  $H^{-1}(\Gamma)$ . In other words,

$$\mathcal{A}^* = O(h) : L^2(\Gamma) \rightarrow L^2(\Gamma),$$

and hence so is the operator  $\mathcal{A}$ . Clearly, the same analysis still holds when  $m \equiv 0$ , using (5.2) instead of (5.1).  $\square$

## 7. EIGENVALUE-FREE REGIONS

Let  $(u_1, u_2)$  be the solution to equation (1.9) and set  $f = u_1|_\Gamma = u_2|_\Gamma$ . Then we can express the restrictions of the normal derivative of  $u_1$  and  $u_2$  in terms of the corresponding Dirichlet-to-Neumann maps, that is,

$$h\partial_\nu u_1|_\Gamma = \mathcal{N}_1(\lambda, m)f, \quad h\partial_\nu u_2|_\Gamma = \mathcal{N}_2(\lambda)f.$$

Therefore,  $\lambda$  is a transmission eigenvalue if  $T(\lambda)f \equiv 0$ , where

$$T(\lambda) = c_1\mathcal{N}_1(\lambda, m) - c_2\mathcal{N}_2(\lambda).$$

We have to show that, for  $\lambda$  belonging to the eigenvalue-free regions of Theorems 1.1 and 1.2, if  $T(\lambda)f \equiv 0$ , then  $f \equiv 0$ . Without loss of generality we may suppose that  $\lambda \in \mathbb{C}^+$  (see Remark 1.3). We will first prove the following

**Lemma 7.1.** *If  $\lambda \in \mathbb{C}^+$  satisfies the condition*

$$(7.1) \quad \text{sign}(m) \text{Im } \lambda \leq 0,$$

then

$$(7.2) \quad \text{sign}(m) \text{Im } \langle c_1\mathcal{N}_1(\lambda, m)f, f \rangle_0 \leq 0.$$

Moreover, there exist constants  $C, \lambda_0 > 0$  such that for all  $\lambda \in \mathbb{C}^+$  satisfying

$$(7.3) \quad |\text{Im } \lambda| \leq C\tau_1(|\lambda|)^{-2}, \quad \text{Re } \lambda \geq \lambda_0,$$

$$(7.4) \quad |\text{Im } \lambda| \leq C\tau_2(|\lambda|)^{-1}, \quad \text{Re } \lambda \geq \lambda_0,$$

we have the estimate

$$(7.5) \quad |\text{Im } \langle c_1\mathcal{N}_1(\lambda, m)f, f \rangle_0| \lesssim |\text{Im } \lambda| |\lambda|^{\ell_2} \tau_1(|\lambda|)^4 \tau_2(|\lambda|)^2 \|f\|_{1,0}^2.$$

*Proof.* The Green formula applied to the second equation in (1.9) gives the identity

$$2\text{Im } \lambda \langle n_2 u_2, u_2 \rangle = \text{Im } \langle c_2 \mathcal{N}_2(\lambda) f, f \rangle_0 = \text{Im } \langle c_1 \mathcal{N}_1(\lambda, m) f, f \rangle_0.$$

Obviously, (7.1) implies (7.2). On the other hand, by (2.6) and (5.1) we have

$$\begin{aligned} \|u_2\| &\lesssim |\lambda|^{\ell_2/2} \tau_2(|\lambda|) (\|f\|_0 + \|\mathcal{N}_2(\lambda)f\|_0) \\ &\lesssim |\lambda|^{\ell_2/2} \tau_2(|\lambda|) (\|f\|_0 + \|\mathcal{N}_1(\lambda, m)f\|_0) \\ &\lesssim |\lambda|^{\ell_2/2} \tau_2(|\lambda|) \tau_1(|\lambda|)^2 \|f\|_{1,0}, \end{aligned}$$

which clearly implies (7.5).  $\square$

To prove Theorem 1.1 we apply Theorem 3.1 with  $u = u_1$  and  $v = 0$ . Note that the conditions (3.8) and (7.2) are equivalent. Therefore, the estimate (3.9) holds for  $\lambda$  belonging to the region (1.10) with a suitable choice of the constant  $C$ . Thus, for such  $\lambda$ , we get  $\|u_1\| = 0$ , which implies  $u_1 \equiv 0$ , and hence  $f \equiv 0$ , as desired.

The proof of Theorem 1.2 is much more complicated. We will need the next two lemmas. Let  $\chi \in C^\infty(T^*\Gamma)$  be of compact support such that  $1 - \chi$  is supported in the region  $\{r_0 \geq \sigma\}$ , where  $\sigma \gg 1$  is a constant to be fixed in the next lemma.

**Lemma 7.2.** *For a suitable choice of  $\sigma$  we have the estimate*

$$(7.6) \quad \|\text{Op}_h(1 - \chi)f\|_{1,0} \lesssim h\|f\|_0$$

for  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$  satisfying (7.3).

*Proof.* Choose a compactly supported function  $\eta \in C^\infty(T^*\Gamma)$  such that  $1 - \eta$  is supported in the region  $\{r_0 \geq \sigma\}$  and  $\chi = 1$  on  $\text{supp } \eta$ . Define  $\rho_j, \mathcal{H}_j, \mathcal{E}_j, j = 1, 2$ , by replacing in the definition of  $\rho, \mathcal{H}, \mathcal{E}$  in Sections 4 and 6 the functions  $c, n$  by  $c_j, n_j$ . Clearly, taking  $\sigma$  big enough we can arrange that the functions  $1 - \chi$  and  $1 - \eta$  are supported in both elliptic regions  $\mathcal{E}_1$  and  $\mathcal{E}_2$ . Since  $Tf = 0$ , we have the identity

$$\begin{aligned} & \text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \chi))f \\ & + \text{Op}_h(1 - \chi)\text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \eta))f - \text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \chi))f \\ & = \text{Op}_h(1 - \chi)\text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \eta))f \\ & = \text{Op}_h(1 - \chi)(T\text{Op}_h(1 - \eta) + \text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \eta)))f \\ & \quad + \text{Op}_h(1 - \chi)T\text{Op}_h(\eta)f. \end{aligned}$$

Since  $(c_1\rho_1 - c_2\rho_2)(1 - \chi)$  and  $(c_1\rho_1 - c_2\rho_2)(1 - \eta)$  belong to  $S^1(\Gamma)$ , the  $h - \Psi$ DO calculus give

$$\text{Op}_h(1 - \chi)\text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \eta)) - \text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \chi)) = O(h) : H^1(\Gamma) \rightarrow L^2(\Gamma).$$

Therefore, using Theorem 6.1 together with Lemma 6.2 we obtain

$$(7.7) \quad \|\text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \chi))f\|_0 \lesssim h\|f\|_{1,0}.$$

On the other hand, we have

$$(c_1\rho_1 + c_2\rho_2)(c_1\rho_1 - c_2\rho_2) = c_1^2\rho_1^2 - c_2^2\rho_2^2 = (c_1^2 - c_2^2)r_0 - z(c_1n_1 - c_2n_2).$$

Hence

$$|(c_1^2 - c_2^2)r_0 - z(c_1n_1 - c_2n_2)| \lesssim \langle \xi' \rangle |c_1\rho_1 - c_2\rho_2|.$$

On the other hand, in view of assumption (1.2) we have  $|c_1^2 - c_2^2| \geq C_0$  with some constant  $C_0 > 0$ . Therefore, taking  $\sigma$  big enough we can arrange that

$$|(c_1^2 - c_2^2)r_0 - z(c_1n_1 - c_2n_2)| \geq C_1\langle \xi' \rangle^2, \quad C_1 > 0,$$

on  $\text{supp}(1 - \eta)$ . Hence

$$(7.8) \quad |c_1\rho_1 - c_2\rho_2| \geq C_2\langle \xi' \rangle, \quad C_2 > 0,$$

on  $\text{supp}(1 - \eta)$ , which implies

$$(7.9) \quad \|\text{Op}_h(1 - \chi)f\|_{1,0} \lesssim \|\text{Op}_h((c_1\rho_1 - c_2\rho_2)(1 - \chi))f\|_0 + h\|f\|_0.$$

By (7.7) and (7.9).

$$\|\text{Op}_h(1 - \chi)f\|_{1,0} \lesssim h\|f\|_{1,0} \lesssim h\|f\|_0 + h\|\text{Op}_h(1 - \chi)f\|_{1,0}.$$

We now absorb the last term in the right-hand side of the above estimate and get (7.6).  $\square$

By (4.1), (7.5) and (7.6) we obtain

$$\|f\|_{1,0} \lesssim |\lambda|^{(\ell_2+1)/2} \tau_1(|\lambda|)^4 \tau_2(|\lambda|) |\text{Im } \lambda|^{1/2} \|f\|_{1,0} + h\|f\|_0.$$

Taking  $h$  small enough we absorb the last term and arrive at the estimate

$$(7.10) \quad \|f\|_{1,0} \leq C|\lambda|^{(\ell_2+1)/2} \tau_1(|\lambda|)^4 \tau_2(|\lambda|) |\text{Im } \lambda|^{1/2} \|f\|_{1,0},$$

which holds for all  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$  satisfying (7.3) and (7.4), where  $C > 0$  is a constant independent of  $\lambda$  and  $N$ . Hence, if

$$(7.11) \quad \lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N), \quad |\text{Im } \lambda| \leq (2C)^{-2} |\lambda|^{-(\ell_2+1)} \tau_1(|\lambda|)^{-8} \tau_2(|\lambda|)^{-2},$$

we can absorb the term in the right-hand side of (7.10) and conclude that  $\|f\|_{1,0} = 0$ , which implies  $f \equiv 0$ . In other words there are no transmission eigenvalues in the region (7.11), as desired.

In what follows we will assume the conditions (1.6) and (1.11) fulfilled and we will show that in this case the factor  $|\lambda|^{1/2}$  in the right-hand side of (7.10) can be removed. Clearly, the condition (1.6) implies  $\mathcal{H}_2 \subset \mathcal{H}_1$  and  $\mathcal{E}_1 \subset \mathcal{E}_2$ . Given a parameter  $0 < \varepsilon \ll 1$ , independent of  $\lambda$ , we can choose functions  $\chi_\varepsilon^-, \chi_\varepsilon^0, \chi_\varepsilon^+ \in C^\infty(T^*\Gamma)$  such that  $\chi_\varepsilon^- + \chi_\varepsilon^0 + \chi_\varepsilon^+ \equiv 1$ ,  $\text{supp } \chi_\varepsilon^- \subset \mathcal{H}_1$ ,  $\text{supp } \chi_\varepsilon^+ \subset \mathcal{E}_1$ ,  $\chi_\varepsilon^0 = 1$  in  $\{|r_0/n_{b,0} - 1| \leq \varepsilon\}$  and  $\chi_\varepsilon^0 = 0$  in  $T^*\Gamma \setminus \{|r_0/n_{b,0} - 1| \leq 2\varepsilon\}$ , where  $n_{b,0} = \frac{n_1}{c_1}|\Gamma$ . Clearly,  $\text{supp } \chi_\varepsilon^+ \subset \mathcal{E}_2$ . Taking  $\varepsilon$  small enough we can also arrange that  $\text{supp}(1 - \chi_\varepsilon^-) \subset \mathcal{E}_2$ . Using this we will prove the following

**Lemma 7.3.** *We have the estimate*

$$(7.12) \quad \|\text{Op}_h(1 - \chi_\varepsilon^-)f\|_{1,0} \lesssim \left( |\lambda|^{\ell_2/2} \tau_1(|\lambda|)^5 \tau_2(|\lambda|) |\text{Im } \lambda|^{1/2} + h^{1/2} \tau_1(|\lambda|)^3 + \varepsilon \right) \|f\|_{1,0}$$

for  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$  satisfying (7.3) and (7.4).

*Proof.* Observe that the condition (1.6) implies the inequality

$$\frac{c_2 n_2 - c_1 n_1}{c_2^2 - c_1^2} < \frac{n_1}{c_1}.$$

Therefore the inequality (7.8) holds on  $\mathcal{E}_1$ . This implies the estimate (7.9) with  $1 - \chi$  replaced by  $\chi_\varepsilon^+$ . On the other hand, since the function  $\chi_\varepsilon^+$  is supported in both elliptic regions, the estimate (6.3) holds with  $1 - \eta$  replaced by  $\chi_\varepsilon^+$ . Thus, in the same way as in the proof of Lemma 7.2, using (5.4) and (7.5) instead of (6.15), we conclude that

$$(7.13) \quad \|\text{Op}_h(\chi_\varepsilon^+)f\|_{1,0} \lesssim \left( |\lambda|^{\ell_2/2} \tau_1(|\lambda|)^5 \tau_2(|\lambda|) |\text{Im } \lambda|^{1/2} + h^{1/2} \tau_1(|\lambda|)^3 \right) \|f\|_{1,0}.$$

Choose a function  $\eta_\varepsilon \in C_0^\infty(T^*\Gamma)$  such that  $\eta_\varepsilon = 1$  in  $\{|r_0/n_{b,0} - 1| \leq 3\varepsilon\}$  and  $\eta_\varepsilon = 0$  in  $T^*\Gamma \setminus \{|r_0/n_{b,0} - 1| \leq 4\varepsilon\}$ . Clearly,  $\eta_\varepsilon = 1$  on  $\text{supp } \chi_\varepsilon^0$ . Moreover, taking  $\varepsilon$  small enough we can arrange that  $\text{supp } \eta_\varepsilon \subset \mathcal{E}_2$ . Since  $Tf = 0$ , we have the identity

$$\begin{aligned} & \text{Op}_h(\chi_\varepsilon^0 \rho_2) f + (\text{Op}_h(\chi_\varepsilon^0) \text{Op}_h(\eta_\varepsilon \rho_2) f - \text{Op}_h(\chi_\varepsilon^0 \rho_2) f) \\ &= \text{Op}_h(\chi_\varepsilon^0) \text{Op}_h(\eta_\varepsilon \rho_2) f = -\text{Op}_h(\chi_\varepsilon^0) \mathcal{N}_2(\lambda) \text{Op}_h(\eta_\varepsilon) f \\ & \quad + \text{Op}_h(\chi_\varepsilon^0) (\mathcal{N}_2(\lambda) \text{Op}_h(\eta_\varepsilon) f + \text{Op}_h(\eta_\varepsilon \rho_2) f) \\ &= \text{Op}_h(\chi_\varepsilon^0) \mathcal{N}_2(\lambda) \text{Op}_h(1 - \eta_\varepsilon) f + \text{Op}_h(\chi_\varepsilon^0) c_2^{-1} c_1 \mathcal{N}_1(\lambda, m) f \\ & \quad + \text{Op}_h(\chi_\varepsilon^0) (\mathcal{N}_2(\lambda) \text{Op}_h(\eta_\varepsilon) f + \text{Op}_h(\eta_\varepsilon \rho_2) f). \end{aligned}$$

The  $h - \Psi$ DO calculus give

$$\begin{aligned} & \text{Op}_h(\chi_\varepsilon^0) \text{Op}_h(\eta_\varepsilon \rho_2) - \text{Op}_h(\chi_\varepsilon^0 \rho_2) = O_\varepsilon(h) : L^2(\Gamma) \rightarrow L^2(\Gamma), \\ & \text{Op}_h(\chi_\varepsilon^0) c_2^{-1} c_1 - c_2^{-1} c_1 \text{Op}_h(\chi_\varepsilon^0) = O_\varepsilon(h) : L^2(\Gamma) \rightarrow L^2(\Gamma). \end{aligned}$$

Therefore, using the estimates (4.11), (5.1), (5.4), (6.3), (6.14) (with  $m \equiv 0$ ) and (7.5), we get

$$(7.14) \quad \|\text{Op}_h(\chi_\varepsilon^0 \rho_2) f\|_0 \lesssim \left( |\lambda|^{\ell_2/2} \tau_1(|\lambda|)^5 \tau_2(|\lambda|) |\text{Im } \lambda|^{1/2} + h^{1/2} \tau_1(|\lambda|)^3 + \varepsilon \right) \|f\|_{1,0}.$$

On the other hand, the condition (1.6) guarantees that  $|\rho_2| \geq C > 0$  on  $\text{supp } \chi_\varepsilon^0$ , provided  $\varepsilon$  is taken small enough. Hence

$$(7.15) \quad \|\text{Op}_h(\chi_\varepsilon^0) f\|_0 \lesssim \|\text{Op}_h(\chi_\varepsilon^0 \rho_2) f\|_0 + h \|f\|_0.$$

Clearly, the estimate (7.12) follows from (7.13), (7.14) and (7.15).  $\square$

By (4.5), (7.5) and (7.12) we obtain

$$\|f\|_{1,0} \lesssim \left( |\lambda|^{\ell_2/2} \tau_1(|\lambda|)^5 \tau_2(|\lambda|) |\text{Im } \lambda|^{1/2} + |\lambda|^{-1/2} \tau_1(|\lambda|)^3 + \varepsilon \right) \|f\|_{1,0}.$$

Taking  $|\lambda|^{-1}$  and  $\varepsilon$  small enough, in view of the assumption (1.11), we can absorb the last two terms to obtain

$$(7.16) \quad \|f\|_{1,0} \leq C|\lambda|^{\ell_2/2} \tau_1(|\lambda|)^5 \tau_2(|\lambda|) |\operatorname{Im} \lambda|^{1/2} \|f\|_{1,0}$$

which holds for all  $\lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N)$  satisfying (7.3) and (7.4), where  $C > 0$  is a constant independent of  $\lambda$  and  $N$ . Hence, if

$$(7.17) \quad \lambda \in \mathbb{C}^+ \setminus \mathcal{L}_N^+(C_N), \quad |\operatorname{Im} \lambda| \leq (2C)^{-2} |\lambda|^{-\ell_2} \tau_1(|\lambda|)^{-10} \tau_2(|\lambda|)^{-2},$$

we can absorb the term in the right-hand side of (7.16) and conclude that  $\|f\|_{1,0} = 0$ , which implies  $f \equiv 0$ . In other words there are no transmission eigenvalues in the region (7.17) in this case, which is the desired conclusion.

## REFERENCES

- [1] N. BURQ, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur*, Acta Math. **180** (1998), 1-29.
- [2] N. BURQ, *Smoothing effect for Schrödinger equations*, Duke Math. J. **123** (2004), 403-427.
- [3] F. CAKONI, D. COLTON AND H. HADDAR, *Transmission eigenvalues*, Notices of the AMS **68** (9) (2021), 1499-1510.
- [4] F. CARDOSO, G. POPOV AND G. VODEV, *Distribution of resonances and local energy decay in the transmission problem. II*, Math. Res. Lett. **6** (1999), 377-396.
- [5] J. FORNEROD AND H-M. NGUYEN, *The Weyl law of transmission eigenvalues and the completeness of generalized transmission eigenfunctions without complementing conditions*, SIAM J. Math. Anal. **55** (2023), no. 4, 3959-3999.
- [6] M. IKAWA, *Decay of solutions of the wave equation in the exterior of several convex bodies*, Annal. Inst. Fourier **38** (1988), 113-146.
- [7] E. LAKSHANOV AND B. VAINBERG, *Ellipticity in the interior transmission problem in anisotropic media*, SIAM J. Math. Anal. **44** (2012), no. 2, 1165-1174.
- [8] E. LAKSHANOV AND B. VAINBERG, *Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem*, Inverse Problems **29** (2013), 104003.
- [9] H-M. NGUYEN AND Q-H. NGUYEN, *Discreteness of interior transmission eigenvalues revisited*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Paper No. 51.
- [10] H-M. NGUYEN AND Q-H. NGUYEN, *The Weyl law of transmission eigenvalues and the completeness of generalized transmission eigenfunctions*, J. Funct. Anal. **281** (2021), 109146.
- [11] V. PETKOV AND G. VODEV, *Asymptotics of the number of the interior transmission eigenvalues*, J. Spectral Theory **7** (2017), 1-31.
- [12] L. ROBBIANO, *Spectral analysis of interior transmission eigenvalues*, Inverse Problems **29** (2013), 104001.
- [13] L. ROBBIANO, *Counting function for interior transmission eigenvalues*, Math. Control Relat. Fields **6** (2016), no. 1, 167-183.
- [14] J. SYLVESTER, *Discreteness of transmission eigenvalues via upper triangular compact operators*, SIAM J. Math. Anal. **44** (2012), no. 1, 341-354.
- [15] G. VODEV, *Transmission eigenvalue-free regions*, Comm. Math. Phys. **336** (2015), 1141-1166.
- [16] G. VODEV, *Transmission eigenvalue-free regions near the real axis*, Pure Appl. Anal. **6** (2) (2024), 611-632.

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