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LONG TIME BEHAVIOUR OF GENERALISED GRADIENT FLOWS VIA OCCUPATIONAL MEASURES

PIERMARCO CANNARSA, WEI CHENG, AND CRISTIAN MENDICO

ABSTRACT. This paper introduces new methods to study the long time behaviour of the generalised gradient flow associated with a solution of the critical equation for mechanical Hamiltonian system posed on the flat torus \mathbb{T}^d . For this analysis it is necessary to look at the critical set of u consisting of all the points on \mathbb{T}^d such that zero belongs to the super-differential of such a solution. Indeed, such a set turns out to be an attractor for the generalised gradient flow. Moreover, being the critical set the union of two subsets of rather different nature, namely the regular critical set and the singular set, we are interested in establishing whether the generalised gradient flow approaches the former or the latter as $t \rightarrow \infty$. One crucial tool of our analysis is provided by limiting occupational measures, a family of measures that are invariant under the generalized flow. Indeed, we show that by integrating the potential with respect to such measures, one can deduce whether the generalised gradient flow enters the singular set in finite time, or it approaches the regular critical set as time tends to infinity.

1. INTRODUCTION

It is well known that viscosity solutions of Hamilton-Jacobi equations on an open domain $\Omega \subset \mathbb{R}^d$

$$H(x, Du(x)) = 0 \quad (x \in \Omega)$$

may fail to be everywhere differentiable (see, for instance, [17]). The set of all the points of non-differentiability of a solution u , called the singular set of u and denoted by $\text{Sing}(u)$, has been the object of several studies. Indeed, understanding the role such a set plays in the underlying optimal control problem is of primary interest for both theory and applications.

Rectifiability results for singular sets were obtained in [26, 27], [24, 25], and [5] showing that $\text{Sing}(u)$ is a countably $(d - 1)$ -rectifiable set. Moreover, the propagation of singularities was addressed in several papers, see [18, 19, 6], [2, 3, 20, 15], [1], and [16, 11, 14]. It was later understood that global propagation of singularities has interesting topological applications, including homotopy equivalence results between different geometric objects (see, [4, 12, 13]).

In particular, for the so-called Hamilton-Jacobi equation of mechanical systems

$$(1.1) \quad \frac{1}{2}|Du(x)|^2 + V(x) = \alpha[0] \quad (x \in \mathbb{T}^d),$$

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where $\alpha[0]$ stands for Mañé's critical value ([23]), it is known that the singularities of any semiconcave solution u propagate along the semi-flow, \mathbf{x}_u , associated with the differential inclusion

$$(1.2) \quad \begin{cases} \dot{\mathbf{x}}(t) \in D^+u(\mathbf{x}(t)), & t \geq 0 \text{ a.e.} \\ \mathbf{x}(0) = x, \end{cases}$$

where D^+u stands for the Dini superdifferential of u (see section 2 for more details). We recall that a well-known consequence of the semiconcavity of u is that problem (1.2) has a unique solution defined on the whole half-line $[0, \infty)$. Moreover, the singularities of u propagate along \mathbf{x}_u in the sense that, for any $x \in \text{Sing}(u)$, $\mathbf{x}_u(t, x) \in \text{Sing}(u)$ for all $t \geq 0$ (see [4] and [1]).

The global dynamics of $\mathbf{x}_u(t, x)$ from the topological point of view was studied in [10] (see also [15]), for a more general Hamiltonian of the form

$$H(x, p) = \frac{1}{2} \langle A(x)(c + p), c + p \rangle + V(x),$$

where $c \in \mathbb{R}^n$, $A(x)$ is a positive definite $n \times n$ matrix smoothly depending on x , and V is of class $C^2(\mathbb{T}^d)$. In particular, an example in the context of monotone twist maps on \mathbb{T}^2 , showing that singularities can approach a Denjoy type Aubry set when the corresponding rotation number is irrational, was constructed.

In this paper, we study the long time behavior of \mathbf{x}_u from a measure-theoretic point of view. For this purpose, for any $x \in \mathbb{T}^d$ and $T > 0$ we consider the individual occupational measure μ_x^T defined by

$$\int_{\mathbb{T}^d} f(y) d\mu_x^T(y) = \frac{1}{T} \int_0^T f(\mathbf{x}_u(t, x)) dt \quad \forall f \in C(\mathbb{T}^d).$$

We begin our analysis by introducing the family $\mathcal{W}_u(x)$ of the *occupational measures* of \mathbf{x}_u , which consists of all the weak limits of μ_x^T along sequences $T_n \rightarrow \infty$. We recall that the notion of occupational measures was used for differential inclusions even when the uniqueness of the solution is not guaranteed, see, for instance, [7, 8]. Moreover, from the classical Aubry-Mather theory, we know that such a set reduces to the set of Mather measures when the characteristics is a minimizing curve, see [9].

The set $\mathcal{W}_u(x)$, together with the singular set of u

$$\text{Sing}(u) = \{x \in \mathbb{T}^d : Du(x) \text{ does not exist}\}$$

and the critical set

$$\text{Crit}(u) = \{x \in \mathbb{T}^d : 0 \in D^+u(x)\},$$

turns out to be an essential tool to study the long time behaviour of the generalised gradient flow \mathbf{x}_u . First of all, we observe that all limiting occupational measures have support in $\text{Crit}(u)$ (Proposition 3.4). Indeed, the Krylov-Bogoliubov argument shows that each measure $\mu \in \mathcal{W}_u(x)$ is invariant under $\mathbf{x}_u(\cdot, x)$.

Then, we proceed to show that the critical set of u is an attractor for the semi-flow $\mathbf{x}_u(t, x)$: more precisely, we prove that the probability for $\mathbf{x}_u(t, x)$ to be arbitrarily close to $\text{Crit}(u)$, with t picked at random in the interval $[0, T]$, tends to 1 as $T \rightarrow \infty$ (Theorem 3.9). In particular, when $\mathcal{W}_u(x)$ reduces to the singleton $\{\delta_y\}$, with δ_y denoting

the Dirac measure centered at y , we prove that y , besides being a critical point of u (Theorem 3.13), is the *approximate limit* of $\mathbf{x}_u(t, x)$ as $t \rightarrow \infty$ (Corollary 3.11).

It is worth noting that the results of this first part of the paper hold true for any semiconcave function on \mathbb{T}^d , not just for weak KAM solutions of (1.1).

Then, since $\text{Crit}(u)$ may contain both regular and singular points of u , we refine the analysis by giving a criterion to decide whether the flow approaches points of the former or latter family of critical points as $t \rightarrow \infty$. For any given point $x \in \mathbb{T}^d$ we show that the following alternative holds:

(1) either

$$\int_{\mathbb{T}^d} V(y) d\mu(y) = \alpha[0] \quad \forall \mu \in \mathcal{W}_u(x),$$

and, for any $\varepsilon > 0$, the probability for $\mathbf{x}_u(t, x)$ to be ε -close to the regular critical set of u , with t picked at random in $[0, T]$, tends to 1 as $T \rightarrow \infty$ (Theorem 4.7),

(2) or

$$\exists \mu \in \mathcal{W}_u(x) \quad \text{such that} \quad \int_{\mathbb{T}^d} V(y) d\mu(y) < \alpha[0],$$

and the generalised gradient flow enters the singular set of u in finite time (Theorem 4.9 and Corollary 4.10).

The results of this paper can be regarded as a first attempt to describe the asymptotic nature of generalised characteristics by looking at the structure of the set of the corresponding occupation measures. We trust that the implications of such a connection, which are mostly to be discovered, will prove to be useful in the future.

This paper is organized as follows. In Section 2, we recall some basic facts about Hamilton-Jacobi equations, viscosity solutions, semiconcave functions, and propagation of singularities. In Section 3, we introduce the notion of occupation measures of the generalized gradient flow \mathbf{x}_u and we prove that the critical set of u is an attractor for \mathbf{x}_u . In Section 4, we give conditions for \mathbf{x}_u to approach either the regular critical set of a solution or the singular set. In Section 5, we discuss the extension of our results to more general eikonal-type equations and open problems.

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2. PRELIMINARIES

We denote by \mathbb{T}^d the flat torus. Let $L \in C^2(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{R})$ be a Tonelli Lagrangian¹ and let H be the associated Hamiltonian, i.e.,

$$H(x, p) = \sup_{v \in \mathbb{R}^d} \{ \langle p, v \rangle - L(x, v) \} \quad (x \in \mathbb{T}^d, p \in \mathbb{R}^d).$$

It is well known (see [22]) that there exists a unique constant $\alpha[0] \in \mathbb{R}$ such that the Hamilton-Jacobi equation

$$(2.1) \quad H(x, Du(x)) = \alpha[0] \quad (x \in \mathbb{T}^d)$$

has a semiconcave viscosity solution. Such a constant is known as the *critical constant* of H and (2.1) is called the *critical equation*.

We recall below the definition of viscosity solutions and semiconcave functions.

Definition 2.1. *We say that a continuous function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ is a viscosity solution to (2.1) if for every $x \in \mathbb{T}^d$ and for every $\varphi \in C^1(\mathbb{T}^d)$*

- (i) *$u - \varphi$ has a maximum at x implies $H(x, D\varphi(x)) \leq \alpha[0]$ (in this case, we say that u is a viscosity subsolution);*
- (ii) *$u - \varphi$ has a minimum at x implies $H(x, D\varphi(x)) \geq \alpha[0]$ (in this case, we say that u is a viscosity supersolution).*

An equivalent definition of viscosity solutions can be given in terms of generalised Dini semi-differentials. We recall that, for any $x \in \mathbb{T}^d$, the closed convex sets

$$D^-u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}$$

$$D^+u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}$$

are called the *sub-differential* and *super-differential* of u at x , respectively. It is well known (see [21]) that a continuous function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ is a viscosity sub-solution of (2.1) if and only if

$$(2.2) \quad H(x, p) \leq \alpha[0] \quad \forall x \in \mathbb{T}^d, \forall p \in D^+u(x),$$

a viscosity super-solution of (2.1) if and only if

$$(2.3) \quad H(x, p) \geq \alpha[0] \quad \forall x \in \mathbb{T}^d, \forall p \in D^-u(x),$$

and a viscosity solution of (2.1) if and only if (2.2) and (2.3) are both satisfied.

Another relevant notion to the topics of this paper is that of semiconcave function.

Definition 2.2. *We say that a continuous function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ is semiconcave with linear modulus (or, briefly, semiconcave) if there exists a constant $C > 0$ such that*

$$\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq \frac{C}{2} \lambda(1 - \lambda)|x - y|^2$$

for any $x, y \in \mathbb{T}^d$ and any $\lambda \in [0, 1]$.

¹We omit recalling this well-known notion because this paper is focussed on the specific class of mechanical systems Lagrangeans.

Any semiconcave function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ is Lipschitz continuous and $D^+u(x)$ is nonempty for all $x \in \mathbb{T}^d$ (see, e.g., [17]).

For a semiconcave viscosity solution u to (2.1), the following sets will play a crucial role in this paper:

- (i) $\text{Sing}(u) = \{x \in \mathbb{T}^d : Du(x) \text{ does not exist}\}$, the *singular set* of u ,
- (ii) $\text{Crit}(u) = \{x \in \mathbb{T}^d : 0 \in D_p H(x, D^+u(x))\}$, the *critical set* of u ,
- (iii) $\text{Crit}^*(u) = \text{Crit}(u) \setminus \text{Sing}(u)$, the *regular critical set* of u .

Notice that, for any $x \in \mathbb{T}^d$,

$$(2.4) \quad x \in \text{Sing}(u) \iff \min_{p \in D^+u(x)} H(x, p) < \alpha[0].$$

Moreover, a critical point x belongs to $\text{Crit}^*(u)$ if and only if $H(x, 0) = \alpha[0]$.

Given a Tonelli Hamiltonian H and a semiconcave function u on \mathbb{T}^d , a Lipschitz curve $\mathbf{x} : [0, T] \rightarrow \mathbb{T}^d$ is called a *generalized characteristic* for the pair (H, u) if

$$(2.5) \quad \dot{\mathbf{x}}(t) \in \text{co } D_p H(\mathbf{x}(t), D^+u(\mathbf{x}(t))), \quad \text{a.e. } t \in [0, T],$$

where “co” denotes the convex hull.

We conclude this section with a list of notations, part of which will be introduced later on.

Table 1. Notation

χ_A	the characteristic function of a set $A \subset \mathbb{R}$ ($= 1$ on A , $= 0$ on $\mathbb{R} \setminus A$)
\mathbb{T}^d	the flat d -dimensional torus
ω_f	the modulus of continuity of $f \in C(\mathbb{T}^d)$
$\alpha[0]$	Mañé’s critical value
δ_x	the Dirac measure centred at x
$D^+u(x)$	the superdifferential of u at x
$p_0(x)$	the element of minimal norm of $D^+u(x)$
$\text{Sing}(u)$	the singular set of u
$\text{Crit}(u)$	the critical set of u
$\text{Crit}^*(u)$	the regular critical set of u
$\mathbf{x}_u(t, x)$	the generalised gradient flow of u
μ_x^T	the (individual) occupational measure of $\mathbf{x}_u(\cdot, x)$
$\mathcal{W}_u(x)$	the family of all limiting occupational measures of $\mathbf{x}_u(\cdot, x)$
$\tau(x)$	the critical time of x

3. GENERALISED GRADIENT FLOW AND OCCUPATIONAL MEASURES

3.1. Generalized gradient flow. Let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be a semiconcave function. It is well known that for $x \in \mathbb{T}^d$ the problem

$$(3.1) \quad \begin{cases} \dot{\mathbf{x}}(t, x) \in D^+u(\mathbf{x}(t, x)), & t \in [0, \infty) \text{ a.e.} \\ \mathbf{x}(0, x) = x. \end{cases}$$

has a unique solution. Such a solution, which is locally Lipschitz continuous in (t, x) on $[0, \infty) \times \Omega$, is called the *generalized gradient flow* of u and is denoted by $\mathbf{x}_u(t, x)$. In fact, $\mathbf{x}_u(t, x)$ is a semi-flow because uniqueness and regularity are guaranteed just for $t \geq 0$. We now give two properties of the generalized gradient flow of interest to this paper.

Proposition 3.1. *Let $x \in \mathbb{T}^d$. Then the following properties hold true.*

(a) $\mathbf{x}_u(\cdot, x)$ has the right derivative $\dot{\mathbf{x}}_u^+(t, x)$ for all $t \in [0, \infty)$ and

$$(3.2) \quad \dot{\mathbf{x}}_u^+(t, x) = p_0(\mathbf{x}_u(t, x)),$$

where

$$(3.3) \quad p_0(y) = \operatorname{argmin}_{p \in D^+u(y)} |p| \quad (y \in \mathbb{T}^d).$$

(b) The right derivative of $u(\mathbf{x}_u(\cdot, x))$ exists for all $t \in [0, \infty)$ and is given by

$$(3.4) \quad \frac{d^+}{dt} u(\mathbf{x}_u(t, x)) = |p_0(\mathbf{x}_u(t, x))|^2.$$

Proof. For point (a) we refer the reader to [4, Theorem 1], where the conclusion is obtained for solutions of eikonal type equations on bounded domains. Since the argument needed to derive (3.2) is of local nature and just uses the differential inclusion in (3.1), the same proof applies to the present case.

In order to prove (b) observe that, since u is Lipschitz, for all $t \geq 0$ we have that

$$\begin{aligned} \frac{d^+}{dt} u(\mathbf{x}_u(t, x)) &= \lim_{\lambda \downarrow 0} \frac{u(\mathbf{x}_u(t + \lambda, x)) - u(\mathbf{x}_u(t, x))}{\lambda} \\ &= \lim_{\lambda \downarrow 0} \frac{u(\mathbf{x}_u(t, x) + \lambda \dot{\mathbf{x}}_u^+(t, x)) - u(\mathbf{x}_u(t, x))}{\lambda}. \end{aligned}$$

Then, recalling that u is semiconcave on Ω , by the representation formula for the directional derivatives of a semiconcave function (see, e.g., [17]) we deduce that

$$\begin{aligned} \lim_{\lambda \downarrow 0} \frac{u(\mathbf{x}_u(t, x) + \lambda \dot{\mathbf{x}}_u^+(t, x)) - u(\mathbf{x}_u(t, x))}{\lambda} &= \min_{p \in D^+u(\mathbf{x}_u(t, x))} \langle p, \dot{\mathbf{x}}_u^+(t, x) \rangle \\ &= \min_{p \in D^+u(\mathbf{x}_u(t, x))} \langle p, p_0(\mathbf{x}_u(t, x)) \rangle = |p_0(\mathbf{x}_u(t, x))|^2, \end{aligned}$$

where the last two equalities follow from (3.2) and (3.3), respectively. \square

Remark 3.2. The following properties of $p_0 : \mathbb{T}^d \rightarrow \mathbb{R}^d$ are easily deduced.

- (a) $x \mapsto \operatorname{argmin}_{p \in D^+u(x)} |p| = p_0(x)$ is single-valued: indeed, $p_0(x)$ is the projection of 0 onto the non-empty convex compact set $D^+u(x)$.
- (b) The function $x \mapsto |p_0(x)|$ is lower semi-continuous and, in particular, Lebesgue measurable: this is a consequence of the upper semi-continuity of the set-valued map $x \mapsto D^+u(x)$ (see, for instance, [17, Proposition 3.3.4]).
- (c) The critical set of u is closed: this follows from (b) above.

Corollary 3.3. *Let $x \in \mathbb{T}^d$. Then:*

- (i) the function $t \mapsto u(\mathbf{x}_u(\cdot, x))$ is nondecreasing on $[0, \infty)$,

(ii) there exists a sequence of positive real numbers $t_k \rightarrow \infty$ such that

$$p_0(\mathbf{x}_u(t_k, x)) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Proof. Point (i) is a direct consequence of (3.4). As for (ii), by integrating (3.4) we obtain

$$\int_0^\infty |p_0(\mathbf{x}_u(t, x))|^2 dt \leq 2\|u\|_\infty.$$

The conclusion follows. \square

3.2. Occupational measures. Let us denote by $\mathcal{P}(\mathbb{T}^d)$ the family of all probability measures on \mathbb{T}^d . We recall that a sequence $\{\mu_k\}_{k \in \mathbb{N}} \subset \mathcal{P}(\mathbb{T}^d)$ *weakly converges* to a measure $\mu \in \mathcal{P}(\mathbb{T}^d)$ (or, $\mu_k \rightharpoonup \mu$) if

$$\lim_{k \rightarrow \infty} \int_{\mathbb{T}^d} f(x) d\mu_k(x) = \int_{\mathbb{T}^d} f(x) d\mu(x) \quad \forall f \in C(\mathbb{T}^d).$$

Let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be a semiconcave function and let \mathbf{x}_u be the semi-flow associated with (3.1). We recall that a measure $\mu \in \mathcal{P}(\mathbb{T}^d)$ is *invariant under \mathbf{x}_u* (or, \mathbf{x}_u -invariant) if

$$\int_{\mathbb{T}^d} f(\mathbf{x}_u(t, x)) d\mu(x) = \int_{\mathbb{T}^d} f(x) d\mu(x) \quad \forall t \geq 0, \forall f \in C(\mathbb{T}^d).$$

We now give a necessary and sufficient condition for invariance under \mathbf{x}_u .

Proposition 3.4. *A measure $\mu \in \mathcal{P}(\mathbb{T}^d)$ is invariant under $\mathbf{x}_u(t, \cdot)$ if and only if*

$$(3.5) \quad \text{spt}(\mu) \subset \text{Crit}(u).$$

Proof. Let $\mu \in \mathcal{P}(\mathbb{T}^d)$ satisfy (3.5). Then, for any $f \in C(\mathbb{T}^d)$ and $t \geq 0$ we have that

$$\int_{\mathbb{T}^d} f(\mathbf{x}_u(t, x)) d\mu(x) = \int_{\text{Crit}(u)} f(\mathbf{x}_u(t, x)) d\mu(x)$$

Now, observe that $\mathbf{x}_u(t, x) \equiv x$ for any $x \in \text{Crit}(u)$ by uniqueness. So,

$$\int_{\mathbb{T}^d} f(\mathbf{x}_u(t, x)) d\mu(x) = \int_{\text{Crit}(u)} f(x) d\mu(x) = \int_{\mathbb{T}^d} f(x) d\mu(x).$$

Thus, μ is \mathbf{x}_u -invariant.

Conversely, let $\mu \in \mathcal{P}(\mathbb{T}^d)$ be invariant under \mathbf{x}_u . Then, (3.4) ensures that

$$0 = \frac{d}{dt} \int_{\mathbb{T}^d} u(\mathbf{x}_u(t, x)) d\mu(x) = \int_{\mathbb{T}^d} |p_0(\mathbf{x}_u(t, x))|^2 d\mu(x) \quad (t \geq 0 \text{ a.e.})$$

Thus, for a.e. $t \geq 0$ we have that $p_0(\mathbf{x}_u(t, \cdot)) = 0$ on the support of μ . In fact, owing to Remark 3.2, we have that $p_0(\mathbf{x}_u(t, \cdot))$ vanishes on the support of μ for all $t \geq 0$. In particular, $p_0(x) \equiv 0$ on the support of μ and $\text{spt}(\mu) \subset \text{Crit}(u)$. \square

For any $x \in \mathbb{T}^d$ and any $T > 0$ the measure $\mu_x^T \in \mathcal{P}(\mathbb{T}^d)$, defined by

$$\int_{\mathbb{T}^d} f(y) d\mu_x^T(y) = \frac{1}{T} \int_0^T f(\mathbf{x}_u(t, x)) dt \quad \forall f \in C(\mathbb{T}^d),$$

is called the *individual occupational measure of $\mathbf{x}_u(\cdot, x)$ on $[0, T]$* .

Definition 3.5. We say that a measure $\mu \in \mathcal{P}(\mathbb{T}^d)$ is an occupational measure of \mathbf{x}_u if there exists a sequence $\{T_k\}_{k \in \mathbb{N}}$, with $T_k \uparrow \infty$, such that

$$(3.6) \quad \mu_x^{T_k} \rightharpoonup \mu \quad (k \rightarrow \infty).$$

We denote by $\mathcal{W}_u(x)$ the family of all occupational measures of \mathbf{x}_u .

By repeating the proof of the Krylov-Bogoliubov theorem, we obtain the following.

Proposition 3.6. Let u be a semiconcave function on \mathbb{T}^d . Then $\mathcal{W}_u(x) \neq \emptyset$ for any $x \in \mathbb{T}^d$. Moreover, any $\mu \in \mathcal{W}_u(x)$ is invariant under \mathbf{x}_u .

Proof. Let $x \in \mathbb{T}^d$ and take any sequence $\{T_k\}_{k \in \mathbb{N}}$, with $T_k \uparrow \infty$. Then Prokhorov's theorem ensures the existence of a subsequence $\{T_{k_j}\}_{j \in \mathbb{N}}$ and a measure $\mu \in \mathcal{P}(\mathbb{T}^d)$ such that

$$(3.7) \quad \mu_x^{T_{k_j}} \rightharpoonup \mu \quad \text{as } j \rightarrow \infty.$$

Next, in order to prove that μ is invariant under \mathbf{x}_u , fix any $f \in C(\mathbb{T}^d)$ and observe that, for any $t \geq 0$,

$$(3.8) \quad \int_{\mathbb{T}^d} [f(\mathbf{x}_u(t, y)) - f(y)] d\mu(y) = \int_{\mathbb{T}^d} f(\mathbf{x}_u(t, y)) (d\mu(y) - d\mu_x^{T_{k_j}}(y)) \\ + \int_{\mathbb{T}^d} [f(\mathbf{x}_u(t, y)) - f(y)] d\mu_x^{T_{k_j}}(y) + \int_{\mathbb{T}^d} f(y) (d\mu_x^{T_{k_j}}(y) - d\mu(y))$$

Now, in view of (3.7) we have that both

$$\int_{\mathbb{T}^d} f(\mathbf{x}_u(t, y)) (d\mu(y) - d\mu_x^{T_{k_j}}(y)) \quad \text{and} \quad \int_{\mathbb{T}^d} f(y) (d\mu_x^{T_{k_j}}(y) - d\mu(y))$$

converge to zero as $j \rightarrow \infty$. Moreover, the semigroup property of the flow yields

$$\int_{\mathbb{T}^d} [f(\mathbf{x}_u(t, y)) - f(y)] d\mu_x^{T_{k_j}}(y) = \frac{1}{T_{k_j}} \int_0^{T_{k_j}} [f(\mathbf{x}_u(t+s, x)) - f(\mathbf{x}_u(s, x))] ds \\ = \frac{1}{T_{k_j}} \int_t^{T_{k_j}+t} f(\mathbf{x}_u(s, x)) ds - \frac{1}{T_{k_j}} \int_0^{T_{k_j}} f(\mathbf{x}_u(s, x)) ds \\ = \frac{1}{T_{k_j}} \int_{T_{k_j}}^{T_{k_j}+t} f(\mathbf{x}_u(s, x)) ds - \frac{1}{T_{k_j}} \int_0^t f(\mathbf{x}_u(s, x)) ds.$$

Therefore,

$$\left| \int_{\mathbb{T}^d} [f(\mathbf{x}_u(t, y)) - f(y)] d\mu_x^{T_{k_j}}(y) \right| \leq \frac{2t \|f\|_\infty}{T_{k_j}} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

In view of (3.8), this shows that μ is invariant under \mathbf{x}_u . \square

In general, $\mathcal{W}_u(x)$ may well consist of more than one element, but the integral of u with respect to all occupational measures of $\mathbf{x}_u(\cdot, x)$ is the same.

Proposition 3.7. *Let u be a semiconcave function on \mathbb{T}^d . Then for any $x \in \mathbb{T}^d$ and any $\mu, \nu \in \mathcal{W}_u(x)$ we have that*

$$\int_{\mathbb{T}^d} u(y) d\mu(y) = \int_{\mathbb{T}^d} u(y) d\nu(y).$$

Proof. Let $\mu_x^{T_k} \rightharpoonup \mu$ and $\mu_x^{S_k} \rightharpoonup \nu$. In view of Corollary 3.3 (a) we have that for all $t > 0$

$$\frac{d}{dt} \left(\frac{1}{t} \int_0^t u(\mathbf{x}_u(s, x)) ds \right) = \frac{1}{t} \left(u(\mathbf{x}_u(t, x)) - \frac{1}{t} \int_0^t u(\mathbf{x}_u(s, x)) ds \right) \geq 0.$$

This yields the existence of the limit

$$\int_{\mathbb{T}^d} u(y) d\mu(y) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(\mathbf{x}_u(s, x)) ds = \int_{\mathbb{T}^d} u(y) d\nu(y)$$

which in turn gives the conclusion. \square

Now, we show that \mathcal{W}_u remains constant along the generalised gradient flow.

Theorem 3.8. *Let u be a semiconcave function on \mathbb{T}^d . Then for any $x \in \mathbb{T}^d$ we have that*

$$(3.9) \quad \mathcal{W}_u(x) = \mathcal{W}_u(\mathbf{x}_u(t, x)) \quad \forall t \geq 0.$$

Proof. Fix $t \geq 0$. Let $\mu \in \mathcal{W}_u(x)$ and let $T_k \uparrow \infty$ be such that $\mu_x^{T_k} \rightharpoonup \mu$ as $k \rightarrow \infty$. For all k large enough, so that $T_k > t$, let us set $S_k = T_k - t$. Then, by the semigroup property of the flow we have that, for all $f \in C(\mathbb{T}^d)$,

$$\begin{aligned} \int_{\mathbb{T}^d} f(y) d\mu_{\mathbf{x}_u(t, x)}^{S_k}(y) &= \frac{1}{S_k} \int_0^{S_k} f(\mathbf{x}_u(s + t, x)) ds \\ &= \frac{1}{S_k} \int_t^{T_k} f(\mathbf{x}_u(s, x)) ds \\ &= \frac{1}{T_k - t} \int_0^{T_k} f(\mathbf{x}_u(s, x)) ds - \frac{1}{S_k} \int_0^t f(\mathbf{x}_u(s, x)) ds \\ &= \frac{T_k}{T_k - t} \int_{\mathbb{T}^d} f(y) d\mu_x^{T_k}(y) - \frac{1}{S_k} \int_0^t f(\mathbf{x}_u(s, x)) ds \xrightarrow{k \rightarrow \infty} \int_{\mathbb{T}^d} f(y) d\mu(y). \end{aligned}$$

Therefore, $\mu \in \mathcal{W}_u(\mathbf{x}_u(t, x))$ and so $\mathcal{W}_u(x) \subset \mathcal{W}_u(\mathbf{x}_u(t, x))$. Since the reverse inclusion can be proved by a similar argument, the proof is complete. \square

Next, we will use occupational measures to prove that the critical set of u is an attractor for the generalised gradient flow with probability one. A finer analysis of the asymptotic behaviour of \mathbf{x}_u , valid only for solutions of equation (4.2), will be presented in Section 4.

Theorem 3.9. *Let u be a semiconcave function on \mathbb{T}^d and let $x \in \mathbb{T}^d$. Then for any $\varepsilon > 0$*

$$(3.10) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{L}^1(\{t \in [0, T] : d_{\text{Crit}(u)}(\mathbf{x}_u(t, x)) \geq \varepsilon\}) = 0.$$

Proof. Let us argue by contradiction assuming that there exist numbers $\varepsilon, \delta > 0$ and a sequence $T_k \uparrow \infty$ such that

$$\frac{1}{T_k} \mathcal{L}^1(\{t \in [0, T_k] : d_{\text{Crit}(u)}(\mathbf{x}_u(t, x)) \geq \varepsilon\}) \geq \delta \quad \forall k \in \mathbb{N}.$$

Without loss of generality we can suppose that $\mu_x^{T_k} \rightharpoonup \mu$ as $k \rightarrow \infty$ for some $\mu \in \mathcal{W}_u(x)$. Therefore, for all $k \in \mathbb{N}$ we have that

$$(3.11) \quad \int_{\mathbb{T}^d} d_{\text{Crit}(u)}(y) d\mu_x^{T_k}(y) = \frac{1}{T_k} \int_0^{T_k} d_{\text{Crit}(u)}(\mathbf{x}_u(t, x)) dt \\ \geq \frac{\varepsilon}{T_k} \mathcal{L}^1(\{t \in [0, T_k] : d_{\text{Crit}(u)}(\mathbf{x}_u(t, x)) \geq \varepsilon\}) \geq \varepsilon \delta.$$

On the other hand, since μ has support in the closed set $\text{Crit}(u)$ by Proposition 3.4,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{T}^d} d_{\text{Crit}(u)}(y) d\mu_x^{T_k}(y) = \int_{\mathbb{T}^d} d_{\text{Crit}(u)}(y) d\mu(y) = 0,$$

in contrast with (3.11). The conclusion follows. \square

Finally, we investigate the interesting case when an occupational measure turns out to be a Dirac measure.

Theorem 3.10. *Let u be a semiconcave function on \mathbb{T}^d . Let $x, \bar{x} \in \mathbb{T}^d$ and let $\{T_k\}_{k \in \mathbb{N}}$ be a real sequence, with $T_k \uparrow \infty$. The following properties are equivalent:*

- (a) $\mu_x^{T_k} \rightharpoonup \delta_{\bar{x}}$ as $k \rightarrow \infty$,
- (b) for any $\varepsilon > 0$

$$(3.12) \quad \lim_{k \rightarrow \infty} \frac{1}{T_k} \mathcal{L}^1(\{t \in [0, T_k] : |\mathbf{x}_u(t, x) - \bar{x}| \geq \varepsilon\}) = 0.$$

Proof. For any $\varepsilon > 0$ and any $T > 0$ let us set

$$F_\varepsilon(T) = \{t \in [0, T] : |\mathbf{x}_u(t, x) - \bar{x}| \geq \varepsilon\} \quad \text{and} \quad G_\varepsilon(T) = [0, T] \setminus F_\varepsilon(T).$$

$(a) \Rightarrow (b)$ Fix any $\varepsilon > 0$. Then we have that

$$\frac{1}{T_k} \int_0^{T_k} |\mathbf{x}_u(t, x) - \bar{x}| dt \geq \frac{\varepsilon}{T_k} \mathcal{L}^1(F_\varepsilon(T_k)).$$

Since the left-hand side converges to $\int_{\mathbb{T}^d} |y - \bar{x}| d\delta_{\bar{x}}(y) = 0$ as $k \rightarrow \infty$, (3.12) follows.

$(b) \Rightarrow (a)$ Let $f \in C(\mathbb{T}^d)$. Then for any $\varepsilon > 0$

$$\frac{1}{T_k} \left| \int_0^{T_k} (f(\mathbf{x}_u(t, x)) - f(\bar{x})) dt \right| \\ \leq \frac{1}{T_k} \int_0^{T_k} |f(\mathbf{x}_u(t, x)) - f(\bar{x})| (\chi_{F_\varepsilon(T_k)}(t) + \chi_{G_\varepsilon(T_k)}(t)) dt \\ \leq \frac{2\|f\|_\infty}{T_k} \mathcal{L}^1(F_\varepsilon(T_k)) + \omega_f(\varepsilon),$$

where $\omega_f(\varepsilon) = \sup_{|y-z| \leq \varepsilon} |f(y) - f(z)|$. Therefore, owing to (3.12),

$$\limsup_{k \rightarrow \infty} \frac{1}{T_k} \left| \int_0^{T_k} (f(\mathbf{x}_u(t, x)) - f(\bar{x})) dt \right| \leq \omega_f(\varepsilon).$$

Since ε is arbitrary and $\omega_f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, the conclusion follows. \square

We now characterize the case when $\mathcal{W}_u(s)$ is atomic, that is $\mathcal{W}_u(x) = \{\delta_{\bar{x}}\}$ for some \bar{x} .

Corollary 3.11. *Let $x, \bar{x} \in \mathbb{T}^d$. The following properties are equivalent:*

- (a) $\mathcal{W}_u(x) = \{\delta_{\bar{x}}\}$,
- (b) $\mu_x^T \rightharpoonup \delta_{\bar{x}}$ as $T \rightarrow \infty$,
- (c) for any $\varepsilon > 0$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{L}^1(\{t \in [0, T] : |\mathbf{x}_u(t, x) - \bar{x}| \geq \varepsilon\}) = 0.$$

Proof. The equivalence between (a) and (b) follows from the definition of $\mathcal{W}_u(x)$. The equivalence between (b) and (c) is a consequence of Theorem 3.10. \square

Remark 3.12. Notice that property (c) above can be expressed saying that \bar{x} is the *approximate limit* of $\mathbf{x}_u(t, x)$ as $t \rightarrow \infty$.

Theorem 3.9 suggests that a point $\bar{x} \in \mathbb{T}^d$ satisfying any of the properties in Corollary 3.11 should be critical for u . This is the object of our next result.

Theorem 3.13. *Let u be a semiconcave function on \mathbb{T}^d and let $x, \bar{x} \in \mathbb{T}^d$. If $\mathcal{W}_u(x) = \{\delta_{\bar{x}}\}$, then $\bar{x} \in \text{Crit}(u)$.*

Proof. Fix any $\varepsilon > 0$. Since $\mathcal{W}_u(x) = \{\delta_{\bar{x}}\}$, Corollary 3.11 ensures that

$$(3.13) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \mathcal{L}^1(G_\varepsilon(t)) = 1 \quad \text{where} \quad G_\varepsilon(t) = \{s \in [0, t] : |\mathbf{x}_u(s, x) - \bar{x}| < \varepsilon\}.$$

Consequently, there exists $t_\varepsilon \geq 1/\varepsilon$ such that $|\mathbf{x}_u(t_\varepsilon, x) - \bar{x}| < \varepsilon$. In fact, $t_\varepsilon \in G_\varepsilon(t)$ for all $t \geq t_\varepsilon$. Moreover, by (3.4), the semigroup property of the flow, and Fubini's theorem for any $T > 0$ we have that

$$\begin{aligned} \frac{1}{T} \int_0^T \left(u(\mathbf{x}_u(t, \mathbf{x}_u(t_\varepsilon, x))) - u(\mathbf{x}_u(t_\varepsilon, x)) \right) dt &= \frac{1}{T} \int_0^T dt \int_0^t |p_0(\mathbf{x}_u(s + t_\varepsilon, x))|^2 ds \\ &= \frac{1}{T} \int_0^T (T - s) |p_0(\mathbf{x}_u(s + t_\varepsilon, x))|^2 ds. \end{aligned}$$

Since $\mathcal{W}_u(x) = \mathcal{W}_u(\mathbf{x}_u(t_\varepsilon, x))$ by Theorem 3.8, our hypothesis forces $\mu_{\mathbf{x}_u(t_\varepsilon, x)}^T \rightharpoonup \delta_{\bar{x}}$ as $T \rightarrow \infty$. So,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (T - s) |p_0(\mathbf{x}_u(s + t_\varepsilon, x))|^2 ds = u(\bar{x}) - u(\mathbf{x}_u(t_\varepsilon, x)) \leq \omega_u(\varepsilon).$$

Therefore, there exists $T_\varepsilon > 0$ such that, for all $T \geq T_\varepsilon$

$$\frac{1}{T} \int_0^T (T - s) |p_0(\mathbf{x}_u(s + t_\varepsilon, x))|^2 ds \leq \omega_u(\varepsilon) + \varepsilon.$$

Now, Lemma A.1 below applied to $f(s) := |p_0(\mathbf{x}_u(s + t_\varepsilon, x))|^2$ yields

$$(3.14) \quad \mathcal{L}^1(K_\varepsilon(T)) \geq \frac{T}{2} - 1 \quad \text{for all} \quad T \geq T_\varepsilon,$$

where

$$K_\varepsilon(T) = \{s \in [0, T] : |p_0(\mathbf{x}_u(s + t_\varepsilon, x))|^2 \leq \omega_u(\varepsilon) + \varepsilon\}.$$

Next, we claim that there exists $S_\varepsilon \geq \max\{t_\varepsilon, T_\varepsilon\}$ such that

$$(3.15) \quad K_\varepsilon(T) \cap (G_\varepsilon(T) - t_\varepsilon) \neq \emptyset \quad \text{for all } T \geq S_\varepsilon$$

where we set $G_\varepsilon(T) - t_\varepsilon = \{s - t_\varepsilon : s \in G_\varepsilon(T)\}$.

Indeed, since

$$\mathcal{L}^1((G_\varepsilon(T) - t_\varepsilon) \cap [0, T]) \geq \mathcal{L}^1(G_\varepsilon(T)) - t_\varepsilon,$$

we have that $K_\varepsilon(T)$ and $(G_\varepsilon(T) - t_\varepsilon) \cap [0, T]$ are subsets of $[0, T]$ satisfying

$$\mathcal{L}^1(K_\varepsilon(T)) + \mathcal{L}^1((G_\varepsilon(T) - t_\varepsilon) \cap [0, T]) \geq \frac{T}{2} + \mathcal{L}^1(G_\varepsilon(T)) - (1 + t_\varepsilon) \quad \forall T \geq T_\varepsilon.$$

Hence, our claim follows noting that, in view of (3.13), the right-hand side of the above inequality can be made strictly larger than T for T large enough.

Finally, appealing to (3.15) with $\varepsilon = 1/k$, one can construct a sequence

$$T_k := s_{1/k} + t_{1/k} \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

with

$$\mathbf{x}_u(T_k, x) \rightarrow \bar{x} \quad \text{and} \quad p_0(\mathbf{x}_u(T_k, x)) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

It follows that $\bar{x} \in \text{Crit}(u)$. □

From the above result we can easily derive a characterisation of the critical set.

Corollary 3.14. *Let u be a semiconcave function on \mathbb{T}^d and let $x \in \mathbb{T}^d$. Then*

(a) $x \in \text{Crit}(u)$ if and only if $\mathcal{W}_u(x) = \{\delta_x\}$.

Moreover, for any $t \geq 0$,

(b) $\mathbf{x}_u(t, x) \in \text{Crit}(u)$ if and only if $\mathcal{W}_u(x) = \{\delta_{\mathbf{x}_u(t, x)}\}$.

Proof. (a): the fact that $\mathcal{W}_u(x) = \{\delta_x\}$ for all $x \in \text{Crit}(u)$ is obvious since $\mathbf{x}_u(t, x) \equiv x$ in this case. The converse is an immediate consequence of Theorem 3.13.

(b): the conclusion follows by combining (a) and Theorem 3.8. □

4. GRADIENT FLOWS OF SOLUTIONS TO HAMILTON-JACOBI EQUATIONS

Given a function $V \in C^2(\mathbb{T}^d)$, consider the Lagrangean

$$L(x, v) = \frac{1}{2}|v|^2 - V(x) \quad (x \in \mathbb{T}^d, v \in \mathbb{R}^d)$$

with the associated Hamiltonian

$$H(x, p) = \frac{1}{2}|p|^2 + V(x) \quad (x \in \mathbb{T}^d, p \in \mathbb{R}^d).$$

We recall that $\alpha[0]$ denotes the critical constant of H .

The following lemma is well known. We give the proof for the reader's convenience.

Lemma 4.1. *The critical constant is given by*

$$(4.1) \quad \alpha[0] = \max_{x \in \mathbb{T}^d} V(x).$$

Moreover, for any semiconcave solution $u : \mathbb{T}^d \rightarrow \mathbb{R}$ of

$$(4.2) \quad \frac{1}{2}|Du(x)|^2 + V(x) = \alpha[0] \quad (x \in \mathbb{T}^d).$$

we have that

$$(4.3) \quad \operatorname{argmax}_{x \in \mathbb{T}^d} V(x) = \operatorname{Crit}^*(u)$$

where $\operatorname{Crit}^*(u) = \operatorname{Crit}(u) \setminus \operatorname{Sing}(u)$ denotes the regular critical set of u .

Proof. Let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be a semiconcave solution of (4.2). For any $x \in \mathbb{T}^d$ and any $p \in D^+u(x)$ we have that

$$\frac{1}{2}|p|^2 + V(x) \leq \alpha[0],$$

which implies that $\max_{\mathbb{T}^d} V \leq \alpha[0]$. Let now $\bar{x} \in \mathbb{T}^d$ be a minimum point of u . Then $0 \in D^-u(\bar{x})$, so that $V(\bar{x}) \geq \alpha[0]$ in view of Definition 2.1. Identity (4.1) follows².

Next, let $\bar{x} \in \mathbb{T}^d$ be a maximum point of V . Then, by (4.1) we have that

$$\frac{1}{2}|p|^2 \leq \alpha[0] - V(\bar{x}) = 0, \quad \forall p \in D^+u(\bar{x}).$$

Hence, $D^+u(\bar{x}) = \{0\}$. So, u is differentiable at \bar{x} and $Du(\bar{x}) = 0$. Thus,

$$\operatorname{argmax}_{x \in \mathbb{T}^d} V(x) \subset \operatorname{Crit}^*(u).$$

Conversely, if $\bar{x} \in \operatorname{Crit}^*(u)$, then

$$\alpha[0] = \frac{1}{2}|Du(\bar{x})|^2 + V(\bar{x}) = V(\bar{x}).$$

So, \bar{x} is a maximum point of V . □

Remark 4.2. In the terminology of weak KAM theory, the set in (4.3) is called the *projected Aubry set*.

For solutions of (4.2) we can develop a more detailed analysis of the long time behaviour of the associated generalized gradient flow \mathbf{x}_u than the one given by Theorem 3.9. In particular we establish conditions for \mathbf{x}_u to approach either the regular critical set or the singular set of u .

We begin by quoting a result by Albano [1] which ensures that $\operatorname{Sing}(u)$ is \mathbf{x}_u -invariant.

Theorem 4.3. *Let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be a semiconcave solution of (4.2) and let $\mathbf{x}_u(t_0, x) \in \operatorname{Sing}(u)$ for some $(t_0, x) \in [0, \infty) \times \mathbb{T}^d$. Then $\mathbf{x}_u(t, x) \in \operatorname{Sing}(u)$ for all $t \in [t_0, \infty)$.*

We now recall the definition of critical time for the generalised gradient flow of a semiconcave function $u : \mathbb{T}^d \rightarrow \mathbb{R}$.

Definition 4.4. For any $x \in \mathbb{T}^d$ we define the *critical time* of x as

$$\tau(x) = \inf \{t \geq 0 : \mathbf{x}_u(t, x) \in \operatorname{Crit}(u)\}.$$

Observe that $\tau(x) \in [0, \infty]$. The following proposition establishes a dichotomy that applies to all the points with a finite critical time.

Proposition 4.5. *Let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be a semiconcave solution of (4.2) and let $x \in \mathbb{T}^d$ be such that $\tau(x) < \infty$. Then the following holds true:*

²Moreover, we have that $\operatorname{argmin}_{\mathbb{T}^d} u \subset \operatorname{argmax}_{\mathbb{T}^d} V$

- (a) $\mathbf{x}_u(\tau(x), x) \in \text{Crit}(u)$,
- (b) $V(\mathbf{x}_u(\tau(x), x)) = \alpha[0]$ if and only if $\mathbf{x}_u(\tau(x), x) \in \text{Crit}^*(u)$,
- (c) $V(\mathbf{x}_u(\tau(x), x)) < \alpha[0]$ if and only if $\mathbf{x}_u(\tau(x), x) \in \text{Sing}(u)$.

Proof. Point (a) follows from the fact that $\text{Crit}(u)$ is closed. Point (b) is a direct consequence of (4.3). In order to prove (c), observe that, in view of (a) and (b),

$$V(\mathbf{x}_u(\tau(x), x)) < \alpha[0] \iff \mathbf{x}_u(\tau(x), x) \in \text{Crit}(u) \setminus \text{Crit}^*(u). \quad \square$$

Remark 4.6. Notice that, if $V(\mathbf{x}_u(\tau(x), x)) = \alpha[0]$, then u is differentiable at $\mathbf{x}_u(t, x)$ for all $t \geq 0$. Indeed, $\mathbf{x}_u(t, x) = \mathbf{x}_u(\tau(x), x) \in \text{Crit}^*(u)$ for all $t \geq \tau(x)$ by (b) above. Moreover, there can be no time $t_0 \in [0, \tau(x))$ such that $\mathbf{x}_u(t_0, x) \in \text{Sing}(u)$, for otherwise the semi-flow would stay in the singular set for all $t \geq t_0$ by global propagation of singularities (Theorem 4.3).

We now proceed to study the set of points with an infinite critical time. The first problem to face is to give a meaning to conditions

$$V(\mathbf{x}_u(\tau(x), x)) = \alpha[0] \quad \text{or} \quad V(\mathbf{x}_u(\tau(x), x)) < \alpha[0]$$

when $\tau(x) = \infty$. Since $\mathcal{W}_u(x) = \{\delta_{\mathbf{x}_u(\tau(x), x)}\}$ when $\tau(x) < \infty$ and

$$\int_{\mathbb{T}^d} V(y) d\delta_{\mathbf{x}_u(\tau(x), x)}(y) = V(\mathbf{x}_u(\tau(x), x)),$$

we guess that occupational measures may be useful for this purpose. Let us set

$$(4.4) \quad M(V) = \underset{x \in \mathbb{T}^d}{\text{argmax}} V(x).$$

Theorem 4.7. *Let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be a semiconcave solution of (4.2) and let $x \in \mathbb{T}^d$ be such that $\tau(x) = \infty$. The following properties are equivalent:*

- (a) for any $\mu \in \mathcal{W}_u(x)$ we have that

$$(4.5) \quad \int_{\mathbb{T}^d} V(y) d\mu(y) = \alpha[0],$$

- (b) for any $\varepsilon > 0$ we have that

$$(4.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{L}^1 \left(\{t \in [0, T] : d_{M(V)}(\mathbf{x}_u(t, x)) \geq \varepsilon\} \right) = 0.$$

Proof. Assume (a) and suppose (b) fails for some $\varepsilon > 0$. Then, for some $\delta > 0$ and some sequence $T_k \uparrow \infty$, we have that

$$(4.7) \quad \frac{1}{T_k} \mathcal{L}^1 \left(\{t \in [0, T_k] : d_{M(V)}(\mathbf{x}_u(t, x)) \geq \varepsilon\} \right) \geq \delta.$$

We can also assume that $\mu_x^{T_k} \rightharpoonup \mu \in \mathcal{W}_u(x)$ as $k \rightarrow \infty$ without loss of generality. Set

$$m(\varepsilon) = \inf \{ \alpha[0] - V(y) : y \in \mathbb{T}^d, d_{M(V)}(y) \geq \varepsilon \}$$

and observe that $0 < m(\varepsilon) \leq 2\|V\|_\infty$. Since for any $k \in \mathbb{N}$

$$m(\varepsilon) \mathcal{L}^1 \left(\{t \in [0, T_k] : d_{M(V)}(\mathbf{x}_u(t, x)) \geq \varepsilon\} \right) \leq \int_0^{T_k} (\alpha[0] - V(\mathbf{x}_u(t, x))) dt,$$

we conclude that

$$\frac{1}{T_k} \mathcal{L}^1 \left(\{t \in [0, T_k] : d_{M(V)}(\mathbf{x}_u(t, x)) \geq \varepsilon\} \right) \leq \frac{1}{m(\varepsilon)} \frac{1}{T_k} \int_0^{T_k} (\alpha[0] - V(\mathbf{x}_u(t, x))) dt.$$

Now,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\alpha[0] - V(\mathbf{x}_u(t, x))) dt = \int_{\mathbb{T}^d} (\alpha[0] - V(y)) d\mu(y) = 0$$

in view of (4.5). This contradicts (4.7) showing that (b) holds true.

Conversely, assume (b). Let $\mu \in \mathcal{W}_u(x)$ and let $T_k \uparrow \infty$ be such that $\mu_x^{T_k} \rightarrow \mu$ as $k \rightarrow \infty$. Fix any $\varepsilon > 0$. Then

$$\begin{aligned} 0 &\leq \frac{1}{T_k} \int_0^{T_k} (\alpha[0] - V(\mathbf{x}_u(t, x))) dt \\ &\leq \max_{d_{M(V)}(y) < \varepsilon} (\alpha[0] - V(y)) + \frac{2\|V\|_\infty}{T_k} \mathcal{L}^1 \left(\{t \in [0, T_k] : d_{M(V)}(\mathbf{x}_u(t, x)) \geq \varepsilon\} \right). \end{aligned}$$

Therefore, taking the limit as $k \rightarrow \infty$ and recalling (4.6) we obtain

$$0 \leq \int_{\mathbb{T}^d} (\alpha[0] - V(y)) d\mu(y) \leq \max_{d_{M(V)}(y) < \varepsilon} (\alpha[0] - V(y)),$$

where the right-hand side of the above inequality tends to zero as $\varepsilon \rightarrow 0$. Since ε is arbitrary the conclusion follows. \square

Remark 4.8.

(1) Owing to Corollary 3.3 (a), we have that $\mathbf{x}_u(\cdot, x)$ cannot approach $\operatorname{argmin}_{\mathbb{T}^d} u$ unless x is itself a minimum point of u . Therefore, if

$$x \in \mathbb{T}^d \setminus \operatorname{argmin}_{\mathbb{T}^d} u \quad \text{and} \quad \operatorname{Crit}^*(u) = \operatorname{argmin}_{\mathbb{T}^d} u,$$

then there exists no occupational measure of \mathbf{x}_u satisfying (4.5).

(2) Another way to formulate (4.6) is to say that for any $\varepsilon > 0$ the probability for $\mathbf{x}_u(t, x)$ to be ε -close to the regular critical set of u , with t picked at random in $[0, T]$, tends to 1 as $T \rightarrow \infty$.

Theorem 4.9. Let $u : \mathbb{T}^d \rightarrow \mathbb{R}$ be a semiconcave solution of (4.2) and set

$$\delta(V) := \max_{\mathbb{T}^d} V - \min_{\mathbb{T}^d} V.$$

Let $x \in \mathbb{T}^d$ be such that $\tau(x) = \infty$. Then, the following properties are equivalent:

(a) there exists $\mu \in \mathcal{W}_u(x)$ such that

$$(4.8) \quad \int_{\mathbb{T}^d} V(y) d\mu(y) < \alpha[0];$$

(b) we have that $\delta(V) > 0$ and there exists a constant $\eta > 0$ such that

$$(4.9) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \mathcal{L}^1 \left(\left\{ t \in [0, T] : \frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 + V(\mathbf{x}_u(t, x)) + \eta \leq \alpha[0] \right\} \right) \geq \frac{\eta}{\delta(V)};$$

(c) there exists a constant $\delta > 0$ such that

$$(4.10) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \mathcal{L}^1 \left(\left\{ t \in [0, T] : d_{M(V)}(\mathbf{x}_u(t, x)) \geq \delta \right\} \right) > 0$$

where $M(V)$ is defined in (4.4).

Proof. (a) \Rightarrow (b): define

$$(4.11) \quad \eta = \frac{1}{3} \int_{\mathbb{T}^d} (\alpha[0] - V(y)) d\mu(y)$$

and observe that $0 < \eta \leq \delta(V)/3$. Hence, $\delta(V) > 0$. Let $T_k \uparrow \infty$ be such that $\mu_x^{T_k} \rightarrow \mu$ as $k \rightarrow \infty$. Since u is bounded and $\mu_x^{T_k} \rightarrow \mu$, for k large enough, say $k \geq k_0$, we have that

$$(4.12) \quad \begin{aligned} \frac{1}{T_k} \int_0^{T_k} \left(\frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 + V(\mathbf{x}_u(t, x)) - \alpha[0] \right) dt \\ = \frac{u(\mathbf{x}_u(T_k, x)) - u(x)}{2T_k} + \frac{1}{T_k} \int_0^{T_k} (V(\mathbf{x}_u(t, x)) - \alpha[0]) dt \leq -2\eta. \end{aligned}$$

This yields

$$\liminf_{k \rightarrow \infty} \frac{1}{T_k} \int_0^{T_k} \left(\frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 + V(\mathbf{x}_u(t, x)) - \alpha[0] \right) dt \leq -2\eta.$$

So, from (4.12) and Lemma A.2 below applied to the function

$$f(t) = \alpha[0] - \frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 - V(\mathbf{x}_u(t, x)) \quad (t \in [0, T_k])$$

with $\delta = \delta(V)$, $\rho = 2\eta$, and $\lambda = \eta$, it follows that

$$\frac{1}{T_k} \mathcal{L}^1 \left(\left\{ t \in [0, T_k] : \frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 + V(\mathbf{x}_u(t, x)) + \eta \leq \alpha[0] \right\} \right) \geq \frac{\eta}{\delta(V) - \eta}$$

for all $k \geq k_0$. The conclusion (4.9) follows.

(b) \Rightarrow (c): let $T_k \uparrow \infty$ be such that

$$(4.13) \quad \lim_{k \rightarrow \infty} \frac{1}{T_k} \mathcal{L}^1 \left(\left\{ t \in [0, T_k] : \frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 + V(\mathbf{x}_u(t, x)) + \eta \leq \alpha[0] \right\} \right) \geq \frac{\eta}{\delta(V)}.$$

In order to prove (4.10) observe that, since $\alpha[0] = \max_{\mathbb{T}^d} V$, by the uniform continuity of V we deduce that, for some $\delta = \delta(\eta) > 0$,

$$\{x \in \mathbb{T}^d : V(x) + \eta \leq \alpha[0]\} \subset \{x \in \mathbb{T}^d : d_{M(V)}(x) \geq \delta\}$$

So, for all $k \in \mathbb{N}$ we obtain the lower bound

$$\begin{aligned} \frac{1}{T_k} \mathcal{L}^1 \left(\left\{ t \in [0, T_k] : d_{M(V)}(\mathbf{x}_u(t, x)) \geq \delta \right\} \right) \\ \geq \frac{1}{T_k} \mathcal{L}^1 \left(\left\{ t \in [0, T_k] : V(\mathbf{x}_u(t, x)) + \eta \leq \alpha[0] \right\} \right) \\ \geq \frac{1}{T_k} \mathcal{L}^1 \left(\left\{ t \in [0, T_k] : \frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 + V(\mathbf{x}_u(t, x)) + \eta \leq \alpha[0] \right\} \right) \end{aligned}$$

which in turn yields the conclusion in view of (4.13).

(c) \Rightarrow (a): let us argue by contradiction assuming that (4.5) holds for all $\mu \in \mathcal{W}_u(x)$. Then, (4.6) must also hold by Theorem 4.7, in contrast with (4.10). \square

The existence of a measure $\mu \in \mathcal{W}_u(x)$ satisfying the strict inequality (4.8) has the following consequence on propagation of singularities.

Corollary 4.10. *Let $x \in \mathbb{T}^d$. If (4.8) holds for some measure $\mu \in \mathcal{W}_u(x)$, then*

$$(4.14) \quad \exists t_0 \in [0, \infty) \text{ such that } \mathbf{x}_u(t, x) \in \text{Sing}(u) \quad \forall t \geq t_0.$$

Proof. Suppose, first, $\tau(x) < \infty$. Then $\mathbf{x}_u(\tau(x), x) \in \text{Crit}(u)$ and $\mathcal{W}_u(x) = \{\delta_{\mathbf{x}_u(\tau(x), x)}\}$ by Proposition 4.5 and Corollary 3.14. Therefore, (4.8) reduces to $V(\mathbf{x}_u(\tau(x), x)) < \alpha[0]$. So, again by Proposition 4.5, $\mathbf{x}_u(\tau(x), x) \in \text{Sing}(u)$. Thus, $\mathbf{x}_u(t, x) \in \text{Sing}(u)$ for all $t \geq \tau(x)$ owing to Theorem 4.3. This yields (4.14) for $t_0 = \tau(x)$.

Now, assume $\tau(x) = \infty$ and let $\eta > 0$ be as in point (b) of Theorem 4.9. Then, (4.13) holds for some sequence $T_k \uparrow \infty$. Owing to (2.4),

$$\left\{ t \in [0, T_k] : \frac{1}{2} |p_0(\mathbf{x}_u(t, x))|^2 + V(\mathbf{x}_u(t, x)) + \eta \leq \alpha[0] \right\} \subset \text{Sing}(u).$$

Therefore,

$$\frac{1}{T_k} \mathcal{L}^1(\{t \in [0, T_k] : \mathbf{x}_u(t, x) \in \text{Sing}(u)\}) \geq \frac{\eta}{\delta(V)}$$

for all $k \in \mathbb{N}$. Thus, (4.10)-(i) follows by taking any

$$t_0 \in \{t \in [0, T_0] : \mathbf{x}_u(t, x) \in \text{Sing}(u)\}$$

and applying Theorem 4.3. \square

Remark 4.11. For any $x \in \mathbb{T}^d$ Proposition 4.5, Theorem 4.7, and Corollary 4.10 provide the following synthetic view of the asymptotic behaviour of $\mathbf{x}_u(t, x)$:

- either equality (4.5) holds for all measures $\mu \in \mathcal{W}_u(x)$, and the generalised gradient flow approaches (in a suitable sense) the regular critical set of u ,
- or the strict inequality (4.8) holds some $\mu \in \mathcal{W}_u(x)$, and the flow enters the singular set of u in finite time.

We conclude this paper with two examples in dimension one. We begin by showing that property (a) of Corollary 3.11 is always true in this case.

Proposition 4.12. *Let u be a semiconcave function on \mathbb{T}^1 . Then for any $x \in \mathbb{T}^1$ there exists a unique $\bar{x} \in \text{Crit}(u)$ such that $\mathcal{W}_u(x) = \{\delta_{\bar{x}}\}$*

Proof. Let $x \in \mathbb{T}^1$. Then, in view of (3.4), the function $t \mapsto u(\mathbf{x}_u(t, x))$ is strictly increasing on the interval $[0, \tau(x))$. Therefore, the function $t \mapsto \mathbf{x}_u(t, x)$ is strictly monotone on $[0, \tau(x))$ and so

$$\mathcal{W}_u(x) = \{\delta_{\bar{x}}\} \quad \text{where} \quad \bar{x} = \begin{cases} \min \{y \in \text{Crit}(u) : y \geq x\} & \text{if } t \mapsto \mathbf{x}_u(t, x) \text{ increasing} \\ \max \{y \in \text{Crit}(u) : y \leq x\} & \text{if } t \mapsto \mathbf{x}_u(t, x) \text{ decreasing} \end{cases}$$

owing to Theorem 3.9. \square

Example 4.13. This example illustrates the interaction among limiting occupational measures, critical points, and singularities. Let $d = 1$ and take

$$V(x) = \cos(2\pi x) \quad (x \in \mathbb{T}^1).$$

Then $\alpha[0] = 1$ and the Hamilton-Jacobi equation (4.2) reads as follows

$$(4.15) \quad \frac{1}{2}|u'(x)|^2 + \cos(2\pi x) = 1 \quad (x \in \mathbb{T}^1).$$

Observe that the periodic extension (with period 1) of the function

$$u(x) = \begin{cases} \int_0^x \sqrt{2(1 - \cos(2\pi y))} dy & 0 \leq x < \frac{1}{2} \\ \int_x^1 \sqrt{2(1 - \cos(2\pi y))} dy & \frac{1}{2} \leq x < 1 \end{cases}$$

is a semiconcave solution to (4.15). For such a solution we have that

$$\text{Crit}^*(u) = \mathbb{Z}, \quad \text{Sing}(u) = \left\{ \frac{1}{2} \right\} + \mathbb{Z}, \quad \text{Crit}(u) = \left\{ 0, \frac{1}{2} \right\} + \mathbb{Z}.$$

Therefore,

$$\mathcal{W}_u(k) = \{\delta_k\} \quad \forall k \in \mathbb{Z}.$$

Moreover, one can compute the limiting occupational measure and critical time at all other points of \mathbb{T}^1 thanks to Remark 4.8 (1). We have that

$$\mathcal{W}_u(x) = \left\{ \delta_{\lceil x \rceil + \frac{1}{2}} \right\} \quad \text{and} \quad \tau(x) = \left| \frac{1}{2} - (x - \lceil x \rceil) \right| \quad \forall x \in \mathbb{T}^1 \setminus \mathbb{Z},$$

where $\lceil x \rceil = \max\{k \in \mathbb{Z} : k \leq x\}$.

Example 4.14. We now give another example where $\tau(x) = \infty$ for some x . Let $d = 1$ and take

$$V(x) = -\sin^2(2\pi x) \quad (x \in \mathbb{T}^1).$$

Since $\alpha[0] = 0$, the critical equation reduces to

$$(4.16) \quad \frac{1}{2}|u'(x)|^2 - \sin^2(2\pi x) = 0 \quad (x \in \mathbb{T}^1).$$

Observe that the projected Aubry set has two components, $\{0\}$ and $\{1/2\}$, and (4.16) admits infinitely many weak KAM solutions. For the smooth solution

$$u(x) = \frac{\sqrt{2}}{2\pi}(1 - \cos(2\pi x)) \quad (x \in \mathbb{T}^1)$$

the associated gradient system is

$$\begin{cases} \dot{\mathbf{x}}(t) = \sqrt{2} \sin(2\pi \mathbf{x}(t)), \\ \mathbf{x}(0) = x_0. \end{cases}$$

Let us consider the solution $\mathbf{x}(\cdot, x_0)$ with initial condition $x = x_0 \in (0, 1/2)$. Then

$$\left[\frac{1}{2\pi} \log \left| \tan(\pi \mathbf{x}(s, x_0)) \right| \right]_0^t = \int_0^t \frac{\mathbf{x}'(s)}{\sin(2\pi \mathbf{x}(s, x_0))} ds = t\sqrt{2}$$

which yields

$$\pi \mathbf{x}(t, x_0) = \arctan \left(e^{2\pi t \sqrt{2}} \tan(\pi x_0) \right).$$

Thus, $\lim_{t \rightarrow +\infty} \mathbf{x}(t, x_0) = 1/2$, $\tau(x_0) = \infty$, and $\mathcal{W}_u(x_0) = \{\delta_{1/2}\}$.

5. EXTENSIONS AND OPEN PROBLEMS

5.1. Extension to eikonal-type equations. All the results of this paper can be extended with minor changes to the eikonal-type equation

$$(5.1) \quad \frac{1}{2} \langle A(x) Du(x), Du(x) \rangle + V(x) = \alpha[0] \quad (x \in \mathbb{T}^d),$$

where $V \in C^2(\mathbb{T}^d)$ and $A(x)$ is a positive definite symmetric $n \times n$ matrix, smoothly depending on $x \in \mathbb{T}^d$. In this case, the role of (3.1) is played by the generalised characteristics system

$$(5.2) \quad \begin{cases} \dot{\mathbf{x}}(t, x) \in A(\mathbf{x}_u(t, x)) D^+ u(\mathbf{x}(t, x)), & t \geq 0 \text{ a.e.} \\ \mathbf{x}(0, x) = x \end{cases}$$

which defines a Lipschitz semi-flow on \mathbb{T}^d . Consequently, instead of (3.4), one has that

$$(5.3) \quad \frac{d^+}{dt} u(\mathbf{x}_u(t, x)) = \langle A(\mathbf{x}_u(t, x)) p_A(\mathbf{x}_u(t, x)), p_A(\mathbf{x}_u(t, x)) \rangle \quad (t \geq 0),$$

where

$$(5.4) \quad p_A(y) = \operatorname{argmin}_{p \in D^+ u(y)} \langle A(y)p, p \rangle \quad (y \in \mathbb{T}^d).$$

The definition of the sets $\operatorname{Crit}(u)$, $\operatorname{Sing}(u)$, and $\operatorname{Crit}^*(u)$ remains unchanged, as well as that of limiting occupational measures. Since the analogue of Theorem 4.3 is already available (see [1]), all the results of Section 3 and Section 4 extend to equation (5.1), in particular Theorem 4.7 and Corollary 4.10.

5.2. Open problem: the time dependent case. A very natural question is to understand how the approach of this paper can be adapted to the time dependent equation

$$(5.5) \quad \frac{\partial u}{\partial t}(t, x) + \frac{1}{2} |Du(t, x)|^2 + V(t, x) = 0 \quad (t, x) \in [0, \infty) \times \mathbb{T}^d,$$

where $V : [0, \infty) \times \mathbb{T}^d$ is of class C^2 . Observe that (5.5) cannot be regarded as a special case of (4.2) for several reasons: the Hamiltonian of (5.5) is not quadratic in the time derivative, the configuration space $[0, \infty) \times \mathbb{T}^d$ is not compact, and the potential V is not assumed to be periodic in time.

5.3. Open problem: nonzero cohomology. In homogenisation problems and weak KAM theory, equation (5.1) often appears as a special case of the family of problems

$$(5.6) \quad \frac{1}{2} \langle A(x)Du(x) + c, Du(x) + c \rangle + V(x) = \alpha[c] \quad (x \in \mathbb{T}^d),$$

where $c \in \mathbb{R}^n$. Unfortunately, even the very basic results of this paper—for example, Proposition 3.1 and Corollary 3.1—seem hard to adapt to the case of $c \neq 0$, for which new ideas will be necessary. Nevertheless, the present theory can serve as a significant example of what one may expect in the general case.

5.4. Open problem: Riemannian manifolds. Unlike the open problems mentioned in 5.2 and 5.3, the results of this paper should not be too hard to generalise to an eikonal-type equation like (5.1) on a compact Riemannian manifold. A crucial point for this plan would be the extension to manifolds of the result ensuring global propagation of singularities. This seems reasonable since the proof in [1] is of local nature and should therefore be easy to adapt to a local chart.

APPENDIX A. TWO TECHNICAL LEMMAS

Lemma A.1. *Let $T, C > 0$ be fixed and let $f : [0, T] \rightarrow [0, \infty)$ be a Lebesgue measurable function such that*

$$(A.1) \quad \frac{1}{T} \int_0^T (T - s)f(s)ds \leq C.$$

Then we have that

$$(A.2) \quad \mathcal{L}^1(\{s \in [0, T] : f(s) \leq C\}) \geq \frac{T}{2} - 1.$$

Proof. Observe that (A.1) yields

$$\begin{aligned} C &\geq \frac{1}{T} \int_0^T (T - s)f(s)(\chi_{\{f > C\}}(s) + \chi_{\{f \leq C\}}(s))dr \\ &\geq \frac{C}{T} \int_0^T (T - s)\chi_{\{f > C\}}(s)ds = \frac{C}{T} \left(\frac{T^2}{2} - \int_0^T (T - s)\chi_{\{f \leq C\}}(s)ds \right) \\ &\geq \frac{C}{T} \left(\frac{T^2}{2} - T\mathcal{L}^1(\{r \in [0, T] : f(s) \leq C\}) \right) \end{aligned}$$

The conclusion follows. □

Lemma A.2. *Let $T, \delta > 0$ be fixed and let $f : [0, T] \rightarrow [0, \delta]$ be a Lebesgue measurable function such that*

$$(A.3) \quad \frac{1}{T} \int_0^T f(t)dt \geq \rho$$

for some $\rho \in (0, \delta)$. Then for any $\lambda \in (0, \rho)$ we have that

$$\frac{1}{T} \mathcal{L}^1(\{t \in [0, T] : f(t) \geq \lambda\}) \geq \frac{\rho - \lambda}{\delta - \lambda}.$$

Proof. Fix $\lambda \in (0, \rho)$ and let

$$\varphi(\lambda) = \mathcal{L}^1(\{t \in [0, T] : f(t) \geq \lambda\}) \quad (t \in [0, T])$$

Then (A.3) ensures that

$$\rho \leq \frac{1}{T} \int_0^T f(t) (\chi_{\{f \geq \lambda\}}(s) + \chi_{\{f < \lambda\}}(s)) ds \leq \frac{\delta}{T} \varphi(\lambda) + \frac{\lambda}{T} (T - \varphi(\lambda)).$$

Thus, $(\delta - \lambda)\varphi(\lambda)/T \geq \rho - \lambda$. □

REFERENCES

- [1] P. Albano. Global propagation of singularities for solutions of Hamilton-Jacobi equations. *J. Math. Anal. Appl.*, 444(2):1462–1478, 2016.
- [2] P. Albano and P. Cannarsa. Structural properties of singularities of semiconcave functions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):719–740, 1999.
- [3] P. Albano and P. Cannarsa. Propagation of singularities for solutions of nonlinear first order partial differential equations. *Arch. Ration. Mech. Anal.*, 162(1):1–23, 2002.
- [4] P. Albano, P. Cannarsa, K. T. Nguyen, and C. Sinestrari. Singular gradient flow of the distance function and homotopy equivalence. *Math. Ann.*, 356(1):23–43, 2013.
- [5] G. Alberti, L. Ambrosio, and P. Cannarsa. On the singularities of convex functions. *Manuscripta Math.*, 76(3-4):421–435, 1992.
- [6] L. Ambrosio, P. Cannarsa, and H. M. Soner. On the propagation of singularities of semi-convex functions. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 20(4):597–616, 1993.
- [7] Z. Artstein. Invariant measures of differential inclusions applied to singular perturbations. *J. Differential Equations*, 152(2):289–307, 1999.
- [8] Z. Artstein. Invariant measures of set-valued maps. *J. Math. Anal. Appl.*, 252(2):696–709, 2000.
- [9] P. Bernard. Young measures, superposition and transport. *Indiana University Mathematics Journal*, 57(1):247–275, 2008.
- [10] P. Cannarsa, Q. Chen, and W. Cheng. Dynamic and asymptotic behavior of singularities of certain weak KAM solutions on the torus. *J. Differential Equations*, 267(4):2448–2470, 2019.
- [11] P. Cannarsa and W. Cheng. Generalized characteristics and Lax-Oleinik operators: global theory. *Calc. Var. Partial Differential Equations*, 56(5):Art. 125, 31, 2017.
- [12] P. Cannarsa, W. Cheng, and A. Fathi. On the topology of the set of singularities of a solution to the Hamilton-Jacobi equation. *C. R. Math. Acad. Sci. Paris*, 355(2):176–180, 2017.
- [13] P. Cannarsa, W. Cheng, and A. Fathi. Singularities of solutions of time dependent Hamilton-Jacobi equations. Applications to Riemannian geometry. *Publ. Math. Inst. Hautes Études Sci.*, 133(1):327–366, 2021.
- [14] P. Cannarsa, W. Cheng, M. Mazzola, and K. Wang. Global generalized characteristics for the Dirichlet problem for Hamilton-Jacobi equations at a supercritical energy level. *SIAM J. Math. Anal.*, 51(5):4213–4244, 2019.
- [15] P. Cannarsa, W. Cheng, and Q. Zhang. Propagation of singularities for weak KAM solutions and barrier functions. *Comm. Math. Phys.*, 331(1):1–20, 2014.
- [16] P. Cannarsa, M. Mazzola, and C. Sinestrari. Global propagation of singularities for time dependent Hamilton-Jacobi equations. *Discrete Contin. Dyn. Syst.*, 35(9):4225–4239, 2015.
- [17] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2004.
- [18] P. Cannarsa and H. M. Soner. On the singularities of the viscosity solutions to Hamilton-Jacobi-Bellman equations. *Indiana Univ. Math. J.*, 36(3):501–524, 1987.
- [19] P. Cannarsa and H. M. Soner. Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications. *Nonlinear Anal., Theory Methods Appl.*, 13(3):305–323, 1989.

- [20] P. Cannarsa and Y. Yu. Singular dynamics for semiconcave functions. *J. Eur. Math. Soc. (JEMS)*, 11(5):999–1024, 2009.
- [21] M. G. Crandall, L. C. Evans, and P.-L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 282(2):487–502, 1984.
- [22] A. Fathi. Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(9):1043–1046, 1997.
- [23] R. Mañé. Lagrangian flows: the dynamics of globally minimizing orbits. *Bol. Soc. Brasil. Mat. (N.S.)*, 28(2):141–153, 1997.
- [24] L. Veselý. On the multiplicity points of monotone operators on separable Banach spaces. *Commentat. Math. Univ. Carol.*, 27:551–570, 1986.
- [25] L. Veselý. On the multiplicity points of monotone operators on separable Banach spaces. II. *Commentat. Math. Univ. Carol.*, 28:295–299, 1987.
- [26] L. Zajicek. On the points of multiplicity of monotone operators. *Commentat. Math. Univ. Carol.*, 19:179–189, 1978.
- [27] L. Zajicek. On the differentiation of convex functions in finite and infinite dimensional spaces. *Czech. Math. J.*, 29:340–348, 1979.

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