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# Internally Driven $\beta$ -plane Plasma Turbulence Using the Hasegawa-Wakatani System

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General problem of plasma turbulence can be formulated as advection of potential vorticity (PV), which handles flow self-organization, coupled to a number of other fields, whose gradients provide free energy sources. Therefore, focusing on PV evolution separates the underlying linear instability from the flow self-organization, and clarifies key spatial scales in terms of balances between various time scales. Considering the Hasegawa-Wakatani model as a minimal, nontrivial model of plasma turbulence where the energy is injected internally by a linear instability, we find that the critical wavenumber  $k_c = C/\kappa$  where  $C$  is the adiabaticity parameter and  $\kappa$  is the normalized density gradient separates the adiabatic (or highly zonostrophic) behavior for large scales from the hydrodynamic behavior at small scales. In the adiabatic range the non-zonal part of the wave-number spectrum goes from  $E(k) \propto \gamma_k U^{-1} k^{-2}$  around the peak to  $E(k) \propto \omega_k^2 k^{-3}$  in the “inertial” range, where  $\gamma_k$  and  $\omega_k$  are the linear growth and frequency and  $U$  is the rms zonal velocity. This proposed spectrum fits very well for the large  $k_c$  case, where the bulk of the spectrum is in the adiabatic range. In contrast for small  $k_c$ , we get the usual forward enstrophy cascade with  $E(k) \propto \epsilon_W^{2/3} k^{-3}$ , where  $\epsilon_W$  is the enstrophy dissipation. In contrast for  $k_c \approx 1$ , the system transitions to hydrodynamic forward enstrophy cascade right after the injection range, with zonal flows at large scales and forward enstrophy cascade at small scales. Note that  $k_c$ , can also be used as a proxy for the scale at which the system switches from wave-dominated (i.e.  $E(k) \propto \omega_k^2 k^{-3}$ ) to hydrodynamic (i.e.  $E(k) \propto \epsilon_W^{2/3} k^{-3}$ ) spectra usually denoted by  $k_\beta$  in geophysical fluid dynamics. It is argued that the ratio  $R_\beta \equiv k_\beta/k_{\text{peak}} \approx k_c/k_{\text{peak}}$  where  $k_{\text{peak}}$  is the peak wave-number can be defined as the zonostrophy parameter, and that the abundance of zonal flows vs. eddies in near and far from “marginality” that is commonly formulated in terms of the Kubo number in plasma problems can also be understood in terms of the zonostrophy parameter, since  $R_\beta$  increases as we approach marginality.

## I. INTRODUCTION

Understanding plasma turbulence and its self-regulation through generation of large scale flows is an important challenge in predicting and controlling turbulent transport in fusion devices. Micro-instabilities, driven by background gradients, inevitably present in a magnetic confinement device, generate collective, fluctuating electric fields that can transport particles, heat and momentum towards the walls. These fluctuations also generate large scale flows, called zonal flows through Reynolds stresses[1]. Zonal flows can then suppress the turbulence that drives them through a mechanism of shear suppression[2], thus allowing the turbulence to self-regulate. This mechanism of flow self-organization is well known, and generally understood in terms of the non-linear dynamics of potential vorticity[3], and is closely related to the similar mechanism of layer formation in geophysical fluid dynamics (GFD) [4, 5].

On the other hand, fluctuations in plasma turbulence, especially in scales where the instability mechanism is active are known to be roughly quasi-linear, where the phases between fluctuations are dictated by linear relations[6]. This justifies our approach of separating the instability mechanism, which is almost linear, from flow self-organization through the PV equation, which is completely nonlinear. Note that a generalization of the QLT is also commonly used in the GFD community[7], ex-

cept that while plasma QLT works better when there are no zonal flows, the GFD type gQLT works best when the zonal flows are the strongest. The current letter, in addition to providing theoretical estimates of the wave-number spectra in different limits, intends to bridge this gap between the two communities.

*$\beta$ -plane model of plasma turbulence:* Consider an equation for the potential vorticity (PV)  $\zeta$ , which can be written in general as:

$$\partial_t \zeta + \hat{\mathbf{z}} \times \nabla \Phi \cdot \nabla \zeta + \kappa \partial_y \tilde{\Phi} = -\mu \zeta + \nu \nabla^2 \zeta, \quad (1)$$

where  $\kappa$  is its normalized background gradient  $\Phi = \bar{\Phi} + \tilde{\Phi}$  where  $\bar{\Phi}$  is the zonally averaged part of the properly normalized electrostatic potential playing the role of the stream function,  $\mu$  and  $\nu$  are large scale friction and small scale dissipation respectively, which serve to regularize the equation and could also be replaced by hypo and hyper viscosities. Note that if one considers a self-consistent, internally driven model such as the Hasegawa-Wakatani system[8, 9] (with the standard modified response), a large scale friction is not necessary for steady state, and we can set  $\mu = 0$ , however if we only consider a linear dispersion relation in order to close (1), the inverse cascade becomes an issue and one has to include large scale friction and hypodiffusion.

The definition of PV in terms of various fields of the system, and its inversion (i.e. writing  $\Phi$  in (1) in terms of  $\zeta$ , or writing  $\zeta$  in terms of  $\Phi$  alone so that we have a sin-

gle field) gives us the details of the physics problem. For example for dissipative drift waves  $\zeta = n - \nabla^2 \Phi$ , where  $n$  is the electron density and the inversion requires that we solve the continuity equation for density along with Eqn. (1). For ion temperature gradient driven (ITG) turbulence, pressure  $P$  also enters the definition of PV through  $\zeta = n - \nabla^2 \Phi - P/\Gamma$  where  $\Gamma$  is the specific heat ratio, and we need to solve an equation for pressure as well, which is in turn coupled to a parallel velocity equation[10]. Even the gyrokinetic equation can be defined in terms of a potential vorticity conservation through the gyrocenter density[11], or the zeroth moment of the gyrokinetic distribution function, where the inversion problem becomes the solution of all the other moments.

It may be possible in some cases to write a linear inversion for the other fields. For example, the Hasegawa-Wakatani system could be inverted approximately using the linear relation  $n_k \rightarrow \sigma_k \Phi_k$  with  $\sigma_k \approx \frac{\kappa k_y + iC}{\omega_k + iC}$  so that:

$$\tilde{\Phi}_k = (\sigma_k + k^2)^{-1} \tilde{\zeta}_k, \quad (2)$$

where  $\omega_k$  is the unstable solution of the linear dispersion relation, which can be written in the inviscid limit as  $\omega_k^2 k^2 + iC\omega_k(1+k^2) = i\kappa k_y C$ , with  $C$  the adiabaticity parameter. Note that substituting (2) into (1) and linearizing, we get back the linear dispersion relation. However linear inversion is problematic in particular because the resulting system lacks the key saturation mechanism *involving the backreaction of the flow on the linear growth*. This can be remedied in part by using a quasi-linear inversion, or eventually, fully nonlinear inversion, which requires solving the continuity equation:

$$\partial_t n + \hat{\mathbf{z}} \times \nabla \Phi \cdot \nabla n + \kappa \partial_y \tilde{\Phi} = C (\tilde{\Phi} - \tilde{n}) + D \nabla^2 n \quad (3)$$

making it necessary to consider the full Hasegawa-Wakatani system [i.e. Eqns. (1) and (3)], even if we are only interested in PV evolution since this is the only way we can infer  $\tilde{\Phi}$  that appears in (1). Note that from the perspective of PV evolution, the energy injection in plasma turbulence is around  $k_y \approx 1$  (or smaller) and displays characteristics of forward cascade either through potential enstrophy cascade carried by waves, or enstrophy cascade through eddies. The inverse energy ‘‘cascade’’ is usually not important, since the scale at which the zonal flows form is usually slightly larger than the injection scale, and they can turn off the underlying instability drive. However if one uses only the linear dispersion relation (instead of the full nonlinear problem), to invert PV, while PV can still flatten, it can not turn off the underlying instability drive, and thus the system becomes pathological, necessitating large scale friction, but the behaviour of the system with linear inversion and large scale friction is different from that of self-consistent saturation without or with little friction.

The key spatial scale in plasma  $\beta$  plane turbulence, is the transition scale between adiabatic and hydrody-

$E(k)$ Spectrum	Time scale balance	range
$U\gamma_k k^{-2}$	$\sqrt{Uk\gamma_k} \sim \tau_{n\ell}^{-1}$	$k \sim k_{\text{peak}} \ll k_c$
$\omega_k^2 k^{-3}$	$\omega_k \sim \tau_{n\ell}^{-1}$	$k_{\text{peak}} < k < k_c$
$\varepsilon^{2/3} k^{-3}$	$\varepsilon^{1/3} \sim \tau_{n\ell}^{-1}$	$k > k_{\text{peak}} \lesssim k_c$

Table I. Limiting forms of the wave-number spectra, corresponding time scale balances [here  $\tau_{n\ell}^{-1} \equiv E^{1/2}(k) k^{3/2}$ ] and the range of wave-numbers for which they are applicable.

namic behavior. For the case of the Hasegawa-Wakatani model, this is a linear scale defined as  $k_c = C/\kappa$ . More generally one can define this scale as the scale where  $\sigma_{kr} \approx \sigma_{ki}$ , (i.e. basically if we define  $\sigma_k \approx 1 + i\delta_k$  this is the scale where  $\delta_k \approx 1$ ). This suggests that for scales larger than this scale, we have the adiabatic behavior (i.e.  $n_k \approx \Phi_k$ ) whereas for scales smaller than this scale, the hydrodynamic behavior follows. In three dimensions, the argument can be extended by noting that  $C$  goes like  $k_{\parallel}^2$  [12]. Note that  $k_c$  is closely related to  $k_{\beta}$ , commonly used in GFD as the scale at which the system switches from inverse cascade to Rossby wave turbulence (recall that, there the energy injection is at small scales). Since the behavior of the spectra are completely different in the adiabatic vs. hydrodynamic regimes (See table I for a detailed classification), we characterize everything with respect to  $k_c$  first.

Considering first  $k_c \gg 1$  (e.g.  $C \gg 1$  with  $\kappa \approx 1$ ), with a general growth rate  $\gamma_k$  which has the most unstable mode that has a finite  $k_y = k_{y0} \approx O(1)$  and vanishing  $k_x$  (the usual case in plasma turbulence), we can discuss some features of the nonlinear saturation. Since strong zonal flows are formed in this state, we can define the usual time scale associated with the zonal flows as  $\tau_Z^{-1} = Uk$  where  $U$  is the rms zonal velocity, and a hybrid time scale of the form  $\tau_{gz}^{-1} \approx (Uk\gamma_k)^{1/2}$ .

We argue that, as long as the peak region is such that  $k_{\text{peak}} < k_c$  it is this hybrid scale which is balanced by the nonlinear time  $\tau_{n\ell}^{-1} \propto E(k)^{1/2} k^{3/2}$  around the peak of the spectrum, yielding the quasi-linear spectrum:

$$E(k) \approx U\gamma_k k^{-2}, \quad (4)$$

valid for  $\gamma_k > 0$ . Note that while the rms velocity  $U$  is an emergent feature, we can estimate it from the momentum balance between the waves and flows, using something like  $U \approx [\omega_k/k_y]_{\text{max}}$ . For example for the standard form of  $\omega_k \propto \kappa k_y / (1+k^2)$ , we get  $U = \kappa$ . However a more general form of the momentum balance, which can be written as  $U \approx [\omega_k/k_y]_{\text{max}} - (\partial_x \bar{n})_{\text{rms}} + (\partial_x \bar{\Omega})_{\text{rms}}$  can not be solved in terms of linear quantities alone.

For  $k_c > 0.1$ , The linear growth peaks around  $k_y \approx 1$  and is fairly localized to wavenumbers around its peak and becomes negative at large  $k$  due to viscosity. If the wave-number where  $\gamma_k < 0$  is smaller than  $k_c$ , we continue with adiabatic wave-turbulence and this time, it

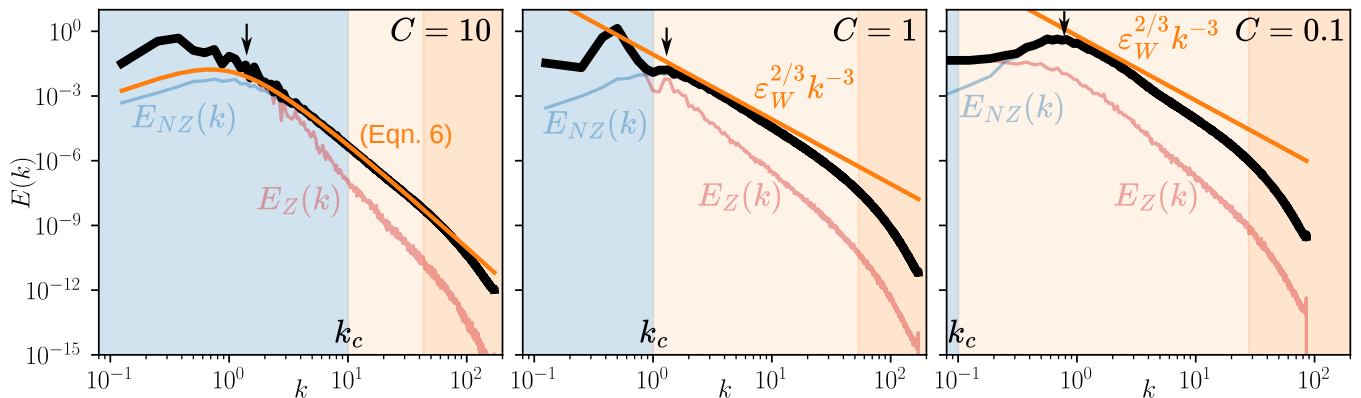


Figure 1.  $E(k)$  in Hasegawa-Wakatani turbulence for  $C = 10$  to  $1$  to  $0.1$  from left to right with  $\kappa = 1$ , so that the transition scale  $k_c$  is varied by two orders of magnitude. Here the blue region is adiabatic, light orange is hydrodynamic and dark orange is dissipative regions, and the wavenumbers for which  $\gamma_k$  is maximum are shown with little black arrows. The solid black curve is the kinetic energy spectra  $E(k)$  computed as  $E(k) = \sum_{k \in k \pm \Delta k} k^2 \Phi_{k_x, k_y}^2$ , averaged over time in the saturated phase of simulations up to  $t = 5000$ , with a padded resolution of  $4096^2$ . The light red curve corresponds to the zonal spectra  $E_Z(k_x) = \sum_{k_x \in k_x \pm \Delta k_x} k_x^2 \Phi_{k_x, 0}^2$  whereas  $E_{NZ}(k) = E(k) - E_Z(k)$ . Note that the solid orange curves corresponding to the theoretical spectra (as given by Eqn. 5, or the Kraichnan-Kolmogorov forward enstrophy cascade with  $\varepsilon^{2/3} k^{-3}$ ) are not multiplied by any additional factors, they naturally fall on the numerical spectra.

is the linear frequency  $\omega_k$  which balances the nonlinear time, giving a  $k$ -spectrum of the form:

$$E(k) \approx \omega_k^2 k^{-3}. \quad (5)$$

The two solutions can actually be put together by using  $\gamma_k \rightarrow \gamma_k + \nu k^2$  (which corresponds to the inviscid part of  $\gamma_k$ , which asymptotes to zero instead of becoming negative), so that we can write:

$$E(k) \approx \frac{\omega_k^2 U (\gamma_k + \nu k^2)}{(\omega_k^2 k^2 + U (\gamma_k + \nu k^2) k^3)}. \quad (6)$$

The spectrum of Eqn. (6), appears to be a very good match to the results of numerical simulations as can be seen in Figure 1. It seems that this form of the wave-number spectrum for plasma turbulence with zonal flows is rather generic in the adiabatic (or high zonostrophy) limit. In the complete opposite limit of  $k_c \ll 1$ , we get 2D Navier-Stokes dynamics with injection around  $k_y \approx 1$ , and a forward cascade of enstrophy, but with no discernible inverse cascade of energy (a key difference to linear inversion, or forced  $\beta$ -plane[13]), where the spectrum in the inertial range has the form  $E(k) \propto \varepsilon_W^{2/3} k^{-3}$ , where the coefficient  $\varepsilon_W^{2/3}$  can be computed directly from enstrophy dissipation  $\varepsilon_W \equiv \sum_k \nu k^4 E(k)$ , which comes mainly from small scales since  $E(k)$  drops slower than  $k^{-4}$ . Note that the faster than  $k^{-3}$  drop of the hydrodynamic spectra in Fig. 1 can be attributed to interactions with large scales, a spectrum of the form  $E(k) \sim k^{-4}/U$ , which may be related to the Saffmann spectrum as a consequence of sharp vorticity gradients[14], seems to be a good fit as well, especially towards the dissipative range.

In between the two limits is  $C \approx 1$ , which is probably the more challenging, and interesting case, in which we

can still argue for the standard forward enstrophy cascade picture for small scales (or  $k^{-4}$  due to sharp vorticity gradients), but since now we have the injection scale and the transition scale  $k_c$  basically on top of each other with zonal flows forming at slightly larger scales, we can have a steady state with large scale zonal flows, more or less governed by adiabatic limit of the equations and the small scale eddies that are governed by 2D hydrodynamics. The stationarity of this state with very strong zonal flows is extremely intriguing. A local linear analysis suggests that the gradient of vorticity provides a feedback loop which reduces the linear growth rate for negative vorticity gradient by flattening the PV profile, and introduces a hypodiffusion like term for positive vorticity gradient, providing a large scale energy sink in the troughs of the zonal velocity profile, which allows the underlying 2D hydrodynamic turbulence to reach a steady state, which explains the steadiness of the fluctuations. The stationarity of large scale zonal flows suggests that the 2D turbulence evolves towards exact solutions of the 2D Euler equations, such as Kida vortices[15], since those can survive in sheared flows without any net momentum transfer, or dipole vortices[16], which are known to be abundant in similar structure dominated turbulent states[17] with depleted nonlinearity due to selective decay[18, 19] relevant also in plasma edge[20]. In other words, in order to have stationary zonal flows with eddies, the eddies must minimize their Reynolds stress on large scale flows.

*The Zonostrophy parameter* A key parameter that defines the strength of the zonal flows to small scale eddies, is the so called zonostrophy parameter  $R_\beta$ . In the context of the usual (say forced)  $\beta$  plane turbulence,  $R_\beta$  is defined as the ratio of the Rhines scale  $L_R$  to the tran-

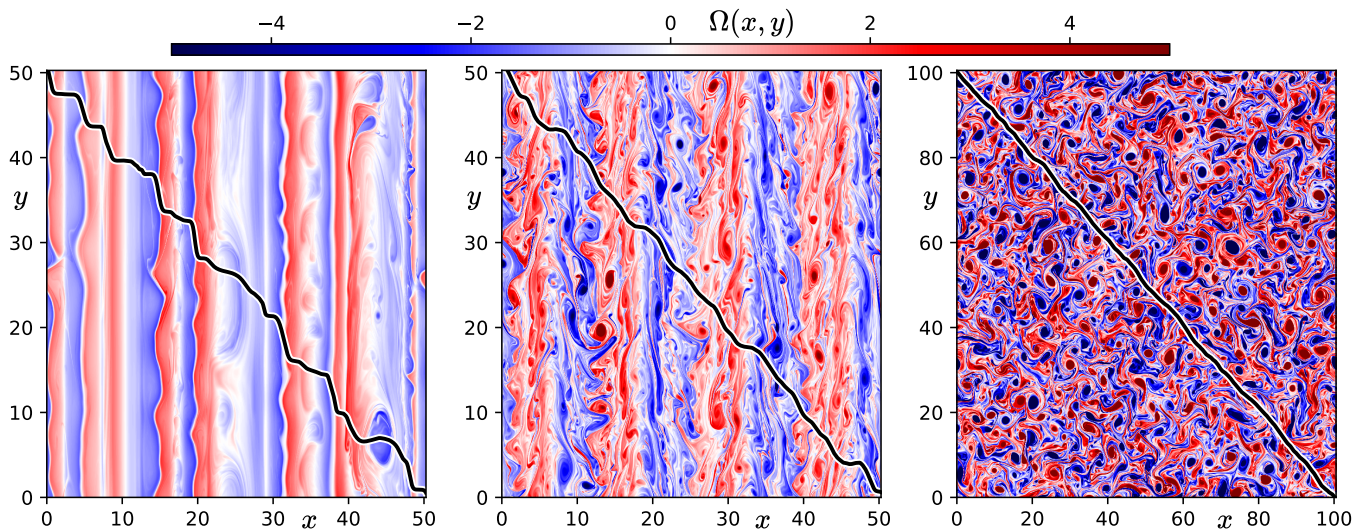


Figure 2. The snapshot of vorticity and the corresponding PV staircase, for  $C = 10$ ,  $C = 1.0$  to  $C = 0.1$  from left to right, for Hasegawa Wakatani turbulence. As we scan  $C$  the transition scale  $k_c = C/\kappa$ , changes also changing the zonostrophy parameter (i.e. here effectively  $R \propto k_{\text{peak}}/k_c$ ). We go from clean staircases for large  $C$  to a corrugated dirt hill to an almost unmodified PV gradient. Comparing the relative overall levels of the zonal to non-zonal spectra shown in Figure 1, around the peak of the nonzonal component shows this trend clearly.

sition scale  $\ell_\beta$  [21], which denotes the scale of transition from 2D inverse cascade at small scales to Rossby wave turbulence at larger scales. Basically if we have a large range of scales between the  $\ell_\beta$  and the  $L_R$ , the zonal flows dominate over eddies, whereas if the two scales are close together it is the other way around.

In the generalized beta plane perspective of the plasma turbulence with internal drive, the zonostrophy parameter can be defined as  $R_\beta \approx k_\beta/k_{\text{peak}}$  or  $k_c/k_{\text{peak}}$ , using  $k_c$  as a proxy for  $k_\beta$ . Snapshots of the vorticity field and the resulting PV staircase in Figure 2 shows clearly that as we change  $k_c$  by two orders of magnitude while keeping  $k_{\text{peak}}$  mostly unchanged, the zonostrophy parameter and the resulting ratio of zonal vs. nonzonal energy around the peak region changes substantially. Note that since  $k_{\text{peak}}$  and  $k_\beta$  are roughly the scales at which the linear growth and linear frequency are balanced by the nonlinear decorrelation time respectively (see table I), if  $\gamma_k \approx \omega_k$  (i.e. far from marginal) around the most unstable mode, the two scales are basically on top of one another, which makes the zonostrophy parameter small. In contrast if  $\gamma_k \ll \omega_k$  for the most unstable mode, the two scales are well separated, which makes  $R_\beta$  rather large. This explains the ubiquitous observation of abundance of zonal flows near marginality (i.e.  $\gamma_k \ll \omega_k$ ), and eddies far from it (i.e.  $\gamma_k \gtrsim \omega_k$ ) in plasma turbulence. The time scale comparison also explains the relation to Kubo number, which is the ratio of the turbulence decorrelation time (i.e. either  $\gamma^{-1}$  or  $\tau_{nl}$ ) to the nonlinear time.

*Results and Conclusions* Plasma turbulence is interpreted as a  $\beta$ -plane model with nonlinear inversion, giving the original dispersion relation when fully linearized,

or a  $\beta$ -plane model with a linear growth rate, when only the inversion is linearized, or the fully self-consistent nonlinear problem when the inversion is fully nonlinear. It was observed that linear inversion while giving a simple  $\beta$  plane model with growth rate from a linear dispersion relation, lacks the key saturation mechanism *involving the backreaction of the flow on the linear growth*. It was noted that  $k_c = C/\kappa$ , the scale at which the linear system switches from adiabatic to hydrodynamic, plays the role of  $k_\beta$  for this system. In the adiabatic limit, a weak wave turbulence spectrum in the form  $E(k) \propto \omega_k^2 k^{-3}$  is shown to be combined with the peak in the form of Eqn. (6), which matches the observed spectra from direct numerical simulations for  $k < k_c$ . In contrast for  $k > k_c$  it was observed that a Kraichnan-Komogorov spectrum of the form  $\varepsilon^{2/3} k^{-3}$  was observed in the “inertial” range. It was argued that the ratio  $k_\beta/k_{\text{peak}}$  can be defined as the zonostrophy parameter  $R_\beta$ , which explains the appearance of zonal flows near marginality and eddies away from it, since  $R_\beta$  increases as we approach marginality.

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