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State dynamics

Each dimension is considered independently. The position x and acceleration a are related according to :

$$\frac{d^2}{dt^2}x = a$$

We introduce the speed v , and rewrite this as a first order linear differential equation on the system state $X = \begin{pmatrix} x \\ v \end{pmatrix}$

$$\frac{d}{dt}x = v$$

$$\frac{d}{dt}v = a$$

$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} (t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} a(t)$$

The observations of the state are made at discrete measurement timepoints separated by a duration $T = 1/F$, where F is the sampling frequency. To obtain

the change in state between successive measurements, we integrate the previous equation, assuming that the acceleration a is constant between measurements:

$$\begin{pmatrix} x \\ v \end{pmatrix} (kT + T) = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} (kT) + \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix} a(kT)$$

We introduce the notations: $x_k = x(kT)$, $v_k = v(kT)$, $a_k = a(kT)$, $X_k = \begin{pmatrix} x_k \\ v_k \end{pmatrix}$.

The discrete time dynamics are then:

$$x_{k+1} = x_k + Tv_k + \frac{T^2}{2}a_k$$

$$v_{k+1} = v_k + Ta_k$$

$$X_{k+1} = \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} X_k + \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix} a_k$$

We introduce the matrix $A = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$ and the vector $B = \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix}$. The dynamics are then given by:

$$X_{k+1} = AX_k + Ba_k \tag{1}$$

Measurements

The position measurement x_k^{meas} is given by:

$$x_k^{meas} = x_k + w_k = CX_k + w_k \tag{2}$$

where $C = (1 \ 0)$ and the position noise w_k is assumed to be Gaussian with zero mean and standard deviation p_{std} .

The acceleration measurement a_k^{meas} is given by:

$$a_k^{meas} = a_k + \nu_k$$

where the acceleration noise ν_k is assumed to be Gaussian with zero mean and standard deviation a_{std} .

State estimator

We define $X_k^{est} = \begin{pmatrix} x_k^{est} \\ v_k^{est} \end{pmatrix}$ as the best estimate of the system state X_k given all the observations $a_{j=1,\dots,k}^{meas}, x_{j=1,\dots,k}^{meas}$ up to timepoint k . Given this estimate X_k^{est} and the acceleration measurement a_k^{meas} , a prediction X_{k+1}^{pred} of the system state at timepoint $k+1$ can be obtained by double integration of the measured acceleration (as in equation 1) :

$$X_{k+1}^{pred} = AX_k^{est} + Ba_k^{meas} \quad (3)$$

The position prediction x_{k+1}^{pred} is then compared to the position measurement x_{k+1}^{meas} , and the position error $x_{k+1}^{meas} - x_{k+1}^{pred}$ is used to adjust the estimate of both position and speed at timepoint $k+1$ using the estimator gain L_k :

$$X_{k+1}^{est} = X_{k+1}^{pred} + L_k (x_{k+1}^{meas} - x_{k+1}^{pred}) = X_{k+1}^{pred} + L_k x_{k+1}^{meas} - L_k C X_{k+1}^{pred} = (1 - L_k C) X_{k+1}^{pred} + L_k x_{k+1}^{meas}$$

Replacing equations 3 and ?? in the above equation, we obtain the following dynamics for the state estimator:

$$X_{k+1}^{est} = (1 - L_k C)(AX_k^{est} + Ba_k^{meas}) + L_k(CX_{k+1} + w_{k+1}) \quad (4)$$

Optimal estimator gain

We introduce the estimation error at timestep k : $\Delta_k = X_k - X_k^{est}$, and the mean squared error $P_k = E(\Delta_k \cdot \Delta_k^T)$, where $E(\cdot)$ denotes the expected value. We wish to find the estimator gain L_k which minimises P_{k+1} .

First, we determine the evolution of the estimation error using equation ??:

$$\begin{aligned} \Delta_{k+1} &= X_{k+1} - X_{k+1}^{est} \\ &= X_{k+1} - (1 - L_k C)(AX_k^{est} + Ba_k^{meas}) - L_k(CX_{k+1} + w_{k+1}) \\ &= (1 - L_k C)(X_{k+1} - AX_k^{est} - Ba_k^{meas}) - L_k w_{k+1} \end{aligned}$$

Replacing equation 1 in the above equation, we obtain:

$$\begin{aligned} \Delta_{k+1} &= (1 - L_k C)(AX_k + Ba_k - AX_k^{est} - Ba_k^{meas}) - L_k w_{k+1} \\ &= (1 - L_k C)(A(X_k - X_k^{est}) + B(a_k - a_k^{meas})) - L_k w_{k+1} \\ &= (1 - L_k C)(A\Delta_k - Bv_k) - L_k w_{k+1} \end{aligned}$$

To obtain the evolution of the mean squared error $P_{k+1} = \mathbb{E}(\Delta_{k+1} \cdot \Delta_{k+1}^T)$, we use the fact that the measurement errors ν_k and w_k are independent from each other and independent of the current estimation error:

$$0 = \mathbb{E}(\Delta_k \cdot \nu_k^T) = \mathbb{E}(\Delta_k \cdot w_k^T) = \mathbb{E}(w_k \cdot \nu_k^T)$$

We then have

$$\begin{aligned} P_{k+1} &= (1 - L_k C) A \mathbb{E}(\Delta_k \cdot \Delta_k^T) A^T (1 - C^T L_k^T) \\ &+ (1 - L_k C) B \mathbb{E}(\nu_k \cdot \nu_k^T) B^T (1 - C^T L_k^T) \\ &+ L_k \mathbb{E}(w_k \cdot w_k^T) L_k^T \\ &= (1 - L_k C) A P_k A^T (1 - C^T L_k^T) + a_{std}^2 (1 - L_k C) B B^T (1 - C^T L_k^T) + p_{std}^2 L_k L_k^T \end{aligned}$$

To find L_k which minimises P_{k+1} , we rewrite the above equation as a second order polynomial in L_k :

$$\begin{aligned} P_{k+1} &= A P_k A^T + a_{std}^2 B B^T + L_k (-C A P_k A^T - a_{std}^2 C B B^T) \\ &+ (-A P_k A^T C^T - a_{std}^2 B B^T C^T) L_k^T + L_k (C A P_k A^T C^T + a_{std}^2 C B B^T C^T + p_{std}^2) L_k^T \end{aligned} \quad (5)$$

To simplify notations, we introduce the symmetrical matrix M_k corresponding to the mean squared error of the predictor:

$$M_k = A P_k A^T + a_{std}^2 B B^T = \mathbb{E} \left((X_k - X_k^{pred}) \cdot (X_k - X_k^{pred})^T \right) \quad (6)$$

Replacing in the equation 26, we obtain:

$$P_{k+1} = M_k - L_k C M_k - M_k C^T L_k^T + L_k (C M_k C^T + p_{std}^2) L_k^T \quad (7)$$

To determine the value of L_k which minimises equation 7, we introduce the scalar r_k corresponding to the sum of the variances of the position measurement and position prediction:

$$r_k = C M_k C^T + p_{std}^2$$

Replacing in equation 7, we obtain:

$$\begin{aligned} P_{k+1} &= M_k + r_k \left(-L_k \frac{C M_k}{r_k} - \frac{M_k C^T}{r_k} L_k^T + L_k L_k^T \right) \\ &= M_k + r_k \left(\left(L_k - \frac{M_k C^T}{r_k} \right) \left(L_k - \frac{M_k C^T}{r_k} \right)^T - \frac{M_k C^T}{r_k} \frac{C M_k}{r_k} \right) \end{aligned}$$

The value of L_k which minimises P_{k+1} is therefore given by:

$$L_k^{opt} = \frac{M_k C^T}{r_k} = \frac{M_k C^T}{C M_k C^T + p_{std}^2} \quad (8)$$

The optimal estimation error is then:

$$P_{k+1}^{opt} = M_k - \frac{M_k C^T C M_k}{r_k} = M_k - \frac{M_k C^T C M_k}{C M_k C^T + p_{std}^2} \quad (9)$$

The optimal prediction error is obtained by injecting equation 9 into equation 6:

$$M_{k+1}^{opt} = A P_{k+1}^{opt} A^T + a_{std}^2 B B^T = A \left(M_k - \frac{M_k C^T C M_k}{C M_k C^T + p_{std}^2} \right) A^T + a_{std}^2 B B^T \quad (10)$$

Steady state

We assume that the recursive equations for the optimal gain (equation 8), estimation error (equation 9) and prediction error (equation 10) have converged to their steady-state values. We wish to determine the steady-state optimal gain, which we write:

$$L = \begin{pmatrix} l_1 \\ \frac{l_2}{T} \end{pmatrix} \quad (11)$$

This can be obtained from the steady state prediction error which we write:

$$M = \begin{pmatrix} m_1 & \frac{m_2}{T} \\ \frac{m_2}{T} & \frac{m_3}{T^2} \end{pmatrix}$$

Indeed, using equation 8, we have:

$$l_1 = \frac{m_1}{m_1 + p_{std}^2} \quad (12)$$

$$l_2 = \frac{m_2}{m_1 + p_{std}^2} \quad (13)$$

We start by determining a system of three equations with three unknowns m_1, m_2, m_3 . Replacing with equations 12 and 13, we then determine a system of two equations with unknowns $l_{1,2}$, which we solve to determine the optimal steady-state gains.

Steady state prediction error

The steady state prediction error M is a fixed point of the recursive equation 10. Therefore, it verifies:

$$\begin{aligned} M &= A \left(M - \frac{MC^T CM}{CMC^T + p_{std}^2} \right) A^T + a_{std}^2 BB^T \\ 0 &= -M + AMA^T - \frac{AMC^T CMA^T}{CMC^T + p_{std}^2} + a_{std}^2 BB^T \end{aligned} \quad (14)$$

We write:

Equation 14 corresponds to a system of three equations with three unknowns m_1, m_2, m_3 . We start by writing out these three equations explicitly.

We have:

$$MA^T = \begin{pmatrix} m_1 & \frac{m_2}{T} \\ \frac{m_2}{T} & \frac{m_3}{T^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & \frac{m_2}{T} \\ \frac{m_2 + m_3}{T} & \frac{m_3}{T^2} \end{pmatrix}$$

Therefore:

$$\begin{aligned} AMA^T &= \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 + m_2 & \frac{m_2}{T} \\ \frac{m_2 + m_3}{T} & \frac{m_3}{T^2} \end{pmatrix} = \begin{pmatrix} m_1 + 2m_2 + m_3 & \frac{m_2 + m_3}{T} \\ \frac{m_2 + m_3}{T} & \frac{m_3}{T^2} \end{pmatrix} \\ -M + AMA^T &= \begin{pmatrix} 2m_2 + m_3 & \frac{m_3}{T} \\ \frac{m_3}{T} & 0 \end{pmatrix} \end{aligned}$$

Moreover:

$$CMA^T = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} m_1 + m_2 & \frac{m_2}{T} \\ \frac{m_2 + m_3}{T} & \frac{m_3}{T^2} \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & \frac{m_2}{T} \end{pmatrix}$$

Therefore:

$$\begin{aligned} AMC^T CMA^T &= (CMA^T)^T CMA^T = \begin{pmatrix} m_1 + m_2 \\ \frac{m_2}{T} \end{pmatrix} \begin{pmatrix} m_1 + m_2 & \frac{m_2}{T} \end{pmatrix} \\ &= \begin{pmatrix} (m_1 + m_2)^2 & (m_1 + m_2) \frac{m_2}{T} \\ (m_1 + m_2) \frac{m_2}{T} & \frac{m_2^2}{T^2} \end{pmatrix} \end{aligned}$$

Finally,

$$BB^T = T^4 \begin{pmatrix} \frac{1}{T} \\ \frac{1}{T} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{T} \end{pmatrix} = T^4 \begin{pmatrix} \frac{1}{4} & \frac{1}{2T} \\ \frac{1}{2T} & \frac{1}{T^2} \end{pmatrix}$$

Replacing all the above into equation 14 and using $CMC^T = m_1$, we obtain:

$$0 = \begin{pmatrix} 2m_2 + m_3 & \frac{m_3}{T} \\ \frac{m_3}{T} & 0 \end{pmatrix} - \frac{1}{m_1 + p_{std}^2} \begin{pmatrix} (m_1 + m_2)^2 & (m_1 + m_2) \frac{m_2}{T} \\ (m_1 + m_2) \frac{m_2}{T} & \frac{m_2^2}{T^2} \end{pmatrix} + (a_{std} T^2)^2 \begin{pmatrix} \frac{1}{4} & \frac{1}{2T} \\ \frac{1}{2T} & \frac{1}{T^2} \end{pmatrix}$$

This gives the following system of three equations with three unknowns:

$$0 = 2m_2 + m_3 - \frac{(m_1 + m_2)^2}{m_1 + p_{std}^2} + \frac{(a_{std}T^2)^2}{4} \quad (15)$$

$$0 = m_3 - \frac{(m_1 + m_2)m_2}{m_1 + p_{std}^2} + \frac{(a_{std}T^2)^2}{2} \quad (16)$$

$$0 = 0 - \frac{m_2^2}{m_1 + p_{std}^2} + (a_{std}T^2)^2 \Rightarrow (a_{std}T^2)^2 = \frac{m_2^2}{m_1 + p_{std}^2} \quad (17)$$

Steady state gain

To obtain a system of two equations with unknowns l_1, l_2 , we start by removing m_3 from the above equations by subtracting equation 16 from equation 15:

$$\begin{aligned} 0 &= 2m_2 + \frac{(m_1 + m_2)}{m_1 + p_{std}^2}(m_2 - m_1 - m_2) - \frac{(a_{std}T^2)^2}{4} \\ &= 2m_2 - m_1 \frac{(m_1 + m_2)}{m_1 + p_{std}^2} - \frac{(a_{std}T^2)^2}{4} \end{aligned} \quad (18)$$

Replacing equation 17 into equation 18:

$$2m_2 - m_1 \frac{(m_1 + m_2)}{m_1 + p_{std}^2} - \frac{1}{4} \frac{m_2^2}{m_1 + p_{std}^2} \quad (19)$$

Dividing by $m_1 + p_{std}^2$ we can replace m_1 and m_2 with the steady state gains l_1 and l_2 using equations 12 and 13:

$$0 = 2l_2 - l_1(l_1 + l_2) - \frac{1}{4}l_2^2 \quad (20)$$

We now need to express l_1 as a function of l_2 . Dividing equation 17 by $m_1 + p_{std}^2$ and replacing with the equation for l_2 (equation 13):

$$l_2^2 = \frac{(a_{std}T^2)^2}{m_1 + p_{std}^2} \quad (21)$$

Using equation 12, we have:

$$\begin{aligned} m_1 &= l_1(m_1 + p_{std}^2) \Rightarrow m_1(1 - l_1) = l_1 p_{std}^2 \Rightarrow m_1 = p_{std}^2 \frac{l_1}{1 - l_1} \\ &\Rightarrow m_1 + p_{std}^2 = p_{std}^2 \left(1 + \frac{l_1}{1 - l_1}\right) = p_{std}^2 \frac{1 - l_1 + l_1}{1 - l_1} = \frac{p_{std}^2}{1 - l_1} \end{aligned} \quad (22)$$

Using equation 18, we can remove m_1 from equation 21:

$$l_2^2 = \frac{(a_{std}T^2)^2}{p_{std}^2} (1 - l_1) \quad (23)$$

Introducing the dimensionless ratio $r = \frac{p_{std}}{a_{std}T^2}$, equation 23 can be rewritten as:

$$\begin{aligned} r^2 l_2^2 &= 1 - l_1 \\ l_1 &= 1 - r^2 l_2^2 \end{aligned} \quad (24)$$

Using equation 24, we can remove l_1 from equation 20:

$$\begin{aligned} 0 &= 2l_2 - (1 - r^2 l_2^2)(1 - r^2 l_2^2 + l_2) - \frac{1}{4} l_2^2 \\ &= 2l_2 - 1 + r^2 l_2^2 - l_2 + r^2 l_2^2 - (r^2 l_2^2)^2 + r^2 l_2^3 - \frac{1}{4} l_2^2 \\ 0 &= r^4 l_2^4 - r^2 l_2^3 + l_2^2 \left(\frac{1}{4} - 2r^2 \right) - l_2 + 1 \end{aligned} \quad (25)$$

Multiplying by r^4 , we find that $x = r^2 l_2$ verifies:

$$0 = x^4 - x^3 + x^2 \left(\frac{1}{4} - 2r^2 \right) - r^2 x + r^4 \quad (26)$$

This polynomial has four roots. To determine the steady-state gain, we must determine the root x^* for which both $l_2 = x^*/r^2$ and $l_1 = 1 - r^2 l_2^2 = 1 - (x^*)^2/r^2$ are real and positive over the whole range of r . We start by decomposing the fourth order polynomial in equation 26 into a product of two second order polynomials:

$$x^4 - x^3 + x^2 \left(\frac{1}{4} - 2r^2 \right) - r^2 x + r^4 = \left(x^2 + x \left(-\frac{1}{2} + 2r \right) + r^2 \right) \left(x^2 - x \left(\frac{1}{2} + 2r \right) + r^2 \right)$$

The first of these second order polynomials has discriminant:

$$\Delta_1 = \left(-\frac{1}{2} + 2r \right)^2 - 4r^2 = \frac{1}{4} - 2r$$

For $r > \frac{1}{8}$, we have $\Delta_1 < 0$ and this first second order polynomial has two complex roots. Therefore, the roots of this polynomial do not correspond to x^* .

Next, we determine the roots of the polynomial $x^2 - x \left(\frac{1}{2} + 2r \right) + r^2$. Its discriminant is:

$$\Delta_2 = \left(\frac{1}{2} + 2r \right)^2 - 4r^2 = \frac{1}{4} + 2r > 0$$

Its roots are:

$$x_{\pm} = \frac{1}{2} \left(\left(\frac{1}{2} + 2r \right) \pm \sqrt{\frac{1}{4} + 2r} \right) = \frac{1}{4} + r \pm \frac{1}{4} \sqrt{1 + 8r}$$

To determine which of these roots corresponds to x^* , we calculate $1 - x_{\pm}^2/r^2$, which must be positive over the whole range of r .

$$\begin{aligned} 1 - \frac{x_+^2}{r^2} &= 1 - \frac{1}{r^2} \left(\frac{1}{16} + r^2 + \frac{1+8r}{16} + \frac{r}{2} + \frac{1}{8} \sqrt{1+8r} + r \frac{1}{2} \sqrt{1+8r} \right) \\ &= -\frac{1}{r^2} \left(-r^2 + \frac{1}{16} + r^2 + \frac{1+8r}{16} + \frac{r}{2} + \frac{1}{8} \sqrt{1+8r} + r \frac{1}{2} \sqrt{1+8r} \right) \\ &= -\frac{1}{r^2} \left(\frac{1}{16} + \frac{1+8r}{16} + \frac{r}{2} + \frac{1}{8} \sqrt{1+8r} + r \frac{1}{2} \sqrt{1+8r} \right) < 0 \end{aligned}$$

Therefore x_+ does not correspond to x^* .

$$\begin{aligned} 1 - \frac{x_-^2}{r^2} &= 1 - \frac{1}{r^2} \left(\frac{1}{16} + r^2 + \frac{1+8r}{16} + \frac{r}{2} - \frac{1}{8} \sqrt{1+8r} - \frac{r}{2} \sqrt{1+8r} \right) \\ &= -\frac{1}{r^2} \left(\frac{1}{16} + \frac{1+8r}{16} + \frac{r}{2} - \frac{1}{8} \sqrt{1+8r} - \frac{r}{2} \sqrt{1+8r} \right) \\ &= -\frac{1}{r^2} \left(\frac{1}{8} + r - \left(\frac{1}{8} + \frac{r}{2} \right) \sqrt{1+8r} \right) = -\frac{1}{8r^2} (1 + 8r - (1 + 4r) \sqrt{1+8r}) \\ &= -\frac{1}{r^2} \sqrt{1+8r} (\sqrt{1+8r} - 1 - 4r) \end{aligned}$$

$$1 - \frac{x_-^2}{r^2} > 0 \Leftrightarrow 1 + 4r > \sqrt{1+8r} \Leftrightarrow 1 + 8r + 16r^2 > 1 + 8r \Leftrightarrow r^2 > 0$$

Therefore, x_- corresponds to x^* and the optimal gains are given by (Figure 1):

$$l_2 = \frac{x_-}{r^2} = \frac{1 + 4r - \sqrt{1+8r}}{4r^2} \quad (27)$$

$$l_1 = 1 - r^2 l_2^2 = \frac{(\sqrt{1+8r})(1+4r) - 1 - 8r}{8r^2} \quad (28)$$

Small p_{std} and small a_{std} limits

In the limit where $p_{std} \rightarrow 0$, we have $r \rightarrow 0$. We do a Taylor expansion:

$$\begin{aligned} l_2 &= \frac{1 + 4r - \sqrt{1+8r}}{4r^2} \approx \frac{1 + 4r - (1 + 4r - (8r)^2/8)}{4r^2} = 2 \\ l_1 &= 1 - r^2 l_2^2 \approx 1 \end{aligned}$$

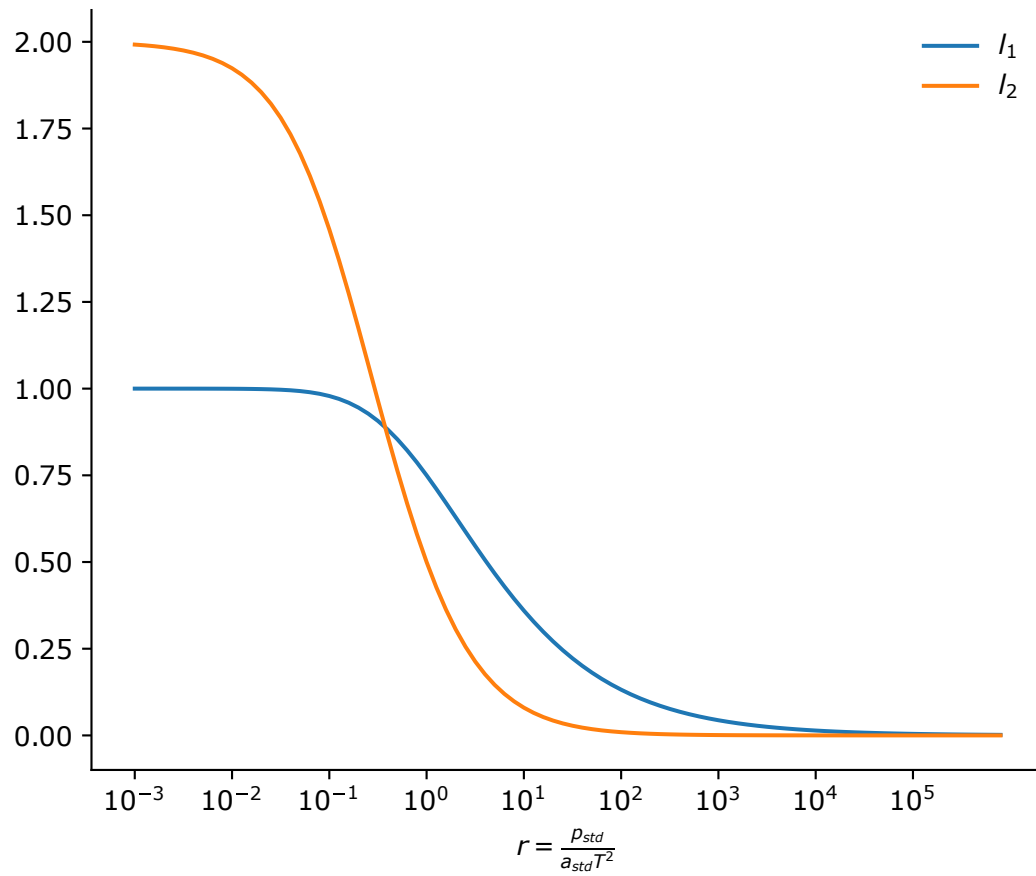


Figure 1: Estimator gains as a function of the ratio of position to acceleration noise

Note that, in this limit, $l_1 = 1$, therefore the position estimate exactly follows the position measurement.

In the limit where $a_{std} \rightarrow 0$, we have $r \rightarrow +\text{inf}$.

$$\begin{aligned} l_2 &= \frac{1 + 4r - \sqrt{1 + 8r}}{4r^2} \approx \frac{4r}{4r^2} = \frac{1}{r} \rightarrow 0 \\ l_1 &= 1 - r^2 l_2^2 \rightarrow 0 \end{aligned}$$

Note that, in this limit, $l_1 = l_2 = 0$, therefore the position measurement is ignored and the position estimate is obtained by double integration of the acceleration measurement.

Estimator transfer function

The estimate is a linear combination of the position and acceleration measurements. To gain insight into how these measurements are combined, we calculate the steady-state transfer function from from position and acceleration measurements to the estimate.

Discrete Fourier transform

We introduce the discrete Fourier transform: the Fourier coefficients (which we denote by a tilde) of a finite sequence $\{s_{n=0,\dots,N-1}\}$ of length N are given by:

$$\forall j \in [0, \dots, N-1], \tilde{s}_j = \sum_{k=0}^{N-1} s_k \exp\left(-i \frac{2\pi j k}{N}\right)$$

The original sequence $\{s_{n=0,\dots,N-1}\}$ can then be obtained from the Fourier coefficients using the inverse discrete Fourier transform:

$$\begin{aligned} \forall k \in [0, \dots, N-1], s_k &= \frac{1}{N} \sum_{j=0}^{N-1} \tilde{s}_j \exp\left(i \frac{2\pi j k}{N}\right) = \frac{1}{N} \sum_{j=0}^{N-1} \tilde{s}_j \left(\exp\left(i \frac{2\pi j}{N}\right)\right)^k \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \tilde{s}_j (z_j)^k \end{aligned}$$

where $z_j = \exp\left(-i \frac{2\pi j}{N}\right)$.

We introduce the discrete Fourier coefficients \tilde{x}_j^{meas} , \tilde{a}_j^{meas} , \tilde{X}_j^{est} of the position measurement, acceleration measurement and estimate. The original signals can

then be obtained using the inverse discrete Fourier transform:

$$\forall k \in [0, \dots, N-1], a_k^{meas} = \frac{1}{N} \sum_{j=0}^{N-1} \tilde{a}_j^{meas} (z_j)^k \quad (29)$$

$$x_k^{meas} = \frac{1}{N} \sum_{j=0}^{N-1} \tilde{x}_j^{meas} (z_j)^k \quad (30)$$

$$X_k^{est} = \frac{1}{N} \sum_{j=0}^{N-1} \tilde{X}_j^{est} (z_j)^k \quad (31)$$

Discrete Fourier transform of the estimator dynamics

To obtain the transfer functions from position and acceleration measurements to the estimate, we need to express \tilde{X}_j^{est} as a linear combination of \tilde{x}_j^{meas} and \tilde{a}_j^{meas} . For this, we first express the steady-state estimator dynamics in terms of the position and acceleration measurements, by replacing equation 2 into equation 4:

$$X_{k+1}^{est} = (1 - LC)(AX_k^{est} + Ba_k^{meas}) + Lx_{k+1}^{meas} \quad (32)$$

To simplify notations, we introduce:

$$\begin{aligned} D &= (1 - LC)A \\ U &= (1 - LC)B \end{aligned}$$

Replacing in equation 32, we obtain:

$$\begin{aligned} X_{k+1}^{est} &= DX_k^{est} + Ua_k^{meas} + Lx_{k+1}^{meas} \\ \Rightarrow X_{k+1}^{est} - DX_k^{est} &= Ua_k^{meas} + Lx_{k+1}^{meas} \end{aligned} \quad (33)$$

We now replace the expressions in equations 29 to 31 into equation 33, using:

$$\begin{aligned} x_{k+1}^{meas} &= \frac{1}{N} \sum_{j=0}^{N-1} \tilde{x}_j^{meas} (z_j)^{k+1} = \frac{1}{N} \sum_{j=0}^{N-1} \tilde{x}_j^{meas} (z_j)^k z_j \\ X_{k+1}^{est} &= \frac{1}{N} \sum_{j=0}^{N-1} \tilde{X}_j^{est} (z_j)^k z_j \\ X_{k+1}^{est} - DX_k^{est} &= Ua_k^{meas} + Lx_{k+1}^{meas} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} (z_j)^k (z_j - D) \tilde{X}_j^{est} = \frac{1}{N} \sum_{j=0}^{N-1} (z_j)^k (U\tilde{a}_j^{meas} + z_j L\tilde{x}_j^{meas}) \end{aligned} \quad (34)$$

For equation 34 to be valid for all k in $[0, \dots, N-1]$, we must have:

$$\forall j \in [0, \dots, N-1], (z_j - D) \tilde{X}_j^{est} = (U\tilde{a}_j^{meas} + z_j L\tilde{x}_j^{meas})$$

We can thus express \tilde{X}_j^{est} as a function of \tilde{a}_j^{meas} and \tilde{x}_j^{meas} :

$$\begin{aligned} \forall j \in [0, \dots, N-1], \tilde{X}_j^{est} &= (z_j - D)^{-1} U \tilde{a}_j^{meas} + z_j (z_j - D)^{-1} L \tilde{x}_j^{meas} \\ &= H_{a \rightarrow X}(z_j) \tilde{a}_j^{meas} + H_{x \rightarrow X}(z_j) \tilde{x}_j^{meas} \end{aligned}$$

where $H_{a \rightarrow X} : z \rightarrow (z - D)^{-1} U$ is the transfer function from acceleration measurement to estimate, and $H_{x \rightarrow X} : z \rightarrow z (z - D)^{-1} L$ is the transfer function from position measurement to estimate.

Calculation of the estimator transfer functions

To obtain explicit expressions for $H_{a \rightarrow X}(z)$ and $H_{x \rightarrow X}(z)$, we start by showing that:

$$(z_j - D)^{-1} = \frac{1}{\det(z_j - D)} (z_j - \det(D)D^{-1}) \quad (35)$$

For this, we write:

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

We use the fact that the inverse of any such 2 by 2 matrix D is given by:

$$D^{-1} = \frac{1}{\det(D)} \begin{pmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{pmatrix}$$

Indeed:

$$\begin{pmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} d_{22}d_{11} - d_{12}d_{21} & 0 \\ 0 & d_{12}d_{21} + d_{22}d_{11} \end{pmatrix} = \det(D) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We likewise have:

$$\begin{aligned} (z_j - D)^{-1} &= \frac{1}{\det(z_j - D)} \begin{pmatrix} z_j - d_{22} & d_{12} \\ d_{21} & z_j - d_{11} \end{pmatrix} \\ &= \frac{1}{\det(z_j - D)} (z_j - \det(D)D^{-1}) \end{aligned}$$

To obtain an expression for $\det(D)$, we write:

$$\begin{aligned} \det(D) &= \det((1 - LC)A) = \det(1 - LC) \det(A) = \det(1 - LC) \\ 1 - LC &= 1 - \begin{pmatrix} l_1 \\ l_2 \\ T \end{pmatrix} (1 \ 0) = \begin{pmatrix} 1 - l_1 & 0 \\ -\frac{l_2}{T} & 1 \end{pmatrix} \end{aligned} \quad (36)$$

$$\Rightarrow \det(D) = \det(1 - LC) = 1 - l_1 \quad (37)$$

To obtain an expression for $\det(z_j - D)$, we write:

$$\det(z_j - D) = (z_j - d_{11})(z_j - d_{22}) - d_{21}d_{12} = z_j^2 - z_j(d_{11} + d_{22}) + \det(D)$$

To obtain an expression for $d_{11} + d_{22}$, we explicitly calculate D (using equation 36):

$$\begin{aligned} D = (1 - LC)A &= \begin{pmatrix} 1 - l_1 & 0 \\ -\frac{l_2}{T} & 1 \end{pmatrix} \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - l_1 & T(1 - l_1) \\ -\frac{l_2}{T} & 1 - l_2 \end{pmatrix} \\ &\Rightarrow d_{11} + d_{22} = 2 - (l_1 + l_2) \\ &\Rightarrow \det(z_j - D) = z_j^2 - z_j(2 - (l_1 + l_2)) + 1 - l_1 \end{aligned} \quad (38)$$

Transfer function from the position measurement

The transfer function from the position measurement is given by:

$$\begin{aligned} H_{x \rightarrow X}(z_j) &= (z_j - D)^{-1}L = \frac{1}{\det(z_j - D)}(z_j - \det(D)D^{-1})L \\ &= \frac{1}{\det(z_j - D)}(z_j - \det(D)A^{-1}(1 - LC)^{-1})L \end{aligned} \quad (39)$$

We have:

$$(1 - LC)^{-1}L = \frac{1}{\det(1 - LC)} \begin{pmatrix} 1 & 0 \\ \frac{l_2}{T} & 1 - l_1 \end{pmatrix} \begin{pmatrix} l_1 \\ \frac{l_2}{T} \end{pmatrix} = \frac{1}{\det(D)} \begin{pmatrix} l_1 \\ \frac{l_2}{T}(l_1 + 1 - l_1) \end{pmatrix} = \frac{1}{\det(D)}L$$

Reinjecting this into equation 39, we have:

$$\begin{aligned} H_{x \rightarrow X}(z_j) &= \frac{1}{\det(z_j - D)}(z_j - A^{-1})L = \frac{1}{\det(z_j - D)} \begin{pmatrix} z_j - 1 & T \\ 0 & z_j - 1 \end{pmatrix} \begin{pmatrix} l_1 \\ \frac{l_2}{T} \end{pmatrix} \\ &= \frac{1}{\det(z_j - D)} \begin{pmatrix} l_1(z_j - 1) + l_2 \\ \frac{l_2}{T}(z_j - 1) \end{pmatrix} \end{aligned} \quad (40)$$

Injecting equation 38 into equation 40, we obtain the transfer function from the position measurement to the position estimate x^{est} (Figure 2.A):

$$H_{x \rightarrow x^{est}}(z_j) = \frac{l_1(z_j - 1) + l_2}{z_j^2 - z_j(2 - (l_1 + l_2)) + 1 - l_1} \quad (41)$$

Likewise, the transfer function from the position measurement to the velocity estimate v^{est} is given by:

$$H_{x \rightarrow v^{est}}(z_j) = \frac{1}{T} \frac{l_2(z_j - 1)}{z_j^2 - z_j(2 - (l_1 + l_2)) + 1 - l_1}$$

Transfer function from the acceleration measurement

The transfer function from the acceleration measurement is given by:

$$\begin{aligned}
H_{x \rightarrow X}(z_j) &= (z_j - D)^{-1} U = \frac{1}{\det(z_j - D)} (z_j - \det(D)D^{-1})(1 - LC)B \\
&= \frac{1}{\det(z_j - D)} (z_j(1 - LC) - \det(D)A^{-1}(1 - LC)^{-1}(1 - LC)) B \\
&= \frac{1}{\det(z_j - D)} \left(z_j \begin{pmatrix} 1 - l_1 & 0 \\ -\frac{l_2}{T} & 1 \end{pmatrix} - (1 - l_1) \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix} \right) B \\
&= \frac{1}{\det(z_j - D)} \begin{pmatrix} (z_j - 1)(1 - l_1) & T(1 - l_1) \\ -z_j \frac{l_2}{T} & z_j - (1 - l_1) \end{pmatrix} \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix} \\
&= \frac{1}{\det(z_j - D)} \begin{pmatrix} \frac{T^2}{2}(1 - l_1)(z_j - 1 + 2) \\ T(-\frac{l_2}{2}z_j + z_j - 1 + l_1) \end{pmatrix} \\
&= \frac{1}{\det(z_j - D)} \begin{pmatrix} \frac{T^2}{2}(1 - l_1)(z_j + 1) \\ T(z_j(1 - \frac{l_2}{2}) - 1 + l_1) \end{pmatrix} \tag{42}
\end{aligned}$$

Injecting equation 38 into equation 42, we obtain the transfer function from the acceleration measurement to the position estimate:

$$H_{a \rightarrow x^{est}}(z_j) = \frac{T^2}{2} \frac{(1 - l_1)(z_j + 1)}{z_j^2 - z_j(2 - (l_1 + l_2)) + 1 - l_1} \tag{43}$$

Likewise, the transfer function from the acceleration measurement to the velocity estimate is given by:

$$H_{a \rightarrow v^{est}}(z_j) = T \frac{z_j(1 - \frac{l_2}{2}) - 1 + l_1}{z_j^2 - z_j(2 - (l_1 + l_2)) + 1 - l_1}$$

Transfer function of double integration

We wish to compare the estimator to the state estimate X^{int} obtained by integration of the acceleration measurement. This state estimate follows the dynamics:

$$X_{k+1}^{int} = AX_k^{int} + Ba_k^{meas}$$

A derivation similar to the one presented above yields the following transfer function from the acceleration measurement to the integrated state:

$$\begin{aligned}
H_{a \rightarrow X^{int}}(z_j) &= (z_j - A)^{-1} B = \frac{1}{\det(z_j - A)} (z - \det(A)A^{-1})B \\
&= \frac{1}{\det(z_j - A)} \begin{pmatrix} z_j - 1 & T \\ 0 & z_j - 1 \end{pmatrix} \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix} = \frac{1}{\det(z_j - A)} \begin{pmatrix} \frac{T^2}{2}(z_j + 1) \\ T(z_j - 1) \end{pmatrix} \\
\det(z_j - A) &= \det \begin{pmatrix} z_j - 1 & -T \\ 0 & z_j - 1 \end{pmatrix} = (z_j - 1)^2
\end{aligned}$$

The transfer function from the acceleration measurement to the position obtained by integration x^{int} is thus:

$$H_{a \rightarrow x^{int}}(z_j) = \frac{T^2}{2} \frac{z_j + 1}{(z_j - 1)^2}$$

The ratio of the estimator and integrator positions (Figure 2.B) is given by:

$$\begin{aligned} \frac{H_{a \rightarrow x^{est}}(z_j)}{H_{a \rightarrow x^{int}}(z_j)} &= (1 - l_1) \frac{(z_j - 1)^2}{z_j^2 - z_j(2 - (l_1 + l_2)) + 1 - l_1} \\ &= (1 - l_1) \frac{(z_j - 1)^2}{(z_j - 1)^2 + z_j(l_1 + l_2) - l_1} \end{aligned} \quad (44)$$

Note that, at low frequencies (i.e. when $z_j \rightarrow 1$), the integrator transfer function $H_{a \rightarrow x^{int}}(z_j)$ diverges whereas the estimator transfer function $H_{a \rightarrow x^{est}}(z_j)$ reaches a finite value $\frac{T^2(1-l_1)^2}{2l_2}$. The ratio of the two thus goes to zero at low frequencies.

The transfer function from the acceleration measurement to the velocity obtained by integration v^{int} is :

$$H_{a \rightarrow v^{int}}(z_j) = \frac{T}{z_j - 1}$$

The ratio of the estimator and integrator velocities is given by:

$$\frac{H_{a \rightarrow v^{est}}(z_j)}{H_{a \rightarrow v^{int}}(z_j)} = (z_j - 1) \frac{z_j(1 - \frac{l_2}{2}) - 1 + l_1}{z_j^2 - z_j(2 - (l_1 + l_2)) + 1 - l_1}$$

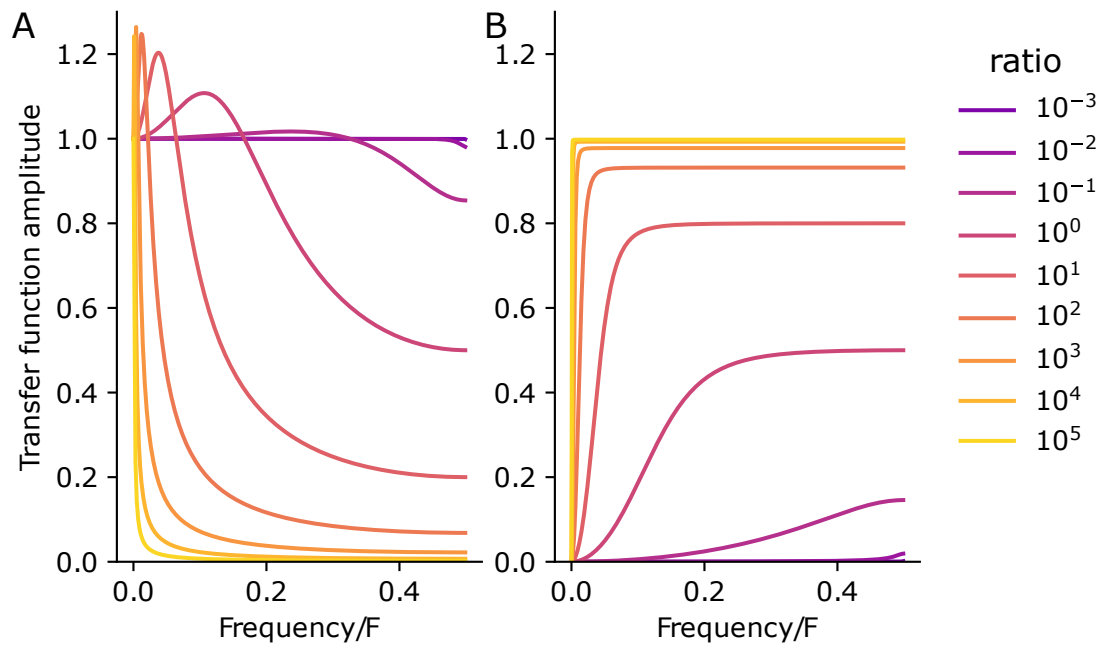


Figure 2: Amplitude of the transfer function (A) from the kinematic to the estimate position, and (B) from the double integral of acceleration to the estimate position, as a function of the frequency divided by the sampling frequency $F = 1/T$, for different values of the noise ratio r