

Cellular automata modelling of direct selling multilevel marketing dynamics

L. FANTI^{1*}

Calista Conseil, France

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This paper deals with the development of a new cellular automata-based model to describe the dynamics of direct selling, multilevel marketing companies. While many cellular automata models have been developed in the field of theoretical marketing, direct selling with multilevel marketing has not been addressed up to now. This specific marketing technique requires to account for both spatial spreading and the internal dynamics of each cell, in which six types of individuals are defined. The associated populations in each cell evolve using a new proposed aggregated-level model made of discrete time equations based on a mix of classical social contagion models and Boltzmann-inspired relaxation-reaction models. The proposed model is able to account for spatial spreading via weak social ties on a Moore neighbourhood, along with internal dynamics (growth, decay and equilibrium) of the six interacting populations in each cell. The model incorporates user-defined parameters that allow to model the dynamics of a given company via parameter fitting.

Key words: Cellular automata, direct selling, multilevel marketing, social contagion model, innovation diffusion model, customer resistance.

1 INTRODUCTION

Cellular automata have been used for a long time in the field of theoretical marketing and innovation diffusion theory. Individual-level numerical

* email: lydiafanti@calistagroupe.com

models have been developed that are able to recover the trends predicted by aggregated-level models such as the Bass model [2, 3] and its very large number of variants, in which a social contagion phenomenon is modelled via a set of ODEs (see [16] for a comprehensive survey). The original Bass model has been mathematically investigated by several authors, e.g. [30], and is very versatile, leading to a huge number of variants developed to account for many effects yielding, e.g. [32, 53].

The individual level models (which correspond to the so-called microscopic level models in statistical physics) are mainly agent based models, implemented on graphs or via cellular automata or graph cellular automata, mostly with the purpose of studying the micro-macro transition, i.e. recovering the aggregated-level models from the more fundamental microscopic models via some statistical averaging technique, e.g. [17, 7, 34, 20, 52]. The same approach is commonly used in classical physics, e.g. by deriving the classical macroscopic continuum mechanics equations from the Boltzmann equation at the mesoscopic level or the Hamilton's equation for molecular dynamics at the microscopic level. Presently, deterministic or probabilistic cellular automata have been observed to recover Bass-type results for many topics, among which innovation diffusion [12], classical marketing [9, 13, 37, 39], word-of-mouth marketing [11], resistance to innovation [29], competition [40, 54], spread dynamics in social networks [23] and multi-information dissemination [41].

While the Bass-type models have been customised to mimic a huge number of effects (competition between products, consumer resistance ...), they are limited to the simulation of the dynamics of a global population without accounting the spatial spreading character of social contagion, therefore restricting their capability to capture fine details of the diffusion among a heterogeneously spatially distributed population.

The present paper deals with the development a cellular automata model able to capture both the spatial spreading character of the social diffusion and the internal dynamics of each cell, with the specific purpose of modelling the dynamics of a Direct Selling Multi-Level Marketing (DS-MLM) company. This type of pyramidal marketing organisation yields the definition of a hierarchy of populations with complex transition rules between them (seven in the present case), then escaping the classical agent-based approaches in which a few population types are accounted only, in a way similar to dynamical system models like the 2-populations KerMack-McKendrick Susceptible-Infective model and its extensions for larger population sets (e.g. [28, 19, 50, 49, 51]). Implementation on graphs has also been addressed with

the same purpose, e.g. [6, 31]. Therefore, it appears that a model that hybridising cellular automata for spatial spreading and aggregated-level modelling for defining transition rules between population types is a promising approach. Only very few models like that have been proposed, e.g. to study the influence of transit media on information spreading in urban area [26].

The paper is organised as follows. The proposed model is discussed in Sect. 2. The single cell dynamics, i.e. the dynamics generated by the transition rules inside an isolated cell by neglecting the influence of its neighbourhood is discussed in Sect. 2, including the analytical analysis of its non-linear steady state equilibrium solution. The influence of the neighbourhood along with the spatial spreading features are investigated in Sect. 4. Concluding remarks are given in Sect. 5.

2 MODEL DESCRIPTION

2.1 Statement of the problem

The present model aims to describe a DS-MLM dynamics, in which a seller manages a group of hosts, each host being expected to organise some parties at home with socially close individuals (friends, family members) and the seller during which goods will be sold. The host is rewarded for the sells made during the party, while the seller will receive a percentage of the total gains made each month during all the parties of his group of hosts, details being governed by the company's policy. The resulting organisation yields a dynamics that may differ a lot from classical marketing based on retailer channel, see [1, 22, 36].

Because sells occur during home parties, the spatial character of the dynamics is very strong for direct selling (see [18] for a discussion of the spatial character of classical marketing before the rise of e-marketing), since hosts will only invite potential customers that live close to their homes. The typical diffusion of this kind of activity is of word-of-mouth type, the spreading occurring through personal social networks of hosts and customers. Motivated customers may organise their own parties and become hosts, and interested hosts may become sellers by signing a contract with the company.

To model this dynamics, it is chosen to define seven types of individuals which span the space of states underlying the cellular automata :

1. The total number of potential customers N , which is assumed to be time independent.

2. The number of active customers, i.e. potential customers who have attended at least one party and bought some goods, $N_{ca}(t)$.
3. The number of lost customers $N_{cl}(t)$, i.e. active customers that get lost. The main reason why this occurs is that their host gets out of business. Lack of satisfaction due to poor quality or too high prices may also play a role [10], but these factors are observed to be less important in the case of DS-MLM (see [35] for a general discussion about consumer resistance). Accounting for consumers that exhibit some resistance has been proposed by several authors, e.g. [5, 10].
4. The number of active hosts, $N_{ha}(t)$, i.e. former active customers who decided to become hosts and who organise parties. Each host has a personal private social network (friends, family) of $Y_{rel} > 0$ individuals and will organise $Y_a > 0$ home parties per year. Each host is assumed to have $Y_c > 0$ active customers in his private network at equilibrium, the equilibrium corresponding to an optimal trade off between many influential parameters such as the motivation of the host, its availability, socio-economical features of its network ... see e.g. [42, 33, 44, 25].
5. The number of lost hosts $N_{hl}(t)$, i.e. active hosts that ceased their activity. The main reason why this occurs is that their referring seller becomes inactive. Negative feedbacks from the customers who did attend their parties might also play a role, but it is shown to be less important [42, 33, 44, 25].
6. The number of active sellers, $N_{sa}(t)$, i.e. former hosts who made the decision to become a seller and recruit some hosts to organise some parties to increase their net income. Each active seller is assumed to have Y_{co} active hosts in his personal network at equilibrium.
7. The number of lost sellers, $N_{sl}(t)$, i.e. sellers who resigned their contract with the company and don't manage their hosts any longer. There are multiple factors that usually lead a seller to resign: loss of attractiveness of the business, conflict with the company management, moving to another place, life accidents ... [42, 33, 44, 25].

The strong spatial character of both the selling dynamics and the business spreading via word of mouth marketing through personal social networks renders the cellular automata modelling very efficient. Therefore, each population will be computed in each cell, leading to the definition of a synchronous

continuous cellular automata with seven types of individuals with transition rules that govern the change of type of individuals at each time iteration.

2.2 Discrete time equation-based aggregated transition rules for a single cell

Since the number of individuals present in each cell is expected to be large (from a few hundreds to tens of thousands), it is chosen to describe the dynamics at the aggregated level rather than at the individual level. Therefore, deterministic equations are used to evaluate the total number of individuals that move to another type at each time iteration, rather than implementing an agent-based model to follow each individual. The deterministic character of the model is due to the fact that the number of individuals is large enough to assume that individuals fluctuations around an ensemble-averaged behavioural model for each individual type will have negligible effects on global statistics of the full system.

The model proposed here is a discrete-time system, which is perfectly coherent with the cellular automata implementation and the monthly character of available data related to the business dynamics modelled here. Consequently, one time iteration will correspond to a one month period in the following. The transition rules implemented in each cell are now described. For the sake of simplicity, the spatial indices related to the cell location are not mentioned, being understood that the dynamics described here is purely local (i.e. limited to the cell interior).

The transition from the state of potential customer to the state of active customer is modelled here as a type of social contagion mostly driven by word-of-mouth mechanisms. Parameters appearing in such a model are meaningful and can be associated to practical psychological, social and economical mechanisms [47, 46, 43]. Consequently, they can be tuned to fit a particular company dynamics using data coming from the targeted company or market, e.g. [8, 4, 40, 27, 24].

The number of potential customers transformed into active customers between iterations n and $(n + 1)$ is evaluated as

$$\Delta_{cp \rightarrow ca}(n) = \alpha_1 \frac{Y_a Y_{rel}}{12} \frac{N_{ha}(n)}{N} (N - N_{ca}(n) - N_{cl}(n) - N_{ha}(n) - N_{hl}(n) - N_{sa}(n) - N_{sl}(n)) \quad (1)$$

where $\alpha_1 \in [0, 1]$ is a company-related parameter, which measures the probability of transforming a contacted individual into an active customer.

This parameter depends in practice on a large number of factors, such as quality and price of the product, but also experience and skills of the host and the seller, company policy and incentives ... [42, 33, 44, 25].

This is a classical social contagion model, referred to as imitator-type diffusion in Bass-type models for innovation diffusion. It is important noting that this imitation model is recovered as a limit at the aggregated level by agent-type models for individual behaviours and graph cellular automata models for social contagion.

The number of active customers becoming active hosts between iterations n and $(n + 1)$ is computed as

$$\Delta_{ca \rightarrow ha}(n) = \omega_1 \left(\frac{N_{ca}(n)}{Y_c} - N_{ha}(n) \right) \quad (2)$$

where the relaxation parameter $\omega_1 > 0$ measures the time needed by the system to relax toward equilibrium. In the present case, it will be a function of the company policy, management efficiency and economic and social parameters (need for an additional income, strength of personal network, acceptance by social environment ...) [33, 25]. From the mathematical point of view, this term is very similar to the Bhatnagar-Gross-Krook relaxation model for the collision term found in Boltzmann's statistical gas kinetic, and was proposed to model word-of-mouth process in [48]. This modelling approach as also the advantage of reflecting the bottom-up evolution of individuals in this pyramidal scheme, in which the number of active customers (and then the total sales revenue of the company) governs the full dynamics, the rest resulting from an adaptation to the sales volume because of the relative reward policy.

Similarly, the number of active hosts becoming active sellers between iterations n and $(n + 1)$ is modelled as

$$\Delta_{ha \rightarrow sa}(n) = \omega_2 \left(\frac{N_{ha}(n)}{Y_{co}} - N_{sa}(n) \right) \quad (3)$$

where $\omega_2 > 0$ is related to the efficiency of the company to recruit the most active hosts as sellers. Here again, this parameter depends a many sociological and economical, but also on the company policy and management.

As mentioned above, the main source of loss of active hosts and active customers in DS-MLM is the fact that a seller will stop its activity. Since active customers and active hosts mostly belong to personal private social networks of active hosts and active sellers, respectively, it is assumed that the resignation of an active seller results in the transformation of its active hosts into lost hosts and the associated active customers into lost customers. This

loss mechanisms differs from the rejection described in classical innovation diffusion theory (e.g. [10]), which mainly originates in negative word-of-mouth, something that exists in the present case but is significantly weaker than loss of personal networks of sellers.

The number of active sellers becoming lost sellers between iterations n and $(n + 1)$ is evaluated as

$$\Delta_{sa \rightarrow sl}(n) = \alpha_2 N_{sa}(n) \quad (4)$$

where $\alpha_2 \in [0, 1]$ is the fraction of active sellers leaving every month. This sink term is similar to the one introduced in the extended Bass model proposed in [45, 21] to account for the existence of several populations, including some individuals which are close to the present "lost" populations. Assuming that the active hosts and active customers are equally distributed between active sellers and active hosts (which is coherent with the ensemble-averaged approach used in the present model), respectively, the number of active hosts and active customers lost between iterations n and $(n + 1)$ are

$$\Delta_{ha \rightarrow hl}(n) = \alpha_2 N_{ha}(n) \quad (5)$$

and

$$\Delta_{ca \rightarrow cl}(n) = \alpha_2 N_{ca}(n) \quad (6)$$

In order to close the model, the fact that every month a fraction $\alpha_3 \in [0, 1]$ of lost customers are recovered as potential customers is taken into account

$$\Delta_{cl \rightarrow cp}(n) = \alpha_3 N_{cl}(n) \quad (7)$$

This term encompasses several practical cases, such as lost customers entering the social networks of some active hosts, decrease in consumer resistance among deceived customers, changes in the populations (individuals moving to another places replaced by new ones) ...

The same rationale is applied to lost hosts and lost sellers, yielding

$$\Delta_{hl \rightarrow cp}(n) = \alpha_4 N_{hl}(n) \quad (8)$$

$$\Delta_{sl \rightarrow cp}(n) = \alpha_5 N_{sl}(n) \quad (9)$$

with $\alpha_4 \in [0, 1]$ and $\alpha_5 \in [0, 1]$. The resulting full discrete time model is

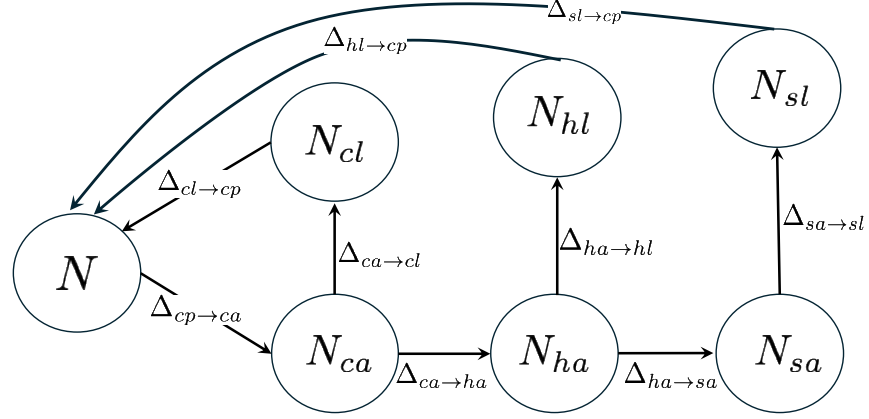


FIGURE 1
Schematic view of transition rules of the cellular automata.

$$N_{ca}(n+1) = N_{ca}(n) + \Delta_{cp \rightarrow ca}(n) - \Delta_{ca \rightarrow cl}(n) \quad (10)$$

$$N_{cl}(n+1) = N_{cl}(n) + \Delta_{ca \rightarrow cl}(n) - \Delta_{cl \rightarrow cp}(n) \quad (11)$$

$$N_{ha}(n+1) = N_{ha}(n) + \Delta_{ca \rightarrow ha}(n) - \Delta_{ha \rightarrow hl}(n) \quad (12)$$

$$N_{hl}(n+1) = N_{hl}(n) + \Delta_{ha \rightarrow hl}(n) - \Delta_{hl \rightarrow cp}(n) \quad (13)$$

$$N_{sa}(n+1) = N_{sa}(n) + \Delta_{ha \rightarrow sa}(n) - \Delta_{sa \rightarrow sl}(n) \quad (14)$$

$$N_{sl}(n+1) = N_{sl}(n) + \Delta_{sa \rightarrow sl}(n) - \Delta_{sl \rightarrow cp}(n) \quad (15)$$

which is illustrated in Fig. 1.

The present model is conservative, i.e. the number total number of individuals is strictly preserved in each cell at every time step and remains equal to N .

The discrete model Eqs. (10) - (15) can be interpreted as a first-order explicit Euler integration of the following continuous ordinary differential equation system (with an integration time step taken equal to 1):

$$\begin{aligned} \frac{dN_{ca}}{dt} &= \beta N_{ha} (N - N_{ca} - N_{cl} - N_{ha} - N_{hl} - N_{sa} - N_{sl}) \\ &\quad - \alpha_2 N_{ca} \end{aligned} \quad (16)$$

$$\frac{dN_{cl}}{dt} = \alpha_2 N_{ca} - \alpha_3 N_{cl} \quad (17)$$

$$\frac{dN_{ha}}{dt} = \omega_1 \left(\frac{N_{ca}}{Y_c} - N_{ha} \right) - \alpha_2 N_{ha} \quad (18)$$

$$\frac{dN_{hl}}{dt} = \alpha_2 N_{ha} - \alpha_4 N_{hl} \quad (19)$$

$$\frac{dN_{sa}}{dt} = \omega_2 \left(\frac{N_{ha}}{Y_{co}} - N_{sa} \right) - \alpha_2 N_{sa} \quad (20)$$

$$\frac{dN_{sl}}{dt} = \alpha_2 N_{sa} - \alpha_5 N_{sl} \quad (21)$$

with

$$\beta = \alpha_1 \frac{Y_a Y_{rel}}{12N} \quad (22)$$

It is observed that the system is nonlinear, since the right-hand side of Eq. (16) exhibits a quadratic term with respect to N_{ha} . This interpretation necessitates some comments. First, the model is intrinsically a discrete time model equipped with coefficients deduced from the analysis of a monthly-refreshed data-basis of the company in practical cases [38]. The continuous time equations are nothing but a limit, for which parameters and coefficients needs to be recalibrated. This re-calibration when switching from the discrete time to the continuous time model has been discussed for the Bass innovation diffusion model by several authors. An additional comment is that in this case, the difference in the results between discrete- and continuous models don't originate in the numerical errors induced by Euler time integration scheme, but to the fact that discrete model coefficients have been calibrated to optimise the discrete time model (and then they account for time integration errors) [30]. Therefore, there is no a priori reason why the best corresponding continuous time model should keep the same values for $\beta, \omega_1, \omega_2, \alpha_2, \alpha_3, \alpha_4$ and α_5 .

2.3 Cellular automata modelling of spatial spreading via weak ties

The discrete time model displayed above is related to a single isolated cell dynamics. In order to model the dynamics and spreading of the activity over a large area, a cellular automata is defined, whose cells are defined to map a given territory, the number of potential customers in each cell $N(i, j)$ (where

indices i and j denote the location of the cell in the computational grid) being evaluated using geographical and sociological data.

In order to account for the neighbouring cells influence on a cell dynamics, it is chosen to account for more distant interactions of hosts in cell with potential customers in the neighbouring cells. This kind of interactions may be interpreted as weak ties [15, 14] (weak because the spatial distance tends to weaken the interaction strength and the capability of a potential customer to attend a selling party far from his home) is known to have an important effect in direct selling, since it allows for the spatial spreading of the activity.

To model this effect, the number of active hosts $N_{ha}(n)$ used to compute $\Delta_{ca \rightarrow ha}$ in the cell (i, j) according to Eq. (2) is replaced by a weighted sum over the Moore neighbourhood $\mathcal{M}(i, j)$ of this cell, yielding

$$\Delta_{ca \rightarrow ha}(i, j) = \omega_1 \left(\frac{N_{ca}^+(i, j, n)}{Y_c} - N_{ha}(i, j, n) \right) \quad (23)$$

with

$$N_{ca}^+(i, j, n) = N_{ca}(i, j, n) + \gamma \sum_{(p, q) \in \mathcal{M}(i, j)} N_{ca}(p, q, n) \quad (24)$$

where $\gamma \in [0, 1]$ is a weighting factor accounting for the decrease of the influence due to the spatial distance.

Using this modification, the system dynamics can be initiated in a cell originally without activity by hosts from its Moore neighbourhood, who will generate some active customers in the cell, which will generate local active hosts and active sellers after some time.

The resulting discrete time cellular automata transition rules are then

$$N_{ca}(i, j, n + 1) = N_{ca}(i, j, n) + \Delta_{cp \rightarrow ca} - \Delta_{ca \rightarrow cl} \quad (25)$$

$$N_{cl}(i, j, n + 1) = N_{cl}(i, j, n) + \Delta_{ca \rightarrow cl} - \Delta_{cl \rightarrow cp} \quad (26)$$

$$N_{ha}(i, j, n + 1) = N_{ha}(i, j, n) + \Delta_{ca \rightarrow ha} - \Delta_{ha \rightarrow hl} \quad (27)$$

$$N_{hl}(i, j, n + 1) = N_{hl}(i, j, n) + \Delta_{ha \rightarrow hl} - \Delta_{hl \rightarrow cp} \quad (28)$$

$$N_{sa}(i, j, n + 1) = N_{sa}(i, j, n) + \Delta_{ha \rightarrow sa} - \Delta_{sa \rightarrow sl} \quad (29)$$

$$N_{sl}(i, j, n + 1) = N_{sl}(i, j, n) + \Delta_{sa \rightarrow sl} - \Delta_{sl \rightarrow cp} \quad (30)$$

3 SINGLE CELL MODEL ANALYSIS

3.1 Steady-state nonlinear equilibrium

A classical step in the analysis of systems (10) - (15) and (16) - (21) is to find their steady-state equilibria. Since coefficients are the same, equilibrium values of the populations will be the same in both discrete and continuous cases.

After some algebra, one finds that there is a trivial solution in which all populations are equal to zero, while the unique non-trivial one is given by (equilibrium values for populations are denoted by the asterisk)

$$N_{sa}^* = \frac{\omega_2}{Y_{co}(\omega_2 + \alpha_2)} N_{ha}^* \quad (31)$$

$$N_{ca}^* = Y_c \left(\frac{\omega_1 + \alpha_2}{\omega_1} \right) N_{ha}^* \quad (32)$$

$$N_{cl}^* = \frac{\alpha_2}{\alpha_3} N_{ca}^* = Y_c \left(\frac{\alpha_2}{\alpha_3} \right) \left(\frac{\omega_1 + \alpha_2}{\omega_1} \right) N_{ha}^* \quad (33)$$

$$N_{hl}^* = \frac{\alpha_2}{\alpha_4} N_{ha}^* \quad (34)$$

$$N_{sl}^* = \frac{\alpha_2}{\alpha_5} N_{sa}^* = \frac{\alpha_2}{\alpha_5} \frac{\omega_2}{Y_{co}(\omega_2 + \alpha_2)} N_{ha}^* \quad (35)$$

$$N_{ha}^* = \frac{N \left(1 - 12 \frac{Y_c \alpha_2 (\omega_1 + \alpha_2)}{\alpha_1 Y_a Y_{rel} \omega_1} \right)}{1 + \frac{\alpha_2}{\alpha_4} + \frac{Y_c (\alpha_3 + \alpha_2) (\omega_1 + \alpha_2)}{\alpha_3 \omega_1} + \frac{\omega_2 (\alpha_5 + \alpha_2)}{Y_{co} \alpha_5 (\omega_2 + \alpha_2)}} \quad (36)$$

The non-trivial solution is reliable if and only if $N_{ha}^* \geq 0$, which corresponds to the following condition deduced from Eq. (36):

$$\alpha_1 Y_a Y_{rel} \omega_1 \geq 12 Y_c \alpha_2 (\omega_1 + \alpha_2) \quad (37)$$

Since Y_a, Y_{rel}, Y_c and α_2 are mainly sociological parameters that escape the company control, the main parameters that can be tuned by the company are α_1 and ω_1 to a lesser extent. The condition for a non-vanishing N_{ha}^* is

$$\alpha_1 > \alpha_1^{crit} = 12 \frac{Y_c \alpha_2 (\omega_1 + \alpha_2)}{Y_a Y_{rel} \omega_1} \quad (38)$$

showing that there is a minimum level attractiveness of the products and marketing efficiency, both measured by α_1 , which is necessary to sustain a

non-vanishing dynamics. A second remark is that the existence of the non-trivial solution can be mainly interpreted as resulting from the balance between the capability to find new active customers (driven by α_1) and the seller-induced loss of active customers (driven by α_2), the later being represented by a quadratic function of α_2 .

Null values of ω_1 and ω_2 yield vanishing solutions, since they correspond to exponentially decreasing N_{ha} and N_{sa} . The same applies for vanishing α_3, α_4 and α_5 , since it would lead to monotonically increasing N_{cl}, N_{hl} and N_{sl} , respectively, which will act as negative reservoirs that will drive the source term $\Delta_{cp \rightarrow ca}$ to zero.

3.2 Illustrative test cases

The dynamics of the single cell cellular automata governed by transition rules (10) - (15) is now illustrated.

The first example deals with the dynamics generated using time-independent parameters, leading to a non-vanishing equilibrium. Time evolution of N_{ca} and N_{cl} are displayed in Fig. 2 in order to illustrate the convergence toward the long-time steady state. It is observed that during a first phase a logistic-type growth is observed, which is coherent with the mathematical expression of $\Delta_{cp \rightarrow ca}$ which is similar to the imitator source term in the Bass equation. But the present model is more versatile, since it predicts a decay phase after the absolute maximum before the equilibrium is reached. This decay is associated to the growth of the lost customer population, the final equilibrium being reached once the later is stabilized.

We now address a management crisis, modelled as a sudden drop of the attractiveness parameter α_1 . In the simulation, α_1 is divided by 2 after 350 iteration, i.e. once the steady equilibrium associated to the original value is almost reached. Results displayed in Fig. 3 show the sudden decay of both N_{ca} and N_{cl} , along with the relaxation toward a new equilibrium with a lower N_{ha}^* , which is coherent with the analysis conducted above.

The sensitivity to the variation in α_1 is illustrated in Fig. 4, which displays results obtained now dividing α_1 by a factor 10 after 350 iterations, corresponding to a very strong management crisis. A very strong monotonous decay is now observed.

4 FULL CELLULAR MODEL MODEL ASSESSMENT

We now illustrate the dynamics of the full cellular automata model. The main difference is the influence of the Moore neighbourhood of each cell

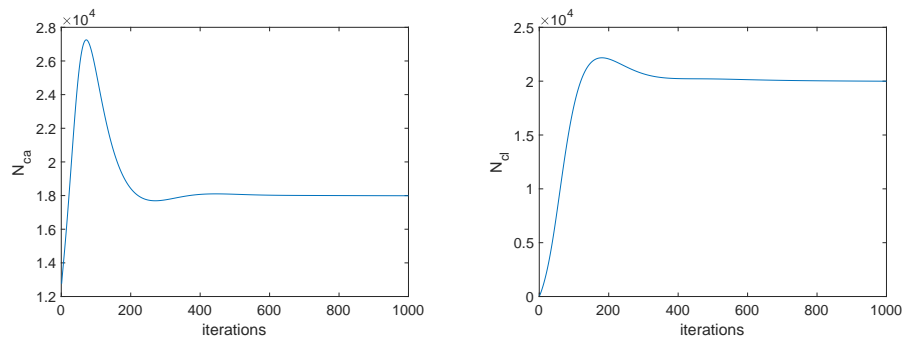


FIGURE 2
Time evolution of the number of active customers N_{ca} (left) and lost customers N_{cl} (right).

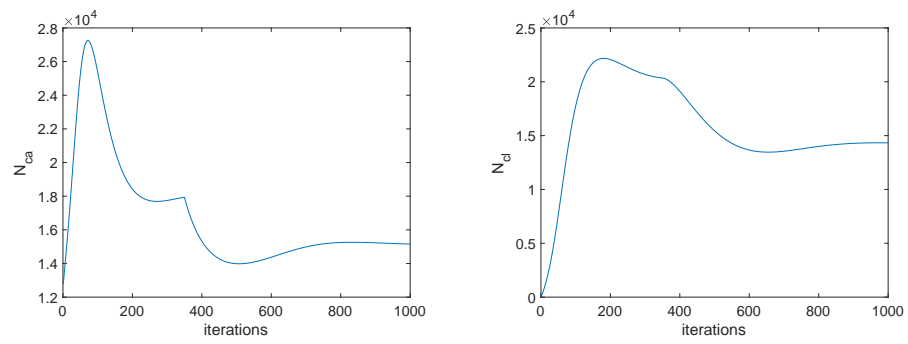


FIGURE 3
Time evolution of the number of active customers N_{ca} and lost customers N_{cl} (right).

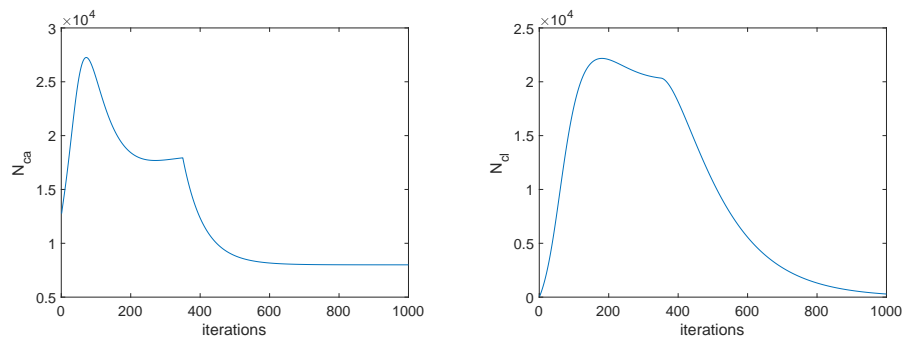


FIGURE 4
Time evolution of the number of active customers N_{ca} and lost customers N_{cl} (right).

on the state transition of potential customer to active customer via interaction with distant active hosts. This influence is parameterised by the weighting parameter γ in Eq. (24), with $\gamma = 0$ corresponding to a null influence (each cell evolves in a fully isolated way). The spatial spreading is obviously governed by γ , a fact that will be illustrated below considering the time evolution of a 100×100 grid with a random distribution of N , with $N(i, j) \in [5000, 10000]$. All other variables are set to 0.

A first simulation is performed with a weak influence with $\gamma = 0.01$. The dynamics is initiated in cell $(i = 50, j = 50)$ by putting $N_{ca}(50, 50) = N(50, 50)/20$ and $N_{ha}(50, 50) = N(50, 50)/200$. Time evolution of the total number of active customers and lost customers on the grid are displayed in Fig. 5, showing a monotonic growth. The spatial distribution of N_{ca} at iteration 1000 is shown in Fig. 6. The spatial distribution reveals that the growth is due to the spatial spreading which increases the total number active cells, i.e. the total number of potential customers, resulting in a dynamic market behavior. A propagation front is observed, which is classical for simulation of spatial spreading via cellular automata, e.g. in fire propagation simulations. The aged active cells are observed to reach an equilibrium, as expected, while new cells contribute by their internal growth until they reach equilibrium and by activating their inactive neighbours.

In order to illustrate the role of γ , a new simulation is performed with $\gamma = 0.1$, i.e. with a strong influence of the neighbours. Results are shown in Figs. 7 and 8. As expected, the activity has spread faster than for the weak interaction case, resulting in a stronger growth.

We now address the competition that may exist between a management crisis, i.e. a drop in the attractiveness parameter α_1 , and the spatial spreading governed by γ . A first (weak) crisis is considered, taking $\gamma = 0.1$ and changing α_1 from the initial value of 0.25 to 0.1 after 350 iterations. Corresponding evolutions are given in Fig. 9. The drop in α_1 is followed by a sudden decrease of the total N_{ca} which is due to relaxation of aged cells toward their new equilibrium values (which are lower, as seen above) and a slower propagation. Once aged cells are at equilibrium, the spatial spreading, which is still active, becomes dominant leading to a new, less fast, monotonic growth phase.

The case of a dramatic crisis in which α drops to 0.01 is now studied, results being plotted in Fig. 10. Here, the spatial spreading is observed to be too weak to balance the crisis effect, leading to a global shutdown of the dynamics.

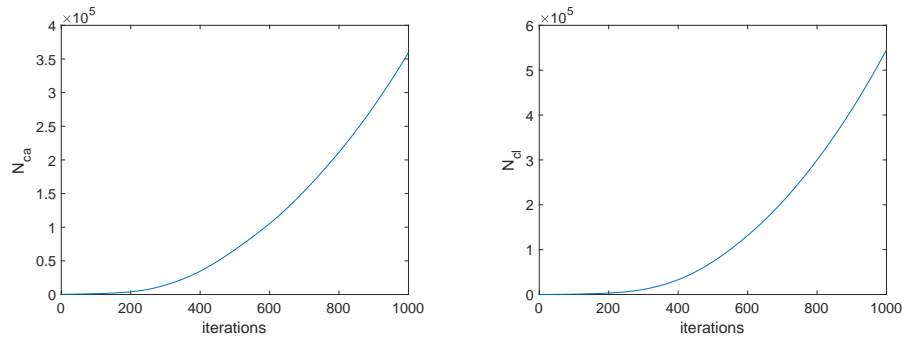


FIGURE 5
 Time evolution of the total number of active customers N_{ca} and lost customers N_{cl} (right) on the grid, with $\alpha_1 = 0.25$ and $\gamma = 0.01$.

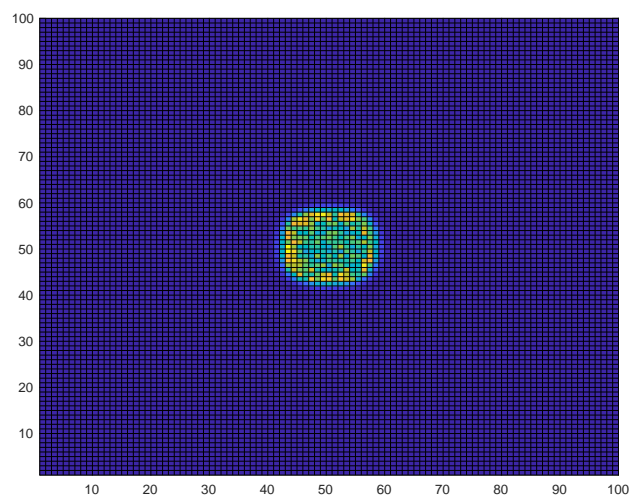


FIGURE 6
Spatial distribution of N_{ca} at iteration 1000, with $\alpha_1 = 0.25$ and $\gamma = 0.01$.

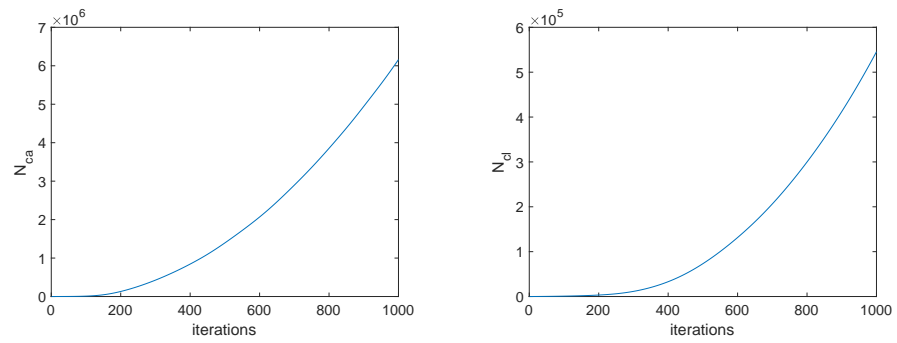


FIGURE 7
 Time evolution of the total number of active customers N_{ca} and lost customers N_{cl} (right) on the grid, with $\alpha_1 = 0.25$ and $\gamma = 0.1$.

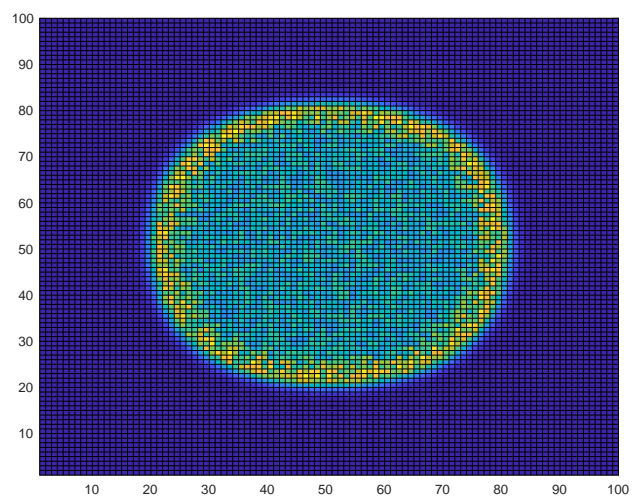


FIGURE 8
Spatial distribution of N_{ca} at iteration 1000, with $\alpha_1 = 0.25$ and $\gamma = 0.1$.

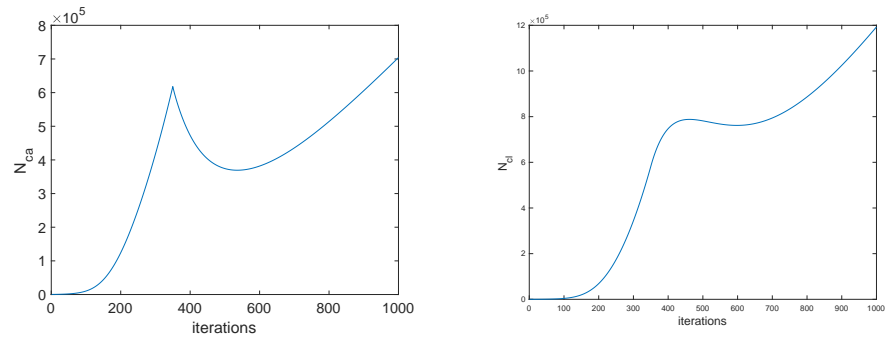


FIGURE 9
 Time evolution of the total number of active customers N_{ca} and lost customers N_{cl} (right) on the grid, with $\alpha_1 = 0.25 \rightarrow 0.1$ and $\gamma = 0.1$.

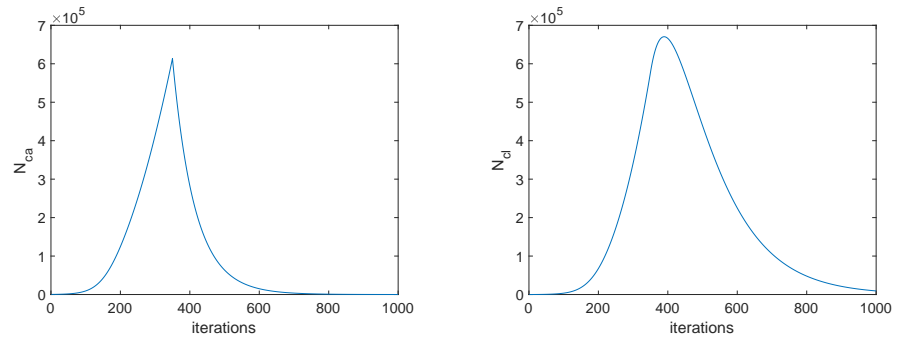


FIGURE 10
 Time evolution of the total number of active customers N_{ca} and lost customers N_{cl} (right) on the grid, with $\alpha_1 = 0.25 \rightarrow 0.01$ and $\gamma = 0.1$.

5 CONCLUDING REMARKS

A new cellular automata model has been proposed to model Direct Selling Multi-Level Marketing dynamics, defining 6 dynamic populations with transition rules based on discrete time equations. The model is based on classical modelling methods for the different mechanisms at play, with parameters that can be tuned to fit a given company/market case. The model is shown to be able to capture many effects, including management/attractiveness crisis (or improvement) and the competition between spatial spreading, which yields a growing dynamic total market, and drop in the attractiveness.

Future works will deal with inclusion of seasonal effects and the use of a distance measure that will account for geographical data and the coupling with a Geographical Information System.

REFERENCES

- [1] Arindam Banerjee and Tanushri Banerjee. (2022). Multi-level marketing in india: Business prospect versus social interactions? some identified research issues for marketers. *Vikalpa*, 47(4):255–273.
- [2] Frank M Bass. (1969). A new product growth for model consumer durables. *Management science*, 15(5):215–227.
- [3] Frank M Bass. (2004). Comments on : a new product growth for model consumer durables the bass model. *Management science*, 50(12_supplement):1833–1840.
- [4] Robert Boehner and Steven C Gold. (2012). Modeling the impact of marketing mix on the diffusion of innovation in the generalized bass model of firm demand. In *Developments in business simulation and experiential learning: Proceedings of the annual ABSEL conference*, volume 39.
- [5] Yu Chang and Tao Zhang. (2019). The effects of product consistency and consumer resistance to innovation on green product diffusion in china. *Sustainability*, 11(9):2702.
- [6] Lei Chen, Xin Bai, Bi Chen, and Jingjing Wang. (2022). Incentives for green and low-carbon technological innovation of enterprises under environmental regulation: From the perspective of evolutionary game. *Frontiers in Energy Research*, 9:793667.
- [7] Laura Di Lucchio and Giovanni Modanese. (2024). The bass diffusion model: agent-based implementation on arbitrary networks. *Mathematical and Computer Modelling of Dynamical Systems*, 30(1):364–384.
- [8] Wellesley Dodds. (1973). An application of the bass model in long-term new product forecasting. *Journal of Marketing Research*, 10(3):308–311.
- [9] Jacob Goldenberg and Sol Efroni. (2001). Using cellular automata modeling of the emergence of innovations. *Technological Forecasting and Social Change*, 68(3):293–308.
- [10] Jacob Goldenberg, Barak Libai, Sarit Moldovan, and Eitan Muller. (2007). The npv of bad news. *International Journal of Research in Marketing*, 24(3):186–200.
- [11] Jacob Goldenberg, Barak Libai, and Eitan Muller. (2001). Talk of the network: A complex systems look at the underlying process of word-of-mouth. *Marketing letters*, 12:211–223.

- [12] Jacob Goldenberg, Barak Libai, and Eitan Muller. (2001). Using complex systems analysis to advance marketing theory development: Modeling heterogeneity effects on new product growth through stochastic cellular automata. *Academy of Marketing Science Review*, 9(3):1–18.
- [13] Jacob Goldenberg, Barak Libai, and Eitan Muller. (2010). The chilling effects of network externalities. *International Journal of Research in Marketing*, 27(1):4–15.
- [14] Mark Granovetter. (1983). The strength of weak ties: A network theory revisited. *Sociological theory*, pages 201–233.
- [15] Mark S Granovetter. (1973). The strength of weak ties. *American journal of sociology*, 78(6):1360–1380.
- [16] Mariangela Guidolin. (2023). *Innovation diffusion models: Theory and practice*. John Wiley & Sons.
- [17] Renato Guseo and Mariangela Guidolin. (2009). Modelling a dynamic market potential: A class of automata networks for diffusion of innovations. *Technological Forecasting and Social Change*, 76(6):806–820.
- [18] Torsten Hagerstrand. (1968). *Innovation diffusion as a spatial process*. The University of Chicago Press.
- [19] Chetan V Hiremath, Priti Pai, and Mathala Juliet. (2010). Mathematical model for network marketing. *Journal of Commerce & Management Thought*, 1(4):380–397.
- [20] Nathan O Hodas and Kristina Lerman. (2014). The simple rules of social contagion. *Scientific reports*, 4(1):4343.
- [21] Andrijana Horvat, Vincenzo Fogliano, and Pieter A Luning. (2020). Modifying the bass diffusion model to study adoption of radical new foods—the case of edible insects in the netherlands. *Plos one*, 15(6):e0234538.
- [22] Sin-Hoon Hum and Mahmut Parlar. (2022). Modeling and optimization of multilevel marketing operations. *Naval Research Logistics (NRL)*, 69(4):581–598.
- [23] Jinshan Li, Zhizuo Chen, and Tao Qin. (2013). Using cellular automata to model evolutionary dynamics of social network. In *11th International Symposium on Operations Research and its Applications in Engineering, Technology and Management 2013 (ISORA 2013)*, pages 1–6. IET.
- [24] Vijay Mahajan. (1986). An evaluation of estimation procedures for new product diffusion models. *Innovation diffusion models of new product acceptance*.
- [25] Albert Makore, Evelyn Madziba, and Talak Moyo. (2023). Factors of multi-level marketing success strategies which motivate participants. *Southern African Business Review*, 27:1–25.
- [26] Krzysztof Małecki, Jarosław Jankowski, and Mateusz Szkwarkowski. (2019). Modelling the impact of transit media on information spreading in an urban space using cellular automata. *Symmetry*, 11(3):428.
- [27] Jérôme Massiani and Andreas Gohs. (2015). The choice of bass model coefficients to forecast diffusion for innovative products: An empirical investigation for new automotive technologies. *Research in transportation economics*, 50:17–28.
- [28] Karismita Medhi and Bapan Kalita. (2023). Multi-level marketing in the context of dynamical system. *Int. J. Adv. Multidisc. Stud.*, 3(3):1071–1076.
- [29] Sarit Moldovan and Jacob Goldenberg. (2004). Cellular automata modeling of resistance to innovations: Effects and solutions. *Technological Forecasting and Social Change*, 71(5):425–442.

- [30] Yair Orbach. (2016). Parametric analysis of the bass model. *Innovative Marketing*, 12(1):29–40.
- [31] James Christopher S Pang and Christopher P Monterola. (2017). Dendritic growth model of multilevel marketing. *Communications in Nonlinear Science and Numerical Simulation*, 43:100–110.
- [32] Philip Parker and Hubert Gatignon. (1994). Specifying competitive effects in diffusion models: An empirical analysis. *International Journal of Research in Marketing*, 11(1):17–39.
- [33] Ioana-Mădălina Purcaru, Ana-Maria Urdea, Cristinel Petrișor Constantin, and Gabriel Brătucu. (2022). Building long-term business sustainability: The influence of experiential marketing on sales representatives loyalty to multi-level marketing systems. *Sustainability*, 14(15):9507.
- [34] Hegselmann Rainer and Ulrich Krause. (2002). Opinion dynamics and bounded confidence: models, analysis and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3).
- [35] Sundaresan Ram and Jagdish N Sheth. (1989). Consumer resistance to innovations: the marketing problem and its solutions. *Journal of consumer marketing*, 6(2):5–14.
- [36] Yaniv Reingewertz. (2021). An economic model of multi-level marketing. *Plos one*, 16(7):e0253700.
- [37] Jean-François Rouhaud. (2000). Cellular automata and consumer behaviour. *European Journal of Economic and Social Systems*, 14(1):37–52.
- [38] Daisuke Satoh. (2001). A discrete bass model and its parameter estimation. *Journal of the Operations Research Society of Japan*, 44(1):1–18.
- [39] Pedro Henrique Triguês Schimit, DF Barros Junior, and SA Araújo. (2017). On estimating the impact of information spreading in a consumer market modeled by probabilistic cellular automata and ordinary differential equations. *Simulation*, 93(12):1063–1072.
- [40] Hyeonju Seol, Gwangman Park, Hakyeon Lee, and Byungun Yoon. (2012). Demand forecasting for new media services with consideration of competitive relationships using the competitive bass model and the theory of the niche. *Technological Forecasting and Social Change*, 79(7):1217–1228.
- [41] Changheng Shao, Fengjing Shao, Xin Liu, Dawei Yang, Rencheng Sun, Lili Zhang, and Kaiwen Jiang. (2024). A multi-information dissemination model based on cellular automata. *Mathematics*, 12(6):914.
- [42] Roongthiwa Sivachan, Kitti Phothikitti, and Rawin Vongurai. (2022). Determinants influencing success in sales performance of multilevel marketing agents in thailand. *Journal of Global Business Review*, 24(2):16–35.
- [43] Mladen Sokele and Luiz Moutinho. (2018). Bass model with explanatory parameters. *Innovative Research Methodologies in Management: Volume I: Philosophy, Measurement and Modelling*, pages 145–164.
- [44] John R Sparks and Joseph A Schenk. (2001). Explaining the effects of transformational leadership: an investigation of the effects of higher-order motives in multilevel marketing organizations. *Journal of Organizational Behavior: The International Journal of Industrial, Occupational and Organizational Psychology and Behavior*, 22(8):849–869.
- [45] John Sterman. (2000). *System Dynamics: systems thinking and modeling for a complex world*. McGraw Hill.
- [46] Christophe Van den Bulte and Stefan Stremersch. (2004). Social contagion and income heterogeneity in new product diffusion: A meta-analytic test. *Marketing Science*, 23(4):530–544.

- [47] Barbara Wejnert. (2002). Integrating models of diffusion of innovations: A conceptual framework. *Annual review of sociology*, 28(1):297–326.
- [48] Jurgen Wöckl. (2007). Modelling advertising and production strategies using cellular automata. In *Proc. EUROSIM 2007*, pages 1–6. ARGESIM.
- [49] Lingling Xia, Guoping Jiang, Yurong Song, and Bo Song. (2015). Modeling and analyzing the interaction between network rumors and authoritative information. *Entropy*, 17(1):471–482.
- [50] Yunpeng Xiao, Diqiang Chen, Shihong Wei, Qian Li, Haohan Wang, and Ming Xu. (2019). Rumor propagation dynamic model based on evolutionary game and anti-rumor. *Nonlinear Dynamics*, 95:523–539.
- [51] Jiuping Xu, Mengxiang Zhang, and Jingneng Ni. (2016). A coupled model for government communication and rumor spreading in emergencies. *Advances in Difference Equations*, 2016:1–15.
- [52] Runxi Zeng and Di Zhu. (2019). A model and simulation of the emotional contagion of netizens in the process of rumor refutation. *Scientific reports*, 9(1):14164.
- [53] Chuan Zhang, Yu-Xin Tian, and Ling-Wei Fan. (2020). Improving the bass model predictive power through online reviews, search traffic and macroeconomic data. *Annals of Operations Research*, 295:881–922.
- [54] Yaming Zhang, Yanyuan Su, Li Weigang, and Haiou Liu. (2018). Rumor and authoritative information propagation model considering super spreading in complex social networks. *Physica A: Statistical Mechanics and its Applications*, 506:395–411.