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# Job Matching and Affirmative Action: The Impact of Transfer Policies\*

Cyril Rouault<sup>†</sup>

[Preliminary version]

## Abstract

Subsidies and taxes are commonly used tools in labor markets to influence employment outcomes. This paper examines job matching within a Kelso-Crawford framework to assess the effects of these transfers on workers' welfare. We show that affirmative action policies, while well-intentioned, may inadvertently reduce the welfare of minority workers. To mitigate this, we identify transfer policies that do not negatively impact a group. A transfer negatively impacts a group if it reduces the utility of at least one member after implementation. Furthermore, we explore various objectives of affirmative action policies, such as increasing minority representation in firms, raising wages, and reducing unemployment among minority workers. For each objective, we propose transfer policies that do not negatively impact minority workers.

**JEL Classification:** C78; D47; D50; J20; J30

**Keywords:** Job matching; Salary; Transfer policy; Market equilibrium; Subsidy; Taxation; Affirmative action

## 1 Introduction

Subsidies and taxes are widely used policy instruments in labor markets to promote employment and reduce inequalities. In the United States, companies receive tax abatements for employing a minimum number of minority workers (Byrnes et al. 1999, Slattery and Zidar 2020). In the United Kingdom, minimum wage laws aim to reduce wage disparities across the population (*Living Wage Laws*, Neumark et al. 2007, Derenoncourt and

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Montialoux 2021). In several OECD countries, subsidies are introduced to promote the employment of the unemployed (Snower 1994). These policies are designed to enhance worker welfare and reduce segregation among different groups. Inspired by a classical framework for studying job markets (Kelso and Crawford 1982), this paper investigates the impact of transfer policies on worker welfare, considering both assignment and salary.

While affirmative action policies are intended to improve outcomes for minority workers, they can have complex economic effects on firms' hiring decisions and wage offers. Subsidies, for example, might increase a worker's desirability, leading firms to adjust salaries to attract these workers within a competitive equilibrium (Shapley and Shubik 1971, Kelso and Crawford 1982, Gul and Stacchetti 1999, Hatfield et al. 2019, Kojima et al. 2020, 2024). Although both assignment and salary are crucial for worker welfare, existing literature predominantly focuses on the former, leaving the latter underexplored. This paper addresses this gap by demonstrating that certain affirmative action policies, such as subsidizing minority workers or taxing majority workers, may paradoxically reduce minority workers' welfare.<sup>1</sup>

To investigate these effects, we develop a model that captures the impact of affirmative action policies on worker welfare in competitive equilibria, where *utility* is a function of both *assignment* and *salary*. In our model, each firm is endowed with a *revenue function* mapping a set of workers it employs to a real number, and a *capacity* representing the maximum number of workers the firm can employ. Transfer policies are modeled through a matrix associating each worker-firm pair with a real number, indicating the *subsidy* or *tax* the firm receives for employing that worker. Firms maximize their *profit*, defined as the sum of their revenue and transfers minus total salaries. A *firm's demand correspondence*, a function of its revenue, transfers, and capacity, corresponds to its profit-maximizing set of workers. In such environments, the *gross substitutes condition* is sufficient for the existence of competitive equilibria (Kelso and Crawford 1982, Gul and Stacchetti 1999).<sup>2</sup> This condition also ensures that the set of equilibria forms a lattice, with *the firm-optimal stable allocation* and *worker-optimal stable allocation* representing the extremities where firms maximize profits and workers maximize utility, respectively. We consider two *mechanisms*: one leading to the firm-optimal stable allocation and the other to the worker-optimal stable allocation.

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<sup>1</sup>Kojima (2012) shows that affirmative action policies harm minority students in a model without transfers for any stable mechanisms.

<sup>2</sup>The gross substitutes condition roughly requires that a set of demanded workers remains demanded after an increase in other workers' salaries.

This paper first examines which *transfer policies do not negatively impact minority workers* at a given mechanism. A transfer *may negatively impact minority workers* if the utility of all minority workers is no lower with the transfer than without it. We establish that only *uniform* transfers—where the transfer amount is the same for each member of the group (minority or majority)—never negatively impact minority workers. When transfers are not uniform, there can exist a market where the transfer reduces minority workers’ welfare. The intuition is that when a larger subsidy is given to one worker over another, that worker becomes more appealing to firms, which may then lower wages or lay off other minority employees to maximize profits. Consequently, subsidies aimed at promoting the employment of unemployed minority workers could inadvertently harm minority workers. A similar issue arises when subsidies are unevenly distributed across different sectors, which is often the case. Thus, targeting subsidies to specific sectors (or a set of firms) may negatively impact minority workers.

Furthermore, we establish the complementarity between uniform taxes and subsidies: if a transfer uniformly subsidizes minority workers, then taxing majority workers by the same amount results in identical worker utility. This equivalence allows for the combination of tax and subsidy policies. For example, if a transfer uniformly subsidizes minority workers by  $t$  and uniformly taxes majority workers by  $t'$ , the resulting worker utility is equivalent to a transfer that subsidizes minority workers or taxes majority workers by  $t + |t'|$ . This result is practically significant, as it suggests cost-effective strategies for affirmative action policies while maintaining their intended effects. Uniform transfers are recommended when the policymaker lacks detailed market information, such as firms’ revenue functions or whether the market equilibrium is the worker-optimal or firm-optimal stable allocation.

When more information is available, we examine three common objectives of affirmative action policies and the transfers that do not negatively impact minority workers. The first objective is to *increase minority workers’ salaries*, applicable in minimum wage policies. The second is to *increase minority representation in firms*, reducing group segregation. The third is to *favor employment for unemployed minority workers*. For each objective, we analyze the impact on the market, including the evolution of firm profits and the welfare of majority workers.

In practice, groups are formed based on various criteria, such as gender, location, or

ethnicity. Since different minority groups may be considered, affirmative action policies may not benefit them uniformly. We show that this situation is similar to implementing a non-uniform transfer, which could negatively impact a minority group. In this context, a uniform tax on majority workers (those not belonging to any minority group) emerges as the only approach ensuring no negative impact on minority workers.

We also explore quota policies as another form of affirmative action. Two types of quotas are typically implemented: the *floor constraint*, which requires firms to employ at least a certain number of minority workers, and the *ceiling constraint*, which limits the number of majority workers firms can employ. Similarly to transfers, we show that *quotas may negatively impact minority workers*. We identify specific quotas that never impact minority workers, ensuring that, based on firm capacities, firms can employ at least as many minority workers as in the allocation without quotas.

## Related Literature

Affirmative action policies have been a subject of extensive debate across disciplines—philosophy, law, and economics—since their inception. This paper sheds new light on their impact on minority populations (Sowell 2004). Historically, these policies were designed to enhance diversity within educational institutions and firms by setting targets for minority representation. Since the institutions failed to reach these targets, sanctions and quotas were introduced for firms and schools (Jencks 1992). Existing literature primarily focuses on feasibility constraints that reflect quota implementations and identifies conditions under which market equilibria can exist (Ehlers et al. 2014; Kominers and Sönmez 2016; Sönmez et al. 2019; Kamada and Kojima 2024). Our contribution builds upon the literature examining affirmative action policies in markets where transfers between agents are possible (Gul et al. 2019, Kojima et al. 2020, 2024, Echenique et al. 2021). While these studies ensure minority representation in firms, they often overlook how these policies affect the welfare of minority workers. In contrast, our paper investigates the impact of affirmative action policies on minority workers’ welfare. In addition, we show that, without imposing quotas, it is possible to incentivize firms to achieve targeted minority representation through the use of taxes and subsidies.

In recent years, new objectives for affirmative action policies have emerged, such as reducing unemployment among minority workers (Snower 1994) and ensuring a minimum wage (Neumark et al. 2007). Surprisingly, these objectives have received limited

attention in the market design literature. We address this gap by exploring how subsidies can be designed to reduce unemployment and increase the wages of minority workers. Snower (1994) emphasizes the importance of subsidies to foster the integration of minority workers into firms, suggesting that firms could receive a portion of unemployment benefits in exchange for providing training to enhance workers' skills.<sup>3</sup> Our model assumes the presence of unemployed workers who are less productive relative to others. We then analyze the necessary subsidies required to ensure these workers are employed in a competitive equilibrium.

Our results also highlight the differences in the impact of affirmative action policies in markets with transfers compared to those without. While Ehlers et al. (2014) show that the set of equilibria (or stable matchings) may be empty when floor constraints are imposed, Kojima et al. (2020) show that if the imposed feasibility constraints are defined by a *generalized interval constraint*, the substitutes condition is preserved, resulting in a non-empty set of market equilibria.

Similarly, Kojima (2012) states the impossibility of ensuring that affirmative action policies do not harm minority students under any stable mechanism in a framework without transfer. Kojima (2012) studies two types of affirmative action policies in education: implementing quotas and increasing school priorities. For the former, we identify quotas that never reduce the welfare of minority workers. For the latter, we show that uniform transfers within groups never negatively impact minority workers.<sup>4</sup> Therefore, we demonstrate that the limitations identified in frameworks without transfer are overcome when transfers are feasible. To our knowledge, this paper is the first to show that affirmative action policies can adversely affect minority workers in a transferable setting and propose solutions to meet various objectives.

Bleemer (2022) provides empirical evidence suggesting that affirmative action policies in education may lead to lower wages for minorities. Our contribution extends this line of inquiry by analyzing the effects of employment policies, minimum wage regulations, and initiatives aimed at reducing segregation among minority workers.

Leonard (1990) and Coate and Loury (1993) examine the impact of policies on minority representation in firms, highlighting that firms often fail to comply with affirmative

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<sup>3</sup>Snower (1994) identifies two significant consequences of unemployment for workers. Psychologically, unemployment causes a growing sense of estrangement from society that intensifies over time. Additionally, it results in a decline in skills, which further diminishes workers' employability.

<sup>4</sup>Since a firm's profit is determined by the sum of its revenue function and the transfer, subsidizing a worker effectively equates to increasing the firm's revenue. Thus, we draw a parallel to increasing priority.

action constraints, leaving minority workers disadvantaged in employment opportunities. In our approach, we propose the implementation of taxes and subsidies to adjust the competitive market equilibrium, ensuring the recruitment of minority workers when firms operate under profit-maximizing behavior.

This paper proceeds as follows. Section 2 introduces the model and provides a leading example. Section 3 examines how subsidies can negatively impact minority workers and proposes a uniform subsidy as a solution. Section 4 explores the relationship between taxes and subsidies, emphasizing the complementarity of uniform subsidies and taxes. Section 5 discusses the application of affirmative action policies. Section 6 highlights the significance of uniform taxes when multiple minority groups are present in the population. Section 7 introduces quota policies and details the design that quotas must have to ensure they do not reduce the welfare of minority workers. Appendix A details the mechanisms discussed. The proofs are collected in Appendix B.

## 2 Model, Definitions, and Transfers

### 2.1 The Model

There are finite sets  $W = \{w_1, w_2, \dots, w_n\}$  and  $F = \{f_1, f_2, \dots, f_m\}$  of *workers* and *firms*. Each worker can be employed by at most one firm. Each firm  $f \in F$  is associated with a *capacity*  $q_f \in \mathbb{N}$ , and we denote the *vector of capacities* as  $\mathbf{q} \equiv (q_f)_{f \in F}$ , which assigns each firm its respective capacity. Firms can hire workers up to their capacity. We assume that  $|W| > \sum_{f \in F} q_f$ .<sup>5</sup>

Agents on both sides of the market derive value from being matched to agents on the other side. Let  $\sigma_{w,f} \in \mathbb{R}$  denote the value that worker  $w$  obtains when employed by firm  $f$ . For each firm  $f \in F$ , we define a *revenue function*  $R_f : 2^W \rightarrow \mathbb{R}$ , which maps a subset of workers to the revenue of firm  $f$  if it hires them. We assume that for each firm  $f$ ,  $R_f$  is *additively separable*, meaning that for any subset of workers  $W' \subset W$ ,  $R_f(W') = \sum_{w \in W'} R_f(w)$ . We refer to  $R_f(w)$  as the *productivity* of  $w$  at firm  $f$ . We assume that for each worker  $w$  and firm  $f$ ,  $R_f(w) + \sigma_{w,f} \geq 0$ .<sup>6</sup>

Without loss of generality, we normalize the value of agents that are unassigned to zero,  $\sigma_{w,\emptyset} = 0$  and  $R_f(\emptyset) = 0$ , where  $\emptyset$  denotes the null firm to which the corresponding

<sup>5</sup>This assumption ensures the existence of unemployed workers.

<sup>6</sup>This restriction, ensuring that workers' marginal product is non-negative, is natural and is discussed by Kelso and Crawford (1982).

agent is assigned. The capacity of the null firm is the number of workers,  $q_\emptyset = |W|$ , implying that all workers have the option of remaining unemployed. We also assume that for any subset  $W' \subseteq W$ ,  $R_\emptyset(W') = 0$ .

A policymaker regulates the market using *transfers*, referred to as *subsidies* or *taxes*. A transfer is a matrix  $\mathbf{T} = (t_{w,f})_{w \in W, f \in F}$ , where each element  $t_{w,f} \in \mathbb{R}$  represents the transfer for worker  $w$  when employed by firm  $f$ . The transfer  $t_{w,f}$  is added to the productivity  $R_f(w)$ .<sup>7</sup> We denote the *productivity with transfer*  $\mathbf{T}$  for worker  $w$  at firm  $f$  by  $R_f^{\mathbf{T}}(w) \equiv R_f(w) + t_{w,f}$ . If  $t_{w,f} < 0$ , we say that  $\mathbf{T}$  *taxes* worker  $w$  at  $f$  by  $t_{w,f}$ , and if  $t_{w,f} > 0$  we say that  $\mathbf{T}$  *subsidizes* worker  $w$  at  $f$  by  $t_{w,f}$ . We extend the definition to subsets of workers. Specifically, transfer  $\mathbf{T}$  *subsidizes the workers*  $W' \subset W$  at  $f$  by  $t$  if for each  $w \in W'$ ,  $\mathbf{T}$  subsidizes worker  $w$  at  $f$  by  $t$ . Similarly, the transfer  $\mathbf{T}$  *taxes the workers*  $W' \subset W$  at  $f$  by  $t$  if for each  $w \in W'$ ,  $\mathbf{T}$  taxes worker  $w$  at  $f$  by  $t$ . A *null transfer* is denoted by  $\mathbf{T}_0$  such that for each worker  $w$  and each firm  $f$ ,  $t_{w,f} = 0$ . Let  $\mathbf{T}$  be the set of all possible transfers.

In this market, we allow for utility transfers between firms and workers, which we refer to as *salaries*. Let  $s_{w,f} \in \mathbb{R}$  denote the salary received by worker  $w \in W$  when working for firm  $f \in F \cup \{\emptyset\}$ .<sup>8</sup> We denote  $\mathbf{s}$  as a *vector of salaries*. We assume that workers are indifferent about which other workers firms hire, and that  $s_{w,\emptyset} = 0$ .

Each worker's *utility function* is assumed to be strictly increasing and continuous in the salaries. Let  $u_w(f, s_{w,f}) \in \mathbb{R}$  be the utility function of worker  $w$  when working for firm  $f$  at salary  $s_{w,f}$ , such that  $u_w(f, s_{w,f}) = \sigma_{w,f} + s_{w,f}$ . Consequently,  $u_w(f, \sigma_{w,f}) = 0$  and  $-\sigma_{w,f}$  is the lowest salary at which worker  $w$  would consider working for firm  $f$ .

If a firm  $f$  hires a subset of workers  $W' \subset W$  while facing a vector of salaries  $\mathbf{s}$ , a transfer  $\mathbf{T}$ , a capacity  $q_f$ , and a revenue function  $R_f$ , its *profit* is given by

$$V_f(W'; \mathbf{s}, R_f, \mathbf{T}) = R_f^{\mathbf{T}}(W') - \sum_{w \in W'} s_{w,f},$$

that is, the firm's revenue with transfers minus salaries of workers it hires. We define the *maximal profit function* as  $\Pi_f(\cdot; R_f, \mathbf{T}, \mathbf{q})$  and the *demand* by  $D_f(\cdot; R_f, \mathbf{T}, \mathbf{q})$  such

<sup>7</sup>Subsidies and taxes are typically determined based on the firm's workforce. By adding the transfer to the worker's productivity, we assume the transfer is an additively separable function. Kojima et al. (2024) provides further discussion on transfers that preserve the substitutability condition.

<sup>8</sup>Note that we allow salaries to be negative.



that for each vector of salaries  $\mathbf{s}$ ,

$$\Pi_f(\mathbf{s}; R_f, \mathbf{T}, \mathbf{q}) = \max\{V_f(W'; \mathbf{s}, R_f, \mathbf{T}) : W' \subset W, |W'| \leq q_f\};$$

$$D_f(\mathbf{s}; R_f, \mathbf{T}, \mathbf{q}) = \{W' \subset W : |W'| \leq q_f \text{ and } V_f(W'; \mathbf{s}, R_f, \mathbf{T}) = \Pi_f(\mathbf{s}; R_f, \mathbf{T}, \mathbf{q})\}.$$

Each element of  $D_f(\mathbf{s}; R_f, \mathbf{T}, \mathbf{q})$  is referred to as a *demand set*.

The *gross substitutes condition*, introduced by Kelso and Crawford (1982), requires that for any given demand set, if the salary of a worker increases, there exists a new demand set, with a corresponding new vector of salaries, that includes workers from the original demand set whose salaries remain unchanged.

**Definition 1.** (Gross Substitutes). A demand  $D_f(\cdot; R_f, \mathbf{T}, \mathbf{q})$  satisfies the gross substitutes condition if, for any two vectors of salaries  $\mathbf{s}, \mathbf{s}'$  if  $\mathbf{s}' \geq \mathbf{s}$  and  $s_w = s'_w$ , then  $w \in D_f(\mathbf{s}; R_f, \mathbf{T}, \mathbf{q})$  implies that  $w \in D_f(\mathbf{s}'; R_f, \mathbf{T}, \mathbf{q})$ .

## 2.2 Assignment Game

A market with salaries is a 6-tuple  $G = (W, F, R, \mathbf{T}, \mathbf{q}, \mathbf{s})$ . Let  $\mathbb{G}$  be the set of all possible markets. Since  $W, F, R$  are fixed throughout this paper, and  $\mathbf{s}$  depends on  $R$ , we simplify the notation to  $G = (\mathbf{T}, \mathbf{q})$ .

An *assignment* is a mapping (or correspondence)  $\mu : W \times F \rightrightarrows W \cup \{\emptyset\} \times F \cup \{\emptyset\}$  satisfying:

- $\mu(w) \in F \cup \{\emptyset\}$ ,
- For any  $w \in W$  and  $f \in F$ , we have  $\mu(w) = f$  if and only if  $w \in \mu(f)$ ,
- $\mu(f) \subseteq W \cup \{\emptyset\}$  and  $|\mu(f)| \leq q_f$ .

We use the notation  $\mu(w) = \emptyset$  to indicate that worker  $w$  is unemployed at  $\mu$ . Given an assignment  $\mu$  and a vector of salaries  $\mathbf{s}$  we define an *allocation* as a pair  $(\mu, \mathbf{s})$ .

An allocation  $(\mu, \mathbf{s})$  is *individually rational for firms* if for each firm  $f \in F$ ,  $V_f(\mu(f); \mathbf{s}, R_f, \mathbf{T}) \geq 0$ . With a slight abuse of notation, we note  $u_w(\mu, \mathbf{s})$  for  $u_w(\mu(w), s_{w, \mu(w)})$ . An allocation  $(\mu, \mathbf{s})$  is *individually rational for workers* if for each worker  $w \in W$ ,  $u_w(\mu, \mathbf{s}) \geq 0$ .

We now introduce the definition of *stability*, which corresponds to the concept of a strict core allocation, as considered by Kelso and Crawford (1982).

**Definition 2.** (Stable Allocation). An allocation  $(\mu, \mathbf{s})$  is stable if it is individually rational for both workers and firms, and there is no firm-worker set  $W' \cup \{f\}$ , with  $W' \subseteq W, |W'| \leq q_f$ , and vector of salaries  $\mathbf{s}'$ , that *block*  $(\mu, \mathbf{s})$  such that:

- $u_w(f, s'_{w, f}) \geq u_w(\mu, \mathbf{s})$ , for each  $w \in W'$ , and
- $V_f(W'; \mathbf{s}', R_f, \mathbf{T}) \geq V_f(\mu(f); \mathbf{s}, R_f, \mathbf{T})$

with strict inequality holding for at least one member of  $W' \cup \{f\}$ .

We introduce the notion of *stable allocation optimality*. When firms' demands satisfy the gross substitutes condition, the set of stable allocations forms a *lattice* with a stable allocation unanimously preferred by firms and a stable allocation unanimously preferred by workers (Roth, 1984). In this paper, we consider the introduction of transfers. Thus, for a market without transfers (i.e.,  $(\mathbf{T}_0, \mathbf{q})$ ), we denote the *worker-optimal stable allocation* by  $(\mu_W, \mathbf{s}_W)$  and the *firm-optimal stable allocation* by  $(\mu_F, \mathbf{s}_F)$ .<sup>9</sup> When a non-null transfer  $\mathbf{T}$  is introduced, we denote the *worker-optimal stable allocation with transfer* by  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$  and the *firm-optimal stable allocation with transfer* by  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ .

Given a market  $(\mathbf{T}, \mathbf{q})$ , the highest utility that workers can achieve at a stable allocation is the utility they receive in  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$  while the lowest utility they can achieve is in  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ . Symmetrically, the highest profit that firms can achieve at a stable allocation is the profit they receive in  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$  while the lowest profit they can achieve is in  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$ .

A *mechanism*  $\varphi$  associates an allocation to each market  $G \in \mathbb{G}$ . We denote the allocation under mechanism  $\varphi$  for the market  $G = (\mathbf{T}, \mathbf{q})$  as  $\varphi(\mathbf{T}, \mathbf{q})$ . In this paper, we focus on two mechanisms. The *worker-optimal stable mechanism*  $\varphi_W$  associates to each market  $G \in \mathbb{G}$  its corresponding worker-optimal stable allocation (i.e., for any  $(\mathbf{T}, \mathbf{q}) \in \mathbb{G}$ ,  $\varphi_W(\mathbf{T}, \mathbf{q}) = (\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$ ). The mechanism is described in Appendix A.1, and corresponds to *the descending salary adjustment process*. The *firm-optimal stable mechanism*  $\varphi_F$  associates to each market  $G \in \mathbb{G}$  its corresponding firm-optimal stable allocation (i.e., for any  $(\mathbf{T}, \mathbf{q}) \in \mathbb{G}$ ,  $\varphi_F(\mathbf{T}, \mathbf{q}) = (\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ ). The mechanism is described in Appendix A.2, and corresponds to *the ascending salary adjustment process*.

### 2.3 Groups of Workers and Transfer

In this paper, we study the impact of affirmative action on groups of the population. A group of workers may be defined based on factors such as gender, ethnicity, qualifications, location, educational background, or any other combination of individual characteristics. Mathematically, we fix a *partition* of  $W$ , denoted by  $\mathcal{P} \subset 2^W \setminus \{\emptyset\}$ , such that  $\cup \mathcal{P} \equiv \cup_{P \in \mathcal{P}} P = W$ , and for any  $P, P' \in \mathcal{P}$  with  $P \neq P'$  it holds that  $P \cap P' = \emptyset$ . Each element of  $\mathcal{P}$  is referred to as a *group*.

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<sup>9</sup>The existence of these allocations in the context of job markets has been extensively studied by Kelso and Crawford (1982), Roth (1984), Roth (1985), Demange and Gale (1985) among others.

## 2.4 Illustrative Example

We illustrate our results and reasoning using an example throughout the paper.

**Example 1.** Consider a market with two firms,  $F = \{f_1, f_2\}$  and five workers,  $W = \{w_1, w_2, w_3, w_4, w_5\}$ . Suppose each firm can employ at most two workers, i.e.,  $\mathbf{q} = (2, 2)$ , and for each worker  $w \in W, \sigma_{w,f} = 0$ . Productivities are given as follows:

Firms	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$
$f_1$	$R_{f_1}(w_1) = 8$	$R_{f_1}(w_2) = 7$	$R_{f_1}(w_3) = 5$	$R_{f_1}(w_4) = 4$	$R_{f_1}(w_5) = 3$
$f_2$	$R_{f_2}(w_1) = 3$	$R_{f_2}(w_2) = 6$	$R_{f_2}(w_3) = 8$	$R_{f_2}(w_4) = 5$	$R_{f_2}(w_5) = 4$

Table 1: Workers' productivities.

The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_4$	$w_1, w_2$	$w_3, w_4$
$s_{w_1} = 0, s_{w_2} = 1$	$s_{w_3} = 0, s_{w_4} = 0$	$s_{w_1} = 5, s_{w_2} = 4$	$s_{w_3} = 4, s_{w_4} = 1$

Table 2: Firm-optimal and worker-optimal stable allocations.

## 3 Subsidy

In this section, we examine the impact of subsidies on workers. Throughout this paper, we restrict the partition  $\mathcal{P}$  to two groups:  $m$ , representing *minority workers*, and  $M$ , representing *majority workers*. These groups are formed arbitrarily. This restriction does not affect the generality of our results, and we relax this assumption in Section 6. As discussed in the introduction, affirmative action policies are designed to favor certain groups within the population. A natural requirement is that when a subsidy is introduced, it should not reduce the welfare of minority workers. We say that a transfer  $\mathbf{T}$  does not *negatively impact minority workers* under  $\varphi$  if, for the market  $G = (\mathbf{T}_0, \mathbf{q})$ , where  $\varphi(\mathbf{T}_0, \mathbf{q}) = (\mu, \mathbf{s})$  and  $\varphi(\mathbf{T}, \mathbf{q}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$ , we have, for each  $w \in m$ :

- $u_w(\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}}) \geq u_w(\mu, \mathbf{s})$ .

A transfer  $\mathbf{T}$  *may negatively impact minority workers* under  $\varphi$  if the condition is not satisfied.

### 3.1 Targeted Subsidy for Minority Workers

Our first analysis focuses on targeted subsidies, which are often used to reduce unemployment among minority workers. Consider Example 1, and suppose that the partition of workers  $\mathcal{P}$  is defined as  $m = \{w_1, w_2, w_5\}$  and  $M = \{w_3, w_4\}$ . Worker  $w_5$  is an unemployed minority worker, and a policymaker introduces a subsidy of 2 for worker  $w_5$  at firm  $f_2$ , meaning  $t_{w_5, f_2} = 2$ , with other elements of  $\mathbf{T}$  set to 0.<sup>10</sup> The new productivities with the transfer are:

Firms	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$
$f_2$	$R_{f_2}^{\mathbf{T}}(w_1) = 3$	$R_{f_2}^{\mathbf{T}}(w_2) = 6$	$R_{f_2}^{\mathbf{T}}(w_3) = 8$	$R_{f_2}^{\mathbf{T}}(w_4) = 5$	$R_{f_2}^{\mathbf{T}}(w_5) = 6$

Table 3: Workers' Productivities at  $f_2$ .

The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$	$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 0, s_{w_2} = 0$	$s_{w_3} = 0, s_{w_5} = 0$	$s_{w_1} = 4, s_{w_2} = 3$	$s_{w_3} = 3, s_{w_5} = 1$

Table 4: Firm-optimal stable allocation and worker-optimal stable allocation with a subsidy of 2 for  $w_5$  at  $f_2$ .

The salary reduction of  $w_1$  and  $w_2$  results from increased competition with  $w_4$ . Without affirmative action policy, worker  $w_5$  remains unemployed. Worker  $w_5$  would accept employment at any firm  $f \in F$  for a salary of  $-\sigma_{w_5, f}$  yielding a profit of  $R_f(w_5) + \sigma_{w_5, f}$  for the firm. With the subsidy, worker  $w_5$  gains employment, but worker  $w_4$  becomes unemployed. Because  $w_4$  has higher productivity without the subsidy, other workers face greater competition and are forced to accept lower wages to retain their jobs.

This example illustrates that when a subsidy targets only certain minority workers, it can lead to negative impacts on the same group.

**Proposition 1.** Suppose  $\mathbf{T}$  subsidizes the workers  $m' \subset m$  at some  $f \in F$ . Then for some market  $G = (\mathbf{T}_0, \mathbf{q})$ ,  $\mathbf{T}$  may negatively impact minority workers under  $\varphi_W$  and  $\varphi_F$ .

We now examine the implementation of a broader policy transfer designed to subsidize all minority workers at a specific firm. Using Example 1, suppose a policymaker

<sup>10</sup>This type of subsidy is commonly used to promote the employment of minority workers.

introduces a subsidy of 2 for each minority worker for the firm  $f_2$ . Formally, for each  $w \in m, t_{w,f_2} = 2$  and 0 for other elements of  $\mathbf{T}$ . The resulting productivities are:

Firms	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$
$f_2$	$R_{f_2}^{\mathbf{T}}(w_1) = 5$	$R_{f_2}^{\mathbf{T}}(w_2) = 8$	$R_{f_2}^{\mathbf{T}}(w_3) = 8$	$R_{f_2}^{\mathbf{T}}(w_4) = 5$	$R_{f_2}^{\mathbf{T}}(w_5) = 6$

Table 5: Workers' productivities at  $f_2$ .

The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$	$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 0, s_{w_2} = 2$	$s_{w_3} = 0, s_{w_5} = 0$	$s_{w_1} = 4, s_{w_2} = 3$	$s_{w_3} = 3, s_{w_5} = 1$

Table 6: Firm-optimal and worker-optimal stable allocations with a subsidy of 2 for  $m$  at  $f_2$ .

The reasoning parallels the case where the transfer only subsidizes worker  $w_5$  at firm  $f_2$ , leading to a reduction in the salaries of  $w_1$  and  $w_2$  in the allocation  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$ . Therefore, when a transfer is directed toward subsidizing minority workers across a subset of firms, it may have adverse effects on minority workers in the worker-optimal stable allocation. Such transfers can be implemented by policymakers to support specific market sectors with the aim of increasing minority worker representation. This example illustrates the critical importance of carefully designing transfer mechanisms in affirmative action policies. By contrast, no decrease in the salaries of minority workers occurs under the firm-optimal stable allocation.

**Proposition 2.** Suppose  $\mathbf{T}$  subsidizes the workers  $m \subset W$  at some  $F' \subset F$  by  $t$ . Then for some market  $G = (\mathbf{T}_0, \mathbf{q})$ ,  $\mathbf{T}$  may negatively impact minority workers under  $\varphi_W$  and does not negatively impact minority workers under  $\varphi_F$ .

*Proof.* See Section B.2. ■

### 3.2 Uniform Subsidy

The negative results highlighted in the previous section prompt a natural question: Is it possible to design a transfer that subsidizes minority workers without adversely affecting their welfare? Our first main theorem establishes that when a transfer uniformly subsidizes minority workers, it ensures that no minority worker experiences a welfare

reduction under either  $\varphi_W$  or  $\varphi_F$ . We say that a transfer  $\mathbf{T}$  *uniformly subsidizes minority workers* if there exists  $t > 0$  such that for each  $w \in m$  and each  $f \in F$ ,  $t_{w,f} = t$  and for each  $w' \in M$ ,  $f \in F$ ,  $t_{w',f} = 0$ .

In Example 1, suppose the transfer  $\mathbf{T}$  uniformly subsidizes minority workers by 2. The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$	$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 0, s_{w_2} = 2$	$s_{w_3} = 0, s_{w_5} = 0$	$s_{w_1} = 6, s_{w_2} = 5$	$s_{w_3} = 3, s_{w_5} = 1$

Table 7: Firm-optimal and worker-optimal stable allocations with a uniform subsidy of 2.

Worker  $w_5$  is now employed by firm  $f_2$ . Moreover, The salary of all minority workers has increased strictly at the worker-optimal stable allocation, and weakly for  $w_1$  at the firm-optimal stable allocation. This observation can be generalized to any market.

**Theorem 1.** If a transfer  $\mathbf{T}$  uniformly subsidizes minority workers, then  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$  and  $\varphi_F$ .

*Proof.* See Section B.3. ■

Theorem 1 is relevant to various practical issues: subsidies introduced for unemployed minority workers or specific market sectors may inadvertently harm employed minority workers. To prevent such negative effects, the subsidy amount must be identical for all minority workers, regardless of their employment status.

## 4 Relation Between Subsidy and Tax

Another common instrument in affirmative action policies is the imposition of taxes. These taxes are designed to penalize firms that fail to meet government-imposed representation requirements. As mentioned in the introduction, firms may receive tax abatements for employing a minimum number of minority workers. In this context, imposing a tax is analogous to reducing the abatement. The key distinction between taxes and subsidies lies in the range of possible amounts. While there may be no mathematical distinction, from an economic perspective, it is important to limit the amount of the tax. If a tax exceeds a worker's productivity, the firm will never employ that worker. Therefore, throughout this paper, we assume that for each worker  $w$  and each firm  $f$ , we have  $R_f(w) + \sigma_{w,f} + t_{w,f} \geq 0$ .

## 4.1 Targeted Tax

This section highlights the difference between taxes and subsidies. When a subsidy targets a subset of workers or firms, it may negatively impact minority workers under the mechanisms  $\varphi_W$  and  $\varphi_F$ . However, when a tax targets a subset of majority workers or firms, minority workers may still be negatively impacted under  $\varphi_W$ , but not under  $\varphi_F$ .

For instance, in Example 1, suppose a transfer  $\mathbf{T}$  is introduced such that  $t_{w_4, f_2} = -2$ , with all other elements of  $\mathbf{T}$  set to zero. The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$	$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 0, s_{w_2} = 2$	$s_{w_3} = 0, s_{w_5} = 0$	$s_{w_1} = 4, s_{w_2} = 3$	$s_{w_3} = 5, s_{w_5} = 1$

Table 8: Firm-optimal and worker-optimal stable allocations with a tax of 2 for  $w_4$  at  $f_2$ .

As with the targeted subsidy,  $w_1$  and  $w_2$  receive lower wages at the worker-optimal stable allocation, while  $w_5$  is now employed. In contrast, majority worker  $w_3$  faces less competition at firm  $f_2$  and may thus enjoy a higher wage. The reasoning behind the salary reduction for  $w_1$  and  $w_2$  follows the same logic as with the subsidy.

The difference emerges when we consider the firm-optimal stable allocation. The salary of  $w_2$  increased. Since firms compete for higher profits in the firm-optimal allocation, the introduction of a tax makes worker  $w_4$  less attractive to firm  $f_2$ . Thus, with a productivity of  $R_{f_2}^{\mathbf{T}}(w_2) = 6$ ,  $w_2$  is more productive than both  $w_4$  and  $w_5$  (with  $R_{f_2}^{\mathbf{T}}(w_4) = 4$  and  $R_{f_2}^{\mathbf{T}}(w_5) = 4$ ). As a result, firm  $f_1$  must offer a salary of 2 to  $w_2$  to retain her, preventing her from joining  $f_2$ . Hence, when a tax is introduced, it does not reduce minority workers' welfare at the firm-optimal stable allocation, even if the tax does not apply to all majority workers.

In Example 1, suppose now that a transfer  $\mathbf{T}$  introduces a tax of 2 for each majority worker at firm  $f_2$ , i.e., for each  $w \in M$ ,  $t_{w, f_2} = -2$ , with all other elements of  $\mathbf{T}$  set to zero. A similar analysis applies here. Worker  $w_3$ 's salary decreases at the worker-optimal stable allocation due to the tax. The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$	$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 0, s_{w_2} = 2$	$s_{w_3} = 0, s_{w_5} = 0$	$s_{w_1} = 4, s_{w_2} = 3$	$s_{w_3} = 3, s_{w_5} = 1$

Table 9: Firm-optimal and worker-optimal stable allocations with a tax of 2 for  $M$  at  $f_2$ .

**Proposition 3.** Suppose

- $\mathbf{T}$  taxes the workers  $M' \subset M$  at some  $f \in F$ , or
- $\mathbf{T}$  taxes the workers  $M \subset W$  at some  $F' \subset F$  by  $t$ .

Then for some market  $G = (\mathbf{T}_0, \mathbf{q})$ ,  $\mathbf{T}$  may negatively impact minority workers under  $\varphi_W$  and does not negatively impact minority workers under  $\varphi_F$ .

*Proof.* See Section B.4. ■

Interestingly, when the firm-optimal stable allocation is considered, taxing majority workers does not negatively impact minority workers. This is because, at the firm-optimal allocation, workers' salaries are influenced by the revenue generated by other workers at competing firms (see Lemma 5). Thus, while subsidies make some workers relatively less productive, leading to a salary decline, taxes do not increase productivity and, in some cases, reduce it, resulting in higher salaries. Despite these differences, both subsidies and taxes can have negative effects on minority workers under  $\varphi_W$ .

## 4.2 Uniform Tax Equivalence

Given that targeted taxes may adversely affect minority workers, we now turn our attention to uniform taxes. A transfer  $\mathbf{T}$  *uniformly taxes majority workers* if there exists a  $t < 0$  such that for each  $w \in M$  and each  $f \in F$ , we have  $t_{w,f} = t$  and for each  $w' \in m, f \in F, t_{w',f} = 0$ .

Consider Example 1, where a transfer  $\mathbf{T}$  is introduced that uniformly taxes majority workers by 2. The stable optimal allocations under this tax are presented in Table 7. Despite the differences between a targeted subsidy and a targeted tax, when applying a uniform tax, the transfer does not negatively impact minority workers under either  $\varphi_W$  and  $\varphi_F$ .

**Theorem 2.** If a transfer  $\mathbf{T}$  uniformly taxes majority workers, then  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$  and  $\varphi_F$ .



*Proof.* See Section B.5. ■

### 4.3 Uniform Tax and Uniform Subsidy Combination

In practice, subsidies and taxes are often implemented simultaneously in a market. Based on our previous results, it follows directly that introducing a targeted subsidy alongside a tax may negatively impact minority workers under both  $\varphi_W$  and  $\varphi_F$ . However, if a transfer uniformly subsidizes minority workers and uniformly taxes majority workers, there is no negative impact on minority workers under either  $\varphi_W$  and  $\varphi_F$ . A transfer  $\mathbf{T}$  *uniformly subsidizes minority workers and uniformly taxes majority workers* if there exist  $t > 0$  and  $t' < 0$  such that for each firm  $f \in F$ ,  $R_f^{\mathbf{T}}(w) = R_f(w) + t$  for each  $w \in m$ , and  $R_f^{\mathbf{T}}(w) = R_f(w) + t'$  and for each  $w' \in M$ .

**Corollary 1.** If a transfer  $\mathbf{T}$  uniformly subsidizes minority workers and uniformly taxes majority workers, then  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$  and  $\varphi_F$ .

As seen in the illustrative example, when a uniform tax of 2 is imposed on majority workers, the optimal stable allocations mirror those obtained when a uniform subsidy of 2 is applied to minority workers. Beyond the absence of a negative impact on minority workers, the effects of a uniform subsidy and a uniform tax are equivalent when their magnitudes are identical. Theorem 3 states that if a transfer uniformly taxes majority workers by  $t$ , then the allocations under  $\varphi_W$  and  $\varphi_F$  are identical to those produced when minority workers are uniformly subsidized by  $t$ .<sup>11</sup>

**Theorem 3.** Consider two transfers  $\mathbf{T}$  and  $\mathbf{T}'$ , where  $\mathbf{T}$  uniformly subsidizes minority workers by  $t$  and  $\mathbf{T}'$  uniformly taxes majority workers by  $t'$ . For any market  $G = (\mathbf{T}_0, \mathbf{q})$  if  $t = |t'|$ , there exist worker-optimal and firm-optimal stable allocations such that  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'})$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'})$ .

*Proof.* See Section B.6. ■

Given this equivalence, uniform taxes and uniform subsidies may serve as complements. In Example 1, consider a transfer  $\mathbf{T}$  where each minority worker  $w \in m$  receives a subsidy of 1 at each  $f \in F$  (i.e.,  $t_{w,f} = 1$ ), and each majority worker  $w' \in M$  is

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<sup>11</sup>Sotomayor (1999) shows that for a given market, multiple worker-optimal and firm-optimal stable allocations can exist (Theorem 1 of Sotomayor 1999). We, therefore, consider that there can be several optimal allocations. However, when there are multiple worker-optimal stable allocations, worker utility is the same for all these allocations. Similarly, when there are multiple firm-optimal stable allocations, worker utility is the same for all of them (Theorem 1 of Sotomayor 1999).

taxed by 1 at each  $f \in F$  (i.e.,  $t_{w',f} = -1$ ). The resulting stable optimal allocations are displayed in Table 7. Corollary 2 states that if a uniform subsidy of  $t$  and a uniform tax of  $t'$  are introduced, then under mechanisms  $\varphi_W$  and  $\varphi_F$ , the resulting allocations are the same as those produced if a uniform subsidy or tax of  $t + |t'|$  had been introduced.

**Corollary 2.** Suppose there exist three transfer  $\mathbf{T}, \mathbf{T}', \mathbf{T}''$  and  $t > 0, t' < 0$  such that:

- $\mathbf{T}$  uniformly subsidizes minority workers by  $t$  and uniformly taxes majority workers by  $t'$ .
- $\mathbf{T}'$  uniformly subsidizes minority workers by  $t + |t'|$ .
- $\mathbf{T}''$  uniformly taxes majority workers by  $t + |t'|$ .

Then, for any market  $G \in \mathbb{G}$ , there exist worker-optimal and firm-optimal stable allocations such that  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'}) = (\mu_W^{\mathbf{T}''}, \mathbf{s}_W^{\mathbf{T}''})$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'}) = (\mu_F^{\mathbf{T}''}, \mathbf{s}_F^{\mathbf{T}''})$ .

The main implication of Theorem 3 and Corollary 2 is that a uniform subsidy has the same impact as a uniform tax. To reduce the cost of the policy, a policymaker may favor the use of a uniform tax. If the policymaker does not have information such as firm revenues for each worker, or the mechanism used in the market, then only the use of a uniform transfer guarantees that the transfer does not negatively impact minority workers. However, if the policymaker has information on revenue functions, we can consider non-uniform transfers to achieve different objectives.

## 5 Affirmative Action Policies with Transfer

In this section, we analyze three key objectives that can be pursued through transfers: increasing the salaries of minority workers, increasing the representation of minority workers in a firm, and reducing unemployment among minority workers. To determine which transfers do not negatively impact minority workers under  $\varphi_W$  and  $\varphi_F$ , we reduce the dimension by fixing the sets of employed minority workers and unemployed minority workers. Given a market  $G \in \mathbb{G}$  and an allocation  $(\mu_W, \mathbf{s}_W)$ , let  $m_E \equiv \{w : w \in m \text{ and } \mu_W(w) \neq \emptyset\}$  represent the set of *employed minority workers* and  $m_U \equiv \{w : w \in m \text{ and } \mu_W(w) = \emptyset\}$  the set of *unemployed minority workers*.<sup>12</sup> Similarly, we define  $W_E \equiv \{w : w \in W \text{ and } \mu_W(w) \neq \emptyset\}$  as the set of *employed workers*, and  $W_U \equiv \{w : w \in$

<sup>12</sup>Considering  $(\mu_W, \mathbf{s}_W)$  is equivalent to consider  $(\mu_F, \mathbf{s}_F)$ . Theorem 5 of Sotomayor (1999) states that the assignment remains unchanged between  $\mu_W$  and  $\mu_F$ ; only salaries differ. The intuition is that workers are assigned to the firm where they generate the highest surplus, subject to the firm's capacity constraints. The difference between firm-optimal and worker-optimal stable allocations lies in the distribution of surplus. We adapt Theorem 5 from Sotomayor (1999) to our model in Lemma 3.

$W$  and  $\mu_W(w) = \emptyset$  as the set of *unemployed workers*. When a transfer  $\mathbf{T}$  is introduced, we define  $m_E^{\mathbf{T}} \equiv \{w : w \in m \text{ and } \mu_W^{\mathbf{T}}(w) \neq \emptyset\}$  as the set of *employed minority workers with transfer  $\mathbf{T}$*  and  $m_U^{\mathbf{T}} \equiv \{w : w \in m \text{ and } \mu_W^{\mathbf{T}}(w) = \emptyset\}$  as the set of *unemployed minority workers with transfer  $\mathbf{T}$* . Similarly, we define  $W_E^{\mathbf{T}} \equiv \{w : w \in W \text{ and } \mu_W^{\mathbf{T}}(w) \neq \emptyset\}$  as the set of *employed workers with transfer  $\mathbf{T}$* , and  $W_U^{\mathbf{T}} \equiv \{w : w \in W \text{ and } \mu_W^{\mathbf{T}}(w) = \emptyset\}$  as the set of *unemployed workers with transfer  $\mathbf{T}$* .

## 5.1 Increasing Salaries of Minority Workers

A common goal of affirmative action policies is to raise the salaries of minority workers. In this section, we focus on transfers that are not uniform across workers within the same group. To ensure that the salaries of minority workers increase, it is crucial to design transfers that do not negatively affect them. As illustrated in Table 8, when a (non-uniform) tax is implemented, the effects cannot be targeted.<sup>13</sup> Therefore, in this section, we consider subsidies to guarantee that the transfers benefit minority workers.

The distinction between employed and unemployed minority workers is critical. As illustrated with the targeted subsidy in Table 4, subsidizing  $w_5$ , an unemployed worker, intensifies market competition among workers. Theorem 4 establishes that when subsidies are allocated exclusively to employed minority workers, and the highest subsidy for each worker is provided to the firm employing them, their salaries will increase by the value of the subsidy in the worker-optimal stable allocation. If subsidies are provided to unemployed minority workers, then employed minority workers must receive an equivalent or larger subsidy to ensure that their salaries are not negatively impacted under  $\varphi_W$ .

**Theorem 4.** Consider a market  $G \in \mathbb{G}$ . Let  $\mathbf{T}$  be a transfer such that for each  $t_{w,f} > 0$ , we have  $w \in m$ , and there is no  $t_{w',f} < 0$  with  $w' \in W$ .

- (i) If for each  $w \in m_E$ ,  $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f})$  and for each  $w' \in m_U$ ,  $f \in F$ ,  $t_{w',f} = 0$ , then  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ . Furthermore,  $\mu_W = \mu_W^{\mathbf{T}}$  and for each  $w \in W$ ,  $s_{w,\mu_W^{\mathbf{T}}(w)}^{\mathbf{T}} = s_{w,\mu_W(w)} + t_{w,\mu_W(w)}$ .
- (ii) If for each  $w \in m_E$ ,  $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f})$ , and  $t_{w,\mu_W(w)} \geq \max_{w' \in m_U, f' \in F}(t_{w',f'})$ , then  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ .

*Proof.* See Section B.7. ■

<sup>13</sup>Considering the worker-optimal stable allocation, the salary of  $w_3$ , a majority worker, increases, while those of  $w_1$  and  $w_2$  decrease.

The intuition behind Theorem 4 (i) is that if a higher subsidy is provided to workers at a different firm, some employees may be incentivized to switch firms in order to increase their utility. This shift in firm creates competition, which can result in a decline in the utility of other workers. To isolate the impact on workers' wages, we impose that unemployed individuals do not receive subsidies. However, we relax this assumption in part (ii). When subsidies are extended to unemployed workers, the salaries of employed minority workers may decrease. To counteract this, if employed workers receive a subsidy of at least the same amount, the reduction in wages caused by subsidies for the unemployed is neutralized, preventing any decrease in their salaries.<sup>14</sup>

Note that subsidies for employed minority workers can vary across firms and individual workers. This allows for the possibility of designing policies that target wage increases for specific workers in certain sectors, while not affecting others. The practical implication of Theorem 4 is that if the policymaker's goal is to raise the wages of minority workers in a particular sector, subsidies should be applied exclusively to workers in that sector.<sup>15</sup> Moreover, this approach mitigates the negative impact on majority workers. In fact, under the worker-optimal stable allocation, and considering the transfer mechanism described in Theorem 4 (i), no worker in the market experiences a decline in utility following the introduction of the subsidy.

However, Theorem 4 holds only under the mechanism  $\varphi_W$ , and the transfers described may negatively impact minority workers under mechanism  $\varphi_F$ .

**Proposition 4.** Consider a market  $G \in \mathbb{G}$ . Let  $\mathbf{T}$  be a transfer such that for each  $t_{w,f} > 0$ , we have  $w \in m$ , and there is no  $t_{w',f} < 0$  with  $w' \in W$ .

- (i) If for each  $w \in m_E$ ,  $t_{w,\mu_W(w)} = \max_f(t_{w,f})$  and for each  $w' \in m_U$ ,  $f \in F$ ,  $t_{w',f} = 0$ , then  $\mathbf{T}$  may negatively impact minority workers under  $\varphi_F$ .
- (ii) If for each  $w \in m_E$ ,  $t_{w,\mu_W(w)} = \max_f(t_{w,f})$ , and  $t_{w,\mu_W(w)} \geq \max_{w' \in m_U, f' \in F}(t_{w',f'})$ , then,  $\mathbf{T}$  may negatively impact minority workers under  $\varphi_F$ .

*Proof.* Consider Example 1, a partition such that  $m = \{w_2, w_4, w_5\}$  and suppose a transfer  $\mathbf{T}$  is introduced with  $t_{w_4, f_2} = 1$ . The salary of  $w_2$  is reduced to zero at  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ . ■

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<sup>14</sup>In Theorem 1, under a uniform subsidy, this condition is trivially met. Theorem 4 extends Theorem 1 by demonstrating that a larger subsidy for employed minority workers will result in an increase in their salaries.

<sup>15</sup>Imposing a sector-specific minimum wage is a direct application of Theorem 4.

Proposition 4 illustrates the distinction between worker-optimal and firm-optimal stable allocations. Subsidizing specific workers at the firms to which they are matched increases the surplus they generate, which, in turn, diminishes the attractiveness of other workers. As the perceived value of workers not employed by the firm declines, firms can reduce the wages of these workers. Proposition 4 highlights the importance of the mechanism used in the market when designing policy interventions. Hence, the policymaker's knowledge of the market mechanism is crucial for the effective design of transfer policies.

## 5.2 Increasing the Representation of Minority Workers in a Firm

As discussed in the introduction, the primary historical goal of affirmative action policies has been to reduce segregation within firms and sectors. In this section, we explore transfers that can increase the representation of minority workers in a firm. A transfer  $\mathbf{T}$  *increases the representation of minority workers in firm  $f$*  under  $\varphi$  if, for a market  $G = (\mathbf{T}_0, \mathbf{q})$ , we have  $\varphi(\mathbf{T}_0, \mathbf{q}) = (\mu, \mathbf{s})$  and  $\varphi(\mathbf{T}, \mathbf{q}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$ , such that  $q_f \geq |\mu^{\mathbf{T}}(f) \cap m| > |\mu(f) \cap m|$  and  $(\mu(f) \cap m) \subset (\mu^{\mathbf{T}}(f) \cap m)$ . We impose that the minority workers employed by firm  $f$  under assignment  $\mu$  remain employed under  $\mu^{\mathbf{T}}$ , and that at least one additional minority worker is hired.<sup>16</sup> By construction, this implies that some majority workers currently employed by  $f$  will be replaced by minority workers following the introduction of the transfer.

For simplicity, without loss of generality, let us consider the case in which a single additional minority worker is hired by firm  $f$ . In this instance, only one majority worker needs to be replaced. We denote these workers as  $w_m$  and  $w_M$ , where  $\mu_W(w_m) \neq f$  and  $\mu_W(w_M) = f$ . For the worker  $w_m$  to be employed by firm  $f$  instead of  $w_M$ ,  $w_m$  must generate a higher profit at  $f$  than the profit generated by  $w_M$ . In a worker-optimal stable allocation, all workers within the firm generate the same profit (see Lemma 4), which leads us to focus on the firm-optimal stable allocation. Consequently, the worker  $w_M$  who will be replaced by  $w_m$  is the majority worker employed by firm  $f$  under the allocation  $(\mu_F, \mathbf{s}_F)$  who produces the lowest profit.

Let  $t_{\rightarrow f} \equiv \min_{w_M \in \mu_F(f) \cap M} (R_f(w_M) - s_{w_M, f}) - (R_f(w_m) + \sigma_{w_m, f}) + R_{\mu_F(w_m)}(w_m) + \sigma_{w_m, \mu_F(w_m)} - \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} (R_{\mu_F(w_m)}(w') + \sigma_{w', \mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F))$  be the

<sup>16</sup>We consider the capacity of firm  $f$  to ensure that there are majority workers employed by  $f$  under assignment  $\mu$ .

subsidy required for  $w_m$  to be hired by  $f$ .<sup>17</sup> The amount of the subsidy depends on both the productivity of the minority worker  $w_m$  at firm  $f$  and the productivity of  $w_m$  at the current firm,  $\mu_F(w_m)$ . The more productive  $w_m$  is at  $\mu_F(w_m)$ , the greater the subsidy required to incentivize the move to firm  $f$ . The underlying intuition is that if the subsidy is insufficient, firm  $\mu_F(w_m)$  will offer a higher salary to retain  $w_m$ . Therefore, firm  $f$  must offer  $w_m$  a sufficiently high wage to ensure that  $\mu_F(w_m)$  has no incentive to retain  $w_m$  and instead finds a more attractive alternative.

**Example 2.** Consider Example 1, a partition such that  $m = \{w_1, w_4\}$  and the objective of the policymaker is for  $w_4$  to be employed by  $f_1$ . Thus, the subsidy is calculated as follows:  $t_{\rightarrow f_1} = R_{f_1}(w_2) - s_{w_2, f_1} - R_{f_1}(w_4) + R_{f_2}(w_4) - \sigma_{w_4, f_2} - (R_{f_2}(w_2) - \sigma_{w_2, f_2} - u_{w_2}(\mu_F, \mathbf{s}_F)) = 7 - 1 - 4 + 5 - (6 - 1) = 2$ . Thus, by subsidizing  $w_4$  by 2 at  $f_1$ ,  $w_4$  replaces  $w_2$  and  $\mathbf{T}$  increases the representation of minority workers in firm  $f$  under  $\varphi_F$ . Suppose a transfer  $\mathbf{T}$  such that  $t_{w_4, f_1} = 2$ , with all other elements of  $\mathbf{T}$  set to zero, is introduced. The stable optimal allocations for this market are given by:

$f_1$	$f_2$
$w_1, w_4$	$w_2, w_3$
$s_{w_1} = 0, s_{w_4} = 0$	$s_{w_2} = 1, s_{w_3} = 0$

$f_1$	$f_2$
$w_1, w_4$	$w_2, w_3$
$s_{w_1} = 3, s_{w_4} = 1$	$s_{w_2} = 2, s_{w_3} = 4$

Table 10: Firm-optimal worker-optimal stable allocations with a subsidy of 2 for  $w_4$  at  $f_1$ .

Table 10 illustrates that minority workers in  $f_1$  can be negatively impacted by this transfer, as the salary of  $w_1$  at the worker-optimal stable allocation is 3. Our next result states that if this subsidy is introduced for all employed minority workers at firm  $f_1$ , then the transfer does not negatively impact minority workers under  $\varphi_W$  and  $\varphi_F$ .

**Theorem 5.** Consider a market  $G \in \mathbb{G}$ . Suppose that  $\mathbf{T}$  is a transfer such that for  $f \in F$ , for each  $w \in m \cap (\mu_F(f) \cup \{w_m\})$ , we have  $t_{w, f} \geq t_{\rightarrow f}$ ; for each  $w' \in m_E \setminus (\mu_F(f) \cup \{w_m\})$ , we have  $t_{w', f} \leq t_{\rightarrow f}$ ; for each  $w' \in W \setminus m_E$ ,  $t_{w', f} = 0$ ; and for each  $f' \in F \setminus \{f\}$ ,  $w \in W$ ,  $t_{w, f'} = 0$ . Then,  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$  and increases the representation of minority workers in firm  $f$  under  $\varphi_W$  and  $\varphi_F$ .

*Proof.* See Section B.8. ■

<sup>17</sup>We demonstrate in Claim 8 that  $t_{\rightarrow f}$  allows  $w_m$  to generate (weakly) more profit than  $w_M$  at firm  $f$ . In certain markets, this subsidy can be reduced by considering workers employed at other firms rather than only those in  $W_U \cup w_M$ . However,  $t_{\rightarrow f}$  applies across all markets.

In contrast to the salary increases discussed in Section 5.1, modifying the assignment of minority workers requires that the transfer be non-uniform and specifically target  $f$ . Theorem 5 asserts that minority workers employed by  $f$  and  $w_m$  must receive a subsidy of at least  $t_{\rightarrow f}$  to not be negatively impacted by the transfer under  $\varphi_W$ . Conversely, other minority workers employed must receive subsidies of at most  $t_{\rightarrow f}$  at  $f$ . The reason for this is that  $w_m$  is chosen from among the minority workers who are not employed by  $f$ . Therefore, some minority workers may require a subsidy lower than  $t_{\rightarrow f}$  to be employed by  $f$ . If they are subsidized by  $t_{\rightarrow f}$ , they will be hired by  $f$ , increasing the representation of minority workers, although  $w_m$  will not be hired by  $f$ . By imposing that minority workers employed by  $f$  be subsidized by at least  $t_{\rightarrow f}$ , we prevent any decrease in their welfare.<sup>18</sup> This argument is used in the proof of Theorem 5.

The proof of Theorem 5 proceeds as follows: we begin by decomposing the transfer  $T$ , into a sum of transfer matrices (utilizing Lemma 1). First, we focus on the subsidy for minority workers employed at  $f$ . By Theorem 4, we know that their salary increases by  $t_{w,f}$  under  $\varphi_W$ . Next we consider a transfer that subsidizes  $w_m$  by  $t_{\rightarrow f}$ . By Lemma 2, we know that the salaries of all workers decrease by at most  $t_{\rightarrow f}$ . Since their salaries have increased by more than  $t_{\rightarrow f}$  we can conclude that the salaries of minority workers at  $f$  are not reduced.

The remaining subsidy for  $w_m$  at  $f$  is then introduced, allowing  $w_m$  to be recruited by  $f$  and being subsidized by  $t_{w_m,f} - t_{\rightarrow f}$ . This, according to Theorem 4, results in an equivalent increase in  $w_m$ 's salary, without affecting the salaries of other workers. Finally, we consider the transfer that subsidizes other minority workers employed elsewhere, capped at  $t_{\rightarrow f}$ . Consequently, the impact on their salaries is limited to at most  $t_{\rightarrow f}$ , ensuring that the salaries of minority workers at  $f$  remain unaffected.

Similar to the policy of increasing salaries, the transfers described in Theorem 5 may negatively impact minority workers under  $\varphi_F$ . In Example 1, consider a partition such that  $m = \{w_1, w_2, w_3\}$  and suppose a transfer  $\mathbf{T}$  is introduced such that  $t_{w_1,f_2} = 7$  and 0 for other elements. The salary of  $w_2$  is 0 under  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ .

Proposition 5 complements Theorem 5 by establishing transfers that increase the representation of minority workers at  $f$ , without imposing any conditions on the welfare

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<sup>18</sup>If a minority worker, other than  $w_m$ , is employed by  $f$  and is subsidized by less than  $t_{\rightarrow f}$ , the salary of the minority workers employed by  $f$  is reduced by less than  $t_{\rightarrow f}$ . Thus the subsidy ensures that their salary does not decrease.

of minority workers. Consequently, some minority workers employed by  $f$  may experience salary reductions following the introduction of  $\mathbf{T}$ .

**Proposition 5.** Consider a market  $G \in \mathbb{G}$  and  $(\mu_F, \mathbf{s}_F)$ . Suppose that  $\mathbf{T}$  is a transfer such that for  $f \in F$ , for each  $w \in m \cap \mu_F(f)$ ,  $t_{w,f} \geq \max(0, R_f(w_M) - s_{w_M,f} - (R_f(w) - (\max_{f' \in F \setminus \{f\}}(R_{f'}(w) + \sigma_{w,f'} - \max_{w' \in W_U \cup \{w_M\}}(R_{f'}(w') + \sigma_{w',f'}))))$ ); for each  $w' \in W \setminus (m \cap (\mu_F(f) \cup \{w_m\}))$ ,  $t_{w',f} = 0$ ; for each  $f' \in F \setminus \{f\}$ ,  $w \in W$ ,  $t_{w,f'} = 0$ ; and  $t_{w_m,f} \geq t_{\rightarrow f}$ . Then,  $\mathbf{T}$  increases the representation of minority workers in firm  $f$  under  $\varphi_W$  and  $\varphi_F$ .

*Proof.* See Section B.9. ■

To maintain their positions at  $f$ , minority workers employed by  $f$  must generate a profit that is higher than that of  $w_M$ . Thus, the transfer described subsidizes a minority worker by at least 0 if the profit they generate at  $f$  is already greater than that generated by  $w_M$ , and by the difference otherwise. Using the same reasoning as we did for  $t_{\rightarrow f}$ , the subsidy must be sufficiently high because the worker  $w_M$  can accept a lower salary and maintain her position at  $f$ .

### 5.3 Reducing Unemployment Among Minority Workers

In this section, we consider a policy aimed at reducing unemployment among minority workers. As illustrated in Table 4, subsidizing certain unemployed workers enhances market competitiveness, which can lead to reduced salaries.

A transfer  $\mathbf{T}$  favors the employment of minority workers under  $\varphi$  if, for a market  $G = (\mathbf{T}_0, \mathbf{q})$ , we have  $\varphi(\mathbf{T}_0, \mathbf{q}) = (\mu, \mathbf{s})$  and  $\varphi(\mathbf{T}, \mathbf{q}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$ , such that  $|m_E^{\mathbf{T}}| > |m_E|$  with  $m_E \subset m_E^{\mathbf{T}}$ . We impose that employed minority workers do not lose their jobs and that some unemployed minority workers gain employment as a result of the transfer. By definition, it follows that if a transfer  $\mathbf{T}$  favors the employment of minority workers, then there exists  $f \in F$  such that  $\mathbf{T}$  increases the representation of minority workers in firm  $f$ . However, there are two major distinctions to note. First, minority workers who are already employed may change firms, provided they remain employed after the transfer. Second, unemployed minority workers must secure employment with a firm, without specification regarding which firm that may be.

Similar to the policy aimed at increasing minority worker representation in a firm, for simplicity, we focus on a single unemployed minority worker who replaces a majority worker. The unemployed minority worker considered is the one requiring the lowest



subsidy to gain employment. We denote these workers as  $w_m$  and  $w_M$  for these workers, respectively, with  $\mu_W(w_M)$  representing the firm. The subsidy required is at least  $t_{m_U \rightarrow} \equiv (R_{\mu_W(w_M)}(w_M) + \sigma_{w_M, \mu_W(w_M)}) - (R_{\mu_W(w_M)}(w_m) + \sigma_{w_m, \mu_W(w_M)})$ .<sup>19</sup>

**Theorem 6.** Consider a market  $G \in \mathbb{G}$ . Suppose that  $\mathbf{T}$  is a transfer such that for each  $w \in m_E$ ,  $t_{w, \mu_W(w)} = \max_{f \in F}(t_{w, f}) \geq t_{m_U \rightarrow}$ ; for each  $w' \in W \setminus m_E \cup \{w_m\}$  for each  $f \in F$ ,  $t_{w', f} = 0$ ; and  $t_{w_m, \mu_W(w_M)} \geq t_{m_U \rightarrow}$ . Then  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$  and favors the employment of minority workers under  $\varphi_W$  and  $\varphi_F$ .

*Proof.* See Section B.10. ■

Theorem 6 states that transfers in which each minority worker employed receives the highest subsidy in their respective firm, exceeding  $t_{m_U \rightarrow}$ , while  $w_m$  is subsidized by at least  $t_{m_U \rightarrow}$  at  $\mu_W(w_M)$  and all other transfers are set to 0, do not negatively impact minority workers under  $\varphi_W$  and favor the employment of minority workers under  $\varphi_F$  and  $\varphi_W$ .

One might argue that the worker  $w_m$  receiving a subsidy of  $t_{m_U \rightarrow}$  will have a utility of 0 in all competitive equilibria with the transfer. Nevertheless, according to Theorem 4, once employed, it is possible to increase  $w_m$ 's salary without diminishing the welfare of other workers. Therefore, it is feasible to promote employment while also establishing a minimum wage for worker  $w_m$ .

For employed minority workers, the argument that the subsidy is highest in the firm where they work follows the same rationale as in Theorem 4: this does not exacerbate competition among workers. By decomposing the transfer, the proof is analogous to that of Theorem 5. The salaries of employed minority workers decrease by at most  $t_{m_U \rightarrow}$ ; thus, by subsidizing them by a larger amount, the transfer does not negatively impact minority workers. While in Theorem 5 the subsidy pertains only to one firm, in Theorem 6, it encompasses all firms employing minority workers. This distinction also illustrates that a uniform subsidy can achieve the desired objective. Indeed, if a transfer  $\mathbf{T}$  that uniformly subsidizes minority workers by  $t_{m_U \rightarrow}$  is introduced, then  $\mathbf{T}$  does not negatively impact minority workers and favors the employment of minority workers under  $\varphi_W$  and  $\varphi_F$ .

The policymaker's objective may be to reduce unemployment without considering changes in the salaries of employed minority workers. However, it is essential to maintain

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<sup>19</sup>The intuition is that the surplus generated by  $w_m$  is at least equal to the surplus generated by  $w_M$ . Therefore, even with a salary of 0,  $\mu_W(w_M)$  is indifferent between  $w_m$  and  $w_M$ . We prove that  $t_{m_U \rightarrow}$  is sufficient to guarantee that  $w_m$  is employed in the proof of Proposition 6.

the employment of minority workers. From Table 4, it is evident that policies subsidizing unemployed minority workers may lead to job losses among other minority workers.<sup>20</sup> Proposition 6 provides the necessary transfer to ensure the continued employment of minority workers while allowing for at least one unemployed minority worker to be hired by a firm.

**Proposition 6.** Consider a market  $G \in \mathbb{G}$ . Suppose that  $\mathbf{T}$  is a transfer such that for each  $w \in m_E \setminus \{\mu_W(\mu_W(w_M))\}$ ,  $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f}) \geq \max(0, t_{m_{U \rightarrow}} - (s_{w,\mu_W(w)} + \sigma_{w,\mu_W(w)}))$ ; for each  $w \in m_E \cap \{\mu_W(\mu_W(w_M))\}$ ,  $t_{w,\mu_W(w_M)} = \max_{f \in F}(t_{w,f}) \geq \max(0, R_{\mu_W(w_M)}(w_M) + \sigma_{w_M,\mu_W(w_M)} - R_{\mu_W(w_M)}(w) - \sigma_{w,\mu_W(w_M)})$ ; for each  $w' \in W \setminus (m_E \cup \{w_m\})$ ,  $f \in F$ ,  $t_{w',f} = 0$ ; and for each  $f \in F \setminus \{\mu_W(w_M)\}$ ,  $t_{w_m,f} = 0$  and  $t_{w_m,\mu_W(w_M)} \geq t_{m_{U \rightarrow}}$ . Then,  $\mathbf{T}$  favors the employment of minority workers under  $\varphi_W$  and  $\varphi_F$ .

*Proof.* See Section B.11. ■

Proposition 6 distinguishes between two cases for minority workers: those employed by  $\mu_W(w_M)$  and those employed by other firms. For workers not employed by  $\mu_W(w_M)$ , it suffices that their utilities exceed  $t_{m_{U \rightarrow}}$ . The surplus generated by a worker at a firm translates into their utility, which corresponds to a share of this surplus. Thus, while subsidizing  $w_m$  may reduce workers' utility, it must remain positive. Consequently, to maintain employment, the utility must decrease by  $t_{m_{U \rightarrow}}$ . For the minority workers employed by  $\mu_W(w_M)$ , it is also essential to ensure that their positions are secure by generating a profit greater than that generated by  $w_M$ . The reasoning mirrors that in Theorem 5.

## 5.4 Sharing the Subsidy

In this section, we investigate how the subsidy is distributed between firms and minority workers. Given a market  $G = (\mathbf{T}_0, \mathbf{q})$ , we define the *cost of a transfer  $\mathbf{T}$  under  $\varphi$* , such that  $\varphi(\mathbf{T}, \mathbf{q}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$ , by  $C_\varphi(\mathbf{T}) \equiv \sum_{w \in m} t_{w,\mu^{\mathbf{T}}(w)}$ . The cost of a transfer that subsidizes minority workers is the sum of the subsidies granted to the firms employing these workers. We define the *share of the cost of the transfer accruing to minority workers* as  $C_\varphi^m(\mathbf{T}) = \sum_{w \in m} u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) - u_w(\mu_W, \mathbf{s}_W)$ .

Considering Theorem 4 (i), it is direct that the transfer described assigns the entire cost of the transfer to the minority workers. The reasoning is that the salary of all employed minority workers increases by the amount of the subsidy granted to the firm

<sup>20</sup>This is exemplified by considering that  $w_4$  is a minority worker in Table 4.

employing them. Thus, given the optimality of the allocation  $(\mu_W, s_W)$  and the conditions of the competitive equilibrium, no firm realizes an increase in profit following the introduction of the transfer.

**Corollary 3.** Consider a market  $G \in \mathbb{G}$ . Suppose that  $\mathbf{T}$  is a transfer such that for each  $w \in m_E$ ,  $t_{w, \mu_W(w)} = \max_{f \in F} (t_{w, f}) \geq 0$ , and for each  $w' \in W \setminus \{m_E\}$ ,  $f \in F$ ,  $t_{w', f} = 0$ . Then,  $C_{\varphi_W}(\mathbf{T}) = C_{\varphi_W}^m(\mathbf{T})$  and for each  $f \in F$ ,  $V_f(\mu_W; \mathbf{s}_W, R_f, \mathbf{T}_0) = V_f(\mu_W^{\mathbf{T}}; \mathbf{s}_W^{\mathbf{T}}, R_f, \mathbf{T})$ .

The crucial aspect in the proof of Theorem 4 is that the workers remain employed by the same firms they were associated with prior to the introduction of the transfer. However, in the other objectives studied, namely increasing the representation of minority workers in a firm and reducing unemployment among minority workers, some firms experience a different workforce composition. Proposition 7 states that this difference results in firms absorbing part of the subsidy, thereby increasing their profits at the worker-optimal stable allocation.

**Proposition 7.** Consider a market  $G \in \mathbb{G}$ . Suppose that  $\mathbf{T}$  subsidizes the workers  $m' \subseteq m$ , and does not taxes any worker. If there exists  $f \in F$  such that for each worker-optimal stable allocation  $|\mu_W^{\mathbf{T}}(f) \cap m| > |\mu_W(f) \cap m|$ , then  $C_{\varphi_W}(\mathbf{T}) > C_{\varphi_W}^m(\mathbf{T})$  and  $V_f(\mu_W^{\mathbf{T}}; \mathbf{s}_W^{\mathbf{T}}, R_f, \mathbf{T}) > V_f(\mu_W; \mathbf{s}_W, R_f, \mathbf{T}_0)$ .

*Proof.* See Section B.12. ■

By subsidizing certain workers, competition within the market is enhanced. Some workers may accept lower wages to retain their jobs, resulting in increased profits for firms. Since the composition of the workforce changes following the introduction of the transfer, it is evident that negotiations between workers and firms lead to an uptick in profits. Thus, if a transfer  $\mathbf{T}$  subsidizes minority workers and increases their representation in a firm or supports their employment, then a portion of the subsidy will contribute to the profits of certain firms.

## 6 Affirmative Action with more than two Groups

In this section, we relax the restrictions on the number of groups. In practice, populations may comprise more than two groups. Multiple groups can be classified as minority groups. Our approach in the previous sections involved considering the union of all minority groups alongside the union of all non-minority groups. While we analyze a policy applicable to all minority groups, it is natural to implement a specific policy for

each individual group. Therefore, we study the impact of transfer policies on various groups.

A transfer  $\mathbf{T}$  does not *negatively impact group  $P$*  under  $\varphi$  if for market  $G = (\mathbf{T}_0, \mathbf{q})$ , we have  $\varphi(\mathbf{T}_0, \mathbf{q}) = (\mu, \mathbf{s})$  and  $\varphi(\mathbf{T}, \mathbf{q}) = (\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}})$ , such that for each  $w \in m$ :

- $u_w(\mu^{\mathbf{T}}, \mathbf{s}^{\mathbf{T}}) \geq u_w(\mu, \mathbf{s})$ .

We say that the transfer  $\mathbf{T}$  *may negatively impact group  $P$*  under  $\varphi$  if the condition is not satisfied. A transfer  $\mathbf{T}$  *uniformly subsidizes (taxes) group  $P$*  if there exists a  $t > 0$  ( $t < 0$ ) such that, for each  $w \in P$  and each  $f \in F$ , we have  $t_{w,f} = t$ , and for each  $w' \in W \setminus P, t_{w',f} = 0$ .

Our previous results highlight the importance of implementing the same subsidy (or tax) for all minority (or majority) workers to avoid a reduction in welfare. Thus, it is clear that if a transfer does not uniformly subsidize all minority groups by the same amount, then it may negatively impact some groups under  $\varphi_W$  and  $\varphi_F$ .

**Corollary 4.** Given a market  $G(\mathbf{T}_0, \mathbf{q}, s)$ , a transfer  $\mathbf{T}$  that uniformly subsidizes group  $P$  may negatively impact group  $P'$  under  $\varphi_W$  and  $\varphi_F$  with  $P' \in \mathcal{P} \setminus \{P\}$ .

Introducing a uniform transfer for one group can still adversely affect other minority groups. Corollary 4 and Theorem 1 illustrate that minority groups must be subsidized by the same amount to prevent a reduction in welfare. Therefore, careful design of transfers is essential when considering multiple groups.

In contrast, if a transfer  $\mathbf{T}$  uniformly taxes a group, then all other groups are not negatively impacted by  $\mathbf{T}$ .

**Corollary 5.** If a transfer  $\mathbf{T}$  uniformly taxes group  $P$ , then  $\mathbf{T}$  does not negatively impact group  $P'$  under  $\varphi_W$  and  $\varphi_F$  with  $P' \in \mathcal{P} \setminus \{P\}$ .

The implication of Corollary 5 is that when considering multiple groups of minority workers, the implementation of a uniform tax should be favored. The distinction between a tax and a subsidy lies in the productivity implications for unemployed workers. With a subsidy, the productivity of unemployed workers increases, thereby intensifying competition in the market. As noted in Theorem 4, if a subsidy targets unemployed workers, it may negatively affect the welfare of other workers. Conversely, a tax decreases the productivity of certain workers without adversely impacting others, making those not taxed uniformly more attractive to firms.

However, as discussed in Section 5, a significant limitation of the tax is its inability to

target effects accurately. Example 3 illustrates how a tax may inadvertently benefit a group that is not classified as a minority.

**Example 3.** Consider Example 1 and a partition such that  $m_1 = \{w_1\}$ ,  $m_2 = \{w_5\}$ ,  $p = \{w_2, w_3\}$  and  $M = \{w_4\}$ . In this partition, groups  $m_1$  and  $m_2$  correspond to minority groups, group  $M$  comprises majority workers, and group  $p$  includes workers who are neither minority nor majority. Suppose a transfer  $\mathbf{T}$  is introduced that uniformly taxes group  $M$ : for each  $f \in F$ , we have  $t_{w_4, f} = -1$  and 0 for all other elements. The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_4$	$w_1, w_2$	$w_3, w_4$
$s_{w_1} = 0, s_{w_2} = 2$	$s_{w_3} = 0, s_{w_4} = 0$	$s_{w_1} = 5, s_{w_2} = 4$	$s_{w_3} = 4, s_{w_4} = 0$

Table 11: Firm-optimal and worker-optimal stable allocations with a uniform tax of 1 for  $w_4$ .

The positive impact of the tax on other workers is limited to group  $p$ , which is not classified as a minority group. The salary of worker  $w_2$  is strictly higher when considering the firm-optimal stable allocation, while the salaries of minority workers,  $w_1$  and  $w_5$ , remain unchanged.

Combining a tax and a subsidy, as presented in Section 4.3, may also lead to a welfare reduction for minority workers. Suppose a transfer  $\mathbf{T}$  is introduced such that for each  $w \in m_2$ ,  $t_{w, f} = 0.5$ , for each  $w \in M$ ,  $t_{w, f} = -0.5$ , and 0 for all other elements of  $\mathbf{T}$ . The stable optimal allocations for this market are given by:

$f_1$	$f_2$	$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$	$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 0, s_{w_2} = 0.5$	$s_{w_3} = 0, s_{w_5} = 0$	$s_{w_1} = 4.5, s_{w_2} = 3.5$	$s_{w_3} = 3.5, s_{w_5} = 1$

Table 12: Firm-optimal and worker-optimal stable allocations with a subsidy of 0.5 for  $m_2$  and a tax of 0.5 for  $M$ .

The salary of  $w_1$ , who is a minority worker, is reduced at the worker-optimal stable allocation.

## 7 Job Matching under Group-Specific Constraints

In this section, we investigate the implementation of other common affirmative action policies, namely *group-specific constraints*. These policies typically impose *ceiling constraints* on firms, limiting the number of workers from specific groups that a firm can employ. Kojima et al. (2020) demonstrate that the gross substitutes condition is preserved for all group-separable revenue functions if and only if the feasibility collection is defined by a *generalized polyhedral constraint*. For the purposes of this analysis, we focus on ceiling constraints and introduce *quotas*, which represent the maximum number of majority workers a firm can employ.

A quota is a vector  $\mathbf{q}^M$ , such that  $\mathbf{q}^M = (q_f^M)_{f \in F}$ , with  $q_f^M \in \mathbb{N}$ , representing the maximum number of majority workers that firm  $f$  can employ. We assume for any  $\mathbf{q}^M$ , it holds that  $q_f \geq q_f^M$  for each  $f \in F$ . A quota  $\mathbf{q}^M$  imposes a *ceiling constraint on majority workers* at firm  $f$  if  $q_f > q_f^M$ . A *null quota* is defined as a quota  $\mathbf{q}_0^M$  such that for each firm  $f \in F$ ,  $q_f^M = q_f$ .

A market with salaries and quotas is a 6-tuple  $G = (W, F, R, \mathbf{q}^M, \mathbf{q}, \mathbf{s})$ . Since  $W$ ,  $F$ , and  $R$  remain fixed throughout this section, and  $\mathbf{s}$  depends on  $R$ , we simplify the notation to  $G = (\mathbf{q}^M, \mathbf{q})$ . To study the impact of quota implementation, we compare optimal stable allocations both with and without quotas. For a market  $G = (\mathbf{q}_0^M, \mathbf{q})$ , with a null quota, we denote  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_W, \mathbf{s}_W)$  the firm-optimal stable allocation and the worker-optimal stable allocation, respectively. When a non null-quota  $\mathbf{q}^M$  is introduced into market  $G$ , this creates a new market denoted as  $(\mathbf{q}^M, \mathbf{q})$ . We then denote the *worker-optimal stable allocation with quota*  $\mathbf{q}^M$  as  $(\mu_W^{\mathbf{q}^M}, \mathbf{s}_W^{\mathbf{q}^M})$  and the *firm-optimal stable allocation with quota*  $\mathbf{q}^M$  as  $(\mu_F^{\mathbf{q}^M}, \mathbf{s}_F^{\mathbf{q}^M})$ .

If a firm  $f$  hires  $W' \subset W$  while facing a vector of salaries  $\mathbf{s}$ , quotas  $\mathbf{q}^M$ , capacity  $q_f$ , and a revenue function  $R_f$ , its *profit with quotas* is  $V_f(W'; \mathbf{s}, R_f, \mathbf{q}^M) = R_f(W') - \sum_{w \in W'} s_{w,f}$  that is, its revenue with transfers minus salaries of workers it hires. We define the *maximal profit function with quotas* by  $\Pi_f(\cdot; R_f, \mathbf{q}^M, \mathbf{q})$  and the *demand with quotas* by  $D_f(\cdot; R_f, \mathbf{q}^M, \mathbf{q})$  such that for each vector of salaries  $\mathbf{s}$ ,

$$\begin{aligned} \Pi_f(\mathbf{s}; R_f, \mathbf{q}^M, \mathbf{q}) &= \max\{V_f(W'; \mathbf{s}, R_f, \mathbf{q}^M) : W' \subset W, |W' \cap M| \leq q_f^M \text{ and } |W'| \leq q_f\}; \\ D_f(\mathbf{s}; R_f, \mathbf{q}^M, \mathbf{q}) &= \{W' \subset W : |W' \cap M| \leq q_f^M, |W'| \leq q_f \text{ and } V_f(W'; \mathbf{s}, R_f, \mathbf{q}^M) = \Pi_f(\mathbf{s}; R_f, \mathbf{q}^M, \mathbf{q})\}. \end{aligned}$$

Each element of  $D_f(\mathbf{s}; R_f, \mathbf{q}^M, \mathbf{q})$  is referred to as a *demand set with quotas*.

**Definition 3.** (Gross Substitutes). A demand  $D_f(\cdot, R_f, \mathbf{q}^M, \mathbf{q})$  satisfies the gross substitutes condition if, for any two vectors of salaries  $\mathbf{s}, \mathbf{s}'$  if  $\mathbf{s}' \geq \mathbf{s}$  and  $s_w = s'_w$ , then  $w \in D_f(\mathbf{s}, R_f, \mathbf{q}^M, \mathbf{q})$  implies that  $w \in D_f(\mathbf{s}', R_f, \mathbf{q}^M, \mathbf{q})$ .

Roughly speaking, the gross substitutes condition with quotas rules out complementarity between workers within a firm. Definition 1 introduces the gross substitutes condition without quotas, indicating that only the demand form changes.

We say that a quota  $\mathbf{q}^M$  does not *negatively impact minority workers* under  $\varphi$  if, for market  $G = (\mathbf{q}_0^M, \mathbf{q})$ , we have  $\varphi(\mathbf{q}_0^M, \mathbf{q}) = (\mu, \mathbf{s})$  and  $\varphi(\mathbf{q}^M, \mathbf{q}) = (\mu^{\mathbf{q}^M}, \mathbf{s}^{\mathbf{q}^M})$ , such that for each  $w \in m$ :

- $u_w(\mu^{\mathbf{q}^M}, \mathbf{s}^{\mathbf{q}^M}) \geq u_w(\mu, \mathbf{s})$ .

A quota  $\mathbf{q}^M$  *may negatively impact minority workers* under  $\varphi$  if the condition is not satisfied. We now demonstrate that implementing quotas, which preserve the gross substitute condition, may negatively impact minority workers under  $\varphi_W$ .

**Example 4.** Consider Example 1 and a partition such that  $m = \{w_1, w_2, w_5\}$  and  $M = \{w_3, w_4\}$ . Suppose that each firm can employ at most one worker from the majority, meaning that  $\mathbf{q}^M = (1, 1)$ . The stable optimal allocations for this market are given by:

$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 0, s_{w_2} = 2$	$s_{w_3} = 0, s_{w_5} = 0$

$f_1$	$f_2$
$w_1, w_2$	$w_3, w_5$
$s_{w_1} = 4, s_{w_2} = 3$	$s_{w_3} = 3, s_{w_5} = 1$

Table 13: Firm-optimal and worker-optimal stable allocations with a quota of 1.

The worker  $w_5$  is now employed by the firm  $f_2$ , achieving representation of minority workers within the firms. However, minority workers  $w_1$  and  $w_2$  experience a reduction in welfare. This decline in salaries for the minority workers arises due to the unemployment of  $w_4$ . Since firm  $f_1$  has not reached its quota, it can recruit  $w_4$  while still adhering to the ceiling constraints. Thus, implementing quotas may inadvertently heighten competition in the market by altering the assignment of workers.

**Proposition 8.** Given a market  $G = (\mathbf{q}_0^M, \mathbf{q})$ , and a quota  $\mathbf{q}^M$  that imposes a ceiling constraint on majority workers for firms,  $\mathbf{q}^M$  may negatively impact minority workers under  $\varphi_W$ .

In Example 4, if firm  $f_1$  reached its quota, it is clear that  $w_4$  could not be recruited, and workers would be immune to the increased competition in the market. Our main

result for this section is as follows: when a quota is implemented that allows a firm to recruit at least as many minority workers as in the market without a quota, then the quota does not negatively impact minority workers under the mechanisms  $\varphi_W$  and  $\varphi_F$ .

**Theorem 7.** Consider a market  $G = (\mathbf{q}_0^M, \mathbf{q})$  and a quota  $\mathbf{q}^M$  such that for each  $f \in F$ ,  $q_f - q_f^M \geq |\mu_W(f) \cap m|$  then  $\mathbf{q}^M$  does not negatively impact minority workers under  $\varphi_W$  and  $\varphi_F$ .

*Proof.* See Section B.13. ■

The incentive for firms when floor quotas and ceiling quotas are imposed is not the same. By imposing a floor quota, the firm is forced to respect it. As a result, it is willing to employ minority workers, even if the profit generated is negative. This maintains the legal feasibility of the allocation. In contrast, when a ceiling quota is imposed, the firm prefers to have vacant positions if a worker generates a negative profit. With transfers, it is possible to guarantee the representation of minority workers, and a minimum salary, which is not possible with quotas. However, we identify transfers and quotas that achieve an objective without reducing the well-being of minority workers.

## 8 Conclusion

This paper examines the impact of transfer policies on workers' well-being within markets for indivisible resources. Focusing on various population groups affected by affirmative action policies, we show that subsidies targeting minority workers can unintentionally reduce their well-being. This is particularly true for policies that subsidize unemployed minority workers or specific market sectors. In contrast, we find that uniform subsidies help mitigate this negative effect. Furthermore, our analysis of taxes on majority workers reveals a relationship and complementarity between these two forms of transfers.

Overall, our findings identify transfer policies that preserve the well-being of minority workers while still achieving the primary objectives of affirmative action initiatives. However, we stress that the competitive equilibrium of the market must be known by policymakers before implementing any transfer. Furthermore, when more than two groups are present in the population, the implementation of subsidies can reduce the well-being of one or more groups. In such cases, it is advisable to treat all minority groups uniformly or to impose a uniform tax on non-minority groups.

Echenique (2012) shows that the matching-with-contracts model proposed by Hatfield and Milgrom (2005), which encompasses various nontransferable utility matching frameworks, can be integrated into the job matching model introduced by Kelso and Crawford



(1982). However, Kojima (2012) shows the impossibility of considering affirmative policy measures that do not negatively impact student welfare within a nontransferable utility framework. Our results suggest that if the set of available contracts is not finite, wage continuity may help overcome this impossibility within the matching-with-contracts model.

We conclude by raising two open questions: First, what types of transfers that tax majority workers can effectively achieve the common goals of affirmative action policies? Second, which transfers can minimize the negative impacts on majority workers while enhancing the positive effects on minority workers?

## A Mechanisms

This section introduces the two mechanisms used in this article. For simplicity and clarity, we present the mechanisms such that the expected utility decreases by 1 at each step, while the proposed wage increases by 1 at each step. This formulation follows the approach used by Kelso and Crawford (1982).

### A.1 Descending Salary Adjustment Processes

This subsection introduces the *Descending Salary Adjustment Processes* with capacity.

*Step 0.* For each worker  $w \in W$ , consider an initial vector of expected utility  $\mathbf{u}_w(0) \equiv (R_f(w) + \sigma_{w,f})_{f \in F}$ . Each worker selects a firm  $f$  such that

$$\mathbf{u}_w(0)(f) = \max_{f \in F} \mathbf{u}_w(0)(f) \geq 0,$$

i.e., the firm that offers the highest expected utility. If there is a tie between firms, a tie-breaking rule is imposed. Each worker  $w$  then proposes a salary of  $\mathbf{u}_w(0)(f) - \sigma_{w,f}$  to that firm.

*Step  $t \geq 1$ .* If a firm  $f$  receives more proposals than its capacity  $q_f$ , it rejects all proposals. For each firm  $f \in F$  that rejects all its proposals, all workers  $w \in W$  reduce their expected utility at that firm by 1, i.e.,

$$\mathbf{u}_w(t)(f) = \mathbf{u}_w(t-1)(f) - 1.$$

Each worker then selects a firm  $f$  such that

$$\mathbf{u}_w(t)(f) = \max_{f \in F} \mathbf{u}_w(t)(f) \geq 0.$$

If there is a tie between firms, a tie-breaking rule is imposed. Each worker  $w$  proposes a salary of  $\mathbf{u}_w(t)(f) - \sigma_{w,f}$  to that firm.

If no such firm exists, the mechanism terminates.

The mechanism continues while at least one proposal is rejected. Since the number of workers and firms is finite, and utility cannot become negative, the mechanism terminates in a finite number of steps. Firms that have not rejected the salary offers accept them and employ the workers who proposed them. The final outcome is the worker-optimal stable allocation.

## A.2 Ascending Salary Adjustment Processes

This section introduces the *Ascending Salary Adjustment Processes* with capacity (Kelso and Crawford, 1982).

*Step 0.* For each firm  $f \in F$ , consider an initial vector of salaries  $\mathbf{s}_f(0) \equiv (\sigma_{w,f})_{w \in W}$ , and let  $W'_f(0) \subset W$  be the subset of workers such that  $W'_f(0) \in D_f(\mathbf{s}_f(0); R_f, \cdot, \mathbf{q}_f)$ , i.e., the subset that maximizes the profit of firm  $f$  under the current vector of salaries and capacity constraint  $\mathbf{q}_f$ . If there is a tie between workers, a tie-breaking rule is imposed. For each worker  $w \in W'_f(0)$ , firm  $f$  offers the salary  $\mathbf{s}_f(0)(w)$  to each worker  $w \in W'(t)$  (i.e.,  $\sigma_{w,f}$ ).

*Step  $t \geq 1$ .* If a worker receives more than one offer, she rejects all offers except her most preferred one, which she tentatively accepts. In case of a tie, a tie-breaking rule is imposed. For each firm  $f$ , if a worker  $w$  rejects the offer, the salary for  $w$  is increased by 1, i.e.,

$$\mathbf{s}_f(t)(w) = \mathbf{s}_f(t-1)(w) + 1.$$

Next, consider the updated vector of salaries  $\mathbf{s}_f(t)$ , and let  $W'(t) \subset W$  be the subset of workers such that  $W'(t) \in D_f(\mathbf{s}_f(t); R_f, \cdot, \mathbf{q}_f)$ , i.e., the subset that maximizes the profit of firm  $f$  at step  $t$ , given the current vector of salaries and capacity constraint  $\mathbf{q}_f$ . If there is a tie, a tie-breaking rule is imposed. Firm  $f$  then offers the updated salary  $\mathbf{s}_f(t)(w)$  to each worker  $w \in W'(t)$ .

The mechanism terminates if no worker rejects an offer.

The process continues until at least one offer is rejected. Since the number of firms is finite and each firm's profit cannot be negative, the mechanism terminates in a finite number of steps. Workers who have not rejected the salary offers accept them and are employed by the firms making the offers. The final outcome is the firm-optimal stable allocation.

## B Proofs

### B.1 Technical Results

Lemma 1 provides a decomposition of the effects of transfers on the utility and profit of agents in the market. The intuition is that for a given market,  $G = (\mathbf{T}_0, \mathbf{q})$  and a transfer  $\mathbf{T}$ , the transfer can be decomposed into several matrices such that their sum equals  $\mathbf{T}$ . We have shown that certain transfers do not negatively impact minority workers. This allows us to implement transfers in the market and analyze their effects on agents.

**Lemma 1.** For any market  $G = (\mathbf{T}_0, \mathbf{q})$ , and any  $\mathbf{T} \in \mathbb{T}$ , suppose  $\mathbb{T}' \subset \mathbb{T}$  such that  $\mathbf{T} = \sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i$ . Then, there exist worker-optimal and firm-optimal stable allocations in the markets  $(\mathbf{T}, \mathbf{q})$  and  $(\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i, \mathbf{q})$  such that:

$$(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i}, \mathbf{s}_W^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i})$$

and

$$(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i}, \mathbf{s}_F^{\sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i}).$$

For convenience, we denote by  $(\mathbf{T}_0, \mathbf{q}) \xrightarrow{\mathbf{T}} (\mathbf{T}, \mathbf{q})$  the *implementation of the transfer*  $\mathbf{T}$  in a market  $(\mathbf{T}_0, \mathbf{q})$ , with  $(\mathbf{T}, \mathbf{q})$  as the resulting market. By decomposing  $\mathbf{T}$ , we have  $\mathbb{T}' \subset \mathbb{T}$  such that  $\mathbf{T} = \sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i$ , and

$$(\mathbf{T}_0, \mathbf{q}) \xrightarrow{\mathbf{T}_1} (\mathbf{T}_1, \mathbf{q}) \xrightarrow{\mathbf{T}_2} ((\mathbf{T}_1 + \mathbf{T}_2), \mathbf{q}) \xrightarrow{\mathbf{T}_3} \dots \xrightarrow{\mathbf{T}_l} \left( \sum_{\mathbf{T}_i \in \mathbb{T}'} \mathbf{T}_i, \mathbf{q}, \mathbf{s} \right).$$

This decomposition primarily facilitates the analysis of the impact of transfers on workers. For instance, if  $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$ , and both  $\mathbf{T}_1$  and  $\mathbf{T}_2$  do not negatively impact minority workers, then  $\mathbf{T}$  does not negatively impact minority workers either. Similarly, it follows that Lemma 2 can capture the maximum impact of each transfer.

**Lemma 2.** For any vector  $\mathbf{T}$ , the following holds:

- If  $\max(\mathbf{T}) > 0$  and  $\min(\mathbf{T}) < 0$ , then for each  $w \in W$ , we have

$$\begin{aligned} |u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})| &\leq \max(\mathbf{T}) - \min(\mathbf{T}), \\ |u_w(\mu_F, \mathbf{s}_F) - u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})| &\leq \max(\mathbf{T}) - \min(\mathbf{T}). \end{aligned}$$

- If  $\max(\mathbf{T}) > 0$  and  $\min(\mathbf{T}) \geq 0$ , then for each  $w \in W$ , we have

$$\begin{aligned} |u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})| &\leq \max(\mathbf{T}), \\ |u_w(\mu_F, \mathbf{s}_F) - u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})| &\leq \max(\mathbf{T}). \end{aligned}$$

- If  $\max(\mathbf{T}) \leq 0$  and  $\min(\mathbf{T}) < 0$ , then for each  $w \in W$ , we have

$$\begin{aligned} |u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})| &\leq -\min(\mathbf{T}), \\ |u_w(\mu_F, \mathbf{s}_F) - u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})| &\leq -\min(\mathbf{T}). \end{aligned}$$

*Proof.* We start with the firm-optimal stable allocation in markets  $(\mathbf{T}_0, \mathbf{q})$  and  $(\mathbf{T}, \mathbf{q})$ . From Lemma 5, we know that for each  $w \in W$ ,

$$u_w(\mu_F, \mathbf{s}_F) = \max \left( 0, \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] \right).$$

Let  $f \equiv \arg \max_{f \in F \setminus \{\mu_F(w)\}} [(R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})]$  and  $w' \equiv \arg \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$ .

It is clear that the utility of worker  $w$  increases when  $R_f(w)$  increases and  $R_f(w')$  decreases, and decreases when  $R_f(w)$  decreases and  $R_f(w')$  increases. Note that as  $R_f(w')$  increases, the worker who generates the lowest profit at  $f$  may change.<sup>21</sup> Also, when  $R_f(w)$  decreases, the maximizing firm  $f$  may change, reducing  $w$ 's utility but to a lesser extent.

Following this reasoning, the utility of  $w$  decreases the most when the firm  $f$  and the worker  $w'$  remain unchanged. We fix  $f$  and  $w'$  throughout the analysis:

- If  $\max(\mathbf{T}) > 0$  and  $\min(\mathbf{T}) < 0$ , suppose that  $t_{w,f} = \min(\mathbf{T})$  and  $t_{w',f} = \max(\mathbf{T})$ . Then, we have

$$\begin{aligned} &|\max(0, [(R_f(w) + \sigma_{w,f}) - (R_f(w') - s_{w',f})]) - \\ &\quad \max(0, [(R_f(w) + t_{w,f} + \sigma_{w,f}) - (R_f(w') + t_{w',f} - s_{w',f})])| \leq t_{w',f} - t_{w,f}. \end{aligned}$$

<sup>21</sup>If the worker  $w'$  is subsidized at  $f$ ,  $R_f(w')$  increases, and there may be another worker who is not subsidized and generates the lowest profit at  $f$ , thus reducing the decrease in  $w$ 's utility.

- If  $\max(\mathbf{T}) > 0$  and  $\min(\mathbf{T}) \geq 0$ , suppose that  $t_{w',f} = \max(\mathbf{T})$ . We know that  $t_{w',f} \geq t_{w,f} \geq 0$ . Then, we have

$$\begin{aligned} & |\max(0, [(R_f(w) + \sigma_{w,f}) - (R_f(w') - s_{w',f})]) - \\ & \max(0, [(R_f(w) + t_{w,f} + \sigma_{w,f}) - (R_f(w') + t_{w',f} - s_{w',f})])| \leq t_{w',f} - t_{w,f} \leq t_{w',f}. \end{aligned}$$

- If  $\max(\mathbf{T}) \leq 0$  and  $\min(\mathbf{T}) < 0$ , suppose that  $t_{w,f} = \min(\mathbf{T})$ . We know that  $t_{w,f} \leq t_{w',f} \leq 0$ . Then, we have

$$\begin{aligned} & |\max(0, [(R_f(w) + \sigma_{w,f}) - (R_f(w') - s_{w',f})]) - \\ & \max(0, [(R_f(w) + t_{w,f} + \sigma_{w,f}) - (R_f(w') + t_{w',f} - s_{w',f})])| \leq t_{w',f} - t_{w,f} \leq -t_{w,f}. \end{aligned}$$

The symmetrical case, where  $t_{w',f} = \min(\mathbf{T})$  and  $t_{w,f} = \max(\mathbf{T})$ , which maximizes the increase in  $w$ 's utility, is omitted.

We now consider the worker-optimal stable allocation in markets  $(\mathbf{T}_0, \mathbf{q})$  and  $(\mathbf{T}, \mathbf{q})$ . First, suppose that  $\max(\mathbf{T}) > 0$  and  $\min(\mathbf{T}) \geq 0$  (i.e.,  $\mathbf{T}$  subsidizes some workers). Without loss of generality, and using Lemma 1, we consider  $\mathbf{T}_1$  such that for  $w \in \{w \in W \mid \exists f \in F, t_{w,f} = \max(\mathbf{T})\}$ , we have for each  $f \in F$ ,  $t_{1_{w,f}} = t_{w,f}$  and 0 for the other elements.

From Theorem 4 (i), we know that if  $t_{1_{w,\mu_W(w)}} = \max(\mathbf{T})$ , then

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) = u_w(\mu_W, \mathbf{s}_W) + t_{1_{w,\mu_W(w)}}.$$

Thus, the impact on  $w$  is at most  $t_{1_{w,\mu_W(w)}} = \max(\mathbf{T})$ , and the utility of other workers is unchanged for each  $w' \in W$  such that for each  $f \in F$ ,  $t_{1_{w',f}} = 0$ . Otherwise, if  $t_{1_{w,\mu_W(w)}} \neq \max(\mathbf{T})$ , we consider two cases:

- **Case 1:** If  $\mu_W(w) = \mu_W^{\mathbf{T}_1}(w)$ . By the optimality and stability of  $(\mu_W, \mathbf{s}_W)$ , we know that for each  $w \in W$  and each  $f \in F$ ,

$$u_w(\mu_W, \mathbf{s}_W) \geq R_f(w) + \sigma_{w,f} - (R_f(w') - s_{w',f}),$$

for any  $w' \in \mu_W(f)$ .<sup>22</sup> Furthermore, from Lemma 4, we know that for each  $w, w' \in \mu_W(w)$ ,

$$R_{\mu_W(w)}(w) - s_{w,\mu_W(w)} = R_{\mu_W(w')}(w) - s_{w',\mu_W(w)}.$$

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<sup>22</sup> $R_f(w) + \sigma_{w,f}$  is the surplus generated by  $w$  at  $f$ , and  $R_f(w') - s_{w',f}$  is the profit obtained by  $f$  for each worker it employs.

Using this reasoning, and since  $\mu_W(w) = \mu_W^{\mathbf{T}_1}(w)$ , it follows that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq R_f(w) + t_{1_{w,f}} + \sigma_{w,f} - (R_f(w') + t_{1_{w',f}} - s_{w',f}),$$

for any  $w' \in \mu_W^{\mathbf{T}_1}(f)$ . By construction of  $\mathbf{T}_1$ , it is direct that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) = u_w(\mu_W, \mathbf{s}_W) + t_{1_{w, \mu_W(w)}} \leq u_w(\mu_W, \mathbf{s}_W) + \max(\mathbf{T}).$$

- **Case 2:** If  $\mu_W(w) \neq \mu_W^{\mathbf{T}_1}(w)$ . Since we know that for each  $w \in W$  and each  $f \in F$ ,

$$u_w(\mu_W, \mathbf{s}_W) \geq R_f(w) + \sigma_{w,f} - (R_f(w') - s_{w',f}),$$

for any  $w' \in \mu_W(f)$ , it follows that

$$R_{\mu_W^{\mathbf{T}_1}(w)}(w) + t_{1_{w, \mu_W^{\mathbf{T}_1}(w)}} + \sigma_{w, \mu_W^{\mathbf{T}_1}(w)} - (R_{\mu_W^{\mathbf{T}_1}(w)}(w') - s_{w', \mu_W^{\mathbf{T}_1}(w)}) \geq u_w(\mu_W, \mathbf{s}_W),$$

for any  $w' \in \mu_W^{\mathbf{T}_1}(f)$ . It follows directly that

$$|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq \max(\mathbf{T}).$$

It remains to show that for each worker  $w' \in W \setminus \{w\}$ , the impact is at most  $\max(\mathbf{T})$ . Since for each  $w' \in W$  and  $f \in F$ ,  $R_f(w') + \sigma_{w',f} \geq 0$ , we know that there exists  $w'' \in W$  such that  $\mu_W(w'') = \mu_W^{\mathbf{T}_1}(w')$  and  $\mu_W^{\mathbf{T}_1}(w'') \neq \mu_W^{\mathbf{T}_1}(w')$ . Therefore, for each  $f \in F \setminus \{\mu_W^{\mathbf{T}_1}(w')\}$ ,

$$u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq R_f(w'') + t_{1_{w'',f}} + \sigma_{w'',f} - (R_f(w''') + t_{1_{w''',f}} - s_{w''',f}),$$

with  $w''' \in \mu_W^{\mathbf{T}_1}(f)$ . By construction of  $\mathbf{T}_1$ , we know that  $t_{1_{w'',f}} = 0$ . Now, consider  $\mu_W(w')$ ,  $u_{w'}(\mu_W, \mathbf{s}_W)$ , and  $u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})$ . It follows that

$$u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq R_{\mu_W(w')}(w') + \sigma_{w', \mu_W(w')} - (R_{\mu_W(w')}(w) + t_{1_{w, \mu_W(w')}} - s_{w, \mu_W(w')}).$$

We now show that

$$|u_{w'}(\mu_W, \mathbf{s}_W) - u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq \max(\mathbf{T}).$$

Using the optimality and stability of  $(\mu_W, \mathbf{s}_W)$ , we have:

$$R_{\mu_W(w')}(w) + t_{1_{w, \mu_W(w')}} - s_{w, \mu_W(w')} \geq R_{\mu_W(w')}(w') - s_{w', \mu_W(w')}.$$

Therefore, since  $t_{1_w, \mu_W(w')} \leq \max(\mathbf{T})$ , it follows directly that

$$|u_{w'}(\mu_W, \mathbf{s}_W) - u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq \max(\mathbf{T}).$$

It follows that for each  $w \in W$ ,

$$|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq \max(\mathbf{T}).$$

The reasoning is repeated for each  $w \in W$ , and we consider  $\mathbf{T}_2$  such that for  $w \in \{w \in W \mid \exists f \in F, t_{w,f} = \max(\mathbf{T} - \mathbf{T}_1)\}$ , we have for each  $f \in F, t_{2_w,f} = t_{w,f}$  and 0 for other elements. Since  $\max(\mathbf{T}) \geq \max(\mathbf{T} - \mathbf{T}_1)$ , we have

$$|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{(\mathbf{T}_1+\mathbf{T}_2)}, \mathbf{s}_W^{(\mathbf{T}_1+\mathbf{T}_2)})| \leq \max(\mathbf{T}).$$

The set  $W$  is finite, and there is a decomposition into  $|W|$  transfer matrices such that  $\sum_{i=1}^{|W|} \mathbf{T}_i = \mathbf{T}$ .

Suppose now that  $\max(\mathbf{T}) \leq 0$  and  $\min(\mathbf{T}) < 0$ . Without loss of generality, and using Lemma 1, we consider a matrix  $\mathbf{T}_1$  such that for each  $w \in \{w \in W \mid \exists f \in F, t_{w,f} = \min(\mathbf{T})\}$ , we have  $t_{1_w,f} = t_{w,f}$  for each  $f \in F$ , and 0 for all other elements. We now consider two cases:

- **Case 1:** If  $\mu_W(w) = \mu_W^{\mathbf{T}_1}(w)$ . By the optimality and stability of  $(\mu_W, \mathbf{s}_W)$ , we know that for each  $w \in W$  and each  $f \in F$ ,

$$u_w(\mu_W, \mathbf{s}_W) \geq R_f(w) + \sigma_{w,f} - (R_f(w') - s_{w',f}),$$

for any  $w' \in \mu_W(f)$ . Using this reasoning, and since  $\mu_W(w) = \mu_W^{\mathbf{T}_1}(w)$ , it follows that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq R_f(w) + t_{1_w,f} + \sigma_{w,f} - (R_f(w') + t_{1_{w'},f} - s_{w',f}),$$

for any  $w' \in \mu_W^{\mathbf{T}_1}(f)$ . By construction of  $\mathbf{T}_1$ , we know that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) = u_w(\mu_W, \mathbf{s}_W) + t_{1_w, \mu_W(w)} \geq u_w(\mu_W, \mathbf{s}_W) + \min(\mathbf{T}).$$

- **Case 2:** If  $\mu_W(w) \neq \mu_W^{\mathbf{T}_1}(w)$ . Since we know that for each  $w \in W$  and each  $f \in F$ ,

$$u_w(\mu_W, \mathbf{s}_W) \geq R_f(w) + \sigma_{w,f} - (R_f(w') - s_{w',f}),$$

for any  $w' \in \mu_W(f)$ , it follows that

$$u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq R_{\mu_W(w)}(w) + t_{1_w, \mu_W(w)} + \sigma_{w, \mu_W(w)} - (R_{\mu_W(w)}(w') + s_{w', \mu_W(w)}^{\mathbf{T}_1}).$$

Hence, we have

$$|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq -\min(\mathbf{T}).$$

Now, it remains to show that for each worker  $w' \in W \setminus \{w\}$ , the impact is at most  $-\min(\mathbf{T})$ . Since for each  $w' \in W$  and  $f \in F$ , we know that  $R_f(w') + t_{w', f} + \sigma_{w', f} \geq 0$ , it follows that there exists a worker  $w' \in W$  such that  $\mu_W(w') = \mu_W^{\mathbf{T}_1}(w)$  and  $\mu_W^{\mathbf{T}_1}(w') \neq \mu_W^{\mathbf{T}_1}(w)$ . Therefore, for each  $f \in F \setminus \{\mu_W^{\mathbf{T}_1}(w')\}$ , we have

$$u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq R_f(w') + t_{1_{w'}, f} + \sigma_{w', f} - (R_f(w'') + t_{1_{w''}, f} - s_{w'', f}),$$

with  $w'' \in \mu_W^{\mathbf{T}_1}(f)$ . By construction of  $\mathbf{T}_1$ , we know that  $t_{1_{w'}, f} = 0$ . Now, consider  $\mu_W(w')$ ,  $u_{w'}(\mu_W, \mathbf{s}_W)$ , and  $u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})$ . It follows that

$$u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) \geq R_{\mu_W(w')}(w') + \sigma_{w', \mu_W(w')} - (R_{\mu_W(w')}(w) + t_{1_w, \mu_W(w')} - s_{w, \mu_W(w')}^{\mathbf{T}_1}).$$

We now show that

$$|u_{w'}(\mu_W, \mathbf{s}_W) - u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq -\min(\mathbf{T}).$$

Indeed, using the optimality and stability of  $(\mu_W, \mathbf{s}_W)$ , we have:

$$R_{\mu_W(w')}(w) + t_{1_w, \mu_W(w')} - s_{w, \mu_W(w')}^{\mathbf{T}_1} \geq R_{\mu_W(w')}(w') - s_{w', \mu_W(w')}.$$

Therefore, since  $t_{1_w, \mu_W(w')} \geq \min(\mathbf{T})$ , it is direct that

$$|u_{w'}(\mu_W, \mathbf{s}_W) - u_{w'}(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq -\min(\mathbf{T}).$$

Thus, for each  $w \in W$ , we have:

$$|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1})| \leq -\min(\mathbf{T}).$$

The reasoning can be repeated for each  $w \in W$ . Next, we consider  $\mathbf{T}_2$  such that for each  $w \in \{w \in W \mid \exists f \in F, t_{w, f} = \min(\mathbf{T} - \mathbf{T}_1)\}$ , we have  $t_{2_{w, f}} = t_{w, f}$  for each  $f \in F$  and 0



for the other elements. Since  $\max(\mathbf{T}) \geq \max(\mathbf{T} - \mathbf{T}_1)$ , we have:

$$|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{(\mathbf{T}_1 + \mathbf{T}_2)}, \mathbf{s}_W^{(\mathbf{T}_1 + \mathbf{T}_2)})| \leq -\min(\mathbf{T}).$$

Since the set  $W$  is finite, there is a decomposition into  $|W|$  transfer matrices such that  $\sum_{i=1}^{|W|} \mathbf{T}_i = \mathbf{T}$ .

We omit the proof for the case when  $\max(\mathbf{T}) > 0$  and  $\min(\mathbf{T}) < 0$ , as it follows directly from the cases where  $\max(\mathbf{T}) > 0$ ,  $\min(\mathbf{T}) \geq 0$ , and  $\max(\mathbf{T}) \leq 0$ ,  $\min(\mathbf{T}) < 0$ . ■

**Lemma 3.** Consider a market  $G \in \mathbb{G}$ . There exists a worker-optimal stable allocation  $(\mu_W, \mathbf{s}_W)$  if and only if there exists a firm-optimal stable allocation  $(\mu_F, \mathbf{s}_W)$  such that  $\mu_W = \mu_F$ .

*Proof.* Theorem 5 of Sotomayor (1999). ■

**Lemma 4.** Consider a market  $G \in \mathbb{G}$ . For each worker  $w \in W$ , such that  $\mu_W(w) \neq \emptyset$ , we have the following:

(i) For any two workers  $w, w' \in \mu_W(\mu_W(w))$ , it holds that:

$$R_{\mu_W(w)}(w) - s_{w, \mu_W(w)} = R_{\mu_W(w)}(w') - s_{w', \mu_W(w)}.$$

(ii) The following equality holds:

$$R_{\mu_W(w)}(w) - s_{w, \mu_W(w)} = \max_{w' \in W \setminus \{\mu_W(\mu_W(w))\}} \left( R_{\mu_W(w)}(w') - \left( u_{w'}(\mu_W, \mathbf{s}_W) - \sigma_{w', \mu_W(w)} \right) \right).$$

*Proof.* (i) If  $w = w'$ , the statement holds trivially. Now suppose  $w \neq w'$ . For contradiction, assume that:

$$R_{\mu_W(w)}(w) - s_{w, \mu_W(w)} \neq R_{\mu_W(w)}(w') - s_{w', \mu_W(w)}.$$

Without loss of generality, assume that:

$$R_{\mu_W(w)}(w) - s_{w, \mu_W(w)} > R_{\mu_W(w)}(w') - s_{w', \mu_W(w)}.$$

Since  $(\mu_W, \mathbf{s}_W)$  is a stable allocation in which workers have the highest possible utility, we know that no worker  $w'' \notin \mu_W(w)$  and salary  $\mathbf{s}'$  can block the allocation  $(\mu_W, \mathbf{s}_W)$ . In particular, firm  $\mu_W(w)$  hires the most productive workers available to it.

If  $R_{\mu_W(w)}(w) - s_{w, \mu_W(w)} > R_{\mu_W(w)}(w') - s_{w', \mu_W(w)}$ , worker  $w$  could potentially receive a higher salary offer, say  $s''_{w, \mu_W(w)} > s_{w, \mu_W(w)}$ , which would increase the utility of  $w$ . Thus, we would have:

$$u_w(\mu_W, \mathbf{s}'') > u_w(\mu_W, \mathbf{s}),$$

contradicting that  $(\mu_W, \mathbf{s}_W)$  is the worker-optimal stable allocation. A similar contradiction arises if we assume  $R_{\mu_W(w)}(w) - s_{w, \mu_W(w)} < R_{\mu_W(w)}(w') - s_{w', \mu_W(w)}$ . Hence, we have:

$$R_{\mu_W(w)}(w) - s_{w, \mu_W(w)} = R_{\mu_W(w)}(w') - s_{w', \mu_W(w)}.$$

(ii) By construction, let us define  $s^*_{w', \mu_W(w)} \equiv u_{w'}(\mu_W, \mathbf{s}_W) - \sigma_{w', \mu_W(w)}$ , for any worker  $w'$  such that  $\mu_W(w') \neq \mu_W(w)$ .

**Claim 1.** For any  $w'$ , it holds that  $u_{w'}(\mu_W, \mathbf{s}_W) = u_{w'}(\mu_W(w), s^*_{w', \mu_W(w)})$ .

*Proof.* By definition of utility, we know that:

$$u_{w'}(\mu_W, \mathbf{s}_W) = \sigma_{w', \mu_W(w')} + s_{w', \mu_W(w')},$$

and:

$$u_{w'}(\mu_W(w), s^*_{w', \mu_W(w)}) = \sigma_{w', \mu_W(w)} + s^*_{w', \mu_W(w)}.$$

Substituting the expression for  $s^*_{w', \mu_W(w)}$ , we get:

$$u_{w'}(\mu_W(w), s^*_{w', \mu_W(w)}) = \sigma_{w', \mu_W(w)} + \left( u_{w'}(\mu_W, \mathbf{s}_W) - \sigma_{w', \mu_W(w)} \right),$$

which simplifies to:

$$u_{w'}(\mu_W(w), s^*_{w', \mu_W(w)}) = u_{w'}(\mu_W, \mathbf{s}_W).$$

■

It follows that

$$\max_{w' \in W \setminus \{\mu_W(\mu_W(w))\}} \left( R_{\mu_W(w)}(w') - \left( u_{w'}(\mu_W, \mathbf{s}_W) - \sigma_{w', \mu_W(w)} \right) \right),$$

represents the highest profit that firm  $\mu_W(w)$  can obtain from hiring another worker  $w'$ , offering a salary  $s^*_{w', \mu_W(w)}$  that provides the same utility, as required by the definition of stability.

If we assume:

$$R_{\mu_W(w)}(w) - s_{w,\mu_W(w)} > \max_{w' \in W \setminus \{\mu_W(\mu_W(w))\}} \left( R_{\mu_W(w)}(w') - \left( u_{w'}(\mu_W, \mathbf{s}_W) - \sigma_{w',\mu_W(w)} \right) \right),$$

then, by the same reasoning as in part (i), there would exist a new vector of salaries that would increase the utility of worker  $w$ , contradicting the optimality of  $(\mu_W, \mathbf{s}_W)$ .

Conversely, if we assume:

$$R_{\mu_W(w)}(w) - s_{w,\mu_W(w)} < \max_{w' \in W \setminus \{\mu_W(\mu_W(w))\}} \left( R_{\mu_W(w)}(w') - \left( u_{w'}(\mu_W, \mathbf{s}_W) - \sigma_{w',\mu_W(w)} \right) \right),$$

then there exists some worker  $w' \in W$  such that a salary  $\mathbf{s}'$  can be offered to  $w'$ , and it holds that:

$$u_{w'}(\mu_W(w), \mathbf{s}'_{w',\mu_W(w)}) > u_{w'}(\mu_W, \mathbf{s}),$$

and firm  $\mu_W(w)$  would earn a higher total profit:

$$V_{\mu_W(w)}(W'; \mathbf{s}', R_{\mu_W(w)}, \mathbf{T}) \geq V_{\mu_W(w)}(\mu_W(\mu_W(w)); \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{T}),$$

contradicting the stability of  $(\mu_W, \mathbf{s}_W)$ .

These two contradictions conclude the proof of part (ii). ■

**Lemma 5.** Consider a market  $G \in \mathbb{G}$ . For each worker  $w \in W$ , the utility  $u_w(\mu_F, \mathbf{s}_F)$  is given by:

$$u_w(\mu_F, \mathbf{s}_F) = \max \left( 0, \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] \right).$$

*Proof.* By definition,  $(\mu_F, \mathbf{s}_F)$  is the stable allocation in which firms have the highest profit.  $\min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$  represents the lowest profit per worker employed by firm  $f$  under the allocation  $(\mu_F, \mathbf{s}_F)$ .  $R_f(w) + \sigma_{w,f}$  represents the total surplus generated by worker  $w$  if  $w$  is employed by firm  $f$ , which constitutes the maximum possible profit that  $f$  could earn from hiring  $w$ .

Now, consider the firm  $f \in F \setminus \{\mu_F(w)\}$  that maximizes the expression  $(R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$ . We examine two cases:

- **Case 1:** If  $(R_f(w) + \sigma_{w,f}) \leq \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$ , the surplus generated by  $w$  at firm  $f$  is less than or equal to the minimum profit generated by any worker  $w' \in \mu_F(f)$ . Since  $f$  is the firm that maximizes the expression  $(R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$ , no other firm can offer a higher salary for  $w$ . Hence,

there is no competition among firms to hire  $w$ , and:

$$(R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \leq 0,$$

which implies that  $u_w(\mu_F, \mathbf{s}_F) = 0$ .

- **Case 2:** If  $(R_f(w) + \sigma_{w,f}) > \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$ , then firm  $f$  has an incentive to hire  $w$  over one of its current workers, say worker  $w' \in \mu_F(f)$  for which  $(R_f(w') - s_{w',f}) = \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$ . Firm  $f$  can offer  $w$  a salary of at most:

$$(R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}).$$

However, due to the stability of the allocation  $(\mu_F, \mathbf{s}_F)$ , we know that  $\mu_F(w) \neq f$ , meaning that  $w$  has no incentive to deviate to  $f$ , and there is no allocation that blocks  $(\mu_F, \mathbf{s}_F)$ . Therefore:

$$u_w(\mu_F, \mathbf{s}_F) \geq \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right].$$

By the optimality of  $(\mu_F, \mathbf{s}_F)$  for firms, and using the same reasoning as in the proof of Lemma 4, we conclude that this inequality holds with equality.

Thus, we have:

$$u_w(\mu_F, \mathbf{s}_F) = \max \left( 0, \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] \right),$$

which completes the proof. ■

## B.2 Proof of Proposition 2

*Proof.* Consider  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ , where  $\mathbf{T}$  subsidizes the minority workers at some  $F' \subset F$  by  $t$ . We proceed by contradiction: suppose there exists a worker  $w \in m$  such that  $u_w(\mu_F, \mathbf{s}_F) > u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ . From Lemma 5, we know that

$$u_w(\mu_F, \mathbf{s}_F) = \max \left( 0, \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] \right).$$

We consider two cases:

- **Case 1:** If  $u_w(\mu_F, \mathbf{s}_F) = 0$ , then, since  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$  is individually rational for workers, we have  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq 0$ . Consequently,  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ .

- **Case 2:** If  $u_w(\mu_F, \mathbf{s}_F) > 0$ , then we have the following inequality:

$$\max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] > \max \left( 0, \max_{f \in F \setminus \{\mu_F^{\mathbf{T}}(w)\}} \left[ (R_f^{\mathbf{T}}(w) + \sigma_{w,f}) - \min_{w' \in \mu_F^{\mathbf{T}}(f)} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}) \right] \right).$$

This implies

$$\max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] > \max_{f \in F \setminus \{\mu_F^{\mathbf{T}}(w)\}} \left[ (R_f^{\mathbf{T}}(w) + \sigma_{w,f}) - \min_{w' \in \mu_F^{\mathbf{T}}(f)} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}) \right].$$

Let  $f = \arg \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right]$  and  $f_{\mathbf{T}} = \arg \max_{f \in F \setminus \{\mu_F^{\mathbf{T}}(w)\}} \left[ (R_f^{\mathbf{T}}(w) + \sigma_{w,f}) - \min_{w' \in \mu_F^{\mathbf{T}}(f)} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}) \right]$ . Also, let  $w' = \arg \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f})$  and  $w'_{\mathbf{T}} = \arg \min_{w' \in \mu_F^{\mathbf{T}}(f_{\mathbf{T}})} (R_{f_{\mathbf{T}}}^{\mathbf{T}}(w') - s_{w',f_{\mathbf{T}}}^{\mathbf{T}})$ . Now we distinguish two subcases:

- **Case 2.1:** If  $f = f_{\mathbf{T}}$ . If  $f \in F'$ , then  $R_f^{\mathbf{T}}(w) = R_f(w) + t$ . If  $w' = w'_{\mathbf{T}}$  and  $w' \in m$ , then  $R_f^{\mathbf{T}}(w') = R_f(w') + t$ . Thus,

$$u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = R_f(w) + t + \sigma_{w,f} - R_f(w') - t + s_{w',f}^{\mathbf{T}}.$$

Since  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$  are firm-optimal stable allocations, it follows that

$$R_f(w) + t + \sigma_{w,f} - R_f(w') - t + s_{w',f}^{\mathbf{T}} \geq R_f(w) + \sigma_{w,f} - R_f(w') + s_{w',f}.$$

Hence,  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ . Similarly, if  $w' \notin m$ , we directly obtain

$$R_f(w) + t + \sigma_{w,f} - R_f(w') + s_{w',f}^{\mathbf{T}} \geq R_f(w) + \sigma_{w,f} - R_f(w') + s_{w',f},$$

which leads to the same conclusion. If  $w' \neq w'_{\mathbf{T}}$ , implying  $w'_{\mathbf{T}} \in M$  and  $w' \in m$ , from the construction of  $u_w(\mu_F, \mathbf{s}_F)$ , we know that

$$R_f(w') + t - s_{w',f} \geq R_f(w'_{\mathbf{T}}) - s_{w'_{\mathbf{T}},f} \geq R_f(w') - s_{w',f}.$$

It follows that

$$R_f(w) + t + \sigma_{w,f} - R_f(w'_{\mathbf{T}}) + s_{w'_{\mathbf{T}},f}^{\mathbf{T}} \geq R_f(w) + \sigma_{w,f} - R_f(w') + s_{w',f}.$$

Therefore, we conclude that

$$u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F).$$

– **Case 2.2:** If  $f \neq f_{\mathbf{T}}$ . In this case, we know that

$$\begin{aligned} (R_{f_{\mathbf{T}}}^{\mathbf{T}}(w) + \sigma_{w,f_{\mathbf{T}}}) - \min_{w' \in \mu_F^{\mathbf{T}}(f_{\mathbf{T}})} (R_{f_{\mathbf{T}}}^{\mathbf{T}}(w') - s_{w',f_{\mathbf{T}}}^{\mathbf{T}}) &\geq \\ (R_f^{\mathbf{T}}(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}). \end{aligned}$$

From Case 2.1, we already know that

$$R_f^{\mathbf{T}}(w) + \sigma_{w,f} - \min_{w' \in \mu_F(f)} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F),$$

thus leading to the conclusion that  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ .

Hence,  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ . ■

### B.3 Proof of Theorem 1

*Proof.* Let  $(\mu_W, \mathbf{s}_W)$  and  $(\mu_F, \mathbf{s}_F)$  be the worker-optimal and firm-optimal stable allocations without transfer  $\mathbf{T}$ , and let  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$  be the worker-optimal and firm-optimal stable allocations with transfer  $\mathbf{T}$ . We have to show that for each  $w \in m$ , we have  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W)$  and  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ , provided that  $\mathbf{T}$  uniformly subsidizes minority workers.

From Proposition 2, we know that a transfer  $\mathbf{T}'$  that subsidizes minority workers at some  $F' \subset F$  by  $t$  does not negatively impact minority workers under the firm-optimal stable allocation  $\varphi_F$ . The proof of Proposition 2 holds directly if  $F' = F$ , implying that a transfer  $\mathbf{T}$  that uniformly subsidizes minority does not negatively impact minority workers under  $\varphi_F$ .

From Theorem 4 (ii), we know that a transfer  $\mathbf{T}'$  that subsidizes minority workers, such that for each  $w \in m_E$ ,  $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f})$  and  $t_{w,\mu_W(w)} \geq \max_{w' \in m_U, f \in F}(t_{w',f})$ , does not negatively impact minority workers under  $\varphi_W$ . Since  $\mathbf{T}$  uniformly subsidizes minority workers, for each  $w \in m_E$ , we have  $t_{w,\mu_W(w)} = \max_{f \in F}(t_{w,f})$  and  $t_{w,\mu_W(w)} = \max_{w' \in m_U, f \in F}(t_{w',f})$ . Therefore, the transfer  $\mathbf{T}$  does not negatively impact

minority workers under  $\varphi_W$ .

Thus, the uniform subsidy  $\mathbf{T}$  ensures that  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W)$  and  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$  for all  $w \in m$ .  $\blacksquare$

#### B.4 Proof of Proposition 3

*Proof.* Consider the allocations  $(\mu_F, \mathbf{s}_F)$ ,  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ , and the transfer  $\mathbf{T}$ . We proceed by contradiction: suppose there exists a worker  $w \in m$  such that  $u_w(\mu_F, \mathbf{s}_F) > u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ . From Lemma 5, we know that

$$u_w(\mu_F, \mathbf{s}_F) = \max \left( 0, \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] \right).$$

We consider two cases:

- **Case 1:**  $u_w(\mu_F, \mathbf{s}_F) = 0$ . Since  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$  is individually rational for workers, we know that  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq 0$ . Therefore,  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ .
- **Case 2:**  $u_w(\mu_F, \mathbf{s}_F) > 0$ . This implies that

$$\begin{aligned} & \max_{f \in F \setminus \{\mu_F(w)\}} \left[ (R_f(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w',f}) \right] > \\ & \max \left( 0, \max_{f \in F \setminus \{\mu_F^{\mathbf{T}}(w)\}} \left[ (R_f^{\mathbf{T}}(w) + \sigma_{w,f}) - \min_{w' \in \mu_F^{\mathbf{T}}(f)} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}) \right] \right). \end{aligned}$$

We use the same notations as in the proof of Proposition 2, namely  $f$ ,  $f_{\mathbf{T}}$ ,  $w'$ , and  $w'_{\mathbf{T}}$ . Now we distinguish two subcases:

- **Case 2.1:** If  $f = f_{\mathbf{T}}$ . Since  $w \in m$ , we know that  $R_f^{\mathbf{T}}(w) = R_f(w)$ . If  $w' = w'_{\mathbf{T}}$ , we know that  $R_f^{\mathbf{T}}(w') \leq R_f(w')$  since  $\mathbf{T}$  taxes some majority workers. Since both  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$  are firm-optimal stable allocations, it follows that

$$R_f(w) + \sigma_{w,f} - R_f(w') + s_{w',f} - \left( R_f(w) + \sigma_{w,f} - R_f^{\mathbf{T}}(w') + s_{w',f}^{\mathbf{T}} \right) > 0.$$

Therefore,

$$R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}} - R_f(w') + s_{w',f} > 0,$$

and

$$R_f(w') + t_{w',f} - s_{w',f}^{\mathbf{T}} - R_f(w') + s_{w',f} > 0,$$

consequently  $t_{w',f} - s_{w',f}^{\mathbf{T}} + s_{w',f} > 0$ .

**Claim 2.**  $s_{w',f}^{\mathbf{T}} \geq s_{w',f} + t_{w',f}$ .

*Proof.* By the optimality of  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ , we know that  $w'$  is the worker employed by  $f$  who generates the lowest profit for the firm (i.e.,  $w' = \operatorname{argmin}_{w \in \mu_F(f)} (R_f(w) - s_{w,f})$ ), and that there is no  $s'_{w',f} < s_{w',f}$  such that  $w'$  would accept to work for  $f$ . Similarly, there is no  $s''_{w',f} < s_{w',f}^{\mathbf{T}}$  such that  $w'$  would accept to work for  $f$ . Since  $\mathbf{T}$  taxes some majority workers, the surplus generated by all workers at all firms is weakly lower. It follows that for each  $w \in W$ , we have  $s_{w',f}^{\mathbf{T}} \geq s_{w',f} + t_{w',f}$ .  $\blacksquare$

By Claim 2, we know that  $s_{w',f}^{\mathbf{T}} \geq s_{w',f} + t_{w',f}$ , and therefore  $t_{w',f} - s_{w',f}^{\mathbf{T}} + s_{w',f} \leq 0$ , which is a contradiction. Hence,  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ .

If  $w' \neq w'_{\mathbf{T}}$ , two cases arise: either  $w' \in \mu_F^{\mathbf{T}}(f)$  or  $w' \notin \mu_F^{\mathbf{T}}(f)$ . Let us show that both cases lead to the same conclusion. If  $w' \notin \mu_F^{\mathbf{T}}(f)$ , it is direct that  $w' \in M$  and is taxed by  $\mathbf{T}$ . Since for each  $w \in W$ ,  $R_f(w) + t_{w,f} + \sigma_{w,f} \geq 0$ , there exists a worker  $w''$  such that  $\mu_F(w'') \neq f$  and  $\mu_F^{\mathbf{T}}(w'') = f$  (i.e.,  $w''$  has replaced  $w'$  at firm  $f$ ). As  $(\mu_F, \mathbf{s}_F)$  is the firm-optimal stable allocation,  $w''$  does not generate a greater profit than  $w'$  at firm  $f$ . Formally,

$$R_f(w') - s_{w',f} \geq R_f(w'') + \sigma_{w'',f} - u_{w''}(\mu_F, \mathbf{s}_F).$$

Since  $\mathbf{T}$  does not subsidize any worker and  $\mu_F^{\mathbf{T}}(w'') = f$ , it follows that  $R_f(w') - s_{w',f} \geq R_f^{\mathbf{T}}(w'') - s_{w'',f}^{\mathbf{T}}$ . Hence,  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ .

If  $w' \in \mu_F^{\mathbf{T}}(f)$ , we know that  $R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}} \geq R_f^{\mathbf{T}}(w'_{\mathbf{T}}) - s_{w'_{\mathbf{T}},f}^{\mathbf{T}}$ . It directly follows that

$$R_f(w) + \sigma_{w,f} - R_f^{\mathbf{T}}(w'_{\mathbf{T}}) + s_{w'_{\mathbf{T}},f}^{\mathbf{T}} \geq R_f(w) + \sigma_{w,f} - R_f(w') + s_{w',f},$$

and thus  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ .

– **Case 2.2:** If  $f \neq f_{\mathbf{T}}$ . Then, we know that

$$\begin{aligned} (R_{f_{\mathbf{T}}}^{\mathbf{T}}(w) + \sigma_{w,f_{\mathbf{T}}}) - \min_{w' \in \mu_F^{\mathbf{T}}(f_{\mathbf{T}})} (R_{f_{\mathbf{T}}}^{\mathbf{T}}(w') - s_{w',f_{\mathbf{T}}}^{\mathbf{T}}) &\geq \\ (R_f^{\mathbf{T}}(w) + \sigma_{w,f}) - \min_{w' \in \mu_F(f)_{\mathbf{T}}} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}). \end{aligned}$$

In Case 2.1, we showed that

$$R_f^{\mathbf{T}}(w) + \sigma_{w,f} - \min_{w' \in \mu_F(f)_{\mathbf{T}}} (R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F).$$



Hence,  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ .

This concludes the proof. Note that the proof is the same if all majority workers are taxed by  $t$  at some firms. ■

## B.5 Proof of Theorem 2

*Proof.* Let  $(\mu_W, \mathbf{s}_W)$  and  $(\mu_F, \mathbf{s}_F)$  be the worker-optimal and firm-optimal stable allocations without the transfer  $\mathbf{T}$ , and let  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$  be the worker-optimal and firm-optimal stable allocations with the transfer  $\mathbf{T}$ . We have to show that for each  $w \in m$ , we have  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W)$  and  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F)$ , provided that  $\mathbf{T}$  uniformly taxes majority workers.

Suppose that the transfer  $\mathbf{T}$  uniformly taxes majority workers by  $t$ , where  $t < 0$ . From Theorem 3, we can instead consider a transfer  $\mathbf{T}'$  which uniformly subsidizes minority workers by  $|t|$ . It follows that there exist worker-optimal and firm-optimal stable allocations with respect to  $\mathbf{T}'$  such that:

$$(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'}) \quad \text{and} \quad (\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'}).$$

From Theorem 1, we know that the transfer  $\mathbf{T}'$  does not negatively impact minority workers under both  $\varphi_W$  and  $\varphi_F$ . Specifically, this implies that the utility of each minority worker  $w \in m$  weakly increases with the transfer  $\mathbf{T}'$ , i.e., for each  $w \in m$ , we have:

$$u_w(\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'}) \geq u_w(\mu_W, \mathbf{s}_W) \quad \text{and} \quad u_w(\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'}) \geq u_w(\mu_F, \mathbf{s}_F).$$

Since the allocations with the transfer  $\mathbf{T}$  and the transfer  $\mathbf{T}'$  are equivalent, it follows that the transfer  $\mathbf{T}$  does not negatively impact minority workers either under  $\varphi_W$  or under  $\varphi_F$ .

Therefore, for each  $w \in m$ , we have:

$$u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W) \quad \text{and} \quad u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq u_w(\mu_F, \mathbf{s}_F),$$

which concludes the proof. ■

## B.6 Proof of Theorem 3

*Proof.* Suppose  $\mathbf{T}$  uniformly subsidizes minority workers by  $t$ , and  $\mathbf{T}'$  uniformly taxes majority workers by  $t'$ , such that  $t = |t'|$ .

We first consider the firm-optimal stable allocation. We use the construction of the ascending salary adjustment process (Appendix A.2).

**Claim 3.** For markets  $(\mathbf{T}, \mathbf{q})$  and  $(\mathbf{T}', \mathbf{q})$ , for each step  $t$  of  $\varphi_F$  and for each firm  $f \in F$ ,  $W'_f(t)$  coincides. In addition, for each  $w \in W'_f(t)$ ,  $s_f(t)(w)$  coincides.

*Proof.* We implicitly assume that the same tie-breaking rules are used in the mechanism. For each firm  $f \in F$ , it is clear that  $\mathbf{s}_f(0)$  is the same in both markets. We now show that the set of workers  $W'_f(0)$  selected by the firm is the same in both markets. Recall that for each  $f \in F$ ,  $R_f^{\mathbf{T}}(w) = R_f(w) + t$  if  $w \in m$ , and  $R_f^{\mathbf{T}}(w) = R_f(w)$  if  $w \in M$ . Similarly,  $R_f^{\mathbf{T}'}(w) = R_f(w)$  if  $w \in m$ , and  $R_f^{\mathbf{T}'}(w) = R_f(w) + t'$  if  $w \in M$ .

By contradiction, suppose  $w \in W'_f(0)$  for market  $(\mathbf{T}, \mathbf{q})$  and  $w \notin W'_f(0)$  for market  $(\mathbf{T}', \mathbf{q})$ , while  $w' \notin W'_f(0)$  for market  $(\mathbf{T}, \mathbf{q})$  and  $w' \in W'_f(0)$  for market  $(\mathbf{T}', \mathbf{q})$ . It follows that in the market  $(\mathbf{T}, \mathbf{q})$ :

$$R_f(w) + t - \mathbf{s}_f(0)(w) > R_f(w') + t - \mathbf{s}_f(0)(w'),$$

and, in the market  $(\mathbf{T}', \mathbf{q})$ :

$$R_f(w') + t' - \mathbf{s}_f(0)(w') > R_f(w) + t' - \mathbf{s}_f(0)(w).$$

It follows that

$$\begin{aligned} R_f(w) + t - \mathbf{s}_f(0)(w) + R_f(w') + t' - \mathbf{s}_f(0)(w') > \\ R_f(w') + t - \mathbf{s}_f(0)(w') + R_f(w) + t' - \mathbf{s}_f(0)(w). \end{aligned}$$

Since  $\mathbf{s}_f(0)$  and  $R_f(w), R_f(w')$  are the same in both markets, we get:

$$t + t' > t + t',$$

which is a contradiction, as  $t = -t'$ .

Therefore, the set of workers  $W'_f(0)$  is the same at step 0. Since the offers received by workers are the same, workers reject the same proposals. The vector of salaries  $\mathbf{s}_f(0)$  then coincides. By induction, and applying the same reasoning to subsequent steps, it follows that the mechanism proceeds identically for both markets.  $\blacksquare$

Hence, we conclude that  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) = (\mu_F^{\mathbf{T}'}, \mathbf{s}_F^{\mathbf{T}'})$ .

Now, we consider the worker-optimal stable allocation. From Lemma 3, we know that  $\mu_F^{\mathbf{T}} = \mu_W^{\mathbf{T}}$ . Since  $\mu_F^{\mathbf{T}} = \mu_F^{\mathbf{T}'}$ , we know that  $\mu_W^{\mathbf{T}} = \mu_W^{\mathbf{T}'}$ . It remains to show that  $\mathbf{s}_W^{\mathbf{T}} = \mathbf{s}_W^{\mathbf{T}'}$ . Using the construction of the descending salary adjustment process (Appendix A.1), we know that  $\varphi_W$  stops at step  $t \geq 0$ , where for each  $w \in W_U$  and each firm  $f \in F$ ,  $\mathbf{u}_w(t)(f) = 0$ .

Thus, we know that there are  $\max_{w' \in W_U} \mathbf{u}_{w'}(0)(f)$  steps in which firm  $f$  has rejected all the offers it has received. Thus, for all workers  $w \in W$ ,  $\mathbf{u}_w(t)(f) = R_f(w) + \sigma_{w,f} - \max_{w' \in W_U} (R_f(w') + \sigma_{w',f})$ .

**Claim 4.** For any  $f \in F$ , let  $w_f^{\mathbf{T}} \equiv \arg \max_{w' \in W_U} (R_f(w') + t_{w',f} + \sigma_{w',f})$  in market  $(\mathbf{T}, \mathbf{q})$ , and  $w_f^{\mathbf{T}'} \equiv \arg \max_{w' \in W_U} (R_f(w') + t'_{w',f} + \sigma_{w',f})$  in market  $(\mathbf{T}', \mathbf{q})$ . Then,  $w_f^{\mathbf{T}} = w_f^{\mathbf{T}'}$ .

*Proof.* The proof is straightforward, considering that  $\mathbf{T}$  uniformly subsidizes minority workers by  $t$  and  $\mathbf{T}'$  uniformly taxes majority workers by  $t'$ , with  $t = |t'|$ .  $\blacksquare$

Hence, for each worker  $w \in \mu_W^{\mathbf{T}}(f)$ , we denote by  $w' \equiv \arg \max_{w' \in W_U} (R_f(w') + t_{w',f} + \sigma_{w',f})$  (i.e., the unemployed worker who has the highest expected utility at step 0 at  $f$ ) for market  $(\mathbf{T}, \mathbf{q})$ . From Claim 4, it follows that for each worker  $w \in \mu_W^{\mathbf{T}}(f)$ ,

$$u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = \mathbf{u}_w(t)(f) = R_f(w) + t_{w,f} + \sigma_{w,f} - (R_f(w') + t_{w',f} + \sigma_{w',f})$$

for market  $(\mathbf{T}, \mathbf{q})$ , and

$$u_w(\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'}) = \mathbf{u}_w(t')(f) = R_f(w) + t'_{w,f} + \sigma_{w,f} - (R_f(w') + t'_{w',f} + \sigma_{w',f})$$

for market  $(\mathbf{T}', \mathbf{q})$ . Since  $R_f(w)$ ,  $R_f(w')$ ,  $\sigma_{w,f}$ , and  $\sigma_{w',f}$  are the same in both markets, we consider four cases:

- **Case 1:** If  $w \in m$  and  $w' \in m$ , then  $t_{w,f} = t_{w',f} = t$  and  $t'_{w,f} = t'_{w',f} = 0$ . It follows that  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = u_w(\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'})$ . Since  $\mu_W^{\mathbf{T}} = \mu_W^{\mathbf{T}'}$ , we conclude that  $s_{w,f}^{\mathbf{T}} = s_{w,f}^{\mathbf{T}'}$ .
- **Case 2:** If  $w \in m$  and  $w' \in M$ , then  $t_{w,f} = t$ ,  $t_{w',f} = 0$ , and  $t'_{w,f} = 0$ ,  $t'_{w',f} = t'$ . It follows that:  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = u_w(\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'})$ . Since  $\mu_W^{\mathbf{T}} = \mu_W^{\mathbf{T}'}$ , we conclude that  $s_{w,f}^{\mathbf{T}} = s_{w,f}^{\mathbf{T}'}$ .
- **Case 3:** If  $w \in M$  and  $w' \in m$ , then  $t_{w,f} = 0$ ,  $t_{w',f} = t$ , and  $t'_{w,f} = t'$ ,  $t'_{w',f} = 0$ . It follows that:  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = u_w(\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'})$ . Since  $\mu_W^{\mathbf{T}} = \mu_W^{\mathbf{T}'}$ , we conclude that  $s_{w,f}^{\mathbf{T}} = s_{w,f}^{\mathbf{T}'}$ .
- **Case 4:** If  $w \in M$  and  $w' \in M$ , then  $t_{w,f} = t_{w',f} = 0$  and  $t'_{w,f} = t'_{w',f} = t'$ . It follows that:  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = u_w(\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'})$ . Since  $\mu_W^{\mathbf{T}} = \mu_W^{\mathbf{T}'}$ , we conclude that

$$s_{w,f}^{\mathbf{T}} = s_{w,f}^{\mathbf{T}'}$$

Using this reasoning for each firm, it follows that  $\mathbf{s}_W^{\mathbf{T}} = \mathbf{s}_W^{\mathbf{T}'}$ . Therefore, we conclude that  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = (\mu_W^{\mathbf{T}'}, \mathbf{s}_W^{\mathbf{T}'})$ . ■

## B.7 Proof of Theorem 4

*Proof.* (i) Consider a market  $G \in \mathbb{G}$ , with the worker-optimal stable allocation  $(\mu_W, \mathbf{s}_W)$  and a transfer  $\mathbf{T}$  such that for each  $t_{w,f} > 0$ , we have  $w \in m$ , there is no  $t_{w',f} < 0$  for any  $w' \in W$ , for each  $w \in m_E$ ,  $t_{w, \mu_W(w)} = \max_f(t_{w,f})$  and for each  $w' \in W \setminus \{m_E\}$  and  $f \in F$ , we have  $t_{w',f} = 0$ .

We know that for each  $w \in W$ , the utility is higher than in any other stable allocation, and that the maximum subsidy for each worker  $w$  is received at  $\mu_W(w)$ . Hence, competition is not intensified among minority workers, and no subsidy is given to unemployed minority workers.

**Claim 5.** Under any transfer  $\mathbf{T}$  described in Theorem 4 (i), for each  $w \in W$ , we have  $\mu_W(w) = \mu_W^{\mathbf{T}}(w)$ .

*Proof.* The proof of Claim 5 proceeds through two claims:

**Claim 6.** For each  $w \in W_U$  at  $(\mu_W, \mathbf{s}_W)$ , we have  $\mu_W^{\mathbf{T}}(w) = \emptyset$ .

*Proof.* For each  $w \in W_U$  at  $(\mu_W, \mathbf{s}_W)$ , we know that for any firm  $f \in F$ , the minimum profit for each firm is greater than the surplus generated by  $w$ , i.e.,

$$\min_{w' \in \mu_W(f)} (R_f(w') - s_{w',f}) \geq R_f(w) + \sigma_{w,f}.$$

Moreover, since there is no  $t < 0$ , for each  $w \in W_U$  and for every  $f \in F$ , we have  $t_{w,f} = 0$ . Therefore, the inequality remains true under transfer  $\mathbf{T}$ :

$$\min_{w' \in \mu_W(f)} (R_f(w') - s_{w',f}) \geq R_f^{\mathbf{T}}(w) + \sigma_{w,f}.$$

This implies that no unemployed worker  $w \in W_U$  is hired under the transfer  $\mathbf{T}$ , so  $\mu_W^{\mathbf{T}}(w) = \emptyset$ . ■

**Claim 7.** For each  $w \in W_E$  at  $(\mu_W, \mathbf{s}_W)$ , we have  $\mu_W(w) = \mu_W^{\mathbf{T}}(w)$ .

*Proof.* There are two cases to consider:

- If  $w \in m_E$ , then  $\max_{f \in F}(t_{w,f}) = t_{w,\mu_W(w)}$ , and since there is no  $t_{w',f'} < 0$ , we know that  $t_{w,\mu_W(w)} \geq 0$ . Since  $(\mu_W, \mathbf{s}_W)$  is the worker-optimal stable allocation, the firm at which  $w$  can achieve the highest utility is  $\mu_W(w)$ .
- If  $w \notin m_E$ , then for each  $f \in F$ , we have  $t_{w,f} = 0$ . Again, since  $(\mu_W, \mathbf{s}_W)$  is the worker-optimal stable allocation, the firm at which  $w$  can achieve the highest utility is  $\mu_W(w)$ .

■

■

For each firm  $f$ , the surplus generated by a worker  $w \in \mu_W(f)$  is given by  $R_f(w) + \sigma_{w,f}$ , and the profit is  $R_f(w) - s_{w,f}$ . With the transfer  $\mathbf{T}$ , the surplus generated by  $w \in \mu_W^{\mathbf{T}}(f)$  is given by  $R_f^{\mathbf{T}}(w) + \sigma_{w,f}$ , and the profit is  $R_f^{\mathbf{T}}(w) - s_{w,f}^{\mathbf{T}}$ . By constructing the allocation  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$ , for each  $w \in W$ , the highest possible utility at a stable allocation is at  $\mu_W(w)$ . By Claim 5, we have  $\mu_W = \mu_W^{\mathbf{T}}$ .

Therefore, the profit generated by each worker  $w \in \mu_W(f)$  for each firm  $f \in F$  must be at least  $R_f(w) - s_{w,f}$ . Since  $(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}})$  is the worker-optimal stable allocation with transfer, it follows that:

$$R_f^{\mathbf{T}}(w) - s_{w,f}^{\mathbf{T}} = R_f(w) - s_{w,f}.$$

For each  $w \in \mu_W(f)$ , we have  $R_f^{\mathbf{T}}(w) + \sigma_{w,f} = R_f(w) + t_{w,f} + \sigma_{w,f}$ . Therefore, for each  $w \in m_E$ , we get:

$$s_{w,\mu_W^{\mathbf{T}}(w)}^{\mathbf{T}} = s_{w,\mu_W(w)} + t_{w,\mu_W(w)}.$$

For other workers  $w' \in W \setminus \{m_E\}$ , we have  $s_{w',\mu_W^{\mathbf{T}}(w')}^{\mathbf{T}} = s_{w',\mu_W(w')}$ , by construction of  $\mathbf{T}$ .

(ii) We now decompose the transfer  $\mathbf{T}$  into two transfers:

- $\mathbf{T}_1$ : For each  $w \in m_E$  and  $f \in F$ , we have  $t_{1_{w,f}} = t_{w,f}$  and 0 for other elements. We know that  $t_{1_{w,\mu_W(w)}} = \max_{f \in F}(t_{w,f})$ .
- $\mathbf{T}_2$ : For each  $w' \in m_U$  and  $f \in F$ , we have  $t_{2_{w',f}} = t_{w',f}$  and 0 for other elements. We know that for each  $w \in m_E$ , we have  $t_{1_{w,\mu_W(w)}} \geq \max(\mathbf{T}_2)$ .

From Theorem 4 (i), we know that for each  $w \in m_E$ , we have:

$$s_{w,\mu_W^{\mathbf{T}}(w)}^{\mathbf{T}} = s_{w,\mu_W(w)} + t_{w,\mu_W(w)}.$$

From Lemma 2, we know that the impact of  $\mathbf{T}_2$  is at most  $\max(\mathbf{T}_2)$ . Since, by Lemma 1, we have  $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$ , it follows that  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ . ■

## B.8 Proof of Theorem 5

*Proof.* In this proof, we consider the firm-optimal stable allocations  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ .<sup>23</sup> First, we show that  $\mathbf{T}$  and  $t_{\rightarrow f}$  affect the assignment of  $w_m$  or another minority worker.

**Claim 8.**  $\mathbf{T}$  increases the representation of minority workers in firm  $f$  under  $\varphi_W$  and  $\varphi_F$ .

*Proof.* Recall that:

$$t_{\rightarrow f} \equiv \min_{w_M \in \mu_F(f) \cap M} (R_f(w_M) - s_{w_M, f}) - (R_f(w_m) + \sigma_{w_m, f}) + R_{\mu_F(w_m)}(w_m) + \sigma_{w_m, \mu_F(w_m)} \\ - \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} (R_{\mu_F(w_m)}(w') + \sigma_{w', \mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F)).$$

By the construction of  $\mathbf{T}$ , there may exist minority workers  $w \in m_E \setminus \mu_F(f)$ , with  $w \neq w_m$ , such that to be employed by firm  $f$ , a subsidy lower than  $t_{\rightarrow f}$  is required. Since transfers as described in Theorem 5 only impose that  $t_{w', f} \leq t_{\rightarrow f}$ , we first show that if there is no such worker, then  $\mu_F(w_m) = f$ .

We need to prove that under  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ , the following inequality holds:

$$R_f(w_m) + t_{\rightarrow f} - s_{w_m, f}^{\mathbf{T}} \geq R_f(w_M) - s_{w_M, f}.$$

By stability, we know that:

$$R_f(w_M) - s_{w_M, f} \geq R_f(w_m) + \sigma_{w_m, f} - u_{w_m}(\mu_F, \mathbf{s}_F).$$

Otherwise, firm  $f$  could offer a higher salary to  $w_m$ , contradicting the stability of  $(\mu_F, \mathbf{s}_F)$ . Hence, we get:

$$R_f(w_M) - s_{w_M, f} - (R_f(w_m) + \sigma_{w_m, f}) \geq -u_{w_m}(\mu_F, \mathbf{s}_F).$$

Next, consider the maximum utility  $w_m$  can obtain at firm  $\mu_F(w_m)$ . The surplus generated by  $w_m$  at  $\mu_F(w_m)$  is  $R_{\mu_F(w_m)}(w_m) + \sigma_{w_m, \mu_F(w_m)}$ . We also consider the best alternative for  $\mu_F(w_m)$ , i.e., the worker who generates the highest profit if  $w_m$  is replaced.

<sup>23</sup>We consider the firm-optimal stable allocation for two main reasons. First, under the worker-optimal stable allocation, all workers employed by a firm generate the same profit (Lemma 4), making it impossible to distinguish between them. Second, under the firm-optimal stable allocation, each firm achieves the highest possible profit in any stable allocation. Therefore, if a worker's salary decreases even slightly, the worker has a better outside option, either by being employed at another firm or by remaining unemployed.

Let us assume that this worker is either unemployed or  $w_M$ . To attract this worker,  $\mu_F(w_m)$  must offer a salary compensating for the worker's current utility. This is given by:

$$\max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} \left( R_{\mu_F(w_m)}(w') + \sigma_{w', \mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F) \right).$$

Therefore, the highest utility that  $w_m$  can expect at  $\mu_F(w_m)$  is:

$$R_{\mu_F(w_m)}(w_m) + \sigma_{w_m, \mu_F(w_m)} - \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} \left( R_{\mu_F(w_m)}(w') + \sigma_{w', \mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F) \right).$$

Thus, we have:

$$\begin{aligned} R_{\mu_F(w_m)}(w_m) + \sigma_{w_m, \mu_F(w_m)} - \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} \left( R_{\mu_F(w_m)}(w') + \sigma_{w', \mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F) \right) &\geq \\ u_{w_m}(\mu_F, \mathbf{s}_F) &\geq - (R_f(w_M) - s_{w_M, f}) + (R_f(w_m) + \sigma_{w_m, f}). \end{aligned}$$

It follows that with the subsidy  $t_{\rightarrow f}$ , firm  $f$  is indifferent between  $w_m$  and  $w_M$ , and the utility of  $w_m$  is weakly higher. By the optimality of  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ , we have  $\mu_W^{\mathbf{T}}(w_m) = f$ .

Next, consider the possibility that there are other workers employed by firms that generate profits higher than those of  $w_m$  at  $\mu_F(w_m)$ . If this is the case for  $w'' \in W \setminus (W_U \cup \{w_M\})$ , then:

$$\begin{aligned} R_{\mu_F(w_m)}(w'') + \sigma_{w'', \mu_F(w_m)} - u_{w''}(\mu_F, \mathbf{s}_F) &> \\ \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w_m\}} \left( R_{\mu_F(w_m)}(w') + \sigma_{w', \mu_F(w_m)} - u_{w'}(\mu_F, \mathbf{s}_F) \right). \end{aligned}$$

By construction, we have  $t_{\rightarrow f} > t'_{\rightarrow f}$ , where  $t'_{\rightarrow f}$  is the subsidy required for  $w''$  to switch firms and be employed by  $f$ . This reduces the overall subsidy needed but does not affect the underlying reasoning.

Finally, if there is a worker  $w \in m_E \setminus \mu_F(f)$  such that:

$$\begin{aligned} t_{w, f} - \min_{w_M \in \mu_F(f) \cap M} (R_f(w_M) - s_{w_M, f}) - (R_f(w) + \sigma_{w, f}) + R_{\mu_F(w)}(w) + \sigma_{w, \mu_F(w)} - \\ \max_{w' \in (W_U \cup \{w_M\}) \setminus \{w\}} \left( R_{\mu_F(w)}(w') + \sigma_{w', \mu_F(w)} - u_{w'}(\mu_F, \mathbf{s}_F) \right) &> t_{w_m, f} - t_{\rightarrow f}, \end{aligned}$$

then  $\mu_F^{\mathbf{T}}(w) = f$ . In both cases, there exists  $w \in m$  such that  $\mu_F(w) \neq f$  and  $\mu_F^{\mathbf{T}} = f$ . By Lemma 3, we know that  $\mu_W(w) \neq f$  and  $\mu_W^{\mathbf{T}} = f$ . ■

It remains to show that minority workers in  $m_E \setminus \{w_m\}$  are not negatively impacted under  $\varphi_W$  and that  $(\mu_F(f) \cap m) \subset (\mu_F^{\mathbf{T}}(f) \cap m)$ .

**Claim 9.**  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ .

*Proof.* We have to show that for each  $w \in m_E$ ,  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W)$ . We have four cases:

- **Case 1:**  $\mu_W(w) = f$  and  $\mu_W^{\mathbf{T}}(w) = f$ . By Lemma 1, we can decompose  $\mathbf{T}$  into three transfers:
  - $\mathbf{T}_1$  is such that for each  $w \in m_E \cap \mu_F(f)$ ,  $t_{1_{w,f}} = t_{w,f}$ , and 0 for other elements.
  - $\mathbf{T}_2$  is such that for each  $w' \in m_E \setminus (\mu_F(f) \cup \{w_m\})$ ,  $t_{2_{w',f}} = t_{w',f}$ ,  $t_{2_{w_m,f}} = t_{\rightarrow f}$ , and 0 for other elements.
  - $\mathbf{T}_3$  is such that  $t_{3_{w_m,f}} = t_{w_m,f} - t_{2_{w_m,f}}$  and 0 for other elements.

From Theorem 4 (i), it follows that  $s_{w,f}^{\mathbf{T}_1} = s_{w,f} + t_{1_{w,f}} \geq s_{w,f} + t_{\rightarrow f}$ . Additionally, by Lemma 2, we have  $|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_2})| \leq t_{\rightarrow f}$ . Since for each  $w \in m_E \cap \mu_F(f)$ ,  $t_{1_{w,f}} \geq t_{\rightarrow f}$ , we consider that  $(\mathbf{T}_1, \mathbf{q}) \xrightarrow{\mathbf{T}_2} ((\mathbf{T}_1 + \mathbf{T}_2), \mathbf{q})$ , hence,  $(\mathbf{T}_1 + \mathbf{T}_2)$  does not negatively impact minority workers under  $\varphi_W$ . According to Claim 8, if there are no minority workers  $w' \in m_E \setminus (\mu_F(f) \cup \{w_m\})$  such that  $\mu_W^{\mathbf{T}}(w') = f$ , then  $\mu_W^{(\mathbf{T}_1 + \mathbf{T}_2)}(w_m) = f$ . It remains to demonstrate that  $\mathbf{T}_3$  does not negatively impact minority workers under  $\varphi_W$ . Considering  $((\mathbf{T}_1 + \mathbf{T}_2), \mathbf{q})$  and introducing  $\mathbf{T}_3$ , since  $\mu_W^{(\mathbf{T}_1 + \mathbf{T}_2)}(w_m) = f$ , by Theorem 4 (i), we conclude that  $\mathbf{T}_3$  does not negatively impact minority workers under  $\varphi_W$ . Consequently, since  $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ , we conclude that  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ .

Thus, the reduction in the welfare of worker  $w$  is at most  $t_{\rightarrow f}$ , and the increase is at least  $t_{\rightarrow f}$ . Therefore,  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W)$ .

- **Case 2:**  $\mu_W(w) \neq f$  and  $\mu_W^{\mathbf{T}}(w) \neq f$ . From Lemma 4, we know that

$$s_{w, \mu_W(w)}^{\mathbf{T}} = R_{\mu_W(w)}^{\mathbf{T}}(w) - \max_{w' \in W \setminus \{\mu_W(\mu_W(w))\}} \left( R_{\mu_W(w)}^{\mathbf{T}}(w') - \left( u_{w'}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) - \sigma_{w', \mu_W^{\mathbf{T}}(w)} \right) \right).$$

Let  $w' \equiv \arg \max_{w' \in W \setminus \{\mu_W(\mu_W(w))\}} \left( R_{\mu_W(w)}^{\mathbf{T}}(w') - \left( u_{w'}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) - \sigma_{w', \mu_W^{\mathbf{T}}(w)} \right) \right)$ . By the construction of  $\mathbf{T}$ , the only firm affected is  $f$ . Thus,  $R_{\mu_W^{\mathbf{T}}(w)}^{\mathbf{T}}(w) = R_{\mu_W(w)}^{\mathbf{T}}(w)$  and  $R_{\mu_W^{\mathbf{T}}(w)}^{\mathbf{T}}(w') = R_{\mu_W(w)}^{\mathbf{T}}(w')$ . As noted in Case 1, the utility of some workers assigned to  $f$  has weakly increased. If  $\mu_W^{\mathbf{T}}(w') = f$ , then  $u_{w'}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_{w'}(\mu_W, \mathbf{s}_W)$ ; otherwise, we have  $u_{w'}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = u_{w'}(\mu_W, \mathbf{s}_W)$ . Hence,  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ .



- **Case 3:**  $\mu_W(w) \neq f$  and  $\mu_W^{\mathbf{T}}(w) = f$ . By Claim 8, it follows directly that  $u_w(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) \geq u_w(\mu_W, \mathbf{s}_W)$ , and thus  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ .
- **Case 4:**  $\mu_W(w) = f$  and  $\mu_W^{\mathbf{T}}(w) \neq f$ . From Claim 8, it follows that  $w \notin m$ , and thus  $\mathbf{T}$  does not negatively impact minority workers under  $\varphi_W$ . ■

From Lemma 3 we know that there exist  $\mu_F^{\mathbf{T}}$  and  $\mu_W^{\mathbf{T}}$  such that  $\mu_F^{\mathbf{T}} = \mu_W^{\mathbf{T}}$ . From Claim 8 and Claim 9, we know that  $|\mu_W^{\mathbf{T}}(f) \cap m| > |\mu_W(f) \cap m|$  and  $(\mu_F(f) \cap m) \subset (\mu_F^{\mathbf{T}}(f) \cap m)$ , which completes the proof. ■

## B.9 Proof of Proposition 5

*Proof.* In this proof, we consider the firm-optimal stable allocations  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ . From Claim 8, we know there exists  $w \in m$  such that  $\mu_F(w) \neq f$  and  $\mu_F^{\mathbf{T}} = f$ . It remains to show that for each  $w \in m \cap \mu_F(f)$ , it holds that  $\mu_F^{\mathbf{T}}(w) = f$ . Given that  $w_M \in M$ , the demand of  $f$ , by the construction of  $\mathbf{T}$  and the optimality of  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ , we have that for each worker  $w \in m \cap \mu_F(f)$ , the condition  $\mu_F^{\mathbf{T}}(w) = f$  holds if

$$R_f^{\mathbf{T}}(w) - s_{w,f}^{\mathbf{T}} \geq R_f(w_M) - s_{w_M,f}.$$

We analyze this condition through two cases for each worker  $w \in m \cap \mu_F(f)$ :

- **Case 1:** If  $R_f(w_M) - s_{w_M,f} > R_f(w) - s_{w,f}$ , we have

$$t_{w,f} \geq R_f(w_M) - s_{w_M,f} - \left( R_f(w) - \max_{f' \in F \setminus \{f\}} \left( R_{f'}(w) + \sigma_{w,f'} - \max_{w' \in W_U \cup \{w_M\}} (R_{f'}(w') + \sigma_{w',f'}) \right) \right).$$

By Lemma 5, we know that

$$s_{w,f}^{\mathbf{T}} \equiv \max_{f' \in F \setminus \{f\}} \left( R_{f'}(w) + \sigma_{w,f'} - \max_{w' \in W_U \cup \{w_M\}} (R_{f'}(w') + \sigma_{w',f'}) \right)$$

is the salary of  $w$  when employed by  $f$ . Consequently, the profit generated by  $w$  at firm  $f$  can be expressed as:

$$R_f(w) - \max_{f' \in F \setminus \{f\}} \left( R_{f'}(w) + \sigma_{w,f'} - \max_{w' \in W_U \cup \{w_M\}} (R_{f'}(w') + \sigma_{w',f'}) \right).$$

Thus, we conclude that

$$R_f^{\mathbf{T}}(w) - s_{w,f}^{\mathbf{T}} \geq R_f(w_M) - u_{w_M}(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}),$$

implying that  $\mu_F(w_M) \neq f$ .

- **Case 2:** If  $R_f(w) - s_{w,f} \geq R_f(w_M) - u_{w_M}(\mu_F, \mathbf{s}_F)$ , then we have  $t_{w,f} = 0$  and

$$R_f^{\mathbf{T}}(w) - s_{w,f}^{\mathbf{T}} \geq R_f(w_M) - u_{w_M}(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}).$$

Therefore, for each  $w \in m \cap \mu_F(f)$ , we conclude that  $\mu_F^{\mathbf{T}}(w) = f$ .

From Lemma 3, we also know that there exists  $\mu_W^{\mathbf{T}}$  such that  $\mu_F^{\mathbf{T}} = \mu_W^{\mathbf{T}}$ , which completes the proof for  $\varphi_W$ . ■

## B.10 Proof of Theorem 6

*Proof.* From Lemma 1 we can we can decompose  $\mathbf{T}$  such that:

- $\mathbf{T}_1$  is such that for each  $w \in m_E$ , we define for each  $f \in F$ ,  $t_{1w,f} = t_{w,f}$ , and 0 for other elements. Since for each  $w \in m_E$ ,  $t_{1w, \mu_W(w)} = \max_{f \in F} (t_{1w,f}) \geq t_{m_U \rightarrow}$ , by Theorem 4, it follows that for each  $w \in m_E$ ,  $u_w(\mu_W^{\mathbf{T}_1}, \mathbf{s}_W^{\mathbf{T}_1}) = u_w(\mu_W, \mathbf{s}_W) + t_{1w, \mu_W(w)}$ .
- $\mathbf{T}_2$  is such that  $t_{2w_m, \mu_W(w_M)} = t_{m_U \rightarrow}$  and 0 for other elements. By Claim 10, it follows that  $\mu_W^{\mathbf{T}_2}(w_m) = \mu_W(w_M)$ . Additionally, from Lemma 2, for each  $w \in m_E$ , we have:

$$|u_w(\mu_W, \mathbf{s}_W) - u_w(\mu_W^{\mathbf{T}_2}, \mathbf{s}_W^{\mathbf{T}_2})| \leq t_{m_U \rightarrow}.$$

- $\mathbf{T}_3$  is such that  $t_{3w_M, \mu_W(w_M)} = t_{m_U, \mu_W(w_M)} - t_{2w_m, \mu_W(w_M)}$  and 0 for other elements. Considering  $\mathbf{T}_3$  alongside  $G = ((\mathbf{T}_1 + \mathbf{T}_2), \mathbf{q})$ , by Theorem 4 (i), we know that  $\mathbf{T}_3$  does not negatively impact minority workers under  $\varphi_W$ .

Hence,  $(\mathbf{T}_0, \mathbf{q}) \xrightarrow{\mathbf{T}_1} (\mathbf{T}_1, \mathbf{q})$  where  $\mathbf{T}_1$  increases the salary of all minority workers by  $t_{w, \mu_F(w)} \geq t_{m_U \rightarrow}$ . Next, we have  $(\mathbf{T}_1, \mathbf{q}) \xrightarrow{\mathbf{T}_2} ((\mathbf{T}_1 + \mathbf{T}_2), \mathbf{q})$  where  $\mathbf{T}_2$  reduces the utility of minority workers by at most  $t_{m_U \rightarrow}$ . Finally,  $((\mathbf{T}_1 + \mathbf{T}_2), \mathbf{q}) \xrightarrow{\mathbf{T}_3} ((\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3), \mathbf{q})$  where  $\mathbf{T}_3$  does not negatively impact minority workers under  $\varphi_W$ .

Thus, we conclude that  $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ , does not negatively impact minority workers under  $\varphi_W$ . Moreover, from Claim 10, we establish that  $\mathbf{T}$  favors the employment of minority workers under both  $\varphi_W$  and  $\varphi_F$ . ■

## B.11 Proof of Proposition 6

*Proof.* In this proof, we consider the firm-optimal stable allocations  $(\mu_F, \mathbf{s}_F)$  and  $(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}})$ . Let us consider the condition under which  $w_m$  is hired by  $\mu_F(w_M)$ . This condition is:

$$R_{\mu_F(w_M)}(w_m) + t_{w, \mu_F(w_M)} + \sigma_{w, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_M) - s_{w_M, \mu_F(w_M)}.$$

In this case, the profit of  $\mu_F(w_M)$  can increase at the firm-optimal stable allocation, and by stability, since  $w_m \in m_U$ ,  $w_m$  is hired by  $\mu_F(w_M)$ . Recall that  $w_M$  is the majority worker generating the lowest profit for firm  $\mu_F(w_M)$  under  $(\mu_F, \mathbf{s}_F)$ . First, we prove Claim 10, which asserts that subsidizing  $w_m$  by  $t_{m_U \rightarrow}$  leads  $\mu_F(w_M)$  to hire  $w_m$ , since the profit generated by  $w_m$  exceeds the profit generated by  $w_M$ , and therefore exceeds the lowest profit generated by any worker, whether majority or minority, at  $\mu_F(w_M)$ .

**Claim 10.**  $R_{\mu_F(w_M)}(w_m) + t_{m_U \rightarrow} + \sigma_{w_m, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_M) - s_{w_M, \mu_F(w_M)}$ .

*Proof.* Recall that  $t_{m_U \rightarrow} \equiv (R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)}) - (R_{\mu_F(w_M)}(w_m) + \sigma_{w_m, \mu_F(w_M)})$ . Thus, we need to show that:

$$R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_m) - s_{w_M, \mu_F(w_M)}.$$

Since  $u_{w_M}(\mu_F, \mathbf{s}_F) = \sigma_{w_M, \mu_F(w_M)} + s_{w_M, \mu_F(w_M)}$  and  $(\mu_F, \mathbf{s}_F)$  is stable, we know that  $u_{w_M}(\mu_F, \mathbf{s}_F) \geq 0$ . Thus, it follows that  $\sigma_{w_M, \mu_F(w_M)} \geq -s_{w_M, \mu_F(w_M)}$ , and consequently:

$$R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_m) - s_{w_M, \mu_F(w_M)}.$$

■

We now turn to proving that, given the transfer  $\mathbf{T}$ , each worker  $w \in m_E$  remains employed, i.e.,  $w \in m_E^{\mathbf{T}}$ .

**Claim 11.**  $u_{w_M}(\mu_F, \mathbf{s}_F) \leq t_{m_U \rightarrow}$ .

*Proof.* From Lemma 5, we know that for each  $w \in W$ ,

$$u_w(\mu_F, \mathbf{s}_F) = \max(0, \max_{f \in F \setminus \{\mu_F(w)\}} [(R_f(w) + \sigma_{w, f}) - \min_{w' \in \mu_F(f)} (R_f(w') - s_{w', f})]).$$

From Claim 10, we know that with a subsidy of  $t_{m_U \rightarrow}$ ,  $w_m$  is employed by  $\mu_F(w_M)$ . Since we consider the firm-optimal stable allocation, it follows that the utility distributed by  $\mu_F(w_M)$  to  $w_M$  is weakly lower than  $t_{m_U \rightarrow}$ . ■

To demonstrate that workers remain employed by the same firm as under  $(\mu_F, \mathbf{s}_F)$ , we examine the firm's profit and the surplus generated, namely  $R_f(w) + \sigma_{w,f}$ . We consider two cases for each  $w \in m_E$ :

- **Case 1:** If  $\mu_F(w) \neq \mu_F(w_M)$ : By stability, each worker is assigned to the firm where she generates the highest surplus, subject to firm capacity constraints. Since we are considering the firm-optimal stable allocation, firms achieve the highest profit. From Claim 11, we know that  $t_{m_U \rightarrow} \geq u_{w_M}(\mu_F, \mathbf{s}_F)$ . There is no firm  $f \in F$  such that  $w_M$  can increase the profit of  $f$  and obtain utility greater than  $u_{w_M}(\mu_F, \mathbf{s}_F)$ . We then differentiate between two sub-cases:
  - **Case 1.1:** If  $u_w(\mu_F, \mathbf{s}_F) - t_{m_U \rightarrow} \geq 0$ . In this case, the subsidy  $t_{m_U \rightarrow}$  reduces  $w$ 's utility, but it remains non-negative (from Lemma 2). Consequently,  $w$  continues to be employed by a firm, either by her original firm  $\mu_F(w)$  or, if her reduced utility makes another firm more attractive, by another firm i.e.,  $\mu_F^{\mathbf{T}}(w) \neq \emptyset$ , and  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq 0$ .
  - **Case 1.2:** If  $u_w(\mu_F, \mathbf{s}_F) - t_{m_U \rightarrow} < 0$ . The subsidy for  $w$  is given by  $t_{m_U \rightarrow} - (s_{w, \mu_F(w)} + \sigma_{w, \mu_F(w)})$ . By Lemma 5 and using the reasoning of Case 1.1,  $w$  is employed i.e.,  $\mu_F^{\mathbf{T}}(w) \neq \emptyset$ , and  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq 0$ .
- **Case 2:** If  $\mu_F(w) = \mu_F(w_M)$ : For  $w$  to remain employed, the surplus she generates must exceed that of  $w_M$ . That is:

$$R_{\mu_F(w_M)}(w) + t_{w, \mu_F(w_M)} + \sigma_{w, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)}.$$

All minority workers employed by  $\mu_F(w_M)$  who generate a surplus higher than  $w_M$  remain employed. For those generating lower surplus, the surplus must be increased to retain them. Additionally, each worker's utility must be positive. We again consider two sub-cases:

- **Case 2.1:** If  $R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w) + \sigma_{w, \mu_F(w_M)}$ , the subsidy for  $w$  is

$$t_{w, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)} - (R_{\mu_F(w_M)}(w) + \sigma_{w, \mu_F(w_M)}).$$

Therefore,  $R_{\mu_F(w_M)}^{\mathbf{T}}(w) + \sigma_{w, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)}$ . By optimality,  $w$  is employed i.e.,  $\mu_F^{\mathbf{T}}(w) \neq \emptyset$ , and  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq 0$ .

- **Case 2.2:** If  $R_{\mu_F(w_M)}(w) + \sigma_{w, \mu_F(w_M)} \geq R_{\mu_F(w_M)}(w_M) + \sigma_{w_M, \mu_F(w_M)}$ , the subsidy for  $w$  is  $t_{w, \mu_F(w_M)} \geq 0$ . Again, by optimality,  $w$  is employed i.e.,  $\mu_F^{\mathbf{T}}(w) \neq \emptyset$ , and  $u_w(\mu_F^{\mathbf{T}}, \mathbf{s}_F^{\mathbf{T}}) \geq 0$ .

From Lemma 3, we know that  $\mu_F = \mu_W$ , completing the proof for  $\varphi_W$ . ■

## B.12 Proof of Proposition 7

*Proof.* We know that  $(\mu_W, \mathbf{s}_W)$  is a worker-optimal stable allocation, and that for all  $w \in W$ , no worker receives a negative transfer, i.e.,  $t_{w,f} \geq 0$ . Consider a firm  $f \in F$  such that  $|\mu_W^{\mathbf{T}}(f) \cap m| > |\mu_W(f) \cap m|$ .

By Lemma 4, we know that for any pair of workers  $w, w' \in \mu_W(f)$ ,  $R_f(w) - s_{w,f} = R_f(w') - s_{w',f}$ . Therefore, there exist a worker  $w_m \notin \mu_W(f)$  such that  $w_m \in \mu_W^{\mathbf{T}}(f)$ , and the following inequality holds:

$$R_f(w_m) + \sigma_{w_m,f} - u_{w_m}(\mu_W, \mathbf{s}_W) < R_f(w) - s_{w,f},$$

where  $w_m$  is newly employed at  $f$  under the allocation  $\mu_W^{\mathbf{T}}$ . With the transfer  $\mathbf{T}$ , there exists a worker  $w \in \mu_W(f)$  such that  $w \notin \mu_W^{\mathbf{T}}(f)$ , and we have:

$$R_f^{\mathbf{T}}(w_m) - s_{w_m,f}^{\mathbf{T}} > R_f^{\mathbf{T}}(w) - s_{w,f}.$$

Since the transfer  $\mathbf{T}$  subsidizes minority workers, it follows that  $R_f^{\mathbf{T}}(w) \geq R_f(w)$ . By Lemma 4, for all workers  $w, w' \in \mu_W^{\mathbf{T}}(f)$ ,  $R_f^{\mathbf{T}}(w) - s_{w,f}^{\mathbf{T}} = R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}}$ . Thus, the profit of firm  $f$  has increased due to the transfer  $\mathbf{T}$ .

Now, consider the surplus generated by each worker  $w$  at firm  $f$ , with and without the transfer  $\mathbf{T}$ . The surplus is defined as  $R_f(w) + \sigma_{w,f}$  without the transfer, and as  $R_f(w) + t_{w,f} + \sigma_{w,f}$  with the transfer. For each  $w \in \mu_W(f)$ , the utility of  $w$  under the original allocation is:

$$u_w(\mu_W, \mathbf{s}_W) = R_f(w) + \sigma_{w,f} - (R_f(w) - s_{w,f}),$$

and for each  $w' \in \mu_W^{\mathbf{T}}(f)$ , the utility under the new allocation becomes:

$$u_{w'}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) = R_f(w') + t_{w',f} + \sigma_{w',f} - (R_f(w') + t_{w',f} - s_{w',f}^{\mathbf{T}}).$$

Since  $R_f^{\mathbf{T}}(w') - s_{w',f}^{\mathbf{T}} > R_f^{\mathbf{T}}(w) - s_{w,f}$ , it follows that:

$$u_{w'}(\mu_W^{\mathbf{T}}, \mathbf{s}_W^{\mathbf{T}}) < u_{w'}(\mu_W, \mathbf{s}_W) + t_{w',f}.$$

Thus,  $C_{\varphi_W}(\mathbf{T}) > C_{\varphi_W}^m(\mathbf{T})$ . ■

### B.13 Proof of Theorem 7

*Proof.* We first consider the worker-optimal stable allocation. By contradiction, suppose there exists a worker  $w \in m$  such that  $u_w(\mu_W, \mathbf{s}_W) > u_w(\mu_W^{\mathbf{q}^M}, \mathbf{s}_W^{\mathbf{q}^M})$ . Given  $(\mu_W, \mathbf{s}_W)$ , we know that there is no  $w' \in W$  such that  $\mu_W(w') \neq \mu_W(w)$  and there exists  $\mathbf{s}'$  with  $u_{w'}(\mu_W(w), s'_{w', \mu_W(w)}) \geq u_{w'}(\mu_W, \mathbf{s}_W)$  and  $V_{\mu_W(w)}((\mu_W(\mu_W(w)) \setminus \{w\}) \cup \{w'\}; \mathbf{s}', R_{\mu_W(w)}, \mathbf{q}_0^M) \geq V_{\mu_W(w)}(\mu_W(\mu_W(w)); \mathbf{s}_W, R_{\mu_W(w)}, \mathbf{q}_0^M)$ , with strict inequality holding for at least one member of  $\{w'\} \cup \{\mu_W(w)\}$ .

By introducing quotas, some majority workers can no longer be employed by a firm under the worker-optimal stable allocation. This increases competition among majority workers. However, since  $q_{\mu_W(w)} - q_{\mu_W(w)}^M \geq |\mu_W(\mu_W(w)) \cap m|$ , it follows that each firm can employ at least as many minority workers as before. Because  $(\mu_W^{\mathbf{q}^M}, \mathbf{s}_W^{\mathbf{q}^M})$  is the worker-optimal stable allocation, we know that there is no  $w' \in W$  such that  $\mu_W^{\mathbf{q}^M}(w') \neq \mu_W^{\mathbf{q}^M}(w)$  and there exists  $\mathbf{s}'$  with  $u_{w'}(\mu_W^{\mathbf{q}^M}(w), s'_{w', \mu_W^{\mathbf{q}^M}(w)}) \geq u_{w'}(\mu_W^{\mathbf{q}^M}, \mathbf{s}_W^{\mathbf{q}^M})$  and

$$V_{\mu_W^{\mathbf{q}^M}(w)}((\mu_W^{\mathbf{q}^M}(\mu_W^{\mathbf{q}^M}(w)) \setminus \{w\}) \cup \{w'\}; \mathbf{s}', R_{\mu_W^{\mathbf{q}^M}(w)}, \mathbf{q}^M) \geq V_{\mu_W^{\mathbf{q}^M}(w)}(\mu_W^{\mathbf{q}^M}(\mu_W^{\mathbf{q}^M}(w)); \mathbf{s}_W^{\mathbf{q}^M}, R_{\mu_W^{\mathbf{q}^M}(w)}, \mathbf{q}^M),$$

with strict inequality holding for at least one member of  $\{w'\} \cup \{\mu_W^{\mathbf{q}^M}(w)\}$ .

Thus, given  $(\mu_W, \mathbf{s}_W)$ , it follows that  $u_w(\mu_W^{\mathbf{q}^M}, \mathbf{s}_W^{\mathbf{q}^M}) \geq u_w(\mu_W, \mathbf{s}_W)$ .

Now we consider the firm-optimal stable allocation. By contradiction, suppose there exists a worker  $w \in m$  such that  $u_w(\mu_F, \mathbf{s}_F) > u_w(\mu_F^{\mathbf{q}^M}, \mathbf{s}_F^{\mathbf{q}^M})$ . Since  $w \in m$ , and knowing that  $q_{\mu_W(w)} - q_{\mu_W(w)}^M \geq |\mu_W(\mu_W(w)) \cap m|$ , and as  $R_{\mu_W(w)}(w) + \sigma_{w, \mu_W(w)}$  is the same in both markets  $(\mathbf{q}_0^M, \mathbf{q})$  and  $(\mathbf{q}^M, \mathbf{q})$ , it follows directly that  $u_w(\mu_F^{\mathbf{q}^M}, \mathbf{s}_F^{\mathbf{q}^M}) \geq u_w(\mu_F, \mathbf{s}_F)$ . ■

## References

- Bleemer, Z. (2022). Affirmative action, mismatch, and economic mobility after california's proposition 209. *The Quarterly Journal of Economics*, 137(1):115–160.
- Byrnes, P., Marvel, M. K., and Sridhar, K. (1999). An equilibrium model of tax abatement: City and firm characteristics as determinants of abatement generosity. *Urban Affairs Review*, 34(6):805–819.

- Coate, S. and Loury, G. C. (1993). Will affirmative-action policies eliminate negative stereotypes? *The American Economic Review*, pages 1220–1240.
- Demange, G. and Gale, D. (1985). The strategy structure of two-sided matching markets. *Econometrica: Journal of the Econometric Society*, pages 873–888.
- Derenoncourt, E. and Montialoux, C. (2021). Minimum wages and racial inequality. *The Quarterly Journal of Economics*, 136(1):169–228.
- Echenique, F. (2012). Contracts versus salaries in matching. *American Economic Review*, 102(1):594–601.
- Echenique, F., Miralles, A., and Zhang, J. (2021). Constrained pseudo-market equilibrium. *American Economic Review*, 111(11):3699–3732.
- Ehlers, L., Hafalir, I. E., Yenmez, M. B., and Yildirim, M. A. (2014). School choice with controlled choice constraints: Hard bounds versus soft bounds. *Journal of Economic theory*, 153:648–683.
- Gul, F., Pesendorfer, W., Zhang, M., et al. (2019). Market design and walrasian equilibrium. *Technical report, Working paper*.
- Gul, F. and Stacchetti, E. (1999). Walrasian equilibrium with gross substitutes. *Journal of Economic theory*, 87(1):95–124.
- Hatfield, J. W., Kominers, S. D., Nichifor, A., Ostrovsky, M., and Westkamp, A. (2019). Full substitutability. *Theoretical Economics*, 14(4):1535–1590.
- Hatfield, J. W. and Milgrom, P. R. (2005). Matching with contracts. *American Economic Review*, 95(4):913–935.
- Jencks, C. (1992). Rethinking social policy: Race, poverty, and the underclass. *Harvard University Press, Cambridge, Massachusetts*.
- Kamada, Y. and Kojima, F. (2024). Fair matching under constraints: Theory and applications. *Review of Economic Studies*, 91(2):1162–1199.
- Kelso, A. S. J. and Crawford, V. P. (1982). Job matching, coalition formation, and gross substitutes. *Econometrica: Journal of the Econometric Society*, pages 1483–1504.
- Kojima, F. (2012). School choice: Impossibilities for affirmative action. *Games and Economic Behavior*, 75(2):685–693.

- Kojima, F., Sun, N., and Yu, N. N. (2020). Job matching under constraints. *American Economic Review*, 110(9):2935–2947.
- Kojima, F., Sun, N., and Yu, N. N. (2024). Job matching with subsidy and taxation. *Review of Economic Studies*, 91(1):372–402.
- Kominers, S. D. and Sönmez, T. (2016). Matching with slot-specific priorities: Theory. *Theoretical Economics*, 11(2):683–710.
- Leonard, J. S. (1990). The impact of affirmative action regulation and equal employment law on black employment. *Journal of Economic Perspectives*, 4(4):47–63.
- Neumark, D., Wascher, W. L., et al. (2007). Minimum wages and employment. *Foundations and Trends® in Microeconomics*, 3(1–2):1–182.
- Roth, A. E. (1984). Stability and polarization of interests in job matching. *Econometrica: Journal of the Econometric Society*, pages 47–57.
- Roth, A. E. (1985). Conflict and coincidence of interest in job matching: some new results and open questions. *Mathematics of Operations Research*, 10(3):379–389.
- Shapley, L. S. and Shubik, M. (1971). The assignment game i: The core. *International Journal of game theory*, 1(1):111–130.
- Slattery, C. and Zidar, O. (2020). Evaluating state and local business incentives. *Journal of Economic Perspectives*, 34(2):90–118.
- Snower, D. J. (1994). Converting unemployment benefits into employment subsidies. *The American Economic Review*, 84(2):65–70.
- Sönmez, T., Yenmez, M. B., et al. (2019). *Affirmative action with overlapping reserves*. Boston College.
- Sotomayor, M. (1999). The lattice structure of the set of stable outcomes of the multiple partners assignment game. *International Journal of Game Theory*, 28:567–583.
- Sowell, T. (2004). *Affirmative action around the world: An empirical study*. Yale University Press.