



HAL
open science

An integrability criterion for a projective limit of Banach distributions

Fernand Pelletier

► **To cite this version:**

Fernand Pelletier. An integrability criterion for a projective limit of Banach distributions. Journal of Geometry and Physics, 2022, 174, pp.104453. 10.1016/j.geomphys.2022.104453 . hal-04752063

HAL Id: hal-04752063

<https://hal.science/hal-04752063v1>

Submitted on 13 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

AN INTEGRABILITY CRITERION FOR A PROJECTIVE LIMIT OF BANACH DISTRIBUTIONS

FERNAND PELLETIER

ABSTRACT. We give an integrability criterion for a projective limit of Banach distributions on a Fréchet manifold which is a projective limit of Banach manifolds. This leads to a result of integrability of projective limit of involutive bundles on a projective sequence of Banach manifolds. This can be seen as a version of Frobenius Theorem in Fréchet setting. As consequence, we obtain a version of the third Lie theorem for a Fréchet-Lie group which is a submersive projective limit of Banach Lie groups. We also give an application to a sequence of prolongations of a Banach Lie algebroid.

2010 MSC: 53C30, 53Z05, 46T05. *secondary* 18A30, 58B20, 58B25.

Keywords: Banach manifold, Fréchet manifold, projective limit of Banach manifolds, projective limit of Banach Lie groups, projective limit of Banach bundles, projective limit of Banach Lie algebroids, Fréchet bundles, integrability of projective limit of Banach distributions, sequence of prolongations of a Banach Lie algebroid, uniformly bounded endomorphisms of a Fréchet space.

1. INTRODUCTION

In classical differential geometry, a *distribution* on a smooth manifold M , is an assignment $\Delta : x \mapsto \Delta_x \subset T_x M$ on M , where Δ_x is a subspace of $T_x M$. This distribution is *integrable* if, for any $x \in M$, there exists an immersed submanifold $f : L \rightarrow M$ such that $x \in f(L)$ and for any $z \in L$, we have $Tf(T_z L) = \Delta_{f(z)}$. On the other hand, Δ is called *involutive* if, for any vector fields X and Y on M tangent to Δ , their Lie bracket $[X, Y]$ is also tangent to Δ .

On a finite dimensional manifold, when Δ is a subbundle of TM , the classical Frobenius Theorem gives an equivalence between integrability and involutivity. In the other cases, the distribution is *singular* and, even under assumptions of smoothness on Δ , in general, the involutivity is not a sufficient condition for integrability (one needs some more additional local conditions). These problems were clarified and resolved essentially in [20] and [19].

In the context of Banach manifolds, the Frobenius Theorem is again true for distributions which are complemented subbundles in the tangent bundle (cf. [16]). For singular Banach distributions closed and complemented (i.e. Δ_x is a complemented Banach subspace of $T_x M$) we also have the integrability property under some natural geometrical conditions (see [5] for instance). In a more general way, weak Banach distributions Δ (i.e. Δ_x which is a Banach subspace of $T_x M$ not necessary complemented), the integrability property is again true under some additional geometrical assumptions (see [18] or [4] for more details).

The proof of this last results is essentially based on the existence of the flow of a local vector field. In a more large infinite dimensional context as distributions on

convenient manifolds or on locally convex manifolds, in general the local flow for a vector field does not exist. Analog results exist in such previous settings: [12], [13], [8], [21], [2], for instance. But essentially, all these integrability criteria are proved under strong assumptions which, either need the existence of a family of vector fields which are tangent and generate locally a distribution and, each one of these vector fields have a local flow, or need the existence of an implicit function theorem in such a setting.

The purpose of this paper is to give an integrability criterion for projective limit of Banach distributions on a Fréchet manifold which is a projective limit of Banach manifolds. The precise assumptions on the distribution are presented in assumptions (*) in Definition 2.1) and this criterion is formulated in a local way in Theorem 2.2 and in a global way in Theorem 2.3. These results are obtained under conditions which permits to use a Theorem of existence and unicity of solution of ODE in a Fréchet space proved in [17], and which can be reformulated in the context of projective limit of Banach spaces (cf. Appendix C). Using such a Theorem, this proof needs, in the one hand, an adaptation of some arguments used in the proof of Theorem 1 in [18] for closed distributions on Banach manifolds, and on the other hand, some properties of the Banach Lie group of "uniformly bounded" automorphisms of a Fréchet space (cf. Appendix B). As application, this criterion permits to obtain a kind of *projective limit of "Banach Frobenius theorem"* for submersive projective limit of involutive bundles on a submersive projective sequence of Banach manifolds (cf. Theorem 2.4). By the way, as consequence, a submersive projective limit of complemented Banach Lie subalgebras of a submersive projective limit of Banach Lie group algebras is the Lie algebra of a Fréchet Lie group (cf. Theorem 3.1). This result can be considered as *a version of the third Lie Theorem for a Fréchet-Lie group* which is a submersive projective limit of Banach Lie groups. We also give an application to a sequence of prolongations of a Banach Lie algebroid (cf. Theorem 3.3). We complete these results, by an example of integrable Fréchet distribution which is a projective limit of non integrable distributions but which satisfies the assumptions (*).

This paper is organized as follows. The first paragraph of the next section describes the context and the assumptions of this criterion of integrability used in Theorem 2.2 and Theorem 2.3. In order to present these Theorems in a more accessible way for a quick reading, the useful definitions and results take place in Appendix A, and we formulate the assumptions with precise references to this Appendix. Theorem 2.3 allows to show that, under some natural conditions, the projective limit E of a submersive projective sequence of involutive subbundles E_i of the tangent bundle TM_i of a submersive projective sequence of Banach manifolds $(M_i)_{i \in \mathbb{N}}$, is a Fréchet involutive and integrable subbundle of the tangent bundle TM of the Fréchet manifold $M = \varprojlim M_i$.

A first application of these results, is the existence of a Fréchet Lie group whose Lie algebra is a submersive projective limit of complemented Banach Lie subalgebra of the a submersive projective limit of Banach Lie groups Lie algebras (cf. Theorem 3.1 in § 3.1). In § 3.2, we give an application of Theorem 2.2 to a sequence of prolongations of a Banach Lie algebroid (see [3]) and we end this paragraph by the announced counter-example of Theorem 2.2 and Theorem 2.3.

The proof of the basic Theorem 2.2 is located in §4.

All properties concerning the set of uniformly bounded endomorphisms of a Fréchet space are developed in Appendix B. Also in a series of other Appendices, we expose all the definitions and results needed in the statements of Theorems and in the proof of Theorem 2.2.

2. AN INTEGRABILITY CRITERION FOR A SUBMERSIVE PROJECTIVE LIMIT OF BANACH DISTRIBUTIONS

2.1. The criterion and its corollaries. *The context needed in this section is detailed in Appendix A.*

Let $(M_i, \delta_i^j)_{j \geq i}$ be a projective sequence Banach manifolds with projective limit $M = \varprojlim M_i$ ¹. In order to give a criterion of integrability for projective limits of local Banach bundles on M under some additional assumptions, we need to introduce some notations.

Let ν (resp. μ) be a norm on a Banach space \mathbb{E} (resp. \mathbb{M}). We denote by $\|\cdot\|^{\text{op}}$ the associated norm on the linear space of continuous linear mappings $\mathcal{L}(\mathbb{E}, \mathbb{M})$.

Then we have:

Definition 2.1. *Let $(M_i, \delta_i^j)_{j \geq i}$ be a projective sequence of Banach manifolds*

where the maps δ_i^j are submersions and $M = \varprojlim M_i$ its projective limit.

A closed distribution Δ on M will be called a submersive projective limit of local anchored bundles if the following property is satisfied:

(*) *For any $x = \varprojlim x_i \in M$, there exists an open neighbourhood $U = \varprojlim U_i$ of x , a submersive projective sequence of anchored Banach bundles $(E_i, \pi_i, U_i, \rho_i)$ ^{2 3 4} fulfilling the following properties for any $z = \varprojlim z_i \in U$:*

1. $\varprojlim \rho_i((E_i)_{z_i}) = \Delta_z$, for any $z \in U$.
2. *The kernel of $(\rho_i)_{z_i}$ is complemented in $(E_i)_{z_i}$ and the range of $(\rho_i)_{z_i}$ is closed, for all $i \in \mathbb{N}$.*
3. *There exists a constant $C > 0$ and a Finsler norm $\|\cdot\|^{E_i}$ (resp. $\|\cdot\|^{M_i}$) on $(E_i)_{|U_i}$ (resp. $TM_i|_{U_i}$) such that:*

$$\forall i \in \mathbb{N}, \|(\rho_i)_{z_i}\|_i^{\text{op}} \leq C, \forall z_i \in U_i.$$
⁵

We have the following criterion of integrability:

Theorem 2.2. *Let M be a projective limit of a submersive projective sequence $(M_i, \delta_i^j)_{j \geq i}$ of Banach manifolds and Δ be a local projective limit of local Banach bundles on M . Assume that, under the property (*), there exists a Lie bracket $[\cdot, \cdot]_i$ on $(E_i, \pi_i, U_i, \rho_i)$ such that $(E_i, \pi_i, U_i, \rho_i, [\cdot, \cdot]_i)$ is a submersive projective sequence of Banach Lie algebroids⁶.*

Then the distribution Δ is integrable and the maximal integral manifold N through

¹cf. section A.4

² **see:** Definition A.13 and Notations A.12 for a submersive projective sequence of Banach bundles

³ **see:** Definition A.15 for an anchored Banach bundle

⁴ a projective sequence of anchored bundle $(E_i, \pi_i, M_i, \rho_i)$ is a projective sequence of Banach bundles which satisfies assumption **(PSBLA 2)** in Definition A.16

⁵ More precisely, $\|(\rho_i)_{z_i}\|_i^{\text{op}} = \sup\{\|(\rho_i)_{z_i}(u)\|^{M_i}, \|u\|^{E_i} \leq 1\}$

⁶ cf. Definition A.16

$x = \varprojlim x_i$ is a closed Fréchet submanifold of M which is a projective limit of the set of maximal leaves N_i of $\rho_i(E_i)$ through x_i in M_i .

The proof of Theorem takes place in section 4.

Now, we have the following consequences of Theorem 2.2:

Theorem 2.3. *Let $(E_i, \pi_i, M_i, \rho_i, [\cdot, \cdot]_i)$ be a submersive projective sequence of split Lie algebroids ⁷. Then we have:*

1. $(E := \varprojlim E_i, \pi := \varprojlim \pi_i, M := \varprojlim M_i, \rho = \varprojlim \rho_i)$ is Fréchet anchored bundle and $\Delta = \rho(E)$ is a closed distribution on M
2. If (ρ_i) satisfies the condition (3) in Definition 2.1, then Δ is integrable and each leaf L of Δ is a projective limit of leaves L_i of Δ_i .

Proof. The property (1) is a consequence of Proposition A.17. From this property, it follows that locally Δ satisfies assumption (1) and (2) of Definition 2.1 so if the assumption (3) is satisfied, the result is a direct consequence of Theorem 2.2. \square

Theorem 2.4. *Let $(M_i, \delta_i^j)_{j \geq i}$ be a submersive projective sequence of Banach manifold and (E_i, π_i, M_i) an involutive subbundle of TM_i where π_i is the restriction of the natural projection $p_{M_i} : TM_i \rightarrow M_i$. Assume that the restriction $T\delta_i^j : E_j \rightarrow E_i$ is a surjective bundle morphism for all $i \in \mathbb{N}$ and $j \geq i$. Then (E_i, π_i, M_i) is a submersive projective sequence of Banach bundles, and $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ is an integrable Fréchet subbundle of TM whose each leaf L of E in M is a projective limit of leaves L_i of E_i in M_i*

Proof. Since $\delta_i^j : M_j \rightarrow M_i$ is a surjective submersion, so is $T\delta_i^j : TM_j \rightarrow TM_i$. If $T\delta_i^j : E_j \rightarrow E_i$ is a surjective morphism, this implies that $T\delta_i^j$ is a submersion onto E_i and so (E_i, π_i, M_i) is a submersive projective sequence of Banach bundles. Let $\iota_i : E_i \rightarrow TM_i$ the natural inclusion and $[\cdot, \cdot]_i$ the restriction of Lie bracket of vector fields to (local) sections of E_i . Then $(E_i, M_i, \iota_i, [\cdot, \cdot]_i)$ is a Banach Lie algebroid and since $T\delta_i^j \circ \iota_j = \iota_i \circ T\delta_i^j$ it follows that $(E_i, M_i, \iota_i, [\cdot, \cdot]_i)$ is a submersive projective sequence of Banach Lie algebroids. Fix some $x = \varprojlim x_i \in M = \varprojlim M_i$. According to Theorem 2.3 we have only to show that the condition (3) of Definition 2.1 is satisfied by ι_i .

Given any norm $\|\cdot\|^{M_i}$ on $T_{x_i}M_i$ we denote by $\|\cdot\|^{E_i}$ the induced norm on the fiber $\{E_i\}_{x_i}$, then, for the associated norm operator we have $\|\{\iota_i\}_{x_i}\|_i^{\text{op}} = 1$. So all the assumptions of Theorem 2.2 are satisfied which ends the proof. \square

3. SOME APPLICATIONS AND COUNTER-EXAMPLE

3.1. Application to submersive projective sequence of Banach Lie groups.

Let $(G_i, \delta_i^j)_{j \geq i}$ be a submersive projective sequence of Banach Lie groups where G_i is modeled on \mathbb{G}_i (cf. Definition A.7). We denote by $\mathbf{L}(G_i)$ the Lie algebra of G_i . Then $\mathbf{L}(G_i) \equiv T_e G_i$ is isomorphic to \mathbb{G}_i . If we set $\bar{\delta}_i^j := T_e \delta_i^j$, then each $\bar{\delta}_i^j$ is

⁷cf. Definition A.16

a surjective linear map from $\mathbf{L}(G_j)$ to $\mathbf{L}(G_i)$ whose kernel is complemented. Consider a sequence \mathfrak{h}_i of complemented Lie sub-algebra of $\mathbf{L}(G_i)$ such that the restriction of $\bar{\delta}_i^j$ to \mathfrak{h}_j is a continuous surjective map. Then $(\mathfrak{h}_i, \bar{\delta}_i^j)_{j \geq i}$ is a submersive projective sequence of Banach Lie algebra and so $\mathfrak{h} = \varprojlim \mathfrak{h}_i$ is a Fréchet Lie algebra (cf. [4] chapter 4). Now from classic results on Banach Lie groups (cf. [16]), by left translation each \mathfrak{h}_i gives rise to a complemented involutive subbundle of \mathcal{H}_i of TG_i and the leaf H_i through the neutral e_i in G_i has a structure of connected Banach Lie group so that the inclusion $\iota_i : H_i \rightarrow G_i$ is a Banach Lie morphism. Note that $\hat{\iota}_i := T_{e_i} \iota_i$ is nothing but else that the inclusion of \mathfrak{h}_i in $\mathbf{L}(G_i)$ and which induces the natural inclusion $\hat{\iota}_i$ of \mathcal{H}_i in TG_i .

Moreover, since $(\mathfrak{h}_i, \bar{\delta}_i^j)_{j \geq i}$ is a submersive projective sequence of Banach Lie algebra and $(G_i, \delta_i^j)_{j \geq i}$ be a submersive a projective sequence of Banach Lie groups, it follows that $(\mathcal{H}_i, G_i, \hat{\iota}_i, [\cdot, \cdot]_i)$ ⁸ is a submersive projective sequence of split Banach Lie algebroids.

On the other hand, from Theorem A.9 the Lie algebra $\mathbf{L}(G)$ of $G = \varprojlim G_i$ is $\varprojlim \mathbf{L}G_i$ and so $\mathfrak{h} = \varprojlim \mathfrak{h}_i$ is a closed complemented Lie subalgebra of $\mathbf{L}(G)$. As in the context of Banach setting, by left translation, \mathfrak{h} gives rise to an involutive Fréchet subbundle \mathcal{H} of TG which is clearly the projective limit of the submersive sequence $(\mathcal{H}_i, G_i, \hat{\iota}_i, [\cdot, \cdot]_i)$. So from Theorem 2.4 by same arguments as in Banach Lie groups, we obtain:

Theorem 3.1. *Let $G = \varprojlim G_i$ be the projective limit of a submersive sequence of Banach Lie groups $(G_i, \delta_i^j)_{j \geq i}$ and for each $i \in \mathbb{N}$, consider a closed complemented Banach Lie subalgebra \mathfrak{h}_i of $\mathbf{L}(G_i)$. Assume that the restriction of $\bar{\delta}_i^j$ to \mathfrak{h}_j is a continuous surjective map. Then there exists a Fréchet Lie group H in G such that $\mathbf{L}(H)$ is isomorphic to \mathfrak{h} and H is the projective limit of $(H_i, \delta_i^j|_{H_j})_{j \geq i}$.*

Remark 3.2. *The reader will also find an application of Theorem 2.2 in the proof of Theorem 8.23 in [4] for submersive projective limit of a sequence of Banach groupoids in [4] which is a kind of generalization of Theorem 3.1 to Lie groupoids setting .*

3.2. Application to sequences of prolongations of a Banach Lie algebroid over a Banach manifold. Consider an anchored Banach bundle $(\mathcal{A}, \pi, M, \rho)$ with typical fiber \mathbb{A} . Let $V\mathcal{A} \subset T\mathcal{A}$ be the vertical subbundle of $p_{\mathcal{A}} : T\mathcal{A} \rightarrow \mathcal{A}$. If $\mathcal{A}_x := \pi^{-1}(x)$ is the fiber over $x \in M$, according to [3], the prolongation $\mathbf{T}\mathcal{A}$ of the anchored Banach bundle $(\mathcal{A}, \pi, M, \rho)$ over \mathcal{A} is the set

$$\{(x, a, b, c), (x, b) \in \mathcal{A}_x, (x, a, c) \in V_{(x,a)}\mathcal{A}\}.$$

This set has a Banach vector bundle $\hat{\mathbf{p}} : \mathbf{T}\mathcal{A} \rightarrow \mathcal{A}$ structure over \mathcal{A} with typical fiber $\mathbb{A} \times \mathbb{A}$ and we have an anchor $\hat{\rho} : \mathbf{T}\mathcal{A} \rightarrow T\mathcal{A}$ given by

$$\hat{\rho}(x, a, b, c) = (x, a, \rho_x(b), c) \in T_{(x,a)}\mathcal{A}.$$

⁸here $[\cdot, \cdot]_i$ denote again the restriction to sections of \mathcal{H}_i of the Lie bracket of vector fields on G_i

From now on, we fix a Banach Lie algebroid $(\mathcal{A}, \pi, M, \rho, [\cdot, \cdot]_{\mathcal{A}})$ such that the typical fiber \mathbb{A} of \mathcal{A} is finite dimensional. By the way, we have a Banach Lie algebroid structure $(\mathbf{T}\mathcal{A}, \hat{\mathbf{p}}, \mathcal{A}, \hat{\rho}, [\cdot, \cdot]_{\mathbf{T}\mathcal{A}})$ (cf. [3] Corollary 44)

We denote $(\mathcal{A}_1, \pi_1, \mathcal{A}_0, \rho_1, [\cdot, \cdot]_1)$ the Banach Lie algebroid $(\mathcal{A}, \pi, M, \rho, [\cdot, \cdot]_{\mathcal{A}})$ over a Banach manifold $\mathcal{A}^0 = M$. Thus we have the following commutative diagram:

$$(3.1) \quad \begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\rho_1} & T\mathcal{A}_0 \\ & \searrow \pi_1 & \swarrow p_{\mathcal{A}_0} \\ & & \mathcal{A}_0 \end{array}$$

According to the notations of Theorem 43 and Corollary 44 in [3], we set $\mathcal{A}_2 = \mathbf{T}\mathcal{A}_1$, $\rho_2 = \hat{\rho}$, $[\cdot, \cdot]_2 = [\cdot, \cdot]_{\mathbf{T}\mathcal{A}_1}$ and $\pi_2 = \hat{\mathbf{p}}$. Then we have the following commutative diagram:

$$(3.2) \quad \begin{array}{ccccc} \mathcal{A}_2 & \xrightarrow{\rho_2} & T\mathcal{A}_1 & & \\ & \searrow \pi_2 & \swarrow p_{\mathcal{A}_1} & \searrow T\pi_1 & \\ & & \mathcal{A}_1 & \xrightarrow{\rho_1} & T\mathcal{A}_0 \\ & & \searrow \pi_1 & \swarrow p_{\mathcal{A}_0} & \\ & & & & \mathcal{A}_0 \end{array}$$

Fix some $x \in \mathcal{A}_0$ and a norm $\|\cdot\|_0$ (resp. $\|\cdot\|_1$) on the fiber $T_x\mathcal{A}_0 \equiv \mathbb{A}_0$ (resp. $\mathcal{A}_x = \pi_1^{-1}(x) \equiv \mathbb{A}_1$). Since the fiber $\mathbf{T}_{(x,a)}\mathcal{A}_1$ (resp. $T_{(x,a)}\mathcal{A}_1$) is isomorphic to $\mathbb{A}_1 \times \mathbb{A}_1$ (resp. $\mathbb{A}_0 \times \mathbb{A}_1$), it follows that $\sup\{\|\cdot\|_1, \|\cdot\|_1\}$ (resp. $\sup\{\|\cdot\|_0, \|\cdot\|_1\}$) gives rise to a norm on $\mathbf{T}_{(x,a)}\mathcal{A}_1$ (resp. $T_{(x,a)}\mathcal{A}_1$). Then for the associated operator norm $\|\cdot\|^{\text{op}}$ we have

$$(3.3) \quad \|(\rho_2)_{(x,a)}\|^{\text{op}} \leq \sup(\|(\rho_1)_x\|^{\text{op}}, 1)$$

By induction, for $i \geq 1$, again according to notations of Theorem 43 and Corollary 44 in [3], we set

$\mathcal{A}^{i+1} = \mathbf{T}^{\mathcal{A}_i}\mathcal{A}_i$, $\rho_{i+1} = \hat{\rho}_i$, $[\cdot, \cdot]_{i+1} = [\cdot, \cdot]_{\mathbf{T}^{\mathcal{A}_i}\mathcal{A}_i}$ and $\pi_{i+1} = \hat{\mathbf{p}}$ and, as before, we have the following commutative diagrams:

$$(3.4) \quad \begin{array}{ccccc} \mathcal{A}_{i+1} & \xrightarrow{\rho_{i+1}} & T\mathcal{A}_i & & \\ & \searrow \pi_{i+1} & \swarrow p_{\mathcal{A}} & \searrow T\pi_i & \\ & & \mathcal{A}_i & \xrightarrow{\rho_i} & T\mathcal{A}_{i-1} \\ & & \searrow \pi_i & \swarrow p_{\mathcal{A}_{i-1}} & \\ & & & & \mathcal{A}_{i-1} \end{array}$$

Also, by same arguments as for (3.3) we obtain:

$$(3.5) \quad \|(\rho_{i+1})_{(x,a_1,\dots,a_{i+1})}\|^{\text{op}} \leq \sup(\|(\rho_i)_{(x,a_1,\dots,a_i)}\|^{\text{op}}, 1)$$

It follows that we have a submersive projective sequence of Banach Lie algebroids $(\mathcal{A}_i, \mathcal{A}_{i-1}, \rho_i, [\cdot, \cdot]_i)$ over a submersive projective sequence of Banach manifolds $(\mathcal{A}_i)_{i \in \mathbb{N}}$ which satisfies the assumptions of Theorem 2.2. Thus, we obtain:

Theorem 3.3. *Under the previous context,*

$$(\mathcal{A} = \varprojlim_{i \geq 1} \mathcal{A}_i, \mathcal{M} = \varprojlim_{j \geq 0} \mathcal{A}_j, \rho = \varprojlim_{i \geq 1} \rho_i, [\cdot, \cdot] = \varprojlim_{i \geq 1} [\cdot, \cdot]_i)$$

is a Fréchet Lie algebroid on the Fréchet manifold \mathcal{M} and the distribution $\rho(\mathcal{A})$ is integrable. Each leaf L is a projective limit of a sequence of leaves of type (L_i) defined by induction in the following way:

L_1 is a leaf of $\rho_1(\mathcal{A}_1)$ and if L_i is a leaf of $\rho_i(\mathcal{A}_i)$ then $L_{i+1} = (\mathcal{A}_i)_{|L_i}$.

3.3. A counter-example. In this subsection we give an Example of an integrable distribution on a Fréchet bundle over a finite dimensional manifold which satisfies the assumptions (*) in Definition 2.1 but is a projective limit of a submersive sequence of Banach **not integrable** distributions.

Let $E = M \times \mathbb{R}^m$ the trivial bundle over a manifold M of dimension n . The set $J^k(E)$ of the k -jets of sections of E over M is a finite dimensional manifold which is a vector bundle $\pi^k : J^k(E) \rightarrow M$ and whose typical fiber is the space $\prod_{j=0}^k \mathcal{L}_{\text{sym}}^j(\mathbb{R}^n, \mathbb{R}^m)$ where $\mathcal{L}_{\text{sym}}^j(\mathbb{R}^n, \mathbb{R}^m)$ is the space of continuous j -linear symmetric mappings $\mathbb{R}^n \rightarrow \mathbb{R}^m$. Then each projection $\pi_k^l : J^l(E) \rightarrow J^k(E)$ defined, for $l \geq k$, by

$$\pi_k^l [j^l(s)(x)] = j^k(s)(x)$$

is a smooth surjection.

Proposition 3.4. ([4]) $(J^k(E), \pi_k^l)$ is a submersive projective sequence of Banach vector bundles and the projective limit

$$J^\infty(E) = \varprojlim J^k(E)$$

can be endowed with a structure of Fréchet vector bundle whose fiber is isomorphic to the Fréchet space $\prod_{j=0}^{\infty} \mathcal{L}_{\text{sym}}^j(\mathbb{R}^n, \mathbb{R}^m)$.

Let s be a section of π on a neighbourhood U of $x \in M$. For $\xi = j^k(s)(x) \in J^k(E)$, the n -dimensional subspace $R(s, x)$ of $T_\xi J^k(E)$ equals to the tangent space at ξ to the submanifold $j^k(s)(U) \subset J^k(\pi)$ is called an *R-plane*.

The Cartan subspace $\mathcal{C}^k(E)$ of $T_\xi J^k(E)$ is the linear subspace spanned by all R -planes $R(s', x)$ such that $j^k(s')(x) = \xi$. So it is the hull of the union of $(j^k(s))_*(x)(T_x M)$ where s is any local section of π around de x .

The set of Cartan subspaces defines a smooth distribution on $J^k(\pi)$ called *Cartan distribution* and denoted by \mathcal{C}^k . Then \mathcal{C}^k is a regular contact distribution and so is not integrable (cf. [14]). We have a projective limit of a submersive sequence of bundles $(\mathcal{C}^k, T\pi_k^l, j^k(E))$ whose projective limit $\mathcal{C} = \varprojlim \mathcal{C}^k$ is called the Cartan distribution on $J^\infty(E)$. In fact \mathcal{C} is integrable (cf. [14]). Note that since \mathcal{C}^k is a subbundle of $TJ^k(E)$, from the proof of Theorem 2.4, the condition (3) of Definition 2.1 is satisfied.

4. PROOF OF THEOREM 2.2

Fix some $x \in M$. According to the property (*) and the Definition 2.1, we can choose a submersive projective sequence of charts $(U_i, \delta_i^j|_{U_j})_{j \geq i}$ and a submersive projective sequence of Banach bundles $(E_i, \lambda_i^j)_{j \geq i}$, such that:

- $U = \varprojlim U_i$ is an open neighbourhood of x in M ;
- if $x = \varprojlim (x_i)$, each U_i is the contractile domain of a chart (U_i, ϕ_i) around x_i in M_i and $(U = \varprojlim (U_i), \phi = \varprojlim (\phi_i))$ is a projective limit chart in M around x .
- the projective sequence of Banach bundles $(E_i, \lambda_i^j)_{j \geq i}$ satisfies the assumption (*) on U .

Step 1: The kernel of ρ_x is complemented.

There exists a trivialization $\tau_i : E_i \rightarrow U_i \times \mathbb{E}_i$ which satisfies the compatibility condition:

$$(4.1) \quad (\delta_i^j \times \overline{\lambda_i^j}) \circ \tau_j = \tau_i \circ \lambda_i^j$$

where $(\mathbb{E}_i, \overline{\lambda_i^j})_{j \geq i}$ is the projective sequence of Banach spaces on which $(E_i, \lambda_i^j)_{j \geq i}$ is modeled.

Under these conditions, without loss of generality we may assume that, for each $i \in \mathbb{N}$, we have

- $x_i \equiv 0 \in \mathbb{M}_i$;
- U_i is an open subset of \mathbb{M}_i and so $TU_i = U_i \times \mathbb{M}_i$;
- $E_i = U_i \times \mathbb{E}_i$.

The projection δ_i^j at point $x_j \equiv 0$ (resp. λ_i^j in restriction to the fiber of E_j over $x_j \equiv 0$) is denoted by d_i^j (resp. ℓ_i^j). The morphism ρ_i in restriction to the fiber E_i over $x_i \equiv 0$ is denoted by r_i and ρ_x is denoted by r , so that $r = \varprojlim r_i$. Now, according to the context of Assumption 1 in Definition 2.1, we have the following result:

Lemma 4.1. *There exists a decomposition $\mathbb{E} = \ker r \oplus \mathbb{F}'$ with the following property:*

if (ν'_n) (resp. (μ_n)) is the gradation on \mathbb{F}' (resp. (μ_n) on \mathbb{M}) induced by the norm $\| \cdot \|_{\mathbb{E}_i}$ (resp. $\| \cdot \|_{\mathbb{M}_i}$) on $(E_i)_{x_i}$ (resp. $T_{x_i} M_i$), then the restriction of r to \mathbb{F}' is a closed uniformly bounded operator according to these gradations ⁹.

Proof. of Lemma 4.1

At first, in such a context we have $T_{x_j} \delta_i^j \equiv d_i^j$ and the following compatibility condition:

$$(4.2) \quad d_i^j \circ r_j = r_i \circ \ell_i^j.$$

We set $\mathbb{F}_i = r_i(\mathbb{E}_i)$ for all $i \in \mathbb{N}$. From Definition 2.1 assumption 1, \mathbb{F}_i is a Banach subspace of \mathbb{E}_i and there exists a decomposition $\mathbb{E}_i = \ker r_i \oplus \mathbb{F}'_i$. Thus the restriction r'_i of r_i to \mathbb{F}'_i is an isomorphism onto \mathbb{F}_i . Now, from (4.2), we have

$$\forall (i, j) \in \mathbb{N}^2 : j > i, \quad d_i^j \circ r'_j = r'_i \circ \ell_i^j.$$

⁹cf. Appendix B

But since each r'_i is an isomorphism, the restriction $(\ell'_i)^j$ of ℓ'_i to \mathbb{F}'_j takes values in \mathbb{F}'_i for all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$. Moreover, as δ'_i is surjective, according to (4.2) again, this implies that $d'_i(\mathbb{F}'_j) = \mathbb{F}'_i$ and so $\ell'_i(\mathbb{F}'_j) = \mathbb{F}'_i$. Since $(\mathbb{E}_i, \ell'_i)_{j \geq i}$ is a projective sequence, this implies that $(\mathbb{F}'_i, (\ell'_i)^j)_{j \geq i}$ is a surjective projective sequence. The vector space $\mathbb{F}' = \varprojlim \mathbb{F}'_i$ is then a Fréchet subspace of \mathbb{E} .

On the other hand, let $(\ell'_i)''$ be the restriction of ℓ'_i to $\ker r_j$. Always from (4.2), we have

$$\forall (i, j) \in \mathbb{N}^2 : j > i, (\ell'_i)''(\ker r_j) \subset \ker r_i.$$

By same argument, it implies that $(\ker r_i, (\ell'_i)'')_{j \geq i}$ is a projective sequence and $\ker r = \varprojlim \ker r_i$. Moreover, since $\mathbb{E}_i = \ker r_i \oplus \mathbb{F}'_i$, it follows that $\mathbb{E} = \ker r \oplus \mathbb{F}'$ and also the restriction r' of r to \mathbb{F}' is obtained as $r' = \varprojlim r'_i$ and r' is an injective continuous operator $r' : \mathbb{F}' \rightarrow \mathbb{M}$ whose range is the closed subspace $\mathbb{F} = \varprojlim \mathbb{F}_i$. It remains to show that r' is uniformly bounded.

According to the context of assumption 2 of Definition 2.1, there exists a constant $C > 0$ and, for each $i \in \mathbb{N}$, we have a norm $\|\cdot\|^{\mathbb{E}_i}$ on \mathbb{E}_i and a norm $\|\cdot\|^{\mathbb{M}_i}$ on \mathbb{M}_i such that $\|r_i\|_i^{\text{op}} \leq C$. As \mathbb{F}'_i is a closed Banach subspace of \mathbb{E}_i , it follows that for the induced norm on \mathbb{F}'_i we have

$$(4.3) \quad \forall i \in \mathbb{N}, \|r'_i\|_i^{\text{op}} \leq C.$$

Set $\ell_i = \varprojlim \ell'_i$ and $d_i = \varprojlim d'_i$. By construction we have $\ell_i(\mathbb{F}') = \mathbb{F}'_i$ and $d_i(\mathbb{M}) = \mathbb{M}_i$. The norm $\|\cdot\|^{\mathbb{E}_i}$ on \mathbb{E}_i induces a norm $\|\cdot\|^{\mathbb{F}'_i}$ on the Banach subspace \mathbb{F}'_i and we get a natural gradation (ν'_i) on \mathbb{F}' given by $\nu'_i(u) = \|\ell_i(u)\|^{\mathbb{F}'_i}$ (cf. Appendix B (B.1)). In the same way, the norm $\|\cdot\|^{\mathbb{M}_i}$ induces a gradation (μ_i) on \mathbb{M} given by $\mu_i(v) = \|d_i(v)\|^{\mathbb{M}_i}$. Now (4.3) implies that r' is uniformly bounded and $r'(\mathbb{F}') = r(\mathbb{E}) = \Delta_x$ is closed by assumption. Therefore, the proof of Lemma 4.1 is complete. \square

Step 2: There exists a neighbourhood $V \subset U$ of x such that the map $\rho' = \rho|_{U \times \mathbb{F}'}$ takes values in $\mathcal{I}\mathcal{H}_b(\mathbb{F}', \mathbb{M})^{10}$ and is K -Lipschitz on V for some $K > 0$.

Since $E_i = U_i \times \mathbb{E}_i$, it follows that $E = U \times \mathbb{E}$ and so $\rho : E \rightarrow TU$ can be seen as a smooth map from U into $\mathcal{H}(\mathbb{E}, \mathbb{M})$. Let ρ' be the restriction of ρ to $U \times \mathbb{F}'$ and so consider ρ' as a smooth map from U to $\mathcal{H}(\mathbb{F}', \mathbb{M})$. From the definition of a Finsler norm, Assumption 2 in Definition 2.1 and Lemma 4.1, the map $x \mapsto \rho'_x$ takes value in $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$.

Lemma 4.2. *There exists a neighbourhood $V_1 \subset V$ of 0 such that the map $\rho' : V_1 \rightarrow \mathcal{H}_b(\mathbb{F}', \mathbb{M})$ is a Lipschitz map, that is:*

there exists $K > 0$ such that

$$\hat{\mu}_i^{\text{op}}(\rho'_z - \rho'_x) \leq K \hat{\mu}_i(z - z'), \quad \forall (z, z') \in V_1^2$$

11

Proof. of Lemma 4.2 Let $(\hat{\nu}'_i)_{i \in \mathbb{N}}$ (resp. $(\hat{\mu}_i)_{i \in \mathbb{N}}$) be the canonical increasing gradation associated to the gradation $(\nu'_i)_{i \in \mathbb{N}}$ on \mathbb{M} (resp. $(\mu_i)_{i \in \mathbb{N}}$) (cf. Appendix (B B.1)). Since the map $x \mapsto \rho'_x$ is a smooth map from U to $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ it follows

¹⁰cf. Appendix B

¹¹cf. Remark B.2

that for each x the differential map $d_x\rho'$ is a continuous linear map from \mathbb{M} to the Banach space $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ and so there exists $i_0 \in \mathbb{N}$ and a constant $A_x > 0$ such that

$$(4.4) \quad \|d_x\rho'(u)\|_\infty \leq A_x \hat{\nu}'_{i_0}(u)$$

for all $u \in \mathbb{F}'$ and so

$$(4.5) \quad \|d_x\rho'(u)\|_\infty \leq A_x \hat{\nu}'_i(u)$$

for all $u \in \mathbb{F}'$ and and $i \geq i_0$ and according to Remark B.2 we set

$$(4.6) \quad \|d_x\rho'\|_i^{\text{op}} := \sup\{\hat{\mu}_i(d_x\rho'(u)) : \hat{\nu}'_i(u) \leq 1\} \leq A_x, \forall i \geq i_0$$

On the other hand, for $1 \leq i < i_0$ we set $C_x^i = \|d_x\rho'\|_i^{\text{op}}$. Then $d_x\rho'$ belongs to $\mathcal{H}_b(\mathbb{M}, \mathcal{H}_b(\mathbb{F}', \mathbb{M}))$ for all $x \in U$ since we have

$$\|d_x\rho'\|_\infty := \sup_{i \in \mathbb{N}} \|d_x\rho'\|_i^{\text{op}} \leq \sup\{A_x, C_x^1, \dots, C_x^{i_0-1}\}.$$

We set $C = \|d_0\rho'\|_\infty$. By continuity, there exists an open neighbourhood V_1 of 0 such that

$$(4.7) \quad \|d_x\rho'\|_\infty \leq 2C$$

By choosing $K = 2C$, from the definition of $\|d_x\rho'\|_\infty$ it implies the announced result. \square

Step 3: Local flow of the vector field $X_u = \rho'(u)$.

Consider a neighbourhood V as announced in step 2. As δ_i is a surjective map, $V_i = \delta_i(V)$ is an open set of U_i and so we have $V = \varprojlim V_i$. For each $u \in \mathbb{F}'$, let $X_u = \rho'(u)$ be the vector field on V . If $u = \varprojlim u_i$ with $u_i \in \mathbb{F}'_i$, then $X_{u_i} = \rho'_i(u_i)$ is a vector field on V_i and $X_u = \varprojlim X_{u_i}$. Now since ρ' takes values in $\mathcal{I}\mathcal{H}_b(\mathbb{F}', \mathbb{M})$, from our assumption, there exists a constant $C > 0$ such that $\|(\rho'_i)_{z_i}\|_i^{\text{op}} \leq C$ for all $z_i \in V_i$. Therefore, if u belongs to \mathbb{F}' and $u_i = \lambda_i(u)$, then $\|X_{u_i}\|^{\mathbb{M}_i} \leq C\|u\|_i^{\mathbb{E}}$ and, from Lemma 4.2 and definition of $\hat{\mu}_i^{\text{op}}$ we have

$$(4.8) \quad \forall (x_i, x'_i) \in (\delta_i(V))^2, \|X_{u_i}(x_i) - X_{u_i}(x'_i)\|^{\mathbb{M}_i} \leq K\|u_i\|_i^{\mathbb{E}}\|x_i - x'_i\|^{\mathbb{M}_i}.$$

Now, recall that we have provided \mathbb{F}' and \mathbb{M} with seminorms $(\hat{\nu}'_n)$ and $(\hat{\mu}_n)$ respectively defined by

$$\nu_n(u) = \|\lambda_i(u)\|_i^{\mathbb{E}_i} \quad \text{and} \quad \mu_n(x) = \|\delta_i(x)\|^{\mathbb{M}_i}.$$

Since $\delta_i(X_u)(x) = X_{u_i}(\delta_i(x))$ and in this way, from (4.8), for all $n \in \mathbb{N}$, we have

$$(4.9) \quad \forall (x, x') \in V^2, \mu_n(X_u(x) - X_u(x')) \leq K\nu_n(u)\mu_n(x - x').$$

Therefore, X_u satisfies the assumption of Corollary C.2. Let $\epsilon > 0$ such that the pseudo-ball

$$B_{\mathbb{M}}(0, 2\epsilon) = \{x \in \mathbb{M}, : \hat{\mu}_{n_i}(x) < 2\epsilon, 1 \leq i \leq k\}$$

is contained in V and set

$$C_1 := \max_{1 \leq i \leq k} \{K\nu_{n_i}(u)\} = K \max_{1 \leq i \leq k} \nu_{n_i}(u);$$

$$C_2 := \sup_{x \in B_{\mathbb{M}}(0, \epsilon)} \left\{ \max_{1 \leq i \leq k} \mu_{n_i}(X(x)) \right\} \leq C \max_{1 \leq i \leq k} \nu_{n_i}(u).$$

By application of Corollary C.2, for any u such that $\max_{1 \leq i \leq k} \nu_{n_i}(u) \leq 1$, there exists $\alpha > 0$ such that

$$\alpha e^{2\alpha K} \leq \frac{\epsilon}{2C},$$

such that the local flow Fl_t^u is defined on the pseudo-ball $B_{\mathbb{M}}(0, \epsilon)$ for all $t \in [-\alpha, \alpha]$ for all u which satisfied the previous inequality

Note that for any $s \in \mathbb{R}$ we have $X_{su} = sX_u$. Therefore, from the classical properties of a flow of a vector field,

there exists $\eta > 0$ such that the local flow Fl_t^u is defined on $[-1, 1]$, for all u in the open pseudo-ball

$$B_{\mathbb{F}'}(0, \eta) := \{u \in \mathbb{F}, : \nu'_{n_i}(u) \leq \eta, \ 1 \leq i \leq k\}.$$

(cf. for instance proof of Corollary 4.2 in [5]).

We set $B_{\mathbb{M}_i}(0, \epsilon) = \delta_i(B_{\mathbb{M}}(0, \epsilon))$. Then X_{u_i} is a vector field on $V_i = \delta_i(V)$ and $\text{Fl}_t^{u_i} := \delta_i \circ (\text{Fl}_t^u) \circ \delta_i$ is the local flow of X_{u_i} which is defined on $B_{\mathbb{M}_i}(0, \epsilon)$ for all $t \in [-1, 1]$ and $\text{Fl}_t^{u_i}(x_i)$ belongs to V_i for all $x_i \in B_{\mathbb{M}_i}(0, \epsilon)$ and $t \in [-1, 1]$ and, from (4.10), we have

$$(4.10) \quad \text{Fl}_t^u = \varprojlim \text{Fl}_t^{u_i}.$$

Step 4: Existence of an integral manifold.

Since $(E_i, \pi_i, U_i, \rho_i, [\cdot, \cdot]_i)$ is a Banach Lie algebroid, the Lie bracket $[X_{u_i}, X_{u'_i}]$ is tangent to Δ_i and so by Definition 3.2 and Lemma 3.6 in [18] we have

$$\forall t \in [-1, 1], \ (T \text{Fl}_t^{u_i})(\Delta_i)_{x_i} = (\Delta_i)_{\text{Fl}_t^{u_i}(x_i)}.$$

Therefore, according to the notations used at the end of step 3, for each $i \in \mathbb{N}$, we set:

$$\forall u_i \in B_{\mathbb{F}'_i}(0, \eta) = \lambda_i(B_{\mathbb{F}'}(0, \eta)), \ \Phi_i(u_i) = (\text{Fl}_1^{u_i})(0);$$

$$\forall u \in B_{\mathbb{F}'}(0, \eta), \ \Phi(u) = (\text{Fl}_1^u)(0);.$$

Lemma 4.3. $\Phi = \varprojlim \Phi_i$ is smooth and there exists $0 < \eta' \leq \eta$ such that the restriction of Φ to $B_{\mathbb{F}'}(0, \eta')$ is injective and $T_u \Phi$ belongs to $\mathcal{I}\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ for all $u \in B_{\mathbb{F}'}(0, \eta')$.

Recall that, for each $i \in \mathbb{N}$, $B_{\mathbb{F}'_i}(0, \eta')$ is an open ball in \mathbb{F}'_i and $\delta_i(V)$ is open neighbourhood of $0 \in \mathbb{M}_i$. From Lemma 4.3, since Φ is injective and each differential map $T_u \Phi$ is injective, it follows that the same is true for each $\Phi_i : B_{\mathbb{F}'_i}(0, \eta') \rightarrow \delta_i(V)$. Thus we can apply the proof of Theorem 1 in [18] for Φ_i . By the way, $\mathcal{U}_i = \Phi_i(B_{\mathbb{F}'_i}(0, \eta'))$ is an integral manifold of $\rho_i(\mathbb{F}'_i)$ and $(\mathcal{U}_i, \Phi_i^{-1})$ is a (global) chart for this integral manifold modeled on \mathbb{F}'_i . As $\Phi = \varprojlim \Phi_i$, $B_{\mathbb{F}'}(0, \eta') = \varprojlim B_{\mathbb{F}'_i}(0, \eta')$ $\Phi_i \circ \lambda_i^j = \delta_i^j \circ \Phi_j$ for all $j \geq i$, it follows that $(\mathcal{U}_i, \delta_i^j)_{j \geq i}$ is a surjective projective sequence and so $\mathcal{U} = \varprojlim \mathcal{U}_i$ is a Fréchet manifold modeled on \mathbb{F}' . This last result clearly **ends the proof of Theorem 2.2.**

Proof. of Lemma 4.3

According to step 3, Φ_i is well defined on $B_{\mathbb{F}'_i}(0, \eta)$ and $\Phi = \varprojlim \Phi_i$.

Now, for every $x \in V$, ρ'_x belongs to $\mathcal{I}\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ and so is injective; after shrinking V , if necessary, there exists some $M > 0$ such that

$$(4.11) \quad \forall x \in V, \|\rho'_x\|_\infty \leq M.$$

Now, by construction, we have $T_0\Phi(u) = \rho'_0(u)$ and so $T_0\Phi$ is injective.

Claim 4.1. *The map $u \mapsto T_u\Phi$ is a smooth map from $B'(0, \eta)$ to $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$.*

Proof. of Claim 4.1

We will use some argument of the proof of Lemma 2.12 of [18]. We fix the index i and, for any $y \in B_{\mathbb{F}'_i}(0, \epsilon)$, $v \in \mathbb{F}'_i$ we set

$$\begin{aligned} &: X_v(y) = \rho'_i(y, v) \\ &: \varphi(t, v) = \text{Fl}_t^{X_v}(0) \\ &: A(t) = \partial_1 \rho'_i(\varphi(t, v), v) \text{ (partial derivative relative to the first variable)} \\ &: B(t) = \rho'_i(\varphi(t, v), \cdot) \end{aligned}$$

Note that A and B are smooth fields on $[0, 1]$ of operators in $\mathcal{L}(\mathbb{M}_i, \mathbb{M}_i)$ and $\mathcal{L}(\mathbb{F}'_i, \mathbb{M}_i)$ respectively. Therefore the differential equation

$$\dot{S} = A \circ S + B$$

has a unique solution S_t with initial condition $S_0 = 0$ given by

$$(4.12) \quad S_t = \int_0^t G_{t-s} \circ B(s) ds$$

where G_t is the unique solution of

$$\dot{G} = A \circ G$$

with initial condition $G_0 = \text{Id}_{\mathbb{M}}$. Given by

$$(4.13) \quad G_t = \text{Id}_{\mathbb{M}} + \int_0^t A \circ G_s ds.$$

Under these notations, from [6], Chapter X § 7, we have $\partial_2 \varphi(t, v)(\cdot) = S_t$.

On the one hand, from the choice of ϵ , $\varphi(t, v)$ belongs to V_i and so from (4.7), we have $\|A(t)\|_i^{\text{op}} \leq K$ for any $t \in [-1, 1]$ and from (4.11) we have $\|B(t)\|_i^{\text{op}} \leq M$. Thus from (4.13), using Gronwal equality, we obtain

$$\|G_t\|_i^{\text{op}} \leq e^K,$$

And so from (4.12) we get

$$\|S_t\|_i^{\text{op}} \leq M e^K$$

This implies that

$$\|\partial_2 \varphi(t, v)\|_i^{\text{op}} \leq M e^K.$$

We set $M_1 = M e^K$. Since $\Phi_i(u_i) = \varphi(1, u_i)$ it follows that:

$$(4.14) \quad \|T_{u_i} \Phi(v_i)\|^{M_i} \leq M_1 \|v_i\|^{E_i}$$

from (4.14) we obtain:

$$(4.15) \quad \mu_i(T_u\Phi(v)) \leq M_1\nu_i(v).$$

But, since $T_u\Phi(\mathbb{F}') = \Delta_{\Phi(u)} \subset \{\Phi(u)\} \times \mathbb{M}$, it follows that the map $u \mapsto T_u\Phi$ can be considered as a continuous linear map from \mathbb{F}' to \mathbb{M} which takes values in $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$. Now as each Φ_i is a smooth map from $B_{\mathbb{F}'_i}(0, \eta')$ to $\delta_i(V)$ and $\Phi = \varprojlim \Phi_i$, this implies that Φ is a smooth map on $B_{\mathbb{F}'}(0, \eta') = \varprojlim B_{\mathbb{F}'_i}(0, \eta')$ to $V = \varprojlim \delta_i(V)$ which ends the proof of the Claim. \square

End of the proof of Lemma 4.2.

At first, from the Claim 4.1, the map $u \mapsto T_u\Phi$ takes values in the Banach space $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$, as in step 2 for ρ , we can show that this map is Lipschitz map on $B_{\mathbb{F}'}(0, \eta)$ for η small enough. As $T_0\Phi = \rho'_0$, from Proposition B.4, it follows that, again for η small enough, $T\Phi$ is injective on $B_{\mathbb{F}}(0, \eta')$, and we have (cf. (4.14))

$$(4.16) \quad \forall u \in B_{\mathbb{F}}(0, \eta'), \forall v \in \mathbb{F}' \quad \mu_i(T_u\Phi(v)) \leq M_1\nu_i(v)$$

using the fact that the range of $T_u\Phi$ is always closed for $u \in B_{\mathbb{F}'}(0, \eta')$. Moreover, for $u \in B_{\mathbb{F}'}(0, \eta')$, as for the relation (B.5) in the proof of Proposition B.4, we obtain:

$$(4.17) \quad \frac{1}{\ell_u} \nu_i(v) \leq \mu_i(T_u\Phi(v)) \leq \ell_u \cdot \nu_n(v)$$

for all $i \in \mathbb{N}$, where $\ell_u = \|T_u\Phi\|_\infty \leq M_1$.

Finally we obtain:

$$(4.18) \quad \forall i \in \mathbb{N}, \quad \frac{1}{M_1} \nu_i(v) \leq \mu_i(T_u\Phi(v)) \leq M_1 \cdot \nu_i(u)$$

Suppose that, for any $0 \leq \eta' \leq \eta$, the restriction of each Φ_i to $B_{\mathbb{F}'_i}(0, \eta)$ is not injective. Consider any pair $(u, v) \in [B_{\mathbb{F}'}(0, \eta)]^2$ such that $u \neq v$ but $\Phi(u) = \Phi(v)$, we set $h = v - u$. For any $\alpha \in \mathbb{M}^*$, we consider the smooth curve $c_\alpha : [0, 1] \rightarrow \mathbb{R}$ defined by:

$$c_\alpha(t) = \langle \alpha, (\Phi(u + th) - \Phi(u)) \rangle.$$

Of course, we have $\dot{c}_\alpha(t) = \langle \alpha, T_{u+th}\Phi(h) \rangle$.

Denote by $]u, v[$ the set of points $\{w = u + th, t \in]0, 1[\}$. As we have $c_\alpha(0) = c_\alpha(1) = 0$, from Rolle's Theorem, there exists $u_\alpha \in]u, v[$ such that

$$(4.19) \quad \langle \alpha, \delta_l(T_{u_\alpha}\Phi(h)) \rangle = 0$$

Note that, for any $t \in \mathbb{R}$, this relation is also true for any th . From our assumption, it follows that, for each $k \in \mathbb{N} \setminus \{0\}$, there exists u_k and v_k in $B_{\mathbb{F}}(0, \frac{\eta}{k})$ so that $u_k \neq v_k$ but with $\Phi(u_k) = \Phi(v_k)$. So from the previous argument, for any $\alpha \in \mathbb{M}^*$, we have

$$(4.20) \quad \langle \alpha, (T_{u_{\alpha,k}}\Phi(h_k)) \rangle = 0$$

for some $u_{\alpha,k} \in]u_k, v_k[$ and $h_k = v_k - u_k$. From (4.20), for any $i \in \mathbb{N}$, any $t \in \mathbb{R}$ and any $\alpha_i \in \mathbb{M}^*_i$, if $\alpha = \delta_i^*(\alpha_i)$, we have:

$$| \langle \alpha, T_0\Phi(th_k) \rangle | = | \langle \alpha_i, \delta_i([T_0\Phi - T_{u_\alpha}\Phi](th_k)) \rangle | = | \langle \alpha_i, [T_0\Phi_i - T_{\delta_i(u_\alpha)}\Phi_i](\lambda_i(th_k)) \rangle |$$

Denote by $\|\cdot\|^{\mathbb{M}_i^*}$ the canonical norm on \mathbb{M}_i^* associated to $\|\cdot\|^{\mathbb{M}_i}$. Then from (4.20) and since $u \mapsto T_u\Phi$ is K_1 -Lipschitz on $B'(0, \eta)$ (for some constant K_1) we obtain:

$$(4.21) \quad \begin{aligned} | \langle \alpha, T_0\Phi(th_k) \rangle | &\leq (\|\alpha_i\|^{\mathbb{M}_i^*} \cdot K_1 \cdot \|\delta_i(u_\alpha)\|^{\mathbb{M}_i} \cdot \|\lambda_i(th_k)\|^{\mathbb{E}_i}) \\ &\leq (\|\alpha_i\|^{\mathbb{M}_i^*} \cdot K_1 \cdot \|\lambda_i(th_k)\|^{\mathbb{E}_i} \frac{\eta}{k}). \end{aligned}$$

Since $u_k \neq v_k$, there must exist at least one integer $i \in \mathbb{N}$ such that $\lambda_i(h_k) \neq 0$. Thus, by taking $t = \frac{u_k - v_k}{v_i(u_k - v_k)}$ in (4.21), we may assume $t = 1$ and $\|\lambda_i(h_k)\|^{\mathbb{E}_i} = 1$.

In this way, for this choice of h_k , we have a 1-form $\bar{\beta}_{k,i}$ on the linear space generated by $\lambda_i(h_k)$ in \mathbb{F}'_i such that $\langle \bar{\beta}_{k,i}, \lambda_i(h_k) \rangle = 1$ and with $(\|\bar{\beta}_{k,i}\|^{\mathbb{E}_i})^* = 1$. From the Hahn-Banach Theorem, we can extend this linear form to a form $\beta_{k,i} \in \mathbb{M}_i^*$ such that $\langle \beta_{k,i}, \lambda_i(h_k) \rangle = 1$ and $(\|\beta_{k,i}\|^{\mathbb{E}_i})^* = 1$. But since each $T_0\Phi$ is injective, this implies that $T_0\Phi_i$ is injective and so the adjoint $T_0^*\Phi_i$ is surjective. This implies that there exists $\alpha_{k,i} \in \mathbb{M}_i^*$ such that $T_0^*\Phi_i(\alpha_{k,i}) = \beta_{k,i}$. Thus from (4.21) we obtain

$$(4.22) \quad \begin{aligned} 1 = | \langle \beta_{k,i}, \lambda_i(h_k) \rangle | &= | \langle \alpha_{k,i}, T_0\Phi_i(\lambda_i(h_k)) \rangle | \\ &= | \langle \delta_i^* \alpha_{k,i}, T_0\Phi(h_k) \rangle | \leq \|\alpha_{k,i}\|^{\mathbb{M}_i^*} K_1 \frac{\eta}{k} \end{aligned}$$

But, on the other hand since the operation "adjoint" is an isometry, from (4.18) we have

$$(4.23) \quad \frac{1}{M_1} \|\alpha_{k,i}\|^{\mathbb{M}_i^*} \leq \|T_0^*\Phi\alpha_{k,i}\|^{\mathbb{M}_i^*} = 1$$

which gives a contradiction with (4.22) for k large enough. \square

APPENDIX A. PROJECTIVE LIMITS

A.1. Projective limits of topological spaces.

Definition A.1. A projective sequence of topological spaces is a sequence

$$\left((X_i, \delta_i^j) \right)_{(i,j) \in \mathbb{N}^2, j \geq i} \quad \text{where}$$

(PSTS 1): For all $i \in \mathbb{N}$, X_i is a topological space;

(PSTS 2): For all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, $\delta_i^j : X_j \rightarrow X_i$ is a continuous map;

(PSTS 3): For all $i \in \mathbb{N}$, $\delta_i^i = Id_{X_i}$;

(PSTS 4): For all $(i, j, k) \in \mathbb{N}^3$ such that $k \geq j \geq i$, $\delta_i^j \circ \delta_j^k = \delta_i^k$.

Notation A.2. For the sake of simplicity, the projective sequence $\left((X_i, \delta_i^j) \right)_{(i,j) \in \mathbb{N}^2, j \geq i}$

will be denoted by $\left(X_i, \delta_i^j \right)_{j \geq i}$.

Remark A.3. Since we have a countable sequence $\left(X_i, \delta_i^j \right)_{j \geq i}$ of topological spaces, according to the properties of bonding maps, the sequence $\left(\delta_i^j \right)_{j \geq i}$ is well defined by the sequence of bonding maps $\left(\delta_i^{i+1} \right)_{i \in \mathbb{N}}$.

An element $(x_i)_{i \in \mathbb{N}}$ of the product $\prod_{i \in \mathbb{N}} X_i$ is called a *thread* if, for all $j \geq i$, $\delta_i^j(x_j) = x_i$.

Definition A.4. *The set $X = \varprojlim X_i$ of all threads, endowed with the finest topology for which all the projections $\delta_i : X \rightarrow X_i$ are continuous, is called the projective limit of the sequence $(X_i, \delta_i^j)_{j \geq i}$.*

A basis of the topology of X is constituted by the subsets $(\delta_i)^{-1}(U_i)$ where U_i is an open subset of X_i (and so δ_i is open whenever δ_i is surjective).

Definition A.5. *Let $(X_i, \delta_i^j)_{j \geq i}$ and $(Y_i, \gamma_i^j)_{j \geq i}$ be two projective sequences whose respective projective limits are X and Y .*

A sequence $(f_i)_{i \in \mathbb{N}}$ of continuous mappings $f_i : X_i \rightarrow Y_i$, satisfying, for all $(i, j) \in \mathbb{N}^2$, $j \geq i$, the coherence condition

$$\gamma_i^j \circ f_j = f_i \circ \delta_i^j$$

is called a projective sequence of mappings.

The projective limit of this sequence is the mapping

$$\begin{aligned} f : X &\rightarrow Y \\ (x_i)_{i \in \mathbb{N}} &\mapsto (f_i(x_i))_{i \in \mathbb{N}} \end{aligned}$$

The mapping f is continuous if all the f_i are continuous (cf. [1]).

A.2. Projective limits of Banach spaces. According to [7], a projective sequence $(\mathbb{E}_i, \delta_i^j)_{j \geq i}$ of Banach spaces is a projective sequence of topological spaces $(\mathbb{E}_i, \delta_i^j)_{j \geq i}$ such that the bonding maps δ_i^j are continuous linear maps.

A.3. Projective limits of differentiable maps. The following proposition (cf. [9], Lemma 1.2 and [4], Chapter 4) is essential

Proposition A.6. *Let $(\mathbb{E}_i, \delta_i^j)_{j \geq i}$ be a projective sequence of Banach spaces whose projective limit is the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}_i$ and $(f_i : \mathbb{E}_i \rightarrow \mathbb{E}_i)_{i \in \mathbb{N}}$ a projective sequence of differentiable maps whose projective limit is $f = \varprojlim f_i$. Then the following conditions hold:*

- (1) *f is smooth in the convenient sense (cf. [15])*
- (2) *For all $x = (x_i)_{i \in \mathbb{N}}$, $df_x = \varprojlim (df_i)_{x_i}$.*
- (3) *$df = \varprojlim df_i$.*

A.4. Projective limits of Banach manifolds and Banach Lie groups.

Definition A.7. [9] *The projective sequence $(M_i, \delta_i^j)_{j \geq i}$ is called projective sequence of Banach manifolds if*

(PSBM 1): *M_i is a manifold modeled on the Banach space \mathbb{M}_i ;*

(PSBM 2): we have a projective sequence of Banach spaces $(\mathbb{M}_i, \overline{\delta}_i^j)_{j \geq i}$ such that:

(PSBM 3): for all $x = (x_i) \in M = \varprojlim M_i$, there exists a projective sequence of local charts $(U_i, \phi_i)_{i \in \mathbb{N}}$ such that $x_i \in U_i$ where one has the relation

$$\phi_i \circ \delta_i^j = \overline{\delta}_i^j \circ \phi_j;$$

(PSBM 4): $U = \varprojlim U_i$ is a non empty open set in M .

Under the assumptions **(PSBM 1)** and **(PSBM 2)** in Definition A.7, the assumptions **(PSBM 3)** and **(PSBM 4)** around $x \in M$ are called *the projective limit chart property* around $x \in M$ and $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ is called a *projective limit chart*.

The projective limit $M = \varprojlim M_i$ has a structure of Fréchet manifold modeled on the Fréchet space $\mathbb{M} = \varprojlim \mathbb{M}_i$ and is called a *PLB-manifold*. The differentiable structure is defined *via* the charts $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$:

- ϕ is a homeomorphism (as projective limit of homeomorphisms) ;
- for any pair $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ and $(V = \varprojlim V_i, \psi = \varprojlim \psi_i)$ of such charts, the associated transition map $(\psi \circ \phi^{-1})|_{\phi(U)} = \varprojlim \left((\psi_i \circ (\phi_i)^{-1})|_{\phi_i(U_i)} \right)$ is a smooth diffeomorphism (in the convenient sense) between appropriate open sets of Fréchet space \mathbb{M} .

Definition A.8. [9] $(G_i, \delta_i^j)_{j \geq i}$ is a projective sequence of Banach Lie groups where G_i is modeled on \mathbb{G}_i if, for all $i \in \mathbb{N}$, there exists a chart (U_i, ϕ_i) around the unity e_i in G_i such that:

(PLBLG 1): $\forall (i, j) \in \mathbb{N}^2 : j \geq i, \delta_i^j(U_j) \subset U_i$;

(PLBLG 2): there exists a projective sequence of Banach spaces $(\mathbb{G}_i, \overline{\delta}_i^j)_{j \geq i}$ such that:

$$\forall (i, j) \in \mathbb{N}^2 : j \geq i, \overline{\delta}_i^j \circ \phi_j = \phi_j \circ \delta_i^j;$$

(PLBLG 3): $\varprojlim \phi_i(U_i)$ is a non empty open set of \mathbb{G} and $\varprojlim U_i$ is open in G according to the projective limit topology.

A projective sequence of Banach Lie groups $(G_i, \delta_i^j)_{j \geq i}$ is *submersive* if each δ_i^j is a surjective submersion.

Theorem A.9. [9] Let $G = \varprojlim G_i$ be a the projective limit of a sequence of Banach Lie groups $(G_i, \delta_i^j)_{j \geq i}$. Then we have the following properties:

- (1) G is a Fréchet Lie group.
- (2) If $\mathbf{L}(G_i)$ is the Lie algebra of G_i then $\mathbf{L}(G) = \varprojlim \mathbf{L}G_i$.
- (3) If \exp_{G_i} is the exponential map for G_i , then $\exp_G = \varprojlim \exp_{G_i}$ is the exponential map of the Fréchet Lie group G .

A.5. Projective limits of Banach vector bundles. Let $(M_i, \delta_i^j)_{j \geq i}$ be a projective sequence of Banach manifolds where each manifold M_i is modeled on the Banach space \mathbb{M}_i .

For any integer i , let (E_i, π_i, M_i) be the Banach vector bundle whose typical fiber is the Banach vector space \mathbb{E}_i .

Definition A.10. $((E_i, \pi_i, M_i), (\xi_i^j, \delta_i^j))_{j \geq i}$, where $\xi_i^j : E_j \rightarrow E_i$ is a morphism of vector bundles, is called a *projective sequence of Banach vector bundles on the projective sequence of manifolds* $(M_i, \delta_i^j)_{j \geq i}$ if there exists a projective sequence of Banach spaces $(\mathbb{E}_i, \lambda_i^j)_{j \geq i}$ such that:

for all (x_i) , we have a projective sequence of trivializations (U_i, τ_i) of (E_i, π_i, M_i) , where $\tau_i : (\pi_i)^{-1}(U_i) \rightarrow U_i \times \mathbb{E}_i$ are local diffeomorphisms, where U_i is an open neighbourhood of x_i in M_i and where $U = \varprojlim U_i$ is a non empty open set in M and, for all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, the following compatibility condition is satisfied

$$\text{(PLBVB): } (\delta_i^j \times \lambda_i^j) \circ \tau_j = \tau_i \circ \xi_i^j.$$

With the previous notations, $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ is called a *projective bundle chart limit*. The triple of projective limits $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ is called a *projective limit of Banach bundles* or **PLB**-bundle for short.

The following proposition generalizes the result of [10] about the projective limit of tangent bundles to Banach manifolds (cf. [7] and [4]).

Proposition A.11. Let $((E_i, \pi_i, M_i), (\xi_i^j, \delta_i^j))_{j \geq i}$ be a projective sequence of Banach vector bundles.

Then $(\varprojlim E_i, \varprojlim \pi_i, \varprojlim M_i)$ is a Fréchet vector bundle.

Notation A.12. For the sake of simplicity, the projective sequence $((E_i, \pi_i, M_i), (\xi_i^j, \delta_i^j))_{j \geq i}$ will be denoted by (E_i, π_i, M_i) .

Definition A.13. A sequence (E_i, π_i, M_i) is called a *submersive projective sequence of Banach vector bundles* if $(M_i, \delta_i^j)_{j \geq i}$ is a submersive projective sequence of Banach manifolds and if around each $x \in M = \varprojlim M_i$, there exists a projective limit chart bundle $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ such that for all $i \in \mathbb{N}$, we have a decomposition $\mathbb{E}_{i+1} = \ker \bar{\lambda}_i^{i+1} \oplus \mathbb{E}'_i$ such that the condition **(PLBVB)** is true.

The projective limit (E, π, M) of a sequence of Banach vector bundles (E_i, π_i, M_i) is called a *submersive projective limit of Banach bundles* or *submersive PLB*-bundle for short.

Now, we have the following result:

Proposition A.14. Let (E_i, π_i, M_i) be a submersive projective sequence of Banach bundles. Then, for each $i \in \mathbb{N}$, the maps $\delta_i = \varprojlim_{j \geq i} \delta_i^j : M \rightarrow M_i$ and $\xi_i = \varprojlim_{j \geq i} \xi_i^j : E \rightarrow E_i$ are submersions.

A.6. Projective limit of Banach Lie algebroids.

Definition A.15. Let $\pi : E \rightarrow M$ be a Banach bundle.

- (1) an anchor is a vector bundle morphism $\rho : E \rightarrow TM$ and (E, ρ) is called an anchored bundle
- (2) An almost Lie bracket $[\cdot, \cdot]_E$ on an anchored bundle E is a sheaf of anti-symmetric bilinear maps

$$[\cdot, \cdot]_{E_U} : \Gamma(E_U) \times \Gamma(E_U) \rightarrow \Gamma(E_U)$$

for any open set $U \subseteq M$ and which satisfies the following properties

(AL 1) the Leibniz identity:

$$\forall (\mathbf{a}_1, \mathbf{a}_2) \in \Gamma(E_U)^2, \forall f \in C^\infty(M), [\mathbf{a}_1, f\mathbf{a}_2]_{E_U} = f \cdot [\mathbf{a}_1, \mathbf{a}_2]_{E_U} + df(\rho(\mathbf{a}_1)) \cdot \mathbf{a}_2.$$

(AL 2) For any open set $U \subseteq M$ and any $(\mathbf{a}_1, \mathbf{a}_2) \in \Gamma(E_U)^2$, the map

$$(\mathbf{a}_1, \mathbf{a}_2) \mapsto [\mathbf{a}_1, \mathbf{a}_2]_{E_U}$$

only depends on the 1-jets of \mathbf{a}_1 and \mathbf{a}_2 .

- (3) An anchored bundle (E, ρ) provided with an almost Lie bracket $[\cdot, \cdot]_E$ which satisfies the Jacobi identity

$$[[\mathbf{a}_1, \mathbf{a}_2]_E, \mathbf{a}_3]_E + [[\mathbf{a}_2, \mathbf{a}_3]_E, \mathbf{a}_1]_E + [[\mathbf{a}_3, \mathbf{a}_1]_E, \mathbf{a}_2]_E = 0$$

$$\forall (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \Gamma(E_U)^3$$

is called a Lie algebroid.

Definition A.16. $(E_i, \pi_i, M_i, \rho_i, [\cdot, \cdot]_i)$ is called a submersive projective sequence of split Lie algebroids if

(PSBLA 1): $(E_i, \xi_i^j)_{j \geq i}$ is a submersive projective sequence of Banach vector bundles $(\pi_i : E_i \rightarrow M_i)_{i \in \mathbb{N}}$ over the projective sequence of manifolds $(M_i, \delta_i^j)_{j \geq i}$;

(PSBLA 2): For all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, one has

$$\rho_i \circ \xi_i^j = T\delta_i^j \circ \rho_j$$

(PSBLA 3): For all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, one has

$$\xi_i^j([\cdot, \cdot]_j) = [\xi_i^j(\cdot), \xi_i^j(\cdot)]_i$$

(PSBLA 4): For all $i \in \mathbb{N}$ and $x_i \in M$ the kernel $\ker(\rho_i)_{x_i}$ is complemented in the fiber E_{x_i} .

Proposition A.17. ([4]) Let $(E_i, \pi_i, M_i, \rho_i, [\cdot, \cdot]_i)$ be a submersive projective sequence of split Lie algebroids. Then $(E := \varprojlim E_i, \pi := \varprojlim \pi_i, M := \varprojlim M_i, \rho = \varprojlim \rho_i)$ is Fréchet anchored bundle and $\Delta = \rho(E)$ is a closed distribution on E

Remark A.18. Under the assumptions of Proposition A.17, unfortunately $[\cdot, \cdot] = \varprojlim [\cdot, \cdot]_i$ does not define a Lie bracket on the set of all local sections of (E, π, M) but only on sections which are projective limits of sections of (E_i, π_i, M_i) . Therefore $(E, \pi, M, \rho, [\cdot, \cdot])$ does not have a Fréchet Lie algebroid structure in general.

APPENDIX B. THE BANACH SPACE $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$

Any Fréchet space \mathbb{F} can be realized as the limit of a surjective projective sequence of Banach spaces $(\mathbb{B}_n, \lambda_n^m)_{m \geq n}$. Following [7], 2.3, we can identify \mathbb{F} with the projective limit of the sequence

$$\left(\widehat{\mathbb{B}}_n = \left\{ x = (x_i) \in \prod_{0 \leq i \leq n} \mathbb{B}_i : \forall j \geq i, x_i = \lambda_i^j(x_j) \right\}, \widehat{\lambda}_n^m = (\lambda_n^m, \dots, \lambda_n^m) \right)_{m \geq n}.$$

We denote by $\lambda_n : \mathbb{F} \rightarrow \mathbb{B}_n$ and $\widehat{\lambda}_n : \mathbb{F} \rightarrow \widehat{\mathbb{B}}_n$ the canonical surjective projections. Let $(\|\cdot\|_n)_{n \in \mathbb{N}}$ be a sequence where $\|\cdot\|_n$ is a norm on \mathbb{B}_n . In this way,

$$\widehat{\|\cdot\|}_n = \sup_{0 \leq i \leq n} \|\cdot\|_i$$

defines a norm on $\widehat{\mathbb{B}}_n$. Then

$$(B.1) \quad \hat{\nu}_n = \widehat{\|\cdot\|}_n \circ \widehat{\lambda}_n \quad (\text{resp. } \nu_n = \|\cdot\|_n \circ \lambda_n)$$

is the semi-norm on \mathbb{F} associated to the sequence $(\widehat{\|\cdot\|}_n)$ (resp. $(\|\cdot\|_n)$). Moreover, we have $\hat{\nu}_n = \max_{0 \leq i \leq n} \nu_i$ and the topology of \mathbb{F} is defined by $(\hat{\nu}_n)$ or (ν_n) .

Let (\mathbb{F}_1, ν_n^1) (resp. (\mathbb{F}_2, ν_n^2)) be a graded Fréchet space.

Recall that a linear map $L : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ is *continuous* if

$$\forall n \in \mathbb{N}, \exists k_n \in \mathbb{N}, \exists C_n > 0 : \forall x \in \mathbb{F}_1, \nu_2^n(L.x) \leq C_n \nu_1^{k_n}(x).$$

The space $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ of continuous linear maps between both these Fréchet spaces generally drops out of the Fréchet category. Indeed, $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ is a Hausdorff locally convex topological vector space whose topology is defined by the family of semi-norms $\{p_{n,B}\}$:

$$p_{n,B}(L) = \sup_{x \in B} \{\nu_n^2(L.x)\}$$

where $n \in \mathbb{N}$ and B is any bounded subset of \mathbb{F}_1 . This topology is not metrizable since the family $\{p_{n,B}\}$ is not countable.

So $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ will be replaced, under certain assumptions, by a projective limit of appropriate functional spaces as introduced in [10].

We denote by $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$ the space of linear continuous maps (or equivalently bounded linear maps because \mathbb{B}_1^n and \mathbb{B}_2^n are normed spaces). We then have the following result ([7], Theorem 2.3.10).

Theorem B.1. *The space of all continuous linear maps between \mathbb{F}_1 and \mathbb{F}_2 which can be represented as projective limits*

$$\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2) = \left\{ (L_n) \in \prod_{n \in \mathbb{N}} \mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n) : \varprojlim L_n \text{ exists} \right\}$$

is a Fréchet space.

For this sequence $(L_n)_{n \in \mathbb{N}}$ of linear maps, for any integer $0 \leq n \leq m$, the following commutative diagram:

$$\begin{array}{ccc} \mathbb{B}_1^n & \xleftarrow{(\delta_1)_n^m} & \mathbb{B}_1^m \\ L_n \downarrow & & \downarrow L_m \\ \mathbb{B}_2^n & \xleftarrow{(\delta_2)_n^m} & \mathbb{B}_2^m \end{array}$$

On $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$, the topology can be defined by the sequence of seminorms p_n given by

$$p_n(L) = \max_{0 \leq k \leq n} \sup \{ \nu_k^2(L.x), x \in \mathbb{F}_1, \nu_k^1(x) = 1 \}$$

so that $(\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2), p_n)$ is a graded Fréchet space.

Remark B.2. For $l \in \{1, 2\}$, given a gradation (ν_n^l) on a Fréchet space \mathbb{F}_l , let \mathbb{B}_l^n be the associated local Banach space and $\delta_l^n : \mathbb{F}_l \rightarrow \mathbb{B}_l^n$ the canonical projection. The quotient norm $\tilde{\nu}_n^l$ associated to ν_n^l is defined by

$$(B.2) \quad \tilde{\nu}_n^l(\delta_n(z)) = \sup \{ \nu_n^l(y) : \delta_n(y) = \delta_n(z) \}.$$

We denote by $(\tilde{\nu}_n^2)^{\text{op}}$ the corresponding operator norm on $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$.

If $L = \varprojlim L_n$ where $L_n : \mathbb{B}_1^n \rightarrow \mathbb{B}_2^n$, then we have

$$(\tilde{\nu}_n^2)^{\text{op}}(L) = \sup \{ \tilde{\nu}_n^2(L_n.x), x \in \mathbb{B}_1^n, \tilde{\nu}_n^1(x) \leq 1 \} = \sup \{ \nu_n^2(L.x), x \in \mathbb{F}_1, \nu^1(x) \leq 1 \}.$$

This implies that

$$p_n(L) = \max_{0 \leq i \leq n} (\tilde{\nu}_i^2)^{\text{op}}(L_n).$$

Definition B.3. Let (\mathbb{F}_1, ν_n^1) and (\mathbb{F}_2, ν_n^2) be graded Fréchet spaces. A linear map $L : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ is called a uniformly bounded operator, if

$$\exists C > 0 : \forall n \in \mathbb{N}, \nu_n(L(x)) \leq C \mu_n(x).$$

We denote by $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ the set of uniformly bounded operators. Of course $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ is contained in $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ and $L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ belongs to $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ if and only if $\|L\|_\infty := \sup_{n \in \mathbb{N}} p_n(L) < \infty$ and so

$$\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2) = [\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)]_b := \{L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2) : \|L\|_\infty < \infty\}$$

When $\mathbb{F} = \mathbb{F}_1 = \mathbb{F}_2$ and $\nu_n^1 = \nu_n^2$ for all $n \in \mathbb{N}$, the set $\mathcal{H}(\mathbb{F}, \mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F}, \mathbb{F})$) is simply denoted by $\mathcal{H}(\mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F})$).

We denote by $\mathcal{IH}_b(\mathbb{F}_1, \mathbb{F}_2)$ (resp. $\mathcal{SH}_b(\mathbb{F}_1, \mathbb{F}_2)$) the set of injective (resp. surjective) operators of $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ with closed range.

Proposition B.4. [4]

- (1) Each operator $L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ has a closed range if and only if, for each $n \in \mathbb{N}$, the induced operator $L_n : \mathbb{B}_1^n \rightarrow \mathbb{B}_2^n$ has a closed range.
- (2) $\mathcal{IH}_b(\mathbb{F}_1, \mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$.
- (3) $\mathcal{SH}_b(\mathbb{F}_1, \mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$.

We will give the sketch of the proof of Point (2) since some arguments used in this proof are also useful for the proof of Theorem 2.2:

Proof. (2) Consider an injective operator $L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$. According to the representation $\mathbb{F}_i = \varprojlim \mathbb{B}_i^n$ as a projective limit of a Banach sequence $(\mathbb{B}_i^n, (\delta_i)_n^m)_{m \geq n}$, we have a sequence of linear operators $L_n : \mathbb{B}_1^n \rightarrow \mathbb{B}_2^n$ such that $L = \varprojlim L_n$ (cf. Theorem B.1). Considering each

$$\mathbb{F}_i = \{(x_n) \in \prod_{n \in \mathbb{N}} \mathbb{B}_i^n : \forall m \geq n, x_n = (\delta_2)_n^m(x_m)\},$$

if $x = (x_n) \in \mathbb{F}_1$, then $L(x) = (L_n(x_n)) \in \mathbb{F}_2$. Thus it is clear that L is injective if and only if L_n is injective for all $n \in \mathbb{N}$.

Now if $L \in \mathcal{I}\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$, then L_n is an isomorphism from \mathbb{B}_1^n onto its range and so we have

$$(B.3) \quad \frac{1}{\ell_n} \tilde{\nu}_n^1(x) \leq \tilde{\nu}_n^2(L_n(x)) \leq \ell_n \tilde{\nu}_n^1(x)$$

for all $x \in \mathbb{B}_1^n$, all $n \in \mathbb{N}$, where $\tilde{\nu}_n^i$ is the quotient norm of ν_n^i on \mathbb{B}_i^n for $i \in 1, 2$, and

$$\ell_n = (\tilde{\nu}_n^2)^{\text{op}}(L) = \sup\left\{\frac{\tilde{\nu}_n^2(L_n(x))}{\tilde{\nu}_n^1(x)} : \tilde{\nu}_n^1(x) \neq 0\right\}.$$

Since δ_n^2 is the canonical projection of \mathbb{F}_2 on \mathbb{B}_2^n and $\nu_n^2 \circ \delta_n = \tilde{\nu}_n^2$, we obtain

$$(B.4) \quad \frac{1}{\ell_n} \nu_n^1(x) \leq \nu_n^2(L(x)) \leq \ell_n \nu_n^1(x)$$

for all $x \in \mathbb{F}_1$ and $n \in \mathbb{N}$. But we have $\ell_n \leq \|L\|_\infty$ and we finally obtain

$$(B.5) \quad \frac{1}{\ell} \nu_n^1(x) \leq \nu_n^2(L(x)) \leq \ell \nu_n^1(x)$$

for all $x \in \mathbb{F}_1$, all $n \in \mathbb{N}$ and where $\ell = \|L\|_\infty$.

Fix some $L \in \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ and set $\ell = \|L\|_\infty$, we consider the open set

$$W = \{T \in \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2), : \|T - L\|_\infty < \frac{\ell}{2}\}$$

Fix some $n \in \mathbb{N}$. For any $x \in \mathbb{F}_1$ and $T \in W$, we have

$$\nu_n^1(x) - \nu_n^1(T(x)) \leq \nu_n^1(T - L)(x) \leq p_n(T - L) \cdot \nu_n^1(x) \leq \|T - L\|_\infty \cdot \nu_n^1(x) \leq \frac{\ell}{2} \nu_n^1(x).$$

This implies that

$$(B.6) \quad \nu_n^2(T(x)) \geq \frac{\ell}{2} \nu_n^1(x).$$

Since (ν_n^i) is a separating sequence of semi-norms, it follows that T is injective.

Now taking in account inequality (B.6) and relation $\tilde{\nu}_n^i = \nu_n^i \circ (\delta_i)_n$, for $T \in W$ and each $n \in \mathbb{N}$, we have

$$\tilde{\nu}_n^2(T_n(x)) \leq \frac{3\ell}{2} \nu_n^1(x) \leq 3\tilde{\nu}_n^2(T_n(x))$$

for all $x \in \mathbb{B}_1^n$. It follows that T_n is closed and so T is closed (cf. 1.). Finally, W is an open neighbourhood of L contained in $\mathcal{I}\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$, which ends the proof of (2). □

From this Proposition one can obtain:

Theorem B.5. ([4])

- (1) The Banach space $\mathcal{H}_b(\mathbb{F})$ has a Banach Lie algebra structure and the set $\mathcal{GH}_b(\mathbb{F})$ of uniformly bounded isomorphisms of \mathbb{F} is open in $\mathcal{H}_b(\mathbb{F})$.
- (2) $\mathcal{GH}_b(\mathbb{F})$ has a structure of Banach Lie group whose Lie algebra is $\mathcal{H}_b(\mathbb{F})$.
- (3) If \mathbb{F} is identified with the projective $\varprojlim \mathbb{B}^n$ we denote by $\exp_n : \mathcal{L}(\mathbb{B}_n) \rightarrow \mathcal{GL}(\mathbb{B}_n)$, then we have a well defined smooth map $\exp := \varprojlim \exp_n : \mathcal{H}_b(\mathcal{F}) \rightarrow \mathcal{GH}_b(\mathbb{F})$ which is a diffeomorphism from an open set of 0 in $\mathcal{H}_b(\mathcal{F})$ onto a neighbourhood of $\text{Id}_{\mathbb{F}}$.

APPENDIX C. A THEOREM OF EXISTENCE OF ODE

The following result is in fact a reformulation in our context of Theorem 1 in [17].

Theorem C.1. *Let \mathbb{F} a Fréchet space realized as the limit of a surjective sequence of Banach spaces $(\mathbb{B}_n, \lambda_n^m)_{m \geq n}$ whose topology is defined by the sequence of seminorms $(\nu_n)_{n \in \mathbb{N}}$. Let I be an open interval in \mathbb{R} and U be an open set of $I \times \mathbb{F}$. Then U is a surjective projective limit of open sets $U_n \subset I \times \mathbb{B}_n$. Consider a smooth map $f = \varprojlim f_n : U \rightarrow \mathbb{F}$, projective limit of maps $f_n : U_n \rightarrow \mathbb{B}_n$ ¹². Assume that for every point $(t, x) \in U$, and every $n \in \mathbb{N}$, there exists an integrable function $K_n > 0$ such that*

$$(C.1) \quad \forall ((t, x), (t, x')) \in U^2, \nu_n(f(t, x) - f(t, x')) \leq K_n(t) \nu_n(x - x').$$

and consider the differential equation:

$$(C.2) \quad \dot{x} = \phi(t, x).$$

- (1) For any $(t_0, x_0) \in U$, there exists $\alpha > 0$ with $I_\alpha = [t_0 - \alpha, t_0 + \alpha] \subset I$, an open pseudo-ball $V = B(x_0, r) \subset U$ and a map $\Phi : I_\alpha \times I_\alpha \times V \rightarrow \mathbb{F}$ such that

$$t \mapsto \Phi(t, \tau, x)$$

is the unique solution of (C.2) with initial condition $\Phi(\tau, \tau, x) = x$ for all $x \in V$.

- (2) V is the projective limit of the open balls V_n of \mathbb{B}_n . For each $n \in \mathbb{N}$, the curve $t \mapsto \lambda_n \circ \Phi(t, \tau, \lambda_n(x))$ is the unique solution $\gamma : I_\alpha \rightarrow \mathbb{B}_n$ of the differential equation $\dot{x}_n = \phi_n(t, x_n)$ with initial condition $\gamma(\tau) = \lambda_n(x)$.

From this theorem we obtain easily:

Corollary C.2. *Let $U = \varprojlim U_n$ be an open subset of \mathbb{F} and $X = \varprojlim X_n : U \rightarrow \mathbb{F}$ a projective limit of smooth maps $X_n : U_n \rightarrow \mathbb{B}_n$. Assume that for every $n \in \mathbb{N}$ we have*

$$(C.3) \quad \forall ((t, x), (t, x')) \in U^2, \nu_n(X(x) - X(x')) \leq K_n \nu_n(x - x').$$

For $x_0 \in U$, let $B(x_0, 2r) = \{x \in \mathbb{F}, : \nu_{n_i}(x - x_0) < 2r, 1 \leq i \leq k\}$ be a pseudo-ball contained in U . Let us set

$$: C_1 = \max_{1 \leq i \leq k} K_{n_i}$$

¹²This means that we have: $\forall m \geq n, \lambda_n^m \circ f_m = f_n \circ (\text{Id}_{\mathbb{R}} \times \lambda_n^m)$

$$: C_2 = \sup_{z \in B(x_0, r)} \left\{ \max_{1 \leq i \leq k} \nu_{n_i}(f(z)) \right\}.$$

Then for any $\alpha > 0$ such that $\alpha e^{2\alpha C_1} \leq \frac{r}{2C_2}$, there exists a neighbourhood $V = B(x_0, r)$ and a smooth map $\phi_\alpha : I_\alpha \times V$ such that $t \mapsto \phi^\alpha(t, x)$ is the unique solution of $\dot{x} = X(x)$ defined on I_α with initial condition $\phi^\alpha(0, x) = x$. Moreover if $V_n = \lambda_n(V)$, consider $\phi_n^\alpha : I_\alpha \times V_n \rightarrow \mathbb{B}_n$ defined by $\phi_n^\alpha = \lambda_n \circ \phi^\alpha$; For each $z \in V_n$, the map $t \mapsto \phi^\alpha(t, z)$ is the unique solution of the differential equation $\dot{x}_n = X_n(x_n)$ defined on I_α with initial condition $\phi^\alpha(0, z) = z$.

Remark C.3. If $X = \varprojlim X_n$ is a smooth vector field defined on an open set $U = \varprojlim U_n$ of \mathbb{F} , which satisfies assumption (C.3), as classically, according to Corollary C.2, the map $\text{Fl}_t^X := \text{Fl}^X(t, \cdot)$ is the local flow of X that is Fl_t^X fulfils the properties of a 1-parameter group:

$$\begin{aligned} &: \text{Fl}_0^X = Id_V \\ &: \text{Fl}_t^X \circ \text{Fl}_s^X = \text{Fl}_{s+t}^X \text{ if } s, t \text{ and } s+t \text{ belong to } I_\alpha. \end{aligned}$$

In particular Fl_t^X is a diffeomorphism from V onto its range and its inverse is Fl_{-t}^X . Moreover $\text{Fl}_t^{X_n} = \lambda_n \circ \text{Fl}_t^X \circ \lambda_n$ is local flow of $X_n = \lambda_n \circ X \circ \lambda_n$ and we have $\text{Fl}_t^X = \varprojlim \text{Fl}_t^{X_n}$.

REFERENCES

- [1] M. Abbati, A. Manià, *On Differential Structure for Projective Limits of Manifolds*, J. Geom. Phys. **29** (1999).
- [2] P. Cabau, F. Pelletier, *Integrability of direct limits of Banach manifolds*, Annales de la Faculté des sciences de Toulouse : Mathématiques, Série **6**, Tome **28** N° **5** (2019).
- [3] P. Cabau, F. Pelletier *Prolongation of convenient Lie algebroids*, arXiv:2007.10657 (2020).
- [4] P. Cabau, F. Pelletier : *Direct and Projective Limits of Geometric Banach Structures (with the assistance and partial collaboration of D. Beltiță)*, In preparation, 2021.
- [5] D. Chillingworth, P. Stefan, *Integrability of singular distributions on Banach manifolds*, Math. Proc. Cambridge Philos. Soc. **79** (1976).
- [6] J. Dieudonné, *Fondement de l'analyse Moderne*, Cahiers Scientifiques, vol. XXVIII, Gauthiers-Villars, Paris, (1967).
- [7] C.T.J. Dodson, G. Galanis, E. Vassiliou, *Geometry in a Fréchet context: a Projective Limit Approach*, Cambridge University Press (2015).
- [8] Jan Milan Eyni *The Frobenius theorem for Banach distributions on infinite-dimensional manifolds and applications in infinite-dimensional Lie theory*, arXiv:1407.3166 (2014).
- [9] G.N. Galanis, *Projective Limits of Banach-Lie groups*, Periodica Mathematica Hungarica **32** (1996).
- [10] G.N. Galanis, *Projective Limits of Banach Vector Bundles*, Portugaliae Mathematica **55** Fasc **1** (1998).
- [11] G.N. Galanis, *Differential and Geometric Structure for the Tangent Bundle of a Projective Limit Manifold*, Rend. Sem. Univ. Padova **112** (2004).
- [12] H. Glöckner, L.R. Lovas, *Frobenius and Stefan-Sussmann theorems for various types of distributions on infinite-dimensional manifolds*, Non published Manuscript.
- [13] S. Hiltunen, *A Frobenius Theorem for Locally Convex Global Analysis*. Monatshefte für Mathematik **129**, (2000).

- [14] D. Igonin, *Notes on symmetries of PDEs and Poisson structures*,
<http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.570.9863&rep=rep1&type=pdf>
- [15] A. Kriegl, P.W. Michor, *The convenient Setting of Global Analysis*,
AMS Mathematical Surveys and Monographs **53** (1997).
- [16] S. Lang, *Differential and Riemannian Manifolds*,
Graduate Texts in Mathematics, **160**, Springer, New York (1995).
- [17] S.G. Lobanov, *Picard's theorem for ordinary differential equations in locally convex spaces*
Izvestiya: Mathematics, Vol **41**, N^o **3**
- [18] F. Pelletier, *Integrability of weak distributions on Banach manifolds*,
Indagationes Mathematicae **23** (2012).
- [19] P. Stefan, *Integrability of systems of vector fields*,
J. London Math. Soc, **2**, 21 (1974).
- [20] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions*,
Trans. Amer. Math. Soc., vol **80** (1973).
- [21] J. Teichmann, *A Frobenius Theorem on Convenient Manifolds*,
Monatshefte für Mathematik **134** (2001).

UNITÉ MIXTE DE RECHERCHE 5127 CNRS, UNIVERSITÉ DE SAVOIE MONT BLANC, LABORATOIRE
DE MATHÉMATIQUES (LAMA), CAMPUS SCIENTIFIQUE, 73370 LE BOURGET-DU-LAC, FRANCE
Email address: fernand.pelletier@univ-smb.fr