Projective limit of a sequence of compatible weak symplectic forms on a sequence of Banach bundles and Darboux Theorem

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PROJECTIVE LIMIT OF A SEQUENCE OF COMPATIBLE WEAK SYMPLECTIC FORMS ON A SEQUENCE OF BANACH BUNDLES AND DARBOUX THEOREM

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Abstract. Given a projective sequence of Banach bundles, each one provided with a of weak symplectic form, we look for conditions under which, the corresponding sequence of weak symplectic forms gives rise to weak symplectic form on the projective limit bundle. Then we apply this results to the tangent bundle of a projective limit of Banach manifolds. This naturally leads to ask about conditions under which the Darboux Theorem is also true on the projective limit of Banach manifolds. We will give some necessary and some sufficient conditions so that such a result is true. Then we discuss why, in general, the Moser's method can not work on projective limit of Banach weak symplectic Banach manifolds without very strong conditions like Kumar 's results ([15]). In particular we give an analog result of Kumar' s one with weaker assumptions and we give an example for which such weaker conditions are satisfied. More generally, we produce examples of projective sequence of weak symplectic Banach manifolds on which the Darboux Theorem is true and an example for which the Darboux Theorem is true on each manifold, but is not true on the projective limit of these manifolds.

2010 MSC: 53D35, 55P35.
Keywords: Banach manifold, Fréchet manifold projective limit of Banach bundles, Fréchet manifold, Fréchet bundles, weak symplectic form, sequence of compatible weak symplectic forms, Darboux theorem, Moser's method.

1. Introduction

In the Banach context, it is well known that a symplectic form can be strong or weak (see Definition 1). The Darboux Theorem was firstly proved for strong symplectic Banach manifolds by Weinstein ([25]). But Marsden ([19]) showed that the Darboux theorem fails for a weak symplectic Banach manifold. However Bambusi [2] found necessary and sufficient conditions for the validity of Darboux theorem for a weak symplectic manifold modelled on a reflexive Banach space (*Darboux-Bambusi Theorem*). The proofs of all these versions of Darboux Theorem were all established by Moser's method.
In a wider context like Fréchet or convenient manifolds, a symplectic form is always weak. Recently, a new approach to differential geometry in Fréchet context was initiated and developed by G. Galanis, C. T. J. Dodson, E. Vassiliou and their collaborators in terms of projective limits of Banach manifolds (see [5] for a panorama of these results). In this situation, P. Kumar, in [15], proves a version of Darboux Theorem, by Moser method, for a projective sequence of weak symplectic manifolds which satisfy the assumptions of the Darboux-Bambusi Theorem but under very strong added conditions on this sequence. On the other hand, a metric approach of differential geometry on Fréchet manifold was firstly introduced by Muller. This concept gives rise to Keller-differentiable calculus as exposed in details by Glockner in [11]. In this way we can consider the so called bounded Fréchet framework (cf. [21]) in which a classical implicit function Theorem is true and a Theorem of existence of local flow can be proved (cf. [6]). In this context Eftekharinasab in [7], proves a version of Darboux Theorem using Moser's method too, and again under very strong assumptions. In fact, when such a Fréchet manifold is also a projective limit of Banach manifolds this result seems to recover Kumar's result.

More generally we can look for conditions under which a family of weak symplectic forms on a projective sequence of Banach bundles gives rise to a weak symplectic form on the projective limit bundle: *this is the first purpose of this paper.*

Of course this result can be applied to projective limits of weak symplectic Banach

manifolds. Thus, this naturally leads to the problem of the existence of a Darboux theorem on the projective limit of weak symplectic Banach manifolds *which is the second purpose of this paper.*

More precisely let $\left(\mathbb{E}_i, \ell_i^j\right)$ be a reductive[1] projective sequence of Banach spaces and $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of (linear) weak symplectic forms ω_i on $\mathbb{E}_i$. We say that $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible symplectic forms if each ℓ_i^{i+1} satisfies

$$\ker \ell_i^{i+1} \cap (\ker \ell_i^{i+1})^\perp = \{0\} \text{ and } (\ell_i^{i+1})^*\omega_i = \omega_{i+1} \text{ in restriction to} (\ker \ell_i^{i+1})^\perp$$

where $(\ker \ell_i^{i+1})^\perp$ is the orthogonal of $\ker \ell_i^{i+1}$ relatively to ω_{i+1} (cf. Defintion 5)

Now consider a reductive projective sequence of Banach bundles $\left(E_i, \lambda_i^j\right)$ over a projective sequence $\left(M_i, \delta_i^j\right)$ of Banach manifolds and let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of weak symplectic forms ω_i on E_i. We say that $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible symplectic forms if, for any $x_i \in M_i$ and $i \in \mathbb{N}$, the sequence $\left((\omega_i)_{x_i}\right)_{i\in\mathbb{N}}$ is a sequence of compatible (linear) weak symplectic forms on the projective sequence $\left(\pi_i^{-1}(x_i), (\lambda_i^j)_{x_j}\right)$ of Banach spaces (cf. Definition 12)

In this context , we have (cf. Theorem 13 and Corollary 14):

Theorem 1. *Consider a reductive[2] projective sequence $\left(E_i, \lambda_i^j\right)$ of Banach bundles over a projective sequence of Banach manifolds $\left(M_i, \delta_i^j\right)$.*

1. *Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of compatible weak symplectic forms on $\left(E_i, \lambda_i^j\right)$. Then $\omega = \varprojlim \omega_i$ is a well defined weak symplectic form on the Fréchet bundle $E = \varprojlim E_i$ over $M = \varprojlim M_i$.*
2. *Conversely, let ω be a weak symplectic form on a projective limit bundles $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ of a submersive[3] projective sequence of Banach bundles$\left(E_i, \lambda_i^j\right)$. Assume that for each $x = \varprojlim x_i$, the map $(\lambda_i)_x : \pi^{-1}(x) \to \pi_i^{-1}(x_i)$ is a symplectic submersion [4]. Then ω induces a weak symplectic 2-form ω_i on E_i which gives rise to a family of compatible weak symplectic forms. Moreover, the 2-form on E defined by this sequence (ω_i) is precisely the given 2-form ω.*
3. *A 2-form ω on a Fréchet bundle, projective limit $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ of a submersive sequence of Banach fibre bundles $\left(E_i, \lambda_i^j\right)$ is a weak symplectic form if and only if there exists a sequence of compatible weak symplectic forms $(\omega_i)_{i\in\mathbb{N}}$ on E_i such that $\omega = \varprojlim \omega_i$.*

As corollary we obtain (cf. Theorem 15):

Theorem 2.

1. *Let $\left(M_i, \lambda_i^j\right)$ be a reduced sequence of Banach manifolds and $(\omega_i)_{i\in\mathbb{N}}$ a sequence of compatible weak symplectic forms. Then $\omega = \varprojlim \omega_i$ is a weak symplectic form on $M = \varprojlim M_i$*
2. *Let ω be a 2-form on a projective limit $M = \varprojlim M_i$ of a submersive sequence of manifolds $\left(M_i, \delta_i^j\right)$. Then ω is a weak symplectic form if and only if each $T_x\delta_i : T_xM \to T_{x_i}M_i$ is symplectic submersion for each $i \in \mathbb{N}$.*

In the context of Theorem 2 Point (1), if $a = \varprojlim a_i \in M$ and if for each i, there exists a Darboux chart around each a_i for ω_i, it seems natural to ask if the same result is true around $a \in M = \varprojlim M_i$ for ω. We have the following sufficient conditions and necessary conditions (cf. Theorm 25):

[1] *i.e.* $\ell_i^j(\mathbb{E}_j)$ is dense in $\mathbb{E}_i$

[2] that is the projective sequence of typical fiber $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)$ is a reduced projective sequence of Banach spaces

[3] cf. Definition 53

[4] cf Definition 9

Theorem 3. *Let $\left(M_i, \delta_i^j\right)$ be a submersive projective sequence of Banach manifolds where M_i is modeled on a reflexive Banach space $\mathbb{M}_i$.*

(1) *Consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of compatible symplectic forms ω_i on M_i and let ω be the symplectic form which is the projective limit of $(\omega_i)_{i\in\mathbb{N}}$ on $M = \varprojlim M_i$. Assume that the following property is satisfied:*
   (D): *There exists a projective limit chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around $a \in M = \varprojlim M_i$ such that, for each $a_i = \delta_i(a) \in M_i$, then (U_i, ϕ_i) is a Darboux chart around a_i for ω_i.*
   Then (U, ϕ) is a Darboux chart around a for ω.

(2) *Let ω be a weak symplectic form on a submersive projective limit $M = \varprojlim M_i$ such that $\delta_i : M \to M_i$ is a symplectic submersion. Assume that there exists a Darboux chart (V, ϕ) around a in M.*
   If ω_i is the symplectic form on M_i induced by ω, then there exists a projective limit chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around a such that the property (D) is satisfied.

In [2], Bambusi has proved a version of Darboux Theorem for a weak symplectic ω on a Banach manifold M modelled on a reflexive Banach space $\mathbb{M}$, around some point $a \in M$, under the following *"Darboux-Bambusi assumptions"*.

Denote by $\widehat{T_xM}$ the Banach space which is the completion of T_xM provided with the norm $||\ ||_{\omega_x}$ associated to some norm $||\ ||$ on T_xM (cf section 2.1). The Banach space $\widehat{T_xM}$ does not depend on this choice of $||\ ||$. Then ω_x can be extended to a continuous bilinear map $\hat{\omega}_x$ on $T_xM \times \widehat{T_xM}$ and $\omega_x^\flat$ becomes an isomorphism from T_xM to $(\widehat{T_xM})^*$. We set

$$\widehat{TM} = \bigcup_{x\in M} \widehat{T_xM} \ \text{ and } \ \widehat{TM}^* = \bigcup_{x\in M} (\widehat{T_xM})^*.$$

Darboux-Bambusi assumptions

(i) *There exists a neighbourhood U of $a \in M$ such that $\widehat{TM}_{|U}$ is a trivial Banach bundle whose typical fibre is the Banach space $(\widehat{T_aM}, ||\ ||_{\omega_a})$;*
(ii) *via some trivialization, ω can be extended to a smooth field of continuous bilinear forms on $TM_{|U} \times \widehat{TM}_{|U}$*[5].

Thus, again in the context of Theorem 2 Point (1), we can ask if there exists a Darboux chart for $\omega = \varprojlim \omega_i$ on the projective limit $M = \varprojlim M_i$ when each M_i satisfies Darboux-Bambusi assumptions. Unfortunately in general the answer is negative (cf. Example 33). About such a question, we explain under which type of conditions the Moser's method in the previous context can be applied [6]. Finally, under (strong) sufficient conditions we have (see Theorem 29 for a more precise statement):

Theorem 4. *Let $M = \varprojlim M_i$ be a submersive projective limit of reflexive Banach manifolds M_i and $(\omega_i)_{i\in\mathbb{N}}$ a compatible sequence of symplectic forms (resp. ω a symplectic form on M such that $\delta_i : M \to M_i$ is a symplectic submersion).*
Consider a point $a = \varprojlim a_i \in M$ and assume that there exists chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around a with the following properties for all $i \in \mathbb{N}$:

(1) *the Darboux-Bambusi assumptions around $a_i = \delta_i(a)$ are satisfied on U_i ;*
(2) *there exist Finsler norms $||\ ||_i$ on $TM_{i|U_i}$ and $||\ ||_i^*$ on $(\widehat{TM_i})^*_{|U_i}$ such that:* [7]

$$\forall x_i = \delta_i(x) \in U_i, \ \ ||((\omega_i)^\flat_{x_i})||_i^{\mathrm{op}} \leq K(x);^{8} \tag{1}$$

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[5] from the skew-symmetry of ω and [2] Lemma 2.8 this assumption implies that $\omega^\flat$ gives rise to bundle isomorphism from $TM_{|U}$ to $(\widehat{TM})^*_{|U}$ (cf. section 2.1)

[6] Note that this discussion is analog to the same type of discussion in [23] in the context of direct limit of weak symplectic manifolds

[7] cf. Definition 28

[8] $||\ ||_i^{\mathrm{op}}$ is the Finsler norm operator from $TM_{i|U_i}$ to $(\widehat{TM_i})^*_{|U_i}$

for some positive continuous function K on U

$$\forall x_i = \delta_i(x) \in U_i,\ ,\ ||(\omega_i - \phi_i^*\Omega_i)^\flat_{x_i}(\mathbf{E}_i)||_i^\sharp \leq C(x) \tag{2}$$

for some positive continuous function C on U, where $\boldsymbol{E}_i$ is an Euler vector fields on M_i[9] and $\Omega_i = (\phi_i^{-1})^*(\omega_i)_a$.

Under these assumptions, there exists a Darboux chart around a in M.

The assumptions of this theorem are weaker than that of those of Kumar's Darboux Theorem in [15] as show the Example 34 and Remark 35 .

Unfortunately, such results require such strong assumptions, that it seems that there are no interesting applications apart basic or well-adapted examples.

We end this paper by some examples for which :

- the Darboux theorem is true on a projective limit of Banach manifolds;
- the Darboux-Bambusi Theorem assumptions are true on each Banach manifold and we have a Darboux chart around some point on the projective limit;
- the Darboux-Bambusi Theorem assumptions are true on each Banach manifold but we have no Darboux chart around some point in the projective limit;
- there exists a projective limit of compatible strong symplectic forms which satisfies the assumptions of Theorem 4.

This work is self contained and and organized as follows :

- ☐ In section 2, we begin by a survey on known results on symplectic forms on a Banach space (§2.1). Then we look for properties of a sequence of compatible (linear) symplectic forms on a projective sequences of Banach spaces (§ 2.2). A precise context of Theorem 1 (resp. Theorem 2) can be found in § 2.3 (resp. §2.4). The proofs of all these results take place in §2.5.
- ☐ Section 3 is devoted to show that under the assumptions of Theorem 2, the Darboux-Bambusi assumptions which are satisfied for a projective limit of weak symplectic manifold are also valid on the projective limit *in an natural adequat sense*. As preliminary for this purpose, the first subsection recall the Moser's method and the Darboux-Bambusi Theorem. In the next subsection, under assumption of Theorem 1 Point (1), for such a projective limit of Banach bundles which satisfy a generalization of Darboux-Bambusi Theorem assumptions, we show that its projective limit has the same properties in an appropriate sense.
- ☐ The discussion on the problem of existence of Darboux charts on a limit of reduced projective sequence of Banach manifolds is exposed in section 4.
- ☐ The proof of Theorem 4 takes place in section 5.
- ☐ Examples and contre-example about the existence of a projective limit of Darboux charts are given in section 6.
- ☐ Finally we end this paper by *a series of Appendices which sumrize all the definitions and properties on projective limits needed in this paper.*

2. Projective limit of a compatible sequence of weak symplectic forms on a projective sequence of Banach bundle

2.1. **Symplectic forms on Banach space.** In this section we recall some well known results on linear symplectic forms on a Banach space (cf. for instance [23]):

Definition 1. *Let $\mathbb{E}$ be a Banach space. A bilinear form ω is said to be weakly non degenerate if* $(\forall Y \in \mathbb{E},\ \omega(X,Y) = 0) \quad \Longrightarrow \quad X = 0$.

Classically, to ω is associated the linear map

$\omega^\flat : \mathbb{E} \longrightarrow \mathbb{E}^*$ defined by $\left(\omega^\flat(X)\right)(Y) = \omega(X,Y)\ ,\ :\ \forall Y \in \mathbb{E}$.

[9] cf. section 5.1 after Definition 28

Clearly, ω is weakly non degenerate if and only if $\omega^\flat$ is injective.
The 2-form ω is called *strongly nondegenerate* if $\omega^\flat$ is an isomorphism.

A fundamental result in finite dimensional linear symplectic space is the following:

If ω is a symplectic form on a finite dimensional vector space $\mathbb{E}$, there exists a vector space $\mathbb{L}$ and an isomorphism $A : \mathbb{E} \to \mathbb{L} \oplus \mathbb{L}^*$ such that $\omega = A^*\omega_{_{\mathbb{L}}}$ where

$$\omega_{_{\mathbb{L}}}((u,\eta),(v,\xi) =< \eta, v > - < \xi, u > \tag{3}$$

This result is in direct relation with the notion of *Lagangian subspace* which is a fundamental tool in the finite dimensional symplectic framework.

In the Banach framework, let ω be a weak symplectic form on a Banach space.

A subspace $\mathbb{F}$ is *isotropic* if $\omega(u,v) = 0$ for all $u, v \in \mathbb{F}$. An isotropic subspace is always closed.
If $\mathbb{F}^{\perp\omega} = \{w \in \mathbb{E} \ : \forall u \in \mathbb{F},\ \omega(u,v) = 0\ \}$ is the orthogonal symplectic space of $\mathbb{F}$, then $\mathbb{F}$ is isotropic if and only if $\mathbb{F} \subset \mathbb{F}^{\perp\omega}$ and is *maximal isotropic if* $\mathbb{F} = \mathbb{F}^{\perp\omega}$. Unfortunately, in the Banach framework, a maximal isotropic subspace $\mathbb{L}$ can be not supplemented. Following Weinstein's terminology ([25]), an isotropic space $\mathbb{L}$ is called a *Lagrangian space* if there exists an isotropic space $\mathbb{L}'$ such that $\mathbb{E} = \mathbb{L} \oplus \mathbb{L}'$. Since ω is strong non degenerate, this implies that $\mathbb{L}$ and $\mathbb{L}'$ are maximal isotropic and then are Lagrangian spaces (see [25]).

Unfortunately, in general, for a given symplectic structure, Lagrangian subspaces need not exist (cf. [12]). Even for a strong symplectic structure on Banach space which is not Hilbertizable, the non existence of Lagrangian subspaces is an open problem to our knowledge. Following [25], a symplectic form ω on a Banach space $\mathbb{E}$ is a *Darboux (linear) form* if there exists a Banach space $\mathbb{L}$ and an isomorphism $A : \mathbb{E} \to \mathbb{L} \oplus \mathbb{L}^*$ such that $\omega = A^*\omega_{\mathbb{L}}$ where $\omega_{\mathbb{L}}$ is defined in (3). Note that in this case $\mathbb{E}$ *must be reflexive.*

Let $\mathbb{E}$ be a Banach space provided with a norm $||\ ||$. We consider a symplectic form ω on $\mathbb{E}$ and let $\omega^\flat : \mathbb{E} \to \mathbb{E}^*$ be the associated bounded linear operator. Following [2] and [14], on $\mathbb{E}$, we consider the norm $||u||_\omega = ||\omega^\flat(u)||^*$ where $||\ ||^*$ is the canonical norm on $\mathbb{E}^*$ associated to $||\ ||$. Of course, we have $||u||_\omega \leq ||\omega^\flat||^{\text{op}}.||u||$ (where $||\omega^\flat||^{\text{op}}$ is the norm of the operator $\omega^\flat$) and so the inclusion of the normed space $(\mathbb{E}, ||\ ||_\omega)$ in $(\mathbb{E}, ||\ ||)$ is continuous. We denote by $\widehat{\mathbb{E}}$ the Banach space which is the completion of $(\mathbb{E}, ||\ ||_\omega)$. Since $\omega^\flat$ is an isometry from $(\mathbb{E}, ||\ ||_\omega)$ to its range in $\mathbb{E}^*$, we can extend $\omega^\flat$ to a bounded operator $\widehat{\omega}^\flat$ from $\widehat{\mathbb{E}}$ to $\mathbb{E}^*$. Assume that $\mathbb{E}$ is *reflexive*. Therefore $\hat{\omega}^\flat$ is an isometry between $\widehat{\mathbb{E}}$ and $\mathbb{E}^*$ ([2] Lemma 2.7). Moreover, $\omega^\flat$ can be seen as a bounded linear operator from $\mathbb{E}$ to $\widehat{\mathbb{E}}^*$ and is in fact an isomorphism ([2] Lemma 2.8). Note that since $\widehat{\mathbb{E}}^*$ *is reflexive*, this implies that $\hat{\mathbb{E}}$ *is also reflexive.*

Remark 2. *If $||\ ||'$ is an equivalent norm of $||\ ||$ on $\mathbb{E}$, then the corresponding $(||\ ||')^*$ and $||\ ||^*$ are also equivalent norm on $\mathbb{E}^*$ and so $||\ ||'_\omega$ and $||\ ||_\omega$ are equivalent norms on $\mathbb{E}$ and so the completion $\widehat{\mathbb{E}}$ depends only on ω and the Banach structure on $\mathbb{E}$ defined by equivalent norms.*

2.2. **Case of projective limit of Banach spaces.** Let ω be a skew-symmetric bilinear form on a Banach space $\mathbb{E}$ and $\mathbb{K}$ a Banach subspace of $\mathbb{E}$. Recall that the *ω-orthogonal subspace* $\mathbb{K}^{\perp\omega}$ is defined by

$$\mathbb{K}^{\perp\omega} = \{x \in \mathbb{E},\ :\ \forall y \in \mathbb{K},\ \omega(x,y) = 0\}.$$

When there is no ambiguity this set is simply denoted $\mathbb{K}^\perp$. Note that since ω is skew-symmetric $\mathbb{K}^\perp = \{x \in \mathbb{E} :\ \forall y \in \mathbb{K},\ \omega(y,x) = 0\}$ and so $(\mathbb{K}^\perp)^\perp = \mathbb{K}$.
If $\mathbb{K}^0 = \{\xi \in \mathbb{E}^* :\ \forall u \in \mathbb{K},\ \xi(u) = 0\}$ is the annihilator of $\mathbb{K}$, then $\mathbb{K}^\perp = (\omega^\flat)^{-1}(\mathbb{K}^0)$.
Given two Banach subspaces $\mathbb{K}$ and $\mathbb{K}'$ of $\mathbb{E}$, the following relations are classical:

- If $\mathbb{K} \subset \mathbb{K}'$ then $\mathbb{K}'^\perp \subset \mathbb{K}^\perp$ and, in particular, for any subspace $\mathbb{K}$, $\mathbb{E}^\perp \subset \mathbb{K}^\perp$.
- $(\mathbb{K} + \mathbb{K}')^\perp = \mathbb{K}^\perp \cap \mathbb{K}'^\perp$.
- $(\mathbb{K} \cap \mathbb{K}')^\perp = \mathbb{K}^\perp + \mathbb{K}'^\perp$.

Let ω (resp. ω') be a skew-symmetric bilinear form on a Banach space $\mathbb{E}$ (resp. $\mathbb{E}'$) and $\ell : \mathbb{E} \to \mathbb{E}'$ a continuous linear map. By analogy with the terminology for Finsler geometry (cf. [1]) we introduce

Definition 3. *We say that ℓ is a weak isometry between ω and ω' if $\ell(E)$ is dense in $\mathbb{E}'$ and we have:*

$$\ker \ell \cap (\ker \ell)^\perp = \{0\} \text{ and } \ell^*\omega' = \omega \text{ in restriction to } (\ker \ell)^\perp \tag{4}$$

Note that the condition "$\ker \ell \cap (\ker \ell)^\perp = \{0\}$" is equivalent to the condition "the restriction of ω to $(\ker \ell)^\perp$ is non degenerate".

Proposition 4. *Let ω (resp. ω') be a weak skew symmetric form on a Banach space $\mathbb{E}$ (resp. $\mathbb{E}'$) and let $\ell : \mathbb{E} \to \mathbb{E}'$ be a continuous linear map. We set $\mathbb{K} = (\ker \ell)$ and denote by $\overline{\omega}$ the restriction of ω to $\mathbb{K}^\perp$. We have the following properties:*

1. *If ℓ is a weak isometry between ω and ω', then $\overline{\omega}$ and $\omega'_{|\ell(\mathbb{E})}$ are non degenerate, and $\ker \ell^*\omega' = \ker \ell$.*
2. *If ω and $\overline{\omega}$ are non degenerate, ℓ is a weak isometry between ω and ω' if and only if $\ell^*\omega' = \omega$ on $(\ker \ell)^\perp$ and, in this case, the restriction of ω' to $\ell(\mathbb{E})$ is non degenerate.*
3. *Let ω'' be a skew symmetric bilinear form on a Banach space $\mathbb{E}''$ and $\ell' : \mathbb{E}' \to \mathbb{E}''$ a continuous linear map. If ℓ (resp. ℓ') is a weak isometry between ω and ω' (resp. ω' and ω'') then $\ell' \circ \ell$ is a weak isometry between ω and ω''.*

Note that if ℓ is a weak symplectic isometry between ω and ω', the restriction $\overline{\ell}$ of ℓ to $(\ker \ell)^\perp$ is an isomorphism and

$$\overline{\ell}^*\omega' = \ell^*\omega'_{|(\ker \ell)^\perp} = \omega_{|(\ker \ell)^\perp} \tag{5}$$

Proof (1) We have $\overline{\omega}(u, v) = 0$ for all $v \in \mathbb{K}^\perp$ if and only if u belongs $\mathbb{K} \cap \mathbb{K}^\perp$ which implies that $\overline{\omega}$ is non degenerate. Since $\overline{\ell}$ is an isomorphism, from (5), it follows that $\omega'_{|\ell(\mathbb{E})}$ is non degenerate. Now, v belongs to $\ker(\ell^*(\omega'))^\flat$ if and only if $\omega'(\ell(u), \ell(v)) = 0$, $\forall v$, if and only if $\ell(u) = 0$.

(2) Assume ω and $\overline{\omega}$ are non degenerate. We must show that $\mathbb{E} = \mathbb{K} \oplus \mathbb{K}^\perp$. But $(\mathbb{K} \oplus \mathbb{K}^\perp)^\perp = \mathbb{K}^\perp \cap \mathbb{K} = \{0\}$ and since ω is non degenerate, it follows that $\{0\}^\perp = \mathbb{E} = \mathbb{K} \oplus \mathbb{K}^\perp$, which ends the proof of Point (2) according to relation (5).

(3) Under the assumptions of Point (3), we have $\mathbb{E} = \ker \ell \oplus (\ker \ell)^\perp$ and $\mathbb{E}' = \ker \ell' \oplus (\ker \ell')^\perp$. Now, since the inclusion of $\ell(E)$ in $\mathbb{E}'$ is continuous, if $\mathbb{K}' := \ker \ell' \cap \ell(\mathbb{E})$, then $(\mathbb{K}')^\perp = (\ker \ell')^\perp \cap \ell(\mathbb{E})$ and so $\ell(E) = \mathbb{K}' \oplus (\mathbb{K}')^\perp$. Let $\overline{\ell}$ be the restriction of ℓ to $(\ker \ell)^\perp$; it is is an isomorphism onto $(\mathbb{K}')^\perp$. If $\overline{\mathbb{K}} = \overline{\ell}^{-1}(\mathbb{K}')$ and $\overline{\mathbb{H}} = \overline{\ell}^{-1}((\mathbb{K}')^\perp)$ then $(\ker \ell)^\perp = \overline{\mathbb{K}} \oplus \overline{\mathbb{H}}$. By construction, $\ker \ell' \circ \ell = \ker \ell \oplus \overline{\mathbb{K}}$ and we have $(\ker \ell \oplus \overline{\mathbb{K}})^\perp = (\ker \ell)^\perp \cap \overline{\mathbb{K}}^\perp = \overline{\mathbb{H}}$. Indeed $\overline{\mathbb{H}}$ is contained in $(\ker \ell)^\perp$, $\overline{\mathbb{H}} = \overline{\ell}^{-1}((\mathbb{K}')^\perp)$ and $(\overline{\ell}^{-1})^*(\omega_{|(\ker \ell)^\perp}) = \omega'_{|\ell(\mathbb{E})}$, this implies that $\overline{\mathbb{H}}$ is the orthogonal of $\overline{\mathbb{K}}$ in $(\ker \ell)^\perp$. Now, the restriction $\overline{\ell' \circ \ell}$ of $\ell' \circ \ell$ to $(\ker(\ell' \circ \ell))^\perp = \overline{\mathbb{H}}$ is an isomorphism. Consider for any $(u, v) \in \overline{\mathbb{H}}^2$ we have:

$$\omega(u, v) = \omega'(\overline{\ell}(u), \overline{\ell}(u)) = \omega''(\overline{\ell' \circ \ell}(u), \overline{\ell' \circ \ell}(v)).$$

So the proof is completed.

□

Definition 5. *Let $\left(\mathbb{E}_i, \ell_i^j\right)$ be a reductive*[10] *projective sequence Banach spaces and $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of (linear) weak symplectic forms ω_i on $\mathbb{E}_i$. We say that $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible symplectic forms if each ℓ_i^{i+1} is a weak isometry between ω_{i+1} and ω_i, for all $i \in \mathbb{N}$*

We then have the following property:

[10] cf. Definition 41

Proposition 6. *Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of compatible symplectic forms on a reduced projective sequence $\left(\mathbb{E}_i,\ell_i^j\right)$. Then if $u=\varprojlim u_i$ and $v=\varprojlim v_i$ in $\mathbb{E}=\varprojlim\mathbb{E}_i$,*

$$\omega(u,v)=\varprojlim\omega_i(u_i,v_i)$$

defines a weak symplectic 2-form on $\mathbb{E}$.

This Proposition is based on the following technical Lemma which can be easily proved by induction

Lemma 7. *Let $\mathbb{E}$ be a projective limit of a reductive projective sequence $\left(\mathbb{E}_i,\ell_i^j\right)$. Assume that, for all $(i,j)\in\mathbb{N}^2$ such that $j\geq i$, the kernel of ℓ_i^j is supplemented.*

(1) *For each $i\in\mathbb{N}$ and each $j\geq i$ we have a decomposition*

$$\mathbb{E}_j=\mathbb{E}_i^j\oplus\mathbb{E}_{i+1}^j\oplus\cdots\oplus\mathbb{E}_{j-1}^j\oplus\ker\ell_{j-1}^j \tag{6}$$

with the following properties for all $j\geq i$
   (a) $\ker\ell_l^j=\mathbb{E}_{l+1}^j\oplus\cdots\oplus\mathbb{E}_{j-2}^j\oplus\ker\ell_{j-1}^j$;
   (b) *the restriction of $(\ell')_l^j$ of ℓ_l^j to $(\mathbb{E}_i^j\oplus\cdots\oplus\mathbb{E}_l^j)$ is injective with dense range in $\mathbb{E}_l$;*
   (c) $\ell_l^j(\mathbb{E}_h^j)$ *is dense in $\mathbb{E}_h^l$ for all $i\leq h\leq l$, is dense in $\ker\ell_{l-1}^l$ for $h=l$ and $\ell_l^j(\mathbb{E}_h^j)=\{0\}$ for $l<h\leq j-1$.*

(2) *Let $\ell_i:\mathbb{E}\to\mathbb{E}_i$ the canonical projection. Then $E=\ker\ell_i\oplus\mathbb{F}_i'$ and the restriction ℓ_i' of ℓ to $\mathbb{E}_i'$ is a continuous map injective map into $\mathbb{E}_i$ with dense range. Moreover, if $||\ ||_i$ is a norm on $\mathbb{E}_i$, then $\nu_i=||\ ||_i\circ\ell_i$ is a semi-norm on E and the restriction of ν_i to $\mathbb{F}_i'$ is a norm and in this case, ℓ_i' is an isometry. In particular, the completion $\overline{\mathbb{F}_i'}$ of $\mathbb{F}_i'$ is isomorphic to $\mathbb{E}_i$.*

(3) *We set $\mathbb{K}_i=\ker\ell_{i-1}^i$ for $i\geq 1$ and $\mathbb{K}_0=\mathbb{E}_0$. If $\mathbb{E}_j'=\prod_{l=0}^{j}\mathbb{K}_l$, then there exist bounding maps $\kappa_i^j:\mathbb{E}_j'\to\mathbb{E}_i'$ with dense range, so that $\left(\mathbb{E}_i',\kappa_i^j\right)$ is a reduced projective sequence. If $\mathbb{E}'=\varprojlim\mathbb{E}_i'$ there exists an injective continuous linear map $\varprojlim\theta_i:\mathbb{E}\to\mathbb{E}'$ with dense range where θ_i is an injective linear map from $\mathbb{E}_i$ into $\mathbb{E}_i'$ with dense range. Moreover, if $\left(\mathbb{E}_i,\ell_i^j\right)$ if a surjective projective sequence, then each θ_i, $i\in\mathbb{N}$ and θ are isomorphisms.*

Remark 8.

(1) *From the properties of the sequence $\left(\ell_i^j\right)$ and Proposition 4 (3), if $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible weak symplectic 2-forms, then ℓ_i^j is a weak isometry between ω_j and ω_i, for all $i\in\mathbb{N}$ and all $j\geq i$.*

(2) *Consider the assumptions of Proposition 6. If $\left(\mathbb{E}_i,\ell_i^j\right)$ is a* ILB *sequence (cf. Appendix B), we have $\ker\ell_i^j=\{0\}$ for all $j\geq i$ and $i\in\mathbb{N}$. Thus $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible symplectic forms on this projective system if and only if, for all $j\geq i$ and $i\in\mathbb{N}$, then $\omega_j=(\lambda_i^j)^*\omega_i$, and ω_0 is symplectic. But in general, if for some pair (i,j), $\ker\ell_i^j\neq\{0\}$, the condition $\omega_j=(\ell_i^j)^*\omega_i$ implies that $\mathbb{E}_j^\perp\neq\{0\}$ and so this does not implies that ω_j is symplectic.*

(3) *In Proposition 6, when $\left(\mathbb{E}_i,\ell_i^j\right)$ is a surjective [11] projective sequence, the symplectic form ω on $\mathbb{E}$ has the property that the induced form on $\ker\ell_i$ is symplectic and so we have $\mathbb{E}=\ker\ell_i\oplus(\ker\ell_i)^\perp$ where $(\ker\ell_i)^\perp$ is the orthogonal of $\ker\ell_i$ (relative to ω).*

As in finite dimension, we introduce:

Definition 9. *Let $\mathbb{E}=\varprojlim\mathbb{E}_i$ a projective limit of a surjective projective sequence $\left(\mathbb{E}_i,\ell_i^j\right)$. Consider a (weak) symplectic form ω on $\mathbb{E}$ such that $\mathbb{E}=\ker\ell_i\oplus(\ker\ell_i)^\perp$. We will say that ℓ_i is a symplectic submersion.*

[11] that is each ℓ_i^j is surjective

Remark 10. *In the context of Definition 9, the restriction of ℓ_i to $(\ker \ell_i)^\perp$ is an isomorphism onto $\mathbb{E}_i$ and so we have a well symplectic form ω_i on $\mathbb{E}_i$ such that $\omega = \ell_i^* \omega_i$ in restriction to $(\ker \ell_i)^\perp$. Thus this definition is analog to the notion of isometric submersion between Finsler manifolds in finite dimension introduced in* [1].

We have the following type of converse of Proposition 6 :

Proposition 11. *Let $\left(\mathbb{E}_i, \ell_i^j\right)$ be a surjective projective sequence of Banach space and $E = \varprojlim E_i$. If ω is a symplectic form on E such that $\ell_i : E \to E_i$ is a symplectic submersion for all $i \in \mathbb{N}$, then ω induces a symplectic form ω_i on E_i. Moreover, $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible symplectic forms and the projective limit associated to this sequence is precisely ω.*

Proof Since for $j \geq i$, $\ell_i = \ell_i^j \circ \ell_j$ this implies $\ker \ell_i = \ker \ell_j \oplus (\ell')_j^{-1}(\ker \ell_i^j)$. Thus we have $\ker \ell_j \subset \ker \ell_i$ and so $(\ker \ell_i)^\perp \subset (\ker \ell_j)^\perp$. As we have seen previously, there exists a (unique) symplectic form ω_j on $\mathbb{E}_j$ such that $\omega = \ell_j^* \omega_j$ on $(\ker \ell_j)^\perp$. Since for any $j \in \mathbb{N}$, the restriction ℓ_j' to $(\ker \ell_j)^\perp$ is an isomorphism onto $\mathbb{E}_j$, we have:
$\omega_j(u_j, v_j) = \omega(\ell_j'(u_j'), \ell_j'(v_j'))$ for all $u_j', v_j' \in (\ker \ell_j)^\perp$ with $u_j = \ell_j'(u_j')$ and $v_j = \ell_j'(v_j')$.

But since $(\ker \ell_i)^\perp \subset (\ker \ell_j)^\perp$, it follows that, for any $u_i', v_i' \in (\ker \ell_i)^\perp$, we have

$$\begin{aligned}\omega(\ell_i'(u_i'), \ell_i'(v_i')) &= \omega(\ell_i^j \circ \ell_j'(u_i'), \ell_i^j \circ \ell_j'(v_i')) \\ &= (\ell_i^j)^* \omega(\ell_j'(u_i'), \ell_j'(v_i').\end{aligned}$$

Thus we obtain

$$(\omega_j) = (\ell_i^j)^* \omega_i \text{ on } \ell_j' \left((\ker \ell_i)^\perp\right).$$

The proof will be completed if we show that $\ell_j' \left((\ker \ell_i)^\perp\right) = (\ker \ell_i^j)^\perp$. But this results follows from $\ell_j'(\ker \ell_i) = \ker \ell_i^j$.

□

2.3. **Case of projective sequence of Banach bundles.**

Definition 12. *Let $\left(E_i, \lambda_i^j\right)$ be a projective sequence of Banach bundles over a projective sequence $\left(M_i, \delta_i^j\right)$ of manifolds and let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of weak symplectic forms ω_i on E_i. If $\mathbb{E}_i$ is the typical fibre of E_i, assume that the following properties are satisfied:*

(RPSBS): *The sequence $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)$ is a reduced projective sequence of Banach spaces.*

We say that $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible symplectic forms if the sequence $((\omega_i)_{x_i})_{i\in\mathbb{N}}$ is a sequence of compatible (linear) weak symplectic forms on the projective sequence $\left(\pi_i^{-1}(x_i), (\lambda_i^j)_{x_j}\right)$ of Banach spaces.

Under the context of this Definition, we have

Theorem 13. *Consider a projective sequence $\left(E_i, \lambda_i^j\right)$ of Banach bundles over a projective sequence of Banach manifolds $\left(M_i, \delta_i^j\right)$ which satisfies the assumption* **(RPSBS)**.

1. *Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of compatible weak symplectic forms on $\left(E_i, \lambda_i^j\right)$. Then $\omega = \varprojlim \omega_i$ is a well defined weak symplectic form on the Fréchet bundle $E = \varprojlim E_i$ over $M = \varprojlim M_i$.*
2. *Conversely, let ω be a weak symplectic form on a projective limit bundles $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ of a submersive[12] projective sequence of Banach bundles (E_i, π_i, M_i). Assume that for each $x = \varprojlim x_i$, the map $(\lambda_i)_x : \pi^{-1}(x) \to \pi_i^{-1}(x_i)$ is a submersion. Then ω induces a weak symplectic 2-form ω_i on E_i which gives rise to a family of compatible weak symplectic forms. Moreover, the 2-form on E defined by this sequence $(\omega_i)_{i\in\mathbb{N}}$ is precisely the given 2-form ω.*

We obtain directly the following Corollary:

[12] cf. Definition 53

Corollary 14. *A 2-form ω on a Fréchet bundle, projective limit $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ of a submersive sequence of Banach fibre bundles $\left(E_i, \lambda_i^j\right)$ is a weak symplectic form if and only if there exists a sequence of compatible weak symplectic forms $(\omega_i)_{i\in\mathbb{N}}$ on E_i such that $\omega = \varprojlim \omega_i$.*

Note that Theorem 1 in the introduction is Theorem 13 joined with Corollary 14.

2.4. **Case of projective limit of weak symplectic Banach manifolds.** By application of Theorem 13 when E_i is the tangent bundle TM_i of a Banach manifold M_i, we obtain the following Theorem which is exactly Theorem 2 in the introduction:

Theorem 15.

(1) *Let $\left(M_i, \lambda_i^j\right)$ be a reduced sequence of Banach manifolds and $(\omega_i)_{i\in\mathbb{N}}$ a sequence of compatible weak symplectic forms. Then $\omega = \varprojlim \omega_i$ is a weak symplectic form on $M = \varprojlim M_i$*
(2) *Let ω be a 2-form on a projective limit $M = \varprojlim M_i$ of a submersive sequence of manifolds $\left(M_i, \delta_i^j\right)$. Then ω is a weak symplectic form if and only if each $T_x\delta_i : T_xM \to T_{x_i}M_i$ is a symplectic submersion for each $i \in \mathbb{N}$.*

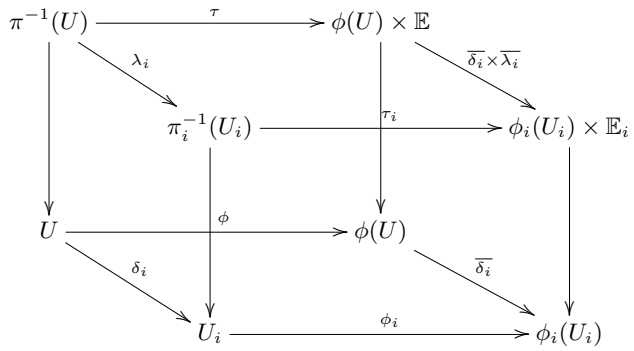
Remark 16.

(1) *Given a submersive sequence $\left(M_i, \delta_i^j\right)$ of manifolds and a weak symplectic form ω_i on each M_i, each map $\delta_i^j : M_j \to M_i$ is a symplectic submersion if the restriction of ω_j is a symplectic form on each fibre of δ_i^j and on the orthogonal symplectic of the vertical bundle of δ_i^j we have $\omega_j = (\delta_i^j)^*\omega_i$.*
(2) *Let $M = \varprojlim M_i$ be a projective limit of a submersive sequence $\left(M_i, \delta_i^j\right)$ of manifolds and ω a weak symplectic form on M. We say that the canonical projection $\delta_i : M \to M_i$ is a symplectic submersion if the restriction of ω to each fibre $\delta^{-1}(x_i)$ is a symplectic form and $\delta_i^*\omega_i = \omega$ on the orthogonal bundle of the vertical bundle of δ_i.*

2.5. **Proofs of results.**

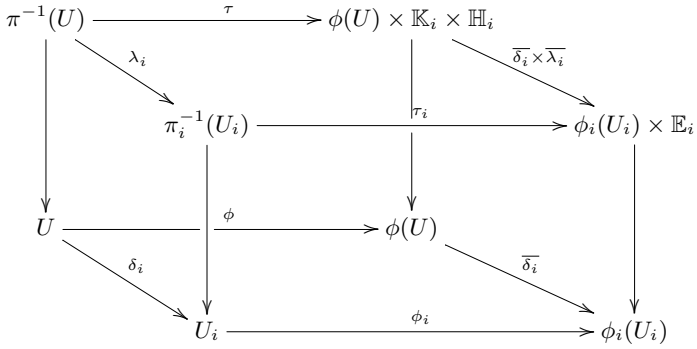
Proof [Proof of Theorem 13] From Proposition 6, we know that ω is well defined. Now ω is a smooth 2-form since it is a projective limit of smooth 2 forms, which ends the proof of (1).
Now let ω be a weak symplectic form on a projective limit bundle $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ which satisfies the assumptions of (2). Given some $x = \varprojlim x_i \in M$, since $(\lambda_i)_x : \pi^{-1}(x) \to \pi_i^{-1}(x_i)$ is a symplectic submersion of symplectic spaces, the restriction $(\lambda_i)'$ of λ_i to $\ker(\lambda_i)_x^\perp$ is an isomorphism on $\pi_i^{-1}(x_i)$ and so $(\omega_i)_{x_i} = \{[(\lambda_i)_x]^{-1}\}^*(\omega_x)_{|\ker(\lambda_i)_x^\perp}$ is a symplectic form on $\pi_i^{-1}(x_i)$. It remains to show that $x_i \mapsto \omega_{x_i}$ is smooth.

Fix some $x = \varprojlim x_i \in M$. There exists $\phi(U) \times \mathbb{E}$ with the following commutative diagram



(7)

Let Ω be the symplectic form on $\phi(U)\times\mathbb{E}$ such that $\omega = \tau^*\Omega$. According to Proposition 54, $\ker \lambda_i$ is a sub-bundle of E. Now, since ω is a smooth symplectic form and the orthogonal $\ker(\lambda_i)_z^\perp$ is a supplemented space of $\ker(\lambda_i)_z$ for all $z \in M$, it follows that $\ker(\lambda_i)_z^\perp$ is a Banach sub-bunlde of E and so the Diagram (7) have the more precise version, after shrinking U if necessary:



(8)

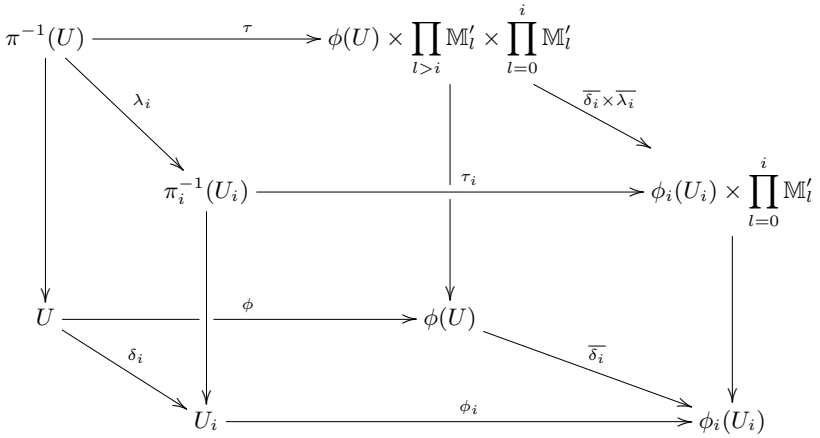
where $\mathbb{K}_i$ is the Kernel of $\overline{\lambda_i}$ and $\mathbb{H}_i$ is the orthogonal of $\mathbb{K}_i$ relative to $\Omega_{\phi(x)}$ over $\phi(x)$. Since the restriction $\overline{\lambda_i}'$ of $\overline{\lambda_i}$ to $\mathbb{H}_i$ is an isomorphism onto $\mathbb{E}_i$ and so $\overline{\delta_i} \times \overline{\lambda_i}'$ is an isomorphism from $\phi(U) \times \mathbb{H}_i$ onto $\phi(U) \times \mathbb{E}_i$. Thus, $(\Omega_i) = [\overline{\delta_i} \times \overline{\lambda_i}']^* \left((\Omega)_{|\phi(U)\times\mathbb{H}_i}\right)$ is a symplectic form on $\phi_i(U_i) \times \mathbb{E}_i)$ and so $\omega_i = \tau_i^*(\Omega)$ is a smooth symplectic form.
The end of the proof follows from Proposition 11.

□

Proof [Proof of Corollary 15] According to the assumption of this Corollary, after applying Theorem 13, the proof will be completed if we prove that the 2-form ω defined by the closed 2-form ω_i is also closed and if ω is a closed 2-form on the projective limit $M = \varprojlim M_i$ (each induced 2 form ω_i induced on M_i is closed). Under the notations of the proof of Theorem 13, we have

: $\mathbb{E}_i = \mathbb{M}_i$ and $\mathbb{E} = \mathbb{M}$;
: $\lambda_i^j = T\delta_i^j$, $\overline{\ell_i^j} = \overline{\delta_i^j}$, $\lambda_i = T\delta_i$;
: $\tau_i = T\phi_i$, $\tau = T\phi$.

We can apply the context of Lemma 7 and so if $\mathbb{M}_i' = \ker \overline{\delta_i^j}$, then $\mathbb{M}_n$ is isomorphic to $\prod_{i=0}^{n} \mathbb{M}_i'$ and so $\mathbb{M} \equiv \prod_{i=0}^{\infty} \mathbb{M}_i'$. According to Diagram 8 in our context, we have



(9)

and $\phi(U)$ is an open set in $\prod_{i=0}^{\infty} \mathbb{M}_i'$ and $\phi_i(U_i)$ is an open set of $\prod_{l=0}^{i} \mathbb{M}_l'$. Thus, $\phi_i(U_i)$ is of type $\prod_{l=0}^{i} U_l'$ where U_l' is an open set of $\mathbb{M}_i'$ and $\phi(U)$ is of type $\prod_{l=0}^{\infty} U_l'$ where U_l' is an open set of $\mathbb{M}_l'$ and with only a finite number of $l \geq i$ for which $U_l \neq \mathbb{M}_l'$.

(1) Assume that ω is a projective limit of the sequence $(\omega_i)_{i\in\mathbb{N}}$. As in the proof of Theorem 13 let Ω_i be the form on $\phi_i(U_i)$ induced by ω_i and we denote by Ω the symplectic form on $\phi(U)$ induced by ω according to the context of Diagram 9. If ι_l be the natural inclusion of U_l' in $\prod_{l=0}^{i} \mathbb{M}_l'$, we set $\overline{\Omega_l} = \iota_i^* \Omega_i$ for $l \geq i$. Note that $\overline{\Omega_l}$ does not depend on the choice of the integer $i \geq l$. As Ω_i is closed , it follows that $\overline{\Omega_l}$ is closed. Note that each subbundle $U_l' \times \mathbb{E}_l'$ is the tangent bundle of U_l'. But, from the construction of ω (and so Ω), if X_1 and X_2 are vector fields on $\phi(U)$ which are tangent to U_{l_1}' and U_{l_2}' respectively, we have $\Omega(X_1, X_2) = 0$ if $l_1 \neq l_2$ and $\Omega(X_1, X_2) = \overline{\Omega_l}(X_1, X_2)$ if $= l_2 = l_1 = l$. This implies that Ω is closed.

(2) Assume that ω is a symplectic form such that $T_x\delta_i : T_xM \to T_{x_i}M_i$ is a symplectic submersion. Then from Theorem 6 (2), ω induces a non degenerate 2-form ω_i on M_i.

Again let Ω (resp. Ω_i) be the 2-form on $\phi(U)$ (resp. $\phi_i(U_i)$) according to the context of Diagram 9. We must show that each Ω_i is closed. Since Ω is the projective limit of the sequence $(\Omega_i)_{i\in\mathbb{N}}$, according to Theorem 13 (2). Thus, as previously, if X_1 and X_2 are vector fields on $\phi(U)$ which are tangent to U_{l_1}' and U_{l_2}' respectively, we have $\Omega(X_1, X_2) = 0$ if $l_1 \neq l_2$ and $\Omega(X_1, X_2) = \overline{\Omega_l}(X_1, X_2)$ if $= l_2 = l_1 = l$. Thus $\Omega_i = \iota_i^* \Omega$ if ι_i is the natural inclusion of $\phi_i(U_i)$ in $\phi(U)$. It follows that each ω_i is closed.

□

3. Weak symplectic forms on a submersive projective sequence of reflexive Banach bundles and Darboux-Bambusi assumption

3.1. **Moser's method and Darboux-Bambusi Theorem.** We recall the following generalization of Moser's Lemma (see [23]).

Let M be a manifold modeled on a reflexive Banach space $\mathbb{M}$. Consider a weak symplectic form ω on M. Then $\omega^\flat : TM \to T^*M$ is an injective bundle morphism. According to section 2.1, we denote by $\widehat{T_xM}$ the Banach space which is the completion of T_xM

provided with the norm $||\ ||_{\omega_x}$ associated to some norm $||\ ||$ on T_xM. The Banach space $\widehat{T_xM}$ does not depend on this choice. Then ω_x can be extended to a continuous bilinear map $\hat{\omega}_x$ on $T_xM \times \widehat{T_xM}$ and $\omega_x^\flat$ becomes an isomorphism from T_xM to $(\widehat{T_xM})^*$. We set

$$\widehat{TM} = \bigcup_{x\in M} \widehat{T_xM} \ \text{ and } \ \widehat{TM}^* = \bigcup_{x\in M} (\widehat{T_xM})^*.$$

Theorem 17 (Moser's Lemma)**.** *Let ω be a weak symplectic form on a Banach manifold M modeled on a reflexive Banach space $\mathbb{M}$. Assume that we have the following properties:*

(i) *There exists a neighbourhood U of $x_0 \in M$ such that $\widehat{TM}_{|U}$ is a trivial Banach bundle whose typical fibre is the Banach space $(\widehat{T_{x_0}M}, ||\ ||_{\omega_{x_0}})$;*
(ii) *via a trivialization, ω can be extended to a smooth field of continuous bilinear forms on $TM_{|U} \times \widehat{TM}_{|U}$.*

Consider a family $\{\omega^t\}_{0\leq t\leq 1}$ of closed 2-forms, smoothly depending on t, with the following properties:

– *$\omega^0 = \omega$ and $\forall t \in [0,1]\,, \omega^t_{x_0} = \omega_{x_0}$;*
– *ω^t can be extended to a smooth field of continuous bilinear forms on $TM_{|U} \times \widehat{TM}_{|U}$.*

Then there exists a neighbourhood V of x_0 such that each ω^t is a symplectic form on V and there exists a family $\{F_t\}_{0\leq t\leq 1}$ of diffeomorphisms F_t from a neighbourhood $V_0 \subset V$ of x_0 to a neighbourhood $F_t(V_0) \subset V$ of x_0 such that $F_0 = \mathrm{Id}$ and $F_t^*\omega^t = \omega$, for all $0 \leq t \leq 1$.

Proof [sketch for more details see [23]] Without loss of generality, we may assume that U is an open neighbourhood of 0 in $\mathbb{M}$ and $\widehat{TM}_{|U} = U \times \widehat{\mathbb{M}}$. Therefore, $U \times \widehat{\mathbb{M}}$ is a trivial Banach bundle modeled on the Banach space $(\widehat{\mathbb{M}}, ||\ ||_{\omega_0})$. Since ω can be extended to a non-degenerate skew symmetric bilinear form (again denoted ω) on $U \times (\mathbb{M} \times \widehat{\mathbb{M}})$ then $\omega^\flat$ is a Banach bundle isomorphism from $U \times \mathbb{M}$ to $U \times \widehat{\mathbb{M}}^*$.
We set $\dot{\omega}^t = \frac{d}{dt}\omega^t$. Since each ω^t is closed for $0 \leq t \leq 1$, we have :

$$d\dot{\omega}^t = \frac{d}{dt}(d\omega^t) = 0$$

and so $\dot{\omega}^t$ is closed. After shrinking U if necessary, from the Poincaré Lemma, there exists a 1-form α^t on U such that $\dot{\omega}^t = d\alpha^t$ for all $0 \leq t \leq 1$. In fact α_t can be given by

$$\alpha^t_x = \int_0^1 s.(\dot{\omega}^t_{sx})^\flat(x)ds.$$

Since at $x = 0$, $(\omega^t_{x_0})^\flat$ is an isomorphism from $\mathbb{M}$ to $\widehat{\mathbb{M}}^*$, there exists a neighbourhood V of 0 such that $(\omega^t_x)^\flat$ is an isomorphism from $\mathbb{M}$ to $\widehat{\mathbb{M}}^*$ for all $x \in V$ and $0 \leq t \leq 1$. In particular, ω^t is a symplectic form on V. Moreover $x \mapsto (\dot{\omega}^t_x)^\flat$ is smooth and takes values in $\mathcal{L}(\mathbb{M}, \widehat{\mathbb{M}}^*)$. We set $X^t_x := -((\omega^t_x)^\flat)^{-1}(\alpha^t_x)$. It is a well defined time dependent vector field and let Fl_t be the flow generated by X^t defined on some neighbourhood $V_0 \subset V$ of 0. As for all $t \in [0,1]$, $\dot{\omega}^t_{x_0} = 0$, then $X^t_{x_0} = 0$. Thus, for all $t \in [0,1]$, $F_t(x_0) = x_0$. As classically, we have

$$\frac{d}{dt}\,\mathrm{Fl}_t^*\,\omega^t = \mathrm{Fl}_t^*(L_{X^t}\omega^t) + \mathrm{Fl}_t^*\,\frac{d}{dt}\omega^t = \mathrm{Fl}_t^*(-d\alpha^t + \dot{\omega}^t) = 0.$$

Thus $\mathrm{Fl}_t^*\,\omega^t = \omega$. □

Now as a Corollary of Theorem 17, we obtain the Bambusi's version of Darboux Theorem ([2], Theorem 2.1).

Theorem 18 (Darboux-Bambusi Theorem)**.** *Let ω be a weak symplectic form on a Banach manifold M modeled on a reflexive Banach space $\mathbb{M}$. Assume that the assumptions (i) and (ii) of Theorem 17 are satisfied. Then there exists a chart (V, F) around x_0 such that $F^*\omega_0 = \omega$ where ω_0 is the constant form on $F(V)$ defined by $(F^{-1})^*\omega_{x_0}$.*

Definition 19. *The chart (V, F) in Theorem 18 will be called a Darboux chart around x_0.*

3.2. **Projective sequence of weak symplectic bundle reflexive Banach bundle with Darboux-Bambusi assumptions.**
Let $\mathbb{E} = \varprojlim \mathbb{E}_i$ be a projective limit of a projective sequence of reflexive Banach spaces $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)$. We can provide each Banach space $\mathbb{E}_i$ with a norm $\| \ \|_i$ such that $||\overline{\lambda_i^{i+1}}||_i^{\mathrm{op}} \leq 1$ for $i \in \mathbb{N}$.

We consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of weak symplectic forms on $\mathbb{E}_i$ and let $\omega_i^\flat : \mathbb{E}_i \to \mathbb{E}_i^*$ be the associated bounded linear operator. According to notations in Remark 2, we consider the norm $||u||_{\omega_i} = ||\omega_i^\flat(u)||_i^*$ where $|| \ ||_i^*$ is the canonical norm on $\mathbb{E}_i^*$ associated to $|| \ ||_i$. We have seen that the inclusion of the Banach space $(\mathbb{E}, || \ ||_i)$ in the normed space $(\mathbb{E}_i, || \ ||_{\omega_i})$ is continuous and we have denoted by $\widehat{\mathbb{E}}_i$ the Banach space which is the completion of $(\mathbb{E}_i, || \ ||_{\omega_i})$. Recall that from Remark 2, the Banach space $\widehat{\mathbb{E}}_i$ does not depend on the choice of the norm $|| \ ||_i$ on $\mathbb{E}_i$. According to section 2.1 (before Remark 2), $\omega_i^\flat$ can be extended to a symplectic submersion between $\hat{\mathbb{E}}_i$ and $\mathbb{E}_i^*$. Moreover, $\omega_i^\flat$ is an isomorphism from $\mathbb{E}_i$ to $\widehat{\mathbb{E}}_i^*$.

Lemma 20.

(1) *The sequence $(\widehat{\mathbb{E}}_i^*)_{i\in\mathbb{N}}$ is a projective sequence of Banach spaces and so $\widehat{\mathbb{E}}^* = \varprojlim \widehat{\mathbb{E}}_i^*$ is well defined. Moreover, if $\overline{\lambda_i^j}$ is surjective and its kernel is split, then the bonding map $\widehat{\overline{\lambda_i^j}} = \widehat{\mathbb{E}}_j^* \to \widehat{\mathbb{E}}_i^*$ also satisfies this assumption.*
(2) *The projective limit $\omega^\flat = \varprojlim \omega_i^\flat$ is well defined and is an isomorphism from $\mathbb{E}$ to $\widehat{\mathbb{E}}^*$*

Proof (1) It is sufficient to show that $\overline{\lambda_i^j}$ and $\omega_i^\flat$ give rise to a map $\widehat{\overline{\lambda_i^j}}$ from $\widehat{\mathbb{E}}_j^*$ into $\widehat{\mathbb{E}}_i^*$ and if $\overline{\lambda_i^j}$ is surjective and with a split kernel so is $\widehat{\overline{\lambda_i^j}}$. Indeed since $\omega_i^\flat$ is an isomorphism from $\mathbb{E}_i$ to $\widehat{\mathbb{E}}_i^*$, the bonding map $\widehat{\overline{\lambda_i^j}} = \omega_i^\flat \circ \overline{\lambda_i^j} \circ (\omega_i^\flat)^{-1}$ satisfied the announced properties in (1).
(2) is obvious.

□

Now we consider a reduced projective sequence (E_i, π_i, M_i) of Banach vector bundles where the typical fibre $\mathbb{E}_i$ is reflexive. The projective limit $E = \varprojlim E_i$ has a structure Fréchet bundle over $M = \varprojlim M_i$ with typical fibre $\mathbb{E} = \varprojlim \mathbb{E}_i$ (cf. Proposition 50).

Consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of compatible weak symplectic forms ω_i on E_i. According to the previous notations, since $\mathbb{E}_i$ is reflexive, we denote by $(\widehat{E_i})_{x_i}$ the Banach space which is the completion of $(E_i)_{x_i}$ provided with the norm $|| \ ||_{(\omega_i)_{x_i}}$. Then $(\omega_i)_{x_i}$ can be extended to a continuous bilinear map $(\hat{\omega}_i)_{x_i}$ on $(E_i)_{x_i} \times (\widehat{E_i})_{x_i}$ and $(\omega_i)_{x_i}^\flat$ becomes an isomorphism from $(E_i)_{x_i}$ to $(\widehat{E_i}_{x_i})$. We set

$$\widehat{E_i} = \bigcup_{x_i \in M_i} (\widehat{E_i}^*)_{x_i}, \quad \widehat{E_i}^* = \bigcup_{x_i \in M_i} (\widehat{E_i}^*)_{x_i}$$

According to the assumption of Theorem 18 we introduce the following terminology:

Definition 21. *Let (E_i, π_i, M_i) be a reduced projective sequence of Banach bundles whose typical fibre $\mathbb{E}_i$ is reflexive. Consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of compatible weak symplectic forms ω_i on E_i. We say that the sequence $(\omega_i)_{i\in\mathbb{N}}$ satisfies the Bambusi-Darboux assumption around $a \in M$ if there exists a projective limit chart $U = \varprojlim U_i$ around a such that:*

- **(i):** *for each $i \in \mathbb{N}$, $(\widehat{E_i})_{|U_i}$ is a trivial Banach bundle;*
- **(ii) :** *for each $i \in \mathbb{N}$, ω_i can be extended to a smooth field of continuous bilinear forms on $(E_i)_{|U_i} \times (\widehat{E_i})_{|U_i}$.*

Under these assumptions we have:

Proposition 22. *Consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of compatible symplectic forms ω_i on E_i which satisfies the Bambusi-Darboux assumption around $a \in M$. Then we have the following properties:*

1. *The projective limit $(\widehat{E}^*)_{|U} = \varprojlim(\widehat{E_i}^*)_{|U_i}$ is well defined and is a trivial Fréchet bundle with typical fibre $\widehat{\mathbb{E}} = \varprojlim\widehat{\mathbb{E}}_i$.*
2. *The sequence $\left(\omega_i^\flat\right)$ of isomorphisms from $E_{i|U_i}$ to $(\widehat{E_i}^*)_{|U_i}$ induces an isomorphism from $E_{|U}$ to $(\widehat{E}^*)_{|U}$.*

Proof (1) From our assumptions, for each i, we have a sequence of trivializations $\widehat{\tau}_i : (\widehat{E_i})_{|U_i} \to U_i \times \widehat{\mathbb{E}}_i$. Thus we obtain a sequence $\widehat{\tau}_i^{-1} : U_i \times \widehat{\mathbb{E}}_i^* \to (\widehat{E}_i^*)_{|U_i}$ of isomorphisms of trivial bundles. Now, from the proof of Lemma 20, we have the bonding map $\widehat{\lambda_i^i} : \widehat{\mathbb{E}}_j^* \to \widehat{\mathbb{E}}_i^*$ and by restriction to U_j we have a bonding map $\delta_i^j : U_j \to U_i$. So we get a bundle morphism $\delta_i^j \times \widehat{\lambda_i^i}$ from $U_j \times \widehat{\mathbb{E}}_j^*$ to $U_i \times \widehat{\mathbb{E}}_i^*$. Now the map

$$\widehat{\tau}_i^{-1} \circ (\delta_i^j \times \widehat{\lambda_i^i}) \circ \widehat{\tau}_j$$

is a bonding map for the projective sequence of trivial bundles $\left((\widehat{E_i}^*)_{|U_i}\right)$. Therefore the projective limits $\widehat{\tau} = \varprojlim\widehat{\tau}_i$ and $\widehat{E}^*_{|U} = \varprojlim(\widehat{E_i}^*)_{|U_i}$ are well defined and $\widehat{\tau}$ is a Fréchet isomorphism bundle from $U \times \widehat{\mathbb{E}^*}$ to $\widehat{E}^*_{|U}$, which ends the proof of (1).

(2) At first, from Proposition 6, then $\omega = \varprojlim\omega_i$ is a 2-form on E. From our assumption, since for each $i \in \mathbb{N}$ we can extend ω_i to a bilinear onto $(E_i)_{|U_i} \times (\widehat{E}_i)_{|U_i}$, this implies that $\omega_i^\flat$ is an isomorphism from $(E_i)_{|U_i}$ to $(\widehat{E_i}^*)_{|U_i}$. Consider the sequence of bonding maps $\left(\widehat{\lambda_i^i}\right)_{i\in\mathbb{N}}$ for the projective sequence $\left((\widehat{E}^*)_{|U_i}\right)$ previously defined. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
U_j \times \mathbb{E}_j & \xrightarrow{\tau_j^{-1}} & (E_j)_{|U_j} & \xrightarrow{\omega_j^\flat} & (\widehat{E}_j^*)_{|U_j} & \xrightarrow{\widehat{\tau}_j} & U_j \times \widehat{\mathbb{E}}_j^* \\
\downarrow{\scriptstyle \delta_i^j \times \ell_i^j} & & \downarrow{\scriptstyle \ell_i^j} & & \downarrow{\scriptstyle \widehat{\ell}_i^j} & & \downarrow{\scriptstyle \delta_i^j \times \widehat{\ell}_i^j} \\
U_i \times \mathbb{E}_i & \xrightarrow[\tau_i^{-1}]{} & (E_i)_{|U_i} & \xrightarrow[\omega_i^\flat]{} & (\widehat{E}_i^*)_{|U_i} & \xrightarrow{\widehat{\tau}_i} & U_i \times \widehat{\mathbb{E}}_i^*
\end{array}$$

It follows that the projective limit $\omega^\flat = \varprojlim\omega_i^\flat$ is well defined and is an isomorphism from $E_{|U}$ to $(\widehat{E}^*)_{|U}$. □

4. Problem of existence of Darboux charts on a strong reduced projective sequence of Banach manifolds

4.1. **Conditions of existence of Darboux charts.** Let $\left(M_i, \delta_i^j\right)$ be a submersive or reductive projective sequence of Banach manifolds where M_i is modeled on a Banach space $\mathbb{M}_i$. We first apply the previous results for $E_i = TM_i$.

Theorem 23.

1. *Consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of compatible weak symplectic forms ω_i on M_i. Then, for each $x \in M$, the projective limit $\omega_x^\flat = \varprojlim(\omega_i)_{x_i}^\flat$ is well defined and is an isomorphism from T_xM to $(\widehat{T_xM})^*$. Moreover $\omega_x(u,v) = \omega_x^\flat(u)(v)$ defines a smooth weak symplectic form on M.*
2. *Let ω be a symplectic form on a submersive projective limit manifold $M = \varprojlim M_i$. For all $i \in \mathbb{N}$, assume that the canonical projection $\delta_i : M \to M_i$ is a symplectic submersion. Then there exists a symplectic form ω_i on M_i such that $\delta_i^*\omega_i = \omega$ in restriction to $(\ker\delta_i)^\perp$ and the sequence $(\omega_i)_{i\in\mathbb{N}}$ is a sequence of compatible weak symplectic forms such that the weak symplectic form which is the projective limit of $(\omega_i)_{i\in\mathbb{N}}$ on M is exactly ω.*

Proof (1) Since $\omega(u,v) = \omega^\flat(u)(v)$, by application of Proposition 22 to $E_i = TM_i$, we obtain that ω is non degenerate. The proof of that ω is closed is formally the same as in the proof of Corollary 15 (1).
(2) is a direct consequence of Corollary 15 (2).

□

As in the Banach context, we introduce the notion of Darboux chart:

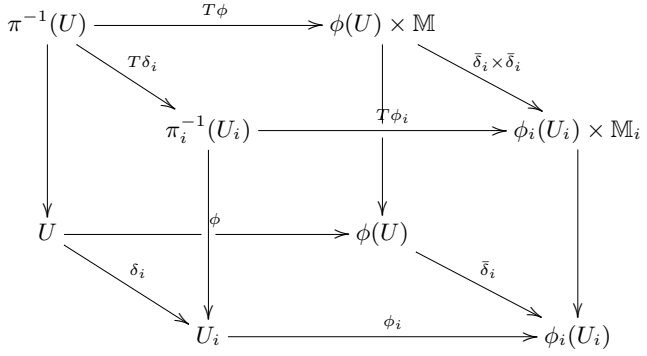
Definition 24. *Let ω be a weak symplectic form on the direct limit $M = \varprojlim M_i$. We say that a chart (V,ψ) around x_0 is a Darboux chart if $\psi^*\omega^0 = \omega$ where ω^0 is the constant form on $\psi(U)$ defined by $(\psi^{-1})^*\omega_{x_0}$.*

We have the following necessary conditions and sufficient conditions of existence of Darboux charts on a submersive projective sequence of Banach manifolds (cf. Theorem 3):

Theorem 25. *Let $\left(M_i, \delta_i^j\right)$ be a submersive or decreasing projective sequence of Banach manifolds where M_i is modeled on a reflexive Banach space $\mathbb{M}_i$.*

(1) *Consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of compatible symplectic forms ω_i on M_i and let ω be the symplectic form which is the projective limit of $(\omega_i)_{i\in\mathbb{N}}$ on $M = \varinjlim M_i$. Assume that the following property is satisfied:*
   (D): *There exists a projective limit chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around x^0 such that, for each $a_i = \delta_i(a) \in M_i$, then (U_i,ϕ_i) is a Darboux chart around a_i for ω_i.*
   Then (U,ϕ) is a Darboux chart around a for ω.
(2) *Let ω be a weak symplectic form on a submersive projective limit $M = \varprojlim M_i$ such that $\delta_i : M \to M_i$ is a symplectic submersion. Assume that there exists a Darboux chart (V,ϕ) around a in M.*
   If ω_i is the symplectic form on M_i induced by ω, then there exists a projective limit chart $(U = \varprojlim U_i, \phi = \varinjlim \phi_i)$ around a such that the property (D) is satisfied.

Proof (1) Assume that the assumption **(D)** is true and that $\left(M_i, \delta_i^j\right)$ is a reduced projective sequence of Banach manifolds. We fix some $a \in M$. We consider a projective limit chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around a such that, if $a_i = \delta_i(a) \in U_i$, then (U_i,ϕ_i) is a Darboux chart around a_i for ω_i. Now we have the following commutative diagram:



(10)

According to this diagram and modulo the diffeomorphisms ϕ and ϕ_i, we may assume that

- U is an open neighbourhood of $a \equiv 0 \in \mathbb{M}$, and U_i is a neighbourhood of $0 \in \mathbb{M}_i$;
- ω is a smooth 2-form on U and ω_i is a constant 2-form on U_i.

Now if $x = \varprojlim x_i \in U$, $u = \varprojlim u_i$ and $v = \varprojlim v_i$, since ω_i is constant on U_i it follows that $(\omega_i)_{x_i}(u_i,v_i)$ is independent of $x_i \in U_i$; so the value

$$\omega_x(u,v) = \varprojlim (\omega_i)_{x_i}(u_i,v_i)$$

is independent of the point x, which ends the proof of (1).

(2) Let ω be a weak symplectic form on $M = \varprojlim M_i$ such that, for all $i \in \mathbb{N}$, $\delta_i : M \to M_i$ is a symplectic submersion. Assume that we have a Darboux chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around a for ω. Fix some $i \in \mathbb{N}$. In the context of Diagram(10), we have $\mathbb{M} \equiv \mathbb{K}_i \times \mathbb{G}_i$ where $\mathbb{K}_i$ is the kernel of $T_0\delta_i$ and $\mathbb{G}_i$ is the orthogonal of $\mathbb{K}_i$ in $\mathbb{M}_i \equiv T_0M$ (cf. Diagram(7) with, for all $i \in \mathbb{N}$, $E_i = TM_i$). Thus again, modulo the diffeomorphisms ϕ and ϕ_i, we may assume that

- $a \equiv 0 \in U \subset \mathbb{K}_i \times \mathbb{G}_i$, $a_i \equiv 0 \in U_i \subset \mathbb{M}_i$;
- ω is a constant 2-form on U and ω_i is a smooth 2-form on U_i.

Recall that the restriction of $\overline{\delta_i}$ to $\mathbb{G}_i$ is an isomorphism onto $\mathbb{M}_i$, thus we may also assume that $\mathbb{G}_i = \mathbb{M}_i$. In this way, we have $\delta_i^*\omega_i = \omega$ in restriction to $\mathbb{G}_i = \mathbb{M}_i$. Thus, with our identification, ω_i is nothing but the restriction of ω to $U_i \times \mathbb{M}_i$ and so ω_i is a constant 2-form on U_i whose value is fixed by the restriction of ω to $\mathbb{M}_i$. □

4.2. **Problem of existence of Darboux chart in general.** In this subsection, we will explain why, even in the context of a submersive projective sequence of weak symplectic Banach manifolds which satisfies the assumption of Theorem 18 , in general, there does not exist any Darboux chart for the induced symplectic form on the projective limit.

Let $\left(M_i, \delta_i^j\right)$ be a projective sequence of Banach manifolds where M_i is modeled on a *reflexive* Banach space $\mathbb{M}_i$. Consider a sequence $(\omega_i)_{i\in\mathbb{N}}$ of compatible weak symplectic forms on M_i. Since $\mathbb{M}_i$ is reflexive, we denote by $\widehat{T_{x_i}M_i}$ the Banach space which is the completion of $T_{x_i}M_i$ provided with the norm $||\ ||_{(\omega_i)_{x_i}}$. Then $(\omega_i)_{x_i}$ can be extended to a continuous bilinear map $(\hat{\omega}_i)_{x_i}$ on $T_{x_i}M_i \times \widehat{T_{x_i}M_i}$ and $(\omega_i)^\flat_{x_i}$ becomes an isomorphism from $T_{x_i}M_i$ to $(\widehat{T_{x_i}M_i})^*$. We set

$$\widehat{TM_i} = \bigcup_{x_i\in M_i} \widehat{T_{x_i}M_i}, \qquad \widehat{TM_i}^* = \bigcup_{x_i\in M_i} (\widehat{T_{x_i}M_i})^*.$$

Then by application of Proposition 22, we have:

Proposition 26. *Let $\left(M_i, \delta_i^j\right)$ be a reduced projective sequence of Banach manifolds whose model is a reflexive Banach space $\mathbb{M}_i$. Consider a sequence (ω_i) of compatible weak symplectic forms ω_i on M_i. Assume that there exists a limit chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around $a \in M = \varprojlim M_i$ such that we have the following assumptions:* [13]

(i) *$(\widehat{TM_i})_{|U_i}$ is a trivial Banach bundle.*
(ii) *ω_i can be extended to a smooth field of continuous bilinear forms on $(TM_i)_{|U_i} \times (\widehat{TM_i})_{|U_i}$ for all $i \in \mathbb{N}$.*

Then $(\widehat{TM}^*)_{|U}$ is a trivial bundle. If ω is the symplectic form defined by the sequence $(\omega_i)_{i\in\mathbb{N}}$, then the morphism

$$\omega^\flat : TM \to T^*M$$

induces an isomorphism from $TM_{|U}$ to $(\widehat{TM})^*_{|U}$.

Note that the context of Proposition 26 covers the particular framework of projective limit of strong symplectic Banach manifolds (M_i, ω_i).

We will expose which arguments are needed to prove a Darboux theorem in the context of reduced projective sequence of Banach manifolds under the assumptions of Proposition 26. In fact, we point out the problems that arise in establishing the existence of a Darboux chart by Moser's method.

Case 1. Assume that $M = \varprojlim M_i$ is a reduced projective limit.
Fix some point $a = \varprojlim a_i \in M$. In the context on Proposition 26, on the projective limit chart (U, ϕ) around a, we can replace U by $\phi(U)$, ω by $\phi^*\omega$ on the open subset $\phi(U)$ of

[13] These assumptions correspond to the Bambusi-Darboux assumptions in Definition 21

the Fréchet space $\mathbb{M}$. Thus, if ω^0 is the constant form on U defined by ω_a, we consider the 1-parameter family

$$\omega^t = \omega^0 + t\overline{\omega},\ \text{with } \overline{\omega} = \omega - \omega^0.$$

Since ω^t is closed and $\mathbb{M}$ is a Fréchet space, by [13] Lemma 33.20, there exists a neighbourhood $V \subset U$ of a and a 1-form α on V such that $d\alpha = \overline{\omega}$ which is given by

$$\alpha_x := \int_0^1 s.\overline{\omega}_{sx}(x,\ )ds. \tag{11}$$

Now, for all $0 \leq t \leq 1$, ω_a^t is an isomorphism from $T_aM \equiv \mathbb{M}$ onto $\widehat{T_aM} \equiv \widehat{\mathbb{M}}^*$. In the Banach context, using the fact that the set of invertible operators is open in the set of operators, after restricting V, we may assume that $(\omega^t)^\flat$ is a field of isomorphisms from $\mathbb{M}$ to $\widehat{\mathbb{M}}^*$. *Unfortunately, this result is not true in the Fréchet setting.* Therefore, the classical proof does not work in this way in general.

Case 2. Assume that $M = \varprojlim M_i$ is submersive projective limit.
According to Theorem 23, assume that the canonical projection $\delta_i : M \to M_i$ is a symplectic submersion, for all $i \in \mathbb{M}$. Then ω induces a symplectic form ω_i on M_i. Therefore, for each i, let α_i be the 1-form induced by α on $\phi_i(U_i \cap V)$. Then we have $\omega_i = d\alpha_i$ and also

$$(\alpha_i)_{x_i} = \int_0^1 s.(\overline{\omega}_i)_{sx_i}(x_i,\ )ds \tag{12}$$

where $\bar{\omega}_i = \omega_i - \omega_i^0$ is associated to the 1-parameter family $\omega_i^t = \omega^i + t\bar{\omega}_i$. We are exactly in the context of the proof of Theorem 17 and so the local flow $\mathrm{Fl}_t^{X_i}$ of $X_i^t = ((\omega_i^t)^\flat)^{-1}(\alpha_i)$ is a local diffeomorphism from a neighbourhood W_i of a_i in V_i and, in this way, we build a Darboux chart around a_i in M_i. Therefore, after restricting each W_i, if necessary, assume that:

(PLDC): (projective limit Darboux chart) We have a projective sequence of such open sets $(W_i)_{n\in\mathbb{N}}$, then on $W = \varprojlim W_i$, the family of local diffeomorphisms $F^t = \varprojlim F_i^t$ is defined on W.

Recall that $\omega^\flat = \varprojlim \omega_i^\flat$ and $\omega^\flat$ is an isomorphism. Thus according to the previous notations, we have a time dependent vector field

$$X^t = ((\omega^t)^\flat)^{-1}(\alpha)$$

and again, we have $L_{X^t}\omega^t = 0$. Of course, if the **(PLDC)** assumption on $(W_n)_{n\in\mathbb{N}}$ is true, then $X^t = \varprojlim X_i^t$. So we obtain a Darboux chart as in the Banach context.

Remark 27. *In fact, under the assumption* **(PLDC)**, *the flow* Fl_t *is the local flow (at time* $t \in [0,1]$*) of* $X^t = \varprojlim X_i^t$ *where* $X_i^t = ((\omega_n^t)^\flat)^{-1}(\alpha_i)$ *(with the previous notations). Unfortunately, according to Remark 45, outside particularity special cases, the "Darboux chart" assumption is not true in general, since, in general,*

$$\bigcap_{j\geq i} \delta_i^j(W_j)$$

is not an open neighbourhood of a.

5. Proof of Theorem 4

We begin by a more precise version of Theorem 4. However we need some preliminaries

5.1. **Preliminaries and results.**

Definition 28.

(1) *Let $\mathbb{E}$ be a Banach space, M be a Banach manifold and $E = M \times \mathbb{E}$ be the trivial bundle over M. A function $||\ || : E \to \mathbb{R}$ is a Finsler structure on E if*
   i) *for each $x \in M$, the map $u \mapsto ||u||_x := ||(x,u)||$ is a norm which defines the topology of $\mathbb{E}$;*
   ii) *given $x_0 \in M$, and $k \geq 1$, there exists a neighbourhood U of x_0 in M such that*
$$\forall x \in U, \forall u \in \mathbb{E},\ \frac{1}{k}||u||_{x_0} \leq ||u||_x \leq k||u||_{x_0}.$$

(2) *A Banach bundle $\pi : E \to M$ with typical fibre $\mathbb{E}$. A map $||\ || : E \to \mathbb{R}$ is called a Finsler norm on E, if for any $x \in M$, there exists a trivilization $\Phi : E_{|U} \to U \times \mathbb{E}$ over an open neighbourhood U of x such that $||\ || \circ \Phi$ is a Finsler structure on $U \times \mathbb{E}$.*

If M is a Banach manifold modeled on $\mathbb{M}$, given a chart (U, ϕ) on M, a trivialization $\Phi : E_{|U} \to U \times \mathbb{E}$, and a norm $\|\ \|$ on $\mathbb{E}$, then we get a canonical Finsler norm on $U \times \mathbb{E}$, (again denoted $\|\ \|$), and so via Φ we can provide $E_{|U}$ with an associated Finsler norm.

Consider a submersive projective sequence (E_i, π_i, M_i) of Banach bundle. For each $x = \varprojlim x_i \in M = \varprojlim M_i$, there exists a projective chart bundle limit $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ and a trivialisation $(U, \Phi = \varprojlim \Phi_i)$ of $E = \varprojlim E_i$ over U where Φ_i is a trivialization of E_i over U_i. If we identify $\mathbb{E}_i$ with the fiber $(E_i)_{x_i}$ then Φ_i is an isomorphism from $(E_i)_{|U_i}$. Given a sequence of norms $((||\ ||_i)_{x_i})_{i\in\mathbb{N}}$ on $(E_i)_{x_i}$ by the previous argument, we obtain an associated Finsler norm $||\ ||_i$ on $(E_i)_{|U_i}$.
Under the assumptions of Proposition 26, given a sequence of norms $((||\ ||_i)_{a_i})_{i\in\mathbb{N}}$ on $T_{a_i}M_i$, as in section 2.1, we denote simply by $||\ ||_i$ the family od associated Finsler metric on $TM_{|U_i}$. Also, can consider the family of norms $(||\ ||_{(\omega_i)_{a_i}})$ on $T_{a_i}M_i$ which gives rise a family of norm $(||\ ||^{\widehat{*}}_{(\omega_i)_{a_i}})$ on $\widehat{T_{a_i}M_i}^*$. Thus the bundle $(\widehat{TM_i})^*_{|U}$ is provided with an associated Finsler norm denoted simply $||\ ||^{\widehat{*}}_i$. Recall that on the set of morphisms $\eta_i : TM_{i|U} \to (\widehat{TM_i})^*_{|U}$ we have an operator norm

$$||\eta_i||^{\mathrm{op}}_i = \sup\{||\eta_i(v)||^{\widehat{*}}_i :\ ||v||_i \leq 1\}$$

The vector field $\mathbf{E}_{\mathbb{M}}$ characterized by

$$\mathbf{E}_{\mathbb{M}}(f)(x) = \frac{d}{dt}_{|t=0} f(e^t x)^{14}$$

is the Euler vector field on $\mathbb{M}$ (in a classical sense). For any chart (U, ϕ) in M then $\mathbf{E} = \phi_*^{-1}(\mathbf{E}_{\mathbb{M}})$ is called an *an Euler vector field* on U. We will always consider such local Euler vector fields.

Here is a precise version of Theorem 4:

Theorem 29. *Let $M = \varprojlim M_i$ be a submersive projective limit of reflexive Banach manifolds M_i and $(\omega_i)_{i\in\mathbb{N}}$ a compatible sequence of symplectic forms (resp. ω a symplectic form on M such that $\delta_i : M \to M_i$ is a symplectic submersion).*
Assume that there exists a chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around a and sequence of norms $(||\ ||_{a_i})_{i\in\mathbb{N}}$ on $T_{a_i}M_i$ such that, for all $i \in \mathbb{N}$, we have:

(DB) *The Darboux-Bambusi assumptions are satisfied on U_i.*

(SN) *if $(||\ ||^{\mathrm{op}}_i)$ is the sequence of Finsler norms operator associated to $(||\ ||_i)_{i\in\mathbb{N}}$ on $TM_{i|U_i}$ and $\left((||\ ||^{\widehat{*}}_i)\right)_{i\in\mathbb{N}}$ on $(\widehat{TM_i})^*_{|U_i}$, there exists a continuous positive map K on U such that*
$$\forall x_i \in U_i,\ \ ||((\omega_i)^\flat_{x_i})||^{\mathrm{op}}_i \leq K(x);$$

[14] the same is true for any locally convex topological vector space

(EN) *consider the Euler vector field* $\mathbf{E}_i = \phi_*^{-1}(\mathbf{E}_{\mathbb{M}_i})$ *on* U_i *then there exists a continuous positive map* C *on* U *such*

$$\forall x_i \in U_i,\ \ ||(\omega_i - \phi_i^*\Omega_i)^\flat_{x_i}(\mathbf{E}_i)||_i^{\widehat{*}} \leq C(x)$$

where Ω_i *is the constant 2-form on* $\phi_i(U_i)$ *defined by* $(\phi_i^{-1})^*\left((\omega_i)_{a_i}\right)$

Then there exists a Darboux chart around a.

5.2. **Proof of Theorem 29.** For the sake of simplicity, since there will have non ambiguity in this proof, the Finsler norm $||\ ||_i^{\widehat{*}}$ will be simply denoted $||\ ||_i^*$.

The context of Theorem 4 and Theorem 29 is exactly the **case 2** *exposed in section 4.2 with the following assumptions :*

(i) there exists a limit chart $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ around a such that $(\widehat{TM_i})_{|U_i}$ is a trivial Banach bundle.

(ii) ω_i can be extended to a smooth field of continuous bilinear forms on $(TM_i)_{|U_i} \times (\widehat{TM_i})_{|U_i}$ for all $i \in \mathbb{N}$.

(iii)

$$\exists K(x) > 0 :\ \forall x_i = \delta_i(x) \in U_i,\ \forall i \in \mathbb{N},\ ||((\omega_i)^\flat_{x_i})||_i^{\mathrm{op}} \leq K(x); \tag{13}$$

$$\exists C(x) > 0 :\ \forall x_i = \delta_i(x) \in U_i,\ \forall i \in \mathbb{N},\ ||(\omega_i - \phi_i^*\Omega_i)^\flat_{x_i}(\mathbf{E}_i)||_i^* \leq C(x) \tag{14}$$

Thus, according Proposition 26, $(\widehat{TM}^*)_{|U} = \varprojlim (\widehat{TM_i})^*_{|U_i}$ is a trivial bundle and $\omega = \varprojlim \omega_i$ and the morphism

$$\omega^\flat : TM \to T^*M$$

induces an isomorphism from $TM_{|U}$ to $(\widehat{TM}^*)_{|U}$.

Since the problem is local, without loss of generality, we can assume that $M_i = U_i$ (resp. $M = U$) and is an open set of $\mathbb{M}_i$ and (resp. $\mathbb{M}$) and $a = 0 \in \mathbb{M}$. By the way we have $TM_i = U_i \times \mathbb{M}_i$ and $(\widehat{TM_i})^*_{|U_i} = U_i \times \widehat{\mathbb{M}_i}^*$. In this context, $x_i \mapsto ((\omega_i)^\flat_{x_i})$ is a smooth map from U_i to $\mathcal{GL}(\mathbb{M}_i, \widehat{\mathbb{M}_i})$ and $x_i \mapsto ((\omega_i)^\flat_{x_i})^{-1}$ is a smooth map from U_i to $\mathcal{GL}(\widehat{\mathbb{M}_i}, \mathbb{M}_i)$.

Now each Finsler norm $||\ ||_i$ on $\mathbb{M}_i$ (resp. $||\ ||_i^*$ on $\widehat{\mathbb{M}_i}^*$) induces a seminorm ν_i on $\mathbb{M}$ (resp. ν_i^* on $\widehat{\mathbb{M}}^*$). Since $(\omega_x^\flat) = \varprojlim((\omega_i)^\flat_{x_i})$ and $((\omega_x^\flat)^{-1} = \varprojlim((\omega_x^\flat)^{-1}$, according to our assumptions, it follows that $(\omega_i)^\flat_x)$ (resp. $((\omega_i)^\flat_x)^{-1}$) belongs to to $\mathcal{GH}_b(\mathbb{M}, \widehat{\mathbb{M}}^*)$ (resp. to $\mathcal{GH}_b(\widehat{\mathbb{M}}^*, \mathbb{M})$) (cf. Appendix F).

But, for all $t \in [0,1]$, we have $\omega_a^t = \omega_a$ so $(\omega_a^t)^\flat$ belongs to $\mathcal{GH}_b(\mathbb{M}, \widehat{\mathbb{M}}^*)$. By compactness of $[0,1]$, as $x \mapsto (\omega_x^t)^\flat$ is a smooth map from U to $\mathcal{H}_b(\mathbb{M}, \widehat{\mathbb{M}}^*)$ and by a continuity argument, from Proposition 58, we may assume that $(\omega_x^t)^\flat$ belongs to $\mathcal{GH}_b(\mathbb{M}, \widehat{\mathbb{M}}^*)$ for all $t \in [0,1]$, after shrinking U if necessary. Thus $((\omega_x^t)^\flat)^{-1}$ exists and belongs to $\mathcal{GH}_b(\widehat{\mathbb{M}}^*, \mathbb{M})$. Let denote by $||\ ||_\infty^{\mathrm{op}}$ the norm on $\mathcal{GH}_b(\mathbb{M}, \widehat{\mathbb{M}}^*)$ and $\mathcal{GH}_b(\widehat{\mathbb{M}}^*, \mathbb{M})$. Thus if $K = ||\omega_0^\flat||_\infty^{\mathrm{op}}$ and $k = ||(\omega_0^\flat)^{-1}||_\infty^{\mathrm{op}}$, again from Proposition 58, by a continuity argument, there exists an open neighbourhood $W = \varprojlim W_i$ of $a \in U$ such that

$$||(\omega_x^t)^\flat||_\infty^{\mathrm{op}} \leq 2K \ \ \forall x \in W \quad t \in [0,1] \tag{15}$$

$$(||((\omega_x^t)^\flat)^{-1}||_\infty^{\mathrm{op}} \leq 2k \ \ \forall x \in W \quad t \in [0,1] \tag{16}$$

On the other hand, from its definition, it follows that the sequence $(\mathbf{E}_i)$ is a projective sequence of vector fields and it projective limit $\mathbf{E} = \varprojlim \mathbf{E}_i$ is Euler vector field on $\mathbb{M}$ in the classical sense. Since in our context have $(\phi_i^*\Omega_i)_{x_i}$ is simply $(\Omega_i)_{x_i} = \omega_{a_i}$ for all $x_i \in U_i$, then $(\omega_i - \phi_i^*\Omega_i)^\flat_{x_i}(\mathbf{E}_i)$ is simply $(\omega_i - \Omega_i)^\flat_{x_i}(\mathbf{E}_i)$ and so we get

$$\varprojlim(\omega_i - \Omega_i)^\flat_{x_i}(\mathbf{E}_i) = (\omega - \Omega)^\flat_x(\mathbf{E}) \tag{17}$$

where obviously $\Omega_x = \varprojlim(\Omega_i)_{x_i} = \omega_a$ for all $x \in U$.
On the other hand by same arguments used to obtain (15), it is clear that from (14), after shrinking W if necessary, we get

$$||(\omega - \Omega_i)^\flat_x(\mathbf{E})||^*_\infty \leq 2C \quad \forall x \in W \tag{18}$$

Note that from the definition of $\mathbf{E}_i$, in our context, we have $\mathbf{E}_i(x_i) = x_i$ and so $\mathbf{E}(x) = x$. Thus the relations (12) and (11) are respectively exactly

$$(\alpha_i)_{x_i} = \int_0^1 (\omega_i - \Omega_i)^\flat_{sx_i}(\mathbf{E}_i)ds; \tag{19}$$

$$\alpha_x = \int_0^1 (\omega - \Omega)^\flat_{sx}(\mathbf{E})ds. \tag{20}$$

Thus from (18), we have

$$||\alpha_x||^*_\infty \leq 2C, \ \forall x \in W \tag{21}$$

where $||\ ||^*_\infty$ is the norm for uniformly bounded elements of $\widehat{\mathbb{M}}^*$.

According to Remark 27, the time dependent vector field $X^t = ((\omega^t)^\flat)^{-1}(\alpha)$ is defined on a star-shaped open set W_1 in W at 0. The following Lemma asserts that X^t satisfies the assumption of Theorem 60 which ends the proof.

Lemma 30. *We can choose the a star-shaped open set V of $0 \in U$ such that X^t is Lipschitz on V that is:*
there exists a a constant $\kappa >$ such that $\nu_i(X^t_x - X^t_{x'}) \leq \kappa\nu_i(x - x')$
for all $x, x' \in V$, all $t \in [0, 1]$ and all $i \in \mathbb{N}$.

Proof Note that from (17), (19) and (20) it follows that

$$\alpha = \varprojlim\alpha_i.$$

Since from (19) the map $x_i \mapsto (\alpha_i)_{x_i}$ is smooth, this implies that $x \mapsto \alpha_x$ is a smooth map from W_1 to $\widehat{\mathbb{M}}^*$ which is uniformly bounded from (21). Therefore this map is κ_1-Lipschitz for some $\kappa_1 > 0$ (cf. [3]) on some star-shaped open W_2 of $0 \in W_1$ that is

$$\nu^*_i(\alpha_x - \alpha_{x'}) \leq \kappa_1\nu_i(x - x'), \ \forall x, x' \in W_2, \ \forall i \in \mathbb{N}$$

From (16), by the same type of arguments for $x \mapsto ((\omega^t_x)^\flat)^{-1}$ we obtain that there exists a star-shaped open neighbourhood of W_3 of $0 \in W_1$ such that this map is κ_2-Lipschitz.
Taking in account (21), we obtain

$$\begin{aligned}\nu_i(X^t_x - X^t_{x'}) &\leq \nu_i\left(((\omega^t_x)^\flat)^{-1}(\alpha_x) - ((\omega^t_{x'})^\flat)^{-1}(\alpha_x)\right) + \nu_i\left(((\omega^t_{x'})^\flat)^{-1}(\alpha_x - \alpha_{x'})\right)\\ &\leq 2C\kappa_2\nu_i(x - x') + 2k\kappa_1\nu_i(x - x')\end{aligned}$$

for any $t \in [0, 1]$ and any $i \in \mathbb{N}$. This ends the proof by taking $V = W_3$ and $\kappa = 2C\kappa_2 + 2k\kappa_1$.

□

6. Examples and contre-example about the existence of a projective limit of Darboux charts

Example 31 (Example of existence of Darboux chart around any point).
According to [23] section 4, the set $\mathsf{L}^p_k(\mathbb{S}^1, M)$ of Sobolev loops of class L^k_p has a Banach structrue manifold and if where (M, ω) is a symplectic manifold, we can provide $\mathsf{L}^p_k(\mathbb{S}^1, M)$ with a weak symplectic form Ω_k and around any $\gamma \in \mathsf{L}^p_k(\mathbb{S}^1, M)$, we have a Darboux chart (cf. [23] Theorem 32). Moreover, $\mathsf{L}^2_k(\mathbb{S}^1, M)$ is a Hilbert space and Ω_k is a strong symplectic form. If we denote by $\mathsf{L}^\infty(\mathbb{S}^1, M)$ the set of smooth loops in M, we have $\mathsf{L}^\infty(\mathbb{S}^1, M) = \varprojlim\mathsf{L}^2_k(\mathbb{S}^1, M)$ and this space is a ILH-manifold. It is easy to see that the sequence of forms $(\Omega_k)_{k\in\mathbb{N}}$ are compatible and since the projective sequence $\left(\mathsf{L}^2_k(\mathbb{S}^1, M)\right)_{k\in\mathbb{N}}$ is reduced, we get a weak symplectic form $\Omega = \varprojlim\Omega_k$ on $\mathsf{L}^\infty(\mathbb{S}^1, M)$. In fact, Ω can be defined directly in the same way as Ω_k on each $\mathsf{L}^p_k(\mathbb{S}^1, M)$.
When $M = \mathbb{R}^{2m}$, consider the canonical (linear) Darboux form ω on $\mathbb{R}^{2m}$. Then we have a global Darboux chart for Ω on $\mathsf{L}^\infty(\mathbb{S}^1, \mathbb{R}^{2m})$ (cf. [16]). Of course, since we also have a

global Darboux chart on each $\mathsf{L}^2_k(\mathbb{S}^1,\mathbb{R}^{2m})$, we then get an example of projective limit of Darboux charts.

Example 32 (**Existence of Darboux chart under Darboux-Bambusi assumptions).** Let $(\mathbb{M}_i)_{i\in\mathbb{N}}$ be a sequence of Banach spaces. Consider the submersive projective sequence of Banach spaces $\left(\overline{\mathbb{M}}_i=\prod_{k=1}^{i}\mathbb{M}_k\right)_{i\in\mathbb{N}^*}$ of Banach spaces where $\bar{\delta}^j_i:\overline{\mathbb{M}}_j\to\overline{\mathbb{M}}_i$ is the canonical projection. Then the projective limit $\overline{\mathbb{M}}$ is the product $\prod_{k=1}^{\infty}\mathbb{M}_k$. On $\overline{\mathbb{M}}$ the projective limit topology is the product topology and it is also the topology of Fréchet manifold.
Now, assume that on each $\mathbb{M}_k$ we have a weak symplectic form ω_k such that, for some $\bar{a}=\varprojlim(a_1,\dots,a_n)\in\overline{\mathbb{M}}$, each symplectic form ω_k satisfies the assumptions (i) and (ii) of Theorem 17 at a_k and for all $k\in\mathbb{N}^*$. Then from this Theorem, around the point $a_k\in\mathbb{M}_k$, we have a Darboux chart (V_k,F_k).
For any $\bar{x}_n:=(x_1,\dots,x_n)\in\overline{\mathbb{M}}_n$ and $\bar{u}_n:=(u_1,\dots,u_n)$, $\bar{v}_n=(v_1,\dots,v_n)$ in $T_{\bar{x}_n}\overline{\mathbb{M}}_n$ we define the 2 form

$$\bar{\omega}_n(\bar{u}_n,\bar{v}_n):=\sum_{k=1}^{n}\omega_k(u_k,v_k).$$

Then $\bar{\omega}_n$ is also a weak symplectic form on $\overline{\mathbb{M}}_n$ and it is easy to see that $(\overline{V}_n,\overline{F}_n)$ is a Darboux chart for $\bar{\omega}_n$ around $\bar{x}_n$. Now it is clear that the sequence $(\bar{\omega}_n)_{n\in\mathbb{N}}$ of weak symplectic forms are compatible and so give rise to a weak symplectic form $\bar{\omega}$ on $\overline{\mathbb{M}}$. Then $(\overline{V}=\varprojlim\overline{V}_n,\varprojlim\overline{F}_n)$ is a Darboux chart around $\bar{a}:=\varprojlim\bar{a}_n$ if V is an open set that is $\overline{V}_n=\overline{M}_n$ for any $n\in\mathbb{N}$ execpt eventually for a finite subset $J\subset\mathbb{N}$. Such a situation occurs in the following contexts:

(1) ω_k is a linear Darboux form on the Banach space $\mathbb{M}_k$ for all $k\in\mathbb{N}$ eventually outside of finite set J (cf. section 2.1).
(2) ω_k is a weak linear symplectic form on the reflexive Banach space $\mathbb{M}_k$ for all $k\in\mathbb{N}^*$ eventually outside of finite set J (cf. [4] Proposition B.3 Point (3))
(3) $\mathbb{H}$ is a separable infinite-dimensional real Hilbert space and we consider:
   - $\mathbb{M}_k=\mathbb{H}$ for each integer $k\in\mathbb{N}$;
   - $S_k:\mathbb{H}\to\mathbb{H}$ is a compact operator with dense range, but proper subset of $\mathbb{H}$, which is self adjoint and positive[15] (such an operator is injective)
   - $\hat{\omega}$ a linear Darboux form on $\mathbb{H}$ and $\omega_k=S_k^*\hat{\omega}$ for at most a finite number of integers and otherwise $S_k=Id_{\mathbb{H}}$.

From the example produced in [19]*, we can obtain the following example for which there is no Darboux chart on a submersive projective limit of symplectic Banach manifolds:*

Example 33 (**Non existence of Darboux chart).** Let $\mathbb{H}$ be a separable infinite-dimensional real Hilbert space endowed with its inner product $<\ ,\ >$. If g is a weak Riemannian metric on $\mathbb{H}$, we may use the trivialization $T\mathbb{H}=\mathbb{H}\times\mathbb{H}$ to define a weak symplectic form ω in the following way ([19]):

$$2\,\omega_{(x,e)}((u,v),(u',v'))=D_xg_x(e,u).u'-D_xg_x(e,u').u+g_x(v',u)-g_x(v,u').$$

Then the operator $\omega^\flat_{(x,e)}:T_{(x,e)}\mathbb{H}\times\mathbb{H}\to T^*_{(x,e)}\mathbb{H}\times\mathbb{H}$ can be written as a matrix of type

$$\frac{1}{2}\begin{pmatrix}\Gamma_{(x,e)} & -g_x^\flat\\ g_x^\flat & 0\end{pmatrix}\tag{22}$$

Since $g_x^\flat$ is always injective by assumption, it follows that $\omega^\flat_{(x,e)}$ is always injective and is surjective if and only $g_x^\flat$ is so. It follows that if Σ is the set of points $x\in\mathbb{H}$ where $g_x^\flat$ is not surjective, then the set of points $(x,e)\in T\mathbb{H}$ where $\omega_{(x,e)}$ is not a strong symplectic

[15]such operators S_k exist since the Hilbert space $\mathbb{H}$ is separable and infinite-dimensional

form is precisely $\Sigma \times \mathbb{H}$.

As at the end of the above Example, let $S : \mathbb{H} \to \mathbb{H}$ be a compact operator with dense range, but proper subset of $\mathbb{H}$, which is self adjoint and positive. Given a fixed $a \in \mathbb{H}$, then $A_x = ||x-a||^2 Id_{\mathbb{H}} + S$ is a smooth field of bounded operators of $\mathbb{H}$ which is an isomorphism for all $x \neq a$ and $A_a(\mathbb{H}) \neq \mathbb{H}$ but $A_a(\mathbb{H})$ is dense in $\mathbb{H}$ (cf. [19]). Then $g_x(e,f) =< A_x(e), f >$ is a weak Riemaniann metric and the associated symplectic form $\omega_{(x,e)}$ is not a strong symplectic form if and only if (x,e) belongs to $\{a\} \times \mathbb{H}$ and, in this case, the range of $\omega^\flat_{(x,e)}$ is dense in $T^*_{(x,e)}(\mathbb{H} \times \mathbb{H}) \equiv \mathbb{H} \times \mathbb{H}$.

For each $k \in \mathbb{N}^*$ and any $x \in \mathbb{H}$ we set

$$(A_k)_x = ||x - \frac{a}{k}||^2 Id_{\mathbb{H}} + S.$$

We consider the Hilbert space $\overline{\mathbb{H}}_n = \prod_{k=1}^{n} \mathbb{H}_k$ where $\mathbb{H}_k = \mathbb{H}$ and provided with the inner product

$$< (x_1, \dots, x_n), (y_1, \dots, y_n) >_n = \sum_{k=1}^{n} < x_k, y_k >$$

As in the previous example, we identify $\overline{\mathbb{H}}_n$ with $\overline{\mathbb{H}}_n \times \{0\}$ in $\overline{\mathbb{H}}_{n+1}$. From now on, we will use the notations introduced in Example 32.
For any $\bar{x}_n = (x_1, \dots, x_n) \in \overline{\mathbb{H}}_n$, we set

$$(\ell_n)_{\bar{x}_n} = \left((A_1)_{x_1}, \dots, (A_n)_{x_n}\right).$$

We denote by $\bar{g}_n$ the Riemannian metric on $\overline{\mathbb{H}}_n$ defined by

$$(\bar{g}_n)_{\bar{x}_n}(\bar{u}_n, \bar{v}_n) =< (\ell_n)_{(x_1,\dots,x_n)}(\bar{u}_n), \bar{v}_n >_n$$

for all $\bar{u}_n$ and $\bar{v}_n$ in $\overline{\mathbb{H}}_n$. Thus we can consider the weak symplectic form $\overline{\omega_n}$ associated to $\overline{g_n}$ as above. Therefore the maximal open set on which $\bar{\omega_n}$ is a strong symplectic form is the open set

$$\mathcal{U}_n = T\overline{\mathbb{H}}_n \setminus \bigcup_{k=1}^{n} (\{\frac{a}{k}\} \times \prod_{k=2}^{n} \mathbb{H}_k) \times \overline{\mathbb{H}}_n$$

By construction, for all $j \geq n$ and $n \in \mathbb{N}^*$, we have

$$\delta^j_n \circ (\ell_j)_{\delta^j_n(\bar{x}_j)} = (\ell_n)_{\bar{x}_n} \circ \delta^j_n.$$

We set $\overline{\mathbb{H}} = \varprojlim \overline{\mathbb{H}}_n$. From all the above considerations, it follows that the sequence $(\bar{\omega_n})_{n\in\mathbb{N}^*}$ is a family of compatible weak symplectic forms which induces a weak symplectic form $\bar{\omega}$ on the Fréchet manifold $T\overline{\mathbb{H}} \equiv \overline{\mathbb{H}} \times \overline{\mathbb{H}}$ since, as in the general case, the cotangent space $T^*_{(\bar{x},\bar{u})}(\overline{\mathbb{H}} \times \overline{\mathbb{H}})$ does not have a Fréchet structure, which implies that $\bar{\omega}^\flat$ can not be surjective.
Now, for each $n \in \mathbb{N}^*$, as $(0,0)$ belongs to the open set $\mathcal{U}_n$, we have a Darboux chart $(\overline{V}_n, \overline{F}_n)$ around $(0,0) \in T\overline{\mathbb{H}}_n$ from the classical Darboux Theorem for strong symplectic Banach manifold (cf. [18] or [25] for instance). Since $\bar{\omega}_n$ is a strong symplectic form on $\overline{V}_n$ we must have $\overline{V}_n \subset \mathcal{U}_n$. But from the definition of $\mathcal{U}_n$, it follows that

$$\delta^n_1(\overline{V}_n) \cap T\overline{\mathbb{H}}_1 \subset \{(x,u) \in \mathbb{H} \times \mathbb{H} \ : \ ||x|| < \frac{1}{n}\}.$$

Therefore, according to Remark 45, the sequence $(\overline{V}_n, \overline{F}_n)$ is not a projective sequence of charts and so there is **no Darboux chart** for $\bar{\omega}$ around $(0,0) \in T\overline{\mathbb{H}}$.

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Example 34 (Existence of Darboux chart under Theorem 29 assumptions).
We again consider an Hilbert space a separable infinite-dimensional real Hilbert space endowed with its inner product $< , >$ and for each $k \in \mathbb{N}^*$ and any $x \in \mathbb{H}$ we set

$$(\Lambda_k)_x = \frac{1}{k^2}(||x||^2 S + Id_{\mathbb{H}})$$

where S is a compact operator with dense range, which is self adjoint and positive. Since the set of $\mathcal{GL}(\mathbb{H})$ automorphisms of $\mathbb{H}$ is open in the Banach space $\mathcal{L}(\mathbb{H})$ of endomorphisms of $\mathbb{H}$, there exists a ball $B(0,2\eta_k)$ such that $\Lambda_k(x)$ belongs to $\mathcal{GL}(\mathbb{H})$ and moreover such that

$$||\Lambda_k(x)||^{\mathrm{op}} \leq \frac{2}{k^2}$$

for all $x \in B(a,2\eta_k)$. Consider a smooth map $\theta_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta_k(t) = t$ for $0 \leq t \leq \eta_k$ and $\theta_k(t) = 0$ for $t > 2\eta_k$ and we set

$$(A_k)_x = \frac{1}{k^2}(\theta_k(||x||^2)S + Id_{\mathbb{H}})$$

Since $|\theta(t)| \leq 2\eta_k$ for all $t \in \mathbb{R}^+$ we obtain

$$||A_k(x)||^{\mathrm{op}} \leq \frac{2}{k^2}. \tag{23}$$

for all $x \in \mathbb{H}$.

Note since θ_k is smooth with compact support, there exist $C_k > 0$ such that $||d\theta_k|| \leq C_k$ and so

$$||d\theta(||x||^2)||^{\mathrm{op}} \leq 4C_k\eta_k \tag{24}$$

for all $x \in \mathbb{H}$.

Now, on $\mathbb{M} = \mathbb{H} \times \mathbb{H}$ provided with the inner product

$$< (x,x'),(y,y') >=< x,x' > + < y,y' >$$

we consider the 1-form

$$(\alpha_k)_{(x,x')}(u,u') = (A_k)_x(u) - (A_k)_{x'}(u)$$

and we set $\omega_k = d\alpha_k$. Then we have
$(\omega_k)_{(x,x')}((u,u'),(v,v')) =$

$$\frac{1}{2k^2}\left((2d\theta_k(||x||^2) < x,u > S(v) + < ((\theta(||x||^2)S + Id_{\mathbb{H}})(v),u > - < ((\theta(||x||^2)S + Id_{\mathbb{H}})(v),u >)\right)$$
$$-\frac{1}{2k^2}\left(-2d\theta_k(||x'||^2) < x',u' > S(v') + < ((\theta(||x'||^2)S + Id_{\mathbb{H}})(v'),u' > - < ((\theta(||x'||^2)S + Id_{\mathbb{H}})(u'),v' >\right)$$

It follows that $(\omega_k)^\flat_x(u,u')$ can be written as the matrix

$$\frac{1}{2k^2}\begin{pmatrix} \left((2d\theta_k(||x||^2) < x,u > S(\ ) + < ((\theta(||x||^2)S + Id_{\mathbb{H}})(\ ),u > - < ((\theta(||x||^2)S + Id_{\mathbb{H}})(u),\ >)\right) \\ \left(2d\theta_k(||x'||^2) < x',u' > S(\ ) - < ((\theta(||x'||^2)S + Id_{\mathbb{H}})(\ ),u' > + < ((\theta(||x'||^2)S + Id_{\mathbb{H}})(u'),\ >\right)\end{pmatrix}. \tag{25}$$

According to (25), (23), and (24) since $\theta_k \equiv 0$ for $t > 2\eta_k$ and we obtain

$$||(\omega_k)^\flat_{(x,x')}||^{\mathrm{op}} \leq \frac{1}{k^2}\left(2 + 4||S||^{\mathrm{op}}\eta_k(C_k+1)\right) \tag{26}$$

Now, from the construction of A_k, for $x = 0$, $(\omega_k)^\flat_{(0,0)}$ is an isomorphism. Thus, for η_k small enough, it follows that $(\omega_k)^\flat_{(x,x')}$ is an isomorphism for all $(x,x') \in \mathbb{M}$ and so ω_k is a strong symplectic form on $\mathbb{M}$.

Note that if we choose a decreasing sequence (η_k) so is the sequence (C_k) and in this way for any $k \geq 1$ we get

$$||(\omega_k)^\flat_{(x,x')}||^{\mathrm{op}} \leq \frac{1}{k^2}\left(2 + 4||S||^{\mathrm{op}}\eta_1(C_1+1)\right) \tag{27}$$

for all $(x,x') \in \mathbb{M}$.

Again from (25), (23), and (24), since $\theta_k \equiv 0$ for $t > 2\eta_k$ and and if (η_k) is decreasing we obtain

$$||(\omega_k)^\flat_{x,y)}(\mathbf{E}(x,x')) - (\omega_k)^\flat_{(00)}(\mathbf{E}(x,x'))||^{\mathrm{op}} \leq \frac{1}{k^2}||S||^{\mathrm{op}}(4\eta_1)^2(C_1+1) \tag{28}$$

for all $(x,x') \in \mathbb{M}$ where $\mathbf{E}$ is the classical Euler vector fields on $\mathbb{M}$.

By analogy with Example 32, we consider the Hilbert space $\overline{\mathbb{M}}_n = \prod_{k=1}^{n} \mathbb{M}_k$ where $\mathbb{M}_k = \mathbb{M}$ for $k = 1, \ldots, n$ and provided with the inner product

$$< (x_1, \ldots, x_n), (y_1, \ldots, y_n) >_n = \sum_{k=1}^{n} < x_k, y_k >$$

From now on, we will use analogous notations introduced in the previous Example 33. For any $\bar{x}_n = (x_1, \ldots, x_n) \in \overline{\mathbb{M}}_n$, here we set

$$(\ell_n)_{\bar{x}_n} = ((A_1)_{x_1}, \ldots, (A_n)_{x_n}) .$$

We can consider the symplectic form $\bar{\omega}_n = (\omega_1, \ldots \omega_n)$ on $\overline{M}_n$ where ω_k is the symplectic form on $\mathbb{M}_k$ associated to (ℓ_k) (as in Example 32) and we choose a decreasing sequence (η_k). At point $\bar{x}_n$ the linear map $(\bar{\omega}_n)^\flat_{\bar{x}_n}$ is the matrix of diagonal blocks $(\omega_k)^\flat_{x_k}$. According to (26), there exists $K > 0$ (independent of n) such that

$$||(\bar{\omega}_n)^\flat_{\bar{x}_n}||^{\mathrm{op}} \leq \left(\sum_{k=1}^{n} \frac{1}{k^2} \right) K \leq \frac{\pi^2}{6} K \tag{29}$$

for all $\bar{x}_n \in \overline{\mathbb{M}}_n$.

On the other hand, denote by $\mathbf{E_k}$ the Euler vector fields on $\mathbb{M}_k$ then the Euler vector field $\overline{\mathbf{E}}_n$ on $\overline{M}_n$ is $(\mathbf{E}_1, \ldots, \mathbf{E}_n)$. Then by analog arguments, from (28), there exists a $C > 0$ such that

$$||(\bar{\omega}_n)^\flat_{\bar{x}_n}(\bar{\mathbf{E}}_n(\bar{x}_n) - (\bar{\omega}_n)^\flat_{\bar{0}}(\bar{\mathbf{E}}_n(\bar{x}_n)||^{\mathrm{op}} \leq \frac{\pi^2}{6} C \tag{30}$$

Now, as in the Example 32, if we set $\overline{\mathbb{M}} = \varprojlim \overline{\mathbb{M}}_n$ it follows that the sequence $(\bar{\omega}_n)_{n \in \mathbb{N}^*}$ is a family of compatible strong symplectic forms which induces a weak symplectic form $\bar{\omega}$ on the Fréchet manifold $\overline{M}$. In particular, the assumptions of Proposition 26 are satisfied around point $\bar{0} \in \overline{M}$ for the global chart $(\overline{M}, Id)$. Moreover, according to (29) and (30), on this chart, the assumptions **(SN)** and **(EN)** are satisfied. It follows that there exists a Darboux chart around $\bar{0} \in \overline{M}$ which is a global chart.

Remark 35. *In* [15] *the required conditions for a sequence of compatible weak symplectic forms can be weakened. Precisely condition (2) is needed only for the sequence* (α_i) *defined by (12). So according to this Remark, it is easy to see that the example 34 satisfies the other assumptions required in Theorem 5.1 in* [15]. *Now consider any finite sequences of weak symplectic manifolds Banach manifold* $(M_1, \sigma_1), \ldots, (M_p, \sigma_p)$ *which satisfies the Darboux-Bambusi Theorem assumptions around some points* $a_i \in M_i$, $i = 1, \ldots, p$ *but not for a global chart (that is the associated vector field* X_i^t *given in the Moser's method is not defined on the whole manifold* M_i *for* $i = 1, \ldots, p$*). For instance, one can take for* M_i *the Sobolev Banach manifold of loops* $\mathsf{L}^p_k(\mathbf{S}, M)$ *where* M *is a finite dimensional symplectic manifold and use Theorem 5.9 in* [23]. *Then by taking* $\mathbf{M}_n = \prod_{i=1}^{k} M_i \times \overline{\mathbb{M}}_n$ *we can build a projective sequence of weak symplectic Banach manifolds for which the assumptions of Theorem 29 or Theorem 4 are satisfied but not the assumptions of Theorem 5.1 of* [15] *since on each* M_i *the vector fields* X_i^t *is not globally defined on* M_i*. Also the condition (2) in* [15] *for compatible sequences of weak symplectic form is not checked either.*

Appendix A. Projective limits of topological spaces

Definition 36. *A projective sequence of topological spaces is a sequence* $\big((X_i, \delta_i^j)\big)_{(i,j) \in \mathbb{N}^2,\ j \geq i}$ *where*

(PSTS 1): *For all* $i \in \mathbb{N}$, X_i *is a topological space;*
(PSTS 2): *For all* $(i, j) \in \mathbb{N}^2$ *such that* $j \geq i$, $\delta_i^j : X_j \to X_i$ *is a continuous map;*
(PSTS 3): *For all* $i \in \mathbb{N}$, $\delta_i^i = Id_{X_i}$*;*
(PSTS 4): *For all* $(i, j, k) \in \mathbb{N}^3$ *such that* $k \geq j \geq i$, $\delta_i^j \circ \delta_j^k = \delta_i^k$.

Notation 37. *For the sake of simplicity, the projective sequence* $\left(\left(X_i,\delta_i^j\right)\right)_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ *will be denoted* $\left(X_i,\delta_i^j\right)$.

An element $(x_i)_{i\in\mathbb{N}}$ of the product $\prod_{i\in\mathbb{N}} X_i$ is called a *thread* if, for all $j\geq i$, $\delta_i^j(x_j)=x_i$.

Definition 38. *The set* $X=\varprojlim X_i$ *of all threads, endowed with the finest topology for which all the projections* $\delta_i : X\to X_i$ *are continuous, is called the projective limit of the sequence* $\left(X_i,\delta_i^j\right)$.

A basis of the topology of X is constituted by the subsets $(\delta_i)^{-1}(U_i)$ where U_i is an open subset of X_i (and so δ_i is open whenever δ_i is surjective).

Definition 39. *Let* $\left(X_i,\delta_i^j\right)$ *and* $\left(Y_i,\gamma_i^j\right)$ *be two projective sequences whose respective projective limits are* X *and* Y.

A sequence $(f_i)_{i\in\mathbb{N}}$ *of continuous mappings* $f_i : X_i\to Y_i$, *satisfying, for all* $(i,j)\in\mathbb{N}^2$, $j\geq i$, *the coherence condition*

$$\gamma_i^j\circ f_j = f_i\circ\delta_i^j$$

is called a projective sequence of mappings.

The projective limit of this sequence is the mapping

$$\begin{array}{rccc} f: & X & \to & Y \\ & (x_i)_{i\in\mathbb{N}} & \mapsto & (f_i(x_i))_{i\in\mathbb{N}} \end{array}$$

The mapping f is continuous if all the f_i are continuous.

Appendix B. Projective limits of Banach spaces

Consider a projective sequence $\left(\mathbb{E}_i,\delta_i^j\right)$ of Banach spaces.

Remark 40. *Since we have a countable sequence of Banach spaces, according to the properties of bonding maps, the sequence* $\left(\delta_i^j\right)_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ *is well defined by the sequence of bonding maps* $\left(\delta_i^{i+1}\right)_{i\in\mathbb{N}}$.

Fix some norm $\|\ \|_i$ on $\mathbb{E}_i$, for all $i\in\mathbb{N}$. If $x=\varprojlim x_i$, then $p_n(x)=\max\limits_{0\leq i\leq n}\|x_i\|_i$ is a semi-norm on the projective limit $\mathbb{F}=\varprojlim\mathbb{E}_n$ which provides a structure of Fréchet space on this vector space (see [5]).

Definition 41. *A projective sequence* $\left(\mathbb{E}_i,\delta_i^j\right)$ *of Banach spaces is called reduced if the range of* δ_i^{i+1} *is dense for all* $i\in\mathbb{N}$.

Definition 42. *Two projective sequences* $\left(\mathbb{E}_i,\delta_i^j\right)$ *and* $\left(\mathbb{E'}_i,{\delta'}_i^j\right)$ *of Banach spaces are called equivalent if there exist isometries* $A_i : \mathbb{E}_i\to\mathbb{E'}_i$ *for all* $i\in\mathbb{N}$ *such that*

$$\delta_i^{i+1} = A_i^{-1}\circ{\delta'}_i^{i+1}\circ A_{i+1}.$$

Of course, any projective sequence $\left(\mathbb{E}_i,\delta_i^j\right)$ of Banach spaces is not reduced and, in general, such a sequence is not equivalent to a reduced one. However, by replacing each $\mathbb{E}_i$ by the closure $\mathbb{E}'_i$ in $\mathbb{E}_i$ of $\delta_i^{i+1}(\mathbb{E}_{i+1})$ and δ_i^{i+1} by the restriction ${\delta'}_i^{i+1}$ of δ_i^{i+1} to $\mathbb{E}'_{i+1}$, we produce a reduced sequence of Banach spaces $\left(\mathbb{E}'_i,{\delta'}_i^j\right)$ such that $\varprojlim\mathbb{E}_i=\varprojlim\mathbb{E}'_i$. Conversely, any Fréchet space provided with a countable family of semi-norms is topologically isomorphic to the projective limit of a reduced projective sequence.

A particular important case of projective limit of a reduced projective sequence of Banach spaces corresponds to the case of a decreasing sequence:

$$\mathbb{E}_0\supset\mathbb{E}_1\supset\cdots\supset\mathbb{E}_i\supset\mathbb{E}_{i+1}\supset\cdots$$

fulfilling, for any $i\in\mathbb{N}$, the properties:

(DecS 1): the inclusion $\iota_i^{i+1}:\mathbb{E}_{i+1}\to\mathbb{E}_i$ is continuous;
(DecS 2): $\mathbb{E}_{i+1}$ is dense in $\mathbb{E}_i$.

Then the projective limit $\varprojlim \mathbb{E}_i$ is the intersection $\bigcap_{i\in\mathbb{N}} \mathbb{E}_i$; it is called an inverse limit of Banach spaces or ILB for short (cf. [22]). In fact, any Fréchet space is an ILB space (cf. Appendix A).

Appendix C. Projective limits of differential maps

The following proposition (cf. [8], Lemma 1.2) is essential

Proposition 43. *Let $\left(\mathbb{E}_i, \delta_i^j\right)$ be a projective sequence of Banach spaces whose projective limit is the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}_i$ and $(f_i : \mathbb{E}_i \to \mathbb{E}_i)_{i\in\mathbb{N}}$ a projective sequence of differential maps whose projective limit is $f = \varprojlim f_i$. Then the following conditions hold:*

1. *f is smooth in the convenient sense (cf. [13])*
2. *For all $x = (x_i)_{i\in\mathbb{N}}$, $df_x = \varprojlim (df_i)_{x_i}$.*
3. *$df = \varprojlim df_i$.*

Appendix D. Projective limits of Banach manifolds

Definition 44. *The projective sequence $\left(M_i, \delta_i^j\right)_{j\geq i}$ is called projective sequence of Banach manifolds if*

- **(PSBM 1):** *M_i is a manifold modeled on the Banach space $\mathbb{M}_i$;*
- **(PSBM 2):** *$\left(\mathbb{M}_i, \overline{\delta_i^j}\right)_{j\geq i}$ is a projective sequence of Banach spaces;*
- **(PSBM 3):** *For all $x = (x_i) \in M = \varprojlim M_i$, there exists a projective sequence of local charts $(U_i, \varphi_i)_{i\in\mathbb{N}}$ such that $x_i \in U_i$ where one has the relation*

$$\varphi_i \circ \delta_i^j = \overline{\delta_i^j} \circ \varphi_j;$$

- **(PSBM 4):** *Under the previous assumptions if $\phi = \varprojlim U_i$ and $U = \varprojlim U_i$ then $\phi(U)$ is an open set of $\mathbb{M} = \varprojlim \mathbb{M}_i$.*

Under the assumptions **(PSBM 1)** and **(PSBM 2)** in Definition 44, the assumptions **(PSBM 3)**] and **(PSBM 4)** around $x \in M$ is called *the projective limit chart property* around $x \in M$ and $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ is called a *projective limit chart*.

The projective limit $M = \varprojlim M_i$ has a structure of Fréchet manifold modeled on the Fréchet space $\mathbb{M} = \varprojlim \mathbb{M}_i$ and is called a PLB-*manifold*. The differentiable structure is defined *via* the charts (U, φ) where $\varphi = \varprojlim \varphi_i : U \to (\varphi_i(U_i))_{i\in\mathbb{N}}$.
φ is a homeomorphism (projective limit of homeomorphisms) and the charts changings $\left(\psi \circ \varphi^{-1}\right)_{|\varphi(U)} = \varprojlim \left(\left(\psi_i \circ (\varphi_i)^{-1}\right)_{|\varphi_i(U_i)}\right)$ between open sets of Fréchet spaces are smooth in the sense of convenient spaces.

Remark 45. *If M is the projective limit of the sequence $\left(M_i, \delta_i^j\right)_{j\geq i}$, then, as a set, M can identified with*

$$\left\{(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} M_i : \ \forall j \geq i,\ x_i = \delta_i^j(x_j)\right\}.$$

Since each M_i is a topological space, we can provide $\prod_{i\in\mathbb{N}} M_i$ with the product topology and so, since each δ_i^j is continuous, it follows that M is a closed subset in $\prod_{i\in\mathbb{N}} M_i$ which can be provided with the induced topology generated by the open sets of type $\prod_{i\in\mathbb{N}} V_i \bigcap M$ where V_i is an open set of M_i for a finite number of indices i and otherwise $V_i = M_i$.
*However, under the previous identification, as Fréchet manifold, the topology on M is not this projective limit topology according to the assumption **(PSBM 4)**. By the way, this topology is generated by all the sets of projective limit of charts $(U = \varprojlim U_i)$ where U_i is a chart domain in M_i for all $i \in \mathbb{N}$. Of course in general such set U is not an open set*

in the projective limit topology. In fact, if $\delta_i = \varprojlim_{j\geq i}\delta_i^j$ *then we have* $U = \bigcap_{i\in\mathbb{N}} \delta_i^{-1}(U_i)$. *Without the assumption* ***(PSBM 4)****,* $\bigcap_{i\in\mathbb{N}} \delta_i^{-1}(U_i)$ *could be reduced to a point.*

The sequence $\left(M_i, \delta_i^j\right)_{j\geq i}$ is called *reduced projective sequence of Banach manifolds* if the sequence $\left(\mathbb{M}_i, \overline{\delta_i^j}\right)$ is a reduced projective sequence of Banach spaces. Then $\delta_i^j(M_j)$ is dense in M_i for all $j \geq i$. We will say that $\left(M_i, \delta_i^j\right)$ is a *reduced projective sequence* and $M = \varprojlim M_i$ is a *reduced* PLB*-manifold*. This situation occurs when the bonding map δ_i^j is a surjective submersion from M_j onto M_i for all $j \geq i$. In this case, we say that $\left(M_i, \delta_i^j\right)$ is a *surjective projective sequence* and $M = \varprojlim \mathbb{M}_i$ is a *surjective* PLB*-manifold*. More particular is the situation:

Definition 46. *The sequence* $\left(M_i, \delta_i^j\right)$ *is called submersive projective sequence of Banach manifolds if*

- **(SPSBM 1):** $\forall (i,j) \in \mathbb{N}^2 : j \geq i,\ \delta_i^j : M_j \to M_i$ *is a surjective submersion;*
- **(SPSBM 2):** *Around each* $x \in M = \varprojlim \mathbb{M}_i$*, there exists a projective limit chart* $\left(U = \varprojlim U_i, \varphi = \varprojlim \varphi_i\right)$*;*
- **(SPSBM 3):** *For all* $i \in \mathbb{N}$*, there exists a decomposition* $\mathbb{M}_i = \ker \bar{\delta}_i^{i+1} \oplus \mathbb{M}_i'$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} U_{i+1} & \xrightarrow{\varphi_{i+1}} & (\ker \bar{\delta}_i^{i+1} \times \mathbb{M}_i') \\ \Big\downarrow{\scriptstyle \delta_i^{i+1}} & & \Big\downarrow{\scriptstyle \bar{\delta}_i^{i+1}} \\ U_i & \xrightarrow{\varphi_i} & \mathbb{M}_i \end{array} \tag{31}$$

Such a chart is called a submersive projective limit chart around x*.*

The projective limit $M = \varprojlim \mathbb{M}_i$ of a submersive projective sequence $\left(M_i, \delta_i^j\right)$ is called a*submersive projective limit of Banach manifolds* or for short a *submersive* PLB*-manifold*. In this case, we have the following results (cf. [3])

Proposition 47. *Let* $\left(M_i, \delta_i^j\right)$ *be a surjective (resp. submersive) projective sequence. Then, for each* $i \in \mathbb{N}$*, the map* $\delta_i : M \to M_i$ *is surjective (resp. is a submersion).*

Under the assumptions of Proposition 47, in fact each $\delta_i^j : M_j \to M_i$ is a surjective submersion for all $j \geq i$ where $(i,j) \in \mathbb{N}^2$.

Another important situation of reduced PLB-manifold, is the case of ILB*-manifold* defined as follows:

Definition 48. *A* PLB*-manifold* $M = \varprojlim \mathbb{M}_i$ *is called* ILB*-manifold if*

- **(ILBM 1):** $\forall i \in \mathbb{N},\ M_{i+1} \subset M_i$*;*
- **(ILBM 2):** $\forall i \in \mathbb{N},\ \delta_i^{i+1} : M_{i+1} \to M_i$ *is the canonical inclusion which is a weak immersion with dense range.*

Note that this definition is stronger than the definition of ILB-manifold in the Omori's sense (see [22]) since we impose the condition **(PSBM4)**. In this case, $M = \bigcap_{i\in\mathbb{N}} M_i$.

Appendix E. Projective limits of Banach vector bundles

Let $\left(M_i, \delta_i^j\right)$ be a projective sequence of Banach manifolds where each manifold M_i is modeled on the Banach space $\mathbb{M}_i$.
For any integer i, let (E_i, π_i, M_i) be the Banach vector bundle whose type fibre is the Banach vector space $\mathbb{E}_i$ where $\left(\mathbb{E}_i, \lambda_i^j\right)$ is a projective sequence of Banach spaces.

Definition 49. $\left((E_i, \pi_i, M_i), \left(f_i^j, \delta_i^j\right)\right)_{j\geq i}$*, where* $f_i^j : E_j \to E_i$ *is a morphism of vector bundles, is called a projective sequence of Banach vector bundles on the projective sequence*

of manifolds $\left(M_i, \delta_i^j\right)$ *if for all* (x_i) *there exists a projective sequence of trivializations* (U_i, τ_i) *of* (E_i, π_i, M_i), *where* $\tau_i : (\pi_i)^{-1}(U_i) \to U_i \times \mathbb{E}_i$ *are local diffeomorphisms, such that* $x_i \in U_i$ *(open in* M_i*) and where* $U = \varprojlim U_i$ *is a non empty open set in* M *where, for all* $(i,j) \in \mathbb{N}^2$ *such that* $j \geq i$, *we have the compatibility condition*

(PLBVB): $\left(\delta_i^j \times \lambda_i^j\right) \circ \tau_j = \tau_i \circ f_i^j$.

With the previous notations, $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ is called a *projective bundle chart limit* The triple of projective limit $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i))$ is called a *projective limit of Banach bundles* or PLB-bundle for short.

The following proposition generalizes the result of [10] about the projective limit of tangent bundles to Banach manifolds.

Proposition 50. *Let* $\left((E_i, \pi_i, M_i), \left(f_i^j, \delta_i^j\right)\right)_{j \geq i}$ *be a projective sequence of Banach vector bundles.*
Then $\left(\varprojlim E_i, \varprojlim \pi_i, \varprojlim M_i\right)$ *is a Fréchet vector bundle.*

Remark that $\mathrm{GL}(\mathbb{E})$ cannot be endowed with a structure of Lie group. So it cannot play the role of structural group. We then consider, as in [9], the generalized Lie group $H^0(\mathbb{E}) = \varprojlim H_i^0(\mathbb{E})$ which is the projective limit of the Banach-Lie groups

$$H_i^0(\mathbb{E}) = \left\{ (h_1, \ldots, h_i) \in \prod_{j=1}^{i} \mathrm{GL}(\mathbb{E}_j) : \lambda_k^j \circ h_j = h_k \circ \lambda_k^j, \text{ for } k \leq j \leq i \right\}.$$

We then obtain the differentiability of the transition functions T.

Example 51. *As a particular case of Proposition 50, we can consider the projective sequence of tangent bundles* $\left((E_i, \pi_i, M_i), \left(T\delta_i^j, \delta_i^j\right)\right)_{j \geq i}$ *of a projective sequence of Banach manifolds* (M_i, δ_i^j). *Thus, if each* M_i *is modeled on the Banach space* $\mathbb{M}_i$, $\left(\varprojlim TM_i, \varprojlim \pi_i, \varprojlim M_i\right)$ *is a Fréchet vector bundle whose typical fibre is* $\mathbb{M} = \varprojlim \mathbb{M}_i$ *with structural group* $H^0(\mathbb{M})$. *As we have already seen, this result was firstly proved in* [10].

Notation 52. *From now on and for the sake of simplicity, the projective sequence of vector bundles* $\left((E_i, \pi_i, M_i), \left(f_i^j, \delta_i^j\right)\right)_{j \geq i}$ *will be denoted* (E_i, π_i, M_i).

As in Appendix D, we introduce

Definition 53. *A sequence* (E_i, π_i, M_i) *is called a submersive projective sequence of Banach vector bundles if* (E_i, π, M_i) *is a submersive projective sequence of Banach manifolds and if around each* $x \in M$, *there exists a projective limit chart bundle* $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ *such that for all* $i \in \mathbb{N}$, *we have a decomposition* $\mathbb{E}_{i+1} = \ker \bar{\lambda}_i^{i+1} \oplus \mathbb{E}'_i$ *such that the condition* (**PLBVB**) *is true.*

The projective limit (E, π, M) of a projective sequence of Banach vector bundles (E_i, π, M_i) is called a *submersive projective limit of Banach bundles* or *submersive* PLB-*bundle* for short.

Now, we have the following result whose proof is similar to Proposition 47:

Proposition 54. *Let* (E_i, π_i, M_i) *be a submersive projective sequence of Banach bundles. Then, for each* $i \in \mathbb{N}$, *the map* $\lambda_i : E \to E_i$ *is a submersion.*

Appendix F. The Banach space $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$

Let $(\mathbb{F}_1, \nu_n^1)$ (resp. $(\mathbb{F}_2, \nu_n^2)$) be a graded Fréchet space.
Recall that a linear map $L : \mathbb{F}_1 \to \mathbb{F}_2$ is *continuous* if

$$\forall n \in \mathbb{N}, \exists k_n \in \mathbb{N}, \exists C_n > 0 : \forall x \in \mathbb{F}_1, \nu_2^n(L.x) \leq C_n \nu_1^{k_n}(x).$$

The space $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ of continuous linear maps between both these Fréchet spaces generally drops out of the Fréchet category. Indeed, $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ is a Hausdorff locally convex topological vector space whose topology is defined by the family of semi-norms $\{p_{n,B}\}$:

$$p_{n,B}(L) = \sup_{x \in B} \left\{ \nu_n^2(L.x) \right\}$$

where $n \in \mathbb{N}$ and B is any bounded subset of $\mathbb{F}_1$. This topology is not metrizable since the family $\{p_{n,B}\}$ is not countable.
So $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ will be replaced, under certain assumptions, by a projective limit of appropriate functional spaces as introduced in [9].

We denote by $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$ the space of linear continuous maps (or equivalently bounded linear maps because $\mathbb{B}_1^n$ and $\mathbb{B}_2^n$ are normed spaces). We then have the following result ([5], Theorem 2.3.10).

Theorem 55. *The space of all continuous linear maps between $\mathbb{F}_1$ and $\mathbb{F}_2$ which can be represented as projective limits*

$$\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2) = \left\{ (L_n) \in \prod_{n \in \mathbb{N}} \mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n) : \varprojlim L_n \text{ exists} \right\}$$

is a Fréchet space.

For this sequence $(L_n)_{n \in \mathbb{N}}$ of linear maps, for any integer $0 \leq n \leq m$, the following diagram is commutative

$$\begin{array}{ccc} \mathbb{B}_1^n & \xleftarrow{(\delta_1)_n^m} & \mathbb{B}_1^m \\ {\scriptstyle L_n}\downarrow & & \downarrow{\scriptstyle L_m} \\ \mathbb{B}_2^n & \xleftarrow{(\delta_2)_n^m} & \mathbb{B}_2^m \end{array}$$

On $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$, the topology can be defined by the sequence of seminorms p_n given by

$$p_n(L) = \max_{0 \leq k \leq n} \sup \left\{ \nu_k^2(L.x), x \in \mathbb{F}_1, \ \nu_k^1(x) \leq 1 \right\}$$

so that $(\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2), p_n)$ is a graded Fréchet space.

Remark 56. *For $l \in \{1, 2\}$, given a graduation $\left(\nu_n^l\right)$ on a Fréchet space $\mathbb{F}_l$, let $\mathbb{B}_l^n$ be the associated local Banach space and $\delta_l^n : \mathbb{F}_l \to \mathbb{B}_l^n$ the canonical projection.*
The quotient norm $\tilde{\nu}_n^l$ associated to ν_n^l is defined by

$$\tilde{\nu}_n^l(\delta_n(z)) = \sup\{\nu_n^l(y) : \ \delta_n(y) = \delta_n(z)\}. \tag{32}$$

We denote by $(\tilde{\nu}_n^2)^{\mathrm{op}}$ the corresponding operator norm on $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$.
If $L = \varprojlim L_n$ where $L_n : \mathbb{B}_1^n \to \mathbb{B}_2^n$, then we have

$$(\tilde{\nu}_n^2)^{\mathrm{op}}(L_n) = \sup\{\tilde{\nu}_n^2(L_n.x), \ \ x \in \mathbb{B}_1^n \ \ \tilde{\nu}_n^1(x) \leq 1\} = \sup\{\nu_n^2(L.x), x \in \mathbb{F}_1, \nu^1(x) \leq 1\}.$$

This implies that

$$p_n(L) = \max_{0 \leq i \leq n} (\tilde{\nu}_i^2)^{\mathrm{op}}(L_n).$$

Definition 57. *Let $(\mathbb{F}_1, \nu_n^1)$ and $(\mathbb{F}_2, \nu_n^2)$ be graded Fréchet spaces. A linear map $L : \mathbb{F}_1 \to \mathbb{F}_2$ is called a uniformly bounded operator, if*

$$\exists C > O : \forall n \in \mathbb{N}, \ \nu_n(L(x)) \leq C \mu_n(x).$$

We denote by $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ the set of uniformly bounded operators. Of course $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ is contained in $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ and $L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ belongs to $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ if and only if $\sup_{n \in \mathbb{N}} p_n(L) < \infty$ and so

$$\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2) = [\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)]_b$$

that is the set of uniformly bounded elements of $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$, relative to the sequence of semi-norms (p_n).

When $\mathbb{F}=\mathbb{F}_1=\mathbb{F}_2$ and $\nu_n^1=\nu_n^2$ for all $n\in\mathbb{N}$, the set $\mathcal{H}(\mathbb{F},\mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F},\mathbb{F})$) is simply denoted $\mathcal{H}(\mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F})$).

We denote by $\mathcal{IH}_b(\mathbb{F}_1,\mathbb{F}_2)$ (resp. $\mathcal{SH}_b(\mathbb{F}_1,\mathbb{F}_2)$) the set of injective (resp. surjective) operators of $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$ with closed range.

Proposition 58. *([3])*

(1) *Each operator $L\in\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2)$ has a closed range if and only if, for each $n\in\mathbb{N}$, the induced operator $L_n:\mathbb{B}_1^n\to\mathbb{B}_2^n$ has a closed range.*
(2) *$\mathcal{IH}_b(\mathbb{F}_1,\mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$.*
(3) *$\mathcal{SH}_b(\mathbb{F}_1,\mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$.*

We are in situation to end this section by the following result:

Theorem 59. *([3])*

(1) *The Banach space $\mathcal{H}_b(\mathbb{F})$ has a Banach-Lie algebra structure and the set $\mathcal{GH}_b(\mathbb{F})$ of uniformly bounded isomorphisms of $\mathbb{F}$ is open in $\mathcal{H}_b(\mathbb{F})$.*
(2) *$\mathcal{GH}_b(\mathbb{F})$ has a structure of Banach-Lie group whose Lie algebra is $\mathcal{H}_b(\mathbb{F})$.*
(3) *If $\mathbb{F}$ is identified with the projective $\varprojlim\mathbb{B}^n$ we denote by $\exp_n:\mathcal{L}(\mathbb{B}_n)\to\mathcal{GL}(\mathbb{B}_n)$, then we a have a well defined smooth map $\exp:=\varprojlim\exp_n:\mathcal{H}_b(\mathbb{F})\to\mathcal{GH}_b(\mathbb{F})$ which is a diffeomorphism from an open set of $0\in\mathcal{H}_b(\mathbb{F})$ onto a neighbourhood of $\mathrm{Id}_{\mathbb{F}}$.*

Appendix G. A theorem of existence of ODE

The following result is in fact a reformulation in our context of Theorem 1 in [17].

Theorem 60. *Let $\mathbb{F}$ a Fréchet space realized as the limit of a surjective projective sequence of Banach spaces $(\mathbb{B}_n,\lambda_n^m)$ whose topology is defined by the sequence of seminorms $(\nu_n)_{n\in\mathbb{N}}$. Let I be an open interval in $\mathbb{R}$ and U be an open set of $I\times\mathbb{F}$. Then U is a surjective projective limit of open sets $U_n\subset I\times\mathbb{B}_n$. Consider a smooth map $f=\varprojlim f_n:U\to\mathbb{F}$, projective limit of maps $f_n:U_n\to\mathbb{B}_n$. [16] Assume that for every point $(t,x))\in U$, and every $n\in\mathbb{N}$, there exists an integrable function $K_n>0$ such that*

$$\forall\left((t,x),(t,x')\right)\in U^2,\ \nu_n(f(t,x)-f(t,x'))\leq K_n(t)\nu_n(x-x'). \tag{33}$$

and consider the differential equation:

$$\dot{x}=f(t,x). \tag{34}$$

(1) *For any $(t_0,x_0)\in U$, there exists $\alpha>0$ with $I_\alpha=[t_0-\alpha,t_0+\alpha]\subset I$, an open pseudo-ball $V=B(x_0,r)\subset U$ and a map $\Phi:I_\alpha\times I_\alpha\times V\to\mathbb{F}$ such that*

$$t\mapsto\Phi(t,\tau,x)$$

is the unique solution of (34) with initial condition $\Phi(\tau,\tau,x)=x$ for all $x\in V$.
(2) *V is the projective limit of the open balls V_n of $\mathbb{B}_n$. For each $n\in\mathbb{N}$, the curve $t\mapsto\lambda_n\circ\Phi(t,\tau,\lambda_n(x))$ is the unique solution $\gamma:I_\alpha\to\mathbb{B}_n$ of the differential equation $\dot{x}_n=\phi_n(t,x_n)$ with initial condition $\gamma(\tau)=\lambda_n(x)$.*

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[16] This means that we have: $\forall m\geq n,\ \lambda_n^m\circ f_m=f_n\circ(Id_{\mathbb{R}}\times\lambda_n^m)$

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