



HAL
open science

Rough Path-Dependent Volatility Models

Léo Parent

► **To cite this version:**

| Léo Parent. Rough Path-Dependent Volatility Models. 2024. hal-04751528

HAL Id: hal-04751528

<https://hal.science/hal-04751528v1>

Preprint submitted on 24 Oct 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Rough Path-Dependent Volatility Models

Léo Parent
PRISM Sorbonne
Paris 1 Panthéon-Sorbonne University
leo.parent@etu.univ-paris1.fr

November 9, 2022

Abstract

This article introduces a family of rough path-dependent volatility (RPDV) models that encompass a wide range of modeling two major empirical features of volatility: its rough behavior and its path dependence. After presenting it in its general form and its connections to other existing models in the literature, we provide a Markovian multi-factor approximation of this family of RPDV models, building upon the works of Abi Jaber (2019). Then, our analysis focuses on some specifications of RPDV models interpretable from an economic point of view, leading to the formulation of different hypotheses about both asset price and volatility formation mechanisms.

Keywords: Volatility modeling, path-dependent volatility, rough volatility, joint price and volatility dynamics.

JEL classification: C15, C32, G10, G12, G13, G14, G17.

1 Introduction

Continuous-time volatility modeling is a central issue in quantitative finance that has given rise to prolific academic literature in recent years. Among these new approaches, two particularly important classes of models emerged: the rough volatility (RV) and the path-dependent volatility (PDV) models.

The rough volatility literature has grown quickly after the seminal article *Volatility is Rough* (Gatheral *et al.* 2018) showing empirical evidence of volatility roughness over a wide range of financial markets. Thus different rough volatility models have emerged among which may be mentioned the rough Bergomi (RB) model (Bayer *et al.* 2016), the rough-Heston (RH) (El Euch and Rosenbaum 2018), and the super-Heston rough (SHR) models (Dandapani *et al.* 2019), or the quadratic rough Heston (QRH) model (Gatheral *et al.* 2020). These have shown strong practical interest to deal with pricing and hedging issues, but also for volatility forecasting (Gatheral *et al.* 2018, El Euch and Rosenbaum 2018, Rosenbaum and Zhang 2022).

In parallel, in recent years there has been a resurgence of interest in PDV models (Guyon 2014, Blanc *et al.* 2017, Gatheral *et al.* 2020, Jacquier and Lacombe 2020) motivated by the fact that empirical volatility clearly "depends on the path followed by the asset price in the recent past" (Guyon and Lekeufack 2023, p.1). This fact already shown by Zumbach works (2009, 2010) has been strongly reaffirmed by the recent article by Guyon and Lekeufack *Volatility Is (Mostly) Path-Dependent* (2023).

Given the significance of these two approaches, the primary objective of the article is to introduce a general class of volatility models that reconciles their respective advantages. For the sake of simplicity, models belonging to this class will be referred to as the rough path-dependent volatility (RPDV) models¹. The secondary objective of the article is to specifically consider the properties of certain models within this class.

¹This nomenclature does not imply that the model family introduced in this article exhaustively encompasses all path-dependent rough volatility models.

The paper is organized as follows. In section 2, we introduce a general form of PDV model that includes several volatility models popular in the academic literature. On that basis, we define the family of RPDV models. Then, using the results of Abi Jaber (2019), we expose a Markovian multi-factor approximation of these models. In section 3, we focus on a parsimonious version of the RPDV models from which different specifications economically consistent are proposed, leading to the formulation of competing hypotheses about price and volatility formation mechanisms.

2 Rough path-dependent volatility models

2.1 A general form of path-dependent volatility model

Let us introduce a general form of path-dependent volatility model defined by the following equation system:

$$\begin{cases} \frac{dP_t}{P_t} = \mu_t dt + \sigma_t dB_t \\ (\sigma_t)^p = \beta_0 + \beta_1^{(+)} (m_{1,t} - \bar{m}_1)_+^{a_1} + \beta_1^{(-)} (\bar{m}_1 - m_{1,t})_+^{a_1} + \beta_2 (m_{2,t})^{p/a_2} \\ m_{1,t} = \int_{I_t} \mathcal{K}_1(t-u) \frac{dP_u}{P_u} + \kappa_1 \int_{I_t} \mathcal{K}_1(t-u) (\theta_{1,u} - m_{1,u}) du, \\ m_{2,t} = \int_{I_t} \mathcal{K}_2(t-u) (\sigma_u)^{a_2} du + \kappa_2 \int_{I_t} \mathcal{K}_2(t-u) (\theta_{2,u} - m_{2,u}) du \end{cases} \quad (1)$$

with B a Brownian motion, I_t the time interval of integration with an upper bound less than or equal to t , where $p, a_1, a_2 \in \{1, 2\}$, and such as θ_j are \mathcal{F}_t -measurable processes, and \mathcal{K}_j are kernel functions continuous decreasing on \mathbb{R}_+ .

In this model, the path dependency of price dynamics is captured through processes m_1 and m_2 . The process m_1 corresponds to a past price trend variable and can be understood as a sort of weighted moving average of past returns. It allows to capture the dependence of the volatility process to the historical price path. The variable m_2 is a historical volatility factor that capture the asset price activity regardless of its trend. It can be viewed as a moving average of the volatility process either $p = 1$, or of the variance process σ^2 if $p = 2$. The impact of these variables on the volatility process depends on the specification of the second equation of (1). In this, the value of the parameter p plays a central role. In case $p = 1$, the multilinear function expressed by the right-hand term of the second equation of (1) defines the volatility, while if $p = 2$ this function defines the variance process.

Furthermore, if θ_1 and θ_2 are deterministic functions, the volatility process is a function of a single source of randomness B : the Brownian motion driving the asset price. The model is then purely path-dependent. Conversely, the functions θ_j can also include an exogenous source of randomness by taking, for example, the following form:

$$\theta_{j,t} dt = \bar{\theta}_{j,t} dt + \sigma_t dZ_t, \quad (2)$$

where $\bar{\theta}_j$ is a deterministic process and Z is a Brownian motion independent of B . In this case, the model is partially path-dependent and corresponds to a stochastic volatility (SV) model in the terminology of Guyon and Lekeufack (2023).

Thanks to its highly flexible form, the model includes a certain number of existing volatility models. Table 1 reports some of them and their respective specifications.

	p	a_1	a_2	$\mathcal{K}_1(\tau)$	$\theta_1(u)$	$\beta_1^{(+)}$	$\beta_1^{(-)}$	\bar{m}_1	$\mathcal{K}_2(\tau)$	$\theta_2(u)$	β_2
Model 1	2	1	–	$\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}$	–	β_1	$-\beta_1$	0	–		–
Model 2	2	2	2	$\gamma_1 e^{-\gamma_1 \tau}$	–	β_1	β_1	0	$\gamma_2 e^{-\gamma_2 \tau}$	$m_{2,u}$	–
Model 3	2	2	–	$\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}$	–	β_1	β_1	\bar{m}_1	–		–
Model 4	1	1	–	$\gamma e^{-\gamma \tau}$	$m_{1,u}$	0	$-\beta_1$	0	–	–	–
Model 5	1	1	2	$(\delta_1 + \tau)^{-\alpha_1}$	$m_{1,u}$	β_1	$-\beta_1$	0	$(\delta_2 + \tau)^{-\alpha_2}$	$m_{2,u}$	β_2

Table 1: Some model specifications that correspond to existing model in stochastic volatility literature.

Model 1 is a specific case of the famous rough Heston model (El Euch and Rosenbaum 2018) in which the price asset and the volatility depend on the same Brownian motion. The variance process is equal to

$$(\sigma_t)^2 = \beta_0 + \beta_1 m_{1,t}$$

$$m_{1,t} = \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \frac{dP_u}{P_u} + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \kappa_1 (\theta_{1,u} - m_{1,u}) du,$$

where $\alpha \in [0.5, 1]$ and

$$\theta_{1,t} dt = (\theta_t - \beta_0) dt + \sigma_t \nu dZ_t,$$

where Z is an independent Brownian motion from B , and θ is an explicit \mathcal{F}_0 -measurable process (El Euch and Rosenbaum 2018). Here thus, when $\nu = 0$, the variance process is linearly dependent on a moving average of past returns with power law weighting.

Model 2 proposed in Blanc *et al.* (2017) corresponds to a limiting model with a microstructural foundation that uses quadratic Hawkes processes. The variance is given by:

$$(\sigma_t)^2 = \beta_0 + \beta_1 (m_{1,t})^2 + \beta_2 m_{2,t},$$

$$m_{1,t} = \gamma_1 \int_0^t e^{\gamma_1(u-t)} \frac{dP_u}{P_u},$$

$$m_{2,t} = \gamma_2 \int_0^t e^{\gamma_2(u-t)} (\sigma_u)^2 du.$$

Here, m_1 and m_2 are respectively an exponentially weighted moving average (EWMA) of past returns and an EWMA of past variance. Note that other similar models based on quadratic Hawkes processes with other kernel type has also been studied in Dandapani *et al.* (2019).

Model 3 corresponds to a quadratic rough Heston model (QRHM) introduced by Gatheral *et al.* (2020) defined as

$$(\sigma_t)^2 = \beta_0 + \beta_1 (\bar{m}_1 - m_{1,t})^2,$$

$$m_{1,t} = \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \frac{dP_u}{P_u} + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} (\theta_1(u) - m_{1,u}) du,$$

with $\bar{m}_1 > 0$. On the contrary to model 1 and 2, it encodes an asymmetry of the feedback effect of the price trend on the volatility. Indeed, for the same absolute value of m_1 , volatility is higher when m_1 is negative than when it is positive.

Model 4 is a specific case of the EWMA HM of Parent (2021), in which

$$\begin{aligned}\sigma_t &= \beta_0 + \beta_1(\bar{m}_1 - m_{1,t})_+, \\ m_{1,t} &= \gamma_1 \int_0^t e^{(u-t)\gamma_1} \frac{dP_u}{P_u},\end{aligned}$$

with $\bar{m}_1 > 0$. Like the QRHM, this model encodes an asymmetry of the feedback effect of the price trend on the volatility but not in exactly the same way. Indeed, in this model, when m_1 falls below \bar{m}_1 , the feedback effect disappears. Another difference with models 1,2 and 3, is that $p = 1$ which means that the volatility process is linearly sensitive to the asset trend process m_1 when $m_{1,t} < \bar{m}_1$.

Model 5 is the PDV model introduced by Guyon and Lekeufack (2023), defined by

$$\begin{aligned}\sigma_t &= \beta_0 + \beta_1 m_{1,t} + \beta_2 \sqrt{m_{2,t}}, \\ m_{1,t} &= \int_{-\infty}^t (\delta_1 + t - u)^{-\alpha_1} \frac{dP_u}{P_u}, \\ m_{2,t} &= \int_{-\infty}^t (\delta_2 + t - u)^{-\alpha_2} (\sigma_u)^2 du.\end{aligned}$$

As in model 2, the volatility dynamics is not only driven by the past price dynamics but also by the variance price path through m_2 . However, contrary to the model of Blanc *et al.*, the volatility is linearly sensitive to m_1 . In addition, it is worth to note that in the case $\delta_1 = \delta_2 = 0$, the model is rough.

These different examples show thus the diversity of approaches to model the path-dependence of volatility which is included in the PDV model (1). Moreover, it is on this basis that the RPDV model family will be defined in the following section.

2.2 The rough path-dependent volatility models

We now define a *rough path-dependent volatility model* as a specification of 1 for which \mathcal{K}_1 and \mathcal{K}_2 are power law kernels of the form

$$\mathcal{K}_j(\tau) = \tau^{-\alpha_j} \tag{3}$$

where α_j is a strictly positive constant.

It may be noted that given this definition, models 1 and 3, and model 5 in the case $\delta_1 = \delta_2 = 0$ (see table 1) belong to the family of the RPDV models. These particular cases share with other RPDV models to be structurally adapted to capture two main features of empirical volatility: its rough behavior and its path dependency.

First, the ability of the model to capture the roughness of volatility is made possible by the power law kernels associated with processes m_1 and m_2 . This characteristic is important to the extent that empirical data tends to show that the rough behavior of volatility is verified for all classes of financial assets, at least at the daily scale (Gatheral *et al.* 2018). Besides this empirical evidence, the rough property of volatility is supported by a strong theoretical argument, that is this property constitutes a necessary condition for the existence of

market impact under the no-arbitrage assumption (Jusselin and Rosenbaum 2020). However, the existence of a market impact on trade is clearly established (Gomes and Waelbroeck 2015).

Second, the path-dependency of the RPDV models is clear given that dynamics of the volatility process are driven (either fully or partially) by past price dynamics through the processes $m_{1,t}$ and $m_{2,t}$. This path-dependence allows capturing important empirical phenomena occurring in the financial market like the Zumbach effect (Gatheral *et al.* 2020), but more generally, the fact that the volatility dynamics are mostly explainable by the past price path (Guyon and Lekeufack 2023).

In addition to their interest considered separately, the relevance of combining these two notions in a parametric model is also justified by the results in Rosenbaum and Zhang (2022), showing that machine learning methods for volatility forecasting tend to confirm that volatility is both rough and path-dependent.

Besides these properties, the model allows the decoupling short- and long-term behavior of volatility, a property whose importance was emphasized notably in Bennedsen *et al.* (2016). For instance, α_1 could take a low value and α_2 a high one. In this example, the long-term behavior of the volatility would be driven by a medium/long-term past price trend while its short-term behavior would be dominated by recent price activity regardless of its trend.

At a more technical level, it should be noted that the existence and uniqueness of the solution to the equation defining the volatility process are not guaranteed for all possible specifications. However, these conditions have been demonstrated for a certain number of cases, among which, for example, the modeling of volatility by a Rough Volterra square-root process (Abi Jaber 2019, 2021). For other specifications of 1, these existence and uniqueness conditions have been demonstrated in the context where \mathcal{K}_j are smooth kernels, but remain an open question for the case where these kernels take the form 3. This is the case, for instance, for a wide range of Volterra processes. For these types of specifications, the lack of an existence and uniqueness proof is not necessarily a problem, particularly if we limit its use as a simulation tool, since power law kernels can be approximated by smooth kernels (Abi Jaber 2019, 2021) as we will see in more detail in the following section (section 2.3). In this context, the theoretical inversion, which involves considering the power-law kernel as an approximation of the actual smooth kernel, can be performed. The rough model then becomes only an ideal-type and a way for parsimoniously approximating the actual model using a kernel with only one parameter α .

2.3 Markovian multi-factor approximation of the RPDV model

The RPDV models are non-Markovian and non-semimartingale, which makes efficient simulation challenging. Accordingly, the purpose of this section is to propose a Markovian multi-factor approximation of RPDV models to address this issue.

2.3.1 Rough kernel approximation

The key idea is to approximate the kernel $\mathcal{K}(\tau) = \tau^{-\alpha}$ by a sum of n exponential kernels defined by

$$\hat{\mathcal{K}}(\tau) = \sum_{i=1}^n w_i e^{-\gamma_i \tau}, \quad (4)$$

where $w_i, \gamma_i > 0 \forall i \in \{1, \dots, n\}$, in order to approximate variables m_1 and m_2 as sums of solutions of stochastic differential equations.

Different methods exist for determining the parameters $(w_i, \gamma_i)_{1 \leq i \leq n}$ in the academic literature (Bochud and Challet 2007, Abi Jaber 2019). Here we adopt the approach proposed by Abi Jaber (2019). This takes as its

starting point that the fractional kernel $\mathcal{K}(\tau)$ may be write as the Laplace transform of a positive measure μ^2

$$\mathcal{K}(\tau) = \int_0^\infty e^{-\gamma\tau} \mu(d\gamma), \quad \mu(d\gamma) = \frac{\gamma^{\alpha-1}}{\Gamma(\alpha)}$$

By approximating μ by a finite sum of Dirac measures $\sum_{i=1}^n w_i \delta_{\gamma_i}$, we obtain a kernel \mathcal{K}_E of the form 4. Then, using a geometric partition of $(w_i, \gamma_i)_{1 \leq i \leq n}$,

$$w_i = \int_{\eta_{i-1}}^{\eta_i} \mu(dx), \quad \gamma_i = \frac{1}{w_i} \int_{\eta_{i-1}}^{\eta_i} x \mu(dx),$$

with $\mu(dx) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} dx$ and $\eta_i = x^{i-\frac{n}{2}}$ we obtain:

$$w_i = \frac{(1-x^{-\alpha})x^{\alpha(i-\frac{n}{2})}}{\alpha\Gamma(\alpha)}, \quad \gamma_i = \frac{\alpha(x^{1+\alpha}-1)x^{i-1-\frac{n}{2}}}{(1+\alpha)(x^\alpha-1)}, \quad (5)$$

with $x > 1$ whose value can be determined by solving

$$x^* = \arg \min_{x>1} \|\hat{\mathcal{K}} - \mathcal{K}\|_{L^2(t_-, t_+)}.$$

With this specification method of $(w_i, \gamma_i)_{1 \leq i \leq n}$, $\hat{\mathcal{K}}$ converges toward \mathcal{K} in the L^2 sense when $n \rightarrow \infty$ (see proof in Abi Jaber 2019).

Note that in Abi Jaber (2019) or Rosenbaum and Zang (2019) t_- is zero. However, it may be consistent to choose a strictly positive value for this parameter, particularly when n is little. Indeed, because of the characteristic of rough type kernel, the very recent past (t close to zero) may represent a too high share of $\|\hat{\mathcal{K}} - \mathcal{K}\|_{L^2(0, t_+)}$ in comparison of its practical importance. For instance, in a context of medium-long term volatility simulation, the divergence between $\mathcal{K}(\tau)$ and $\hat{\mathcal{K}}(\tau)$ for a very small value of τ (ex.: τ is equal to one second) does not matter much, particularly if the discretization timestep is proportionally large (of the order of an hour or more). In such a case, it is coherent to set $t_- > 0$, because it allows choosing a value of x which minimizes the difference between $\mathcal{K}(\tau)$ and $\hat{\mathcal{K}}(\tau)$ for the range of values of τ which really impacts the quality of the estimation in the context considered.

2.3.2 Markovian approximation of the RPDV models

Using results exposed in section 2.3.1, the RPDV models may be approximated through the following multi-factor model Markovian in variables:

$$\left\{ \begin{array}{l} \frac{dP_t}{P_t} = \mu_t dt + \sigma_t dB_t, \\ \sigma_t = \left(\beta_0 + \beta_1^{(+)} (m_{1,t} - \bar{m}_1)_+^{\alpha_1} + \beta_1^{(-)} (\bar{m}_1 - m_{1,t})_+^{\alpha_1} + \beta_2 (m_{2,t})^{p/\alpha_2} \right)^{1/p}, \\ dM_{1,t} = \mathbf{1}_n \cdot \left(\frac{dP_t}{P_t} + \kappa_1 (\theta_1(t) - m_{1,t}) dt \right) - \Lambda_1 \odot M_{1,t} dt, \\ dM_{2,t} = \mathbf{1}_n \cdot \left((\sigma_t)^{\alpha_2} + \kappa_2 (\theta_2(t) - m_{2,t}) \right) dt - \Lambda_2 \odot M_{2,t} dt, \\ m_{1,t} = \langle W_1, M_{1,t} \rangle, \\ m_{2,t} = \langle W_2, M_{2,t} \rangle, \end{array} \right.$$

²The considered kernel being $\mathcal{K}(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}$ in the original article, the measure μ differs according to the kernel specificity.

where $\mathbf{1}_n$ is a $n \times 1$ vector of ones, W_j the vector of weights $(w_{j,i})_{1 \leq i \leq n}$ and Λ_j the vector of discount coefficients $(\gamma_{j,i})_{1 \leq i \leq n}$, with $(w_{j,i}, \gamma_{j,i})_{1 \leq i \leq n}$ given by 5), such as

$$W_j = \begin{bmatrix} w_{j,1} \\ \dots \\ w_{j,n} \end{bmatrix}, \quad \Lambda_j = \begin{bmatrix} \gamma_{j,1} \\ \dots \\ \gamma_{j,n} \end{bmatrix}.$$

As shown in appendix A, we have

$$m_{1,t} = \int_0^t \underbrace{\sum_{i=1}^n w_i e^{-\gamma_{1,i}(t-u)}}_{\hat{\kappa}_1(t-u)} \left(\frac{dP_u}{P_u} + \kappa_1 (\theta_{1,u} - m_{1,u}) du \right),$$

and

$$m_{2,t} = \int_0^t \underbrace{\sum_{i=1}^n w_i e^{-\gamma_i(t-u)}}_{\hat{\kappa}_2(t-u)} \left((\sigma_t)^{a_2} + \kappa_2 (\theta_{2,u} - m_{2,u}) \right) du.$$

This results in a Markovian model in variables that approximates the RPDV models through smooth kernels.

3 Some specifications of the RPDV model with interesting theoretical implications

The definition given previously to a RPDV model is deliberately very general in order to include a lot of possible specifications. The aim of this section is to focus on a more parsimonious specification of this model understandable from an economic point of view, which is defined by:

$$\begin{aligned} \sigma_t &= \beta_0 + \beta_1 m_{1,t}, \\ m_{1,t} &= \int_{I_t} (t-u)^{-\alpha} \left(\frac{dP_u}{P_u} + \kappa_1 (\theta_{1,u} - \sigma_u) du \right), \\ \theta_{1,t} dt &= \left(\theta_t - \frac{r_t}{\kappa_1} \right) dt + \nu \sigma_t dW_t \end{aligned}$$

where W is an independent Brownian motion from B , $\alpha, \beta_0 > 0$, $\beta_1 < 0$, $\kappa_1, \nu \in \mathbb{R}$, and with r the free risk rate. For the sake of clarity, we rewrite the expression of the volatility as:

$$\sigma_t = \beta_0 + \int_{I_t} (t-u)^{-\alpha} \left((\kappa(\theta_u - \sigma_u) + \beta r_u) du + \left(\bar{\nu} \sigma_t dW_t - \beta \frac{dP_u}{P_u} \right) \right), \quad (6)$$

with $\beta = -\beta_1$, $\kappa = \kappa_1 \beta_1$ and $\bar{\nu} = \nu \beta_1 \kappa_1$. Equivalently, by introducing a Brownian motion Z such that $dZ_t = \rho dW_t + \sqrt{1-\rho^2} dB_t$, and setting $\beta = -\bar{\beta} \rho$ and $\nu = \bar{\beta} \sqrt{1-\rho^2}$, we can also express the volatility process as follows:

$$\sigma_t = \beta_0 + \beta_1 \int_{I_t} (t-u)^{-\alpha} \left((\kappa(\theta_u - \sigma_u) + \beta(r_u - \mu_u)) du + \bar{\beta} \sigma_t dZ_u \right).$$

An initial observation is that in the case where $\bar{\nu} = 0$ and θ is a deterministic function, the model is purely path-dependent. Conversely, if $\bar{\nu}$ is nonzero, the model is only partially path-dependent and includes an exogenous source of randomness.

Moreover, the proposed model is a stochastic Volterra function when $I_t = [0 : t]$. While this type processes with a power-law kernel are commonly used in the volatility literature, the use of this type of Volterra processes in this context is largely confined to modeling variance through a Volterra square-root or Volterra Heston process (Abi Jaber 2019). In these modelings, therefore, $p = 2$ and $a_1 = 1$. Conversely, in the considered model, $p = a_1 = 1$, which makes the term m_1 homogenous to the volatility.

Another important element, this RPDV model does not include a feedback effect of the volatility through a process m_2 . This choice may seem questionable in view of the results of Guyon and Lekeufack (2023) which seem to show that this type of historical volatility factor explains, even before the price trend process, most of the historical and implied volatility. However, this apparent need to incorporate this type of factor can be largely caused by a spurious relationship induced by the exogenous randomness generated by the Brownian motion W . Consequently, the results of Guyon and Lekeufack, do not delegitimize this type of modeling.

On the contrary, the introduction of this model has an interest in proposing a credible parsimonious alternative to existing volatility models, capable of reproducing the main part of the empirical phenomena which characterize the volatility dynamics, but adopting a different modeling approach.

With this perspective in mind, we consider in this section certain specifications of 6 that are economically consistent, and we delve into the assumptions implied by these specifications. This has the dual objective of defining coherent parsimonious RPDV models and proposing different hypotheses about price and volatility formation mechanisms consistent with empirical data. To this end, we will start by looking at the case where θ is a constant, and on this basis, we will expose the conditions of compatibility of the parameters so that the model remains coherent from an economic point of view. Using these results, we then consider two cases where the value of θ is time-dependent with different assumptions about the determinant of price and volatility dynamics.

In addition, in the following considered model specifications, we will assume the drift of the asset price follows a quadratic relationship of the form³

$$\mu_t = r_t + \lambda_1 \sigma_t + \lambda_2 (\sigma_t)^2, \quad (7)$$

where λ_1 and λ_2 are positive or zero volatility and variance risk premium, respectively. This type of asset price drift, defined by a risk-free rate component and a risk premium component, aligns with a modeling approach similar to that of the intertemporal capital asset pricing model proposed by Merton (1973), under the assumption that the volatility of the considered asset is linked by a linear relationship with the instantaneous covariance of its returns with the returns of the market portfolio (covariance, which is assumed to be positive). About the quadratic form, this allows for modeling a potential convex relationship between the level of volatility and the risk premium.

Under this assumption, the volatility may be rewritten as the following Brownian semistationary (Barndorff-Nielsen and Schmiegel 2009) process:

$$\sigma_t = \beta_0 + \int_{I_t} (t-u)^{-\alpha} f_u(\sigma_u) du - \beta \int_{I_t} (t-u)^{-\alpha} \sigma_u dZ_u,$$

where

$$f_t(\sigma_t) = \kappa \theta_t - (\kappa + \beta \lambda_1) \sigma_t - \beta \lambda_2 (\sigma_t)^2. \quad (8)$$

From the equation 8, we introduce the notion of *attraction volatility* defined as a strictly positive real number $\bar{\sigma}$ which is a zero of the function f (i.e. $f(\bar{\sigma}) = 0$) and such that f is decreasing continuously in the neighborhood of $\bar{\sigma}$. This notion will facilitate the description of the behavior of the volatility dynamics for the different model specifications. Furthermore, for all considered specifications, $f_t(0) \geq 0$ and f_t is concave decreasing on an interval $[\bar{\sigma} : +\infty]$ with $\bar{\sigma} < +\infty$, $\forall t \in \mathbb{R}_+$, in order to ensure strict positivity and non-divergence of σ_t .

³The assumption of constant risk premia is lifted in section 3.3.2.

3.1 The function θ as a constant

The θ function plays a central role in the dynamics of volatility. A natural specification is to consider it as a constant.

The first option is to assume that θ is strictly positive. In such a case, for $f(0) > 0$, κ must also be positive. Under this specification, because of the term $\kappa(\theta - \sigma_t)$, θ could seem to correspond to the long-term volatility level towards which the volatility process tends to revert like in the Heston (Heston 1993) or OU (Schöbel and Zhu 1999) models. However, except in the risk-neutral measure hypothesis (i.e. when λ_1 and λ_2 are zero), the volatility level towards which the volatility tends to revert is not equal to θ . Indeed, in addition to the difference between θ and σ_t , another component plays a mean reverting role: the drift of the price μ_t defined by equation 7. It induces a risk premium effect $-\beta(\lambda_1\sigma_t + \lambda_2(\sigma_t)^2)$ which is a negative continuously decreasing function of the volatility on \mathbb{R}_+ .

The second option is to assume that θ and κ are negative. The underlying hypothesis is the existence of positive volatility feedback through $-\kappa\sigma_t$. At first sight, under this specification, the model may seem to be divergent due to the combination of positive feedback from the volatility process and the characteristics of the power law kernel. However, due to the risk premium effect mentioned above, this is not the case under appropriate specifications of the drift process μ_t , which are detailed in the rest of this section.

The price drift, therefore, plays a central role in the dynamics of volatility, regardless of the value taken by θ . Let us, therefore, consider in turn the cases $\lambda_2 = 0$ and $\lambda_2 > 0$.

3.1.1 The case $\lambda_2 = 0$

If we consider the case $\lambda_2 = 0$, f is defined by⁴:

$$f(\sigma_t) = \kappa\theta - (\kappa + \beta\lambda_1)\sigma_t. \quad (9)$$

So that σ_t not to explode, $-\kappa \leq \beta\lambda_1$. In addition, κ and θ must share the same sign in order that $f(0) \geq 0$. If κ is positive, the condition $-\kappa \leq \beta\lambda_1$ is clearly satisfied. However, this inequality is more interesting for $\kappa < 0$. In this case, it means concretely that the risk premium effect that puts downward pressure on volatility must be greater than the positive volatility feedback. As long as this condition is met, regardless the κ value, f decrease linearly with the volatility and admit unique constant attraction volatility given by:

$$\bar{\sigma} = \frac{\kappa\theta}{\kappa + \beta\lambda_1}. \quad (10)$$

The attraction volatility results thus from two effects: a volatility feedback effect and a risk premium effect. If $\kappa > 0$ both ones share the same direction. Conversely, if $\kappa < 0$, these two effects are antagonistic. In both cases, f can be rewritten as follows:

$$f(\sigma_t) = (\kappa + \beta\lambda_1)(\bar{\sigma} - \sigma_t), \quad (11)$$

and therefore,

$$\sigma_t = \beta_0 + \bar{\kappa} \int_{I_t} (t-u)^{-\alpha} (\bar{\sigma} - \sigma_u) du - \beta \int_{I_t} (t-u)^{-\alpha} \sigma_u dZ_u, \quad (12)$$

with $\bar{\kappa} = \kappa + \beta\lambda_1$. Equation 12 highlights thus the mean-reverting component that pushes the volatility process towards the attraction level $\bar{\sigma}$. Thus, the greater the risk premium λ_1 , the lower the attraction volatility and the higher the mean-reverting force $\bar{\kappa}$.

⁴We denote f without a time index when its time dependency only depends on σ_t .

The $\theta = 0$ case is specific since it does not admit an attraction volatility. The volatility is then equal to

$$\sigma_t = \beta_0 - \int_{I_t} (t-u)^{-\alpha} \sigma_u (\beta dZ_u - \bar{\kappa} du).$$

It is interesting to note that if $\bar{\kappa} = 0$ ($\kappa = -\beta\lambda_1$), the volatility becomes an affine function of a power-law moving average of the past returns adjusted to its deterministic component. In economic terms, it may be interpreted as the fact that the risk premiums and the instantaneous volatility are known by the market and, therefore, only stochastic fluctuations (modeled by Brownian dynamics) have an impact on the dynamics of volatility. Moreover, it may be noted that, by property of the stochastic integral, $\mathbb{E}[\sigma_t] = \beta_0$.

3.1.2 The case $\lambda_2 > 0$

Now consider the behavior of f under the assumption $\lambda_2 > 0$. Given this configuration, f admits as unique attraction volatility

$$\bar{\sigma} = \frac{\kappa + \beta\lambda_1 - \sqrt{(\kappa + \beta\lambda_1)^2 + 4\beta\lambda_2\kappa\theta}}{-2\beta\lambda_2}. \quad (13)$$

and f may be rewritten as (see appendix B):

$$f(\sigma_t) = (\kappa + \beta\lambda_1)(\bar{\sigma} - \sigma_t) + \beta\lambda_2(\bar{\sigma}^2 - (\sigma_t)^2). \quad (14)$$

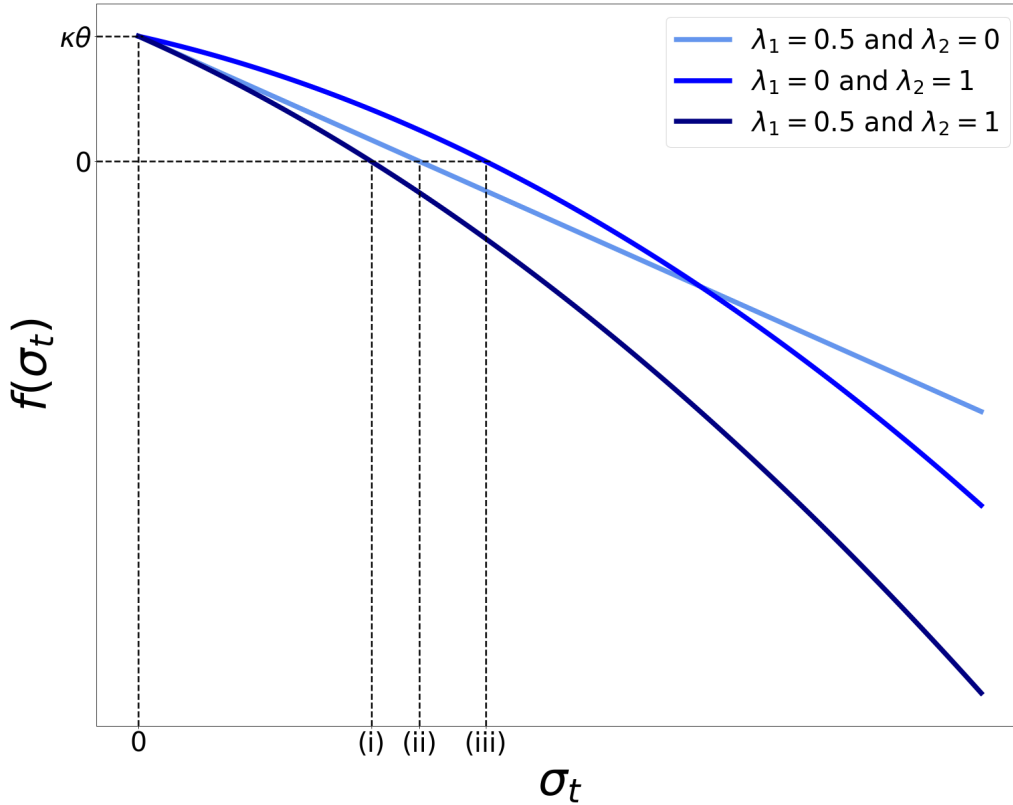


Figure 1: Plot of $f(\sigma_t)$ when κ and θ are strictly positive with different hypotheses about the price drift μ_t , with respectively (i) $= \frac{\sqrt{(\kappa + \beta\lambda_1)^2 + 4\kappa\theta\beta\lambda_2} - \kappa - \beta\lambda_1}{2\beta\lambda_2}$, (ii) $= \frac{\kappa\theta}{\kappa + \beta\lambda_1}$, and (iii) $= \frac{\sqrt{\kappa^2 + 4\kappa\theta\beta\lambda_2} - \kappa}{2\beta\lambda_2}$.

The existence of a non-zero variance risk premium, therefore, led to the emergence of a second mean-reverting

term $-\beta\lambda_2(\bar{\sigma}^2 - (\sigma_t)^2)$ - which depends linearly on the differential between the attraction variance $\bar{\sigma}^2$ and the variance process $(\sigma_t)^2$. However, two cases must be treated separately: $-\kappa \leq \beta\lambda_1$ and $-\kappa > \beta\lambda_1$. In the first case ($-\kappa \leq \beta\lambda_1$), f is continuously decreasing in volatility (on \mathbb{R}_+), but contrary to the case defined by equation 9, this relationship is strictly concave. On a practical level, this direct consequence of the quadratic form of the drift process reduces the probability of occurrence of very high levels of volatility in comparison with the pure volatility premium hypothesis (i.e. $\lambda_2 = 0$).

However, if $-\kappa > \beta\lambda_1$, the behaviour of f change dramatically as shown in table 2.

σ_s	0	$-\frac{\kappa + \beta\lambda_1}{2\beta\lambda_2}$	$\frac{\kappa + \beta\lambda_1 - \sqrt{(\kappa + \beta\lambda_1)^2 + 4\beta\lambda_2\kappa\theta}}{-2\beta\lambda_2}$	$+\infty$
$f'(\sigma_s)$	+	0	-	-
$f(\sigma_s)$	0	$\kappa\theta + \frac{(\kappa + \beta\lambda_1)^2}{4\beta\lambda_2}$	0	$-\infty$

Figure 2: The table of variation of $f(\sigma_t)$ when $\lambda_2 > 0$ and $-\kappa > \beta\lambda_1$.

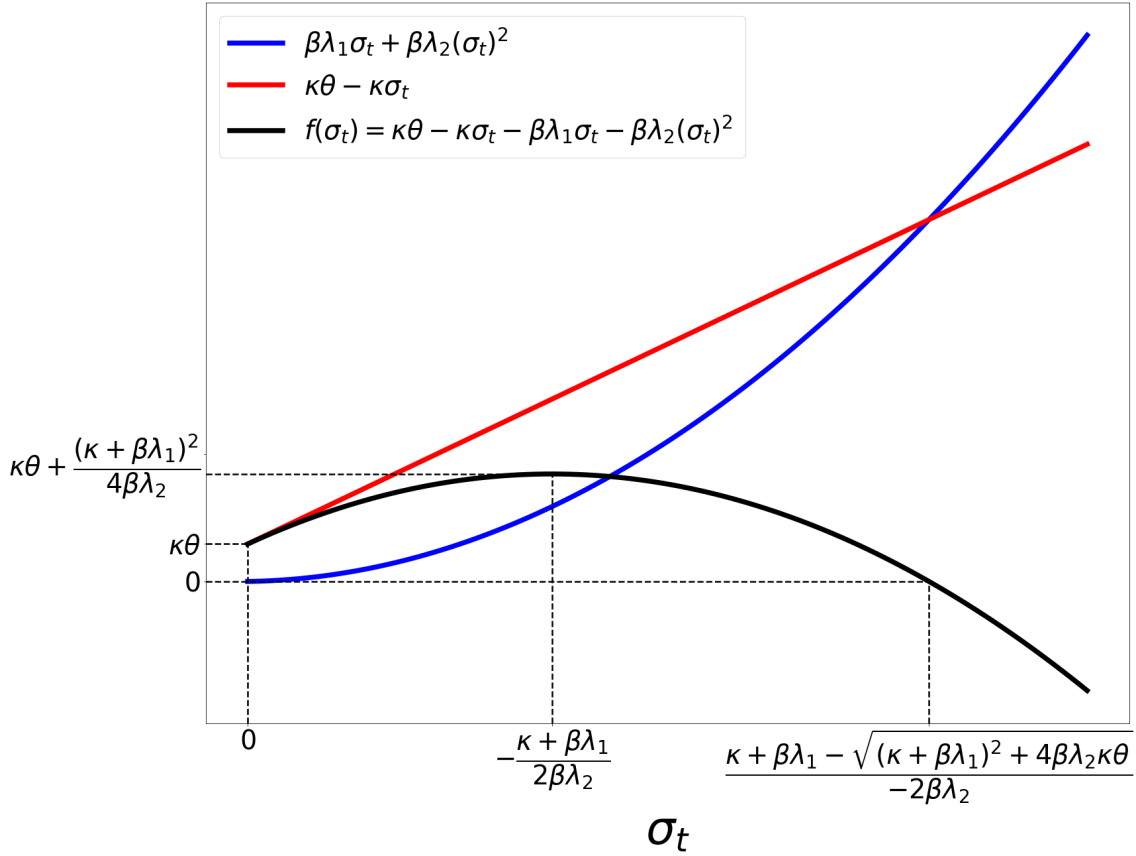


Figure 3: Plot of $f(\sigma_t)$ and its 2 components $\kappa\theta - \kappa\sigma_t$ and $\beta\lambda_1\sigma_t + \beta\lambda_2(\sigma_t)^2$, when $-\kappa > \beta\lambda_1$.

Thus, in this specification, f is no more homogeneous but concave downward because of the antagonistic effects of the positive volatility feedback and the risk premium effect (see figure 3). Indeed, the volatility

feedback component $-\kappa\lambda_1$ is linearly increasing in volatility while the risk premium effect $-\beta(\lambda_1\sigma_t + \lambda_2(\sigma_t)^2)$ is strictly concave decreasing of volatility. Therefore, when the volatility is between 0 and the attraction volatility (equation 13), the volatility feedback effect dominates and f is positive. Conversely, when σ_t is greater than the attraction volatility, the risk premium effect dominates and f is positive.

The study of these different specifications of 6 emphasizes the importance of assumptions relating to the drift process on the mechanisms that push back the volatility towards an attraction level. More precisely, it has been shown through equations 10 and 13 that the level of attraction volatility is a decreasing function of risk premia. In other words, all things being equal, the higher the risk premia are, the lower the mean volatility. Conversely, the mean-reversion speed is an increasing function of risk premia (equations 11 and 14). Consequently, the higher the risk premia, the lower the dispersion of the volatility distribution. Broadly speaking, it has shown that even when we limit to the case θ as constant, the model covers various competing assumptions about volatility dynamics.

In sections 3.2 and 3.3, we will consider hypotheses in which θ is time-dependent.

3.2 The function θ as a path-dependent process

A possible alternative to understanding θ as a constant is to specify it as a path-dependent process. The idea is to take inspiration from the structure of the super-Heston raw volatility model (SHRV) introduced in Dapani *et al.* (2019) in the specification of θ . This SHRV model is defined by:

$$\begin{aligned}(\sigma_t)^2 &= (\sigma_0)^2 + \beta \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \sigma_u dZ_u + \kappa \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} (\theta_u - (\sigma_u)^2) du, \\ \theta_t &= \bar{\theta}_t + (X_t)^2, \\ X_t &= \int_0^t K(t-u) \frac{dP_u}{P_u},\end{aligned}$$

with $\bar{\theta}$ a deterministic function, K a kernel function, and Z a Brownian motion independent on B^5 . Therefore, the process X and thus θ are path-dependent processes allowing to capture of the strong Zumbach effect. Concretely, θ which plays the role of attraction variance (the level towards which the process of variance tends), is a function of the square of a moving average of past returns (i.e. $(X_t)^2$). Due to the independence between W and B , the path dependency is only captured by this variable θ . On this point, the SHR model and the RPDV differ.

Indeed, for a RPDV model of the form 6, the path dependency of θ adds to the path dependency depending on

$$\int_{I_t} (t-u)^{-\alpha} \frac{dP_u}{P_u}. \quad (15)$$

There is then a mechanism of entanglement of path dependencies. The question then arises about the interest to introduce this new layer of complexity. One of the main potential motivations is better accounting of the time-reversal asymmetry that characterizes financial time series (Zumbach 2009, 2010). The aim is then to model the fact that a strong increase in the volatility is generally originated by a price drop through a fall of 15, but that even when 15 go back to its level anterior to the volatility spike, the volatility tends to stay higher due to a latency phenomenon. More generally, it allows make the attraction volatility as a path-dependent process and thus explains what is generally apprehended as exogenous structural breaks of

⁵Note also that in the case where $\langle dB_t, dZ_t \rangle \neq 0$ the SHRV model is a RPDV model.

the volatility formation mechanism. If $\kappa > 0$, a possible specification of θ in line with this objective is

$$\theta_t = \bar{\theta} + |X_t|^{\frac{1}{a}}, \quad (16)$$

$$X_t = \int_{I_t} K(t-u) \left(\frac{dP_u}{P_u} \right)^a, \quad (17)$$

with $a \in \{1, 2\}$, $\bar{\theta} > 0$ and K a kernel whose integral on \mathbb{R}_+ is convergent (for instance an exponential kernel).

First, note that since θ is necessarily positive and given results obtained in section 3.1, the existence and unicity of the attraction volatility is guaranteed at each time t . Moreover, this attraction volatility is given by equation 10 if $\lambda_2 = 0$ and by equation 13 if $\lambda_2 > 0$ (we just replace θ by $\theta(t)$ defined by 16)⁶. Whatever the assumption about risk premia, by the relationship between θ and the attraction volatility, the expression 16 makes the level towards which volatility tends a function of a process dependent on the historical price path: X . The nature of this process depends on the value taken by the parameter a .

If $a = 1$, X is a moving average of returns like the term 15, and may be apprehended as a past price trend. Consequently, the volatility process becomes linearly dependent on a moving average of the absolute value of a moving average of past returns through the term⁷:

$$\kappa \int_{I_t} (t-u)^{-\alpha} \underbrace{\left| \int_{-\infty}^u K(u-s) \frac{dP_s}{P_s} \right|}_{X_u} du. \quad (18)$$

This entanglement of moving averages has several implications. First, it allows the introduction of inertia in the impact of past price trends on the volatility level coherent with the empirical time reversal asymmetry mentioned above. Second, it implies that the longer a price trend the higher 18. Another important element, on the contrary to the term which is linearly sensitive to a past price trend, θ is linearly sensitive to the absolute value of a past price trend. Therefore, a positive price trend such as $X_t > 0$ increases $\theta(t)$ and by extension the attraction volatility. Consequently, a strong positive price trend produces two antagonistic effects on the volatility level: on the one hand it decreases the volatility component 15, on the other hand, it increases the component 18. Conversely, in a negative price trend circumstance, both component puts upwards pressure on the volatility process. This asymmetry of the impact of the price trend on the volatility process is consistent with the empirical relationship emphasized by Parent (2021) that links the EWMA of past returns and the volatility level observable in stock markets.

If $a = 2$, X corresponds to a weighted moving average of the realized variance. Therefore, X is no more a price trend variable but a variable of asset price activity like m_2 in 1. The volatility process is thus linearly sensitive to

$$\kappa \int_0^t (t-u)^{-\alpha} \sqrt{\underbrace{\int_0^u K(u-s)(\sigma_s)^2 ds}_{X_u}} du. \quad (19)$$

Through this term, in a similar way to the case $a = 1$, the volatility process is subject to a force of inertia that depends on the duration of the volatility trend due to the fact that X is a moving average. Consequently, a short volatility spike does not increase the attraction volatility in the same proportion as a longer period

⁶For instance, if $\lambda_2 = 0$, the attraction volatility is

$$\bar{\sigma}_t = \frac{\kappa \left(\bar{\theta} + |X_t|^{\frac{1}{a}} \right)}{\kappa + \beta \lambda_1}.$$

⁷The volatility may be written as

$$\sigma_t = \beta_0 + \int_{I_t} (t-u)^{-\alpha} \left(\kappa \left(\bar{\theta} + |X_u| \right) - (\kappa + \beta \lambda_1) \sigma_u - \beta \lambda_2 (\sigma_u)^2 \right) du + \bar{\beta} \int_{I_t} (t-u)^{-\alpha} \sigma_u dZ_u.$$

of high volatility. Furthermore, this specification is particularly suited to capture time-reversal asymmetry. Indeed, even after a strong rebound following a negative trend causing a period of high volatility, the volatility process tends to remain higher than normal for some time due to the force of inertia induced by the term 19.

Thus, if the case $a = 1$ and $a = 2$ imply different hypotheses about the determinant of the attraction volatility, both allow make the RPDV model 6 strongly time-reversal asymmetry.

3.3 The function θ as a long-run coherence operator between price and value

A third way to apprehend θ is to link it to the long-term consistency between "fundamental" value and price. The idea is thus to model the hypothesis formulated in 1986 by Black that there are "forces tending to cause price to return to value", a hypothesis consistent with the empirical work of Campbell and Shiller (1998, 2005) showing that fundamental indicators (like the dividend per share or the price/earnings ratio for the stock market) tend to revert towards a long-term level in the long run, which prevents a total disconnection between the price and the fundamentals. θ will thus take the role of coherence operator between price and value, allowing to reconcile this empirical evidence while preserving the hypothesis on price drift as a quadratic function of volatility (equation 7).

For this purpose, we assume, first, the existence and the uniqueness of attraction volatility denoted $\bar{\sigma}_t$, and second that the following conditions are met:

$$r_t + \lambda_1 \bar{\sigma}_t + \lambda_2 (\bar{\sigma}_t)^2 \begin{cases} \leq q_t & \text{for } V_t \leq P_t, \\ = q_t & \text{for } P_t = V_t, \\ \geq q_t & \text{for } V_t \geq P_t, \end{cases} \quad (20)$$

where V is the value, q its growth rate such as $q_t \geq r_t$ ⁸. Concretely, these conditions mean that the drift tends to revert toward: (1) a level superior or equal to the growth rate of the value when the price is inferior to the value, (2) the growth rate of the value when the price is equal to the value, (3) a level inferior or equal to the growth rate of the value when the price is superior to the value, and these adjustments pass by a variation of the level towards which the volatility tends.

A hypothesis consistent with 20 is to assume that

$$\lambda_1 \bar{\sigma}_t + \lambda_2 (\bar{\sigma}_t)^2 = F(V_t, P_t)(q_t - r_t), \quad (21)$$

where F is a function that respects $\forall (V, P) \in \mathbb{R}_+^2$, $F(V, P) > 0$ and such as

$$F(V, P) \begin{cases} \leq 1 & \text{for } V_t < P_t, \\ = 1 & \text{for } P_t = V_t, \\ \geq 1 & \text{for } V_t > P_t. \end{cases} \quad (22)$$

This supplementary condition ensures a positive risk price, an important property since market players are supposed to be risk-averse. Obviously, its definition remaining quite general, F admits a multiplicity of specifications.

The simplest case satisfying conditions 22 is $F(V, P) = 1$, $\forall (V, P) \in \mathbb{R}_+^2$. Within this framework, the attraction volatility does not depend on V_t and P_t . This property is interesting from a practical since it dispenses with determining the value V : only its growth rate q is required to specify both θ and the attraction

⁸If the value follows an ordinary stochastic differential equation, the value dynamics are thus given by:

$$\frac{dV_t}{V_t} = q_t dt.$$

volatility. In consequence, the differential between price and value has no impact on the drift of the asset price.

In all other specifications of F , the term $F(V, P)$ is a path-dependent component of the volatility process. Indeed, it implies necessarily that the right-hand term in equation 21 whom the volatility process depends is greater in case $V > P$ than in case $V < P$. The particularity of this path-dependence is that it is not purely endogenous, but linked with the joint past path of price and value. It implies the existence of a mechanism for an increase in the drift when $V_t > P_t$ and a decrease in it when $V_t < P_t$. Obviously, the introduction of such a mechanism must be done in accordance with the conditions previously defined, in particular the existence of attraction volatility (by definition strictly positive). For instance, a simple specification that respects conditions 22 is to define F as the ratio value-on-price, i.e.

$$F(V_t, P_t) = \frac{V_t}{P_t}. \quad (23)$$

The impact of this specification on the dynamics of the volatility process differs significantly depending on the drift components ensuring that the equality 21 holds.

We will consider below two competing hypotheses. In the first one, risk premia are constant and equality 21 is guaranteed through a variation of the attraction volatility. Conversely, in the second hypothesis, the attraction volatility is constant and equation 21 is realized at each instant by risk premia adjustments.

3.3.1 The attraction volatility as adjustment variable

Let us begin with the case where risk premia are constants, and the attraction volatility constitutes the adjustment variable allowing equation 21 to be respected. Again, the cases $\lambda_2 = 0$ and $\lambda_2 > 0$ needs to be considered separately.

If $\lambda_2 = 0$, as shown in section 3.1.1, it exists attraction volatility only if $-\kappa < \beta\lambda_1$. Under this condition, the attraction volatility is defined by:

$$\bar{\sigma}_t = \frac{F(V_t, P_t)(q_t - r_t)}{\lambda_1}, \quad (24)$$

and $\theta(t)$ (using equation 10), by

$$\theta(t) = \frac{F(V_t, P_t)(q_t - r_t)(\kappa + \beta\lambda_1)}{\kappa\lambda_1}. \quad (25)$$

In this case, there is so a linear relationship with $F(V_t, P_t)(q_t - r_t)$ for both $\theta(t)$ and the attraction volatility. Therefore, for a given pair (V, P) , the attraction volatility is an increasing linear function of the differential between the growth rate of value and the risk-free rate. Thus, for a constant free-risk rate of the value, a variation Δq of the growth rate of the value results in a variation $\Delta\bar{\sigma}$ of the attraction value. Symmetrically, for a constant growth rate of the value, a variation Δr of the risk-free rate results in a variation $-\Delta\bar{\sigma}$ of the attraction value. Of course, the independence between q and r is not required.

In case $\lambda_2 > 0$, the attraction volatility is defined by:

$$\bar{\sigma} = \frac{-\lambda_1 + \sqrt{(\lambda_1)^2 + 4F(V_t, P_t)(q_t - r_t)\lambda_2}}{2\lambda_2}, \quad (26)$$

and $\theta(t)$ is given (using equation 13) by⁹:

$$\theta(t) = \frac{\beta F(V_t, P_t)(q_t - r_t)\lambda_2 + 0.5\kappa \left(\sqrt{(\lambda_1)^2 + 4F(V_t, P_t)(q_t - r_t)\lambda_2} - \lambda_1 \right)}{\kappa\lambda_2}. \quad (27)$$

⁹See appendix C.

In addition, to ensure that $f_u(0) > 0$, the following inequality must be respected

$$F(V_t, P_t)(q_t - r_t) > \frac{\kappa}{\beta\lambda_2} \left(\lambda_1 + \frac{\kappa}{\beta} \right). \quad (28)$$

The hypothesis then makes then the attraction volatility a strictly concave increasing function of $F(V_t, P_t)(q_t - r_t)$. It follows that for a given pair (V, P) , a movement of the differential $q - r$ has more impact on the attraction volatility when $q - r$ is low. In the same way, a variation of $F(V_t, P_t)$ has more impact on the attraction volatility when volatility is low. Moreover, it should be emphasized that the condition 28 is a strong constraint when F is different from $F(V, P) = 1 \forall (V, P) \in \mathbb{R}_+^2$. For instance, the hypothesis of F as the value-on-price ratio is not compatible with $\lambda_2 > 0$ due to this condition, because in such case, the inequality 28 has a non-zero probability to be violated.

In both considered assumptions relating to risk premia, all things being equal, for all specifications of F different of $F(V, P) = 1 \forall (V, P) \in \mathbb{R}_+^2$ the attraction volatility is strictly higher when $V_t > P_t$ than in the situation $V_t < P_t$. The price adjustment to value in the long run pass then through variations of the attraction volatility that impact the volatility and thus, due to the positive relation between μ_t and σ_t , the asset price drift. For instance, if F corresponds to the value-to-price ratio (equation 23), the higher the price relative to the value, the lower the attraction volatility is. In this specification of F , a significant drop in price that is not due to a change in the fundamentals generates an increase in the attraction value as long as the value-to-price ratio has not returned to its original level. This increase is higher in absolute terms the more that the value-to-price ratio is high. For example, under the hypothesis $\lambda_2 = 0$ and $\lambda_1 > 0$, an instant 20% price drop increases attraction volatility by 25%, while a 50% price drop increases it by 100%. Furthermore, as in the hypotheses considered in section 3.2, this mechanism joined to the negative relationship between past price trends and the volatility process makes the model structurally adapted to capture a strong reversal asymmetry.

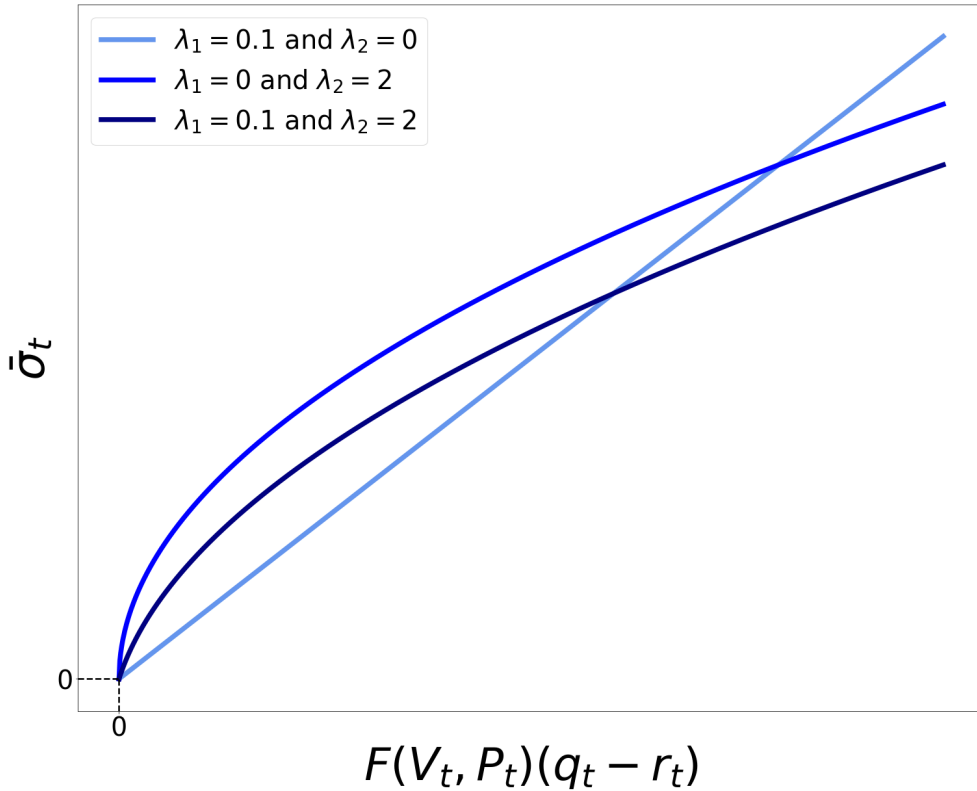


Figure 4: The relationship between the attraction volatility $\bar{\sigma}_t$ and $F(V_t, P_t)(q_t - r_t)$ under different hypothesis about risk premia values.

3.3.2 Risk premia as adjustment variables

An alternative hypothesis is that the attraction volatility is constant, and the risk premia variables. Risk premia become then adjustment variables guaranteeing equation 20. This reversal has strong theoretical implications. Indeed, under this hypothesis, the implied risk aversion of the market becomes endogenous. The price of the risk is no more the result of exogenous market participant preferences but a consequence on the one hand of the difference between the growth rate of the value and the risk-free rate, on the other hand of the long-run level of volatility. In other words, the price of the risk becomes a variable socially constituted by the interaction between agents that treat information related to fundamentals, and thus cannot be deduced by the aggregation of individual risk aversions.

To understand the practical implications of this reversal, let us start by considering the value taken by risk premia in this hypothesis and its implication on the volatility dynamics. To this end, we deal here only the case where the ratio $\lambda_{1,t}$ on $\lambda_{1,t} + \lambda_{2,t}$ is constant $\forall t$, i.e.

$$w = \frac{\lambda_{1,t}}{\lambda_{1,t} + \lambda_{2,t}}.$$

Under this assumption, the drift may be write as

$$\mu_t = r_t + \lambda_t \mathcal{R}_t$$

with \mathcal{R} the measure of risk priced by the market, given by

$$\mathcal{R}_t = (w + (1 - w)\sigma_t)\sigma_t.$$

and λ_t the risk premium associated.

Equation 21 may then be rewritten as

$$\lambda_t(w + (1 - w)\bar{\sigma})\bar{\sigma} = F(V_t, P_t)(q_t - r_t).$$

It follows that

$$\lambda_t = \frac{F(V_t, P_t)(q_t - r_t)}{(w + (1 - w)\bar{\sigma})\bar{\sigma}},$$

which implies $\lambda_{1,t} = w\lambda_t$ and $\lambda_{2,t} = (1 - w)\lambda_t$. Consequently, the price of the risk given by λ is both linearly dependent of the differential between the growth rate of value and the risk-free rate and of $F(V_t, P_t)$. From an economic perspective, this means that an increase (resp. decrease) of the differential between the growth rate of value and the risk-free rate increase (resp. decrease) the implied risk aversion. Similarly, if F is a strictly increasing function of the differential between price and value, an increase (resp. decrease) in the value/price ratio also increases (resp. decreases) the implicit risk aversion.

Besides, if θ were constant, an increase in the risk premium would result in a reduction of the attraction volatility (equations 10 and 13). However, as mentioned above, we here assume constant attraction volatility. Accordingly, θ is defined by

$$\theta(t) = \frac{\bar{\sigma}}{\kappa}(\kappa + \beta\lambda_{1,t} + \bar{\sigma}\beta\lambda_{2,t}),$$

and preserves thus the attraction volatility invariance by neutralizing the impact of risk premia variations on the attraction volatility level. However, this invariance does not imply neutrality of the risk premium λ level on the volatility dynamics. Indeed, as already remarked in section 3.1.2, the function f may be written as

$$f_t(\sigma_t) = (\kappa + \beta\lambda_{1,t})(\bar{\sigma} - \sigma_t) + \beta\lambda_{2,t}(\bar{\sigma}^2 - (\sigma_t)^2),$$

and therefore, the reverting force that pushes the volatility process towards its attraction level is an increasing function of risk premia. Due to this relationship, all things being equal, the higher the risk premium λ (for

a given value of w) the lower the dispersion of the volatility distribution. In this manner, an increase in the risk premium, which can be originated by an increase of the differential between the growth rate of value and the risk-free rate or an increase of the value-on-price ratio (or both), tends to reduce the randomness of the volatility.

4 Conclusion

This paper introduced a class of models referred to as RPDV models, which reconcile two major empirical features of volatility: its rough behavior and its path dependency. Additionally, the article has delved deeper into certain properties of some models within this class.

To achieve this, section 2 initiated by outlining a general form of path-dependent model, which encompasses several significant volatility models found in the current academic literature. Based on this foundation, we then defined what we have referred to as the family of RPDV models. In this modeling framework, volatility is a function of two stochastic processes, which can be seen respectively as a type of price trend process and a type of volatility trend process. Being neither Markovian nor semimartingale since their properties actually define them as a rough models, RPDV models are not very tractable for simulations. To overcome this issue, we therefore introduced a Markovian multi-factor approximation of RPDV models using Abi Jabers findings (2019).

In the section 3, we focused on different specifications belonging to a rather parsimonious sub-family of RPDV models, in which the price drift is a quadratic function of the volatility process. A key idea that emerged from these analyses is the central importance of the price drift on the volatility dynamics induced by the actual structure of the RPDV models. This drift determines central properties such as the level towards which the volatility tends to revert (the attraction volatility) or the speed of mean reversion, and so ultimately the volatility distribution. More precisely, attraction volatility is a decreasing function of risk premia while the speed of mean reversion of these premia is increasing. Furthermore, the study of these model specifications made it possible to formulate competing hypotheses on the mechanisms of asset price and volatility formation that each underlies. Thus, having exposed the simpler specifications in which both attraction volatility and mean-reverting force were constant, we considered a specification in which the attraction volatility level is itself a path-dependent process leading to an entanglement of path dependencies. This assumption makes it possible to reinforce the time reversal asymmetry property of the process, which is an important empirical characteristic of financial data (Zumbach 2009, 2010). The third hypothesis involved considering the price drift as an operator of long-term consistency between price and value. It has been shown within this framework, that depending on the sub-hypotheses adopted, either the attraction volatility level or the speed of mean-reversion of the volatility process then becomes an increasing function of both the difference between the rate of growth of the value and the risk-free rate, and the differential between price and value. Therefore, the choice between these sub-hypotheses also has implications in terms of time reversal asymmetry and volatility distributions.

This papers exploration of the competing specifications of the RPDV model thus opens up avenues of reflection for future theories of asset price and volatility dynamics consistent with empirical data.

5 Acknowledgments

I would like to thank my thesis supervisor Jean-Paul Laurent for his support and useful advice in preparing this paper, as well as Julien Guyon for its feedback

References

- [1] Abi Jaber E. (2019). Lifting the Heston model. *Quantitative Finance*, 19(12), 1995-2013.
- [2] Abi Jaber E., and El Euch O. (2019). Multifactor approximation of rough volatility models. *SIAM Journal on Financial Mathematics*, 10(2), 309-349.
- [3] Abi Jaber E., Cuchiero C., Larsson M., and Pulido S. (2021). A weak solution theory for stochastic Volterra equations of convolution type. *The Annals of Applied Probability*, 31(6), 2924-2952.
- [4] Abi Jaber E., Larsson M., and Pulido S. (2019). Affine volterra processes.
- [5] Barndorff-Nielsen O. E., and Schmiegel J. (2009). Brownian semistationary processes and volatility/intermittency. *Advanced financial modelling*, 8, 1-26.
- [6] Bayer C., Friz P., and Gatheral J. (2016). Pricing under rough volatility. *Quantitative Finance*, 16(6), 887-904.
- [7] Bennedsen M., Lunde A., and Pakkanen M.S. (2016). Decoupling the short-and long-term behavior of stochastic volatility. arXiv preprint arXiv:1610.00332.
- [8] Black F. (1986). Noise. *The journal of finance*, 41(3), 528-543.
- [9] Blanc P., Donier J., and Bouchaud J.P. (2017). Quadratic Hawkes Processes for Financial Prices. *Quantitative Finance*, 17(2), 171-188.
- [10] Bochud T. and Challet D. (2007). Optimal approximations of power laws with exponentials: application to volatility models with long memory. *Quantitative Finance*, 7(6), 585-589.
- [11] Campbell J.Y. et Shiller R.J. Valuation ratios and the long-run stock market outlook: An update. *Advances in Behavioral Finance*, 2005, vol. 2, p. 173-201.
- [12] Cont R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1(2), 223.
- [13] Dandapani A., Jusselin P., and Rosenbaum M. (2019). From Quadratic Hawkes Processes to Super-Heston Rough Volatility Models with Zumbach Effect. *arXiv preprint arXiv:1907.06151*, 2019.
- [14] Dissanaike G. (1997). Do Stock Market Investors Overreact? *Journal of Business Finance & Accounting*, 24(1), 27-50.
- [15] El Euch O., and Rosenbaum M. (2018). Perfect Hedging in Rough Heston Models. *Annals of Applied Probability*, 28(6), 3813-3856.
- [16] El Euch O. and Rosenbaum M. (2019). The Characteristic Function of Rough Heston Models. *Mathematical Finance*, 29(1), 3-38.
- [17] El Euch O., (2018). *Quantitative Finance under rough volatility*, PhD diss., Sorbonne université.
- [18] Gatheral J., Jaisson T., and Rosenbaum M. (2018). Volatility is Rough. *Quantitative Finance*, 18(6), 933-949.
- [19] Gatheral J., Jusselin P., and Rosenbaum M. (2020). The Quadratic Rough Heston Model and the Joint S&P500/VIX Smile Calibration Problem. *arXiv preprint arXiv:2001.01789*.
- [20] Gomes C., and Waelbroeck H. (2015). Is market impact a measure of the information value of trades? Market response to liquidity vs. informed metaorders. *Quantitative Finance*, 15(5), 773-793.
- [21] Guyon J. (2014). Path-dependent volatility. *Risk Magazine*
- [22] Guyon J., and Lekeufack J. (2023). Volatility Is (Mostly) Path-Dependent. *Quantitative Finance*.

- [23] Hobson D.G., and Rogers L.C. (1998). Complete Models with Stochastic Volatility. *Mathematical Finance*, 8(1), 27-48.
- [24] Jusselin P., and Rosenbaum M. (2020). Noarbitrage implies powerlaw market impact and rough volatility. *Mathematical Finance*, 30(4), 1309-1336.
- [25] Mandelbrot B.B., and Hudson R. L. (2005). *The (mis)behavior of markets: a fractal view of risk, ruin, and reward*. Basic Books.
- [26] Merton R.C. (1973). An intertemporal capital asset pricing model. *Econometrica: Journal of the Econometric Society*, 867-887.
- [27] Parent L. (2023). The EWMA Heston model. *Quantitative Finance*, 23(1), 71-93.
- [28] Rosenbaum M., Zhang J. (2022). On the universality of the volatility formation process: when machine learning and rough volatility agree. *arXiv* preprint arXiv:2206.14114.
- [29] Schöbel R., and Zhu J. (1999). Stochastic volatility with an OrnsteinUhlenbeck process: an extension. *Review of Finance*, 3(1), 23-46.
- [30] Zumbach G., and Lynch P. (2001). Heterogeneous Volatility Cascade in Financial Markets. *Physica A: Statistical Mechanics and its Applications*, 298(3-4), 521-529.
- [31] Zumbach G. (2009). Time Reversal Invariance in Finance. *Quantitative Finance*, 9(5), 505-515.
- [32] Zumbach G. (2010), Volatility Conditional on Price Trends. *Quantitative Finance* 10.4 (2010): 431-442.

Appendix A Stochastic differential equations for Markovian approximation of the RPDV model

We want to solve the following SDE:

$$dm_{1,t}^{(i)} = \frac{dP_t}{P_t} + \left(\kappa_1 \theta_{1,t} - \kappa_1 m_{1,t} - \gamma_i m_{1,t}^{(i)} \right) dt,$$

To consider the dynamics of $m_{1,t}^{(i)}$, we set $g(m_{1,t}^{(i)}, t) = e^{\gamma_i t} m_{1,t}^{(i)}$ and apply the Itô lemma:

$$dg(m_{1,t}^{(i)}, t) = e^{\gamma_i t} \left(\frac{dP_t}{P_t} + \kappa_1 (\theta_{1,t} - m_{1,t}) dt \right).$$

Consequently, if $m_{1,0}^{(i)} = 0$

$$m_{1,t}^{(i)} = \int_0^t e^{-\gamma_i(t-u)} \left(\frac{dP_u}{P_u} + \kappa_1 (\theta_{1,u} - m_{1,u}) du \right)$$

Thus:

$$\sum_{i=1}^n w_{1,i} m_{1,t}^{(i)} = \int_0^t \underbrace{\sum_{i=1}^n w_{1,i} e^{-\gamma_{1,i}(t-u)}}_{\hat{\kappa}_1(t-u)} \left(\frac{dP_u}{P_u} + \kappa_1 (\theta_{1,u} - m_{1,u}) du \right)$$

Analogously, with

$$dm_{2,t}^{(i)} = \left((\sigma_t)^{a_2} + \kappa_2 \theta_{2,t} - \kappa_2 m_{2,t} - \gamma_i m_{2,t}^{(i)} \right) dt,$$

by applying same steps, we obtain:

$$\sum_{i=1}^n w_{2,i} m_{2,t}^{(i)} = \int_0^t \sum_{i=1}^n w_{2,i} e^{-\gamma_{2,i}(t-u)} \left((\sigma_u)^{a_2} + \kappa_2 (\theta_{2,u} - m_{2,u}) \right) du.$$

Appendix B Highlighting the mean-reverting component of the volatility process

Using expression of the attraction volatility 13, we have:

$$\begin{aligned} \theta &= \frac{(2\beta\lambda_2\bar{\sigma} + \kappa + \beta\lambda_1)^2 - (\kappa + \beta\lambda_1)^2}{4\beta\lambda_2\kappa} \\ \theta &= \frac{4(\beta\lambda_2\bar{\sigma})^2 + 4\beta\lambda_2\bar{\sigma}(\kappa + \beta\lambda_1)}{4\beta\lambda_2\kappa} \\ \theta &= \frac{\beta\lambda_2\bar{\sigma}^2}{\kappa} + \frac{\bar{\sigma}(\kappa + \beta\lambda_1)}{\kappa} \\ \theta &= \frac{\bar{\sigma}}{\kappa} (\kappa + \beta\lambda_1 + \beta\lambda_2\bar{\sigma}). \end{aligned}$$

Replacing θ in equation 8, we then obtain

$$f(\sigma_t) = (\kappa + \beta\lambda_1)(\bar{\sigma} - \sigma_t) + \beta\lambda_2(\bar{\sigma}^2 - (\sigma_t)^2).$$

Appendix C Zero of the function f in polynomial drift hypothesis

The unique positive solution to¹⁰

$$y_t + \lambda_1 \bar{\sigma} + \lambda_2 \bar{\sigma}^2 = 0,$$

is

$$\bar{\sigma}_t = \frac{-\lambda_1 + \sqrt{(\lambda_1)^2 + 4(q_t - r_t)\lambda_2}}{2\lambda_2}.$$

In addition, from equation 13 in section 3.1.2, we now that

$$\bar{\sigma}_t = \frac{\sqrt{(\kappa + \beta\lambda_1)^2 + 4\kappa\beta\lambda_2\theta_t} - \kappa - \beta\lambda_1}{2\beta\lambda_2}.$$

By combining the two previous equalities, we have:

$$\begin{aligned} \frac{\sqrt{(\kappa + \beta\lambda_1)^2 + 4\kappa\beta\lambda_2\theta_t} - \kappa - \beta\lambda_1}{2\beta\lambda_2} &= \frac{-\lambda_1 + \sqrt{(\lambda_1)^2 + 4y_t\lambda_2}}{2\lambda_2} \\ \sqrt{(\kappa + \beta\lambda_1)^2 + 4\kappa\beta\lambda_2\theta_t} - \kappa - \beta\lambda_1 &= -\beta\lambda_1 + \beta\sqrt{(\lambda_1)^2 + 4y_t\lambda_2} \\ (\kappa + \beta\lambda_1)^2 + 4\kappa\beta\lambda_2\theta_t &= \left(\kappa + \beta\sqrt{(\lambda_1)^2 + 4y_t\lambda_2}\right)^2 \\ (\kappa + \beta\lambda_1)^2 + 4\kappa\beta\lambda_2\theta_t &= \kappa^2 + \beta^2((\lambda_1)^2 + 4y_t\lambda_2) + 2\kappa\beta\sqrt{(\lambda_1)^2 + 4y_t\lambda_2} \\ 4\kappa\beta\lambda_2\theta_t &= -2\kappa\beta\lambda_1 + \beta^2(4y_t\lambda_2) + 2\kappa\beta\sqrt{(\lambda_1)^2 + 4y_t\lambda_2} \\ \theta_t &= \frac{\beta y_t \lambda_2 + 0.5\kappa \left(\sqrt{(\lambda_1)^2 + 4y_t\lambda_2} - \lambda_1\right)}{\kappa\lambda_2}. \end{aligned}$$

It is clear that if κ is positive, θ_t is also positive $\forall y_t > 0$ and therefore $\theta_t\kappa > 0$ which ensures that $f(\sigma_t) > 0$. If κ is negative, to ensure that $\theta_t\kappa \geq 0$:

$$\begin{aligned} \beta y_t \lambda_2 + 0.5\kappa \left(\sqrt{(\lambda_1)^2 + 4(q_t - r_t)\lambda_2} - \lambda_1\right) &> 0 \\ \left(\frac{\beta\lambda_2}{0.5\kappa} y_t - \lambda_1\right)^2 - (\lambda_1)^2 - 4\lambda_2 y_t &> 0 \\ \frac{4(\beta\lambda_2)^2}{\kappa^2} (y_t)^2 - 4\lambda_2 \left(\frac{\beta\lambda_1}{\kappa} + 1\right) y_t &> 0 \\ 4\lambda_2 y_t \left(\frac{\beta^2\lambda_2}{\kappa^2} y_t - \left(\frac{\beta\lambda_1}{\kappa} + 1\right)\right) &> 0 \end{aligned}$$

Consequently,

$$y_t > \frac{\kappa}{\beta\lambda_2} \left(\lambda_1 + \frac{\kappa}{\beta}\right).$$

¹⁰In the considered context set out in section 3.3,

$$y_t = F(V_t, P_t)(q_t - r_t).$$