



**HAL**  
open science

# BLOWING-UPS OF PRIMITIVE MULTIPLE SCHEMES

Jean-Marc Drézet

► **To cite this version:**

| Jean-Marc Drézet. BLOWING-UPS OF PRIMITIVE MULTIPLE SCHEMES. 2024. <hal-04750558>

**HAL Id: hal-04750558**

**<https://hal.science/hal-04750558v1>**

Preprint submitted on 23 Oct 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# BLOWING-UPS OF PRIMITIVE MULTIPLE SCHEMES

JEAN-MARC DRÉZET

RESUME. A primitive multiple scheme is a Cohen-Macaulay scheme  $\mathfrak{X}$  such that the associated reduced scheme  $X = \mathfrak{X}_{red}$  is smooth, irreducible, and that  $\mathfrak{X}$  can be locally embedded in a smooth variety of dimension  $\dim(X) + 1$ . If  $\mathcal{I}_X$  is the ideal sheaf of  $X$ ,  $L = \mathcal{I}_X/\mathcal{I}_X^2$  is a line bundle on  $X$ , called the *associated line bundle* of  $\mathfrak{X}$ . The simplest example is the trivial primitive multiple scheme of multiplicity  $n$  associated to a line bundle  $L$  on  $X$ : it is the  $n$ -th infinitesimal neighborhood of  $X$ , embedded in the line bundle  $L^*$  by the zero section.

A subscheme  $Z$  of  $\mathfrak{X}$  is called *good* if  $Z_{red}$  is smooth and connected, and if  $d = \text{codim}(Z)$  and  $\mathcal{I}_{Z,\mathfrak{X}}$  is the ideal sheaf of  $Z$ , then for every closed point  $z \in Z$ ,  $\mathcal{I}_{Z,\mathfrak{X},z}$  can be generated by  $d$  elements.

Two kinds of subschemes  $Z$  of  $\mathfrak{X}$  will be considered: the closed smooth subschemes of  $X$ , seen as subschemes of  $\mathfrak{X}$ , and the good subschemes. In the two cases, the blowing-up  $\mathbf{B}_{Z,\mathfrak{X}}$  of  $\mathfrak{X}$  along  $Z$  is a primitive multiple scheme of multiplicity  $n$ , and its underlying smooth scheme is the blowing-up  $\mathbf{B}_{Z_{red},X}$  of  $X$  along  $Z_{red}$ . Additional results are obtained in the case of hypersurfaces or points of  $X$ .

We treat the case of  $X = \mathbb{P}_2$ , with  $Z$  a single point  $P$ . Let  $\pi : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$  be the blowing-up of  $\mathbb{P}_2$  along  $P$ . We find all primitive double schemes  $\mathbb{X}, \mathbb{Y}$ , with  $\mathbb{Y}_{red} = \tilde{\mathbb{P}}_2$ ,  $\mathbb{X}_{red} = \mathbb{P}_2$ , such that there is a morphism  $\mathbb{Y} \rightarrow \mathbb{X}$  inducing  $\pi$  and an isomorphism  $\mathbb{Y} \setminus \pi^{-1}(P) \rightarrow \mathbb{X} \setminus \{P\}$ . We obtain in this way the list of all K3-carpet with underlying smooth variety  $\tilde{\mathbb{P}}_2$ .

## SUMMARY

1. Introduction	1
2. Preliminaries	5
3. Primitive multiple schemes	6
4. Blowing-up of subschemes of primitive multiple schemes	11
5. Blowing-up of good zero-dimensional subschemes of primitive double schemes	17
6. Blowing-up of an hypersurface	22
7. Primitive double structures on $\tilde{\mathbb{P}}_2$	26
References	34

Mathematics Subject Classification : 14D20, 14B20

## 1. INTRODUCTION

A *primitive multiple scheme* is a Cohen-Macaulay scheme  $\mathfrak{X}$  over  $\mathbb{C}$  such that:

- $\mathbf{X}_{red} = X$  is a smooth connected variety,
- for every closed point  $x \in X$ , there exists a neighborhood  $U$  of  $x$  in  $\mathbf{X}$ , and a smooth variety  $S$  of dimension  $\dim(X) + 1$  such that  $U$  is isomorphic to a closed subscheme of  $S$ .

We call  $X$  the *support* of  $\mathbf{X}$ . It may happen that  $\mathbf{X}$  is quasi-projective, and in this case it is projective if  $X$  is.

For every closed subscheme  $Z \subset \mathbf{X}$ , let  $\mathcal{J}_Z$  (or  $\mathcal{J}_{Z,x}$ ) denote the ideal sheaf of  $Z$  in  $\mathbf{X}$ . For every positive integer  $i$ , let  $\mathbf{X}_i$  be the closed subscheme of  $\mathbf{X}$  corresponding to the ideal sheaf  $\mathcal{J}_{\mathbf{X}}^i$ . The smallest integer  $n$  such that  $\mathbf{X}_n = \mathbf{X}$  is called the *multiplicity* of  $\mathbf{X}$ . For  $1 \leq i \leq n$ ,  $\mathbf{X}_i$  is a primitive multiple scheme of multiplicity  $i$ ,  $L = \mathcal{J}_{\mathbf{X}}/\mathcal{J}_{\mathbf{X}_2}$  is a line bundle on  $X$ , and we have  $\mathcal{J}_{\mathbf{X}_i}/\mathcal{J}_{\mathbf{X}_{i+1}} = L^i$ . We call  $L$  the line bundle on  $X$  *associated* to  $\mathbf{X}$ . If  $n = 2$ ,  $\mathbf{X}$  is called a *primitive double scheme*.

For any  $L \in \text{Pic}(X)$ , the *trivial primitive scheme* of multiplicity  $n$ , with induced smooth variety  $X$  and associated line bundle  $L$  on  $X$  is the  $n$ -th infinitesimal neighborhood of  $X$ , embedded by the zero section in the dual bundle  $L^*$ , seen as a smooth variety.

The primitive multiple curves were defined in [14], [2]. Primitive double schemes were parameterized and studied in [3] and [13]. More results on primitive multiple curves can be found in [7], [8], [9], [4], [19], [20], [21]. Some primitive double schemes are studied in [1], [15] and [16]. The case of varieties of any dimension, and higher multiplicity, is studied in [10], where the following subjects were treated:

- construction and parametrization of primitive multiple schemes,
- obstructions to the extension of a vector bundle on  $\mathbf{X}_m$  to  $\mathbf{X}_{m+1}$ ,
- obstructions to the extension of a primitive multiple scheme of multiplicity  $n$  to one of multiplicity  $n + 1$ .

In [11], the construction and properties of fine moduli spaces of vector bundles on primitive multiple schemes are described. In [12] other types of sheaves are studied.

Let  $\mathbf{Y}$  be a noetherian scheme. If  $Z \subset \mathbf{Y}$  is a closed subscheme, let  $\pi_{Z,\mathbf{Y}} : \mathbf{B}_{Z,\mathbf{Y}} \rightarrow \mathbf{Y}$  be the blowing-up of  $\mathbf{Y}$  along  $Z$ .

Let  $X$  be a smooth irreducible variety,  $n \geq 2$  be an integer and  $\mathbf{X}$  a primitive multiple scheme of multiplicity  $n$  such that  $\mathbf{X}_{red} = X$  and with associated line bundle  $L \in \text{Pic}(X)$ .

### 1.1. THE CASE OF SUBSCHEMES OF $X$

Let  $Y \subset X$  a nonempty closed irreducible smooth subvariety, such that  $Y \neq X$ . Let

$$\tilde{Y} = \pi_{Y,X}^{-1}(Y), \quad \tilde{Y}_n = \pi_{Y,\mathbf{X}}^{-1}(Y).$$

We will prove (cf. 4.1)

**1.1.1. Theorem:**  $\mathbf{B}_{Y,\mathbf{X}}$  is a primitive multiple scheme of multiplicity  $n$ ,  $(\mathbf{B}_{Y,\mathbf{X}})_{red} = \mathbf{B}_{Y,X}$ , its associated line bundle is

$$\mathcal{O}_{\mathbf{B}_{Y,\mathbf{X}}}(\tilde{Y}_n)|_{\mathbf{B}_{Y,\mathbf{X}}} \otimes \pi_{Y,\mathbf{X}}^*(L) = \mathcal{O}_{\mathbf{B}_{Y,\mathbf{X}}}(\tilde{Y}) \otimes \pi_{Y,\mathbf{X}}^*(L),$$

The exceptional divisor  $\tilde{Y}_n$  is a trivial primitive multiple scheme of multiplicity  $n$ , such that  $(\tilde{Y}_n)_{red} = \tilde{Y}$ , and with associated line bundle  $(\mathcal{O}_{\mathbf{B}_{Y,X}}(\tilde{Y}) \otimes \pi_{Y,X}^*(L))|_{\tilde{Y}}$ .

The proof follows the construction of blowing-ups given in [17], II-2. If  $Y$  is a hypersurface, we give another proof, inspired by [3] and [8], using the fact that  $\mathbf{B}_{Y,X} = X$ .

**1.1.2. The case of double schemes** – Suppose that  $\mathcal{X}$  is of multiplicity 2. Let  $\mathcal{X}'$  another double primitive scheme such that  $\mathcal{X}'_{red} = X$ , not isomorphic to  $\mathcal{X}$ . We will prove that if  $\text{codim}(Y, X) \geq 2$ , then  $\mathbf{B}_{Y,\mathcal{X}}$  and  $\mathbf{B}_{Y,\mathcal{X}'}$  are not isomorphic (proposition 4.1.3).

Suppose that  $Y$  is a hypersurface. To  $\mathcal{X}$  one can associate  $\sigma \in H^1(X, T_X \otimes L)$ , such that  $\mathbb{C}.\sigma$  characterizes  $\mathcal{X}$  (cf. 3.2). We have a canonical map  $\phi_Y : H^1(X, T_X \otimes L) \rightarrow H^1(X, T_X \otimes L \otimes \mathcal{O}_X(Y))$ . Using the second proof the preceding theorem for hypersurfaces, we prove that the element of  $H^1(X, T_X \otimes L \otimes \mathcal{O}_X(Y))$  associated to  $\mathbf{B}_{Y,\mathcal{X}}$  is  $\phi_Y(\sigma)$ .

## 1.2. THE CASE OF GOOD SUBSCHEMES OF $\mathcal{X}$

Suppose that  $\mathcal{X}$  is of multiplicity  $n$ . Let  $Z \subset \mathcal{X}$  be a good subscheme of dimension  $d$ . Let  $Y = Z_{red}$  and  $\tilde{Z}_n = \pi_{Z,\mathcal{X}}^{-1}(Z)$ . We will prove (cf. 4.2)

**1.2.1. Theorem:**  $\mathbf{B}_{Z,\mathcal{X}}$  is a primitive multiple scheme of multiplicity  $n$ ,  $(\mathbf{B}_{Z,\mathcal{X}})_{red} = \mathbf{B}_{Y,\mathcal{X}}$ , and its associated line bundle is  $\pi_{Y,\mathcal{X}}^*(L)$ .

The exceptional divisor  $\tilde{Z}_n$  is a primitive multiple scheme of multiplicity  $n$ ,  $(\tilde{Z}_n)_{red} = \tilde{Y} = \pi_{Y,\mathcal{X}}^{-1}(Y)$ , and its associated line bundle is  $\pi_{Y,\mathcal{X}}^*(L)|_{\tilde{Y}}$ .

The proof is very similar to that of theorem 4.1.1. The two cases are related: the blowing-up of  $\mathbf{B}_{Z,\mathcal{X}}$  along  $\tilde{Y} = \pi_{Y,\mathcal{X}}^{-1}(Y) \subset \mathbf{B}_{Y,\mathcal{X}}$  is  $\mathbf{B}_{Y,\mathcal{X}}$  (proposition 4.3.1).

*Zero-dimensional good subschemes* – Suppose that  $\mathcal{X}$  is of multiplicity 2. Let  $P \in X$ . Let  $\Gamma_P$  be the set of good subschemes  $Z \subset \mathcal{X}$  such that  $Z_{red} = \{P\}$ . Then  $\Gamma_P$  has a natural structure of affine space, with associated vector space  $(T_X \otimes L)_P$ . Hence, given  $Z, Z' \in \Gamma_P$ , we can define the “difference”  $Z - Z' \in (T_X \otimes L)_P$ .

Let  $\text{ev}_L : \mathcal{O}_X \otimes H^0(X, T_X \otimes L) \rightarrow T_X \otimes L$  be the evaluation morphism, and  $H_{L,P} = \text{im}(\text{ev}_{L,P}) \subset (T_X \otimes L)_P$ .

In 5.2, we determine when  $\mathbf{B}_{Z,\mathcal{X}}$  and  $\mathbf{B}_{Z',\mathcal{X}}$  are isomorphic. When  $\mathcal{X}$  is not trivial, it is the case if and only if  $Z - Z' \in H_{L,P}$  (theorem 5.2.2).

## 1.3. DOUBLE STRUCTURES ON $\mathbb{P}_2$

Let  $P \in \mathbb{P}_2$ . Let  $\pi : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$  be the blowing-up of  $\mathbb{P}_2$  along  $P$ , and  $\tilde{P} = \pi^{-1}(P)$  the exceptional divisor.

According to [10], there exists only one non trivial primitive double scheme  $\mathbb{X}$  such that  $\mathbb{X}_{red} = \mathbb{P}_2$ , and its associated line bundle is  $\mathcal{O}_{\mathbb{P}_2}(-3)$ .

Let  $\mathbb{Y}$  be a primitive double scheme such that  $\mathbb{Y}_{red} = \mathbb{P}_2$ , and with associated line bundle  $\mathcal{O}_{\mathbb{P}_2}(m)$ . We will give the list of all primitive double schemes  $\tilde{\mathbb{Y}}$  such that

- $\tilde{\mathbb{Y}}_{red} = \tilde{\mathbb{P}}_2$ ,
- there exists a morphism  $\pi_{\tilde{\mathbb{Y}}} : \tilde{\mathbb{Y}} \rightarrow \mathbb{Y}$  inducing  $\pi_{P, \tilde{\mathbb{P}}_2} : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$ , and such that its restriction  $\tilde{\mathbb{Y}} \setminus \tilde{P} \rightarrow \mathbb{Y} \setminus \{P\}$  is an isomorphism.

In this case, if  $\mathbb{L}$  is the line bundle on  $\tilde{\mathbb{P}}_2$  associated to  $\tilde{\mathbb{Y}}$ , there is an integer  $p$  such that  $\mathbb{L} = \pi_{P, \tilde{\mathbb{P}}_2}^*(\mathcal{O}_{\mathbb{P}_2}(m)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(p\tilde{P})$ . We will prove

**1.3.1. Theorem:** *We have  $p \geq 0$ .*

**1** – *If  $p = 0$ , the schemes  $\tilde{\mathbb{Y}}$  are the blowing-ups of  $\mathbb{Y}$  along the good subschemes  $Z$  such that  $Z_{red} = \{P\}$ . If  $m \geq -1$ , all these schemes are isomorphic. If  $m \leq -2$  they are all distinct.*

**2** – *If  $p = 1$ , these schemes form a family  $(\tilde{\mathbb{Y}}_{1, \alpha})_{\alpha \in \mathbb{C}}$ ,  $\tilde{\mathbb{Y}}_{1, 0}$  being the blowing-up of the schemes of **1** along  $\tilde{P}$ , and also the blowing-up of  $\mathbb{Y}$  along  $P$ . If  $\mathbb{Y} = \mathbb{X}$ , these schemes are all distinct. If  $\mathbb{Y}$  is trivial,  $\tilde{\mathbb{Y}}_{1, 0}$  is trivial, and all the  $\tilde{\mathbb{Y}}_{1, \alpha}$ ,  $\alpha \neq 0$  are isomorphic and non trivial.*

**3** – *If  $p \geq 2$ , there is only one such scheme  $\tilde{\mathbb{Y}}_p$ . If  $p = 2$ ,  $\tilde{\mathbb{Y}}_p$  is the blowing-up of the schemes of **2** along  $\tilde{P}$ . If  $p > 2$ ,  $\tilde{\mathbb{Y}}_p$  is the blowing-up of  $\tilde{\mathbb{Y}}_{p-1}$  along  $\tilde{P}$ .*

*K3-carpets* – We consider the schemes  $\tilde{\mathbb{X}}_{1, \alpha}$ ,  $\alpha \in \mathbb{C}$ , of theorem 1.3.1, 2 (when  $\mathbb{Y}$  is the non trivial double scheme  $\mathbb{X}$ ). The line bundle associated to  $\tilde{\mathbb{X}}_{1, \alpha}$  is  $\pi_{P, \tilde{\mathbb{P}}_2}^*(\mathcal{O}_{\mathbb{P}_2}(-3)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(\tilde{P}) = \omega_{\tilde{\mathbb{P}}_2}$ . It follows that  $\tilde{\mathbb{X}}_{1, \alpha}$  is a *K3-carpet* (cf. [11], [15]).

**1.3.2. Theorem:** **1** –  *$\tilde{\mathbb{X}}_{1, \alpha}$  is quasi-projective if and only  $\alpha \in \mathbb{Q}$  and  $\alpha > 0$ .*

**2** – *If  $\alpha \notin \mathbb{Q}$ ,  $\text{Pic}(\tilde{\mathbb{X}}_{1, \alpha}) = \{\mathcal{O}_{\tilde{\mathbb{X}}_{1, \alpha}}\}$ .*

**3** – *If  $\alpha \in \mathbb{Q}$ ,  $\text{Pic}(\tilde{\mathbb{X}}_{1, \alpha}) \simeq \mathbb{Z}$ .*

If instead of  $\mathbb{X}$  we take for  $\mathbb{Y}$  the trivial double scheme with associated line bundle  $\mathcal{O}_{\mathbb{P}_2}(-3)$ , we obtain two K3-carpets, the trivial scheme  $\tilde{\mathbb{Y}}_{1, 0}$  (the blowing-up of  $\mathbb{Y}$  along  $P$ ), and  $\tilde{\mathbb{Y}}_{1, \alpha}$ ,  $\alpha \neq 0$ . We have  $\text{Pic}(\tilde{\mathbb{Y}}_{1, \alpha}) \simeq \mathbb{Z}$ , generated by an extension of  $\pi^*(\mathcal{O}_{\mathbb{P}_2}(1))$  to  $\tilde{\mathbb{Y}}_{1, \alpha}$ .

#### 1.4. OUTLINE OF THE PAPER

In Chapter 2, we give several preliminary definitions and technical results that will be used in the next chapters.

In Chapter 3 we recall the definitions and properties of primitive multiple schemes and of good subschemes. We show also how to define morphisms between primitive double schemes.

In Chapter 4, the blowing-ups are described, and theorems 1.1.1 and 1.2.1 are proved.

In Chapter 5, we give supplementary results on blowing-ups of  $\mathfrak{X}$  along zero-dimensional good subschemes, when  $\mathfrak{X}$  is of multiplicity 2.

In Chapter 6, we give another proof of theorem 1.1.1, when  $Y$  is an hypersurface.

In Chapter 7, we treat the case of  $X = \mathbb{P}_2$ , and prove the results of 1.3.

**Terminology:** A *scheme* is a noetherian separated scheme over  $\mathbb{C}$ .

## 2. PRELIMINARIES

### 2.1. CANONICAL CLASS OF A VECTOR BUNDLE

**2.1.1. Definition** – Let  $Z$  be a scheme over  $\mathbb{C}$  and  $L$  a line bundle on  $Z$ . To  $L$  one associates an element  $\nabla_0(L)$  of  $H^1(Z, \Omega_Z)$ , called the *canonical class* of  $L$ . If  $Z$  is smooth and projective, and  $L = \mathcal{O}_Z(Y)$ , where  $Y \subset Z$  is a smooth hypersurface, then  $\nabla_0(L)$  is the cohomology class of  $Y$ .

Let  $(Z_i)_{i \in I}$  be an open cover of  $Z$  such that  $L$  is defined by a cocycle  $(\theta_{ij})$ ,  $\theta_{ij} \in \mathcal{O}_Z(Z_{ij})^*$ . Then  $\left(\frac{d\theta_{ij}}{\theta_{ij}}\right)$  is a cocycle which represents  $\nabla_0(L)$ .

### 2.2. ČECH COHOMOLOGY AND TRIVIALIZATIONS

Let  $X$  be a scheme over  $\mathbb{C}$  and  $E$  a vector bundle on  $X$ ,  $r = \text{rk}(E)$ . Let  $(U_i)_{i \in I}$  be an open cover of  $X$  such that we have trivializations:

$$\alpha_i : E|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i} \otimes \mathbb{C}^r.$$

Let

$$\alpha_{ij} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{ij}} \otimes \mathbb{C}^r \xrightarrow{\cong} \mathcal{O}_{U_{ij}} \otimes \mathbb{C}^r,$$

so that we have the relation  $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ .

**2.2.1.** Let  $n$  be a positive integer, and for every sequence  $(i_0, \dots, i_n)$  of distinct elements of  $I$ ,  $\sigma_{i_0 \dots i_n} \in H^0(U_{i_0 \dots i_n}, \mathcal{O}_X \otimes \mathbb{C}^r)$ . Let

$$\theta_{i_0 \dots i_n} = \alpha_{i_0}^{-1} \sigma_{i_0 \dots i_n} \in H^0(U_{i_0 \dots i_n}, E).$$

The family  $(\theta_{i_0 \dots i_n})$  represents an element of  $H^n(X, E)$  if the cocycle relations are satisfied: for every sequence  $(i_0, \dots, i_{n+1})$  of distinct elements of  $I$ ,

$$\sum_{k=0}^n (-1)^k \theta_{i_0 \dots \widehat{i_k} \dots i_{n+1}} = 0,$$

which is equivalent to

$$\alpha_{i_0 i_1} \sigma_{i_1 \dots i_{n+1}} + \sum_{k=1}^{n+1} (-1)^k \sigma_{i_0 \dots \widehat{i_k} \dots i_{n+1}} = 0 .$$

For  $n = 1$ , this gives that elements of  $H^1(X, E)$  are represented by families  $(\sigma_{ij})$ ,  $\sigma_{ij} \in H^0(U_{ij}, \mathcal{O}_X \otimes \mathbb{C}^r)$ , such that

$$\alpha_{ij} \sigma_{jk} + \sigma_{ij} - \sigma_{ik} = 0 .$$

In Čech cohomology, it is generally assumed that  $\theta_{ji} = -\theta_{ij}$ . This implies that  $\sigma_{ji} = -\alpha_{ji} \sigma_{ij}$ .

**2.2.2.** Let  $F$  be a vector bundle on  $X$ . Similarly, an element of  $H^n(X, F \otimes E)$  is represented by a family  $(\mu_{i_0 \dots i_n})$ , with  $\mu_{i_0 \dots i_n} \in H^0(U_{i_0 \dots i_n}, F \otimes \mathbb{C}^r)$ , satisfying the relations

$$(I_F \otimes \alpha_{i_0 i_1})(\mu_{i_1 \dots i_{n+1}}) + \sum_{k=1}^{n+1} (-1)^k \mu_{i_0 \dots \widehat{i_k} \dots i_{n+1}} = 0 .$$

The corresponding element of  $H^0(U_{i_0 \dots i_n}, E \otimes F)$  is  $\theta_{i_0 \dots i_n} = (I_F \otimes \alpha_{i_0})^{-1}(\mu_{i_0 \dots i_n})$ .

For  $n = 1$ , this gives that elements of  $H^1(X, F \otimes E)$  are represented by families  $(\sigma_{ij})$ ,  $\sigma_{ij} \in H^0(U_{ij}, F \otimes \mathbb{C}^r)$ , such that

$$(I_F \otimes \alpha_{ij}) \sigma_{jk} + \sigma_{ij} - \sigma_{ik} = 0 .$$

If  $E$  is a line bundle, then  $\alpha_{ij} \in \mathbb{C}^*$ , and the cocycle relation is  $\alpha_{ij} \sigma_{jk} + \sigma_{ij} - \sigma_{ik} = 0$ .

The family  $(\sigma_{ij})$  represents 0 if and only if there exists, for every  $i \in I$ ,  $\rho_i \in H^0(U_i, F \otimes \mathbb{C}^r)$ , such that  $\sigma_{ij} = \rho_i - (I_F \otimes \alpha_{ij})(\rho_j)$ .

**2.2.3.** Let  $D$  be a line bundle on  $X$ , and  $s \in H^0(X, D)$ . Let  $h \in H^1(X, E \otimes F)$ , represented (according to 2.2.2) by a family  $(h_{ij})$ ,  $h_{ij} \in H^0(X, F \otimes \mathbb{C}^r)$ , satisfying the relations  $\alpha_{ij} h_{jk} + h_{ij} - h_{ik} = 0$ . The section  $s$  induces a map

$$\phi_s : H^1(X, E \otimes F) \longrightarrow H^1(X, E \otimes F \otimes D).$$

Let  $\rho_i : D|_{U_i} \rightarrow \mathcal{O}_{U_i}$  be trivializations, and  $s_i = \rho_i s \in \mathbb{C}$ . Then  $(\rho_{ij} \alpha_{ij})$  is a cocycle that defines the vector bundle  $E \otimes D$ . We have  $s_i h_{ij} \in H^0(X, F \otimes \mathbb{C}^r)$ , and using the method of 2.2.2 (with  $E$  replaced with  $E \otimes D$ ),  $\phi_s(h)$  is represented by the family  $(s_i h_{ij})$ .

### 3. PRIMITIVE MULTIPLE SCHEMES

#### 3.1. DEFINITION AND CONSTRUCTION

Let  $X$  be a smooth connected variety, and  $d = \dim(X)$ . A *multiple scheme with support  $X$*  is a Cohen-Macaulay scheme  $\mathfrak{X}$  such that  $\mathfrak{X}_{red} = X$ . If  $\mathfrak{X}$  is quasi-projective we say that it is a *multiple variety with support  $X$* . In this case  $\mathfrak{X}$  is projective if  $X$  is.

Let  $n$  be the smallest integer such that  $\mathfrak{X} = X^{(n-1)}$ ,  $X^{(k-1)}$  being the  $k$ -th infinitesimal neighborhood of  $X$ , i.e.  $\mathcal{J}_{X^{(k-1)}} = \mathcal{J}_X^k$ . We have a filtration  $X = X_1 \subset X_2 \subset \dots \subset X_n = \mathfrak{X}$  where

$X_i$  is the biggest Cohen-Macaulay subscheme contained in  $\mathfrak{X} \cap X^{(i-1)}$ . We call  $n$  the *multiplicity* of  $Y$ .

We say that  $\mathfrak{X}$  is *primitive* if, for every closed point  $x$  of  $X$ , there exists a smooth variety  $S$  of dimension  $d + 1$ , containing a neighborhood of  $x$  in  $\mathfrak{X}$  as a locally closed subscheme. In this case,  $L = \mathcal{J}_X/\mathcal{J}_{X_2}$  is a line bundle on  $X$ ,  $X_j$  is a primitive multiple scheme of multiplicity  $j$  and we have  $\mathcal{J}_{X_j} = \mathcal{J}_X^j$ ,  $\mathcal{J}_{X_j}/\mathcal{J}_{X_{j+1}} = L^j$  for  $1 \leq j < n$ . We call  $L$  the line bundle on  $X$  *associated* to  $\mathfrak{X}$ . The ideal sheaf  $\mathcal{J}_{X,\mathfrak{X}}$  can be viewed as a line bundle on  $X_{n-1}$ .

Let  $P \in X$ . Then there exist elements  $y_1, \dots, y_d, t$  of  $m_{S,P}$  whose images in  $m_{S,P}/m_{S,P}^2$  form a basis, and such that for  $1 \leq i < n$  we have  $\mathcal{J}_{X_i,P} = (t^i)$ . In this case the images of  $y_1, \dots, y_d$  in  $m_{X,P}/m_{X,P}^2$  form a basis of this vector space.

Even if  $X$  is projective, we do not assume that  $\mathfrak{X}$  is projective.

The simplest case is when  $\mathfrak{X}$  is contained in a smooth variety  $S$  of dimension  $d + 1$ . Suppose that  $\mathfrak{X}$  has multiplicity  $n$ . Let  $P \in X$  and  $f \in \mathcal{O}_{S,P}$  a local equation of  $X$ . Then we have  $\mathcal{J}_{X_i,P} = (f^i)$  for  $1 < j \leq n$  in  $S$ , in particular  $\mathcal{J}_{\mathfrak{X},P} = (f^n)$ , and  $L = \mathcal{O}_X(-X)$ .

For any  $L \in \text{Pic}(X)$ , the *trivial primitive variety* of multiplicity  $n$ , with induced smooth variety  $X$  and associated line bundle  $L$  on  $X$  is the  $n$ -th infinitesimal neighborhood of  $X$ , embedded by the zero section in the dual bundle  $L^*$ , seen as a smooth variety.

If  $U \subset X$  is open, we note  $U^{(n)}$  the corresponding open subscheme of  $X_n$ .

**3.1.1. Construction of primitive multiple schemes** – Let  $\mathfrak{X}$  be a primitive multiple scheme of multiplicity  $n$ ,  $X = \mathfrak{X}_{red}$ . Let  $\mathbf{Z}_n = \text{spec}(\mathbb{C}[t]/(t^n))$ . Then for every closed point  $P \in X$ , there exists an open neighborhood  $U$  of  $P$  in  $X$  such that

- There exists a section  $\mathcal{O}_X(U) \rightarrow \mathcal{O}_{\mathfrak{X}}(U^{(n)})$  of the restriction map  $\mathcal{O}_{\mathfrak{X}}(U^{(n)}) \rightarrow \mathcal{O}_X(U)$ ,
- $L|_U$  is trivial.

Then there exists a commutative diagram

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ U^{(n)} & \xrightarrow{\cong} & U \times \mathbf{Z}_n \end{array}$$

i.e.  $\mathfrak{X}$  is locally trivial ([8], théorème 5.2.1, corollaire 5.2.2).

It follows that we can construct a primitive multiple scheme of multiplicity  $n$  by taking an open cover  $(U_i)_{i \in I}$  of  $X$  and gluing the varieties  $U_i \times \mathbf{Z}_n$  (with automorphisms of the  $U_{ij} \times \mathbf{Z}_n$  leaving  $U_{ij}$  invariant).

Let  $(U_i)_{i \in I}$  be an affine open cover of  $X$  such that we have trivializations

$$\delta_i : U_i^{(n)} \xrightarrow{\cong} U_i \times \mathbf{Z}_n,$$

and  $\delta_i^* : \mathcal{O}_{U_i \times \mathbf{Z}_n} \rightarrow \mathcal{O}_{U_i^{(n)}}$  the corresponding isomorphism. Let

$$\delta_{ij} = \delta_j \delta_i^{-1} : U_{ij} \times \mathbf{Z}_n \xrightarrow{\cong} U_{ij} \times \mathbf{Z}_n .$$

Then  $\delta_{ij}^* = \delta_i^{*-1} \delta_j^*$  is an automorphism of  $\mathcal{O}_{U_i \times Z_n} = \mathcal{O}_X(U_{ij})[t]/(t^n)$ , such that for every  $\phi \in \mathcal{O}_X(U_{ij})$ , seen as a polynomial in  $t$  with coefficients in  $\mathcal{O}_X(U_{ij})$ , the term of degree zero of  $\delta_{ij}^*(\phi)$  is the same as the term of degree zero of  $\phi$ .

**3.1.2. The ideal sheaf of  $X$**  – There exists  $\alpha_{ij} \in \mathcal{O}_X(U_{ij})[t]/(t^{n-1})$  such that  $\delta_{ij}^*(t) = \alpha_{ij}t$ . Let  $\alpha_{ij}^{(0)} = \alpha_{ij}|_X \in \mathcal{O}_X(U_i)$ . For every  $i \in I$ ,  $\delta_i^*(t)$  is a generator of  $\mathcal{J}_{X, \mathcal{X}|U^{(n)}}$ . So we have local trivializations

$$\begin{aligned} \lambda_i : \mathcal{J}_{X, \mathcal{X}|U_i^{(n-1)}} &\longrightarrow \mathcal{O}_{U_i^{(n-1)}} \\ \delta_i^*(t) &\longmapsto 1 \end{aligned}$$

Hence  $\lambda_{ij} = \lambda_i \lambda_j^{-1} : \mathcal{O}_{U_{ij}^{(n-1)}} \rightarrow \mathcal{O}_{U_{ij}^{(n-1)}}$  is the multiplication by  $\delta_j^*(\alpha_{ij})$ . It follows that  $(\delta_j^*(\alpha_{ij}))$  (resp.  $(\alpha_{ij}^{(0)})$ ) is a cocycle representing the line bundle  $\mathcal{J}_{X, \mathcal{X}}$  (resp.  $L$ ) on  $X_{n-1}$  (resp.  $X$ ).

### 3.2. THE CASE OF DOUBLE SCHEMES

We suppose that  $n = 2$ . Let  $\alpha_i : L|_{U_i} \rightarrow \mathcal{O}_{U_i}$  be isomorphisms such that  $\alpha_{ij} = \alpha_i \circ \alpha_j^{-1}$  on  $U_{ij}$ . Then we have the following description of  $\delta_{ij}^*$ :

- there is a derivation  $D_{ij}$  of  $\mathcal{O}_X(U_{ij})$  such that  $\delta_{ij}^*(\beta) = \beta + D_{ij}(\beta)t$  for every  $\beta \in \mathcal{O}_X(U_{ij})$ .
- $\delta_{ij}^*(t) = \alpha_{ij}t$ .

The relation  $\delta_{ij}^* \delta_{jk}^* = \delta_{ik}^*$  is equivalent to  $D_{ij} + \alpha_{ij} D_{jk} = D_{ik}$ , i.e.  $(D_{ij})$  is a cocycle in the sense of 2.2.2. We can view  $D_{ij}$  as section of  $T_X|_{U_{ij}}$ , the family  $(\alpha_i^{-1} \otimes D_{ij})$  represents an element  $\lambda$  of  $H^1(X, T_X \otimes L)$ , and  $\mathbb{C}\lambda$  is independent of the choice of the  $\delta_{ij}^*$  and  $\alpha_i$ . We will note  $\mathbb{C}\lambda = \sigma(X_2) \in \mathbb{P}(H^1(X, T_X \otimes L)) \cup \{0\}$ .

Let  $\mathcal{X}, \mathcal{X}'$  primitive double schemes,  $\phi : \mathcal{X}_{red} \rightarrow X$ ,  $\phi' : \mathcal{X}'_{red} \rightarrow X$  isomorphisms. We say that  $\mathcal{X}, \mathcal{X}'$  are *isomorphic* if there is an isomorphism  $\Phi : \mathcal{X} \rightarrow \mathcal{X}'$  such that  $\phi = \Phi_{red} \circ \phi'$ . In this case the associated line bundles on  $X$  of  $\mathcal{X}, \mathcal{X}'$  are isomorphic. Let  $\mathbf{DS}(X, L)$  be the set of isomorphism classes of primitive double schemes  $\mathcal{X}$  such that  $\mathcal{X}_{red} \simeq X$  and with associated line bundle  $L$ .

According to [10] and [3], two primitive double schemes  $X_2, X'_2$ , with underlying smooth variety  $X$  and associated line bundle  $L$  are isomorphic (over  $X$ ) if and only if  $\sigma(X_2) = \sigma(X'_2)$ . So there is a canonical bijection  $\mathbf{DS}(X, L) \simeq \mathbb{P}(H^1(X, T_X \otimes L)) \cup \{0\}$ .

Another family  $(D'_{ij})$  of derivations defines (with  $(\alpha_{ij})$ ) the same double scheme as  $(D_{ij})$  if and only if there exist  $\tau \in \mathbb{C}^*$ , and for every  $i$  a derivation  $T_i$  of  $\mathcal{O}_X(U_i)$  such that  $D'_{ij} = \tau D_{ij} + T_i - \alpha_{ij} T_j$ .

**3.2.1. Change of trivialization** – Let  $i_0 \in I$  and  $\theta$  an automorphism of  $U_{i_0} \times \mathbf{Z}_2$ , defined by a derivation  $D$  of  $\mathcal{O}_X(U_{i_0})$ , i.e.  $\theta^*(u) = u + D(u)t$  for every  $u \in \mathcal{O}_X(U_{i_0})$ , and  $\theta^*(t) = t$ . Now let  $\mu_i = \delta_i$  if  $i \in I, i \neq i_0$ , and  $\mu_{i_0} = \theta \circ \delta_{i_0}$ . We have  $\mu_{jk}^* = \delta_{jk}^*$  if  $j, k \in I, j, k \neq i_0$ , and  $\mu_{i_0k}^* = \theta^{*-1} \delta_{i_0k}^*$ ,  $\mu_{ji_0}^* = \delta_{ji_0}^* \theta^*$ . With these new trivializations,  $X_2$  is defined by the same cocycle  $(\alpha_{ij})$ , and the family  $E_{ij}$  of derivations, with  $E_{jk} = D_{jk}$  if  $j, k \in I, j, k \neq i_0$ ,

$$E_{i_0k} = D_{i_0k} - D, \quad E_{ji_0} = D_{ji_0} + \alpha_{ji_0} D.$$

### 3.3. EXTENSION OF LINE BUNDLES IN HIGHER MULTIPLICITY

Suppose that  $X_n$  can be extended to a primitive multiple scheme  $X_{n+1}$  of multiplicity  $n + 1$ . We have an exact sequence of coherent sheaves on  $X_n$

$$0 \longrightarrow L^n \longrightarrow \Omega_{X_{n+1}|X_n} \longrightarrow \Omega_{X_n} \longrightarrow 0 ,$$

corresponding to  $\sigma_n \in \text{Ext}_{\mathcal{O}_{X_n}}^1(\Omega_{X_n}, L^n)$ . Let  $\mathbb{F}$  be a line bundle on  $X_n$ . Let  $\nabla_0(\mathbb{F}) \in H_{\mathcal{O}_{X_n}}^1(X_n, \Omega_{X_n})$  be the canonical class of  $\mathbb{F}$  (cf. 2.1.1). Then we have

$$\sigma_n \nabla_0(\mathbb{F}) \in \text{Ext}_{\mathcal{O}_{X_n}}^2(\mathbb{F}, \mathbb{F} \otimes L^n) = H^2(X, L^n) .$$

From [10], theorem 7.1.2,  $\mathbb{F}$  can be extended to a line bundle on  $X_{n+1}$  if and only if  $\sigma_n \nabla_0(\mathbb{F}) = 0$ .

### 3.4. GOOD SUBSCHEMES

Let  $Z$  a subscheme of  $\mathfrak{X} = X_n$ . We say that  $Z$  is *good* if  $Z_{red}$  is smooth and connected, and if  $d = \text{codim}(Z)$  and  $\mathcal{J}_{Z,\mathfrak{X}}$  is the ideal sheaf of  $Z$ , then for every closed point  $z \in Z$ ,  $\mathcal{J}_{Z,\mathfrak{X},z}$  can be generated by  $d$  elements.

**3.4.1. Proposition:** *Let  $Z$  be a subscheme of  $\mathfrak{X}$  such that if  $Z_{red}$  is smooth and connected of dimension  $d$ . Then  $Z$  is good if and only if, for every closed point  $z \in Z$ , there exists an open neighborhood  $U \subset X$  of  $z$  such that there is a trivialization  $\rho : U^{(n)} \rightarrow U \times \mathbf{Z}_n$ , and generators  $y_1, \dots, y_d$  of  $\mathcal{J}_{Z_{red},X}(U)$ , which are also generators of the ideal sheaf of  $\rho(Z \cap U^{(n)})$ .*

*Proof.* If the condition is satisfied, then  $Z$  is good. Conversely, suppose that  $Z$  is good. There exists an affine neighborhood  $U \subset X$  of  $z$  such that

- $\mathcal{J}_{Z,\mathfrak{X}}(U)$  is generated by  $d$  elements  $z_1, \dots, z_d$ .
- The restriction of the tangent bundle of  $X$  on  $U$  is free.
- There exists a trivialization  $\theta : U^{(n)} \rightarrow U \times \mathbf{Z}_n$ .

We can write, for  $1 \leq i \leq d$ ,  $\theta^{*-1}(z_i) = y_i + a_i t$ , with  $y_i \in \mathcal{J}_{Z_{red},X}(U)$   $a_i \in \mathcal{O}_X(U)[t]/(t^{n-1})$ . Then  $y_1, \dots, y_d$  generate  $\mathcal{J}_{Z_{red},X}(U)$ , and for every  $z' \in Z_{red} \cap U$ , the images of  $y_1, \dots, y_d$  in  $m_{X,z'}/m_{X,z'}^2$  are linearly independent (where  $m_{X,z'} \subset \mathcal{O}_{X,z'}$  is the maximal ideal). According to [10], 5, there exists an automorphism  $\phi$  of  $U \times \mathbf{Z}_n$  leaving  $U$  invariant and such that  $\phi^*(y_i) = y_i + a_i t$  for  $1 \leq i \leq d$ . We can then take  $\rho = \theta \circ \phi$ .  $\square$

**Remarks: 1** – It is necessary to consider only smooth subvarieties  $Z_{red} \subset X$  in proposition 3.4.1: suppose that  $\mathfrak{X} = \mathbb{C} \times \mathbf{Z}_2 = \text{spec}(\mathbb{C}[X_0]) \times \mathbf{Z}_2$ , and let  $f \in \mathbb{C}[X_0]$  be a non zero polynomial multiple of  $X_0^2$ . Let  $Z \subset \mathfrak{X}$  be the subscheme defined by the ideal  $(f + t) \subset \mathbb{C}[X_0][t]/(t^2)$ . Let  $U \subset \mathbb{C}$  be a neighborhood of 0 and  $\rho$  an automorphism of  $U \times \mathbf{Z}_2$  leaving  $U$  invariant. Then, according to 3.2, there exists a derivation  $D$  of  $\mathcal{O}_{\mathbb{C}}(U)$  such that, for every  $\alpha \in \mathcal{O}_{\mathbb{C}}(U)$  we have  $\rho^*(\alpha) = \alpha + D(\alpha)t$ . Hence  $\rho^*(f) = f + D(f)t$ . Since  $D(f)$  vanishes at 0, it is impossible to obtain  $\rho^*(f) = f + t$ . Hence proposition 3.4.1 is false for  $Z_{red} \subset \mathbb{C}$ .

**2** – If  $d = \dim(X) - 1$ ,  $Z$  is a Cartier divisor, and  $\mathcal{J}_{Z,\mathfrak{X}}$  is a locally free sheaf. If  $\mathfrak{X}$  is the non projective double scheme with smooth underlying variety  $\mathbb{P}_2$ , such varieties  $Z$  of dimension 1 do not exist, since  $\mathcal{O}_{\mathfrak{X}}$  is the only line bundle on  $\mathfrak{X}$ .

### 3.5. MORPHISMS OF PRIMITIVE DOUBLE SCHEMES

**3.5.1. Derivations** – Let  $A, B$  be commutative  $\mathbb{C}$ -algebras, and  $f : A \rightarrow B$  a morphism. A map  $\theta : A \rightarrow B$  is called a  $f$ -derivation if

- For every  $\epsilon \in \mathbb{C}$ ,  $\theta(\epsilon) = 0$ .
- For every  $a, a' \in A$ ,  $\theta(a + a') = \theta(a) + \theta(a')$ .
- For every  $a, a' \in A$ ,  $\theta(a.a') = f(a)\theta(a') + f(a')\theta(a)$ .

**3.5.2. Morphisms of primitive double rings** – Let  $A, B$  be commutative  $\mathbb{C}$ -algebras, and  $A[2] = A[t]/(t^2)$ ,  $B[2] = B[t]/(t^2)$ . Let  $\phi : A \rightarrow B$ ,  $\Phi : A[2] \rightarrow B[2]$  be morphisms of  $\mathbb{C}$ -algebras such that  $\Phi((t)) \subset (t)$  and  $\phi$  is induced by  $\Phi$ . We have

$$\begin{aligned}\Phi(\alpha) &= \phi(\alpha) + \gamma(\alpha)t \quad \text{for every } \alpha \in A, \\ \Phi(t) &= \tau.t,\end{aligned}$$

where  $\gamma : A \rightarrow B$  is a  $\phi$ -derivation and  $\tau \in B$ .

**3.5.3. Morphisms of primitive double schemes** – Let  $X, Y$  be smooth varieties,  $X_0, X_1$  (resp.  $Y_0, Y_1$ ) open subsets of  $X$  (resp.  $Y$ ) such that  $X = X_0 \cup X_1$  (resp.  $Y = Y_0 \cup Y_1$ ). Let  $\mathcal{X}, \mathcal{Y}$  be primitive double schemes such that  $\mathcal{X}_{red} = X$  and  $\mathcal{Y}_{red} = Y$ . Assume that we have trivializations

$$\lambda_i : X_i^{(2)} \longrightarrow X_i \times \mathbf{Z}_2, \quad \mu_i : Y_i^{(2)} \longrightarrow Y_i \times \mathbf{Z}_2,$$

for  $i = 1, 2$ , i.e. isomorphisms

$$\lambda_i^* : \mathcal{O}_X(X_i)[t]/(t^2) \longrightarrow \mathcal{O}_{\mathbb{X}}(X_i), \quad \mu_i^* : \mathcal{O}_Y(Y_i)[t]/(t^2) \longrightarrow \mathcal{O}_{\mathbb{Y}}(Y_i).$$

Let

$$\begin{aligned}\lambda &= \lambda_1 \lambda_0^{-1} : X_{01} \times \mathbf{Z}_2 \rightarrow X_{01} \times \mathbf{Z}_2, & \lambda^* &= \lambda_0^{*-1} \lambda_1^* : \mathcal{O}_X(X_{01})[t]/(t^2) \rightarrow \mathcal{O}_X(X_{01})[t]/(t^2), \\ \mu &= \mu_1 \mu_0^{-1} : Y_{01} \times \mathbf{Z}_2 \rightarrow Y_{01} \times \mathbf{Z}_2, & \mu^* &= \mu_0^{*-1} \mu_1^* : \mathcal{O}_Y(Y_{01})[t]/(t^2) \rightarrow \mathcal{O}_Y(Y_{01})[t]/(t^2),\end{aligned}$$

We have

$$\begin{aligned}\lambda^*(a) &= a + D_X(a)t \quad \text{for every } a \in \mathcal{O}_X(X_{01}), & \lambda^*(t) &= \alpha t, \\ \mu^*(b) &= b + D_Y(b)t \quad \text{for every } b \in \mathcal{O}_Y(Y_{01}), & \mu^*(t) &= \beta t,\end{aligned}$$

where  $D_X$  (resp.  $D_Y$ ) is a derivation of  $\mathcal{O}_X(X_{01})$  (resp.  $\mathcal{O}_Y(Y_{01})$ ),  $\alpha \in \mathcal{O}_X(X_{01}^*)$  and  $\beta \in \mathcal{O}_Y(Y_{01}^*)$ .

Let  $\phi : Y \rightarrow X$  be a morphism such that  $\phi(Y_0) \subset X_0$  and  $\phi(Y_1) \subset X_1$ . Let  $\Delta_i : Y_i^{(2)} \rightarrow X_i^{(2)}$  be a morphism inducing  $\phi$ . Let

$$\Theta_i = \lambda_i \Delta_i \mu_i^{-1} : Y_i \times \mathbf{Z}_2 \longrightarrow X_i \times \mathbf{Z}_2.$$

Then we have

$$\Theta_i^*(a) = \phi^*(a) + \gamma_i(a)t \quad \text{for every } a \in \mathcal{O}_X(X_i), \quad \Theta_i^*(t) = \tau_i.t,$$

where  $\gamma_i : \mathcal{O}_X(X_i) \rightarrow \mathcal{O}_Y(Y_i)$  is a  $\phi^*$ -derivation and  $\tau_i \in \mathcal{O}_Y(Y_i)^*$ .

We have  $\Delta_0 = \Delta_1$  on  $Y_{01}^{(2)}$  if and only if  $\Theta_0^* \lambda^* = \mu^* \Theta_1^*$  on  $X_{01} \times \mathbf{Z}_2$ . This is equivalent to

$$\gamma_0 + \tau_0 \phi^* D_X = D_Y \phi^* + \beta \gamma_1 \quad \text{on } X_{01} \text{ and } \phi^*(\alpha) \tau_0 = \beta \tau_1.$$

Hence it is equivalent to find an extension  $\mathcal{Y} \rightarrow \mathcal{X}$  of  $\phi$ , and to find  $\phi^*$ -derivations  $\gamma_0, \gamma_1, \tau_i \in \mathcal{O}_Y(Y_i)^*$  satisfying the preceding equations. The first equation concerns  $\phi^*$ -derivations on  $X_{01}$ , one has to verify that  $\gamma_i$  sends  $\mathcal{O}_X(X_i)$  to  $\mathcal{O}_Y(Y_i)$ .

Suppose now that  $X_0 = Y_0, \phi|_{Y_0} = I_{Y_0}$  and  $\Delta_0 = I_{Y_0^{(2)}}$ . We have then  $\gamma_0 = 0$  and  $\tau_0 = 1$ . The preceding equations become

$$(1) \quad \beta\gamma_1 = D_X - D_Y \quad \text{on } X_{01} \text{ and } \alpha = \beta\tau_1 .$$

#### 4. BLOWING-UP OF SUBSCHEMES OF PRIMITIVE MULTIPLE SCHEMES

(cf. [17], II-2)

Let  $X$  be a smooth irreducible variety,  $n \geq 2$  be an integer and  $X_n$  a primitive multiple scheme of multiplicity  $n$  such that  $(X_n)_{red} = X$  and with associated line bundle  $L \in \text{Pic}(X)$ .

If  $Z \subset X_n$  is a closed subscheme, let  $\pi_{Z, X_n} : \mathbf{B}_{Z, X_n} \rightarrow X_n$  be the blowing-up of  $X_n$  along  $Z$ , and  $\tilde{Z}_n = \pi_{Z, X_n}^{-1}(Z)$ .

##### 4.1. BLOWING-UPS OF REDUCED SUBSCHEMES

Let  $Y \subset X$  be a smooth closed subvariety. Let  $\pi_{Y, X} : \mathbf{B}_{Y, X} \rightarrow X$  be the blowing-up of  $X$  along  $Y$  and  $\tilde{Y} = \pi_{Y, X}^{-1}(Y)$ . We can view  $Y$  as a closed subscheme of  $X_n$ . So we have also  $\pi_{Y, X_n} : \mathbf{B}_{Y, X_n} \rightarrow X_n$ , the blowing-up of  $X_n$  along  $Y$ , and  $\tilde{Y}_n = \pi_{Y, X_n}^{-1}(Y)$ . We will prove

**4.1.1. Theorem:**  $\mathbf{B}_{Y, X_n}$  is a primitive multiple scheme of multiplicity  $n$ ,  $(\mathbf{B}_{Y, X_n})_{red} = \mathbf{B}_{Y, X}$ , its associated line bundle is

$$\mathcal{O}_{\mathbf{B}_{Y, X_n}}(\tilde{Y}_n)|_{\mathbf{B}_{Y, X}} \otimes \pi_{Y, X}^*(L) = \mathcal{O}_{\mathbf{B}_{Y, X}}(\tilde{Y}) \otimes \pi_{Y, X}^*(L) ,$$

The exceptional divisor  $\tilde{Y}_n$  is a primitive multiple scheme of multiplicity  $n$ , such that  $(\tilde{Y}_n)_{red} = \tilde{Y}$ , and with associated line bundle  $(\mathcal{O}_{\mathbf{B}_{Y, X}}(\tilde{Y}) \otimes \pi_{Y, X}^*(L))|_{\tilde{Y}}$ .

*Proof.* Let  $U = \text{spec}(A)$  be an affine irreducible smooth variety,  $V \subset U$  a closed subvariety and  $\mathcal{J}_V = \mathcal{J}_{V, U}$  its ideal sheaf. Let  $U_n = U \times \mathbf{Z}_n$ . We have  $V \subset U \subset U_n$  and  $\mathcal{O}_{U_n}(U) = A[t]/(t^n)$ . We will first prove the theorem when  $X = U, Y = V$  and  $X_n = U_n$ , i.e.:  $\mathbf{B}_{V, U_n}$  is the trivial primitive multiple scheme of multiplicity  $n$  such that  $(\mathbf{B}_{V, U_n})_{red} = \mathbf{B}_{V, U}$ , and with associated line bundle  $\mathcal{O}_{\mathbf{B}_{V, U_n}}(\tilde{V})|_{\mathbf{B}_{V, X}}$ .

**Step 1 – Description of  $\mathbf{B}_{V, U}$**  – We have  $\mathbf{B}_{V, U} = \text{Proj}(S)$ , where  $S = S(V, U)$  is the graduate ring

$$S = \bigoplus_{p \geq 0} \mathcal{J}_V(U)^p .$$

The ideal sheaf  $\mathcal{J}_{U, U_n}$  of  $U$  in  $U_n$  is generated by  $t$ , and we have

$$\mathcal{J}_{V, U_n} = \mathcal{J}_V + \mathcal{J}_{U, U_n} .$$

We have  $\mathbf{B}_{V,U_n} = \text{Proj}(\mathbb{S}_n)$ , where  $\mathbb{S}_n = \mathbb{S}_n(V, U)$  is the graduate ring

$$\mathbb{S}_n = \bigoplus_{p \geq 0} \mathcal{J}_{V,U_n}(U)^p,$$

and

$$\mathcal{J}_{V,U_n}(U)^p = \bigoplus_{\max(p-n,0) \leq k \leq p} \mathcal{J}_V(U)^k t^{p-k}.$$

Let  $f \in \mathcal{J}_V(U)$ ,  $f \neq 0$ . Let  $S_f$  be the corresponding localized ring, and  $S_{(f)} \subset S_f$  the subring of elements of degree 0. Then  $D_+^S(f) = \{\mathfrak{p} \in \text{Proj}(S); f \notin \mathfrak{p}\}$  is an affine open subset of  $\mathbf{B}_{V,U}$ , and  $D_+^S(f) \simeq \text{spec}(S_{(f)})$ . We have

$$S_{(f)} = \sum_{p \geq 0} S_{(f),p}, \quad \text{with} \quad S_{(f),p} = \left\{ \frac{\sigma_p}{f^p}; \sigma_p \in \mathcal{J}_V(U)^p \right\}.$$

The ideal  $I_{V,p} \subset S_{(f)}$  of  $V \cap D_+^S(f)$  is

$$I_{V,f} = \sum_{p \geq 0} S'_{(f),p}, \quad \text{with} \quad S'_{(f),p} = \left\{ \frac{\mu_p}{f^p}; \mu_p \in \mathcal{J}_V(U)^{p+1} \right\}.$$

We have an isomorphism  $S_{(f)} \rightarrow I_{V,f}$ , which sends  $\alpha$  to  $f\alpha$ . Let  $f_1, \dots, f_p$  be generators of  $\mathcal{J}_V(U)$ . Then  $(D_+^S(f_i))_{1 \leq i \leq p}$  is an open cover of  $\mathbf{B}_{V,U}$ , and with respect to this cover,  $\left(\frac{f_j}{f_i}\right)$  is a cocycle that defines the line bundle  $\mathcal{O}_{\mathbf{B}_{V,U}}(-\tilde{V})$ .

**Step 2** – *Description of  $\mathbf{B}_{V,U_n}$*  – Similarly, we have  $f \in \mathcal{J}_{V,U_n}(U)$ , and we can define the affine open subset  $D_+^{\mathbb{S}_n}(f)$  of  $\mathbf{B}_{V,U_n}$ . Let  $\mathbb{S}_{n,f}$  be the corresponding localized ring, and  $\mathbb{S}_{n,(f)} \subset \mathbb{S}_{n,f}$  the subring of elements of degree 0. Then  $D_+^{\mathbb{S}_n}(f) \simeq \text{spec}(\mathbb{S}_{n,(f)})$ . We have

$$\mathbb{S}_{n,(f)} = \sum_{p \geq 0} \mathbb{S}_{n,(f),p}, \quad \text{with} \quad \mathbb{S}_{n,(f),p} = \left\{ \frac{\theta_p}{f^p}; \theta_p \in \mathcal{J}_{V,U_n}(U)^p \right\}.$$

It follows that

$$\mathbb{S}_{n,(f)} = S_{(f)}[z]/(z^n), \quad \text{with} \quad z = \frac{t}{f}.$$

We can do the same if  $f \in \mathcal{J}_V(W)$ , where  $W \subset U$  is an open subset. We obtain an isomorphism  $D_+^{\mathbb{S}_n}(f) \simeq D_+^S(f) \times \mathbf{Z}_n$  inducing the identity on  $D_+^S(f)$ .

**Step 3** – *The primitive multiple scheme structure on  $\mathbf{B}_{V,U_n}$*  – Let  $f_1, \dots, f_p$  be generators of  $\mathcal{J}_V(U)$ . Then  $(D_+^S(f_i))_{1 \leq i \leq p}$  (resp.  $(D_+^{\mathbb{S}_n}(f_i))_{1 \leq i \leq p}$ ) is an open cover of  $\mathbf{B}_{V,U}$  (resp.  $\mathbf{B}_{V,U_n}$ ). We define the primitive scheme structure of  $\mathbf{B}_{V,U_n}$  by taking the isomorphisms

$$\delta(f_i) : D_+^{\mathbb{S}_n}(f_i) \longrightarrow D_+^S(f_i) \times \mathbf{Z}_n$$

such that  $\delta(f_i)^* : S_{(f_i)}[z]/(z^n) \longrightarrow \mathbb{S}_{n,(f_i)}$   $\delta(f_i)^*(\alpha) = \alpha$  for every  $\alpha \in S_{(f_i)}$  and  $\delta(f_i)^*(z) = \frac{t}{f_i}$ . It follows that, if  $\delta_{ij} = \delta(f_j) \circ \delta(f_i)^{-1}$ , we have  $\delta_{ij}^*(\alpha) = \alpha$  for every  $\alpha \in S_{(f_i)} \cap S_{(f_j)}$ , and  $\delta_{ij}^*(z) = \frac{f_i}{f_j} z$ . It is easily verified that  $\delta_{ij}^* \delta_{jk}^* = \delta_{ik}^*$ . This proves that  $\mathbf{B}_{V,U_n}$  is a trivial primitive

multiple scheme of multiplicity  $n$  such that  $(\mathbf{B}_{V,U_n})_{red} = \mathbf{B}_{V,U}$ . Moreover we have a canonical obvious commutative square

$$\begin{array}{ccc} \mathbf{B}_{V,U} & \xrightarrow{\pi_{V,U}} & U \\ \downarrow & & \downarrow \\ \mathbf{B}_{V,U_n} & \xrightarrow{\pi_{V,U_n}} & U_n \end{array}$$

**Step 4** – *The ideal sheaf  $\mathcal{J}_{\mathbf{B}_{V,U}}$  of  $\mathbf{B}_{V,U}$  in  $\mathbf{B}_{V,U_n}$*  – On  $D_+^{\mathbb{S}_n}(f)$ , according to step 1, we have  $\mathcal{J}_{\mathbf{B}_{V,U}} = \sum_{j \geq 1} I_j$ , with

$$I_j = \left\{ \frac{t\sigma_j}{f^j} ; \sigma_j \in \mathcal{J}_V(U)^{j-1} \right\}.$$

It follows that we have a well defined isomorphism  $\mathcal{O}_{D_+^{\mathbb{S}_{n-1}}(f)} \rightarrow \mathcal{J}_{\mathbf{B}_{V,U}|D_+^{\mathbb{S}_n}(f)}$ , sending  $\frac{\sigma_j}{f^j}$  to  $\frac{t\sigma_j}{f^{j+1}}$ . In particular, we have trivializations  $\alpha_i : \mathcal{O}_{D_+^{\mathbb{S}_{n-1}}(f_i)} \rightarrow \mathcal{J}_{\mathbf{B}_{V,U}|D_+^{\mathbb{S}_n}(f_i)}$  for  $1 \leq i \leq p$ . It follows that the automorphism  $\alpha_{ij}^*$  of  $\mathcal{O}(D_+^{\mathbb{S}_{n-1}}(f_i) \cap D_+^{\mathbb{S}_{n-1}}(f_j))$  is the multiplication by  $\frac{f_i}{f_j}$ . According to step 1, it follows that the line bundle on  $\mathbf{B}_{V,U}$  associated to  $\mathbf{B}_{V,U_n}$  is  $\mathcal{O}_{\mathbf{B}_{V,U}}(\tilde{V})$ .

**Step 5** – *Automorphisms* – Let  $\Phi$  be an automorphism of  $U \times \mathbf{Z}_n$  leaving  $U$  invariant. Then there exists  $\alpha \in A[t]/(t^{n-1})$ , invertible, such that  $\Phi^*(t) = \alpha t$ . Let  $\tilde{\Phi}$  (resp.  $\tilde{\Phi}_n$ ) be the automorphism of  $\mathbf{B}_{V,U}$  (resp.  $\mathbf{B}_{V,U_n}$ ) induced by  $\Phi$ . Then  $\tilde{\Phi}^*(\mathcal{J}_V(U)) \subset \mathcal{J}_V(U)$ , and  $\tilde{\Phi}^*(f) \in \mathbb{S}_{n,f}$  is an invertible element of degree 1.

We have  $\tilde{\Phi}(D_+^{\mathbb{S}}(f)) = D_+^{\mathbb{S}}(f)$  and  $\tilde{\Phi}_n(D_+^{\mathbb{S}_n}(f)) = D_+^{\mathbb{S}_n}(f)$ . Let  $\tilde{\Phi}^f$  (resp.  $\tilde{\Phi}_n^f$ ) be the corresponding automorphism of  $D_+^{\mathbb{S}}(f)$  (resp.  $D_+^{\mathbb{S}_n}(f)$ ).

Let  $\frac{\sigma_p}{f^p} \in \mathbb{S}_{n,(f)}$ . We have

$$\tilde{\Phi}_n^{f*} \left( \frac{\sigma_p}{f^p} \right) = \frac{\Phi^*(\sigma_p)}{\Phi^*(f)^p}$$

and  $\tilde{\Phi}_n^{f*}(t) = \alpha t$ .

**Step 6** – *The primitive multiple scheme structure on  $\mathbf{B}_{Y,X_n}$*  – Let  $(U_i)_{i \in I}$  be a finite open affine cover of  $X$  such that for every  $i \in I$  there is an isomorphism  $\lambda_i : U_i^{(n)} \rightarrow U_i \times \mathbf{Z}_n$  leaving  $U_i$  invariant. We have then  $\lambda_{ik}^*(t) = \alpha_{ik} t$ , where  $\alpha_{ik} \in \mathcal{O}_X(U_{ik})[t]/(t^{n-1})$  is invertible. The line bundle  $L \in \text{Pic}(X)$  associated to  $X_n$  is defined by the cocycle  $(\alpha_{ik}|_{U_{ik}})$ .

Let  $\tilde{\lambda}_i : \mathbf{B}_{Y \cap U_i, U_i^{(n)}} \rightarrow \mathbf{B}_{Y \cap U_i, U_i \times \mathbf{Z}_n}$  be the induced isomorphism. We have

$\pi_{Y \cap U_i, U_i^{(n)}} = \pi_{Y \cap U_i, U_i \times \mathbf{Z}_n} \circ \tilde{\lambda}_i$ . Then from step 2

- $\mathbf{B}_{Y \cap U_i, U_i^{(n)}}$  is a primitive multiple scheme of multiplicity  $n$ , with associated smooth scheme  $\mathbf{B}_{Y \cap U_i, U_i}$ .
- We have  $\mathbf{B}_{Y \cap U_i, U_i} = \pi_{Y,X}^{-1}(U_i)$  and  $\pi_{Y \cap U_i, U_i}$  is the restriction of  $\pi_{X,Y}$ . We have a similar statement for  $\mathbf{B}_{Y \cap U_i, U_i^{(n)}}$ .

Hence  $\mathbf{B}_{Y,X_n}$  is a primitive multiple scheme of multiplicity  $n$ , and  $(\mathbf{B}_{Y,X_n})_{red} = \mathbf{B}_{Y,X}$ .

*The open cover* – More precisely, for every  $i \in I$ , let  $\mathbf{S}_i = S(Y \cap U_i, U_i)$ ,  $\Sigma_i = \mathbb{S}_n(Y \cap U_i, U_i)$  and  $(\mathbf{f}_{ij})_{j \in J_i}$  a finite sequence of generators of  $\mathbb{J}_{Y \cap U_i, U_i}(U_i)$ . Let

$$T_{ij} = D_+^{\mathbf{S}_i}(\mathbf{f}_{ij}) \subset \mathbf{B}_{Y \cap U_i, U_i}, \quad \mathbb{T}_{ij} = D_+^{\Sigma_i}(\mathbf{f}_{ij}) \subset \mathbf{B}_{Y \cap U_i, U_i \times \mathbf{Z}_n},$$

and  $Z_{ij} = \tilde{\lambda}_i^{-1}(\mathbb{T}_{ij}) \subset \mathbf{B}_{Y,X_n}$ . We have  $(\mathbb{T}_{ij})_{red} = (Z_{ij})_{red} = T_{ij}$ , and  $(T_{ij})_{i \in I, j \in J_i}$  is an open cover of  $\mathbf{B}_{Y,X}$ .

*The trivializations* – We have seen in step 3 that

$$\delta(\mathbf{f}_{ij}) : \mathbb{T}_{ij} \longrightarrow T_{ij} \times \mathbf{Z}_n$$

is an isomorphism. Then

$$\rho_{ij} = \delta(\mathbf{f}_{ij}) \circ \tilde{\lambda}_i : Z_{ij} \longrightarrow T_{ij} \times \mathbf{Z}_n$$

is an isomorphism. To summarize,  $(T_{ij})_{i \in I, j \in J_i}$  (resp.  $(Z_{ij})_{i \in I, j \in J_i}$ ) is an affine open cover of  $\mathbf{B}_{Y,X}$  (resp.  $\mathbf{B}_{Y,X_n}$ ), and the  $\rho_{ij}$  are the trivializations that define the primitive multiple scheme structure on  $\mathbf{B}_{Y,X_n}$ . The restriction of  $\rho_{ij}$ ,

$$Z_{ij} \cap (X_n \setminus Y) \longrightarrow (T_{ij} \cap (X \setminus Y)) \times \mathbf{Z}_n,$$

is  $\lambda_i|_{Z_{ij} \cap (X_n \setminus Y)}$ .

Let  $i, k \in I$ ,  $j \in J_i$  and  $l \in J_k$ . Let

$$W_{ij,kl} = D_+^{\Sigma_i}(\mathbf{f}_{i,j}) \cap D_+^{\Sigma_k}(\mathbf{f}_{k,l}) \subset \mathbf{B}_{Y \cap U_i, U_i \times \mathbf{Z}_n}.$$

Then

$$\tilde{\lambda}_i^{-1}(W_{ij,kl}) = \tilde{\lambda}_k^{-1}(W_{kl,ij}) = Z_{ij} \cap Z_{kl} = Z_{ij,kl},$$

and  $\rho_{ij,kl}^* = \rho_{ij}^{*-1} \rho_{kl}^*$  is an automorphism of  $\mathcal{O}_{\mathbf{B}_{Y,X}}(T_{ij,kl})[t]/(t^n)$ .

*The associated line bundle* – We have  $\rho_{ij,kl}^*(t) = \alpha_{ik}t$ , and

$$\rho_{ij,kl}^* \left( \frac{t}{\mathbf{f}_{kl}} \right) = \alpha_{ik} \frac{\mathbf{f}_{ij}}{\lambda_{ik}^*(\mathbf{f}_{kl})} \frac{t}{\mathbf{f}_{ij}}.$$

The coefficient  $\alpha_{ik} \frac{\mathbf{f}_{ij}}{\lambda_{ik}^*(\mathbf{f}_{kl})}$  restricted to  $T_{ij,kl}$  is  $\alpha_{ik|U_{ik}} \frac{\mathbf{f}_{ij}}{\mathbf{f}_{kl}}$ . It follows that the line bundle on  $\mathbf{B}_{Y,X}$  associated to  $\mathbf{B}_{Y,X_n}$  is

$$\mathcal{O}_{\mathbf{B}_{Y,X_n}}(\tilde{Y}_n)|_{\mathbf{B}_{Y,X}} \otimes \pi_{Y,X}^*(L) = \mathcal{O}_{\mathbf{B}_{Y,X}}(\tilde{Y}) \otimes \pi_{Y,X}^*(L).$$

The last statement, about the primitive multiple structure of  $\tilde{Y}_n$ , is an easy consequence of the previous description of  $\mathbf{B}_{Y,X_n}$ .  $\square$

**4.1.2. The case of double schemes** – We suppose that  $n = 2$  and keep the notations of the proof of theorem 4.1.1. Let  $i, k \in I$ . From 3.2,  $\lambda_{ik}^*$  is an automorphism of  $\mathcal{O}_X(U_{ik})[t]/(t^2)$ , and there exists a derivation  $D_{ik}$  of  $\mathcal{O}_X(U_{ik})$  such that, for every  $u \in \mathcal{O}_X(U_{ik})$  we have  $\lambda_{ik}^*(u) = u + D_{ik}(u)t$ .

Let  $j \in J_i$ ,  $l \in J_k$ . Then  $\rho_{ij,kl}^*$  is an automorphism of  $\mathcal{O}_{\mathbf{B}_{Y,X}}(T_{ij,kl})[z]/(z^2)$ , and there is a derivation  $E_{ij,kl}$  of  $\mathcal{O}_{\mathbf{B}_{Y,X}}(T_{ij,kl})$  such that, for every  $v \in \mathcal{O}_{\mathbf{B}_{Y,X}}(T_{ij,kl})$ , we have  $\rho_{ij,kl}^*(v) = v + E_{ij,kl}(v)z$ .

Let  $\Sigma = \{f_{ij}^m f_{kl}^n ; m, n \geq 0\}$ . It is a multiplicative system. Let  $R_{ij,kl} = S(U_{ik}, V)_{\Sigma}$ , and  $R_{(ij,kl)}$  the ring of elements of degree zero in  $R_{ij,kl}$ . We have

$$R_{(ij,kl)} = \sum_{m,n \geq 0} I_{mn}, \quad \text{with } I_{mn} = \left\{ \frac{\sigma_{m+n}}{\mathbf{f}_{ij}^m \mathbf{f}_{kl}^n} ; \sigma_{m+n} \in \mathcal{J}_V(U_{ik})^{m+n} \right\},$$

and

$$T_{ij,kl} = \text{spec}(R_{(ij,kl)}).$$

We have then  $E_{ij,kl} = \mathbf{f}_{ij} D_{ik}$  (the factor  $f_{ij}$  comes from the fact that  $z = \frac{t}{\mathbf{f}_{ij}}$  by step 2).

In this way one obtains a linear map

$$\xi_Y : H^1(X, T_X \otimes L) \longrightarrow H^1(\mathbf{B}_{Y,X}, T_{\mathbf{B}_{Y,X}} \otimes \pi_{Y,X}^*(L) \otimes \mathcal{O}_{\mathbf{B}_{Y,X}}(\tilde{Y})),$$

that sends the element of  $H^1(X, T_X \otimes L)$  represented by  $(D_{ij})$  to the element of  $H^1(\mathbf{B}_{Y,X}, T_{\mathbf{B}_{Y,X}} \otimes \pi_{Y,X}^*(L) \otimes \mathcal{O}_{\mathbf{B}_{Y,X}}(\tilde{Y}))$  represented by  $(E_{ij,kl})$ . This map induces the one

$$\mathbf{DS}(X, L) \longrightarrow \mathbf{DS}(\mathbf{B}_{Y,X}, \pi_{Y,X}^*(L) \otimes \mathcal{O}_{\mathbf{B}_{Y,X}}(\tilde{Y}))$$

which sends  $X_2$  to  $\mathbf{B}_{Y,X_2}$ .

**4.1.3. Proposition:** *Suppose that  $\text{codim}(Y, X) \geq 2$ . Then  $\xi_Y$  is injective.*

*Proof.* Suppose that  $(E_{ij,kl})$  represents 0. Then for every  $i \in I$ ,  $j \in J_i$ , there exists a derivation  $F_{ij}$  of  $\mathcal{O}_{\mathbf{B}_{Y \cap U_i, U_i}}(D_+^{\mathbf{S}_i}(\mathbf{f}_{ij}))$  such that

$$E_{ij,kl} = F_{ij} - \alpha_{ik} \frac{\mathbf{f}_{ij}}{\mathbf{f}_{kl}} F_{kl},$$

i.e.  $D_{ik} = \frac{1}{\mathbf{f}_{ij}} F_{ij} - \alpha_{ik} \frac{1}{\mathbf{f}_{kl}} F_{kl}$ . In particular, if  $k = i$  we have  $\frac{1}{\mathbf{f}_{ij}} F_{ij} = \frac{1}{\mathbf{f}_{il}} F_{il}$  for every  $j, l \in J_i$ .

Hence there exists a derivation  $G_i$  of  $\mathcal{O}_{\mathbf{B}_{Y \cap U_i, U_i}}(\mathbf{B}_{Y \cap U_i, U_i})$  such that  $\frac{1}{\mathbf{f}_{ij}} F_{ij} = G_i|_{D_+^{\mathbf{S}_i}(\mathbf{f}_{ij})}$  for every  $j \in J_i$ . We have  $U_i \setminus Y \subset \mathbf{B}_{Y \cap U_i, U_i}$ , hence  $G_i$  induces a derivation of  $\mathcal{O}_X(U_i \setminus Y)$ , and since  $\text{codim}(Y, X) \geq 2$ , it is also a derivation of  $\mathcal{O}_X(U_i)$ . We have  $D_{ik} = G_i - \alpha_{ik} G_k$ . It follows that  $(D_{ik})$  represents 0 in  $H^1(X, T_X \otimes L)$ .  $\square$

## 4.2. BLOWING-UP OF GOOD SUBSCHEMES

Let  $Z \subset X_n$  be a good subscheme (cf. 3.4) of dimension  $d$ . Let  $Y = Z_{red}$  and  $\tilde{Z}_n = \pi_{Z, X_n}^{-1}(Z)$ .

**4.2.1. Theorem:**  $\mathbf{B}_{Z, X_n}$  is a primitive multiple scheme of multiplicity  $n$ ,  $(\mathbf{B}_{Z, X_n})_{red} = \mathbf{B}_{Y, X}$ , and its associated line bundle is  $\pi_{Y, X}^*(L)$ .

The exceptional divisor  $\tilde{Z}_n$  is a primitive multiple scheme of multiplicity  $n$ ,  $(\tilde{Z}_n)_{red} = \tilde{Y} = \pi_{Y, X}^{-1}(Y)$ , and its associated line bundle is  $\pi_{Y, X}^*(L)|_{\tilde{Y}}$ .

*Proof.* It is very similar to that of theorem 4.1.1. We will only give indications. We use the notations of the proof of this theorem.

From proposition 3.4.1, there is an open affine cover  $(U_i)_{i \in I}$  of  $X$  such that, for every  $i \in I$ , there is a trivialization  $\delta_i : U_i^{(n)} \rightarrow U_i \times \mathbf{Z}_n$ ,  $y_{i,1}, \dots, y_{i,d} \in \mathcal{J}_{Y,X}(U_i)$  that generate  $\mathcal{J}_{Y \cap U_i, U_i}$  and  $\mathcal{J}_{\delta_i(Z \cap U_i^{(n)}), U_i \times \mathbf{Z}_n}$ .

Let  $U \subset X$  be a smooth open subset, and  $V \subset U$  a closed smooth connected subvariety of codimension  $d$ , such that there exist generators  $y_1, \dots, y_d$  of  $\mathcal{J}_{V,U}$ . Let  $U_n = U \times \mathbf{Z}_n$ . We have  $U_n = \text{spec}(\mathcal{O}_X(U)[t]/(t^n))$ . Let  $Z \subset U_n$  be the subscheme corresponding to the ideal  $(y_1, \dots, y_d) \subset \mathcal{O}_X(U)[t]/(t^n)$ .

We have  $\mathbf{B}_{Z,U_n} = \text{Proj}(\mathbb{B}_n)$ , where  $\mathbb{B}_n = \mathbb{B}_n(Z, U)$  is the graduate ring

$$\mathbb{B}_n = \bigoplus_{p \geq 0} \mathcal{J}_{Z,U_n}(U)^p,$$

and

$$\mathcal{J}_{Z,U_n}(U)^p = (y_1, \dots, y_d)^p.$$

As in step 2 of the proof of theorem 4.1.1, if  $f \in \mathcal{J}_{Z,U_n}(U)$ , we can define the affine open subset  $D_+^{\mathbb{B}_n}(f)$  of  $\mathbf{B}_{Z,U_n}$ . Let  $\mathbb{B}_{n,f}$  be the localized ring, and  $\mathbb{B}_{n,(f)} \subset \mathbb{B}_{n,f}$  the subring of elements of degree 0. Then  $D_+^{\mathbb{B}_n}(f) \simeq \text{spec}(\mathbb{B}_{n,(f)})$ . We have

$$\mathbb{B}_{n,(f)} = \sum_{p \geq 0} \mathbb{B}_{n,(f),p}, \quad \text{with} \quad \mathbb{B}_{n,(f),p} = \left\{ \frac{\theta_p}{f^p}; \theta_p \in \mathcal{J}_{Z,U_n}(U)^p \right\}.$$

It follows that  $\mathbb{B}_{n,(f)} = S_{(f)}[t]/(t^n)$ . We obtain an isomorphism  $D_+^{\mathbb{B}_n}(f) \simeq D_+^S(f) \times \mathbf{Z}_n$  inducing the identity on  $D_+^S(f)$ .

Using the open cover  $(D_+^{\mathbb{B}_n}(y_i))_{i \in I}$ , it is easy to see that we have a canonical isomorphism  $\mathbf{B}_{Z,U_n} \simeq \mathbf{B}_{Y,U} \times \mathbf{Z}_n$ . It is now straightforward to follow the same way as in steps 5,6 of the proof of theorem 4.1.1, and finish the proof of theorem 4.2.1.  $\square$

### 4.3. SUCCESSIVE BLOWING-UPS

Let  $Z \subset X_n$  be a good subscheme (cf. 3.4). Let  $Y = Z_{red}$ , and  $\tilde{Y} = \pi_{Y,X}^{-1}(Y) \subset \mathbf{B}_{Y,X}$ . Then  $\tilde{Y}$  is a Cartier divisor of  $\mathbf{B}_{Y,X}$ . Hence  $\mathbf{B}_{\tilde{Y}, \mathbf{B}_{Y,X}} = \mathbf{B}_{Y,X}$ .

The following result is an easy consequence of the descriptions of the blowing-ups in theorems 4.1.1 and 4.2.1.

**4.3.1. Proposition:** *The blowing-up of  $\mathbf{B}_{Z,X_n}$  along  $\tilde{Y}$  is canonically isomorphic to  $\mathbf{B}_{Y,X_n}$ , i.e.  $\mathbf{B}_{\tilde{Y}, \mathbf{B}_{Z,X_n}} \simeq \mathbf{B}_{Y,X_n}$ .*

5. BLOWING-UP OF GOOD ZERO-DIMENSIONAL SUBSCHEMES OF PRIMITIVE DOUBLE SCHEMES

We use the notations of 4. We suppose that  $d = \dim(X) \geq 2$ , that  $Y$  is a single point:  $Y = \{P\}$ , and that  $n = 2$ . There exists a finite open affine cover  $(U_i)_{i \in I}$  of  $X$  such that

- There is a unique  $i_0 \in I$  such that  $P \in U_{i_0}$ , and  $\cup_{i \in I, i \neq i_0} U_i = X \setminus \{P\}$ .
- $\mathcal{J}_{\{P\}, U_{i_0}}$  is generated by  $d$  elements  $y_1, \dots, y_d$ , and  $\Omega_{X|U_{i_0}}$  by  $dy_1, \dots, dy_d$ .
- $L$  is defined by a cocycle  $(\alpha_{ij})$ , with  $\alpha_{ij} \in \mathcal{O}_X(U_{ij})^*$ .
- For every  $i \in I$ , there is an isomorphism  $\delta_i : U_i^{(2)} \rightarrow U_i \times \mathbf{Z}_2$  (where  $U_i^{(2)}$  is the open subscheme of  $X_2$  corresponding to  $U_i$ ), inducing the identity on  $U_i$ . There is a derivation  $D_{ij}$  of  $\mathcal{O}_X(U_{ij})$  such that, for every  $u \in \mathcal{O}_X(U_{ij})$ ,  $\delta_{ij}^*(u) = u + D_{ij}(u)t$ . We have  $\delta_{ij}^*(t) = \alpha_{ij}t$ .

We consider the open affine cover of  $\mathbf{B}_{P,X}$  which consists of the  $U_i$ ,  $i \in I$ , and the  $D_+^{\mathbb{B}^2}(y_k) \subset \mathbf{B}_{P,U_{i_0}}$ ,  $1 \leq k \leq d$ . Let  $J = I \cup \{i_1, \dots, i_d\}$ ,  $V_i = U_i$  if  $i \in I$  and  $V_{i_k} = D_+^{\mathbb{B}^2}(y_k)$ . The line bundle  $\pi_{P,X}^*(L)$  on  $\mathbf{B}_{P,X}$  is defined by the cocycle  $(\beta_{ij})_{i,j \in J}$ ,  $\beta_{ij} \in \mathcal{O}_{\mathbf{B}_{P,X}}(V_{ij})^*$ , with  $\beta_{ij} = \alpha_{ij}$  if  $i, j \in I$ ,  $\beta_{ki_0} = \alpha_{ki_0}$ ,  $\beta_{i_m i_n} = 1$  if  $k \in I$ ,  $1 \leq m, n \leq d$ .

We will study the blowing-ups of  $X_2$  along good subschemes  $Z$  such that  $Z_{red} = \{P\}$ . From theorem 4.2.1,  $\mathbf{B}_{Z,X_2}$  is a primitive multiple scheme of multiplicity 2,  $(\mathbf{B}_{Z,X_2})_{red} = \mathbf{B}_{P,X}$ , and its associated line bundle is  $\pi_{P,X}^*(L)$ . The exceptional divisor  $\tilde{P} = \pi_{P,X}^{-1}(P)$  is a projective space of dimension  $d - 1$ . Let  $\tilde{P}_2 = \pi_{Z,X_2}^{-1}(P)$ . It is the trivial primitive multiple scheme of multiplicity 2, such that  $(\tilde{P}_2)_{red} = \tilde{P}$ , and with associated line bundle  $\mathcal{O}_{\tilde{P}}$ .

5.1. PARAMETRIZATION OF GOOD ZERO-DIMENSIONAL SUBSCHEMES

We will prove

**5.1.1. Proposition:** *Let  $\Gamma_P$  be the set of good subschemes  $Z \subset X_2$  such that  $Z_{red} = \{P\}$ . Then  $\Gamma_P$  has a natural structure of affine space, with associated vector space  $(T_X \otimes L)_P$ .*

Let  $\mathfrak{m}_P$  be the maximal ideal of  $\mathcal{O}_{X,P}$ . Let  $a_1, \dots, a_d \in \mathcal{O}_{X,P}$ ,  $I(a_1, \dots, a_d) = (y_1 + ta_1, \dots, y_d + ta_d)$ , and  $Z_0(a_1, \dots, a_d)$  the corresponding subscheme of  $U_{i_0} \times \mathbf{Z}_2$ . The good subschemes  $Z \subset U_{i_0} \times \mathbf{Z}_2$  such that  $Z_{red} = \{P\}$  are precisely the  $Z_0(a_1, \dots, a_d)$ .

Let  $Z(a_1, \dots, a_d) = \delta_{i_0}^{-1}(Z_0(a_1, \dots, a_d)) \subset X_2$ . It is the subscheme defined by the ideal sheaf  $\delta_{i_0}^*(I(a_1, \dots, a_d))$ . The good subschemes  $Z \subset X_2$  such that  $Z_{red} = \{P\}$  are the  $Z(a_1, \dots, a_d)$ .

**5.1.2. Lemma:** *Let  $a_1, a'_1, \dots, a_d, a'_d \in \mathcal{O}_{X,P}$ . Then  $I(a_1, \dots, a_d) \simeq I(a'_1, \dots, a'_d)$  if and only if  $a_i - a'_i \in \mathfrak{m}_P$  for  $1 \leq i \leq d$ .*

*Proof.* Suppose that  $I(a_1, \dots, a_d) \simeq I(a'_1, \dots, a'_d)$ . Then there exists an invertible  $(d \times d)$ -matrix  $M = (\theta_{ij})$  of elements of  $\mathcal{O}_{U_{i_0} \times \mathbf{Z}_2, P}$  such that

$$\begin{pmatrix} y_1 + ta'_1 \\ \vdots \\ y_d + ta'_d \end{pmatrix} = M \cdot \begin{pmatrix} y_1 + ta_1 \\ \vdots \\ y_d + ta_d \end{pmatrix}.$$

We can write  $\theta_{ij} = \lambda_{ij} + t\mu_{ij}$ , with  $\lambda_{ij}, \mu_{ij} \in \mathcal{O}_{X, P}$ . We have then

$$y_i = \sum_{j=1}^d \lambda_{ij} y_j, \quad a'_i = \sum_{j=1}^d \lambda_{ij} a_j + \sum_{j=1}^d \mu_{ij} y_j.$$

The Koszul relations imply that we have  $\lambda_{ii} = 1$ , and that  $\lambda_{ij} \in \mathfrak{m}_P$  if  $i \neq j$ . It follows that  $a_i - a'_i \in \mathfrak{m}_P$ .

Conversely, if  $a_i - a'_i \in \mathfrak{m}_P$  for  $1 \leq i \leq d$ , we can write  $a'_i = a_i + \sum_{j=1}^d \rho_{ij} y_j$ ,  $\rho_{ij} \in \mathcal{O}_{X, P}$ . We have then

$$y_i + ta'_i = y_i + ta_i + \sum_{j=1}^d t\rho_{ij}(y_j + ta_j),$$

hence  $I(a_1, \dots, a_d) \simeq I(a'_1, \dots, a'_d)$ . □

It follows from lemma 5.1.2 that the map

$$\begin{aligned} \mathbb{C}^d &\longrightarrow \Gamma_P \\ (\zeta_1, \dots, \zeta_d) &\longmapsto Z(\zeta_1, \dots, \zeta_d) \end{aligned}$$

is a bijection.

Let  $U \subset X$  be an open affine neighborhood of  $P$ , and  $D$  a derivation of  $\mathcal{O}_X(U)$ , that we can view as a section of  $T_X|_U$ . Let  $D_P \in T_{X, P}$  its value at  $P$ . Let  $a_1, \dots, a_d \in \mathcal{O}_{X, P}$ . Then

$D(a_1, \dots, a_d) = \sum_{i=1}^d a_i \frac{\partial}{\partial y_i}$  is a derivation of  $\mathcal{O}_X(U)$ , where  $U$  is a suitable neighborhood of  $P$ .

We have

$$D(a_1, \dots, a_d)_P = \sum_{i=1}^d a_i(P) \left( \frac{\partial}{\partial y_i} \right)_P.$$

According to lemma 5.1.2,  $Z(a_1, \dots, a_d) = Z(a'_1, \dots, a'_d)$  if and only if  $D(a_1, \dots, a_d)_P = D(a'_1, \dots, a'_d)_P$ .

If  $Z \in \Gamma_P$ , and  $Z = Z(a_1, \dots, a_d)$ , let

$$\Delta_P(Z, (y_i)_{1 \leq i \leq d}, \delta_{i_0}) = D(a_1, \dots, a_d)_P \otimes \delta_{i_0}^*(t)(P) \in (T_X \otimes L)_P.$$

**5.1.3. Lemma:**  $\Delta_P(Z, (y_i)_{1 \leq i \leq d}, \delta_{i_0})$  does not depend on  $(y_i)_{1 \leq i \leq d}$ .

*Proof.* Let  $z_1, \dots, z_d$  be generators of  $\mathfrak{m}_P$ . We have  $(y_1, \dots, y_d) = (z_1, \dots, z_d)M$ , where  $M = (\mu_{ij})_{1 \leq i, j \leq d}$  is an invertible  $(d \times d)$ -matrix of elements of  $\mathcal{O}_{X,P}$ . We have

$$\frac{\partial y_i}{\partial z_k} = \mu_{ki} + \sum_{j=1}^d \frac{\partial \mu_{ji}}{\partial z_k} z_j,$$

hence

$$\left( \left( \frac{\partial}{\partial z_1} \right)_P, \dots, \left( \frac{\partial}{\partial z_d} \right)_P \right) = \left( \left( \frac{\partial}{\partial y_1} \right)_P, \dots, \left( \frac{\partial}{\partial y_d} \right)_P \right) \cdot {}^t M(P).$$

Let  $(b_1, \dots, b_d) = (a_1, \dots, a_d)M(P)^{-1}$ . Then

$$(y_1 + a_1 t, \dots, y_d + a_d t) = (z_1 + b_1 t, \dots, z_d + b_d t)$$

in  $\mathcal{O}_{X,P}[t]/(t^2)$ . We have

$$\begin{aligned} \Delta(Z, (z_i)_{1 \leq i \leq d}, \delta_{i_0}) &= \left( \left( \frac{\partial}{\partial z_1} \right)_P, \dots, \left( \frac{\partial}{\partial z_d} \right)_P \right) \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ b_d \end{pmatrix} \otimes \delta_{i_0}^*(t)(P) \\ &= \left( \left( \frac{\partial}{\partial y_1} \right)_P, \dots, \left( \frac{\partial}{\partial y_d} \right)_P \right) \cdot {}^t M(P) \cdot {}^t M(P)^{-1} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ a_d \end{pmatrix} \otimes \delta_{i_0}^*(t)(P) \\ &= \Delta(Z, (y_i)_{1 \leq i \leq d}, \delta_{i_0}). \end{aligned}$$

□

So we will omit the parameter  $(y_i)_{1 \leq i \leq d}$ , and note  $\Delta(Z, (y_i)_{1 \leq i \leq d}, \delta_{i_0}) = \Delta(Z, \delta_{i_0})$ . The following lemma ends the proof of proposition 5.1.1.

**5.1.4. Lemma** *Let  $Z, Z' \in \Gamma_P$  and  $\phi : U_{i_0}^{(2)} \rightarrow U_{i_0} \times \mathbf{Z}_2$  an isomorphism leaving  $U_{i_0}$  invariant. Then we have*

$$\Delta_P(Z', \delta_{i_0}) - \Delta_P(Z, \delta_{i_0}) = \Delta_P(Z', \phi) - \Delta_P(Z, \phi).$$

*Proof.* Let  $\theta = \phi \circ \delta_{i_0}^{-1}$ , which is an automorphism of  $U_{i_0} \times \mathbf{Z}_2$ . We have

$$\theta^{*-1}(y_i) = y_i + \epsilon_i t \text{ for } 1 \leq i \leq d, \quad \theta^{*-1}(t) = \alpha t,$$

with  $\epsilon_i \in \mathcal{O}_X(U_{i_0})$ ,  $\alpha \in \mathcal{O}_X(U_{i_0})^*$ . Suppose that  $Z = Z(a_1, \dots, a_d)$ . Then

$$Z = \delta_{i_0}^{-1}(Z(a_1, \dots, a_d)) = \phi^{-1}(\theta(Z(a_1, \dots, a_d))),$$

and  $\theta(Z(a_1, \dots, a_d))$  is the subscheme whose ideal is generated by  $\theta^{*-1}(y_i + \epsilon_i t)$ ,  $1 \leq i \leq t$ . We have

$$\theta^{*-1}(y_i + \epsilon_i t) = y_i + (\epsilon_i + \alpha a_i) t, \quad \text{hence } \delta_{i_0}^*(y_i + \epsilon_i t) = \phi^*(y_i + (\epsilon_i + \alpha a_i) t),$$

and  $\phi^*(t) = \delta_{i_0}^*(\theta^*(t)) = \frac{1}{\alpha} \delta_{i_0}^*(t)$ . It follows that

$$\Delta_P(Z, \phi) = \left( \sum_{i=1}^d (\epsilon_i(P) + \alpha(P) a_i(P)) \left( \frac{\partial}{\partial y_i} \right)_P \right) \otimes \left( \frac{1}{\alpha(P)} \delta_{i_0}^*(t)(P) \right).$$

We have a similar formula for  $Z' = Z(a'_1, \dots, a'_d)$ , and the result follows easily.  $\square$

From proposition 5.1.1 and the preceding lemmas, we have  $Z = Z'$  if and only if  $\Delta_P(Z, \delta_{i_0}) = \Delta_P(Z', \delta_{i_0})$ .

**5.1.5. The case of a trivial primitive double scheme** – If  $X_2$  is trivial, we can choose the trivializations such that  $D_{ij} = 0$  for every  $i, j \in I$ . We obtain a canonical identification  $\Gamma_P = (T_X \otimes L)_P$ .

In this case  $X_2$  is the second infinitesimal neighborhood of  $X$  in the line bundle  $L^*$  (viewed as a smooth variety, and  $X \subset X_2$  defined by the zero section of  $L^*$ ). We have a canonical projection  $p : X_2 \rightarrow X$ .

Let  $Z_0 \subset X_2$  be the good subscheme corresponding to  $0 \in (T_X \otimes L)_P$ , it is  $p^{-1}(P)$ . Then  $\mathbf{B}_{Z_0, X_2}$  is the trivial primitive scheme with associated smooth variety  $\mathbf{B}_{P, X}$  and associated line bundle  $\pi_{P, X}^*(L)$ .

## 5.2. BLOWING-UPS

**5.2.1. Lemma:** *Let  $E$  be a derivation of  $U_{i_0} \setminus \{P\}$ . Then  $U$  can be extended uniquely to a derivation of  $U_{i_0}$ . It can be extended to a derivation of  $\pi_{P, \mathbf{B}_{Z, X_2}}^{-1}(U_{i_0})$  if and only if  $E_P = 0$ .*

*Proof.* Since  $\dim(X) \geq 2$ ,  $E$  can be extended to  $U_{i_0}$ . It can be extended to  $\pi_{P, \mathbf{B}_{Z, X_2}}^{-1}(U_{i_0})$  if and only, for  $1 \leq i \leq d$ , if it be extended to  $D_+^{\mathbb{B}^n}(y_i)$ . For every  $\sigma \in \mathcal{J}_{P, U_{i_0}}$ , we have

$$E\left(\frac{\sigma}{y_i}\right) = \frac{E(\sigma)y_i - E(y_i)\sigma}{y_i^2}.$$

Hence  $E$  can be extended to  $D_+^{\mathbb{B}^n}(y_i)$  if and only if  $E(\sigma)y_i - E(y_i)\sigma \in \mathcal{J}_{P, U_{i_0}}^2$ , for every  $\sigma \in \mathcal{J}_{P, U_{i_0}}$ . It is easy to see that this happens if and only if  $E_P = 0$ .  $\square$

Let  $\text{ev}_L : \mathcal{O}_X \otimes H^0(X, T_X \otimes L) \rightarrow T_X \otimes L$  be the evaluation morphism, and  $H_{L, P} = \text{im}(\text{ev}_{L, P}) \subset (T_X \otimes L)_P$ . Let  $\Psi_{L, P} : T_{X_P} \otimes L \rightarrow (T_{X, P} \otimes L)/H_{L, P}$  be the quotient map. Let  $Z, Z' \in \Gamma_P$ . Then  $\mathbf{B}_{Z, X_2}$  and  $\mathbf{B}_{Z', X_2}$  are primitive double schemes with associated smooth variety  $\mathbf{B}_{P, X}$ . We say that they are *isomorphic* if there exists an isomorphism  $\mathbf{B}_{Z, X_2} \simeq \mathbf{B}_{Z', X_2}$  inducing the identity on  $\mathbf{B}_{P, X}$ .

**5.2.2. Theorem:** *If  $X_2$  is trivial, we choose the trivializations such that  $D_{ij} = 0$  for every  $i, j \in I$ . Let  $Z, Z' \in \Gamma_P$ .*

**1** – *If  $X_2$  is not trivial,  $\mathbf{B}_{Z, X_2}$  and  $\mathbf{B}_{Z', X_2}$  are isomorphic if and only if  $\Delta_P(Z, \delta_{i_0}) - \Delta_P(Z', \delta_{i_0}) \in H_{L, P}$ .*

**2** – *If  $X_2$  is trivial,  $\mathbf{B}_{Z, X_2}$  and  $\mathbf{B}_{Z', X_2}$  are isomorphic if and only if  $\mathbb{C} \cdot \Psi_{L, P}(\Delta_P(Z', \delta_{i_0})) = \mathbb{C} \cdot \Psi_{L, P}(\Delta_P(Z, \delta_{i_0}))$ .*

*Proof.* According to 4.2, the double structures on  $\mathbf{B}_{Z, X_2}$  and  $\mathbf{B}_{Z', X_2}$  are defined by families  $(E_{ij})_{i,j \in J}$ ,  $(E'_{ij})_{i,j \in J}$ , where  $E_{ij}$  and  $E'_{ij}$  are derivations of  $\mathcal{O}_X(V_{ij})$ . These families define the elements  $h_Z, h_{Z'}$  of  $H^1(X, T_{\mathbf{B}_{P,X}} \otimes L)$  associated to  $\mathbf{B}_{Z, X_2}$  and  $\mathbf{B}_{Z', X_2}$  respectively.

Suppose that  $Z = Z(a_0, \dots, a_d)$ ,  $Z' = Z(a'_1, \dots, a'_d)$ . Let  $D = D(a_0, \dots, a_d)$ ,  $D' = D(a'_0, \dots, a'_d)$ . Suppose that  $Z \neq Z'$ . Then  $D_P \neq D'_P$ . We can assume that  $D'_P \neq 0$ . By adding to  $D$  a suitable derivation  $F$  such that  $F_P = 0$ , we can assume that, for every  $\lambda \in \mathbb{C}$ , we have  $D \neq \lambda D'$ .

Let  $\theta$  be the automorphism of  $U_{i_0} \times \mathbf{Z}_2$  defined by the derivation  $D$ , i.e.  $\theta^*(u) = u + D(u)t$  for every  $u \in \mathcal{O}_X(U_{i_0})$ , and  $\theta^*(t) = t$ . We have  $\theta^*(y_i) = y_i + a_i t$  for  $1 \leq i \leq d$ . Hence  $\theta \circ \delta_{i_0}(Z)$  is the subscheme defined by the ideal  $(y_1, \dots, y_d)$ . If instead of  $\delta_{i_0}$  we take the trivialization  $\theta \circ \delta_{i_0}$ , and do a similar change for  $Z'$ , we obtain by 3.2.1 and the proof of theorem 4.2.1 that

$$\begin{aligned} E_{ij} &= E'_{ij} = D_{ij} \quad \text{if } i, j \in I \setminus \{i_0\}, & E_{i_m i_n} &= E'_{i_m i_n} = 0 \quad \text{if } 1 \leq m, n \leq d, \\ E_{ki_m} &= D_{ki_0} + \alpha_{ki_0} D, & E'_{ki_m} &= D_{ki_0} + \alpha_{ki_0} D' \quad \text{if } k \in I \setminus \{i_0\}, 1 \leq m \leq d. \end{aligned}$$

From 3.2,  $\mathbf{B}_{Z, X_2}$  and  $\mathbf{B}_{Z', X_2}$  are isomorphic if and only if there exists  $\lambda \in \mathbb{C}^*$  such that  $h_Z = \lambda h_{Z'}$ , i.e. if and only if there exist, for every  $j \in J$ , a derivation  $G_j$  of  $\mathcal{O}_{\mathbf{B}_{P,X}}(V_j)$  such that  $E_{ij} = \lambda E'_{ij} + G_i - \beta_{ij} G_j$  on  $V_{ij}$ , for all  $i, j \in J$ . These relations are equivalent to  $G_{i_m} = G_{i_n}$  on  $V_{i_m i_n}$ , for every  $i, j \in I \setminus \{i_0\}$

$$(2) \quad (1 - \lambda) D_{ij} = G_i - \alpha_{ij} G_j \quad \text{on } U_{ij},$$

and for  $k \in I \setminus \{i_0\}$ ,  $1 \leq m \leq d$ ,

$$(3) \quad (1 - \lambda) D_{ki_0} = \alpha_{ki_0} (\lambda D' - D) + G_k - \alpha_{ki_0} G_{i_m} \quad \text{on } V_{ki_m}.$$

The derivations  $G_{i_m}$  do not depend on  $m$ , and can be extended to a derivation  $\Theta$  on  $U_{i_0}$ , as well as on  $\pi_{P, \mathbf{B}_{Z, X_2}}^{-1}(U_{i_0})$ , and by lemma 5.2.1 we have  $\Theta_P = 0$ .

Suppose that  $\lambda = 1$ . Then from (2), and the fact that  $\cup_{i \in I, i \neq i_0} U_i = X \setminus \{P\}$ , there exists  $\Phi \in H^0(X, T_X \otimes L)$  such that  $G_i \otimes \alpha_i^{-1} = \Phi|_{U_i \otimes L}$  for every  $i \in I \setminus \{i_0\}$ . From (3), we have

$$D' - D - \Theta + \Phi \otimes \alpha_{i_0} = 0,$$

which implies that  $\Delta_P(Z, \delta_{i_0}) - \Delta_P(Z', \delta_{i_0}) \in H_{L,P}$ .

Suppose that  $\lambda \neq 1$ . Let  $G_{i_0} = \Theta + D - \lambda D'$ , and for every  $i \in I$ ,  $F_i = \frac{1}{1 - \lambda} G_i$ . Then we have, for every  $i, j \in I \setminus \{i_0\}$

$$D_{ij} = F_i - \alpha_{ij} F_j \quad \text{on } U_{ij},$$

and, for every  $k \in I \setminus \{i_0\}$ ,  $1 \leq m \leq d$

$$D_{ki_0} = F_k - \alpha_{ki_0} F_{i_0} \quad \text{on } V_{ki_m}.$$

Hence  $D_{ki_0} = F_k - \alpha_{ki_0} F_{i_0}$  on  $U_{ki_0}$ . It follows that  $X_2$  is trivial. But in this case we have assumed that  $D_{ij} = 0$  for every  $i, j \in I$ . Hence there exists  $\Phi \in H^0(X, T_X \otimes L)$  such that  $G_i \otimes \alpha_i^{-1} = \Phi|_{U_i \otimes L}$  for every  $i \in I$ , and we have

$$\lambda D' - D - \Theta + \Phi \otimes \alpha_{i_0} = 0,$$

which implies that  $\mathbb{C} \cdot \Psi_{L,P}(\Delta_P(Z', \delta_{i_0})) = \mathbb{C} \cdot \Psi_{L,P}(\Delta_P(Z, \delta_{i_0}))$ .

The converse is immediate. □

## 6. BLOWING-UP OF AN HYPERSURFACE

## 6.1. PRIMITIVE MULTIPLE RINGS

Let  $R$  be an entire noetherian ring and  $K$  its fraction field.

Let  $n$  be a positive integer and  $R[n] = R[t]/(t^n)$ , which we call the *primitive multiple ring of multiplicity  $n$*  associated to  $R$ .

An element  $u$  of  $R[n]$  can be written in an unique way as

$$u = \sum_{i=0}^{n-1} u_i t^i ,$$

with  $u_i \in R$  for  $0 \leq i \leq n$ . Then  $u$  is a zero divisor if and only if  $u_0 = 0$ , i.e. if  $u$  is a multiple of  $t$ , and it is invertible if and only if  $u_0$  is invertible.

Let  $S_n$  be the multiplicative system of non zero divisors of  $R[n]$ , i.e.  $S_n = \{u \in R[n]; u_0 \neq 0\}$ . It is easy to see that  $S_n^{-1}R[n] = K[t]/(t^n) = K[n]$  (the primitive multiple ring of multiplicity  $n$  associated to  $K$ ).

Suppose that  $n > 1$ . Let

$$\begin{array}{ccc} \sigma_n^R : R[n] & \longrightarrow & R[n-1] , & \rho_n^R : R[n] & \longrightarrow & R \\ u & \longmapsto & \sum_{i=0}^{n-2} u_i t^i & u & \longmapsto & u_0 . \end{array}$$

which are morphisms of rings.

Let  $\Gamma_n(R)$  be the set of morphisms  $R \rightarrow R[n]$  such that  $\rho_n^R \circ \phi = I_R$ , i.e. such that  $\phi(a)_0 = a$  for every  $a \in R$ , and  $\Sigma_n(R)$  the set of endomorphisms  $\Theta$  of  $R[n]$  such that  $\rho_n^R \circ \Theta = \rho_n^R$ .

Let  $\epsilon \in R[n-1]$ ,  $\phi \in \Gamma_n(R)$ , and

$$\begin{array}{ccc} \Theta_n^R(\phi, \epsilon) : R[n] & \longrightarrow & R[n] \\ u = \sum_{i=0}^{n-1} u_i t^i & \longmapsto & \sum_{i=0}^{n-1} \phi(u_i) \epsilon^i t^i . \end{array}$$

**6.1.1. Proposition:** **1** –  $\Theta_n^R(\phi, \epsilon)$  is an endomorphism.

**2** – For every  $\Theta \in \Sigma_n(R)$ , there exist a unique  $\phi \in \Gamma_n(R)$  and a unique  $\epsilon \in R[n-1]$  such that  $\Theta = \Theta_n^R(\phi, \epsilon)$ .

**3** –  $\Theta_n^R(\phi, \epsilon)$  is injective if and only if  $\epsilon_0 \neq 0$ .

**4** –  $\Theta_n^R(\phi, \epsilon)$  is surjective if and only if it is an isomorphism, if and only if  $\epsilon_0$  is invertible.

*Proof.* **1** is immediate.

Now we prove **2**. We have  $\Theta(t)^n = \Theta(t^n) = 0$ , hence  $\Theta(t)$  is a multiple of  $t$ :  $\Theta(t) = \epsilon t$ , for a unique  $\epsilon \in R[n-1]$ . If we take  $\phi = \Theta|_R$ , it is immediate that  $\Theta = \Theta_n^R(\phi, \epsilon)$  and that  $\phi$  is unique.

Suppose that  $\epsilon_0 = 0$ , i.e.  $\epsilon$  is a multiple of  $t$ . Then  $\Theta_n^R(\phi, \epsilon)(t^{n-1}) = 0$ , hence  $\Theta_n^R(\phi, \epsilon)$  is not injective. Conversely, suppose that  $\epsilon_0 \neq 0$ . We will prove that  $\Theta_n^R(\phi, \epsilon)$  is injective by induction on  $n$ . It is obvious for  $n = 1$ . Suppose that it is true for  $n - 1 \geq 1$ . Let  $\Theta = \Theta_n^R(\phi, \epsilon)$ ,  $\Theta' = \Theta_{n-1}^R(\sigma_n^R \circ \phi, \sigma_n^R(\epsilon)) \in \Sigma_{n-1}(R)$ . Then  $\Theta'$  is injective, and we have  $\Theta' = \rho_n^R \circ \Theta|_{R[n-1]}$ . Let  $u \in R[n]$ ,  $u = v + u_{n-1}t^{n-1}$ ,  $v \in R[n-1]$ . Suppose that  $\Theta(u) = 0$ . Then  $\Theta(v) = -u_{n-1}\epsilon^{n-1}t^{n-1}$ , hence  $\Theta'(v) = 0$ , and  $v = 0$ . So we have  $\Theta(u) = u_{n-1}\epsilon^{n-1}t^{n-1} = 0$ . Since  $\epsilon_0 \neq 0$ ,  $u_{n-1} = 0$  and  $u = 0$ . Hence  $\Theta$  is injective. This proves **3**. The proof of **4** is similar.  $\square$

Let  $\phi \in \Gamma_n(R)$  (resp.  $\Theta \in \Sigma_n(R)$ ). Then  $\phi$  naturally induces  $\bar{\phi} \in \Gamma_n(K)$  (resp.  $\bar{\Theta} \in \Sigma_n(K)$ ) such that  $\bar{\phi}|_R = \phi$  (resp.  $\bar{\Theta}(R[n]) \subset R[n]$  and  $\bar{\Theta}|_{R[n]} = \Theta$ ). Let  $\epsilon \in R[n]$ . We have

$$\Theta_n^K(\bar{\phi}, \epsilon) = \overline{\Theta_n^R(\phi, \epsilon)},$$

and it is an isomorphism if and only if  $\epsilon_0 \neq 0$ .

For every  $\lambda = \sum_{i=0}^{n-1} \lambda_i t^i \in R[n]$  and  $\alpha \in R[n]$ , let  $\lambda[\alpha t] = \sum_{i=0}^{n-1} \lambda_i \alpha^i t^i$ . If  $\phi \in \Gamma_n(R)$ , let

$$\begin{aligned} \phi[\alpha t] : R &\longrightarrow R[n] \\ a &\longmapsto \phi(a)[\alpha t], \end{aligned}$$

which also belongs to  $\Gamma_n(R)$ .

Let  $y \in R$ ,  $y \neq 0$ . Let  $i_R : R \rightarrow R[n]$  be the inclusion, and

$$\chi_y^R = \Theta(i_R, y).$$

We have  $\overline{\chi_y^R} = \chi_y^K$ .

**6.1.2. Proposition:** *Let  $x \in R$  such that  $x \neq 0$ , and  $\alpha \in R[n]$ . Let  $\Theta = \Theta_n^R(\phi, \epsilon) \in \Sigma_n(R)$  and*

$$\Psi = \chi_{\alpha x}^K \circ \bar{\Theta} \circ \chi_{1/x}^K \in \Sigma_n(K).$$

*Let  $\phi(x) = x + \mu t$ , with  $\mu \in R[n-1]$ . Then we have  $\Psi = \overline{\Theta(\phi', \epsilon')}$ , with*

$$(4) \quad \phi' = \phi[\alpha x t], \quad \epsilon' = \frac{\alpha \epsilon[\alpha x t]}{1 + \alpha \mu[\alpha x t] t}.$$

*Moreover, if  $\Theta$  is an isomorphism and  $\alpha$  is invertible, then  $\Theta(\phi', \epsilon')$  is also an isomorphism.*

*Proof.* Let  $a \in R$ . We have

$$\Psi(a) = \chi_{\alpha x}^K \circ \bar{\Theta}(a) = \chi_{\alpha x}^K(\phi(a)) = \phi[a][\alpha x t] \in R[n],$$

and

$$\Psi(t) = \chi_{\alpha x}^K \circ \bar{\Theta}\left(\frac{1}{x}t\right) = \chi_{\alpha x}^K\left(\frac{\epsilon}{\phi(x)}t\right) = \frac{\epsilon[\alpha x t]\alpha x}{\phi(x)[\alpha x t]}t.$$

We have  $\phi(x)[\alpha x t] = x + \mu[\alpha x t]\alpha x t = x(1 + \alpha \mu[\alpha x t]t)$ . Hence

$$\epsilon' = \frac{\epsilon[\alpha x t]\alpha x}{\phi(x)[\alpha x t]} = \frac{\alpha \epsilon[\alpha x t]}{1 + \alpha \mu[\alpha x t]t} \in R[n].$$

We have  $\epsilon'_0 = \alpha\epsilon_0$ , hence by proposition 6.1.1, if  $\Theta$  is an isomorphism and  $\alpha$  is invertible,  $\epsilon_0$  is invertible and so is  $\epsilon'_0$ , so  $\Theta(\phi', \epsilon')$  is an isomorphism.  $\square$

**Remark:** suppose that  $R$  is a  $\mathbb{C}$ -algebra. All the rings considered in 6.1 are also  $\mathbb{C}$ -algebras, and we can restrict ourselves to morphisms of  $\mathbb{C}$ -algebras, and obtains the same results.

## 6.2. BLOWING-UP OF HYPERSURFACES

Let  $X$  be a smooth, projective and irreducible variety. Let  $\mathfrak{X}$  be a primitive multiple scheme of multiplicity  $n \geq 2$ , with underlying smooth variety  $X$ , and associated line bundle  $L$  on  $X$ . Let  $(U_i)_{i \in I}$  be an affine open cover of  $X$  such that we have trivializations

$$\delta_i : U_i^{(n)} \xrightarrow{\simeq} U_i \times \mathbf{Z}_n,$$

(cf. 3.1.1). Let

$$\delta_{ij} = \delta_j \delta_i^{-1} : U_{ij} \times \mathbf{Z}_n \xrightarrow{\simeq} U_{ij} \times \mathbf{Z}_n .$$

Then  $\delta_{ij}^* = \delta_i^{*-1} \delta_j^*$  is an automorphism of  $\mathcal{O}_{U_i \times \mathbf{Z}_n} = \mathcal{O}_X(U_{ij})[t]/(t^n)$ , such that for every  $\phi \in \mathcal{O}_X(U_{ij})[t]/(t^n)$ , seen as a polynomial in  $t$  with coefficients in  $\mathcal{O}_X(U_{ij})$ , the term of degree zero of  $\delta_{ij}^*(\phi)$  is the same as the term of degree zero of  $\phi$ .

Let  $H \subset X$  be a hypersurface (i.e. a closed subscheme such that its ideal sheaf  $I_H = I_{H, \mathfrak{X}}$  is invertible). We can choose the open cover  $(U_i)_{i \in I}$  such that, for every  $i \in I$ ,  $I_{H \cap U_i}$  is generated by  $x_i \in \mathcal{O}_X(U_i)$ . We have  $\mathbf{B}_{H, X} = X$ , and  $\mathbf{B}_{H, \mathfrak{X}}$  is a primitive multiple scheme with underlying smooth variety  $X$  and associated line bundle  $L \otimes \mathcal{O}_X(H)$ .

We now use 6.1, with  $R = \mathcal{O}_X(U_{ij})$ . If  $i, j \in I$ ,  $i \neq j$ , let

$$\theta_{ij} = \chi_{x_i}^K \circ \overline{\delta_{ij}^*} \circ \chi_{1/x_j}^K .$$

On  $U_{ij}$  we can write  $x_i = \beta_{ij} \cdot x_j$ , with  $\beta_{ij} \in \mathcal{O}_X(U_{ij})$  invertible. The family  $(\beta_{ij})$  is a cocycle defining the line bundle  $\mathcal{O}_X(H)$ . By proposition 6.1.2, we have  $\theta_{ij} = \bar{\tau}$ , where  $\tau$  is an automorphism of  $\mathcal{O}_X(U_{ij})[t]/(t^n)$ . We can write  $\tau = \gamma_{ij}^*$ , where  $\gamma_{ij}$  is an automorphism of  $U_{ij} \times \mathbf{Z}_n$ . The family  $(\gamma_{ij}^*)$  satisfies the cocycle relations  $\gamma_{ij}^* \gamma_{jk}^* = \gamma_{ik}^*$ , and defines a primitive multiple scheme  $\mathfrak{X}'$  such that  $\mathfrak{X}'_{red} = X$ . By formula (4),  $L(H)$  is the line bundle on  $X$  associated to  $\mathfrak{X}'$ .

We have commutative diagrams

$$\begin{array}{ccc} \mathcal{O}_X(U_{ij})[t]/(t^n) & \xrightarrow{\delta_{ij}^*} & \mathcal{O}_X(U_{ij})[t]/(t^n) \\ \downarrow \chi_{x_j} & & \downarrow \chi_{x_i} \\ \mathcal{O}_X(U_{ij})[t]/(t^n) & \xrightarrow{\gamma_{ij}^*} & \mathcal{O}_X(U_{ij})[t]/(t^n) \end{array}$$

which define a morphism  $\pi : \mathfrak{X}' \rightarrow \mathfrak{X}$  which the identity on  $X$ , and such that its restriction to the open subset of  $\mathfrak{X}$  corresponding to  $X \setminus H$  is an isomorphism to the similar open subset of  $\mathfrak{X}'$ .

**6.2.1. Proposition:** *The morphism  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  is the blowing-up of  $\mathcal{X}$  along  $H$ .*

*Proof.* We must prove that  $\pi$  satisfies the following universal property: let  $Y$  be a scheme and  $f : Y \rightarrow \mathcal{X}$  a morphism such that  $\mathcal{J} = f^{-1}(I_{H,\mathcal{X}}) \cdot \mathcal{O}_Y$  is an invertible sheaf of ideals on  $Y$ . Then there exists a unique morphism  $\tilde{f} : Y \rightarrow \mathcal{X}'$  such that  $\pi \circ \tilde{f} = f$ .

Let  $y \in Y$  be a closed point in the support of  $\mathcal{J}$  and suppose that  $f(y) \in U_i$ . Let  $U \in Y$  be an affine open neighborhood of  $y$  such that  $f(U) \subset U_i$  and that  $\mathcal{J}|_U$  is generated by  $z \in \mathcal{J}(U)$ . It suffices to prove the claim for  $f|_U$ , since by unicity, all the obtained morphisms  $U \rightarrow \mathcal{X}'$  can be glued to define  $\tilde{f}$ . So we can assume that  $f$  is a morphism  $U \rightarrow U_i \times \mathbf{Z}_n$ .

The morphism  $\pi^* : \mathcal{O}_X(U_i)[t]/(t^n) \rightarrow \mathcal{O}_X(U_i)[t]/(t^n)$  is defined by: for every  $a \in \mathcal{O}_X(U_i)$ ,  $\pi^*(a) = a$ , and  $\pi^*(t) = x_i t$ . We have  $I_{H,U_i \times \mathbf{Z}_n} = (x_i, t)$ . Now consider  $f^* : \mathcal{O}_X(U_i)[t]/(t^n) \rightarrow \mathcal{O}_Y(U)$ . We can write  $f^*(x_i) = \alpha z$ ,  $f^*(t) = \beta z$ , with  $\alpha, \beta \in \mathcal{O}_Y(U)$ .

We will show that  $\alpha$  is invertible. We have  $f^*(t^n) = 0 = \beta^n z^n$ , hence  $\beta^n = 0$ . If  $\alpha$  is not invertible, the ideal  $(\alpha)$  can be included in a maximal ideal  $\mathfrak{m} \neq \mathcal{O}_Y(U)$  of  $\mathcal{O}_Y(U)$ . Since  $\mathfrak{m}$  is prime and  $\beta^n = 0$ , we have  $\beta \in \mathfrak{m}$ . By the definition of  $\mathcal{J}$ , there exist  $\lambda, \mu \in \mathcal{O}_Y(U)$  such that  $\lambda f^*(x_i) + \mu f^*(t) = z$ , i.e.  $\lambda \alpha + \mu \beta = 1 \in \mathfrak{m}$ . This is impossible, so  $\alpha$  is invertible.

It follows that there one and only one possible definition for  $\tilde{f}^* : \mathcal{O}_X(U_i)[t]/(t^n) \rightarrow \mathcal{O}_Y(U) : \tilde{f}^*(a) = f^*(a)$  for every  $a \in \mathcal{O}_X(U_i)$ , and  $\tilde{f}^*(t) = \beta \alpha^{-1} t$ .  $\square$

### 6.3. THE CASE OF DOUBLE SCHEMES

We suppose that  $n = 2$  and keep the notations of the proof of theorem 4.1.1. Let  $i, j \in I$ . From 3.2,  $\lambda_{ij}^*$  and  $\gamma_{ij}^*$  are automorphisms of  $\mathcal{O}_X(U_{ij})[t]/(t^2)$ , and there exists derivations  $D_{ij}, E_{ij}$  of  $\mathcal{O}_X(U_{ij})$  such that, for every  $u \in \mathcal{O}_X(U_{ij})$  we have

$$\lambda_{ij}^*(u) = u + D_{ij}(u)t, \quad \gamma_{ij}^*(u) = u + E_{ij}(u)t.$$

According to 3.2, the family  $(D_{ij})$  (resp.  $(E_{ij})$ ) represents  $A \in H^1(X, T_X \otimes L)$  (resp.  $\mathbb{A} \in H^1(X, T_X \otimes L \otimes \mathcal{O}_X(H))$ ), and  $\mathbb{C}.A$  (resp.  $\mathbb{C}.\mathbb{A}$ ) is the element of  $\mathbf{DS}(X, L)$  (resp.  $\mathbf{DS}(X, L \otimes \mathcal{O}_X(H))$ ) corresponding to  $\mathcal{X}$  (resp  $\mathcal{X}' = \mathbf{B}_{H,\mathcal{X}}$ ).

Let  $\phi_H : H^1(X, T_X \otimes L) \rightarrow H^1(X, T_X \otimes L \otimes \mathcal{O}_X(H))$  be the canonical map.

**6.3.1. Proposition:** *We have  $\mathbb{A} = \phi_H(A)$ .*

*Proof.* This follows easily from 2.2.3 and proposition 6.1.2.  $\square$

## 7. PRIMITIVE DOUBLE STRUCTURES ON $\tilde{\mathbb{P}}_2$

### 7.1. THE MAIN RESULT

Let  $P \in \mathbb{P}_2$ . Let  $\pi : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$  be the blowing-up of  $\mathbb{P}_2$  along  $P$ , and  $\tilde{P} = \pi^{-1}(P)$  the exceptional divisor.

According to [10], there exists only one non trivial primitive double scheme  $\mathbb{X}$  such that  $\mathbb{X}_{red} = \mathbb{P}_2$ , and its associated line bundle is  $\mathcal{O}_{\mathbb{P}_2}(-3)$ .

Let  $\mathbb{Y}$  be a primitive double scheme such that  $\mathbb{Y}_{red} = \mathbb{P}_2$ , and with associated line bundle  $\mathcal{O}_{\mathbb{P}_2}(m)$ . We will give the list of all primitive double schemes  $\tilde{\mathbb{Y}}$  such that

- $\tilde{\mathbb{Y}}_{red} = \tilde{\mathbb{P}}_2$ ,
- there exists a morphism  $\pi_{\tilde{\mathbb{Y}}} : \tilde{\mathbb{Y}} \rightarrow \mathbb{Y}$  inducing  $\pi_{P, \mathbb{P}_2} : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$ , and such that its restriction  $\tilde{\mathbb{Y}} \setminus \tilde{P} \rightarrow \mathbb{Y} \setminus \{P\}$  is an isomorphism.

In this case, if  $\mathbb{L}$  is the line bundle on  $\tilde{\mathbb{P}}_2$  associated to  $\tilde{\mathbb{Y}}$ , there is an integer  $p$  such that  $\mathbb{L} = \pi_{P, \mathbb{P}_2}^*(\mathcal{O}_{\mathbb{P}_2}(m)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(p\tilde{P})$ . We will prove

**7.1.1. Theorem:** *We have  $p \geq 0$ .*

**1** – *If  $p = 0$ , the schemes  $\tilde{\mathbb{Y}}$  are the blowing-ups of  $\mathbb{Y}$  along the good subschemes  $Z$  such that  $Z_{red} = \{P\}$ . If  $m \geq -1$ , all these schemes are isomorphic. If  $m \leq -2$  they are all distinct.*

**2** – *If  $p = 1$ , these schemes form a family  $(\tilde{\mathbb{Y}}_{1, \alpha})_{\alpha \in \mathbb{C}}$ ,  $\tilde{\mathbb{Y}}_{1, 0}$  being the blowing-up of the schemes of **1** along  $\tilde{P}$ , and also the blowing-up of  $\mathbb{Y}$  along  $P$ . If  $\mathbb{Y} = \mathbb{X}$ , these schemes are all distinct. If  $\mathbb{Y}$  is trivial,  $\tilde{\mathbb{Y}}_{1, 0}$  is trivial, and all the  $\tilde{\mathbb{Y}}_{1, \alpha}$ ,  $\alpha \neq 0$  are isomorphic and non trivial.*

**3** – *If  $p \geq 2$ , there is only one such scheme  $\tilde{\mathbb{Y}}_p$ . If  $p = 2$ ,  $\tilde{\mathbb{Y}}_p$  is the blowing-up of the schemes of **2** along  $\tilde{P}$ . If  $p > 2$ ,  $\tilde{\mathbb{Y}}_p$  is the blowing-up of  $\tilde{\mathbb{Y}}_{p-1}$  along  $\tilde{P}$ .*

According to 5, if  $m \leq -2$ , the set of schemes  $\tilde{\mathbb{Y}}$  such that  $p = 0$  has a natural structure of affine space with associated vector space  $T_{\mathbb{P}_2, P} \otimes \mathcal{O}_{\mathbb{P}_2}(m)_P$ .

We will give the proof only for  $\mathbb{X}$ , the non trivial double scheme. The other cases are similar and simpler.

### 7.2. DEFINITION AND PROPERTIES OF THE NON PROJECTIVE DOUBLE PLANE $\mathbb{X}$

Let  $V$  be a complex vector space of dimension 3,  $\mathbb{P}_2 = \mathbb{P}(V)$  and  $\mathbb{P}_2^* = \mathbb{P}(V^*)$ . Let  $Q$  be the universal quotient bundle on  $\mathbb{P}_2^*$ . We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2^*}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_2^*} \otimes V^* \longrightarrow Q \longrightarrow 0 .$$

We fix a basis of  $V$ , so that we can define points of  $V$  or  $\mathbb{P}_2$  using 3 coordinates. For  $0 \leq i \leq 2$ , let  $U_i = \{(x_0, x_1, x_2) \in \mathbb{P}_2 ; x_i \neq 0\}$ . We will take the indices in  $\mathbb{Z}/3\mathbb{Z}$ , so that  $x_3 = x_0, x_4 = x_1, \dots$

**7.2.1.** *The double projective plane*  $\mathbb{X}$  (cf. [10], 8) – We use the notations of 7.3 and 3.1. For  $0 \leq i \leq 2$ ,  $\Omega_{\mathbb{P}_2}(U_i)$  is generated by  $d\left(\frac{x_{i+1}}{x_i}\right)$ ,  $d\left(\frac{x_{i+2}}{x_i}\right)$ . The corresponding dual basis of  $T_{\mathbb{P}_2}(U_i)$  is  $\left(\frac{\partial}{\partial(x_{i+1}/x_i)}, \frac{\partial}{\partial(x_{i+2}/x_i)}\right)$ . Then  $\mathbb{X}$  is constructed as in 3.1.1, 3.2, using the open cover  $(U_i)_{0 \leq i \leq 2}$ . We take

$$\alpha_{i,i+1} = \left(\frac{x_i}{x_{i+1}}\right)^3, \quad D_{i,i+1} = \frac{x_i}{x_{i+1}} \frac{\partial}{\partial(x_{i+2}/x_i)}.$$

So we have isomorphisms  $\delta_i : U_i^{(2)} \rightarrow U_i \times \mathbf{Z}_2$  (where  $U_i^{(2)}$  is the open subscheme of  $\mathbb{X}$  corresponding to  $U_i$ ). Let

$$\delta_{ij} = \delta_j \delta_i^{-1} : U_{ij} \times \mathbf{Z}_2 \xrightarrow{\simeq} U_{ij} \times \mathbf{Z}_2,$$

and  $\delta_{ij}^* = \delta_i^{*-1} \delta_j^*$  is the corresponding automorphism of  $\mathcal{O}_{\mathbb{P}_2}(U_i)[t]/(t^2)$ .

The line bundle on  $\mathbb{P}_2$  associated to  $\mathbb{X}$  is  $\mathcal{O}_{\mathbb{P}_2}(-3)$ .

### 7.3. THE BLOWING-UP OF A POINT OF $\mathbb{P}_2$

Let  $P = (0, 0, 1) \in \mathbb{P}_2$  and  $\mathbb{U} = \mathbb{P}_2 \setminus \{P\}$ . Let  $\pi : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$  be the blowing-up of  $\mathbb{P}_2$  along  $P$ . Let  $\mathbb{P} \subset \mathbb{P}_2^*$  be the line whose points are the lines of  $\mathbb{P}_2$  that contain  $P$ . We have

$$\tilde{\mathbb{P}}_2 = \{(E, Q) \in \mathbb{P} \times \mathbb{P}_2 ; Q \in E\},$$

and  $\pi = \pi_{P, \mathbb{P}_2} : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$  is the second projection. Let  $\Psi : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}$  be the first one. It is a projective bundle:  $\tilde{\mathbb{P}}_2 = \mathbb{P}(Q_{|\mathbb{P}}^*)$ . Let  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(1)$  be the corresponding line bundle, and  $\tilde{P} = \pi^{-1}(P)$  the exceptional line. We have (cf. [17], V, 2)

$$\mathcal{O}_{\tilde{\mathbb{P}}_2}(\tilde{P}) \simeq \Psi^*(\mathcal{O}_{\mathbb{P}}(-1)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(1), \quad \pi^*(\mathcal{O}_{\mathbb{P}_2}(1)) \simeq \mathcal{O}_{\tilde{\mathbb{P}}_2}(1).$$

**7.3.1.** *An open cover* – We have an isomorphism  $D : \mathbb{P}_1 \rightarrow \mathbb{P}$  that associates to  $(\alpha_0, \alpha_1) \in \mathbb{P}_1$  the line of equation  $\alpha_0 x_0 - \alpha_1 x_1 = 0$ . For  $i = 0, 1$ , let  $V_i = \{(\alpha_0, \alpha_1) \in \mathbb{P} ; \alpha_i \neq 0\}$ . We have

$$\psi^{-1}(V_0) = \{(D(1, \lambda), (\lambda x_1, x_1, x_2)) \in \tilde{\mathbb{P}}_2 ; \lambda \in \mathbb{C}, (x_1, x_2) \in \mathbb{P}_1\} \simeq \mathbb{C} \times \mathbb{P}_1,$$

$\psi^{-1}(V_0) \cap \mathbb{U} = U_1$ , and a similar description of  $\psi^{-1}(V_1)$ . Let  $W_0 = U_0$ ,  $W_1 = U_1$ ,

$$\begin{aligned} W_2 &= \{(E, (x_0, x_1, x_2)) \in \psi^{-1}(V_0) ; x_2 \neq 0\} \\ &= \{(D(1, \lambda), (\lambda \mu, \mu, 1)) ; \lambda, \mu \in \mathbb{C}\} \simeq \mathbb{C}^2, \\ W_3 &= \{(E, (x_0, x_1, x_2)) \in \psi^{-1}(V_1) ; x_2 \neq 0\} \\ &= \{(D(\eta, 1), (\nu, \eta \nu, 1)) ; \eta, \nu \in \mathbb{C}\} \simeq \mathbb{C}^2. \end{aligned}$$

We have  $W_0 \cup W_1 \cup W_2 \cup W_3 = \tilde{\mathbb{P}}_2$ , and

$$\begin{aligned} W_{02} &= U_{012}, \quad W_{03} = U_{02}, \quad W_{12} = U_{12}, \quad W_{13} = U_{012}, \quad W_{23} \simeq \mathbb{C} \times \mathbb{C}^*, \\ W_{012} &= W_{013} = W_{023} = W_{123} = U_{012}. \end{aligned}$$

We will use the variables  $\lambda = \frac{x_0}{x_1}$ ,  $\mu = \frac{x_1}{x_2}$ .

With the notations of 5, we have  $i_0 = 2$ ,  $y_1 = \mu$ ,  $y_2 = \nu = \lambda\mu$ ,  $W_2 = D_+^{\mathbb{B}_2}(y_1)$  and  $W_3 = D_+^{\mathbb{B}_2}(y_2)$ .

**7.3.2. Line bundles** – With respect to the open cover  $(W_i)_{0 \leq i \leq 3}$ , the line bundle  $\pi^*(\mathcal{O}_{\mathbb{P}_2}(m)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(p\tilde{P})$  is represented by the cocycle  $(\beta_{ij})_{0 \leq i < j \leq 3}$ , with

$$\begin{aligned} \beta_{01} &= x_0^{-m} x_1^m = \lambda^{-m}, & \beta_{12} &= x_1^{-p-m} x_2^{m+p} = \mu^{-p-m}, \\ \beta_{02} &= x_0^{-m} x_1^{-p} x_2^{m+p} = \lambda^{-m} \mu^{-p-m}, & \beta_{23} &= x_0^{-p} x_1^p = \lambda^{-p}, \\ \beta_{03} &= x_0^{-p-m} x_2^{m+p} = \lambda^{-p-m} \mu^{-p-m}, & \beta_{13} &= x_0^{-p} x_1^{-m} x_2^{m+p} = \lambda^{-p} \mu^{-p-m}. \end{aligned}$$

#### 7.4. MORPHISMS TO $\mathbb{X}$

We use here the results of 3.5.

**7.4.1. Rings and derivations** – Let  $\mathbb{C}(\lambda, \mu)$  be the field of rational functions in the variables  $\lambda$ ,  $\mu$  (of 7.3.1). If  $a_1, \dots, a_n \in \mathbb{C}(\lambda, \mu)$ , let  $\mathbf{A}(a_1, \dots, a_n)$  be the sub- $\mathbb{C}$ -algebra of  $\mathbb{C}(\lambda, \mu)$  generated by  $a_1, \dots, a_n$ . We have

$$\begin{aligned} \mathcal{O}_{\mathbb{P}_2}(U_2) &= \mathbf{A}(\mu, \lambda\mu), & \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2) &= \mathbb{C}[\lambda, \mu], & \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3) &= \mathbf{A}\left(\frac{1}{\lambda}, \lambda\mu\right), \\ \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{12}) &= \mathbf{A}\left(\lambda, \mu, \frac{1}{\mu}\right), & \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{23}) &= \mathbf{A}\left(\lambda, \frac{1}{\lambda}, \mu\right), & \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{03}) &= \mathbf{A}\left(\frac{1}{\lambda}, \lambda\mu, \frac{1}{\lambda\mu}\right), \\ \mathcal{O}_{\mathbb{P}_2}(U_{012}) &= \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{012}) &= \mathbf{A}\left(\lambda, \frac{1}{\lambda}, \mu, \frac{1}{\mu}\right). \end{aligned}$$

We will write the derivations  $\mathcal{D}$  of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_i)$  (or  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{ij})$ ,  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{012})$ ) as

$$\mathcal{D} = U \cdot \frac{\partial}{\partial \lambda} + V \cdot \frac{\partial}{\partial \mu},$$

where  $U, V \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{012})$ . There will be conditions on  $U$  and  $V$ , depending on the open subset where they are defined. For example suppose that we consider derivations of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{12})$ . Then  $\mathcal{O}_{\mathbb{P}_2}(W_{12}) = \mathbf{A}\left(\lambda, \mu, \frac{1}{\mu}\right)$ . So  $\mathcal{D}$  is a derivation of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{12})$  if and only if  $\mathcal{D}(\lambda), \mathcal{D}(\mu) \in \mathcal{O}_{\mathbb{P}_2}(W_{12})$ , i.e.  $U, V \in \mathcal{O}_{\mathbb{P}_2}(W_{12})$ . No verification is required for  $W_{02}$  and  $W_{13}$ , since  $W_{02} = W_{13} = U_{012}$ .

**7.4.2. Construction of double structures** – We want to define primitive double schemes  $\mathbb{Y}$ , such that  $\mathbb{Y}_{red} = \tilde{\mathbb{P}}_2$ , and such that there exists a morphism  $\phi : \mathbb{Y} \rightarrow \mathbb{X}$  with  $\phi_{red} = \pi : \tilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$ , and such that  $\phi$  induces an isomorphism between the open subschemes of  $\mathbb{Y}$ ,  $\mathbb{X}$  over  $\mathbb{U} = \mathbb{P}_2 \setminus \{P\}$ .

The associated line bundle will be of the form  $L = \pi^*(\mathcal{O}_{\mathbb{P}_2}(-3)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(p\tilde{P})$ , with associated cocycle  $(\beta_{ij})$  as in 7.3.2, with  $m = -3$ .

We will use the open cover  $(W_i)_{0 \leq i \leq 3}$  of  $\tilde{\mathbb{P}}_2$  to build them as in 3. For this we need to define a derivation  $E_{ij}$  of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{ij})$ , for  $0 \leq i < j \leq 3$ , such that  $(\beta_i^{-1} \otimes E_{ij})_{0 \leq i < j \leq 3}$  represents the associated element of  $H^1(\tilde{\mathbb{P}}_2, T_{\tilde{\mathbb{P}}_2} \otimes L)$ . In particular we have the relations  $E_{ik} = E_{ij} + \beta_{ij} E_{jk}$ .

So we have, as in 7.2 for  $\mathbb{X}$ , for  $0 \leq i \leq 3$ , isomorphisms  $\rho_i : W_i^{(2)} \rightarrow W_i \times \mathbf{Z}_2$ , with  $\rho_0 = \delta_0$ ,  $\rho_1 = \delta_1$ .

We have  $E_{01} = D_{01} = -\lambda^2 \mu^2 \frac{\partial}{\partial \mu}$ . We will take

$$E_{12} = A \frac{\partial}{\partial \lambda} + B \frac{\partial}{\partial \mu}, \quad E_{13} = C \frac{\partial}{\partial \lambda} + D \frac{\partial}{\partial \mu}.$$

Hence we have

$$E_{23} = \beta_{21}(E_{13} - E_{12}) = \mu^{p-3} \left( (C - A) \frac{\partial}{\partial \lambda} + (D - B) \frac{\partial}{\partial \mu} \right),$$

$$E_{03} = E_{01} + \beta_{01} E_{13} = \lambda^3 C \frac{\partial}{\partial \lambda} + (\lambda^3 D - \lambda^2 \mu^2) \frac{\partial}{\partial \mu}.$$

The fact that  $E_{ij}$  is a derivation of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{ij})$  will impose conditions on  $A, B, C$  and  $D$ , necessary and sufficient to define  $\mathbb{Y}$ . The existence on  $\phi$  will also impose other conditions.

**7.4.3. Morphisms  $\mathbb{Y} \rightarrow \mathbb{X}$**  - We have

$$\pi(W_0) = U_0, \quad \pi(W_1) = U_1, \quad \pi(W_2) \subset U_2, \quad \pi(W_3) \subset U_2,$$

and  $\pi : W_i \rightarrow U_i$  is the identity for  $i = 0, 1$ . Let

$$\Delta_0 : W_0^{(2)} \rightarrow U_0^{(2)}, \quad \Delta_1 : W_1^{(2)} \rightarrow U_1^{(2)}$$

be the identities. We will define

$$\Delta_2 : W_2^{(2)} \rightarrow U_2^{(2)}, \quad \Delta_3 : W_3^{(2)} \rightarrow U_2^{(2)},$$

(inducing  $\pi$ ), such that  $\Delta_1$  and  $\Delta_2$  (resp.  $\Delta_1$  and  $\Delta_3$ ) coincide on  $W_{12}^{(2)}$  (resp.  $W_{13}^{(2)}$ ). It is easy to see that this describes completely a morphism  $\phi : \mathbb{Y} \rightarrow \mathbb{X}$ .

With the notations of 3.5.3, we have

$$\lambda_0 = \delta_1 : U_1^{(2)} \xrightarrow{\simeq} U_1 \times \mathbf{Z}_2, \quad \lambda_1 = \delta_2 : U_2^{(2)} \xrightarrow{\simeq} U_2 \times \mathbf{Z}_2,$$

and  $\lambda^* = \lambda_0^{*-1} \lambda_1^* = \delta_{12}^*$  is an automorphism of  $\mathcal{O}_{\mathbb{P}_2}(U_{12})[t]/(t^2)$ . We have

$$\lambda^*(a) = a + D_X(a)t \quad \text{for every } a \in \mathcal{O}_{\mathbb{P}_2}(U_{12}),$$

Where  $D_X = \frac{x_1}{x_2} \frac{\partial}{\partial(x_0/x_1)}$  (with respect to the variables  $\frac{x_2}{x_1}, \frac{x_0}{x_1}$ ) and  $\lambda^*(t) = \alpha t$ , with

$$\alpha = \frac{x_1^3}{x_2^3} = \mu^3. \quad \text{With respect to the variables } \lambda, \mu \text{ we have } D_X = \mu \frac{\partial}{\partial \lambda}.$$

We have similarly, with  $\mu_0 = \rho_1, \mu_1 = \rho_2, \mu^* = \rho_{12}^*$ ,

$$\mu^*(b) = b + D_Y(b)t \quad \text{for every } b \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{12}),$$

where  $D_Y$  is the derivation  $E_{12}$  of the definition on  $\mathbb{Y}$ , and  $\mu^*(t) = \beta t$ , with  $\beta = \beta_{12} = \mu^{-p+3}$ .

Let  $A_2 = \lambda_1 \Delta_2 \mu_1^{-1} = \delta_2 \Delta_2 \rho_2^{-1} : W_2 \times \mathbf{Z}_2 \rightarrow U_2 \times \mathbf{Z}_2$ . We have

$$A_2^*(a) = \pi^*(a) + \gamma_2(a)t$$

for every  $a \in \mathcal{O}_X(U_2)$ , where  $\gamma_2 : \mathcal{O}_{\mathbb{P}_2}(U_2) \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2)$  is a  $\pi^*$ -derivation, and  $A_2^*(t) = \tau_2 t$ , with  $\tau_2 \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2)$ .

From (1) we have  $\tau_2 = \mu^p$ . It follows that  $p \geq 0$ . From (1) also we have

$$\gamma_2 = \mu^{p-3} \left( (\mu - A) \frac{\partial}{\partial \lambda} - B \frac{\partial}{\partial \mu} \right).$$

We have  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2) = \mathbb{C}[\lambda, \mu]$  and  $\mathcal{O}_{\mathbb{P}_2}(U_2) = \mathbf{A}(\lambda\mu, \mu)$ . So the fact that  $\gamma_2$  has values in  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2)$  is equivalent to  $\gamma_2(\mu), \gamma_2(\lambda\mu) \in \mathbb{C}[\lambda, \mu]$ . These conditions imply also that  $E_{12}$  is a derivation of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{12})$ .

We obtain similar results for  $A_3$ . We have

$$A_3^*(a) = \pi^*(a) + \gamma_3(a)t$$

for every  $a \in \mathcal{O}_X(U_2)$ , where  $\gamma_3 : \mathcal{O}_{\mathbb{P}_2}(U_2) \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$  is a  $\pi^*$ -derivation, and  $A_3^*(t) = \tau_3.t$ , with  $\tau_3 \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$ . From (1) we have  $\tau_3 = \lambda^p \mu^p$ . From (1) we have

$$\gamma_3 = \lambda^p \mu^{p-3} \left( (\mu - C) \frac{\partial}{\partial \lambda} - D \frac{\partial}{\partial \mu} \right).$$

**7.4.4. Conditions on  $A, B, C, D$  and consequences** – We will study the conditions imposed on  $A, B, C$  and  $D$  by the facts that  $\gamma_i(\mathcal{O}_{\mathbb{P}_2}(U_2)) \subset \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_i)$  for  $i = 2, 3$ , and that the  $E_{ij}$  are derivations of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{ij})$ .

**7.4.5. Lemma:** *We can suppose that  $B = \mu^{-p+3}B_0$ , where  $B_0$  is any element of  $\mathbb{C}[\lambda, \mu]$ , and that  $A$  is of the form  $A = \mu - \mu^{-p+2}R(\lambda)$ , where  $R$  is a polynomial.*

*Proof.* We have  $\mu \in \mathcal{O}_{\mathbb{P}_2}(U_2)$  and  $\gamma_2(\mu) = -\mu^{p-3}B \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2)$ , hence we can write

$B = \mu^{-p+3}B_1$ , where  $B_1 \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2)$ . But  $B_1 \frac{\partial}{\partial \mu}$  is a derivation of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_2)$  and  $\beta_{12} = \mu^{-p+3}$ ,

hence by 3.2 we can replace  $E_{12}$  with  $E_{12} - \mu^{-p+3}(B_1 - B_0) \frac{\partial}{\partial \mu}$ , i.e. we can assume that  $B = \mu^{-p+3}B_0$ .

We have  $\gamma_2(\lambda\mu) = \mu^{p-2}(\mu - A)$ , hence we can write  $\mu^{p-2}(\mu - A) = P(\lambda, \mu)$ , where  $P$  is a polynomial, i.e.  $A = \mu - \mu^{-p+2}P(\lambda, \mu)$ . By 3.2, we can replace  $A$  with  $A + \mu^{-p+3}Q(\lambda, \mu)$ , where  $Q$  is a polynomial. The result for  $A$  follows immediately.  $\square$

**7.4.6. Lemma:** *Let  $D = U \frac{\partial}{\partial \lambda} + V \frac{\partial}{\partial \mu}$ , with  $U, V \in \mathbb{C}(\lambda, \mu)$ . Then  $D$  is a derivation of*

$\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$  *if and only there are  $\epsilon, \tau \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$  such that  $U = \lambda^2\epsilon$ ,  $V = \frac{1}{\lambda}\tau - \lambda\mu\epsilon$ .*

*Proof.* This follows easily from the fact that  $D$  is a derivation of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$  if and only if  $D(1/\lambda), D(\lambda\mu) \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$ .  $\square$

**7.4.7. Lemma:** *We can suppose that*

$$C = \mu + \lambda^{-p} \mu^{-p+2} \left( -c_0 \lambda + S \left( \frac{1}{\lambda} \right) \right), \quad D = \lambda^{-p} \mu^{-p+3} c_0,$$

*where  $c_0 \in \mathbb{C}$  and  $S$  is a polynomial.*

*Proof.* We have  $\gamma_3(\mu) = -\lambda^p \mu^{p-3} D \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$  and  $\gamma_3(\lambda\mu) = \lambda^p \mu^{p-3}(\mu(\mu - C)) - \lambda D \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$ . Hence

$$C = \mu - \lambda^{-p} \mu^{-p+3} \left( -\frac{\lambda}{\mu} \gamma_3(\mu) + \frac{1}{\mu} \gamma_3(\lambda\mu) \right), \quad D = -\lambda^{-p} \mu^{-p+3} \gamma_3(\mu).$$

By 3.2, we can replace  $C$  with  $C + \lambda^{-p} \mu^{-p+3} \cdot \lambda^2 \epsilon$  and  $D$  with  $D + \lambda^{-p} \mu^{-p+3} \left( \frac{1}{\lambda} \tau - \lambda \mu \epsilon \right)$ , with  $\epsilon, \tau \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$ . We get

$$C = \mu + \lambda^{-p} \mu^{-p+3} \left( \lambda^2 \epsilon + \frac{\lambda}{\mu} \gamma_3(\mu) - \frac{1}{\mu} \gamma_3(\lambda\mu) \right), \quad D = \lambda^{-p} \mu^{-p+3} \left( -\gamma_3(\mu) + \frac{1}{\lambda} \tau - \lambda \mu \epsilon \right).$$

Since  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3) = \mathbf{A}\left(\frac{1}{\lambda}, \lambda\mu\right)$ , we can take  $\tau, \epsilon$  such that  $-\gamma_3(\mu) + \frac{1}{\lambda} \tau - \lambda \mu \epsilon = c_0 \in \mathbb{C}$ . We have then  $D = \lambda^{-p} \mu^{-p+3} c_0$ . We can still replace  $\epsilon$  with  $\epsilon + \frac{1}{\lambda} \theta$  and  $\tau$  with  $\tau + \lambda \mu \theta$ , with  $\theta \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_3)$ . We have

$$\lambda^2 \epsilon + \frac{\lambda}{\mu} \gamma_3(\mu) - \frac{1}{\mu} \gamma_3(\lambda\mu) = -\frac{\lambda}{\mu} c_0 + \frac{1}{\mu} (\lambda \mu \theta + \tau - \gamma_3(\lambda\mu)).$$

We can then choose  $\theta$  such that  $\lambda \mu \theta + \tau - \gamma_3(\lambda\mu)$  is a polynomial in  $\frac{1}{\lambda}$ . □

Let  $R_0$  (resp.  $S_0$ ) be the constant term of the polynomial  $R(\lambda)$  (resp.  $S\left(\frac{1}{\lambda}\right)$ ). Recall that we have  $p \geq 0$ .

**7.4.8. Lemma:** **1** – If  $p \geq 2$  then  $R = S = c_0 = 0$ .

**2** – If  $p = 1$ , then  $S = 0$  and  $R = c_0$ .

**3** – If  $p = 0$ , then  $R = R_0 + c_0 \lambda$ ,  $S = S_0$  and  $R_0 = -S_0$ .

*Proof.* We have  $E_{23}(\lambda) = \mu^{p-3}(C - A) \in \mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{23}) = \mathbf{A}\left(\lambda, \frac{1}{\lambda}, \mu\right)$ . The result follows immediately that, from lemmas 7.4.5 and 7.4.7, we have

$$\mu^{p-3}(C - A) = \frac{1}{\mu} \left( -c_0 \lambda^{-p+1} + \lambda^{-p} S\left(\frac{1}{\lambda}\right) + R(\lambda) \right).$$

□

The fact that  $E_{03}$  is a derivation of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(W_{03})$  does not bring other conditions.

### 7.5. PROOF OF THEOREM 7.1.1

Suppose that  $p = 0$ . Then we have, according to lemma 7.4.8,

$$E_{12} = (\mu + \mu^2(-c_0\lambda - R_0)) \frac{\partial}{\partial \lambda} + \mu^3 c_0 \frac{\partial}{\partial \mu},$$

and the same formula for  $E_{13}$ . With the notations and context of 5.2, we have  $i_0 = 2$ ,  $y_1 = \mu$ ,  $y_2 = \lambda\mu$ ,  $J = \{0, 1, i_1, i_2\}$ ,  $(V_0, V_1, V_{i_1}, V_{i_2}) = (U_0, U_1, W_2, W_3)$ . If  $Z = Z(a_1, a_2)$ , and  $(E'_{ij})_{i,j \in J}$  is the family of derivations that define  $\mathbf{B}_{Z, \mathbb{X}}$ , we have

$$E'_{1, i_1} = \mu \frac{\partial}{\partial \lambda} + \mu^3 D(a_1, a_2).$$

Using the preceding formulas, we have

$$\mu^3 D(a_1, a_2) = \mu^3 \left( a_1 \frac{\partial}{\partial y_1} + a_2 \frac{\partial}{\partial y_2} \right) = \mu^2 (-a_1 \lambda + a_2) \frac{\partial}{\partial \lambda} + \mu^3 a_1 \frac{\partial}{\partial \mu}.$$

Hence we have  $E_{12} = E'_{12}$  if  $a_1 = c_0$   $a_2 = -R_0$ . With simpler other checks, we see easily that the primitive double schemes of lemma 7.4.8, 3, are the  $\mathbf{B}_{Z, \mathbb{X}}$ .

The other cases ( $p = 1, 2$ ) are similar. The scheme  $\tilde{\mathbb{Y}}_{1, \alpha}$  of theorem 7.1.1, 2, corresponds to  $c_0 = \alpha$  in lemma 7.4.8.

### 7.6. THE DOUBLE SCHEMES $\tilde{\mathbb{X}}_{1, \alpha}$

We will study the schemes  $\tilde{\mathbb{X}}_{1, \alpha}$ ,  $\alpha \in \mathbb{C}$ , of theorem 7.1.1, 2 (when  $\mathbb{Y}$  is the non trivial double scheme  $\mathbb{X}$ ). The line bundle associated to  $\tilde{\mathbb{X}}_{1, \alpha}$  is  $\pi_{P, \mathbb{P}_2}^*(\mathcal{O}_{\mathbb{P}_2}(-3)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(\tilde{P}) = \omega_{\tilde{\mathbb{P}}_2}$ . It follows that  $\tilde{\mathbb{X}}_{1, \alpha}$  is a *K3-carpet* (cf. [11], [15]).

**7.6.1. Theorem:** **1** –  $\tilde{\mathbb{X}}_{1, \alpha}$  is quasi-projective if and only  $\alpha \in \mathbb{Q}$  and  $\alpha > 0$ .

**2** – If  $\alpha \notin \mathbb{Q}$ ,  $\text{Pic}(\tilde{\mathbb{X}}_{1, \alpha}) = \{\mathcal{O}_{\tilde{\mathbb{X}}_{1, \alpha}}\}$ .

**3** – If  $\alpha \in \mathbb{Q}$ ,  $\text{Pic}(\tilde{\mathbb{X}}_{1, \alpha}) \simeq \mathbb{Z}$ .

We will first describe  $H^1(\tilde{\mathbb{P}}_2, \Omega_{\tilde{\mathbb{P}}_2})$  with usual Čech cohomology, using the open cover  $(W_i)_{0 \leq i \leq 3}$ .

We have  $H^1(\tilde{\mathbb{P}}_2, \Omega_{\tilde{\mathbb{P}}_2}) \simeq \mathbb{C}^2$ , and it is generated by  $\nabla_0(\pi^*(\mathcal{O}_{\mathbb{P}_2}(1)))$  and

$\nabla_0(\mathcal{O}_{\tilde{\mathbb{P}}_2}(-\tilde{P}))$  (cf. 2.1). With respect to  $(W_i)_{0 \leq i \leq 3}$ ,

–  $\nabla_0(\pi^*(\mathcal{O}_{\mathbb{P}_2}(1)))$  is represented by the cocycle  $(u_{ij})$ , with

$$u_{01} = -\frac{d\lambda}{\lambda}, \quad u_{12} = -\frac{d\mu}{\mu}, \quad u_{23} = 0.$$

–  $\nabla_0(\mathcal{O}_{\tilde{\mathbb{P}}_2}(-\tilde{P}))$  is represented by the cocycle  $(v_{ij})$ , with

$$v_{01} = 0, \quad v_{12} = \frac{d\mu}{\mu}, \quad v_{23} = \frac{d\lambda}{\lambda}.$$

For the canonical product  $H^1(\tilde{\mathbb{P}}_2, \Omega_{\tilde{\mathbb{P}}_2}) \times H^1(\tilde{\mathbb{P}}_2, \Omega_{\tilde{\mathbb{P}}_2}) \rightarrow H^2(\tilde{\mathbb{P}}_2, \omega_{\tilde{\mathbb{P}}_2}) \simeq \mathbb{C}$ , we have

$$\nabla_0(\pi^*(\mathcal{O}_{\mathbb{P}_2}(1)))^2 = 1, \quad \nabla_0(\mathcal{O}_{\tilde{\mathbb{P}}_2}(-\tilde{P}))^2 = -1, \quad \nabla_0(\pi^*(\mathcal{O}_{\mathbb{P}_2}(1))).\nabla_0(\mathcal{O}_{\tilde{\mathbb{P}}_2}(-\tilde{P})) = 0.$$

From 7.4, the scheme  $\tilde{\mathbb{X}}_{1,\alpha}$  is defined, with respect to  $(W_i)_{0 \leq i \leq 3}$ , by the family  $(E_{ij})$  of derivations, with

$$E_{01} = -\lambda^2 \mu^2 \frac{\partial}{\partial \mu}, \quad E_{12} = \mu(1-\alpha) \frac{\partial}{\partial \lambda}, \quad E_{23} = -\frac{\alpha}{\lambda} \frac{\partial}{\partial \mu}.$$

This family defines also  $\sigma_\alpha \in H^1(\tilde{\mathbb{P}}_2, \Omega_{\tilde{\mathbb{P}}_2})$  (which defines  $\tilde{\mathbb{X}}_{1,\alpha}$  according to 3.2), in the sense of 2.2.

**7.6.2. Lemma:** *We have  $\sigma_\alpha = \nabla_0(\pi^*(\mathcal{O}_{\mathbb{P}_2}(1))) + \alpha \nabla_0(\mathcal{O}_{\tilde{\mathbb{P}}_2}(-\tilde{P}))$ .*

*Proof.* On the open subsets  $W_i$  we have the following trivializations  $\alpha_i : \omega_{\tilde{\mathbb{P}}_2|W_i} \rightarrow \mathcal{O}_{W_i}$  of  $\tilde{\mathbb{P}}_2$  :

$$\alpha_0 : d\left(\frac{x_2}{x_0}\right) \wedge d\left(\frac{x_1}{x_0}\right) \mapsto 1, \quad \alpha_1 : d\left(\frac{x_0}{x_1}\right) \wedge d\left(\frac{x_2}{x_1}\right) \mapsto 1, \quad \alpha_2 : d\lambda \wedge d\mu \mapsto 1, \quad \alpha_3 : d\nu \wedge d\eta \mapsto 1,$$

and, using the variables  $\lambda, \mu$ :

$$\alpha_0^{-1} : 1 \mapsto -\frac{1}{\lambda^3 \mu^2} d\lambda \wedge d\mu, \quad \alpha_1^{-1} : 1 \mapsto -\frac{1}{\mu^2} d\lambda \wedge d\mu, \quad \alpha_2^{-1} : 1 \mapsto d\lambda \wedge d\mu, \quad \alpha_3^{-1} : 1 \mapsto -\frac{1}{\lambda} d\lambda \wedge d\mu.$$

In usual Čech cohomology,  $\sigma_\alpha$  is defined by  $E'_{ij} = \alpha_i^{-1} E_{ij}$  :

$$E'_{01} = -\frac{1}{\lambda^3 \mu^2} .d\lambda \wedge d\mu \otimes (-\lambda^2 \mu^2 \frac{\partial}{\partial \mu}), \quad E'_{12} = -\frac{1}{\mu^2} .d\lambda \wedge d\mu \otimes \mu(1-\alpha) \frac{\partial}{\partial \lambda},$$

$$E'_{23} = d\lambda \wedge d\mu \otimes \left(-\frac{\alpha}{\lambda} \frac{\partial}{\partial \mu}\right).$$

Now we apply the canonical isomorphism  $\omega_{\tilde{\mathbb{P}}_2} \otimes T_{\tilde{\mathbb{P}}_2} \simeq \Omega_{\tilde{\mathbb{P}}_2}$  and we get

$$E_{01} = -\frac{1}{\lambda} .d\lambda, \quad E_{12} = \frac{\alpha-1}{\mu} .d\mu, \quad E_{23} = \frac{\alpha}{\lambda} .d\lambda.$$

The result follows from the preceding description of  $\nabla_0(\pi^*(\mathcal{O}_{\mathbb{P}_2}(1)))$  and  $\nabla_0(\mathcal{O}_{\tilde{\mathbb{P}}_2}(-\tilde{P}))$ .  $\square$

Now we prove theorem 7.6.1. A line bundle  $\mathbb{F}$  on  $\tilde{\mathbb{X}}_{1,\alpha}$  is ample if and only if  $\mathbb{F}|_{\tilde{\mathbb{P}}_2}$  is ample (by [18], proposition 4.2). Hence  $\tilde{\mathbb{X}}_{1,\alpha}$  is quasi-projective if and only if some ample line bundle on  $\tilde{\mathbb{P}}_2$  can be extended to a line bundle to  $\tilde{\mathbb{X}}_{1,\alpha}$ .

Let  $F$  be a line bundle on  $\tilde{\mathbb{P}}_2$ . According to 3.3,  $F$  can be extended to a line bundle on  $\tilde{\mathbb{X}}_{1,\alpha}$  if and only if  $\sigma_\alpha . \nabla_0(F) = 0$ . Let  $F_{mn} = \pi^*(\mathcal{O}_{\mathbb{P}_2}(m)) \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(-n\tilde{P})$ . Then we have  $\sigma_\alpha . \nabla_0(F_{mn}) = m - n\alpha$ .

The ample line bundles on  $\tilde{\mathbb{P}}_2$  are the  $F_{mn}$  such that  $m, n > 0$ . It follows that  $\tilde{\mathbb{X}}_{1,\alpha}$  is quasi-projective if and only if there exist positive integers  $m, n$  such that  $m - n\alpha = 0$ . This proves 1.

According to [11], theorem 1.2.1, if  $\mathbb{F}, \mathbb{F}'$  are line bundles on  $\widetilde{\mathbb{X}}_{1,\alpha}$ , then  $\mathbb{F} \simeq \mathbb{F}'$  if and only if  $\mathbb{F}|_{\widetilde{\mathbb{P}}_2} \simeq \mathbb{F}'|_{\widetilde{\mathbb{P}}_2}$ . In other words, two extensions to  $\widetilde{\mathbb{X}}_{1,\alpha}$  of a line bundle on  $\widetilde{\mathbb{P}}_2$  are isomorphic. This implies **2** and **3**.

## REFERENCES

- [1] Bangere, P., Mukherjee, J., Raychaudhry, D. *K3 carpets on minimal rational surfaces and their smoothing*. Internat. J. Math. 32 (2021), no. 6, Paper No. 2150032
- [2] Bănică, C., Forster, O. *Multiple structures on space curves*. In: Sundararaman, D. (Ed.) Proc. of Lefschetz Centennial Conf. (10-14 Dec. Mexico), Contemporary Mathematics 58, AMS, 1986, 47-64.
- [3] Bayer, D., Eisenbud, D. *Ribbons and their canonical embeddings*. Trans. of the Amer. Math. Soc., 1995, 347-3, 719-756.
- [4] Chen, D., Kass, J. L. *Moduli of generalized line bundles on a ribbon*. J. of Pure and Applied Algebra 220 (2016), 822-844.
- [5] Drézet, J.-M. *Déformations des extensions larges de faisceaux*. Pacific Journ. of Math. 220, 2 (2005), 201-297.
- [6] Drézet, J.-M. *Variétés de modules alternatives*. Annales de l'Institut Fourier 49 (1999), 57-139.
- [7] Drézet, J.-M. *Faisceaux cohérents sur les courbes multiples*. Collect. Math. 2006, 57-2, 121-171.
- [8] Drézet, J.-M. *Paramétrisation des courbes multiples primitives* Adv. in Geom. 2007, 7, 559-612.
- [9] Drézet, J.-M. *Faisceaux sans torsion et faisceaux quasi localement libres sur les courbes multiples primitives*. Mathematische Nachrichten, 2009, 282-7, 919-952.
- [10] Drézet, J.-M. *Primitive multiple schemes*. European J. of Math. 7 (2021), 985-1045.
- [11] Drézet, J.-M. *Moduli of vector bundles on primitive multiple schemes*. International Journal of Mathematics, Vol. 34, No. 7 (2023) 2350035.
- [12] Drézet, J.-M. *Coherent sheaves on primitive multiple schemes*. <https://arxiv.org/abs/2404.07639> .
- [13] Eisenbud, D., Green, M. *Clifford indices of ribbons*. Trans. of the Amer. Math. Soc., 1995, 347-3, 757-765. scientifiques et industrielles 1252, Hermann, Paris (1964).
- [14] Ferrand, D. *Courbes gauches et fibrés de rang 2*. C.R. Acad. Sci. Paris, 281 (1977), 345-347.
- [15] Gallego, F.J., González, M., Purnaprajna, B.P. *K3 double structures on Enriques surfaces and their smoothings*. J. of Pure and Applied Algebra 212 (2008), 981-993.
- [16] González, M. *Smoothing of ribbons over curves*. J. reine angew. Math. 591 (2006), 201-235.
- [17] Hartshorne, R. *Algebraic geometry*. Grad. Texts in Math., Vol. 52, Springer (1977).
- [18] Hartshorne, R. *Ample vector bundles*. Publ. Math. IHES, 29 (1966), 319-350.
- [19] Savarese, M. *On the irreducible components of the compactified Jacobian of a ribbon*. <https://arxiv.org/abs/1803.05360> .
- [20] Savarese, M. *Coherent Sheaves on Ribbons and their Moduli*. <https://arxiv.org/abs/1902.08510> .
- [21] Savarese, M. *Generalized line bundles on primitive multiple curves and their moduli*. <https://arxiv.org/abs/1902.09463> .

INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE, CASE 247, 4 PLACE JUSSIEU, F-75252 PARIS, FRANCE

*Email address:* jean-marc.drezet@imj-prg.fr