

Finite rotations on \mathbb{Z}^2 : hierarchical structure and bijectivity analysis

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Rotations on \mathbb{R}^2

Definition (Rotation)

A rotation on \mathbb{R}^2 is defined by a mapping:

$$\left| \begin{array}{l} \mathcal{R}_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ x \mapsto M_\theta \cdot x \end{array} \right.$$

where M_θ is a rotation matrix:

$$M_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Any \mathcal{R}_θ is **bijective**, **isometric** and **orientation-preserving**.

Rotations on \mathbb{Z}^2

Definition (Discretized rotation)

A discretized rotation on \mathbb{Z}^2 is defined as

$$R_\theta := D \circ \mathcal{R}_\theta|_{\mathbb{Z}^2}$$

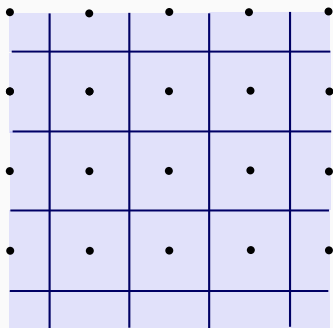
where $D : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$ is a discretization (rounding) function.

Remark: Most often, $\mathcal{R}_\theta(\mathbf{x}) \notin \mathbb{Z}^2$ for $\mathbf{x} \in \mathbb{Z}^2$.

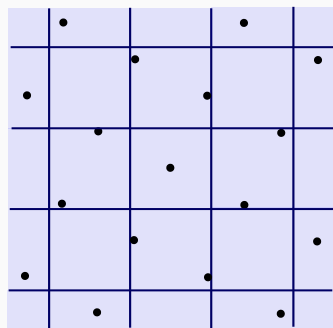
Loss of the bijectivity

Those defects would be caused by the **loss of the bijectivity** of the discretized function:

$$R_\theta := D \circ \mathcal{R}_{\theta|\mathbb{Z}^2}$$



Initial grid



Its discretized rotation
(non-bijective)

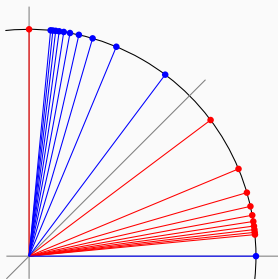
Bijjective rotations on \mathbb{Z}^2

Proposition (Nouvel, Rémila, 2004)

A **discrete rotation** R_θ is **bijjective** iff $\theta \in \mathbb{B}$ with

$$\mathbb{B} = \left\{ \theta \in \mathbb{U} \mid \sin \theta \in \left\{ \pm \frac{2p(p+1)}{2p^2+2p+1}, \pm \frac{2p+1}{2p^2+2p+1} \mid p \in \mathbb{N} \right\} \right\}$$

where \mathbb{U} is the set of all rotation angles.



Finite rotations on \mathbb{Z}^2

Instead of \mathbb{Z}^2 , let us consider the discrete rotation R_θ restricted to the **discretized Euclidean ball of radius ρ** in \mathbb{Z}^2 , i.e.

$$B^\rho = \{\mathbf{q} \in \mathbb{Z}^2 \mid \|\mathbf{q}\|_2 \leq \rho\}$$

defined as $R_\theta^\rho = (R_\theta)|_{B^\rho}$, called a **ρ -rotation**.

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Questions:

- **What are the bijective ρ -rotations?**

Are there rotation angles that are not in \mathbb{B} (i.e. bijective discrete rotations), whose ρ -rotations are bijective?

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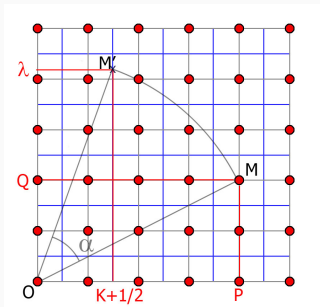
Are there rotation angles that are not in \mathbb{B} (i.e. bijective discrete rotations), whose ρ -rotations are bijective?

- **What is the combinatorial structure of bijective ρ -rotations?**

Hinge angles

Definition (Nouvel, 2006)

An angle α is a **hinge angle** for a discrete point $(p, q) \in \mathbb{Z}^2$ if the result of its rotation by α is a point on the half-grid.



We say that α is a hinge angle if there exist $p, q, k \in \mathbb{Z}$ such that

$$p \cos \alpha - q \sin \alpha = k + \frac{1}{2}$$

or

$$p \sin \alpha + q \cos \alpha = k + \frac{1}{2}$$

Properties of hinge angles

Property (Nouvel, 2006; Thibault, 2009)

- Hinge angles are **dense** in \mathbb{R} .

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- Hinge angles are **not rational rotation angles**.
- Each hinge angle α is **represented by a triple of integers** (p, q, k) such that

$$\cos \alpha = \frac{p\lambda + q(k + \frac{1}{2})}{p^2 + q^2}$$
$$\sin \alpha = \frac{p(k + \frac{1}{2}) - q\lambda}{p^2 + q^2}$$

where $\lambda = \sqrt{p^2 + q^2 - (k + \frac{1}{2})^2}$ and $k < \sqrt{p^2 + q^2}$.

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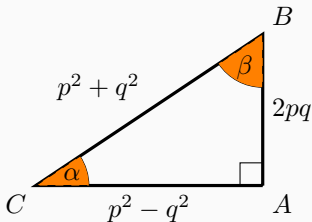
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- The **comparison** between hinge angles (and rational rotation angles) can be made in constant time by using only integers.
- For a discretized ball B^n , we have $8n^3$ hinge angles.

Rational rotations

Definition

Rational rotations are defined by **primitive Pythagorean triples** $(p^2 - q^2, 2pq, p^2 + q^2)$ where $p, q \in \mathbb{Z}$, $p > q$ and $p - q$ is odd, such that they are pairwise coprime. Then $\cos \alpha = \frac{p^2 - q^2}{p^2 + q^2}$ and $\sin \alpha = \frac{2pq}{p^2 + q^2}$.



Rotation space subdivision by hinge angles

Let us consider the (finite) subset of hinge angles h_j^ρ induced by the discretized ball B^ρ :

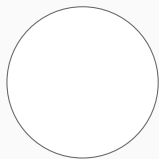
$$\mathbb{H}^\rho = \{h_j^\rho\}_{j=0}^{\sigma^\rho-1}$$

The set \mathbb{U} of all rotation angles is subdivided by

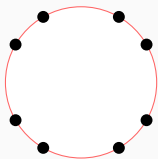
$$\mathbb{S}_\bullet^\rho = \left\{ \{h_j^\rho\} \right\}_{j=0}^{\sigma^\rho-1}$$

$$\mathbb{S}_{\circ-\circ}^\rho = \left\{ \left(h_j^\rho, h_{(j+1)[\sigma^\rho]}^\rho \right) \right\}_{j=0}^{\sigma^\rho-1}$$

$$\mathbb{S}_{\circ-\circ}^\rho = \left\{ \left[h_j^\rho, h_{(j+1)[\sigma^\rho]}^\rho \right] \right\}_{j=0}^{\sigma^\rho-1}$$



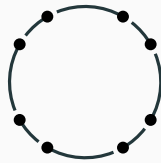
\mathbb{U}



\mathbb{S}_\bullet^ρ



$\mathbb{S}_{\circ-\circ}^\rho$



$\mathbb{S}_{\circ-\circ}^\rho$

Hierarchical structure of finite rotations on \mathbb{Z}^2

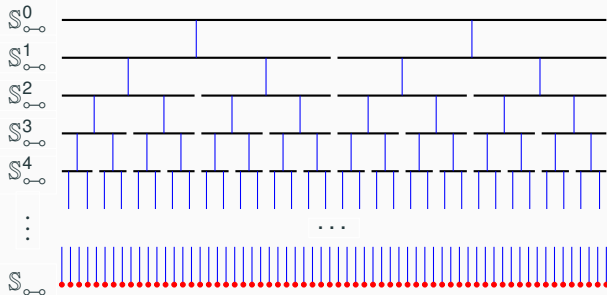
Each segment of $\mathbb{S}_{\circ\circ}^\rho$ (resp. $\mathbb{S}_{\bullet\circ}^\rho$) models a specific ρ -rotation R_θ^ρ .

Let

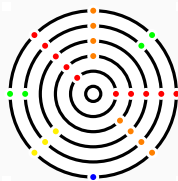
$$\mathbb{S}_{\circ\circ} = \{\{\theta\} \mid \theta \in \mathring{U}\} \quad \mathbb{S}_{\circ\circ}^* = \mathbb{S}_{\circ\circ} \cup \bigcup_{\rho \in \mathbb{N}^{\frac{1}{2}}} \mathbb{S}_{\circ\circ}^\rho$$

$\mathbb{S}_{\circ\circ}$ is in bijection with $\mathring{U} = U \setminus H$ and is a partition of \mathring{U} .

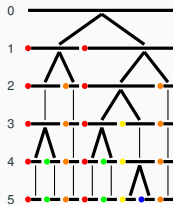
The Hasse diagram $(\mathbb{S}_{\circ\circ}^*, \triangleleft)$ of the ordered set $(\mathbb{S}_{\circ\circ}^*, \subseteq)$ is a **partition tree**.



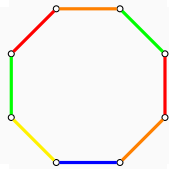
Combinatorial space of finite rotations on \mathbb{Z}^2 : Watershed tree



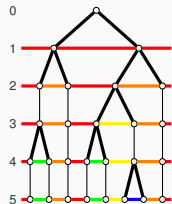
$(\mathbb{S}_{\circ-\circ}^{\rho}, \mathbb{S}_{\bullet}^{\rho})$
 $\rho \in \llbracket 0, 5 \rrbracket$



Partition tree



Saliency graph



Watershed tree

The combinatorial space of all the ρ -rotations for $0 \leq \rho \leq \mu$ can be seen as the watershed tree (Najman, Schmitt, 1996) of the saliency graph $(\mathfrak{G}^{\mu}, \Delta)$.

Bijjective finite rotations on \mathbb{Z}^2

There are two types of **bijjective ρ -rotations** R_θ^ρ :

- B_1 : the restrictions to B^ρ of **bijjective discrete rotations** R_θ
- B_2 : the **injective mappings** from B^ρ to $R_\theta(B^\rho)$ which are not in B_1

Bijjective finite rotations on $\mathbb{Z}^2: B_1$

Definition

A ρ -rotation R_θ^ρ is **bijjective** iff $\theta \in \mathbb{B}^\rho$ with

$$\mathbb{B}^\rho = \left\{ \theta \in \mathbb{U} \mid \sin \theta \in \left\{ \pm \frac{2\rho(\rho+1)}{2\rho^2+2\rho+1}, \pm \frac{2\rho+1}{2\rho^2+2\rho+1} \mid \rho \in \mathbb{N} \wedge \rho \leq 2\rho \right\} \right\}$$

Remark: $\mathbb{B}^\rho \cap \mathbb{H}^\rho = \emptyset$.

Remark: $|\mathbb{B}^\rho| = O(\rho)$ while $|\mathbb{H}^\rho| = O(\rho^3)$.

Remark: The exact comparison of elements of $\mathbb{B}^\rho \cup \mathbb{H}^\rho$ can be made in constant time.

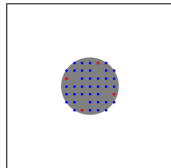
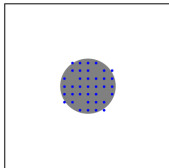
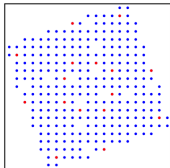
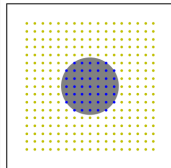
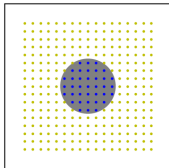
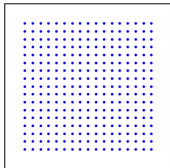
Injective finite rotations on $\mathbb{Z}^2: B_1 \cup B_2$

Definition

A ρ -rotation R_θ^ρ is **injective** iff we have

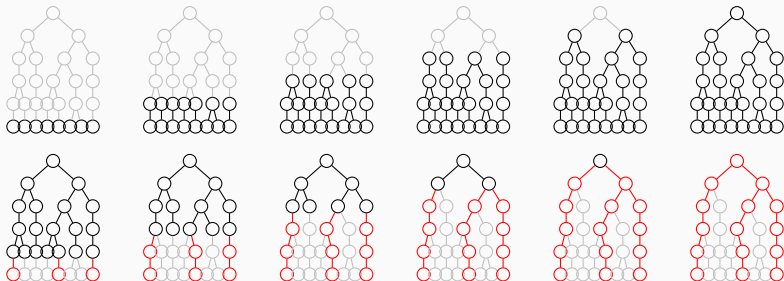
$$\forall \mathbf{p}, \mathbf{q} \in B^\rho, (R_\theta^\rho(\mathbf{p}) = R_\theta^\rho(\mathbf{q})) \implies (\mathbf{p} = \mathbf{q})$$

Example: After ρ -rotation with $\rho = \infty, \sqrt{12}, \sqrt{13}$, each point is either **injective (bijective)** or **non-injective**.



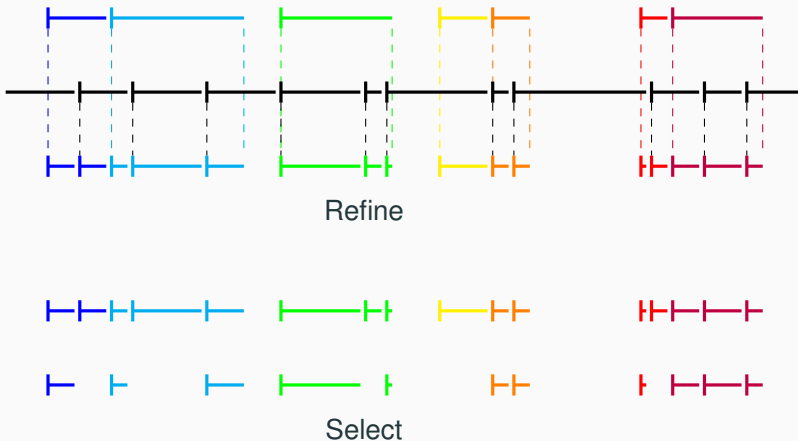
Building the (bijective) finite rotations on \mathbb{Z}^2 (bottom-up)

1. Build the watershed tree of $(\mathbb{S}_{\rightarrow}^{\rho}, \mathbb{S}_{\bullet}^{\rho})$, $\rho \in \llbracket 0, \mu \rrbracket$ with a time cost $O(\mu^3 \log \mu)$
2. Determine the bijective rotations in the tree from the leaves up to the root:
 - B_1 is verified with a time cost $O(\mu^3 \log \mu)$ for the leaves and $O(\mu^3)$ for the other vertices.
 - B_2 cannot be verified efficiently with a bottom-up strategy.



Building the (bijective) finite rotations on \mathbb{Z}^2 (top-down)

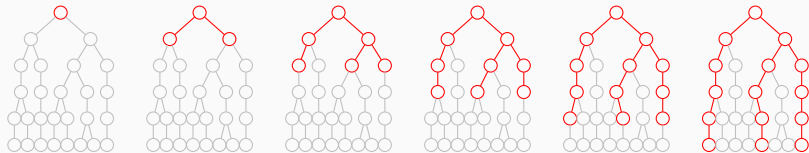
Build only the bijective ρ -rotations, from $\rho = 0$ down to $\rho = \mu$ by following an iterative “refine and select” paradigm.



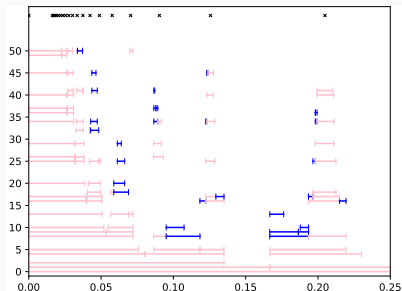
Building the (bijective) finite rotations on \mathbb{Z}^2 (top-down) (cont.)

The time cost for building the combinatorial space depends on the family of considered ρ -rotations:

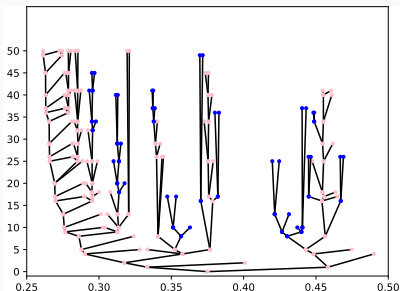
- $O(\mu^3 \log \mu)$ for all the ρ -rotations
- $O(\mu^3)$ for all the bijective ρ -rotations (B_1)
- $O(\mu^{k+2})$ for all the injective ρ -rotations (B_2) with $1 \leq k \leq 3$ (not yet known!)



Hierarchical structure of the bijective rotations on \mathbb{Z}^2



Rotations defined by their intervals

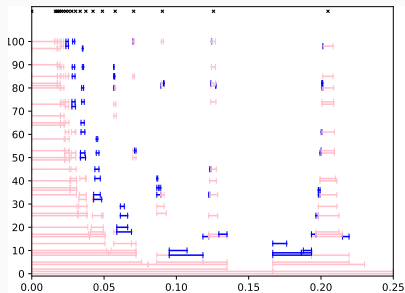


Inclusion links between intervals

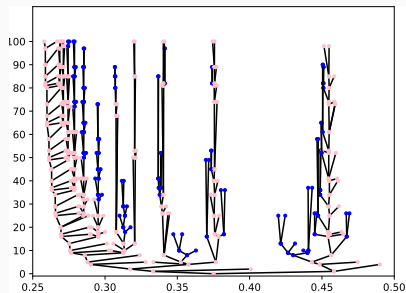
Bijective and injective ρ -rotations for angles in $[0, \frac{\pi}{4}]$ where $\rho \in [0, \sqrt{50}]$:

B_1 in pink; B_2 in blue.

Hierarchical structure of the bijective rotations on \mathbb{Z}^2



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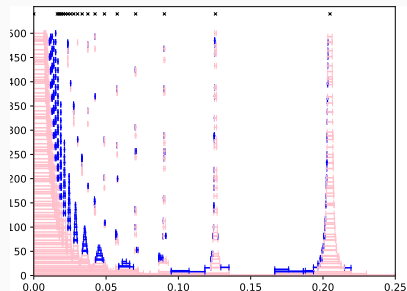


Inclusion links between intervals

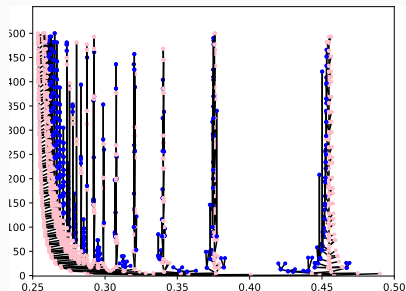
Bijective and injective ρ -rotations for angles in $[0, \frac{\pi}{4}]$ where $\rho \in [0, \sqrt{100}]$:

B_1 in pink; B_2 in blue.

Hierarchical structure of the bijective rotations on \mathbb{Z}^2



Rotations defined by their intervals

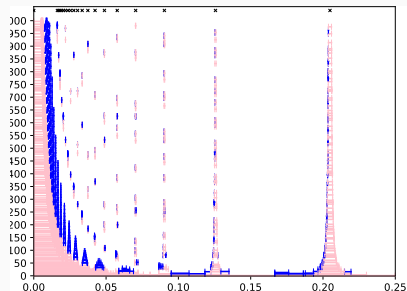


Inclusion links between intervals

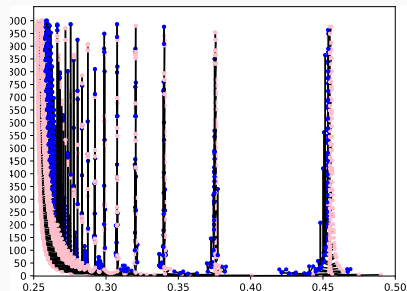
Bijective and injective ρ -rotations for angles in $[0, \frac{\pi}{4}]$ where $\rho \in [0, \sqrt{500}]$:

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Hierarchical structure of the bijective rotations on \mathbb{Z}^2



Rotations defined by their intervals

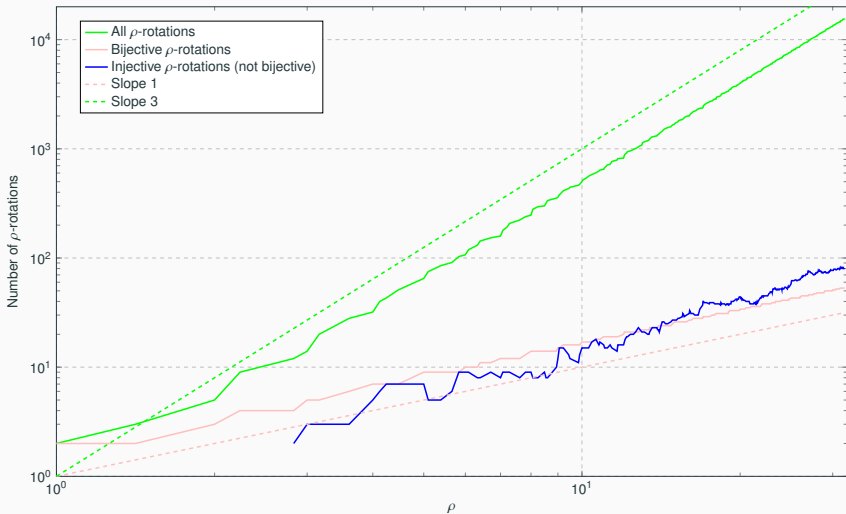


Inclusion links between intervals

Bijective and injective ρ -rotations for angles in $[0, \frac{\pi}{4}]$ where $\rho \in [0, \sqrt{1000}]$:

B_1 in pink; B_2 in blue.

Complexity analysis



Sizes of the families of all ρ -rotations, bijective and injective ones (log-log scale) for $\rho \in [\sqrt{1}, \sqrt{1000}]$.

Complexity analysis



Sizes of the families of bijective and injective ρ -rotations for $\rho \in [\lceil \sqrt{1} \rceil, \lceil \sqrt{1000} \rceil]$.

Conclusion and perspectives

Conclusion

We investigated:

- The hierarchical structure of the finite (and infinite) rotations on \mathbb{Z}^2 based on hinge angles
- The bijective rotations either as the restrictions of bijective rotations on \mathbb{Z}^2 or injective rotations on the Euclidean balls
- Two algorithmic schemes for building the hierarchical structure of finite discrete rotations

Perspectives

- More study on the evolution of injective cases
- Generalization of the top-down algorithm for a non-Euclidean-ball region