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Galton-Watson processes in dynamical environments

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2024

Abstract

We define a model of Galton-Watson processes in dynamical environments where the environment evolves according to a dynamical system (\mathbb{X}, T) . Three behaviours are possible: uniformly subcritical, critical, and uniformly supercritical. We study the extinction probability q in the uniformly supercritical case. In particular, we investigate the regularity of this application as a function of the environment x . In the critical case, we study the set of bad environments N (where the probability of extinction is one), which is T -invariant. We give its Hausdorff dimension in some cases.

Introduction

We define a model of Galton-Watson processes in dynamical environments. These are Galton-Watson processes where the reproduction law changes between generations. We give a discrete-time dynamical system defined on a compact space (\mathbb{X}, T) and a continuous law of reproduction μ . Defined on \mathbb{X} , the reproduction law at generation n is $\mu_{T^n x}$. These processes are special cases of Galton-Watson processes in varying environments introduced by Jagers [Jag74] and studied in particular in [Agr75, AR08, Ker20].

To study this model, we use the probability generating function of the law of reproduction μ , defined on $\mathbb{X} \times [0, 1]$ as:

$$\varphi(x, s) := \sum_{k=0}^{+\infty} \mu_x(k) s^k.$$

The extinction probability q is a function defined on \mathbb{X} . When q is equal to one, there is almost certain extinction of the process, and we say that $x \in \mathbb{X}$ is a bad environment. We will then denote N the set of bad environments. However, since N is a T -invariant set, using the point of view of ergodic theory, we can ask under what condition the measure of N by an ergodic probability T on \mathbb{X} is zero or one. The work of Athreya and Karlin [AK71b], under some integrability assumptions, answers this question. When ν be an T -ergodic probability on \mathbb{X} ,

$$\nu(N) = 0 \text{ if and only if } \mathbb{E}_\nu(\log m(x)) > 0,$$

where $m(x)$ is the expectation of the law μ_x for $x \in \mathbb{X}$.

Three behaviours are possible. The process is:

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- **uniformly subcritical** if the set N is of full measure according to all ergodic probability measures,
- **critical** if the set N is of measure zero for some ergodic probability measure and of full measure for another,
- **uniformly supercritical** if the set N is of measure zero according to all ergodic probability measures.

In the critical case, when (\mathbb{X}, T) is an expanding map of the circle or a two-dimensional Anosov, by adapting the work of Keller and Otani [KO13], we can look at the Hausdorff dimension of the set N (Theorem 1.3.2).

The probability of extinction q is a solution of the functional equation:

$$q(x) = \varphi(x, q(Tx)).$$

We then say that q is an invariant graph. The regularity of invariant graphs of skew product systems has been studied in [Sta97, Sta99, HNW02, WW18] and the regularity of the probability of extinction of Galton-Watson processes in varying environments has been studied in [AR08]. In the supercritical case and assuming some assumptions of integrability, one may wonder whether $x \in \mathbb{X} \mapsto q(x)$ inherits the regularity of $x \in \mathbb{X} \mapsto \mu_x$. Continuity will be well preserved by q (Theorem 1.3.5). An essential tool in the proof is the semi-uniform ergodic theorem [SS00, Theorem 1.9]. We also get bounds on the Hölder regularity of q (Theorem 1.3.8).

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1 Model and results

In this section, we will present the different Galton-Watson models found in the literature, introduce the Galton-Watson model in dynamical environments, present the results of this article, and observe the link between Galton-Watson processes in dynamical environments and skew products.

1.1 Galton-Watson processes: classical case and random environments

We will begin by defining Galton-Watson processes in three well-known cases: the classical case, in variable environments, and in random environments. These models will enable us to define the Galton-Watson processes in dynamical environments.

1.1.1 Galton-Watson processes

A Galton-Watson process is a stochastic process used to describe the evolution of a population. It is a discrete-time process where each individual born in generation n dies at time $n + 1$ and produces a random number of offspring at time $n + 1$ who live, die, and reproduce in the same way independently. In the classical case, the reproduction law governing the number of descendants of an individual (a random variable with support in \mathbb{N}) is the same for all individuals. We will assume that the population at generation zero always equals one. Bienaymé [Bie45] in 1845, then Galton and Watson [WG75] in 1875 introduced this model to describe the evolution of surnames in a population.

Let $\mu \in \mathcal{P}(\mathbb{N})$ be a probability measure on \mathbb{N} . The Galton-Watson process associated with the law of reproduction μ is the sequence of random variables $(Z_n)_{n \in \mathbb{N}}$ defined recursively by:

$$\begin{cases} Z_0 &= 1, \\ Z_{n+1} &= \sum_{k=1}^{Z_n} Y_{n,k} \text{ for all } n \in \mathbb{N}, \end{cases} \quad (1.1)$$

where $(Y_{n,k})_{(n,k) \in \mathbb{N}^2}$ is a family of independent random variables such that $Y_{n,k}$ is distributed according to μ for all $n, k \in \mathbb{N}$. For all $n \in \mathbb{N}$, Z_n is a random variable representing the size of the population at the n th generation. We define the extinction set as:

$$\text{Ext} := \bigcup_{n \geq 0} \{Z_n = 0\}.$$

Let $q := \mathbb{P}(\text{Ext})$ the probability of extinction. To calculate this, we introduce the generating function of a probability measure $\mu \in \mathcal{P}(\mathbb{N})$, defined by:

$$\varphi_\mu(s) = \sum_{k=0}^{+\infty} s^k \mu(k) \text{ for all } s \in [0, 1].$$

φ_μ is analytic on $[0, 1)$, convex (strictly convex if $\mu(\{0, 1\}) < 1$), $\varphi_\mu(0) = \mu(0)$, and $\varphi_\mu(1) = 1$. Additionally, μ has a moment of order $k \in \mathbb{N}^*$ if and only if the k -th derivative of φ_μ has a finite limit at 1. In particular, $m := \varphi'_\mu(1)$ is the expectation of μ . Then, the following theorem gives us the probability of extinction:

Theorem 1.1.1. *The probability of extinction q is the smallest $s \in [0, 1]$ such that $\varphi_\mu(s) = s$. In particular, if $m \in [0, 1]$ and $\mu \neq \delta_1$, then $q = 1$, and if $m \in (1, +\infty]$, then $q < 1$.*

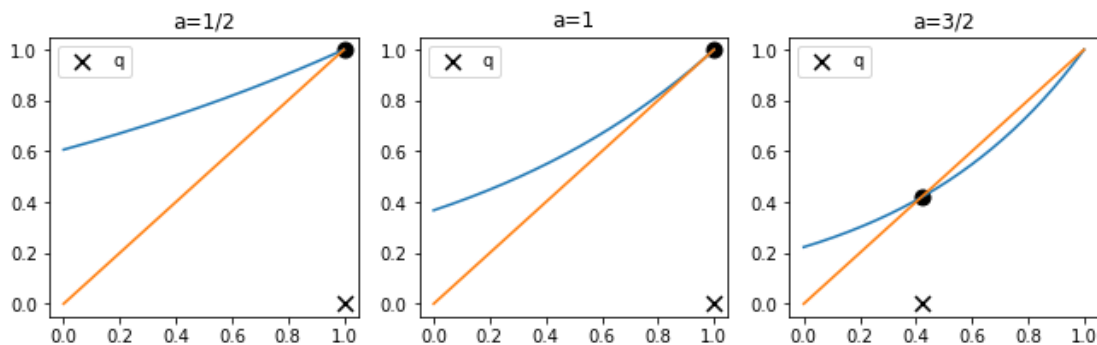


Figure 1: Probability-generating function for the Poisson distribution with parameter a , and the probability of extinction q of the associated Galton-Watson process.

1.1.2 Galton-Watson processes in varying environments

We now assume that the law of reproduction remains the same within a generation but evolves between generations. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of probability measures on \mathbb{N} . The Galton-Watson process in varying environments $(Z_n)_{n \in \mathbb{N}}$ associated with $(\mu_n)_{n \in \mathbb{N}}$ is defined recursively by Equation (1.1) where $(Y_{n,k})_{(n,k) \in \mathbb{N}^2}$ is a family of independent random variables such that $Y_{n,k}$ is distributed according to μ_n for all $n, k \in \mathbb{N}$. Jagers introduced this model [Jag74].

Kersting [Ker20] gives conditions for almost certain extinction or strictly positive survival probability as a function of the first and second moments of the laws of reproduction.

1.1.3 Galton-Watson processes in random environments

We assume that the sequence of environments is random and independent of reproduction. The most studied case is when $(\mu_n)_{n \in \mathbb{N}}$ is i.i.d. according to a probability law ν on $\mathcal{P}(\mathbb{N})$. We then have two levels of independent randomness: environments and offspring.

This model was introduced by Smith and Wilkinson [SW69, Smi68]. They discussed the probability of extinction. Athreya and Karlin [AK71b, AK71a] reformulated and generalized the model, studied the probability of extinction, and gave limit theorems for the Galton-Watson processes. Geiger, Kersting, and Vatutin [AGKV05] have studied the extinction time where extinction is almost certain. More generally, Galton-Watson processes with $(\mu_n)_{n \in \mathbb{N}}$ a Markov process have also been studied [GLLP19, SW71, LPY10].

1.2 Galton-Watson processes: dynamical environments

We define the model of Galton-Watson in dynamical environments from Galton-Watson processes in varying environments. We will then define the probability generating function φ , the probability of extinction q , and the set of bad environments N .

1.2.1 Presentation

In this article, we consider $(\mathbb{X}, \mathcal{B}(\mathbb{X}), T)$ a topological discrete-time dynamical system, where:

- (\mathbb{X}, d) is a compact metric space equipped with its Borel algebra $\mathcal{B}(\mathbb{X})$,
- $T : \mathbb{X} \rightarrow \mathbb{X}$ is a continuous map.

We denote by $\mathcal{P}_T(\mathbb{X})$ the set of T -invariant probabilities on \mathbb{X} endowed with the weak*-topology, and by $\mathcal{E}_T(\mathbb{X})$ the set of T -ergodic probabilities on \mathbb{X} .

We equip $\mathcal{P}(\mathbb{N})$ with the topology generated by ℓ^1 and its associated Borel algebra. Additionally, let:

$$\mu : \begin{cases} \mathbb{X} & \rightarrow \mathcal{P}(\mathbb{N}) \\ x & \mapsto \mu_x \end{cases}$$

and assume (H1).

Hypothesis 1 (H1).

- a) $x \in \mathbb{X} \mapsto \mu_x$ is continuous.
- b) $\mu_x \neq \delta_0$ for all $x \in \mathbb{X}$.

For each $x \in \mathbb{X}$, let $(Z_n(x))_{n \in \mathbb{N}}$ be the Galton-Watson process in varying environments associated with $(\mu_{T^n x})_{n \in \mathbb{N}}$. In other words, $(Z_n(x))_{n \in \mathbb{N}}$ is defined recursively by Equation (1.1) where $(Y_{n,k})_{(n,k) \in \mathbb{N}^2}$ is a family of independent random variables such that for all $(n,k) \in \mathbb{N}^2$, $Y_{n,k}$ distributed according to $\mu_{T^n x}$. The family of random variables $(Z_n(x))_{n \in \mathbb{N}}$ is the Galton-Watson process associated with the discrete-time dynamical system $(\mathbb{X}, \mathcal{B}(\mathbb{X}), T)$, the reproduction law μ , and the point $x \in \mathbb{X}$.

Example 1.2.1. *This example is related to the other Galton-Watson models presented in Subsection 1.1.*

- If $x \in \mathbb{X}$ is a fixed point of the transformation T , then the process $(Z_n(x))_{n \in \mathbb{N}}$ is a classical Galton-Watson process with a reproduction law μ_x .
- If (\mathbb{X}, T) is a Bernoulli shift, μ depends only on the first variable, and $x \in \mathbb{X}$ is chosen according to an invariant distribution, then the process $(Z_n(x))_{n \in \mathbb{N}}$ is a Galton-Watson process in random i.i.d. environments.

Example 1.2.2 will be used several times in the rest of this article to illustrate our results.

Example 1.2.2. *Let:*

- $\mathbb{X} := \mathbb{R}/\mathbb{Z}$ with the Borel algebra $\mathcal{B}(\mathbb{R}/\mathbb{Z})$ and the usual distance on the circle,
- $T := \begin{cases} \mathbb{R}/\mathbb{Z} & \rightarrow \mathbb{R}/\mathbb{Z} \\ x & \mapsto 2x \text{ modulo } 1 \end{cases}$,
- For all $\lambda \in \mathbb{R}$, $\mu_{\lambda,x} := \begin{cases} \mathbb{R}/\mathbb{Z} & \rightarrow \mathcal{P}(\mathbb{N}) \\ x & \mapsto \text{Pois}(e^{\lambda - \cos(2\pi x)}) \end{cases}$.

For each $\lambda \in \mathbb{R}$, this defines a Galton-Watson process in dynamical environments which satisfies (H1). In this example, we will add an index λ to all the objects defined for the study of this model to show the dependency on the parameter λ .

1.2.2 Extinction and generating function

In dynamical environments, the law of reproduction depends on the environment, so the generating functions of the law of reproduction depend on it too. Propositions of this sub-subsection are proven in Section 2.

Definition 1.2.3. *The probability generating function of the law of reproduction μ is defined on $\mathbb{X} \times [0, 1]$ as:*

$$\varphi(x, s) := \sum_{k=0}^{+\infty} \mu_x(k) s^k = \mathbb{E}[s^{Y(x)}],$$

where $Y(x)$ is distributed according to μ_x .

For any $n \in \mathbb{N}$, the probability generating function of the distribution of $Z_n(x)$ is:

$$\varphi^{(n)}(x, s) := \sum_{k=0}^{+\infty} \mu_x^{(n)}(k) s^k = \mathbb{E}[s^{Z_n(x)}],$$

where $x \in \mathbb{X}$, $s \in [0, 1]$, and $\mu_x^{(n)}$ is the distribution of $Z_n(x)$.

For any $x \in \mathbb{X}$ and $s \in [0, 1]$,

$$\begin{aligned} \varphi^{(0)}(x, s) &= s, \\ \varphi^{(1)}(x, s) &= \varphi(x, s). \end{aligned}$$

More generally, we relate the probability generating function of the population at successive times:

Proposition 1.2.4. *For any $x \in \mathbb{X}$, $n, k \in \mathbb{N}$, and $s \in [0, 1]$:*

$$\varphi^{(n+k)}(x, s) = \varphi^{(k)}(x, \varphi^{(n)}(T^k x, s)). \quad (1.2)$$

The probability generating function allows us to define the probability of extinction.

Definition 1.2.5. *We define the probability of extinction q for all $x \in \mathbb{X}$, by:*

$$q(x) := \lim_{n \rightarrow \infty} \nearrow \varphi^{(n)}(x, 0) = \lim_{n \rightarrow \infty} \nearrow \mathbb{P}(Z_n(x) = 0).$$

A function that verifies the functional equation of Proposition 1.2.6 is called an invariant graph of our system. The constant function equal to 1 is a solution to this functional equation.

Proposition 1.2.6. *Assume (H1), for each $x \in \mathbb{X}$:*

$$q(x) = \varphi(x, q(Tx)). \quad (1.3)$$

As in the classical case, the expectation of the laws of reproduction allows us to express the probability of extinction of the process.

Definition 1.2.7. *For all $x \in \mathbb{X}$, let:*

$$m(x) := \partial_s \varphi(x, 1) \in (0, +\infty].$$

For all $x \in \mathbb{X}$, $m(x)$ is the expectation of μ_x .

Definition 1.2.8. *The set of bad environments is $N := \{x \in \mathbb{X} : q(x) = 1\}$.*

The set N is the set of environments where the process $\{Z_n(x), n \in \mathbb{N}\}$ almost surely goes extinct, i.e., where $\mathbb{P}\left(\bigcup_{n \geq 0} \{Z_n(x) = 0\}\right) = 1$.

1.3 Main results

We present the main results of this article on the regularity of the invariant graph q and on the measure and the topology of the set of bad environments N .

1.3.1 The set of bad environments

Measure of the set of bad environments

The set N is difficult to describe precisely (see Lemma 3.2.2). We will begin our study by using the point of view of ergodic theory and, therefore, by calculating the measure of this set by an invariant measure (and even ergodic without losing generality through the ergodic decomposition). The set N is T -invariant (see Proposition 2.3.1), so its measure according to an ergodic measure equals 0 or 1. Corollary 3.1.2 and Theorem 3.1.3 (from Athreya and Karlin [AK71b]) provide criteria to determine whether or not there is almost sure extinction of the process in the case where ν is an ergodic measure.

Theorem 1.3.1. *Assume (H1). Let $\nu \in \mathcal{E}_T(\mathbb{X})$. Then $\nu(N) = 0$ if and only if $\mathbb{E}_\nu(\log m(x)) > 0$.*

Theorem 1.3.1 is proved in Subsection 3.1. We can define:

$$\lambda_{\min} := \inf_{\nu \in \mathcal{P}_T(\mathbb{X})} \mathbb{E}_\nu[\log m],$$

$$\lambda_{\max} := \sup_{\nu \in \mathcal{P}_T(\mathbb{X})} \mathbb{E}_\nu[\log m].$$

By Theorem 1.3.1, the process is uniformly subcritical if $\lambda_{\max} \leq 0$, critical if $\lambda_{\min} \leq 0 < \lambda_{\max}$, and uniformly supercritical if $\lambda_{\min} > 0$.

Dimension of the set of bad environments

Topologically, in the critical case, we only know that the set N is different from the empty set and \mathbb{X} . Theorem 1.3.2 and Theorem 1.3.3 give the Hausdorff dimension of the set N in two cases.

Hypothesis 2 (H2).

1. There exists $\alpha \in (0, 1]$ such that $x \in \mathbb{X} \mapsto \mu_x$ is α -Hölder continuous for the l^1 norm on $\mathcal{P}(\mathbb{N})$.
2. $\mu_x(\{0, 1\}) < 1$ for all $x \in \mathbb{X}$.
3. Critical case: $\lambda_{\min} < 0 < \lambda_{\max}$.
4. The second moment of $(\mu_x)_{x \in \mathbb{X}}$ is uniformly bounded: $\sup_{x \in \mathbb{X}} \sum_{k=0}^{+\infty} k^2 \mu_x(k) < +\infty$.

Theorem 1.3.2. *Assume (H2), that \mathbb{X} is a two-dimensional compact Riemannian manifold, and that T a topologically mixing C^2 -Anosov diffeomorphism. Denote $T_x \mathbb{X} = E^s(x) \oplus E^u(x)$ the splitting decomposition of the tangent fibre over $x \in \mathbb{X}$ into its stable and unstable subspaces.*

$$\dim_H(N) = \begin{cases} 2 & \text{if } \mathbb{E}_{\nu_{SRB}}[\log m] \leq 0, \\ \max_{\nu \in \mathcal{P}_T(\mathbb{X})} \left\{ \frac{h_T(\nu)}{\nu(\log \|dT|_{E^u}\|)} : \int_{\mathbb{X}} \log m(x) d\nu(x) = 0 \right\} + 1 & \text{else.} \end{cases} \quad (1.4)$$

ν_{SRB} is the unique Sinai-Ruelle-Bowen measure of T characterized by the variational principle (see [Bow75, Section 4B]):

$$h_T(\nu_{SRB}) - \nu_{SRB}(\log \|dT|E^u\|) = \sup_{\nu \in \mathcal{P}_T(\mathbb{X})} (h_T(\nu) - \nu(\log \|dT|E^u\|)) = 0.$$

Theorem 1.3.3. *Assume (H2), that $\mathbb{X} = \mathbb{R}/\mathbb{Z}$, and that T is a \mathcal{C}^2 uniformly expanding transformation (there exists $\kappa > 1$ such that for all $x \in \mathbb{R}/\mathbb{Z}$, $|T'(x)| > \kappa$).*

$$\dim_H(N) = \begin{cases} 1 & \text{if } \mathbb{E}_{\nu_{Leb}}[\log m] \leq 0, \\ \max_{\nu \in \mathcal{P}_T(\mathbb{X})} \left\{ \frac{h_T(\nu)}{\nu(\log T')} : \int_{\mathbb{R}/\mathbb{Z}} \log m(x) d\nu(x) = 0 \right\} & \text{else.} \end{cases} \quad (1.5)$$

ν_{Leb} is the unique invariant probability measure absolutely continuous with respect to the Lebesgue measure (see [KS69]). Moreover (see [Bal00, Exercise 2.7]),

$$h_T(\nu_{Leb}) - \nu_{Leb}(\log T') = \sup_{\nu \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})} (h_T(\nu) - \nu(\log T')) = 0.$$

Theorem 1.3.2 and Theorem 1.3.3 are proved in Subsection 3.2

1.3.2 Regularity of the invariant graph

The question arises whether the regularity of the reproduction law μ leads to regularity in the invariant graph q in the uniformly supercritical case.

Remark 1.3.4. *In the case of Example 1.2.2, the dependence on λ of the probability of extinction inherits the regularity in λ of the probability generating function [AR08, Theorem 2.2].*

Continuity in the supercritical case

Theorem 1.3.5 allows us to see how continuity is preserved by q in the supercritical case.

Hypothesis 3 (H3).

- a) $x \in \mathbb{X} \mapsto \mu_x$ is continuous.
- b) $\mu_x \notin \{\delta_0, \delta_1\}$ for all $x \in \mathbb{X}$.
- c) Uniformly supercritical case: $\lambda_{\min} > 0$.
- d) There exists $\tilde{\mu}$ a positive measure on \mathbb{N} , which has a first moment, and which stochastically dominates all the $(\mu_x)_{x \in \mathbb{X}}$: for all $k \in \mathbb{N}$ and $x \in \mathbb{X}$, $\mu_x([k, +\infty)) \leq \tilde{\mu}([k, +\infty))$.

Theorem 1.3.5. *Under (H3), q is continuous.*

Theorem 1.3.5 is proved in Subsection 4.1.

Remark 1.3.6. *The condition “There exists $\varepsilon > 0$ such that the $(1+\varepsilon)$ moment of $(\mu_x)_{x \in \mathbb{X}}$ is uniformly bounded: $\sup_{x \in \mathbb{X}} \sum_{k=0}^{+\infty} k^{1+\varepsilon} \mu_x(k) < +\infty$ ” implies the condition (H3)d).*

Indeed, assume this condition. Let $\tilde{\mu}$ the measure on \mathbb{N} defined by:

$$\tilde{\mu}(k) = \sup_{x \in \mathbb{X}} \mu_x([k, +\infty)) - \sup_{x \in \mathbb{X}} \mu_x([k+1, +\infty)) \geq 0 \quad \forall k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$. By Markov's inequality and because $\sup_{x \in \mathbb{X}} \sum_{k=0}^{+\infty} k^{1+\varepsilon} \mu_x(k) < +\infty$,

$$\begin{aligned} \tilde{\mu}([k, +\infty)) &= \sup_{x \in \mathbb{X}} \mu_x([k, +\infty)) - \lim_{i \rightarrow +\infty} \sup_{x \in \mathbb{X}} \mu_x([i, +\infty)) \\ &= \sup_{x \in \mathbb{X}} \mu_x([k, +\infty)). \end{aligned}$$

Moreover, $\tilde{\mu}$ has a first moment:

$$\begin{aligned} \sum_{k=1}^{+\infty} k \tilde{\mu}(k) &= \sum_{k=1}^{+\infty} \sup_{x \in \mathbb{X}} \mu_x([k, +\infty)) \\ &\leq \sup_{x \in \mathbb{X}} \sum_{i=1}^{+\infty} i^{1+\varepsilon} \mu_x(i) \sum_{k=1}^{+\infty} \frac{1}{k^{1+\varepsilon}} \text{ by Markov's inequality,} \\ &< +\infty. \end{aligned}$$

Hölder continuity in the supercritical case

Theorem 1.3.8 gives an estimate on the Hölder regularity of q .

Definition 1.3.7. *The Lyapunov exponent of the transformation T is defined as:*

$$\lambda_u := \lim_{n \rightarrow \infty} \frac{1}{n} \log(\|T^n\|_{Lip}) \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

The Lyapunov exponent of the invariant graph q in the fibre is defined as:

$$\lambda_F := \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in \mathbb{X}} \log(\partial_s \varphi^{(n)}(x, q(T^n x))). \quad (1.6)$$

The Lyapunov exponent λ_u is well defined (and different from $+\infty$ when T is Lipschitz) by Fekete's subadditive lemma because $\|\cdot\|_{Lip}$ is an algebra norm and λ_F is well defined by Lemma 4.2.2. λ_u controls the speed at which two orbits move apart. In opposition, λ_F allows us to control the speed at which the probability of extinction at the n th generation converges towards the invariant graph q . The Lyapunov exponent of the invariant graph $\mathbf{1}$ in the fibre is λ_{\max} .

Let $\alpha \in (0, 1]$.

Hypothesis 4 (H4(α)).

- a) $\mu_x \notin \{\delta_0, \delta_1\}$ for all $x \in \mathbb{X}$.
- b) For all $\nu \in \mathcal{E}_T(\mathbb{X})$, $\mathbb{E}_\nu[\log m(x)] > 0$ (uniformly supercritical case).
- c) There exists $\tilde{\mu}$ a positive measure on \mathbb{N} which has a first moment such that for all $k \in \mathbb{N}$ and $x \in \mathbb{X}$, $\mu_x([k, +\infty)) \leq \tilde{\mu}([k, +\infty))$.
- d) $x \in \mathbb{X} \mapsto \mu_x$ is in $\mathcal{C}^{0,\alpha}(\mathbb{X})$ for the ℓ^1 norm (α -Hölder continuous).
- e) T is Lipschitz.
- f) $\lambda_u \leq 0$ or $(\lambda_u > 0 \text{ and } \alpha < -\frac{\lambda_F}{\lambda_u})$.
- g) $q(x) > 0$ for all $x \in \mathbb{X}$.

Let $\alpha \in (0, 1]$. Then $(H4(\alpha)) \implies (H3) \implies (H1)$.

Under $(H4(\alpha))$, λ_F is negative (see Lemma 4.2.6). $H4(\alpha)g$ is equivalent to “ $\{x \in \mathbb{X} : \mu_x(0) = 0\}$ does not contain any T -invariant subsets”.

The ratio $-\frac{\lambda_F}{\lambda_u}$ compares the convergence speed towards the invariant graph and the separation of two orbits by T . This ratio controls the Hölder seminorm of $\varphi^{(n)}$. In the supercritical case, under some integrability assumptions, we have the continuity of q , but we need $(H4(\alpha))e$ and f) to have the Hölder continuity.

Theorem 1.3.8. *Let $\alpha \in (0, 1]$. Under $(H4(\alpha))$, q is α -Hölder continuous.*

Corollary 1.3.9 asserts that the convergence of the extinction probability to the invariant graph is pointwise, by definition, but also Hölder.

Corollary 1.3.9. *Let $\alpha \in (0, 1]$. Assume $(H4(\alpha))$. The sequence of functions $(x \mapsto \varphi^{(n)}(x, 0))_{n \in \mathbb{N}}$ converges in the β -Hölder norm to q for all $0 < \beta < \alpha$.*

Theorem 1.3.8 and Corollary 1.3.9 are proven in Sub-subsection 4.2.2.

1.4 Relationship with skew products

The cocycle relation

$$\varphi^{(n)}(x, s) = \varphi(x, \varphi^{(n)}(Tx, s)).$$

verified by the probability generating function does not allow us to define a skew product transformation. However, assuming that the transformation T is invertible, we can reduce our model to a skew product model. We define Ψ , by:

$$\Psi : \begin{cases} \mathbb{X} \times [0, 1] & \rightarrow \mathbb{X} \times [0, 1] \\ (x, s) & \mapsto (T^{-1}x, \varphi(x, s)) \end{cases} .$$

Then for all $x \in \mathbb{X}$, $n \in \mathbb{N}$, and $s \in [0, 1]$,

$$\Psi^n(x, s) = (T^{-n}x, \varphi^{(n)}(T^{-(n-1)}x, s)). \quad (1.7)$$

Equation (1.7) provides a link with the model of Keller and Otani in [KO13]. In their article, they consider Θ a two-dimensional compact Riemannian manifold, $T : \Theta \mapsto \Theta$ a topologically mixing \mathcal{C}^2 -Anosov diffeomorphism and $g : \Theta \rightarrow (0, \infty)$ a Hölder continuous function. Moreover, for all $t \in \mathbb{R}$, they define a skew product transformation:

$$T_t : \begin{cases} \Theta \times \mathbb{R}^+ & \rightarrow \Theta \times \mathbb{R}^+ \\ (\theta, x) & \mapsto (T\theta, f_t(\theta, x)) \end{cases} .$$

And they define a fibre function,

$$f_t : \begin{cases} \Theta \times \mathbb{R}^+ & \rightarrow \mathbb{R}^+ \\ (\theta, x) & \mapsto e^{-t}g(\theta)h(x) \end{cases} ,$$

where $h \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$ is strictly concave with $h(0) = 0$, $h'(0) = 1$, and $\lim_{x \rightarrow \infty} \frac{h(x)}{x} = 0$. For $n \geq 2$, they define:

$$f_t^n : \begin{cases} \Theta \times \mathbb{R}^+ & \rightarrow \mathbb{R}^+ \\ (\theta, x) & \mapsto f_t(T^{n-1}\theta, f_t^{n-1}(\theta, x)) \end{cases} .$$

For all $t \in \mathbb{R}$, there exists $M_t > 0$ such that $f_t(\theta, M_t) < M_t$ for all $\theta \in \Theta$. Moreover, they define for $\theta \in \mathbb{R}$,

$$\varphi_t(\theta) := \lim_{n \rightarrow \infty} f_t^n(T^{-n}\theta, M_t).$$

Finally; they define $N_t = \{\theta \in \Theta : \varphi_t(\theta) = 0\}$ for $t \in \mathbb{R}$.

We fix the parameter t in [KO13] to provide a link to our article. We can then give the following partial correspondence table:

[KO13]	this article
Θ, T^{-1}	\mathbb{X}, T
f_t	φ
f_t^n	$\varphi^{(n)}$
N_t	N
g_t	m
φ_t	q

The link between φ_t and q comes in particular from the fact that they are maximal (respectively minimal) solutions of the respective equations $f_t(\theta, \varphi(\theta)) = \varphi(T\theta)$ and $q(x) = \varphi(x, q(Tx))$. This link gives the motivation to Theorem 1.3.2. Our model doesn't need to assume that T is invertible. We consider the dynamics of T , contrary to Keller and Otani, who study the dynamics of T^{-1} . In our model, see Theorem 1.3.3, we only need an unstable direction. Whereas the Keller and Otani model has a stable and an unstable direction, only the stable direction of T is important for the dynamic (because the stable direction of T is the unstable direction of T^{-1}).

Example 1.4.1 (Example 1.2.2). *The model of Example 1.2.2 depends on a parameter λ , which can be identified with the parameter $-t$ in [KO13]. We have the same property in both articles: $\log g_t = \log g - t$ and $\log m_\lambda = \log m + \lambda$. In both models, we try to observe the bifurcation of the Hausdorff dimension of N_t (respectively N_λ) in t (respectively in λ), see Example 3.2.7.*

2 Elementary properties

We prove some elementary results on probability generating functions in the case of the model, on the invariant graph, and on the set of bad environments (defined in Subsubsection 1.2.2).

2.1 Probability generating functions

We study the regularity properties of generating functions and the links between generating functions across generations.

Proof of Proposition 1.2.4. We will show that for all $x \in \mathbb{X}$, $n \in \mathbb{N}$, and $s \in [0, 1]$,

$$\varphi^{(n+1)}(x, s) = \varphi^{(n)}(x, \varphi(T^n x, s)).$$

Let $x \in \mathbb{X}$, $n \in \mathbb{N}$, and $s \in [0, 1]$. Using that $\{Y_{n,k}, k \in \mathbb{N}\}$ is i.i.d. and independent of

$Z_n(x)$,

$$\begin{aligned}
\varphi^{(n+1)}(x, s) &= \mathbb{E}[s^{Z_{n+1}(x)}] \\
&= \mathbb{E}[s^{\sum_{k=1}^{Z_n(x)} Y_{n,k}}] \\
&= \mathbb{E}[\mathbb{E}[s^{\sum_{k=1}^{Z_n(x)} Y_{n,k}} | Z_n(x)]] \\
&= \mathbb{E}[\mathbb{E}[s^{Y_{n,1}}]^{Z_n(x)}] \\
&= \mathbb{E}[\varphi(T^n x, s)^{Z_n(x)}] \\
&= \varphi^{(n)}(x, \varphi(T^n x, s)).
\end{aligned}$$

The proposition follows by induction. □

Corollary 2.1.1. *Under (H1), for all $n \in \mathbb{N}$, $\varphi^{(n)}$ is continuous on $\mathbb{X} \times [0, 1]$.*

Proof. Let $x, y \in \mathbb{X}$ and $s, t \in [0, 1]$. Then,

$$\begin{aligned}
|\varphi(x, s) - \varphi(y, t)| &\leq |\varphi(x, s) - \varphi(x, t)| + |\varphi(x, t) - \varphi(y, t)| \\
&\leq |\varphi(x, s) - \varphi(x, t)| + \sum_{k=0}^{+\infty} |\mu_x(k) - \mu_y(k)| t^k \\
&\leq |\varphi(x, s) - \varphi(x, t)| + \sum_{k=0}^{+\infty} |\mu_x(k) - \mu_y(k)| \text{ since } t \in [0, 1] \\
&= |\varphi(x, s) - \varphi(x, t)| + \|\mu_x - \mu_y\|_1.
\end{aligned}$$

Continuity of φ follows from the continuity of $x \in \mathbb{X} \mapsto \mu_x$ and $s \in [0, 1] \mapsto \varphi(x, s)$ for all $x \in \mathbb{X}$. For $n \in \mathbb{N}$, continuity of $\varphi^{(n)}$ follows by induction and Proposition 1.2.4. □

For any $x \in \mathbb{X}$, $s \mapsto \varphi(x, s)$ is \mathcal{C}^∞ on $[0, 1]$. By monotonicity, its derivative and second derivative are well-defined at one if we admit the value $+\infty$. We denote by $\partial_s \varphi$ (respectively $\partial_s^2 \varphi$) the derivative (respectively the second derivative) of the function φ with respect to the second variable. We use the convention $\log(0) = -\infty$ and $\log(+\infty) = +\infty$.

Lemma 2.1.2. *For all $x \in \mathbb{X}$, $n, k \in \mathbb{N}$, and $s \in [0, 1]$:*

$$\log \partial_s \varphi^{(n+k)}(x, s) = \log \partial_s \varphi^{(k)}(T^n x, s) + \log \partial_s \varphi^{(n)}(x, \varphi^{(k)}(T^n x, s)).$$

Proof. We differentiate Equation (1.2) with respect to the second variable. □

Corollary 2.1.3. *For all $x \in \mathbb{X}$, $s \in [0, 1]$, and $n \in \mathbb{N}$,*

$$\begin{aligned}
\log \partial_s \varphi^{(n)}(x, s) &= \sum_{k=0}^{n-1} \log \partial_s \varphi(T^k x, \varphi^{(n-1-k)}(T^{k+1} x, s)) \\
&= \sum_{k=0}^{n-1} \log \partial_s \varphi(T^{n-k-1} x, \varphi^{(k)}(T^{n-k} x, s)).
\end{aligned} \tag{2.1}$$

Proof. We prove the second equality by induction on $n \in \mathbb{N}$. For $n = 0$, the result holds for all $x \in \mathbb{X}$ and $s \in [0, 1]$.

Assume that the result is true at rank $n \in \mathbb{N}$ for all $x \in \mathbb{X}$ and $s \in [0, 1]$. Let $x \in \mathbb{X}$ and $s \in [0, 1]$.

$$\begin{aligned} \log \partial_s \varphi^{(n+1)}(x, s) &= \log \partial_s \varphi^{(n)}(Tx, s) + \log \partial_s \varphi(x, \varphi^{(n)}(Tx, s)) \text{ by Lemma 2.1.2,} \\ &= \sum_{k=0}^{n-1} \log \partial_s \varphi(T^{n-k}x, \varphi^{(k)}(T^{n-k+1}x, s)) + \log \partial_s \varphi(x, \varphi^{(n)}(Tx, s)) \\ &= \sum_{k=0}^n \log \partial_s \varphi(T^{n+1-k-1}x, \varphi^{(k)}(T^{n+1-k}x, s)). \end{aligned}$$

This completes the induction. The first equality is obtained immediately by changing the index in the sum. \square

Proposition 2.1.4 gives a sufficient condition for $\partial_s \varphi$ to be continuous.

Proposition 2.1.4. *Assume (H1) and that there exists $\tilde{\mu}$ a positive measure on \mathbb{N} which has a first moment such that for all $k \in \mathbb{N}$ and $x \in \mathbb{X}$,*

$$\mu_x([k, +\infty)) \leq \tilde{\mu}([k, +\infty)).$$

Then $\partial_s \varphi$ is continuous on $\mathbb{X} \times [0, 1]$. In particular, m is continuous on \mathbb{X} .

Proof. Let $x, y \in \mathbb{X}$ and $s, t \in [0, 1]$. Then,

$$\begin{aligned} |\partial_s \varphi(x, s) - \partial_s \varphi(y, t)| &\leq |\partial_s \varphi(x, s) - \partial_s \varphi(x, t)| + |\partial_s \varphi(x, t) - \partial_s \varphi(y, t)| \\ &\leq |\partial_s \varphi(x, s) - \partial_s \varphi(x, t)| + \sum_{k=0}^{+\infty} k |\mu_x(k) - \mu_y(k)| t^{k-1} \\ &\leq |\partial_s \varphi(x, s) - \partial_s \varphi(x, t)| + \sum_{k=0}^{+\infty} k |\mu_x(k) - \mu_y(k)|. \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{k=0}^{+\infty} k |\mu_x(k) - \mu_y(k)| &= \sum_{k=0}^{+\infty} \sum_{i=k+1}^{+\infty} |\mu_x(i) - \mu_y(i)| \\ &\leq \sum_{k=0}^{+\infty} [\mu_x([k+1, +\infty)) + \mu_y([k+1, +\infty))] \\ &\leq 2 \sum_{k=0}^{+\infty} \tilde{\mu}([k+1, +\infty)) \\ &< +\infty. \end{aligned}$$

By the dominated convergence theorem, continuity follows from the continuity of $x \in \mathbb{X} \mapsto \mu_x$ and for all $x \in \mathbb{X}$, $s \in [0, 1] \mapsto \partial_s \varphi(x, s)$. \square

Example 2.1.5 (Example 1.2.2). *We can explicitly compute the probability generating function. Let $\lambda \in \mathbb{R}$. Then, for all $x \in \mathbb{R}/\mathbb{Z}$ and $s \in [0, 1]$,*

$$\begin{aligned} \varphi_\lambda(x, s) &= \exp\left(e^{\lambda - \cos 2\pi x} (s - 1)\right), \\ \partial_s \varphi_\lambda(x, s) &= e^{\lambda - \cos 2\pi x} \varphi_\lambda(x, s), \\ \log m_\lambda(x) &= \lambda - \cos 2\pi x. \end{aligned} \tag{2.2}$$

The product form of the derivative (2.2) is specific to Poisson distributions and provides a property similar to [KO13] (see Example 1.4.1).

2.2 The invariant graph

We study the probability of extinction q and justify the term invariant graph. Lemma 2.2.1 is a stronger version of Proposition 1.2.6.

Lemma 2.2.1. *Assume (H1).*

- $q(x) = \varphi^{(k)}(x, q(T^k x))$ for all $x \in \mathbb{X}$ and $k \in \mathbb{N}$,
- q is lower semi-continuous.

Proof.

- Let $x \in \mathbb{X}$ and $k \in \mathbb{N}$.

$$\begin{aligned} q(x) &= \lim_{n \rightarrow \infty} \varphi^{(n+k)}(x, 0) \\ &= \lim_{n \rightarrow \infty} \varphi^{(k)}(x, \varphi^{(n)}(T^k x, 0)) \text{ by Corollary 1.2.4} \\ &= \varphi^{(k)}(x, \lim_{n \rightarrow \infty} \varphi^{(n)}(T^k x, 0)) \text{ by continuity of } \varphi^{(k)} \\ &= \varphi^{(k)}(x, q(T^k x)). \end{aligned}$$

- Since $q(x) = \lim_{n \rightarrow \infty} \nearrow \varphi^{(n)}(x, 0) = \sup_{n \in \mathbb{N}} \varphi^{(n)}(x, 0)$, q is lower semi-continuous as a supremum of continuous functions. \square

We measure the convergence speed towards the invariant graph with the function F .

Definition 2.2.2. *We define F for all $x \in \mathbb{X}$ by:*

$$F(x) := \log \partial_s \varphi(x, q(Tx)).$$

Lemma 2.2.3 measures the convergence speed to the invariant graph as a Birkhoff sum.

Lemma 2.2.3. *Assume (H1), for all $x \in \mathbb{X}$ and $n \in \mathbb{N}$:*

$$\log \partial_s \varphi^{(n)}(x, q(T^n x)) = \sum_{k=0}^{n-1} F(T^k x). \quad (2.3)$$

Proof. Let $x \in \mathbb{X}$ and $n \in \mathbb{N}$. Apply Corollary 2.1.3 with $s = q(T^n x)$. We obtain:

$$\begin{aligned} \log \partial_s \varphi^{(n)}(x, q(T^n x)) &= \sum_{k=0}^{n-1} \log \partial_s \varphi(T^k x, \varphi^{(n-1-k)}(T^{k+1} x, q(T^n x))) \\ &= \sum_{k=0}^{n-1} \log \partial_s \varphi(T^k x, q(T^{k+1} x)) \text{ by Lemma 2.2.1.} \quad \square \end{aligned}$$

Example 2.2.4 (Example 1.2.2). *By Lemma 2.2.1, q_λ is the smallest solution of the functional equation:*

$$q_\lambda(x) = \exp(e^{\lambda - \cos 2\pi x} (q_\lambda(Tx) - 1)) \text{ for all } x \in \mathbb{R}/\mathbb{Z}.$$

Remark 2.2.5. *The scale of the first graph is different from the others.*

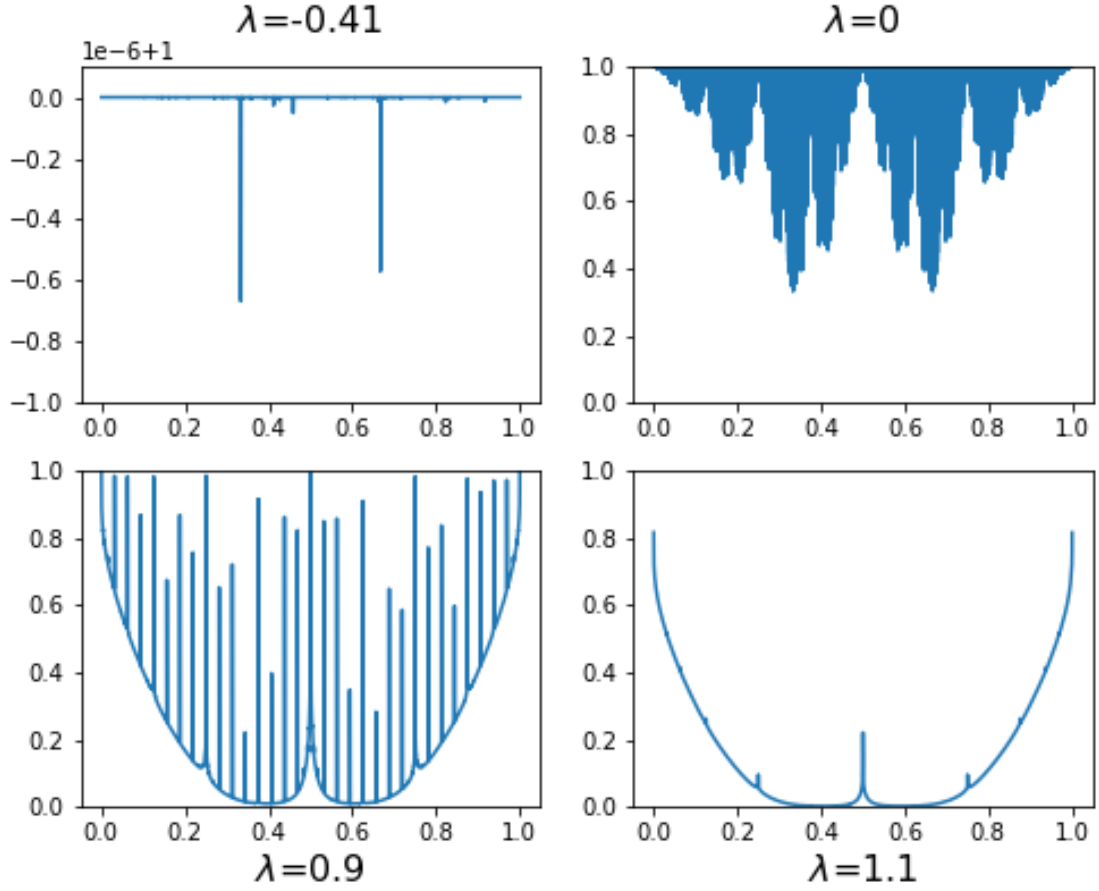


Figure 2: Plot of the probability of extinction q_λ for some $\lambda \in \mathbb{R}$.

2.3 The set of bad environments

We study the set of bad environments, a T invariant and G_δ set.

Proposition 2.3.1. *Assume (H1). Then $N = T^{-1}N$. In particular if $\nu \in \mathcal{E}_T(\mathbb{X})$, then $\nu(N) \in \{0, 1\}$.*

Proof. Let $\nu \in \mathcal{E}_T(\mathbb{X})$.

$$\begin{aligned}
 N &= \{x \in \mathbb{X} : q(x) = 1\}. \\
 &= \{x \in \mathbb{X} : \varphi(x, q(Tx)) = 1\} \text{ by Lemma 2.2.1.} \\
 &= \{x \in \mathbb{X} : q(Tx) = 1\} \text{ because for all } x \in \mathbb{X}, \mu_x \neq \delta_0 \\
 &= T^{-1}N.
 \end{aligned}$$

As $\nu \in \mathcal{E}_T(\mathbb{X})$ and N is T -invariant, $\nu(N) \in \{0, 1\}$. □

Lower semi-continuity of q allows us to show that N is a G_δ set.

Lemma 2.3.2. *Assume (H1). Then N is a G_δ set.*

Proof. For any $k \in \mathbb{N}$, the set $\{x \in \mathbb{X} : q(x) > 1 - \frac{1}{k}\}$ is open since q is lower semi-

continuous. Moreover,

$$\begin{aligned} N &= \{x \in \mathbb{X} : q(x) = 1\} \\ &= \bigcap_{k \in \mathbb{N}} \left\{ x \in \mathbb{X} : q(x) > 1 - \frac{1}{k} \right\} \text{ since } q \leq 1. \end{aligned} \quad \square$$

Example 2.3.3 (Example 1.2.2). *Assume (H1) and that T is a expanding map of \mathbb{R}/\mathbb{Z} , then N is either empty or N is a dense G_δ set. Indeed, if N is non-empty, then there exists $x \in \mathbb{R}/\mathbb{Z}$ such that $x \in N_\lambda$. By Proposition 2.3.1, $T^{-1}N \subset N$, so*

$$\bigcup_{n \in \mathbb{N}} T^{-n}(x) \subset \bigcup_{n \in \mathbb{N}} T^{-n}N \subset N.$$

And $\bigcup_{n \in \mathbb{N}} T^{-n}(x)$ is dense in \mathbb{R}/\mathbb{Z} as T is expanding.

In particular, in the case of Example 1.2.2, for all $\lambda \in \mathbb{R}$, N_λ is either empty or N_λ is a dense G_δ set.

3 The set of bad environments

In this section, we study the metric and topological properties of the set of bad environments.

3.1 Measure of the set of bad environments

It is sometimes difficult to say anything about the process $(Z_n(x))_{n \in \mathbb{N}}$ for all $x \in \mathbb{X}$. We can then use the point of view of ergodic theory by choosing x according to an invariant (and even ergodic without losing generality) measure. In this case, the dynamic environment $(\mu_{T^n x})_{n \in \mathbb{N}}$ is a stationary and ergodic process that allows us to use Athreya and Karlin's results [AK71b]. They obtained a criterion to determine whether the measure of the set N is zero or full when considering an ergodic measure.

We consider $\nu \in \mathcal{E}_T(\mathbb{X})$. As seen in Proposition 2.3.1, $\nu(N) \in \{0, 1\}$. We must determine whether $\nu(N)$ is 0 or 1.

Theorem 3.1.1. [AK71b, Theorem 1] *If $\nu(N) = 0$ and $\mathbb{E}_\nu[\log m(x)]^+ < \infty$, then:*

- $\mathbb{E}_\nu|\log m(x)| < \infty$ and $\mathbb{E}_\nu[\log m(x)] > 0$.
- $\mathbb{E}_\nu \left| \log \frac{1-q(x)}{1-q(Tx)} \right| < \infty$ and $\mathbb{E}_\nu \left[\log \frac{1-q(x)}{1-q(Tx)} \right] = 0$.

By contrapositive, we deduce Corollary 3.1.2:

Corollary 3.1.2. [AK71b, Corollary 1] *If $\mathbb{E}_\nu[\log m(x)]^+ < \infty$ and $\mathbb{E}_\nu[\log m(x)]^+ \leq \mathbb{E}_\nu[\log m(x)]^- \leq \infty$, then $\nu(N) = 1$.*

Moreover, we have the following reciprocal:

Theorem 3.1.3. [AK71b, Theorem 3] *If $\mathbb{E}_\nu[-\log(1 - \varphi(x, 0))] < \infty$ and $\mathbb{E}_\nu[\log m(x)]^- < \mathbb{E}_\nu[\log m(x)]^+ \leq \infty$, then $\nu(N) = 0$.*

Proof of Theorem 1.3.1. Corollary 3.1.2 and Theorem 3.1.3 (from the results of Athreya and Karlin [AK71b]) provide criteria to determine whether or not there is almost sure extinction in the case where ν is an ergodic measure. Specifically, $\nu(N) = 0$ if and only if $\mathbb{E}_\nu[\log m(x)] > 0$. Indeed, the two integrability hypotheses are verified here:

- $\mathbb{E}_\nu[-\log(1 - \varphi(x, 0))] < \infty$ because for all $x \in \mathbb{X}$, $\varphi(x, 0) = \mu_x(0) < 1$ and μ is continuous on the compact \mathbb{X} .
- $\mathbb{E}_\nu[\log m(x)]^- < \infty$ or $\mathbb{E}_\nu[\log m(x)]^+ < \infty$ because for all $x \in \mathbb{X}$, $m(x) \geq 1 - \mu_x(0) > 0$ and μ is continuous on the compact \mathbb{X} . \square

3.2 Dimension of the set of bad environments

In Theorem 1.3.2 and Theorem 1.3.3, we give the Hausdorff dimension of the set of bad environments in the critical case on two types of systems.

Definition 3.2.1. For each $x \in \mathbb{X}$,

$$\Gamma(x) := \liminf_{n \in \mathbb{N}^*} \frac{1}{n} \log \partial_s \varphi^{(n)}(x, 1) = \liminf_{n \in \mathbb{N}^*} \frac{1}{n} \sum_{k=0}^{n-1} \log m(T^k x).$$

In Lemma 3.2.2, we observe that $\Gamma(x)$ is the right quantity to know if x is in the set of bad environments.

Lemma 3.2.2. Assume (H1) and that $C := \sup_{x \in \mathbb{X}} \sum_{k=0}^{+\infty} k^2 \mu_x(k) < +\infty$. For each $x \in \mathbb{X}$:

- if $x \in N$, then $\Gamma(x) \leq 0$,
- if $x \notin N$, then $\Gamma(x) \geq 0$.

Proof.

- Let $x \in \mathbb{X}$ such that $\Gamma(x) > 0$. As for any $n \in \mathbb{N}$, $m(T^n x) \leq C$ and $\partial_s^2 \varphi(T^n x, 1) \leq C$. Since μ is continuous, and $\mu_y(0) < 1$ for every $y \in \mathbb{X}$ (which is compact), there exists $c > 0$ such that for every $n \in \mathbb{N}$, $m(T^n x) > c$. Since $\Gamma(x) > 0$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that for every $n \geq N$, $\partial_s \varphi^{(n)}(x, 1) \geq e^{\delta n}$.

Thus, the sequence given by:

$$\frac{1}{\partial_s \varphi^{(n-1)}(x, 1)} \left(\frac{\partial_s^2 \varphi(T^n x, 1)}{m(T^n x)^2} + \frac{1 - m(T^n x)}{m(T^n x)} \right)$$

is the general term of a convergent series. Hence, the assumption [Ker20, A] and the condition [Ker20, vii Theorem 1] are satisfied, so $q(x) < 1$, i.e. $x \notin N$.

- Let $x \in \mathbb{X}$ such that $\Gamma(x) < 0$. There exists an extracted sequence $(n_k)_{k \in \mathbb{N}}$ and $\delta > 0$ such that for all $k \in \mathbb{N}$,

$$\partial_s \varphi^{(n_k)}(x, 1) \leq e^{-\delta n_k}.$$

Then for each $k \in \mathbb{N}$,

$$\begin{aligned} q(x) &\geq \varphi^{(n_k)}(x, 0) \\ &= 1 - \int_0^1 \partial_s \varphi^{(n_k)}(x, t) dt \\ &\geq 1 - \sup_{t \in [0, 1]} \partial_s \varphi^{(n_k)}(x, t) \\ &\geq 1 - e^{-\delta n_k} \xrightarrow{k \rightarrow +\infty} 1. \end{aligned}$$

Thus, $q(x) = 1$, i.e. $x \in N$. \square

By Proposition 2.1.4, under the assumptions of Lemma 3.2.3, m is continuous. But, if μ is Hölder continuous, we need a stronger integrability hypothesis that m inherits the Hölder regularity of μ .

Lemma 3.2.3. *Assume (H1), that there exists $\varepsilon > 0$ such that $\sup_{x \in \mathbb{X}} \sum_{k=0}^{+\infty} k^{1+\varepsilon} \mu_x(k) < +\infty$, and that $x \in \mathbb{X} \mapsto \mu_x$ is α -Hölder continuous for the ℓ^1 norm (for $\alpha \in (0, 1]$). Then m is $\alpha \frac{\varepsilon}{1+\varepsilon}$ Hölder continuous.*

Proof. Let $x, y \in \mathbb{X}$, by Hölder's inequality,

$$\begin{aligned} |m(x) - m(y)| &\leq \sum_{k=0}^{+\infty} k |\mu_x(k) - \mu_y(k)| \\ &\leq \left(\sum_{k=0}^{+\infty} k^{1+\varepsilon} |\mu_x(k) - \mu_y(k)| \right)^{1/(1+\varepsilon)} \left(\sum_{k=0}^{+\infty} |\mu_x(k) - \mu_y(k)| \right)^{\varepsilon/(1+\varepsilon)} \\ &\leq C^{1/(1+\varepsilon)} \|\mu_x - \mu_y\|_1^{\varepsilon/(1+\varepsilon)} \\ &\leq C^{1/(1+\varepsilon)} |\mu|_{\alpha}^{\varepsilon/(1+\varepsilon)} d(x, y)^{\alpha\varepsilon/(1+\varepsilon)}. \quad \square \end{aligned}$$

Proof of Theorem 1.3.2. In the case where \mathbb{X} is a two-dimensional compact Riemannian manifold, the proof of Theorem 1.3.2 is the same as the proof of [KO13, Theorem 2] (which uses results from [Bar08, Section 10 and 12]) using the correspondence table from Section 1.4, Lemma 3.2.2 and the Hölder continuity of m (Lemma 3.2.3). The results of [Bar08] were originally proved in [BS01, BSS02, BV06]. However, we study the dynamic of T , unlike Keller and Otani, who study the dynamics of T^{-1} . We will therefore use $u := \log \|dT|E^u\|$ and for all $\lambda \in (\lambda_{\min}, \lambda_{\max})$,

$$D(\lambda) = \max \left\{ \frac{h_T(\nu)}{\nu(\log \|dT|E^u\|)} : \nu \in \mathcal{P}_T(\mathbb{X}) \text{ and } \int_{\mathbb{X}} \log m(x) d\nu(x) = \lambda \right\}.$$

If T is a \mathcal{C}^2 uniformly expanding transformation of \mathbb{R}/\mathbb{Z} , the system is conjugated to a one-sided full shift of finite type (using a Markov partition). The entropy function is upper semi-continuous, as T is expansive. Thus, the proofs of [KO13, Theorem 2] and [Bar08] can be adapted by considering only the dilating direction. We will therefore use for all $\lambda \in (\lambda_{\min}, \lambda_{\max})$,

$$D(\lambda) = \max \left\{ \frac{h_T(\nu)}{\nu(\log T')} : \nu \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z}) \text{ and } \int_{\mathbb{R}/\mathbb{Z}} \log m(x) d\nu(x) = \lambda \right\}.$$

The maximum of the function D is reached by:

$$\lambda = \int_{\mathbb{R}/\mathbb{Z}} \log m(x) d\nu_{Leb}(x)$$

and is equal to one where ν_{Leb} is the unique invariant probability measure absolutely continuous with respect to the Lebesgue measure (see [KS69]). To adapt [KO13, Lemma 2], unlike Keller and Otani, removing a dimension is unnecessary because our model has only an unstable dimension. Results of [Bar08, Chapter 10] (10.1.4, 10.3.1, and 10.1.6) and [Bar08, Example 7.2.5] are applicable to our model. This gives us Lemma 3.2.4 and Lemma 3.2.5. For $\lambda \in (\lambda_{\min}, \lambda_{\max})$, let:

$$S_{\lambda} = \left\{ x \in \mathbb{R}/\mathbb{Z} : \frac{1}{n} \sum_{i=0}^{n-1} \log m(T^i x) \xrightarrow{n \rightarrow +\infty} \lambda \right\}.$$

Lemma 3.2.4. [KO13, Lemma 2] For $\lambda \in (\lambda_{\min}, \lambda_{\max})$, $\dim_H(S_\lambda) = D(\lambda)$.

Moreover, the function D is analytic, but we only need the continuity of D to prove Theorem 1.3.2.

Lemma 3.2.5. [KO13, Lemma 3] The function $\lambda \in (\lambda_{\min}, \lambda_{\max}) \mapsto D(\lambda)$ is continuous.

Lemma 3.2.4 and Lemma 3.2.5 allow us to prove Theorem 1.3.2. By Lemma 3.2.2, if $\lambda \in (\lambda_{\min}, 0)$, then $S_\lambda \subset N$. Let $\lambda_c := \mathbb{E}_{\nu_{Leb}}[\log m]$.

- Case $\lambda_c \leq 0$. Let $\lambda \in (\lambda_{\min}, \lambda_c)$. So,

$$\dim_H(S_\lambda) \leq \dim_H(N) \leq 1.$$

Therefore, by Lemma 3.2.4 and Lemma 3.2.5,

$$\dim_H(S_\lambda) = D(\lambda) \xrightarrow{\lambda \rightarrow \lambda_c^-} D(\lambda_c) = 1.$$

Thus,

$$\dim_H(N) = 1.$$

- Case $\lambda_c > 0$. Let $\lambda \in (\lambda_{\min}, 0)$. So,

$$\dim_H(S_\lambda) \leq \dim_H(N),$$

Therefore, by Lemma 3.2.4 and Lemma 3.2.5,

$$\dim_H(S_\lambda) = D(\lambda) \xrightarrow{\lambda \rightarrow 0^-} D(0).$$

Let,

$$N^+ = \left\{ x \in \mathbb{R}/\mathbb{Z} : \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log m(T^i x) \leq 0 \right\}.$$

By Lemma 3.2.2, $N \subset N^+$. There remains to bound the dimension of N^+ by $D(0)$ using the lower pointwise dimension.

Lemma 3.2.6. [KO13, Lemma 4] There exists $\nu \in \mathcal{P}(\mathbb{R}/\mathbb{Z})$ such that for all $x \in N^+$,

$$\underline{d}_\nu(x) := \liminf_{r \rightarrow 0} \frac{\log \nu(\mathcal{B}(x, r))}{\log r} \leq D(0).$$

Proof. Let $p \geq 2$ be the degree of T . Consider a Markov partition of \mathbb{R}/\mathbb{Z} , and let σ be the associated one-sided topological Markov Chain defined on $\{1, \dots, p\}^{\mathbb{N}}$. We also consider the coding map $\chi : \{1, \dots, p\}^{\mathbb{N}} \rightarrow \mathbb{R}/\mathbb{Z}$ obtained from the Markov partition. Define the following functions on $\{1, \dots, p\}^{\mathbb{N}}$:

$$\begin{aligned} \phi &:= \log m \circ \chi, \\ d &:= \log T' \circ \chi. \end{aligned}$$

Let $D := D(0)$ and $a(w) := q\phi(w) - Dd(w)$ where $q \in \mathbb{R}$. As T is topologically mixing, there exists a unique $q \in \mathbb{R}$ and a unique equilibrium measure $\tilde{\nu}$ of a such

that $P_\sigma(a) = 0$ and $\int_{\{1, \dots, p\}^{\mathbb{N}}} \phi(w) d\tilde{\nu}(w) = 0$. Moreover, $\tilde{\nu}$ is a Gibbs measure and $q < 0$ (see [Bar08, Lemma 12.3.3] and the proof of [KO13, Lemma 3]).

For all $n \in \mathbb{N}$, for all $(i_0, \dots, i_n) \in \{1, \dots, p\}^{n+1}$, we define the cylinder:

$$C_{i_0, \dots, i_n} := \{i_0\} \times \dots \times \{i_n\} \times \{1, \dots, p\}^{\mathbb{N}}.$$

As $\tilde{\nu}$ is a Gibbs measure and $P(a) = 0$, there exists a constant $K > 0$ such that $n \in \mathbb{N}$, for all $(i_0, \dots, i_n) \in \{1, \dots, p\}^{n+1}$, and for all $\omega \in C_{i_0, \dots, i_n}$:

$$\frac{1}{K} < \frac{\tilde{\nu}(C_{i_0, \dots, i_n})}{\exp \sum_{i=0}^{n-1} a(\sigma^i \omega)} < K. \quad (3.1)$$

Let $\nu := \tilde{\nu} \circ \chi^{-1}$ a measure on \mathbb{R}/\mathbb{Z} defined on the Markov partition and

$$\Omega_0 := \left\{ w \in \{1, \dots, p\}^{\mathbb{N}}, \liminf_{n \rightarrow \infty} \sum_{n=0}^{+\infty} \phi(\sigma^n w) \leq 0 \right\}.$$

Let $\delta > 0$, $w \in \Omega_0$, and $x = \chi(w) \in \mathbb{R}/\mathbb{Z}$, there exists an increasing sequence $(n_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for all $k \in \mathbb{N}$,

$$q \sum_{i=0}^{n_k} \phi(\sigma^i w) \geq q n_k \delta. \quad (3.2)$$

Let $(r_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that for all $k \in \mathbb{N}$,

$$-\sum_{i=0}^{n_k} d(\sigma^i \omega) > \log r_k \text{ and } -\sum_{i=0}^{n_k+1} d(\sigma^i \omega) \leq \log r_k. \quad (3.3)$$

As $d > 0$ and continuous,

$$-(n_k + 1) \inf d > \log r_k \text{ and } -(n_k + 2) \sup d \leq \log r_k. \quad (3.4)$$

In particular, as $n_k \xrightarrow[k \rightarrow \infty]{} +\infty$, $r_k \xrightarrow[k \rightarrow \infty]{} 0$. Let $(i_n)_{n \in \mathbb{N}} \in \{1 \dots p\}^{\mathbb{N}}$ be the sequence such that for all $n \in \mathbb{N}$, $\omega \in C_{i_0, \dots, i_n}$. Combining (3.2) and (3.1) we obtain, there exists $\rho > 1$ such that for all $k \in \mathbb{N}$,

$$\begin{aligned} \nu(\mathcal{B}(x, \rho r_k)) &\geq \tilde{\nu}(C_{i_0, \dots, i_{n_k}}) \\ &\geq \frac{1}{K} \exp \left(q \sum_{i=0}^{n_k} \phi(\sigma^i \omega) - D \sum_{i=0}^{n_k} d(\sigma^i \omega) \right) \\ &\geq \frac{1}{K} \exp \left(q n_k \delta - D \sum_{i=0}^{n_k} d(\sigma^i \omega) \right) \end{aligned}$$

So by (3.3) and by (3.4):

$$\nu(\mathcal{B}(x, \rho r_k)) \geq \frac{1}{K} r_k^D \exp \left(q \delta \left(\frac{-\log r_k}{\sup d} - 2 \right) \right).$$

Thus,

$$\liminf_{r \rightarrow 0} \frac{\log \nu(\mathcal{B}(x, r))}{\log r} \leq D - \frac{q \delta}{\sup d}.$$

As the result is true for all $\delta > 0$,

$$\liminf_{r \rightarrow 0} \frac{\log \nu(\mathcal{B}(x, r))}{\log r} \leq D.$$

Moreover, $N^+ = \chi(\Omega_0)$, which concludes the proof of the lemma. \square

So, by [Bar08, Theorem 2.1.5], the bound of the lower pointwise dimension obtained in Lemma 3.2.6 gives the dimension of Hausdorff of N :

$$D(0) \leq \dim_H(N) \leq \dim_H(N^+) \leq D(0). \quad \square$$

Example 3.2.7 (Example 1.2.2). *The Lebesgue measure on \mathbb{R}/\mathbb{Z} is ergodic. Thus, Theorem 1.3.2 implies:*

$$\dim_H(N_\lambda) = \begin{cases} 1 & \text{if } \lambda \in (-\frac{1}{2}, 0], \\ \max_{\nu \in \mathcal{P}_T(\mathbb{X})} \left\{ \frac{h_T(\nu)}{\log 2} : \int_{\mathbb{R}/\mathbb{Z}} \cos(2\pi x) d\nu(x) = \lambda \right\} & \text{if } \lambda \in [0, 1). \end{cases}$$

Moreover, $\lambda \in (-\frac{1}{2}, 1) \mapsto \dim_H(N_\lambda)$ is continuous (see Lemma 3.2.5).

3.3 Application to the example of the doubling map

We examine the implications of Athreya and Karlin's results in the case of Example 1.2.2. Let $\nu \in \mathcal{E}_T(\mathbb{R}/\mathbb{Z})$ and $\lambda \in \mathbb{R}$. Then,

$$\mathbb{E}_\nu[\log m_\lambda(x)] = \lambda - \mathbb{E}_\nu[\cos 2\pi x].$$

Therefore, by Theorem 1.3.1, $\nu(N_\lambda) = 1$ if and only if $\lambda \leq \mathbb{E}_\nu[\cos 2\pi x]$ (and $\nu(N_\lambda) = 0$ if and only if $\lambda > \mathbb{E}_\nu[\cos 2\pi x]$).

As \mathbb{R}/\mathbb{Z} is a compact set, inf and sup are reached. Then for all $\lambda \in \mathbb{R}$, if:

- $\lambda \leq \lambda_{\min}$, then for all $\nu \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})$, $\nu(N_\lambda) = 1$,
- $\lambda_{\min} < \lambda \leq \lambda_{\max}$, there exists $\nu_0, \nu_1 \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})$ such that $\nu_0(N_\lambda) = 1$ and $\nu_1(N_\lambda) = 0$,
- $\lambda > \lambda_{\max}$, then for all $\nu \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})$, $\nu(N_\lambda) = 0$.

We determine the values of λ_{\min} and λ_{\max} . For any $\nu \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})$, $\mathbb{E}_\nu[\cos 2\pi x] \in [-1, 1]$. $\delta_0 \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})$ and $\mathbb{E}_{\delta_0}[\cos 2\pi x] = 1$, then $\lambda_{\max} = 1$. By [BS94, Bou00], the atomic ergodic measure $\nu_0 = \frac{1}{2}\delta_{\frac{1}{3}} + \frac{1}{2}\delta_{\frac{2}{3}}$ is the invariant measure that minimizes the quantity $\mathbb{E}_\nu[\cos 2\pi x]$ (see Figure 2, first graph) and $\mathbb{E}_{\nu_0}[\cos 2\pi x] = \frac{1}{2} \cos \frac{2\pi}{3} + \frac{1}{2} \cos \frac{4\pi}{3} = -\frac{1}{2}$. Thus $\lambda_{\min} = -\frac{1}{2}$.

If $\lambda > 1 = \lambda_{\max}$, then for any $x \in \mathbb{R}/\mathbb{Z}$, the laws $\text{Pois}(e^{\lambda - \cos 2\pi x})$ stochastically dominate the law $\text{Pois}(e^{\lambda - 1})$, which has an expectation $\lambda - 1$ strictly greater than 1. Thus, by coupling, we can conclude that $N_\lambda = \emptyset$.

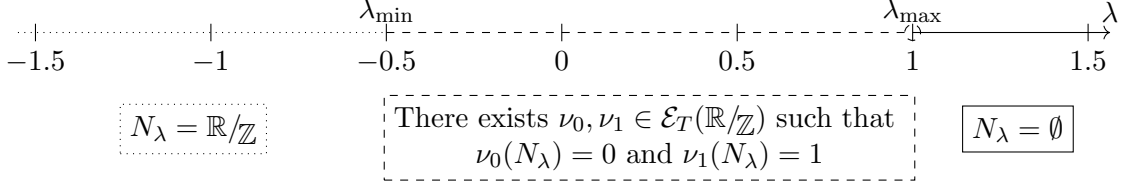
If $\lambda \leq 1 = \lambda_{\max}$, then there exists $\nu \in \mathcal{E}_T(\mathbb{R}/\mathbb{Z})$ such that $\nu(N_\lambda) = 1$. In particular, N_λ is non-empty, so by Example 2.3.3, N_λ is a dense G_δ . Lemma 3.2.2 allow us that for $\lambda < \lambda_{\min} = -\frac{1}{2}$, $N_\lambda = \mathbb{R}/\mathbb{Z}$. Indeed, let $\lambda < \lambda_{\min} = -\frac{1}{2}$. $\log m_\lambda$ is continuous, and for all $\nu \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})$,

$$\mathbb{E}_\nu[\log m_\lambda(x)] \leq \lambda_{\min} - \lambda < 0.$$

By the semi-uniform ergodic theorem [SS00, Theorem 1.9], there exists $\varepsilon > 0$ and $N \in \mathbb{N}^*$ such that for all $x \in \mathbb{R}/\mathbb{Z}$ and $n \geq N$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \log m_\lambda(T^k x) < \varepsilon.$$

By the contrapositive of Lemma 3.2.2, $x \in N_\lambda$ for all $x \in \mathbb{R}/\mathbb{Z}$. We summarize the different regimes of this example on the following graduated line:



For $\lambda = \lambda_{min} = -\frac{1}{2}$, by Theorem 1.3.1, $\nu(N_\lambda) = 1$ for all $\nu \in \mathcal{P}_T(\mathbb{R}/\mathbb{Z})$ but we don't know if $N_\lambda = \mathbb{R}/\mathbb{Z}$.

4 Regularity of the invariant graph

In this section, we study the continuity (Theorem 1.3.5) and Hölder continuity (Theorem 1.3.8) of the invariant graph in the supercritical case.

4.1 Continuity in the supercritical case

We will now examine the uniformly supercritical case, i.e., for all $\nu \in \mathcal{E}_T(\mathbb{X})$, $\mathbb{E}_\nu[\log m(x)] > 0$. With some additional assumptions, the function q is continuous. We already know that q is lower semi-continuous by Lemma 2.2.1. Now, we will express q as an infimum of continuous functions to make q upper semi-continuous and hence continuous. We start by proving that the function q is upper-bounded by a constant strictly less than 1.

Lemma 4.1.1. *Assume (H3). There exists $K < 1$ and $N \in \mathbb{N}$ such that for all $x \in \mathbb{X}$, the sequence $(\varphi^{(nN)}(x, K))_{n \in \mathbb{N}}$ is decreasing and converges to $q(x)$.*

Proof. For all $\nu \in \mathcal{P}_T(\mathbb{X})$,

$$\int_{\mathbb{X}} \log m(x) d\nu(x) > 0.$$

Thus, there exists $\varepsilon > 0$ such that for all $\nu \in \mathcal{P}_T(\mathbb{X})$,

$$\int_{\mathbb{X}} \log m(x) d\nu(x) > \varepsilon$$

because of the compactness of $\mathcal{P}_T(\mathbb{X})$ (since \mathbb{X} is compact), and because m is continuous by Proposition 2.1.4. By the semi-uniform ergodic theorem [SS00, Theorem 1.9], there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in \mathbb{X}$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \log m(T^k x) > \frac{\varepsilon}{2}.$$

In particular, for all $x \in \mathbb{X}$,

$$\partial_s \varphi^{(N)}(x, 1) > \exp\left(\frac{\varepsilon N}{2}\right) > 1.$$

By induction and Proposition 2.1.4, $\partial_s \varphi^{(N)}$ is uniformly continuous on the compact $\mathbb{X} \times [0, 1]$. Thus, there exists $K < 1$ such that for all $x \in \mathbb{X}$ and $t \in [K, 1]$,

$$\partial_s \varphi^{(N)}(x, t) > 1.$$

Let $x \in \mathbb{X}$, $\varphi^{(N)}(x, 1) = 1$, so $\varphi^{(N)}(x, K) \leq K$. Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned} \varphi^{((n+1)N)}(x, K) &= \varphi^{(Nn)}(x, \varphi^{(N)}(T^{nN}x, K)) \text{ by Proposition 1.2.4} \\ &\leq \varphi^{(Nn)}(x, K) \end{aligned}$$

because $s \in [0, 1] \mapsto \varphi^{(nN)}(x, s)$ is increasing and $\varphi(T^{nN}x, K) \leq K$.

Thus, the sequence $(\varphi^{(nN)}(x, K))_{n \in \mathbb{N}}$ is decreasing. Moreover, for all $n \in \mathbb{N}$, $\varphi^{(nN)}(x, 0) \leq \varphi^{(nN)}(x, K)$, so $q(x) \leq K$.

For $n \in \mathbb{N}$, $\varphi^{(nN)}(x, 0) \leq \varphi^{(Nn)}(x, K)$ because $s \in [0, 1] \mapsto \varphi^{(nN)}(x, s)$ is increasing. Since $\varphi^{(nN)}(x, 0)$ converges to $q(x)$, we only need to show that $\varphi^{(nN)}(x, K) - \varphi^{(nN)}(x, 0) \rightarrow 0$.

We will prove this using [AK71b, Theorem 5]. This theorem states that given a sequence of reproduction laws, the probability generating function associated with the population size at the n th generation converges to a function g for all $s \in [0, 1]$. Moreover, since there exists $c < 1$ such that for all $y \in \mathbb{X}$, $\mu_y(1) < c$ (because μ is continuous, and for all $y \in \mathbb{X}$, $\mu_y(1) < 1$), then for all $s \in [0, 1]$, $g(s) = g(0)$. Therefore, $\varphi^{(nN)}(x, K) - \varphi^{(nN)}(x, 0) \xrightarrow{n \rightarrow +\infty} 0$. \square

Lemma 4.1.1 allows us to prove Theorem 1.3.5.

Proof of Theorem 1.3.5. By Lemma 2.2.1, q is lower semi-continuous. According to Lemma 4.1.1, $q(x) = \inf_{n \in \mathbb{N}} \varphi^{(Nn)}(x, K)$, and for all $n \in \mathbb{N}$, $x \in \mathbb{X} \mapsto \varphi^{(nN)}(x, K)$ is continuous. Thus, q is upper semi-continuous, and consequently, q is continuous. \square

By Lemma 4.1.1, for all $a \in [0, 1)$, $(\varphi^{(n)}(\cdot, a))_{n \in \mathbb{N}}$ converges pointwise to q . Corollary 4.1.2 shows that the convergence is uniform.

Corollary 4.1.2. *Assume (H3). For each $a \in [0, 1)$, the sequence of functions $(\varphi^{(n)}(\cdot, a))_{n \in \mathbb{N}}$ converges uniformly to q .*

Proof. We define a function Φ as:

$$\Phi : \begin{cases} \mathcal{C}(\mathbb{X}, [0, 1]) & \rightarrow \mathcal{C}(\mathbb{X}, [0, 1]) \\ f & \mapsto (x \in \mathbb{X} \mapsto \varphi(x, f(Tx))) \end{cases} .$$

Φ is uniformly continuous. Indeed, let $\varepsilon > 0$. The function $(x, s) \in \mathbb{X} \times [0, 1] \mapsto \varphi(x, s)$ is uniformly continuous. Hence, there exists $\eta > 0$ such that for all $x \in \mathbb{X}$ and $s, t \in [0, 1]$ with $|s - t| \leq \eta$, we have $|\varphi(x, s) - \varphi(x, t)| \leq \varepsilon$. Let $f, g \in \mathcal{C}(\mathbb{X}, [0, 1])$ such that $\|f - g\|_\infty \leq \eta$. Then, for all $x \in \mathbb{X}$, $|f(Tx) - g(Tx)| \leq \eta$. Thus, for all $x \in \mathbb{X}$, $|\varphi(x, f(Tx)) - \varphi(x, g(Tx))| \leq \varepsilon$, i.e. $\|\Phi(f) - \Phi(g)\|_\infty \leq \varepsilon$. Therefore, Φ is uniformly continuous.

Let $a \in [0, 1)$, $0 < K < 1$ defined as in Lemma 4.1.1, and $\tilde{K} \in [K, 1)$ such that $\tilde{K} > a$. The sequence of continuous and increasing functions $(\varphi^{(n)}(\cdot, 0))_{n \in \mathbb{N}}$ (respectively

the sequence of continuous and decreasing functions $(\varphi^{(nN)}(\cdot, \tilde{K}))_{n \in \mathbb{N}}$ converges pointwise to the continuous (by Theorem 1.3.5) function q . Moreover, since \mathbb{X} is compact, by Dini's theorem, the sequences $(\varphi^{(n)}(\cdot, 0))_{n \in \mathbb{N}}$ and $(\varphi^{(nN)}(\cdot, \tilde{K}))_{n \in \mathbb{N}}$ converge uniformly to q . The uniform convergence of $(\varphi^{(nN)}(\cdot, \tilde{K}))_{n \in \mathbb{N}}$ to q can be expressed as:

$$\Phi^{nN}(\tilde{\mathbb{K}}) \xrightarrow{n \rightarrow +\infty} q$$

(where $\tilde{\mathbb{K}}$ is the constant function equal to \tilde{K}). For every $k \in \llbracket 0, N-1 \rrbracket$, by the continuity of Φ and by Lemma 2.2.1:

$$\Phi^{nN+k}(\tilde{\mathbb{K}}) \xrightarrow{n \rightarrow +\infty} \Phi^k(q) = q.$$

Thus, $(\varphi^{(n)}(\cdot, \tilde{K}))_{n \in \mathbb{N}}$ converges uniformly to q . By squeezing, $(\varphi^{(n)}(\cdot, a))_{n \in \mathbb{N}}$ converge uniformly to q . \square

4.2 Hölder regularity in the supercritical case

We will now see in the case where the invariant graph is continuous whether the Hölder regularity of the reproduction function μ is preserved by the invariant graph q .

Assume (H3) and that $q(x) > 0$ for all $x \in \mathbb{X}$. By Proposition 2.1.4, m is continuous, and by Theorem 1.3.5, q is continuous. Denote $\beta := \sup_{x \in \mathbb{X}} m(x) < +\infty$, $K := \sup_{x \in \mathbb{X}} q(x) < 1$ and $\kappa := \inf_{x \in \mathbb{X}} q(x) > 0$.

4.2.1 Lyapunov exponent in the fibre

In this section, we will study the Lyapunov exponent in the fibre

$$\lambda_F = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in \mathbb{X}} \log(\partial_s \varphi^{(n)}(x, q(T^n x))).$$

We start by showing in Lemma 4.2.1 and Lemma 4.2.2 that λ_F is well defined:

Lemma 4.2.1. *Assume (H3) and that $q(x) > 0$ for all $x \in \mathbb{X}$. Then $x \in \mathbb{X} \mapsto F(x) = \log \partial_s \varphi(x, q(Tx))$ takes finite values and is continuous.*

Proof. q is continuous, and for all $x \in \mathbb{X}$, $0 < \kappa < q(x) < K < 1$. Then by Proposition 2.1.4, $x \in \mathbb{X} \mapsto \partial_s \varphi(x, q(Tx))$ is continuous. All we have to do is check that $\partial_s \varphi(x, q(Tx)) > 0$ for all $x \in \mathbb{X}$. Let $x \in \mathbb{X}$.

$$\begin{aligned} \partial_s \varphi(x, q(Tx)) &= \sum_{k=1}^{+\infty} k \mu_x(k) q(Tx)^{k-1} \\ &\geq \sum_{k=1}^{+\infty} \mu_x(k) \kappa^{k-1} \\ &> 0 \text{ because } \mu_x(0) < 1. \end{aligned} \quad \square$$

Lemma 4.2.2 proves that λ_F is well defined using Fekete's subadditive lemma.

Lemma 4.2.2. *Assume (H3) and that $q(x) > 0$ for all $x \in \mathbb{X}$. Then $(\frac{1}{n} \sup_{x \in \mathbb{X}} \log(\partial_s \varphi^{(n)}(x, q(T^n x))))_{n \in \mathbb{N}}$ has a finite limit.*

Proof. Let $x \in \mathbb{X}$ and $n, m \in \mathbb{N}$, by Lemma 2.2.3:

$$\frac{1}{n} \log(\partial_s \varphi^{(n)}(x, q(T^n x))) = \frac{1}{n} \sum_{k=0}^{n-1} F(T^k x).$$

And,

$$\begin{aligned} \sup_{x \in \mathbb{X}} \sum_{k=0}^{n+m-1} F(T^k x) &\leq \sup_{x \in \mathbb{X}} \sum_{k=0}^{n-1} F(T^k x) + \sup_{x \in \mathbb{X}} \sum_{k=n}^{n+m-1} F(T^k x) \\ &\leq \sup_{x \in \mathbb{X}} \sum_{k=0}^{n-1} F(T^k x) + \sup_{x \in \mathbb{X}} \sum_{k=0}^{m-1} F(T^k x). \end{aligned}$$

Then $\left(\sup_{x \in \mathbb{X}} \sum_{k=0}^{n-1} F(T^k x) \right)_{n \in \mathbb{N}}$ is subadditive, so by Fekete's subadditive lemma, $\frac{1}{n} \sup_{x \in \mathbb{X}} \log(\partial_s \varphi^{(n)}(x, q(T^n x)))$ has a limit in $\{-\infty\} \cup \mathbb{R}$. Moreover, F is continuous on \mathbb{X} (by Lemma 4.2.1) so the limit is finite. \square

Proposition 4.2.3 and Lemma 4.2.4 give other equivalent definitions of the Lyapunov exponent in the fibre.

Proposition 4.2.3. *Assume (H3) and that $q(x) > 0$ for all $x \in \mathbb{X}$. Then for all $0 < a < 1$,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \partial_s \varphi^{(n)}(x, a) = \lambda_F.$$

Proof. Let $\varepsilon > 0$ and $\delta := \min(\frac{\varepsilon}{2}, a)$. The function $\log \partial_s \varphi$ is uniformly continuous on the compact $\mathbb{X} \times [\delta, 1]$. Thus, there exists $\eta > 0$ such that for all $x \in \mathbb{X}$ and $s, t \in [\delta, 1]$ with $|s - t| \leq \eta$, $|\log \partial_s \varphi(x, s) - \log \partial_s \varphi(x, t)| \leq \varepsilon$.

By Corollary 4.1.2, there exists $m \in \mathbb{N}$ such that $\sup_{x \in \mathbb{X}} |q(x) - \varphi^{(m+k)}(x, a)| < \min(\eta, \delta)$ for all $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $x \in \mathbb{X}$. Using Lemma 2.1.2 and Lemma 2.2.1,

$$\log \partial_s \varphi^{(n+m)}(x, a) = \log \partial_s \varphi^{(m)}(T^n x, a) + \log \partial_s \varphi^{(n)}(x, \varphi^{(m)}(T^n x, a)) \quad (4.1)$$

And,

$$\begin{aligned} \log \partial_s \varphi^{(n+m)}(x, q(T^{n+m} x)) &= \log \partial_s \varphi^{(m)}(T^n x, q(T^{n+m} x)) + \log \partial_s \varphi^{(n)}(x, \varphi^{(m)}(T^n x, q(T^{n+m} x))) \\ &= \log \partial_s \varphi^{(m)}(T^n x, q(T^{n+m} x)) + \log \partial_s \varphi^{(n)}(x, q(T^n x)). \end{aligned} \quad (4.2)$$

There exists $C > 0$ such that the continuous function $(x, s) \in \mathbb{X} \times [\delta, 1] \mapsto \log \partial_s \varphi^{(m)}(x, s)$ is bounded by C .

Using Corollary 2.1.3, Lemma 2.2.1, and by Proposition 1.2.4,

$$\begin{aligned}
& \left| \log \partial_s \varphi^{(n)}(x, \varphi^{(m)}(T^n x, a)) - \log \partial_s \varphi^{(n)}(x, q(T^n x)) \right| \\
&= \sum_{k=0}^{n-1} \left| \log \partial_s \varphi(T^k x, \varphi^{(n-1-k)}(T^{k+1} x, \varphi^{(m)}(T^n x, a))) \right. \\
&\quad \left. - \log \partial_s \varphi(T^k x, \varphi^{(n-1-k)}(T^{k+1} x, q(T^n x))) \right| \\
&= \sum_{k=0}^{n-1} \left| \log \partial_s \varphi(T^k x, \varphi^{(n+m-1-k)}(T^{k+1} x, a)) \right. \\
&\quad \left. - \log \partial_s \varphi(T^k x, q(T^{k+1} x)) \right| \\
&\leq n\varepsilon \text{ since } |\varphi^{(n+m-1-k)}(T^{k+1} x, a) - q(T^{k+1} x)| \leq \eta \quad (4.3) \\
&\quad \text{and } \varphi^{(n+m-1-k)}(T^{k+1} x, a), q(T^{k+1} x) \in [\delta, 1].
\end{aligned}$$

Hence, by Equality 4.1, Equality 4.2, and Inequality 4.3,

$$\begin{aligned}
& \left| \limsup_{n \rightarrow +\infty} \frac{1}{n} \sup_{x \in \mathbb{X}} \log \partial_s \varphi^{(n)}(x, a) - \lambda_F \right| \\
&= \left| \limsup_{n \rightarrow +\infty} \frac{1}{n} \sup_{x \in \mathbb{X}} \log \partial_s \varphi^{(n)}(x, a) - \lim_{n \rightarrow +\infty} \frac{1}{n} \sup_{x \in \mathbb{X}} \log \partial_s \varphi^{(n)}(x, q(T^n x)) \right| \\
&\leq \limsup_{n \rightarrow +\infty} \frac{1}{n+m} \left| \sup_{x \in \mathbb{X}} \log \partial_s \varphi^{(n+m)}(x, a) - \sup_{x \in \mathbb{X}} \log \partial_s \varphi^{(n+m)}(x, q(T^{n+m} x)) \right| \\
&= \limsup_{n \rightarrow +\infty} \frac{1}{n+m} \left| \sup_{x \in \mathbb{X}} (\log \partial_s \varphi^{(m)}(T^n x, a) + \log \partial_s \varphi^{(n)}(x, \varphi^{(m)}(T^n x, a))) \right. \\
&\quad \left. - \sup_{x \in \mathbb{X}} (\log \partial_s \varphi^{(m)}(T^n x, q(T^{n+m} x)) + \log \partial_s \varphi^{(n)}(x, q(T^n x))) \right| \\
&\leq \limsup_{n \rightarrow +\infty} \frac{1}{n+m} \left(\sup_{x \in \mathbb{X}} \left| \log \partial_s \varphi^{(n)}(x, \varphi^{(m)}(T^n x, a)) - \log \partial_s \varphi^{(n)}(x, q(T^n x)) \right| \right. \\
&\quad \left. + \sup_{x \in \mathbb{X}} \left| \log \partial_s \varphi^{(m)}(T^n x, a) \right| + \sup_{x \in \mathbb{X}} \left| \log \partial_s \varphi^{(m)}(T^n x, q(T^{n+m} x)) \right| \right) \\
&\leq \limsup_{n \rightarrow +\infty} \frac{1}{n+m} (n\varepsilon + 2C) = \varepsilon \quad (4.4)
\end{aligned}$$

Likewise,

$$\left| \liminf_{n \rightarrow +\infty} \frac{1}{n} \sup_{x \in \mathbb{X}} \log \partial_s \varphi^{(n)}(x, a) - \lambda_F \right| \leq \varepsilon \quad (4.5)$$

The conclusion of the proposition follows from Inequality 4.4 and Inequality 4.5. \square

Lemma 4.2.4 allows us to express λ_F without using limits.

Lemma 4.2.4. *Assume (H3) and that $q(x) > 0$ for all $x \in \mathbb{X}$. Then*

$$\lambda_F = \sup_{\nu \in \mathcal{P}_T(\mathbb{X})} \int_{\mathbb{X}} F(x) d\nu(x).$$

Proof. The function F is continuous (by Lemma 4.2.1). By semi-uniform ergodic theorem [SS00, Theorem 1.9] and by Equation (2.3), we have:

$$\lambda_F \leq \sup_{\nu \in \mathcal{P}_T(\mathbb{X})} \int_{\mathbb{X}} F(x) d\nu(x). \quad (4.6)$$

Since $\mathcal{P}_T(\mathbb{X})$ is compact, the supremum in (4.6) is reached by an ergodic measure that we denote $\nu_0 \in \mathcal{E}_T(\mathbb{X})$. By Birkhoff's ergodic theorem applied to the continuous function F and by Equation (2.3), for ν_0 -almost every $x \in \mathbb{X}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \partial_s \varphi^{(n)}(x, q(T^n x)) = \int_{\mathbb{X}} F(x) d\nu_0(x).$$

Thus, for ν_0 -almost every $x \in \mathbb{X}$,

$$\lambda_F \geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \partial_s \varphi^{(n)}(x, q(T^n x)) = \sup_{\nu \in \mathcal{P}_T(\mathbb{X})} \int_{\mathbb{X}} F(x) d\nu(x). \quad \square$$

Proposition 4.2.5 shows that λ_F controls the exponential convergence to the invariant graph.

Proposition 4.2.5. *Assume (H3) and that $q(x) > 0$ for all $x \in \mathbb{X}$. Then for all $a \in [0, 1]$,*

$$\limsup_{n \rightarrow \infty} \frac{\log \|\varphi^{(n)}(\cdot, a) - q\|_{\infty}}{n} \leq \lambda_F$$

Proof. Let $a \in [0, 1]$, $\tilde{K} := \max(a, K) \in (0, 1)$, $x \in \mathbb{X}$, and $n \in \mathbb{N}$. Using that $t \in [0, 1] \mapsto \partial_s \varphi^{(n)}(x, t)$ and $t \in [0, 1] \mapsto \varphi^{(n)}(x, t)$ are non-decreasing,

$$|\varphi^{(n)}(x, a) - q(x)| \leq \varphi^{(n)}(x, \tilde{K}) - \varphi^{(n)}(x, 0) \leq \partial_s \varphi^{(n)}(x, \tilde{K}) \tilde{K}$$

Therefore, by Proposition 4.2.3,

$$\limsup_{n \rightarrow \infty} \frac{\log \|\varphi^{(n)}(\cdot, a) - q\|_{\infty}}{n} \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in \mathbb{X}} \log \partial_s \varphi^{(n)}(x, \tilde{K}) = \lambda_F \quad \square$$

Lemma 4.2.6 shows that convergence towards the invariant graph is exponential.

Lemma 4.2.6. *Assume (H3) and that $q(x) > 0$ for all $x \in \mathbb{X}$. Then $\lambda_F < 0$.*

Proof. Let $x \in \mathbb{X}$. Using Lemma 2.2.1, Jensen's inequality, and that for all $x \in \mathbb{X}$, $t \in [0, 1] \mapsto \partial_s \varphi(x, t)$ is non-decreasing:

$$\begin{aligned} \log(1 - q(x)) &= \log(1 - \varphi(x, q(Tx))) \\ &= \log \left(\int_{q(Tx)}^1 \partial_s \varphi(x, t) dt \right) \\ &= \log(1 - q(Tx)) + \log \left(\int_{q(Tx)}^1 \partial_s \varphi(x, t) \frac{dt}{1 - q(Tx)} \right) \\ &\geq \log(1 - q(Tx)) + \int_{q(Tx)}^1 \log(\partial_s \varphi(x, t)) \frac{dt}{1 - q(Tx)} \\ &\geq \log(1 - q(Tx)) + \log(\partial_s \varphi(x, q(Tx))). \end{aligned} \quad (4.7)$$

Let $\nu \in \mathcal{P}_T(\mathbb{X})$. As $\mathbb{E}_{\nu}[\log m(x)] > 0$, there exists $A \in \mathcal{B}(\mathbb{X})$ such that for all $x \in A$, $\mu_x(\{0, 1\}) < 1$ and $\nu(A) > 0$. Thus, for all $x \in A$, $t \mapsto \partial_s \varphi(x, t)$ is increasing and Inequation (4.7) is strict. We integrate Inequation (4.7) according to ν . Using that ν is T -invariant,

$$\int_{\mathbb{X}} \partial_s \varphi(x, q(Tx)) d\nu(x) < 0.$$

Thus, by Lemma 4.2.4, $\lambda_F < 0$ because $\mathcal{P}_T(\mathbb{X})$ is compact. \square

4.2.2 Proof of the Hölder regularity of the invariant graph

We can now proceed to the proof of Theorem 1.3.8. Lemma 4.2.7 shows that φ inherits the Hölder regularity from μ .

Lemma 4.2.7. *Let $\alpha \in (0, 1]$, assume $(H4(\alpha))$. Then φ is α -Hölder continuous of seminorm smaller than $|\mu|_\alpha$ in the first variable uniformly in the second variable. In other words, for all $s \in [0, 1]$, $|\varphi(\cdot, s)|_\alpha \leq |\mu|_\alpha$.*

Proof. For all $s \in [0, 1]$ and $x, y \in \mathbb{X}$,

$$\begin{aligned} |\varphi(x, s) - \varphi(y, s)| &\leq \sum_{k=0}^{+\infty} s^k |\mu_x(k) - \mu_y(k)| \\ &\leq \|\mu_x - \mu_y\|_1 \text{ because } s \in [0, 1], \\ &\leq |\mu|_\alpha d(x, y)^\alpha \text{ because } \mu \text{ is } \alpha\text{-Hölder continuous.} \quad \square \end{aligned}$$

Then Lemma 4.2.8 shows that for all $n \in \mathbb{N}$, $\varphi^{(n)}$ inherits the Hölder regularity from φ .

Lemma 4.2.8. *Let $\alpha \in (0, 1]$, assume $(H4(\alpha))$. For all $n \in \mathbb{N}$, the function $\varphi^{(n)}$ is α -Hölder continuous in the first variable uniformly in the second variable. In other words, there exists $C_n > 0$ such that for all $s \in [0, 1]$, $|\varphi^{(n)}(\cdot, s)|_\alpha \leq C_n$.*

Proof. We prove this result by induction on $n \in \mathbb{N}$. For $n = 0$, the result is true.

Assuming the result is true for $n \in \mathbb{N}$. Let $s \in [0, 1]$ and $x, y \in \mathbb{X}$. Using Lemma 4.2.7 and the induction hypothesis,

$$\begin{aligned} |\varphi^{(n+1)}(x, s) - \varphi^{(n+1)}(y, s)| &= |\varphi(x, \varphi^{(n)}(Tx, s)) - \varphi(y, \varphi^{(n)}(Ty, s))| \\ &\leq |\varphi(x, \varphi^{(n)}(Tx, s)) - \varphi(y, \varphi^{(n)}(Tx, s))| \\ &\quad + |\varphi(y, \varphi^{(n)}(Tx, s)) - \varphi(y, \varphi^{(n)}(Ty, s))| \\ &\leq |\mu|_\alpha d(x, y)^\alpha + \beta |\varphi^{(n)}(Tx, s) - \varphi^{(n)}(Ty, s)| \\ &\leq |\mu|_\alpha d(x, y)^\alpha + \beta C_n d(Tx, Ty)^\alpha \\ &\leq |\mu|_\alpha d(x, y)^\alpha + \beta C_n \|T\|_{Lip}^\alpha d(x, y)^\alpha \\ &\leq C_{n+1} d(x, y)^\alpha \text{ where } C_{n+1} = |\mu|_\alpha + \beta C_n \|T\|_{Lip}^\alpha. \end{aligned}$$

Thus, $\varphi^{(n+1)}$ is α -Hölder continuous (with seminorm C_{n+1}) in the first variable uniformly in the second. \square

Controlling the seminorm of α -Hölder continuity of $\varphi^{(n)}(\cdot, s)$ allows us to obtain, by taking the limit, that the function q is also α -Hölder continuous.

Proof of Theorem 1.3.8. Let $\varepsilon > 0$ such that $\lambda_F + \varepsilon + \alpha(\lambda_u + \varepsilon) < 0$. Such an ε exists since $\lambda_u \leq 0$ or $(\lambda_u > 0 \text{ and } \alpha < -\frac{\lambda_F}{\lambda_u})$. There exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\frac{1}{n} \log(\|T^n\|_{Lip}) < \lambda_u + \varepsilon$. There exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ and $s \in [0, K]$, $\sup_{x \in \mathbb{X}} \frac{1}{n} \log(\partial_s \varphi^{(n)}(x, s)) < \lambda_F + \varepsilon$ by Proposition 4.2.3. Let $N = \max(N_0, N_1)$, $A_N = e^{N(\lambda_F + \varepsilon + \alpha(\lambda_u + \varepsilon))}$, and C_N defined as in Lemma 4.2.8. We have $A_N < 1$. Start by proving Lemma 4.2.9, which controls the composition of continuous Hölder functions:

Lemma 4.2.9. *Let $f \in \mathcal{C}(\mathbb{X}, [0, K])$. Suppose that f is α -Hölder continuous. Then $x \in \mathbb{X} \mapsto \varphi^{(N)}(x, f(T^N x))$ is α -Hölder continuous, and $|\varphi^{(N)}(\cdot, f(T^N \cdot))|_\alpha \leq |f|_\alpha A_N + C_N$.*

Proof. Let $x, y \in \mathbb{X}$.

$$\begin{aligned}
|\varphi^{(N)}(x, f(T^N x)) - \varphi^{(N)}(y, f(T^N y))| &\leq |\varphi^{(N)}(x, f(T^N x)) - \varphi^{(N)}(y, f(T^N x))| \\
&\quad + |\varphi^{(N)}(y, f(T^N x)) - \varphi^{(N)}(y, f(T^N y))| \\
&\leq C_N d(x, y)^\alpha + e^{N(\lambda_F + \varepsilon)} |f(T^N x) - f(T^N y)| \\
&\leq C_N d(x, y)^\alpha + e^{N(\lambda_F + \varepsilon)} |f|_\alpha d(T^N x, T^N y)^\alpha \\
&\leq C_N d(x, y)^\alpha + e^{N(\lambda_F + \varepsilon)} |f|_\alpha e^{N\alpha(\lambda_u + \varepsilon)} d(x, y)^\alpha.
\end{aligned}$$

Thus, $x \in \mathbb{X} \mapsto \varphi^{(N)}(x, f(T^N x))$ is α -Hölder continuous, and

$$|\varphi^{(N)}(\cdot, f(T^N \cdot))|_\alpha \leq |f|_\alpha A_N + C_N. \quad \square$$

Lemma 4.2.9 implies Corollary 4.2.10, allowing us to control the seminorm α -Hölder of $\varphi^{(nN)}$. Corollary 4.2.10 and Lemma A.0.2 allow us to check the Hölder regularity of q .

Corollary 4.2.10. *For all $n \in \mathbb{N}^*$, $x \in \mathbb{X} \mapsto \varphi^{(nN)}(x, 0)$ is α -Hölder continuous with a seminorm less than $\frac{C_N}{1-A_N}$.*

Proof. Prove this result by induction on $n \in \mathbb{N}^*$. For $n = 1$, the result is true by Lemma 4.2.8 and since $C_N \leq \frac{C_N}{1-A_N}$.

Assuming the result is true at rank $n \in \mathbb{N}^*$. Then for all $x \in \mathbb{X}$, $\varphi^{(N(n+1))}(x, 0) = \varphi^{(N)}(x, \varphi^{(nN)}(T^N x, 0))$. We can apply Lemma 4.2.9 with the function $x \mapsto \varphi^{(nN)}(x, 0)$, which is α -Hölder continuous by the induction hypothesis. Thus, $x \mapsto \varphi^{(N(n+1))}(x, 0)$ is α -Hölder continuous. Moreover,

$$\begin{aligned}
|\varphi^{(N(n+1))}(\cdot, 0)|_\alpha &\leq A_N |\varphi^{(nN)}(\cdot, 0)|_\alpha + C_N \text{ by Lemma 4.2.9} \\
&\leq A_N \frac{C_N}{1-A_N} + C_N \text{ by induction hypothesis.} \\
&= \frac{C_N}{1-A_N}.
\end{aligned}$$

Therefore, $x \in \mathbb{X} \mapsto \varphi^{((n+1)N)}(x, 0)$ is α -Hölder continuous with a seminorm less than $\frac{C_N}{1-A_N}$. \square

Now, we can return to the proof of Theorem 1.3.8. By definition, for all $x \in \mathbb{X}$, $q(x) = \lim_{n \rightarrow \infty} \varphi^{(n)}(x, 0)$. Moreover, by Corollary 4.2.10, for all $n \in \mathbb{N}^*$, $x \mapsto \varphi^{(nN)}(x, 0)$ is α -Hölder continuous, and $|\varphi^{(nN)}(\cdot, 0)|_\alpha \leq \frac{C_N}{1-A_N}$. Thus, by Lemma A.0.2, q is α -Hölder continuous. \square

Now that we have proven that the function q is α -Hölder continuous, we will show that the convergence of the sequence of functions $(x \mapsto \varphi(x, 0))_{n \in \mathbb{N}}$ to q is not only pointwise (as defined by default) but also a Hölder convergence.

Proof of Corollary 1.3.9. Let $0 < \beta < \alpha$, $t = \frac{\beta}{\alpha} \in (0, 1)$, and $N \in \mathbb{N}$, $A_N < 1$ and $C_N \geq 0$ defined as in Theorem 1.3.8 and Lemma 4.2.8. By Corollary 4.2.10, for all $n \in \mathbb{N}$, $|\varphi^{(nN)}(\cdot, 0)|_\alpha \leq \frac{C_N}{1-A_N}$ and by the proof of Lemma 4.2.8, $|\varphi^{(n+1)}(\cdot, 0)|_\alpha \leq |\mu|_\alpha + \beta |\varphi^{(n)}(\cdot, 0)|_\alpha \|T\|_{lip}^\alpha$. In particular, $(|\varphi^{(n)}(\cdot, 0)|_\alpha)_{n \in \mathbb{N}}$ is bounded by a constant denoted C . Moreover, by Proposition A.0.3, for all $n \in \mathbb{N}$:

$$\begin{aligned}
|q - \varphi^{(n)}(\cdot, 0)|_\beta &\leq (2\|q - \varphi^{(n)}(\cdot, 0)\|_\infty)^{1-t} |q - \varphi^{(n)}(\cdot, 0)|_\alpha^t \\
&\leq (2\|q - \varphi^{(n)}(\cdot, 0)\|_\infty)^{1-t} (|q|_\alpha + |\varphi^{(n)}(\cdot, 0)|_\alpha)^t \\
&\leq (2\|q - \varphi^{(n)}(\cdot, 0)\|_\infty)^{1-t} (|q|_\alpha + C)^t.
\end{aligned}$$

Since $(\varphi^{(n)}(\cdot, 0))_{n \in \mathbb{N}}$ converges uniformly to q by Corollary 4.1.2, it follows that $|q - \varphi^{(n)}(\cdot, 0)|_\beta$ converges to 0 as n tends to $+\infty$. Thus, $(\varphi^{(n)}(\cdot, 0))_{n \in \mathbb{N}}$ converges in the β -Hölder norm to q . \square

A Annex: Hölder spaces on a compact set

In this section, let (X, d) be a compact metric space and $(Y, \|\cdot\|)$ be a normed vector space.

Definition A.0.1. Let $f \in \mathcal{C}(X, Y)$ and $0 < \alpha \leq 1$. f is said to be α -Hölder if there exists $C \geq 0$ such that for all $x, y \in X$:

$$\|f(x) - f(y)\| \leq Cd(x, y)^\alpha.$$

We denote $\mathcal{C}^{0,\alpha}(X, Y)$ the space of α -Hölder functions. For $f \in \mathcal{C}^{0,\alpha}(X)$, let:

$$|f|_\alpha := \inf_{x, y \in X: x \neq y} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha},$$

which is a seminorm on $\mathcal{C}^{0,\alpha}(X, Y)$.

For $f \in \mathcal{C}^{0,\alpha}(X)$, let

$$\|f\|_\alpha := \|f\|_\infty + |f|_\alpha$$

which is a norm on $\mathcal{C}^{0,\alpha}(X, Y)$.

Lemma 4.2.9 gives a condition that the limit inherits the Hölder continuity of a sequence of functions.

Lemma A.0.2. Let $\alpha \in (0, 1]$ and $(f_n)_{n \in \mathbb{N}} \in \mathcal{C}^{0,\alpha}(X, Y)^\mathbb{N}$. Assume that $(f_n)_{n \in \mathbb{N}}$ converges pointwise to a function f and $\sup_{n \in \mathbb{N}} |f_n|_\alpha < +\infty$. Then $f \in \mathcal{C}^{0,\alpha}(X)$ and $|f|_\alpha \leq \sup_{n \in \mathbb{N}} |f_n|_\alpha$.

Proof. Let $C := \sup_{n \in \mathbb{N}} |f_n|_\alpha$. For $x, y \in X$ and any $n \in \mathbb{N}$:

$$\|f_n(x) - f_n(y)\| \leq |f_n|_\alpha d(x, y)^\alpha \leq Cd(x, y)^\alpha.$$

By the triangle inequality:

$$\|f(x) - f(y)\| \leq Cd(x, y)^\alpha + \|f_n(x) - f(x)\| + \|f_n(y) - f(y)\|.$$

Therefore, by the pointwise convergence:

$$\|f(x) - f(y)\| \leq Cd(x, y)^\alpha.$$

Thus, $|f|_\alpha \leq C$. □

Proposition A.0.3 is a result of interpolation on Hölder spaces.

Proposition A.0.3. Let $0 < \beta < \alpha \leq 1$, $t = \frac{\beta}{\alpha} \in (0, 1)$, and $f \in \mathcal{C}^{0,\alpha}(X, Y)$. Then $|f|_\beta \leq (2\|f\|_\infty)^{1-t}|f|_\alpha^t$.

Proof. For $x, y \in X$:

$$\begin{aligned} \|f(x) - f(y)\| &\leq 2\|f\|_\infty \\ \text{and } \|f(x) - f(y)\| &\leq |f|_\alpha d(x, y)^\alpha. \end{aligned}$$

Therefore,

$$\|f(x) - f(y)\| \leq (2\|f\|_\infty)^{1-t}|f|_\alpha^{1-t}d(x, y)^{t\alpha} = (2\|f\|_\infty)^{1-t}|f|_\alpha^t d(x, y)^\alpha.$$

Thus, $|f|_\beta \leq (2\|f\|_\infty)^{1-t}|f|_\alpha^t$. □

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