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Generalized Morse theory for tubular neighborhoods

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Abstract

We define a notion of Morse function and establish Morse theory-like theorems over offsets of any compact set in a Euclidean space at regular values of their distance function. Using non-smooth analysis and tools from geometric measure theory, we prove that the homotopy type of the sublevel sets of these Morse functions changes at a critical value by gluing exactly one cell around each critical point.

Keywords — Morse Theory, Non-smooth Analysis, Geometric Measure Theory, CW-Complexes.

MSC Classification — 57R10, 26B12, 49Q15.

1 Introduction

In his celebrated book *Morse Theory* [11], Milnor describes the changes in topology of the closed sublevel sets $X_c := f^{-1}(-\infty, c]$ when c increases for C^2 functions $f : X \rightarrow \mathbb{R}$ over a compact C^2 manifold satisfying certain generic conditions, which he calls *Morse functions*. In this setting, Milnor shows that topological changes only happen at a finite number of reals called *critical values* determined by the values the function f takes at *critical points*, which are the points where the differential of f vanishes. Around a critical point x with critical value $c = f(x)$, the topology of the sublevel sets $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by gluing a cell (i.e.,

a set homeomorphic to the closed unit ball of a Euclidean space) around x when ε is small enough. More precisely, a smooth function $f : X \rightarrow \mathbb{R}$ is said to be Morse when its Hessian is non-degenerate at every critical point. In this case the previous considerations can be summarized by the two fundamental results of Morse theory, which we call *Morse theorems*:

- Let $a < b \in \mathbb{R}$. If the interval $[a, b]$ does not contain any critical value of f , X_a has the same homotopy type as X_b . This is the constant homotopy type lemma.
- Around a critical value c of f , the homotopy type of $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by gluing a cell around each critical point $x_i \in f^{-1}(c)$ when ε is small enough, where the dimension of the cell is the index of the Hessian of f at x_i . This is the handle attachment lemma.

Since then, several works on Morse theory have aimed at broadening the class of sets and adapting the definition of Morse functions for which the Morse theorems hold, leading to the extension of Morse theory to smooth functions on stratified sets lying inside a Riemannian manifold in the sense of the monography by Goresky & MacPherson [8]. In this case, the handle attachment lemma becomes weaker as the homotopy type of $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by gluing what the authors call the local Morse data of the critical point x , which is not necessarily a cell, around x . In this setting, there can be strictly more homological events than critical points of the corresponding Morse function. In 1989, Fu [5] proved Morse theorems for smooth functions any compact set X with a $C^{1,1}$ -hypersurface boundary and more generally to sets with positive reach in Euclidean spaces. His reasoning is the main inspiration for the present article, as we adapt his proofs to our setting using non-smooth analysis. Other works extended the Morse theorems to some classes of non-smooth functions on manifolds, such as piecewise smooth functions [1] and min-type functions, which are the functions that can be locally written as the minimum of a finite number of smooth functions [7].

Our contribution is as follows. We prove that the Morse theorems extend to smooth functions on any offsets of a subset of \mathbb{R}^d at a regular value of its distance function. Such sets are Lipschitz domains which are not necessarily smooth nor stratified, and we call them *complementary regular sets*. Contrary to stratified Morse theory [8], which was the only previous extension of Morse theory to a class containing sets with possibly non-convex singularities, we prove that the classical handle attachment lemma does hold for complementary regular sets, with topological changes of the sublevel set filtration of a Morse function consisting in a cell being glued around each critical point. While it is hopeless to extend the Morse theorems to general compact sets, which can be topologically wayward, this

result shows that Morse theorems do typically hold when one replaces the compact set with an arbitrarily small tubular neighborhood.

Theorem 1.1. *Let Y be a compact subset of \mathbb{R}^d and $\varepsilon > 0$ be a regular value of the distance function to Y . Let $X = Y^\varepsilon$ be the ε -offset of Y . Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function such that $f|_X$ admits only non-degenerate critical points.*

Then for every regular value c of $f|_X$, $X_c := X \cap f^{-1}(-\infty, c]$ has the homotopy type of a CW-complex with one cell per critical point with value less than c , whose dimension is determined by the index of the Hessian of $f|_X$ and the curvatures of X .

Outline

In Section 2, we define the objects used throughout this article.

- In Section 2.1 we define and illustrate the basic tools of our study. This includes the reach_μ and the reach of a compact subset of \mathbb{R}^d , eroded sets X^{-r} for any positive real r and any $X \subset \mathbb{R}^d$, Clarke gradients of locally Lipschitz functions, normal and tangent cones of a compact set with positive reach.
- In Section 2.2 we define the unit normal bundle of sets with positive reach and the normal bundle of their complement set. We describe how local curvatures of such sets are related to their normal bundle.
- Section 2.3 gives the definitions and notations of critical points and Hessian for a restricted function $f|_X$ for sets with positive reach used by Fu in [5]. We will use the same definitions of critical points, Hessians and non-degeneracy for the class of *complementary regular sets* defined in Section 3.1.
- Section 2.4 focuses on properties of locally Lipschitz functions. We build a retraction between sublevel sets of such functions assuming a bound from below on the distance to zero of their Clarke gradient.
- In Section 2.5 we establish a link between the normal bundle of a set X and the Clarke gradient of its distance function d_X . This crucial step allows us to use results from non-smooth analysis on assumptions about critical points of $f|_X$.

Section 3 articulates the previous results to establish the main theorem.

- In Section 3.1 we define the class of *complementary regular sets*, which are the sets verifying the assumptions needed in our reasoning through the remainder of the section to prove Morse theory results. We prove that X is a complementary regular set if and only if it is an offset of some compact set Y at a positive regular value of d_Y .
- In Section 3.2 we describe how to build functions $f_{r,c}$ such that the sublevel sets $X_c^{-r} = X^{-r} \cap f_{r,c}^{-1}(-\infty, c]$ and X_c have the same homotopy type when c is a regular value and $r > 0$ is small enough. To that end we consider some locally Lipschitz functions and prove that they verify the assumptions needed in the theorems of Section 2.4. The retractions obtained are used to build a homotopy equivalence between X_c^{-r} and X_c .
- In Section 3.3 we show that in between critical values, the topology of sublevel sets does not change. This is done by applying Section 2.4 using computations from the previous section.
- Section 3.4 describes the topological changes happening around a critical value as long as it has only one corresponding critical point which is non-degenerate. We adapt the proof from Fu [5] to our setting, circumventing the problem of considering sets with reach 0 using non-smooth analysis. We then extend this result to critical values with a finite number of corresponding critical points which are all non-degenerate.

2 Definitions and useful lemmas

2.1 Preliminaries

We fix $d \in \mathbb{N}$ to be the dimension of the Euclidean space in which our objects will be included. A function on \mathbb{R}^d will be called *smooth* when it is C^2 . The canonical scalar product over \mathbb{R}^d will be denoted by $\langle \cdot, \cdot \rangle$, and $B(x, r)$ will denote the closed ball of radius r centered in $x \in \mathbb{R}^d$. The inclusion (proper or not) of a set into another will be denoted by \subset .

For any subset X of \mathbb{R}^d , $\text{int}(X)$ denotes the interior of X while \overline{X} denotes its closure, both for the topology of \mathbb{R}^d induced by the Euclidean distance. Throughout this paper, we define the *complement set* of X as the closure of the classical complement set and denote it by ${}^{\neg}X := \overline{\mathbb{R}^d \setminus X} = \mathbb{R}^d \setminus \text{int}(X)$.

Let A be a subset of \mathbb{R}^d . Its *distance function* is $d_A : x \mapsto \inf\{\|x - a\| \mid a \in A\}$. This function is 1-Lipschitz over \mathbb{R}^d and thus differentiable almost everywhere

for the Lebesgue measure. For any positive real r and for any subset X of \mathbb{R}^d , define the r and $-r$ tubular neighborhoods (respectively *offsets* and *counter offsets*) of X (see Figure 1, left) as follows:

$$\begin{aligned} X^r &:= \left\{ x \in \mathbb{R}^d \mid d_X(x) \leq r \right\} \\ X^{-r} &:= \left\{ x \in \mathbb{R}^d \mid d_{-X}(x) \geq r \right\}. \end{aligned}$$

The *Hausdorff distance* $d_H(A, B)$ between two subsets A, B of \mathbb{R}^d is the infimum of the set of $t \in \mathbb{R}^+$ such that $B \subset A^t$ and $A \subset B^t$. It is also equal to $\|d_A - d_B\|_\infty = \sup_{x \in \mathbb{R}^d} |d_A(x) - d_B(x)|$, the infinity norm between d_A and d_B . The Hausdorff distance yields a topology on the set of compact subsets of \mathbb{R}^d . One easily checks that given X a compact subset of \mathbb{R}^d , the set equality $\overline{\text{int}(X)} = X$ is equivalent to the Hausdorff convergence of the arbitrarily small counter-offsets to X itself, i.e., $\lim_{r \rightarrow 0^+} X^{-r} = X$.

A *cone* A in \mathbb{R}^d is a set stable under multiplication by a positive number. Given any subset B of \mathbb{R}^d , we denote by *Cone* B the smallest cone containing B , defined as the image of $[0, \infty) \times B$ by the scalar multiplication map $(\lambda, x) \mapsto \lambda x$. We denote by *Conv* B the *convex hull* of B . The *dimension* of a cone or a convex set is the dimension of the vector space it spans. The *polar cone* or *dual cone* of a set $B \subset \mathbb{R}^d$, denoted by B° , is the convex cone defined by:

$$B^\circ := \{u \in \mathbb{R}^d \mid \langle u, b \rangle \leq 0 \quad \forall b \in B\}.$$

The polar cone operation is idempotent on convex cones, as it notably verifies the identity $(B^\circ)^\circ = \text{Conv}(\text{Cone } B)$ for any subset B of \mathbb{R}^d .

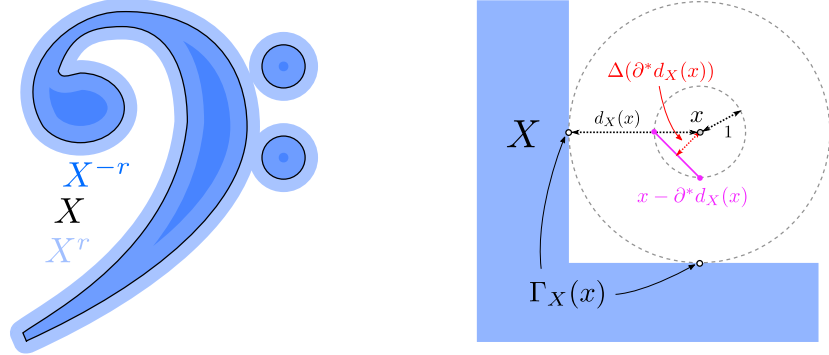
Given a subset X of \mathbb{R}^d , its *distance to 0* measures how far it is from intersecting $\{0\}$. It is defined by $\Delta(X) := \inf \{\|x\| \mid x \in X\} = d_X(0)$.

Given a locally Lipschitz function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, the *Clarke gradient of f at x* is the convex hull of limits of the form $\lim_{n \rightarrow \infty} \nabla f(x + h_n)$ where h_n is a sequence converging to 0 such that the gradient of f exists at $x + h_n$ for every $n \in \mathbb{N}$. We denote the Clarke gradient of f at x by $\partial^* f(x)$. In particular, if $f = d_X$ and if x lies outside of X , it is known (e.g., [3]) that $-\partial^* d_X(x)$ is the convex hull of the directions to the points $z \in X$ such that $d_X(x) = \|x - z\|$:

$$\partial^* d_X(x) := \text{Conv} \left(\left\{ \frac{x - z}{\|x - z\|} \mid z \in \Gamma_X(x) \right\} \right)$$

where $\Gamma_X(x)$ is the set of *closest points* to x in X (Figure 1, right). Elements of $\Gamma_X(x)$ will be denoted by the letter ξ . In particular, we denote by $\xi_X(x)$ the closest point to x in X when $\Gamma_X(x)$ is a singleton.

We say that $x \in \mathbb{R}^d$ is a *critical point* of a locally Lipschitz function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ when $0 \in \partial^* \phi(x)$, which is equivalent to $\Delta(\partial^* \phi(x)) = 0$. A number $c \in \mathbb{R}$ is called a *critical value* of ϕ when $\phi^{-1}(\{c\})$ contains a critical point, and a *regular value* of ϕ otherwise.



A bass clef X inflated (X^r) and eroded (X^{-r})

Clarke gradient of d_X outside of X

FIGURE 1: Offsets of X and Clarke gradient of d_X outside of X .

Given μ in $(0, 1]$, the μ -reach of a subset X of \mathbb{R}^d is defined by:

$$\text{reach}_\mu(X) := \sup \left\{ s \in \mathbb{R} \mid d_X(x) \leq s \implies \Delta(\partial^* d_X(x)) \geq \mu \right\}. \quad (2.1)$$

Equivalently, having $\text{reach}_\mu(X) > 0$ means that in a certain neighborhood of X , the cosines of the half-angles between two closest points in X are bounded from below by μ . This definition coincides with the classical one found in geometric inference as $\Delta(\partial^* d_X(x))$ is the norm of the generalized gradient $\nabla d_X(x)$ defined by Lieutier in [10].

Throughout this article, when no value of μ has been fixed, for any closed $X \subset \mathbb{R}^d$, having a positive μ -reach means that there exists $\mu \in (0, 1]$ with $\text{reach}_\mu(X) > 0$. The class of sets having a positive μ -reach is certainly broad, intuitively containing stratified sets without concave cusps. A corollary from Fu [6, Lemma 1.6] is that for any subanalytic set $X \subset \mathbb{R}^d$, the set of values $r > 0$ such that X^r has not a positive μ -reach is finite.

The *reach* of a subset X of \mathbb{R}^d is a quantity that was first studied by Federer in [4] and that coincides with $\text{reach}_1(X)$. It is the largest number t such that $d_X(x) < t$ implies that x has a unique closest point in X . The class of sets with positive reach notably contains convex sets and submanifolds of Euclidean spaces.

Geometric properties of such sets have been studied for a long time, and we refer the reader to [13] for a broad overview.

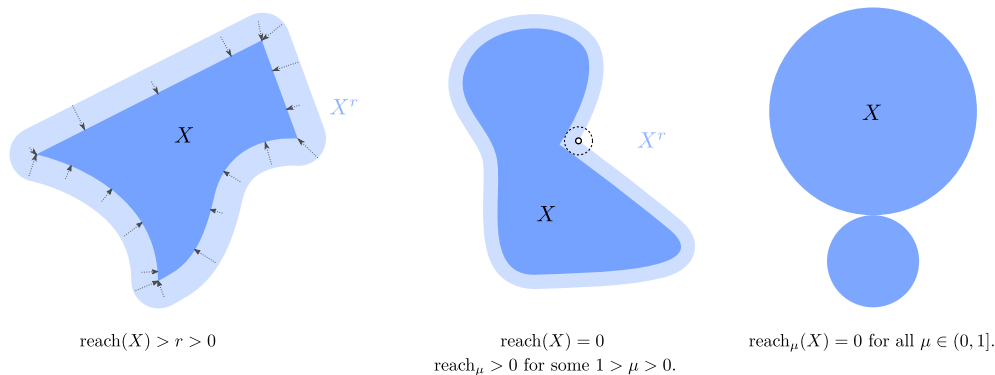


FIGURE 2: Sets with particular reach_μ .

When X has a positive μ -reach the complement sets of small offsets of X have positive reach:

Theorem 2.1 (Reach of complements of offsets [2, 4.1]). *Let X be a compact subset of \mathbb{R}^d , $\mu \in (0, 1]$ and $0 < r < \text{reach}_\mu(X)$. Then $\text{reach}(\complement(X^r)) \geq \mu r$.*

The *tangent cone* of X at x , $\text{Tan}(X, x)$ is defined as the cone generated by the limits $\lim_{n \rightarrow \infty} \frac{x_n - x}{\|x_n - x\|}$, where the sequence $(x_n)_{n \in \mathbb{N}}$ belongs in X , converges to x and never takes the value x . In that case, we say that u is *represented* by the sequence $(x_n)_{n \in \mathbb{N}}$. When $X \subset \mathbb{R}^d$ has positive reach, the set $\text{Tan}(X, x)$ is a convex cone.

When X has positive reach, we define its *normal cone* at x , denoted by $\text{Nor}(X, x)$, as the set dual to the tangent cone at x :

$$\text{Nor}(X, x) := \text{Tan}(X, x)^\circ.$$

It is related to the projection to the closest point in X function ξ_X by the following characterisation, for any $0 < t < \text{reach}(X)$:

$$\text{Nor}(X, x) \cap \mathbb{S}^{d-1} = \left\{ u \in \mathbb{S}^{d-1} \mid \xi_X(x + tu) = x \right\}.$$

If $X \subset \mathbb{R}^d$ has positive reach, we say that X is *fully dimensional* when $\text{Tan}(X, x)$ has dimension d for every $x \in \partial X$. This is equivalent to having the set equality $\overline{\text{int}(\text{Tan}(X, x))} = \text{Tan}(X, x)$ for all $x \in \partial X$. In particular, a Lipschitz domain of \mathbb{R}^d is always fully dimensional.

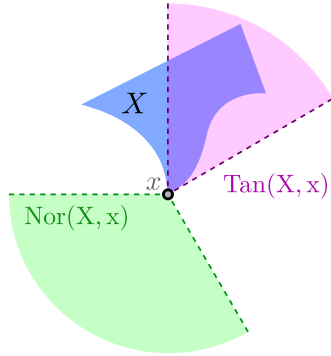


FIGURE 3: Tangent and normal cones of X at x when $\text{reach}(X) > 0$.

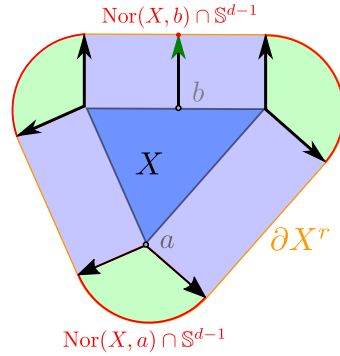


FIGURE 4: Some unit normal cones (in red) when $0 < r < \text{reach}(X)$.

2.2 Normal bundles

We are now in position to define the *normal bundle* of sets with positive reach or whose complement sets have positive reach.

Definition 2.2 (Sets admitting a normal bundle). *Let $X \subset \mathbb{R}^d$. When $\complement X := \mathbb{R}^d \setminus X$ has positive reach and is a Lipschitz domain, define*

$$\text{Nor}(X, x) := -\text{Nor}(\complement X, x).$$

This definition is consistent in case both $X, \complement X$ have positive reach.

If either $\text{reach}(X) > 0$ or both $\text{reach}(X) > 0$ and X is a Lipschitz domain, we say that X admits a normal bundle $\text{Nor}(X)$ with

$$\text{Nor}(X) := \bigcup_{x \in \partial X} \{x\} \times (\text{Nor}(X, x) \cap \mathbb{S}^{d-1}).$$

Normal bundles have intrinsic dimension $(d - 1)$, in the following sense.

Proposition 2.3 (Normal bundles are Lipschitz submanifolds of $\mathbb{R}^d \times \mathbb{S}^{d-1}$). *When either X or $\complement X$ has positive reach, $\text{Nor}(X)$ is a $(d - 1)$ -Lipschitz submanifold of $\mathbb{R}^d \times \mathbb{S}^{d-1}$. As a consequence, pairs $(x, n) \in \text{Nor}(X)$ are regular \mathcal{H}^{d-1} -almost everywhere on $\text{Nor}(X)$, i.e., the tangent cone $\text{Tan}(\text{Nor}(X), (x, n))$ is a vector space of dimension $(d - 1)$.*

Proof. Assume $\text{reach}(X) > 0$ and let $0 < r < \text{reach}(X)$. The map $\text{Nor}(X) \rightarrow \partial X^r, (x, n) \mapsto (x + rn)$ is bilipschitz and ∂X^r is a C^1 $(d - 1)$ -submanifold of \mathbb{R}^d by the implicit function theorem. Otherwise, $\text{Nor}(X) = \rho(\text{Nor}(\complement X))$ is the image of a Lipschitz submanifold by the bilipschitz map $\rho : (x, n) \mapsto (x, -n)$. \square

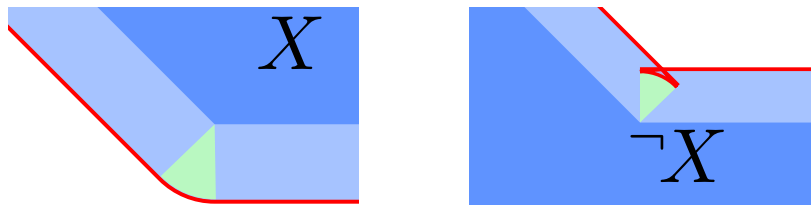


FIGURE 5: Normal bundles (in red) of a set of positive reach (left) and its complement set $\neg X$.

The construction of $\text{Nor}(X)$ stems from the more general concept of *normal cycle* of a set [14, 6]. While we do not need to write our hypothesis using this more involved language, in our case the normal bundle is the support of a $(d-1)$ -Legendrian cycle over $\mathbb{R}^d \times \mathbb{S}^{d-1}$, whose tangent spaces' structure is already known.

Proposition 2.4 (Tangent spaces of normal bundles [12]). *Let X be a compact set admitting a normal bundle $\text{Nor}(X)$. Then for any regular pair $(x, n) \in \text{Nor}(X)$, there exist*

- A family $\kappa_1, \dots, \kappa_{d-1}$ in $\mathbb{R} \cup \{\infty\}$ called principal curvatures at (x, n) ;
- A family $b_1, \dots, b_{d-1} \in \mathbb{R}^d$ of vectors orthogonal to n called principal directions at (x, n) such that the family $\left(\frac{1}{\sqrt{1+\kappa_i^2}} b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}} b_i \right)_{1 \leq i \leq d-1}$ form an orthonormal basis of $\text{Tan}(\text{Nor}(X), (x, n))$.

Moreover, principal curvatures are unique up to permutations.

These principal curvatures coincide with the ones found in differential geometry as eigenvalues of the second fundamental form. Indeed, assume that $X \subset \mathbb{R}^d$ is bounded by a $C^{1,1}$ -hypersurface, i.e the boundary of X is an hypersurface such that the Gauss map $x \in \partial X \mapsto n(x) \in \mathbb{S}^{d-1}$ is Lipschitz. The pair $(x, n(x)) \in \text{Nor}(X)$ is regular if and only if n is differentiable at x [5]. In that case, its differential is symmetric and its eigenvalues counted with multiplicity (resp. orthonormal basis of eigenvectors) are the principal curvatures (resp. principal directions) at $(x, n(x))$.

2.3 Critical points and Hessians for $f|_X$

In [5], Fu defines a notion of Morse functions over sets of positive reach and prove the Morse theorems. The sections of this paper focusing on generalized Morse

theory form the basis of the reasoning in Section 3. We recall below the definitions of critical points of smooth function restricted to a compact set, Hessians and non-degenerate critical points of restricted functions. We will use these definitions as they naturally extend to any set admitting a normal bundle. The projection $\mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ onto the first factor is denoted by π_0 .

Definition 2.5 (Critical points and Hessian). *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be smooth and X be a set of \mathbb{R}^d admitting a normal bundle.*

- *Let $(x, n) \in \text{Nor}(X)$ be a regular pair as in Proposition 2.4. The second fundamental form $\mathbb{I}_{x,n}$ of X at (x, n) is defined as the bilinear form on $\pi_0(\text{Tan}(\text{Nor}(X), (x, n)))$ such that for every pair $(u, v), (u', v')$ in $\text{Tan}(\text{Nor}(X), (x, n))$,*

$$\mathbb{I}_{x,n}(u, u') := \langle u, v' \rangle. \quad (2.2)$$

Taking (b_i) an orthonormal basis of $\pi_0(\text{Tan}(\text{Nor}(X), (x, n)))$ consisting of all principal directions with finite associated principal curvatures, this definition is equivalent to:

$$\mathbb{I}_{x,n}(b_i, b_j) := \kappa_i \delta_{i,j} \quad (2.3)$$

and generalizes the classical fundamental form obtained when X has a smooth boundary;

- *We say that $x \in X$ is a critical point of $f|_X$ when $\nabla f(x) \in -\text{Nor}(X, x)$;*
- *We say that $c \in \mathbb{R}$ is a critical value of $f|_X$ when $f^{-1}(c)$ contains at least a critical point of $f|_X$. Otherwise, c is a regular value of $f|_X$;*
- *If x is a critical point of $f|_X$ with $\nabla f(x) \neq 0$, let $n := \frac{-\nabla f(x)}{\|\nabla f(x)\|}$. When (x, n) is a regular pair, the Hessian of f restricted to X at x denoted by $H_x f|_X$ is defined over $\pi_0(\text{Tan}(N_X, (x, n)))$ by:*

$$H_x f|_X(u, u') := H_x f(u, u') + \|\nabla f(x)\| \mathbb{I}_{x,n}(u, u');$$

- *The index of this Hessian is the dimension of the largest subspace on which $H_x f|_X$ is negative definite;*
- *We say that a critical point x of $f|_X$ is non-degenerate when $\nabla f(x) \neq 0$, (x, n) is a regular pair of $\text{Nor}(X)$, and the Hessian $H_x f|_X$ is not degenerate;*
- *$f|_X$ is said to be Morse when its critical points are non-degenerate.*

Using these definitions, Fu proved the Morse theorems for sets with positive reach.

Theorem 2.6 (*Generalized Morse theory for sets with positive reach*). Let X be a compact subset of \mathbb{R}^d with positive reach and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function such that $f|_X$ is Morse with at most one critical point per level set.

Then for any regular value $c \in \mathbb{R}$, X_c has the homotopy type of a CW-complex with one λ_p cell for each critical point p such that $f(p) < c$, where

$$\lambda_p = \text{Index of } Hf|_X \text{ at } p.$$

2.4 Clarke gradients and approximate flows

We use a classical tool in the analysis of Lipschitz function called the *Clarke Gradient*. We recall its definition and study some of its properties. We refer the reader to the original article from Clarke [3] for the properties we do not prove.

Definition 2.7 (Clarke gradients of locally Lipschitz functions). Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally Lipschitz function. Its Clarke gradient at x is the subset of \mathbb{R}^d defined as the convex hull of limits of the form $\nabla\phi(x+h)$, $h \rightarrow 0$.

$$\partial^* \phi(x) := \text{Conv} \left(\lim_{i \rightarrow \infty} \nabla\phi(x_i) \mid x_i \in \mathbb{R}^d \rightarrow x, \phi \text{ differentiable at } x_i \text{ for all } i \right).$$

Every time we will refer to the explicit definition of the Clarke gradient, the fact that ϕ needs to be differentiable at any x_i will be implied.

Here are some basic properties of the Clarke gradient.

Proposition 2.8 (Basic properties of the Clarke Gradient). Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally Lipschitz function.

- By Rademacher's theorem, $\partial^* \phi(x)$ is non-empty for all x ;
- When ϕ is smooth around x , we have

$$\partial^* \phi(x) = \{\nabla\phi(x)\};$$

- If ϕ is R -Lipschitz around x , $\partial^* \phi(x) \subset B(0, R)$.

A key property of the Clarke gradient of any locally Lipschitz function is its upper semi-continuity [3, Definition 1.1], leading to the following proposition.

Proposition 2.9 (Semi-continuity of Clarke gradients). *Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally Lipschitz function. If a sequence $(x_i)_{i \in \mathbb{N}}$ converges to x , we have*

$$\liminf_{i \rightarrow \infty} \Delta \left(\partial^* \phi(x_i) \right) \geq \Delta \left(\partial^* \phi(x) \right).$$

Assuming $\partial^* \phi(x)$ stays uniformly away from 0, we are able to build deformation retractions between the sublevel sets of ϕ using approximations of what would be the flow of $-\phi$ had it been smooth.

Proposition 2.10 (Approximate inverse flow of a Lipschitz function). *Let $a < b \in \mathbb{R}$. Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz function on $\overline{\phi^{-1}(a, b)}$. Assume that*

$$\inf \{ \Delta(\partial^* \phi(x)) \mid x \in \phi^{-1}(a, b) \} = \mu > 0.$$

Then for every $\varepsilon > 0$, there exists a continuous function

$$C_\phi : \begin{cases} [0, 1] \times \phi^{-1}(\infty, b] & \rightarrow \phi^{-1}(-\infty, b] \\ (t, x) & \mapsto C_\phi(t, x) \end{cases}$$

such that

- *For any $s > t$ and x such that $C(s, x) \in \phi^{-1}(a, b]$, we have*

$$\phi(C_\phi(s, x)) - \phi(C_\phi(t, x)) \leq -(s - t)(b - a)$$
- *For any $t \in [0, 1]$ and $x \in \phi^{-1}(\infty, a]$, we have $C_\phi(t, x) = x$*
- *For any $x \in \phi^{-1}(-\infty, b]$, the map $s \mapsto C_\phi(s, x)$ is $\frac{b-a}{\mu-\varepsilon}$ -Lipschitz.*

In particular, $C_\phi(1, \cdot)$ is a deformation retraction between $\phi^{-1}(-\infty, a]$ and $\phi^{-1}(-\infty, b]$.

Proof. A weaker form of this claim can be found in section D of [9]. Here the constants have been optimized and the proposition generalized to Lipschitz functions. For the sake of completeness, we display a full proof.

Let $\varepsilon > 0$ and let $x \in \phi^{-1}(a, b]$. By semi-continuity of the Clarke gradient we can consider B_x an open ball centered in x such that $\partial^* \phi(y) \subset \partial^* \phi(x)^\varepsilon$ for any $y \in B_x$. Since $\partial^* \phi(x)$ is a closed convex set, there is a unique point $W(x)$ in $\partial^* \phi(x)$ realising the distance to 0 i.e., $\|W(x)\| = \Delta(\partial^* \phi(x))$. This is the closest point to 0 in $\partial^* \phi(x)$. From the convexity of $\partial^* \phi(x)$, we have:

$$\forall u \in \partial^* \phi(x), \langle u, W(x) \rangle \geq \|W(x)\|^2. \quad (2.4)$$

The family $\{B_x\}_{x \in \phi^{-1}(a,b)}$ is an open covering of $\phi^{-1}(a,b]$. By paracompactness, there exists a locally finite partition of unity $(\rho_i)_{i \in I}$ subordinate to this family, i.e., such that the support of each ρ_i is included in one of the balls $B(x_i)$ with $x_i \in \phi^{-1}(a,b]$. Use them to define the vector field V as a smooth interpolation of normalized $-W$:

$$V(y) := - \sum_{i \in I} \rho_i(y) \frac{W(x_i)}{\|W(x_i)\|}. \quad (2.5)$$

Obviously $\|V(x)\| \leq 1$ and V is locally Lipschitz. Now by classical results there is a flow C of V defined on a maximal open domain \mathbb{D} in $\mathbb{R}^+ \times \phi^{-1}(a,b]$. For any $x \in \phi^{-1}(a,b]$ and any $\zeta \in \partial^* \phi(x)$, we have:

$$\left\langle \frac{\partial}{\partial t} C(0,x), \zeta \right\rangle \leq - \sum_{i \in I} \rho_i(x) (\|W(x_i)\| - \varepsilon) \leq -\mu + \varepsilon. \quad (2.6)$$

Define \mathbb{D}_x via $(\mathbb{R}^+ \times \{x\}) \cap \mathbb{D} =: \mathbb{D}_x \times \{x\}$ the maximal subset of \mathbb{R}^+ for which the flow starting at x is defined. The set \mathbb{D}_x is connected in \mathbb{R}^+ and we put $s_x = \sup \mathbb{D}_x$, assuming this is finite. Now the trajectory $C(\cdot, x)$ is 1-Lipschitz, meaning that the curve $s \mapsto C(s, x)$ is rectifiable. We can thus define $C(s_x, x)$ as the endpoint of this curve, that is, $C(s_x, x) = \lim_{s \rightarrow s_x} C(s, x)$.

The function $\phi(C(\cdot, x)) : \overline{\mathbb{D}_x} \rightarrow [a, b]$ is Lipschitz and thus differentiable almost everywhere. Let (s, x) be in \mathbb{D} with $\phi(C(\cdot, x))$ differentiable at s . Since we have $C(s+h, x) = C(s, C(h, x))$, we can assume $s = 0$ without loss of generality. Since $C(\cdot, x)$ has non-vanishing gradient $V(x)$ at 0, ϕ has a directional derivative $\phi'(x, V(x))$ in direction $V(x)$. From the work of Clarke [3, Proposition 1.4] we know that when the directional derivative exists, the Clarke gradients acts like a maxing support set, that is:

$$\phi'(x, V(x)) \leq \max \left\{ \langle \zeta, V(x) \rangle \mid \zeta \in \partial^* \phi(x) \right\} \leq -\mu + \varepsilon \quad (2.7)$$

Any Lipschitz function is absolutely continuous, thus when $s \leq t \in \mathbb{D}_x$ we can integrate the previous inequality to obtain:

$$\phi(C(s, x)) - \phi(C(t, x)) \leq -(\mu - \varepsilon)(s - t) \quad (2.8)$$

This yields $\phi(C(s_x, x)) = a$. This also implies s_x needs to be finite, since reaching $\phi^{-1}(a)$ only takes a finite time. More precisely we have $s_x \leq \frac{b-a}{\mu-\varepsilon}$ for all $x \in \phi^{-1}(a,b]$.

We extend the flow to $\mathbb{R}^+ \times \phi^{-1}(-\infty, b]$ by putting

$$C(t, x) := \begin{cases} C(\min(t, s_x), x) & \text{when } a < \phi(x) \leq b, \\ x & \text{else.} \end{cases}$$

We will now show that C is continuous at every point $(s, x) \in \mathbb{R}^+ \times \phi^{-1}(-\infty, b]$. C is obviously continuous inside its original domain \mathbb{D} . C is continuous inside $\mathbb{R}^+ \times \phi^{-1}(-\infty, a)$ since in this set $C(t, x) = x$. We now turn our attention to the other points. Let k be a Lipschitz constant for ϕ over $\overline{\phi^{-1}(a, b]}$.

Let $x \in \phi^{-1}(a, b]$ and let $s \geq s_x$. Let $c > 0$. For every $\delta > 0$, there exists $\rho_x(\delta) > 0$ such that for all $y \in B(x, \rho_x(\delta))$, $s_y > s_x - c$ (i.e the original flow starting at y is well-defined at time $s_x - c$) and $|\phi(C(t, y)) - \phi(C(t, x))| \leq \delta$ for any $t \in [0, s_x - c]$. This implies $\phi(C(s_x - c, y)) \leq a + \delta + kc$, which yields $s_y \leq s_x - c + \frac{kc + \delta}{\mu - \varepsilon}$. And finally, for any (y, t) such that $|s - t| \leq c$ and $\|y - x\| \leq \rho_x(\delta)$, we have:

$$\begin{aligned} \|C(y, t) - C(x, s)\| &\leq \|C(\min(t, s_y), y) - C(s_x - c, y)\| \\ &\quad + \|C(s_x - c, y) - C(s_x - c, x)\| \\ &\quad + \|C(s_x - c, x) - C(s_x, x)\| \\ &\leq \frac{\delta + kc}{\mu - \varepsilon} + \delta + c. \end{aligned}$$

The only case left is when $\phi(x) = a$. Then $C(s, x) = x$ for all $s \in \mathbb{R}^+$. Since $u \mapsto \max(a, \phi(u))$ is k -lipschitz, we have $s_y \leq \frac{k\|x - y\|}{\mu - \varepsilon}$. We can write:

$$\|C(s, y) - C(s, x)\| \leq \|C(s, y) - y\| + \|y - x\| \leq \left(\frac{k}{\mu - \varepsilon} + 1\right) \|x - y\|.$$

and thus C is continuous at (s, x) . Finally we reparametrize C to obtain $C_\phi(t, x) = C\left(\frac{(b-a)t}{\mu - \varepsilon}, x\right)$ which yields an homotopy such that $\phi^{-1}(-\infty, a]$ is a strong deformation retraction of $\phi^{-1}(-\infty, b]$. \square

2.5 Relating normal cones to Clarke gradients of distance functions

We prove several results on tangent cones of compact sets of \mathbb{R}^d verifying weak regularity assumptions, leading to Theorem 2.15 which relates normal cones to the Clarke gradient of the distance function. These assumptions are verified by all *complementary regular sets* defined in Section 3.1, which is the class for which we will prove the Morse theorems.

Lemma 2.11 (Tangent cone of the boundary). *Let $X \subset \mathbb{R}^d$. Then for every $x \in \partial X$,*

$$\text{Tan}(\partial X, x) = \text{Tan}(X, x) \cap \text{Tan}(\lrcorner X, x).$$

Proof. The cone $\text{Tan}(\partial X, x)$ being included in both $\text{Tan}(X, x)$ and $\text{Tan}(\complement X, x)$, we have to prove that $\text{Tan}(X, x) \cap \text{Tan}(\complement X, x)$ is included in $\text{Tan}(\partial X, x)$.

Let $u \in \text{Tan}(X, x) \cap \text{Tan}(\complement X, x)$ be of norm 1. Take a sequence x_n (resp. $\complement x_n$) in X (resp. $\complement X$) representing u , i.e., such that

$$\begin{aligned} x_n &= x + \|x_n - x\| (u + o(1)) \\ \complement x_n &= x + \|\complement x_n - x\| (u + o(1)). \end{aligned}$$

The segment $[x_n, \complement x_n]$ has to intersect ∂X , which means that there exists a $\lambda_n \in [0, 1]$ such that $\partial x_n = \lambda_n x_n + (1 - \lambda_n) \complement x_n$ belongs in ∂X . This yields

$$\partial x_n - x = (\lambda_n \|x_n - x\| + (1 - \lambda_n) \|\complement x_n - x\|) (u + o(1))$$

Taking the norm of this equality yields

$$\|\partial x_n - x\| = (\lambda_n \|x_n - x\| + (1 - \lambda_n) \|\complement x_n - x\|) + o(\|\partial x_n - x\|).$$

This quantity is strictly positive when n is large enough, and we have

$$\partial x_n - x = \|\partial x_n - x\| (u + o(1))$$

meaning that u is represented by the sequence ∂x_n , which lies in ∂X . \square

Lemma 2.12 (Complement of tangent cones are tangent cones of complements). *Let $X \subset \mathbb{R}^d$ be a closed set such that $\complement X$ has positive reach. For any $x \in \partial \complement X$, we have:*

$$\complement \text{Tan}(\complement X, x) = \text{Tan}(X, x).$$

Proof. Since $\text{Tan}(X, x) \cup \text{Tan}(\complement X, x) = \mathbb{R}^d$, we know that $\complement \text{Tan}(\complement X, x) \subset \text{Tan}(X, x)$. We will show the opposite inclusion by proving that $\text{Tan}(X, x) \cap \text{int}(\text{Tan}(\complement X, x)) \cap \mathbb{S}^{d-1} = \emptyset$.

Let u be a unit vector in $\text{Tan}(X, x) \cap \text{int}(\text{Tan}(\complement X, x))$ and let $v \in \text{Nor}(\complement X, x)$. Then by definition, we have $\text{Nor}(\complement X, x) = \text{Tan}(\complement X, x)^\circ$, which yields

$$\langle u, v \rangle \leq 0. \tag{2.9}$$

We also have, for every $\lambda \in (0, \text{reach}(\complement X))$,

$$\text{int}(B(x + \lambda v, \lambda)) \cap \complement X = \emptyset. \tag{2.10}$$

Since $u \in \text{int}(\text{Tan}(\complement X, x))$, there exists a $\lambda' \in (0, \text{reach}(X))$ small enough such that $u + \lambda' v \in \text{Tan}(\complement X, x)$. Now let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\complement X$ representing $u + \lambda' v$. We will prove that y_n cannot be in $\complement X$ for large n . We can write

$$y_n = x + a_n (u + \lambda' v + \omega_n)$$

with $\|\omega_n\| \rightarrow 0$ and $a_n \rightarrow 0^+$. By 2.9, this leads to

$$\|y_n - x - \lambda v\|^2 \leq \lambda^2 - 2\lambda\lambda' a_n + o(a_n).$$

When n is large enough, this quantity is strictly smaller than λ^2 , which contradicts 2.10. \square

Lemma 2.13 (Tangent cone stability under addition with $\partial^* d_X(x)$). *Let $X \subset \mathbb{R}^d$, $x \in \partial X$ and $u \in \partial^* d_X(x)^\circ$. Then for all $h \in \text{Tan}(X, x)$, $u + h \in \text{Tan}(X, x)$.*

Proof. We use Clarke's [3] characterization of the dual cone to the Clarke gradient:

$$\partial^* d_X(x)^\circ = \left\{ u \mid \lim_{\substack{x_h \rightarrow x \\ x_h \in X}} \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} d_X(x_h + \delta u) = 0 \right\}. \quad (2.11)$$

Consider the following modulus of continuity:

$$\omega_u(\varepsilon, \lambda) := \sup_{\substack{x_h \in X \\ \|x - x_h\| \leq \varepsilon}} \sup_{0 < \delta \leq \lambda} \frac{d_X(x_h + \delta u)}{\delta}.$$

When u belongs to $\partial^* d_X(x)^\circ$, by Clarke's characterization 2.11 we have

$$\lim_{\substack{\varepsilon \rightarrow 0^+ \\ \lambda \rightarrow 0^+}} \omega_u(\varepsilon, \lambda) = 0.$$

Now let $x_i \rightarrow x$ be a sequence representing any unit vector $h \in \text{Tan}(\partial X, x)$. Put $\varepsilon_i = \|x - x_i\|$ and consider the sequence $x_i + \varepsilon_i u$. Take ξ_i in $\Gamma_X(x_i + \varepsilon_i u)$, that is, a point in X realizing the distance of $x_i + \varepsilon_i u$ to X . By the definition of ω_u , we have:

$$\|\xi_i - x_i - \varepsilon_i u\| = d_X(x_i + \varepsilon_i u) \leq \varepsilon_i \omega(\varepsilon_i, \varepsilon_i).$$

Thus we can write

$$\xi_i - x = \varepsilon_i(h + o(1) + u + O(\omega(\varepsilon_i, \varepsilon_i))) = \varepsilon_i(u + h + o(1))$$

which shows that ξ_i is a sequence in X representing $u + h$. \square

Lemma 2.14 (Relationship between normal cones and Clarke gradients). *Let $X \subset \mathbb{R}^d$ such that $\text{reach}(\lrcorner X) > 0$. Then if $\text{Tan}(\lrcorner X, x)$ has full dimension, we have:*

$$\partial^* d_X(x)^\circ \subset -\text{Tan}(\lrcorner X, x)$$

Proof. Let $u \in \partial^* d_X(x)^\circ$. By Lemma 2.13 we know that

$$u + \text{Tan}(X, x) \subset \text{Tan}(X, x)$$

which is equivalent to

$$u + \mathbb{R}^d \setminus \text{Tan}(X, x) \supset \mathbb{R}^d \setminus \text{Tan}(X, x).$$

By Lemma 2.12 we have $\text{Tan}(\text{int} X, x) = \text{Tan}(\text{int} X, x)$. Along with the full dimensionality condition, taking the closure of the previous inclusion yields:

$$u + \text{Tan}(\text{int} X, x) \supset \text{Tan}(\text{int} X, x)$$

which implies that u belongs in $-\text{Tan}(\text{int} X)$. \square

We are now in position to relate normal cones of a set X to the Clarke gradient of d_X under weak regularity assumptions.

Theorem 2.15 (Normal cones and the Clarke gradient of the distance function). *Let $X \subset \mathbb{R}^d$ be such that $\text{reach}(\text{int} X) > 0$ and such that $\text{int} X$ is fully dimensional. Let $x \in \partial \text{int} X$. Then the normal cone of X at x is determined by the Clarke gradient of d_X :*

$$\text{Nor}(X, x) = \text{Cone } \partial^* d_X(x).$$

Proof. Let $\text{reach}(\text{int} X) > r > 0$. First remark that we have

$$\begin{aligned} \partial^* d_{X-r}(x) &= - \text{Conv} \left\{ \frac{x-z}{\|x-z\|} \mid z \in X^{-r} \text{ with } d_X^{-r}(x) = \|z-x\| \right\} \\ &= - \text{Conv} \{ u \in \mathbb{S}^{d-1} \mid d_{\text{int} X}(x+ru) = r \} \\ &= - \text{Conv} \left(\text{Nor}(\text{int} X, x) \cap \mathbb{S}^{d-1} \right). \end{aligned}$$

On the other hand, by definition the Clarke gradient of d_{X-r} at x is determined locally by the gradients around x in every direction:

$$\partial^* d_{X-r}(x) = \text{Conv} \left\{ \lim_{i \rightarrow \infty} \nabla d_{X-r}(x_i) \mid (x_i) \in (\mathbb{R}^d)^{\mathbb{N}} \text{ converging to } x \right\}.$$

Now compare to the Clarke gradient of d_X at x , for which the gradient contributing only come from directions outside of X [3]:

$$\partial^* d_X(x) = \text{Conv} \left(\{0\} \cup \left\{ \lim_{i \rightarrow \infty} \nabla d_X(x_i) \mid (x_i) \xrightarrow{i \rightarrow \infty} x \text{ with } d_X(x_i) > 0 \right\} \right).$$

Note that in both definition we implicitly require x_i to be points where d_X is differentiable. On those points the gradients of d_X and $d_{X^{-r}}$ coincide, yielding

$$\text{Cone } \partial^* d_X(x) \subset \text{Cone } \partial^* d_{X^{-r}}(x) = -\text{Nor}(\cap X, x). \quad (2.12)$$

The other inclusion $-\text{Nor}(\cap X, x) \subset \text{Cone } \partial^* d_X(x)$ is Lemma 2.14 after applying the dual cone operation. \square

3 Morse theory for complementary regular sets

In this section, we use the previous tools and propositions to infer the two Morse theorems when X is *complementary regular* (cf. Section 3.1) and f is Morse (in the sense of Definition 2.5). In this setting, the eroded sets X^{-r} converge to X in the Hausdorff sense when r tends to 0 and they are $C^{1,1}$ by the implicit function theorem when $0 < r < \text{reach}(\cap X)$.

Our approach is as follows. Let $c \in \mathbb{R}$. Consider a family of functions $f_{r,c}$ converging to f as r tends to 0, in a way we will later precise. When $r = 0$, our notations are consistent with $f_{0,c} = f$. Consider the sublevel sets:

$$X_c = X \cap f^{-1}(-\infty, c] \quad \text{and} \quad X_c^{-r} := X^{-r} \cap f_{r,c}^{-1}(-\infty, c].$$

They are the zero sublevel sets of the following functions:

$$\phi = d_X + \max(f - c, 0) \quad \text{and} \quad \phi_r = d_{X^{-r}} + \max(f_{r,c} - c, 0).$$

- In Section 3.1, we define the regularity condition required on sets $X \subset \mathbb{R}^d$ for which we prove the Morse Theorems. Such sets are called *complementary regular*. We describe some of their properties and show that these sets are exactly sets of the form Y^ε , where Y is a compact subset of \mathbb{R}^d and the map $\Delta \circ \partial^* d_Y$ is strictly positive over $d_Y^{-1}(\{\varepsilon\})$, i.e X is the offset of a set at a regular value of its distance function.
- In Section 3.2, we take f_r as f precomposed with a uniformly bounded smooth function. If c is a regular value of $f|_X$, we prove that there exists a $K > 0$ such that there exists a retraction of any tubular neighborhood $(X_c^{-r})^K$ onto X_c for any $r > 0$ small enough. We prove a technical lemma to ensure that we can build an approximate inverse flow of ϕ_r^c using Proposition 2.10.

- In Section 3.3 we study the case $r = 0$ and prove that for $\varepsilon > 0$ small enough, the set X_{c+a} is a deformation retract of X_{c+b} for any $a \leq b \in [-\varepsilon, \varepsilon]$ when c is a regular value, also using Proposition 2.10. As a consequence, we obtain the constant homotopy lemma.
- In Section 3.4 we let c be a critical value and assume there is only one critical point x in $f^{-1}(c)$, which is non-degenerate. We show that for any $\varepsilon > 0$ small enough the change in topology between $X_{c+\varepsilon}$ and $X_{c-\varepsilon}$ is determined by the curvature of X at the pair $(x, \frac{\nabla f(x)}{\|\nabla f(x)\|})$ and the Hessian of $f|_X$ at x . We prove this by considering $f_{r,c}$ to be f translated with magnitude r in the direction $-\nabla f(x)$. We extend this result to the case where the level set $f^{-1}(c)$ has a finite number of critical points changes by considering a modified, more involved $f_{r,c}$ which depends on the different critical points of $f^{-1}(c)$.

3.1 Complementary regular sets and their properties

In this section, we define the class of *complementary regular sets* which are the subsets of \mathbb{R}^d for which we will prove the Morse theorems. We describe some of their properties and prove that they are exactly offsets of compact subset of \mathbb{R}^d at a regular value.

Definition 3.1 (Complementary regular sets). *We say that a compact subset X of \mathbb{R}^d is a complementary regular set when it verifies the following three conditions:*

$$(A_1) \quad \overline{\text{int}(X)} = X;$$

$$(A_2) \quad \exists \mu \in (0, 1] \text{ such that } \text{reach}_\mu(X) > 0;$$

$$(A_3) \quad \text{reach}(\neg X) > 0.$$

Lemma 3.2 (Tangent cones of complementary regular sets contain a ball). *Let $\mu \in (0, 1]$ and let X be complementary regular with $\text{reach}_\mu(X) > 0$. Let $x \in \partial X$. Then $\text{Tan}(\neg X, x)$ contains a ball of radius μ centered around a unit vector.*

Proof. By [2, Section 3], we know that for each $0 < r < \text{reach}_\mu(X)$ there exists a point x_r such that $d_X(x_r) = r$ and $\|x_r - x\| \leq \frac{r}{\mu}$. Let r_n be any sequence converging to 0 and consider a sequence x_n such that $\|x_n - x\| \leq \frac{r_n}{\mu}$ and $d_X(x_n) = r_n$. Extracting a subsequence we can assume that $\frac{x_n - x}{\|x_n - x\|}$ converges to a unit vector $u \in \text{Tan}(\neg X, x)$, i.e we have

$$x_n = x + \varepsilon_n(u + o(1)). \tag{3.1}$$

where $\varepsilon_n = \|x_n - x\| \rightarrow 0^+$. Now let $v \in \mathbb{R}^d$ be in the unit ball. The sequence $x_n + \mu\varepsilon_n v$ lies in $\ulcorner X$ for any n , while we have

$$x_n + \varepsilon_n \mu v = x + \varepsilon_n(u + \mu v + o(1)) \quad (3.2)$$

which implies that $u + \mu v$ belongs in $\text{Tan}(\ulcorner X, x)$. \square

Corollary 3.3 (Normal cones of $\ulcorner X$ are thin). *Let $\mu \in (0, 1]$ and let X be complementary regular with $\text{reach}_\mu(X) > 0$. Let $x \in \partial X$. Then $\Delta(\text{Conv}(\text{Nor}(\ulcorner X, x) \cap \mathbb{S}^{d-1})) \geq \mu$.*

Proof. By the previous lemma, take a unit vector u such that $B(u, \mu) \subset \text{Tan}(\ulcorner X, x)$. This yields the opposite inclusion on their dual cones $\text{Nor}(X, x) \subset B(u, \mu)^\circ$. Take any unit vector $w \in B(u, \mu)^\circ$. For any $v \in \mathbb{S}^{d-1}$, we have

$$0 \geq \langle w, u + \mu v \rangle = \langle u, w \rangle + \mu \langle w, v \rangle.$$

Letting $v = w$, we see that any such w lies in the half space $H_u^{-\mu} = \{u' \in \mathbb{R}^d \mid \langle u, u' \rangle \leq -\mu\}$ which is a convex set such that $\Delta(H_u^{-\mu}) \geq \mu$. \square

Lemma 3.4 (Characterization of complementary regular sets). *Let X be a compact subset of \mathbb{R}^d and let $\mu \in (0, 1]$. Then the three conditions*

$$(A_1) \quad \overline{\text{int}(X)} = X;$$

$$(A'_2) \quad \text{reach}_\mu(X) > 0;$$

$$(A_3) \quad \text{reach}(\ulcorner X) > 0;$$

are equivalent to the existence of $\varepsilon, \delta > 0$ and of a compact subset Y of \mathbb{R}^d such that $X = Y^\varepsilon$ with $\inf\{\Delta(\partial^ d_Y(x)) \mid d_Y(x) \in [\varepsilon, \varepsilon + \delta]\} \geq \mu$. The quantity $\text{reach}_\mu(X)$ is the supremum of δ such that the previous inequality holds.*

Proof. On the one hand, assume the conditions (A_i) are true for $i \in \{1, 2, 3\}$. Then for any $0 < r < \text{reach}(\ulcorner X)$ we have $(X^{-r})^r = X$ thanks to (A_1) . Further assuming that $r < \text{reach}_\mu(X)$, any such X^{-r} will provide a suitable Y with $\varepsilon = r$. Now let $\delta \in (0, \text{reach}_\mu(X))$. For any $x \in \mathbb{R}^d$ such that $d_X(x) > 0$ we have $d_{X^{-r}} = d_X + r$ on a neighborhood of x . Thus we have:

$$\mu \leq \inf\{\Delta(\partial^* d_{X^{-r}}(x)) \mid d_{X^{-r}}(x) \in (r, r + \delta]\}.$$

We now bound $\Delta(\partial^* d_{X^{-r}}(x))$ from below for points x such that $d_{X^{-r}}(x) = r$. Those points are exactly the set $\partial^\ulcorner X$ when $r < \text{reach}(\ulcorner X)$. For such x , we have

$\partial^* d_{X^{-r}}(x) = -\text{Conv}(\text{Nor}(\cdot X, x) \cap \mathbb{S}^{d-1})$, and Corollary 3.3 yields the desired bound $\Delta(\partial^* d_{X^{-r}}(x)) \geq \mu$.

On the other hand, if there exist $\varepsilon > 0, Y \subset \mathbb{R}^d$ such that $X = Y^\varepsilon$ with $\inf\{\Delta(\partial^* d_Y(x)) \mid d_Y(x) \in [\varepsilon, \varepsilon + \delta]\} \geq \mu$, then by Clarke's Lipschitz local inversion theorem, the set X is a Lipschitz domain of \mathbb{R}^d , which implies that $\overline{\text{int}(X)} = X$ (condition (A_1)). Since $d_X = d_Y - \varepsilon$ around any point at distance to Y strictly greater than ε , by definition of the μ -reach we have $\text{reach}_\mu(X) \geq \delta$, implying condition (A_2) , and that $\text{reach}_\mu(X)$ is equal to the supremum of such δ . Finally, by lower semi-continuity of the Clarke gradient and compactness of Y , there exists a $\sigma > 0$ such that

$$\inf\{\Delta(\partial^* d_Y(x)) \mid d_Y(x) \in [\varepsilon - \sigma, \varepsilon + \delta]\} \geq \frac{\mu}{2} \quad (3.3)$$

which yields $\text{reach}(\cdot X) \geq \sigma \frac{\mu}{2} > 0$ by Theorem 2.1 combined with the equality $(Y^{\varepsilon - \sigma})^\sigma = X$, and condition (A_3) is verified. \square

Theorem 3.5 (Complementary regular sets are offsets of sets with regular value). *A set is complementary regular if and only if it is the offset of a compact set at a regular value of its distance function.*

Proof. This is a consequence of the previous lemma along with the semi-continuity of the Clarke gradient, since if $\text{reach}_\mu(X) > 0$ and $X = Y^\varepsilon$, there is a $\sigma > 0$ such that on $d_Y^{-1}[\varepsilon - \sigma, \varepsilon + \sigma]$, $\Delta(\partial^* d_Y)$ is greater than $\frac{\mu}{2}$ and thus positive. From the set equality $d_Y^{-1}(\varepsilon, \varepsilon + \sigma] = d_X^{-1}(0, \sigma]$ and the fact that in this set $\partial^* d_Y$ and $\partial^* d_X$ coincide, we have the desired result. \square

3.2 Building a deformation retraction between X_c and its smooth surrogate

For the remainder of this section we let $X \subset \mathbb{R}^d$ be a complementary regular set and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function. We also let η be a smooth function $\mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\|\eta\|_\infty \leq 1$.

Definition 3.6 (Closed sublevel sets). *Let $X \subset \mathbb{R}^d$ be a complementary regular set and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function. Let c be a regular value of $f|_X$ and let f_r be f precomposed by the translation by $r\eta$:*

$$f_r : x \mapsto f(x + r\eta(x)).$$

We define the smooth surrogates for X_c set as:

$$X_c^{-r} := X^{-r} \cap f_r^{-1}(-\infty, c]$$

and non-negative, locally Lipschitz functions

$$\phi^c := d_X + \max(f - c, 0) \quad \text{and} \quad \phi_r^c := d_{X^{-r}} + \max(f_r - c, 0).$$

verifying $X_c = (\phi^c)^{-1}(0)$ and $X_c^{-r} = (\phi_r^c)^{-1}(0)$. When the value of c is clear from the context, we write ϕ_r instead of ϕ_r^c to ease notations.

When c is a regular value, the following convergence of sublevel sets holds.

Lemma 3.7 (Hausdorff convergence of sublevel sets). *Let X be a complementary regular set, let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be smooth and let c be a regular value of $f|_X$. Then in the Hausdorff topology we have:*

$$\lim_{r \rightarrow 0} X_c^{-r} = X_c.$$

Proof. Since $\|\eta\| \leq 1$, we have $X_c^{-r} \subset (X_c)^r$ for any $r > 0$. Now assuming that there is no Hausdorff convergence, there is a point $x \in X$ and a real $t > 0$ such that $B(x, t)$ and X_c^{-r} have empty intersection for any $r > 0$ small enough. Let u be in $\text{Tan}(X, x)$. Since $X = \overline{\text{int}(X)}$, remark that there is a sequence $x_n \in \text{int}(X)$ representing u , i.e such that $x_n = x + \varepsilon_n(u + o(1))$ with the sequence ε_n in $\mathbb{R}^+ \setminus 0$ converging to 0. For n big enough, x_n lies in $B(x, t)$ and since it is in the interior of X , x_n belongs to X^{-r} for any $0 < r < d_X(x_n)$. Since it does not belong in X_c^{-r} , we have $f_r(x_n) > c$. When r goes to 0 this implies $c \leq f(x_n)$. Since $f(x) \leq c$, we have by first-order expansion $\langle \nabla f(x), u \rangle \geq 0$ for every u in $\text{Tan}(X, x)$, which amounts to the following inclusion in a half-space:

$$\text{Tan}(X, x) \subset -\nabla f(x)^\circ. \tag{3.4}$$

Combined with the fact that $\text{Tan}(X, x)$ is the complement set of the convex cone $\text{Tan}(\neg X, x)$, this yields $\text{Tan}(\neg X, x) = \nabla f(x)^\circ$ i.e $\text{Cone}(\nabla f(x)) = \text{Nor}(\neg X, x)$ which contradicts the fact that c is a regular value. \square

The following lemma gives a uniform lower bound on $\Delta \circ \partial^* \phi_r$ over neighborhoods of X_c^{-r} when r tends to 0 and c is a regular value.

Lemma 3.8 (Non vanishing $\partial^* \phi_r$ around a regular value). *Let c be a regular value of $f|_X$. Then there exists a positive constant α such that for any sequences of positive reals $(r_i), (K_i)$ such that $r_i, K_i \rightarrow 0^+$, and any sequence (x_i) of points within $\phi_{r_i}^{-1}(0, K_i]$ for all $i \in \mathbb{N}$, we have:*

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \alpha.$$

Proof. The map $\phi_{r_i} = d_{X-r_i} + \max(0, f_{r_i} - x)$ is the sum of a lipschitz function and the positive part of a smooth function. We distinguish seven cases to compute the Clarke gradient $\partial^* \phi_{r_i}(x_i)$, each with different contributions from d_{X-r_i} and $\max(0, f_{r_i} - c)$. By extracting subsequences we can assume that the sequence (x_i) lies in one of these cases. They are depicted in Figure 6. In fact, we will show that for any such sequence, we have:

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \min(\mu, \sigma, \kappa) > 0 \quad (3.5)$$

where

- $\kappa := \inf_{f^{-1}(c) \cap X} \|\nabla f\|$ is a positive quantity because c is a regular value of $f|_X$.
- $\mu \leq \inf_{t \rightarrow 0} \{\Delta(\partial^* d_X(x)) \mid 0 < d_X(x) < t\}$ is positive by hypothesis.
- $\sigma := \inf_{x \in \partial X \cap f^{-1}(c)} \Delta(A_x)$ where $x \mapsto A_x$ is the upper semi-continuous set-valued map defined by:

$$A_x := \left([0, 1] \cdot \partial^* d_X(x) + \{\nabla f(x)\}\right) \cup \left(\partial^* d_X(x) + [0, 1] \cdot \{\nabla f(x)\}\right).$$

For any point $x \in \partial X$, keep in mind that from Theorem 2.15 we have the identity

$$\text{Cone } \partial^* d_X(x) = \text{Nor}(X, x)$$

which means that any direction in $\partial^* d_X(x)$ is a direction in $\text{Nor}(X, x)$. The constant σ is positive when c is a regular value of $f|_X$. The set $\partial X \cap f^{-1}(c)$ is compact and the map $x \mapsto \Delta(A_x)$ is lower semi-continuous. Assume that σ is zero. Then there is be a point $x \in \partial X \cap f^{-1}(c)$ with $\Delta(A_x) = 0$. This means that the direction of $\nabla f(x)$ meets $\text{Nor}(X, x)$, and thus c is a critical value of $f|_X$.

Idea behind the proof. For each of the seven cases cases, we will show that $\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i))$ is greater than one among σ, κ, μ , depending on the contributions of d_{X-r_i} and f_{r_i} . Computations will show that $\partial^* \phi_{r_i}(x_i)$ either lies close to $\nabla f(x_i)$, $\partial^* d_X(x_i)$ or close to be inside A_{x_i} , each being bounded away from zero respectively by the non-vanishing of κ, μ and σ .

To ease some notations we write $\nu(x) := \frac{x}{\|x\|}$ and $\|\nabla f_{r_i} - \nabla f\|_{\infty, X^1} =: \varepsilon_i$ the infinity norm of $\nabla f_{r_i} - \nabla f$ over the 1-offset of X ¹. Remark that by elementary computations we have $\varepsilon_i = O(r_i)$.

¹We could have taken the infinity norm over any bounded neighborhood of X without altering the line of reasoning.

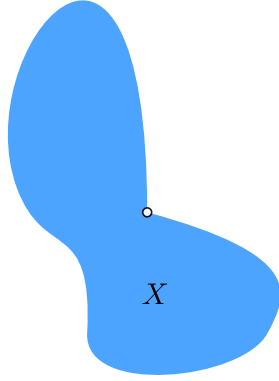
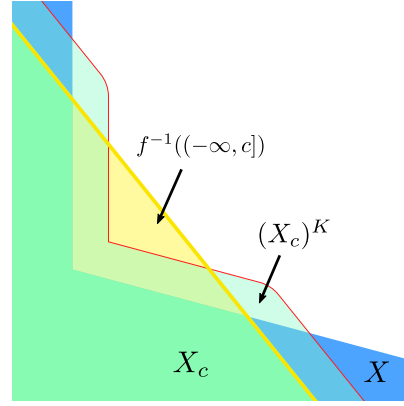


Illustration. X is a compact of \mathbb{R}^2 with $\text{reach}_\mu(X) > 0$ for some $\mu > 0$.



Zoomed-in depiction of $X_c = X \cap f^{-1}(-\infty, c]$ and a tubular neighborhood $(X_c)^K$, $K > 0$ where f is a linear form.

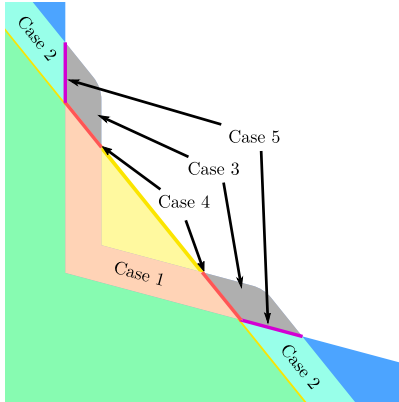
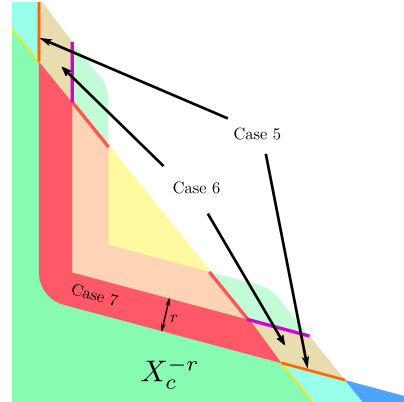


Illustration of cases 1 to 5 when $r = 0$. Cases 1 to 4 are defined independently of r .



Cases 5, 6 and 7 when $r > 0$.

FIGURE 6: Illustration of the 7 cases of Lemma 3.8.

Case 1. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) < c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i)$ with $0 < d_X(x_i) < K_i + d_H(X^{-r_i}, X)$ which tends to 0 as $i \rightarrow \infty$. By the μ -reach hypothesis, we have

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \mu > 0. \quad (3.6)$$

Case 2. $x_i \in \text{int}(X^{-r_i})$.

Then $\partial^* \phi_{r_i}(x_i) = \{\nabla f_{r_i}(x_i)\}$ and $0 < f_{r_i}(x_i) - c \leq K_i$. As such, we have the inclusion $\partial^* \phi_{r_i}(x_i) \subset \{\nabla f(x_i)\}^{\varepsilon_i}$ and we obtain

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \kappa > 0. \quad (3.7)$$

Case 3. $d_{X-r_i}(x_i) > r_i$ and $f_{r_i}(x_i) > c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i) + \nabla f_{r_i}(x_i) \subset (A_{x_i})^{\varepsilon_i}$, which yields

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0. \quad (3.8)$$

Case 4. $d_{X-r_i}(x_i) > r_i$ and $f_{r_i}(x_i) = c$.

First remark that since $d_{X-r_i}(x_i) > r_i$ we have $\partial^* d_{X-r_i}(x_i) = \partial^* d_X(x_i)$, $d_X(x_i) \rightarrow 0$ since $\lim_{r \rightarrow 0} X^{-r} = X$, and $d_X(x_i) > 0$. Now without loss of generality by extracting we can assume x_i converges to a point x in $\partial X \cap f^{-1}(c)$.

Now $\nabla f_{r_i}(x_i)$ has to be non-zero for i big enough as $\varepsilon_i = O(r_i)$ and

$$\liminf_{i \rightarrow \infty} \|\nabla f(x_i)\| \geq \inf_{x \in X \cap f^{-1}(c)} \|\nabla f(x)\| = \kappa$$

which yields that the set $\{y \mid f_{r_i}(y) \neq c\}$ has density 1 at x_i by the local inverse function theorem. As the Clarke gradient can be computed in a set of density 1 at x_i (see [3]), we have have for any x_i where $\nabla f_{r_i}(x_i) \neq 0$:

$$\partial^* \phi_{r_i}(x_i) = \text{Conv} \left\{ \lim_{n \rightarrow \infty} \nabla \phi_{r_i}(z_n) \mid z_n \rightarrow x_i, f_{r_i}(z_n) \neq c \right\}.$$

We can decompose this set as

$$\partial^* \phi_{r_i}(x_i) = \text{Conv}(A_+ \cup A_-)$$

where

$$\begin{aligned} A_+ &:= \left\{ \lim_{n \rightarrow \infty} \nabla \phi_{r_i}(z_n) \mid z_n \rightarrow x_i, f_{r_i}(z_n) > c \right\} \\ A_- &:= \left\{ \lim_{n \rightarrow \infty} \nabla \phi_{r_i}(z_n) \mid z_n \rightarrow x_i, f_{r_i}(z_n) < c \right\}. \end{aligned}$$

Now only d_{X-r_i} contributes to the gradients of A_- whereas f_{r_i} also contributes in A_+ . Thus any point in $\text{Conv}(A_+ \cup A_-)$ can be written as $u + \lambda \nabla f_{r_i}(x)$ where $u \in \partial^* d_{X-r_i}(x_i) = \partial^* d_X(x_i)$ and $\lambda \in [0, 1]$. This yields finally

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \Delta(A_x) \geq \sigma > 0. \quad (3.9)$$

Case 5. $x_i \in \partial X^{-r_i}$ and $f_{r_i}(x_i) > c$.

If $r_i > 0$, then $\partial^* d_{X^{-r_i}}(x_i)$ is the convex set generated by 0 and the direction normal to X^{-r_i} at x_i , that is $[0, 1] \cdot \nu(\xi_{\neg X}(x_i) - x_i)$. Note that this direction belongs in the normal cone $\text{Nor}(X, \xi_{\neg X}(x_i))$ as illustrated in Figure 7. Adding the contribution of f_{r_i} we obtain

$$\partial^* \phi_{r_i}(x_i) \subset (A_{\xi_{\neg X}(x_i)})^{\varepsilon_i}.$$

If $r_i = 0$, then $\partial^* \phi_{r_i}(x_i) = [0, 1] \cdot \partial^* d_X(x_i) + \nabla f_{r_i}(x_i)$ and we obtain

$$\partial^* \phi_{r_i}(x_i) \subset (A_{x_i})^{\varepsilon_i}.$$

Either way,

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \Delta(A_x) \geq \sigma > 0. \quad (3.10)$$

Now the remaining cases fit inside sequences of points (x, r) such that $0 < d_{X^{-r}}(x) \leq r$. Remark that $\text{reach}(X^{-r}) \geq r$. If $d_{X^{-r}}(x) < r$ we know that x has only one closest point $\xi_{X^{-r}}(x)$ in X , which yields $\partial^* d_{X^{-r}}(x) = \{\nu(x - \xi_{X^{-r}}(x))\}$. If $d_{X^{-r}}(x) = r$, x belongs to ∂X and the Clarke gradient $\partial^* d_{X^{-r}}(x)$ is $\text{Conv}(\text{Nor}(X, x) \cap \mathbb{S}^{d-1})$ which is $\text{Conv}(\text{Cone } \partial^* d_X(x) \cap \mathbb{S}^{d-1})$ by Theorem 2.15. These considerations are illustrated in Figure 7 with $0 < d_{X^{-r}}(x_1) < r$ and $d_{X^{-r}}(x_2) = r$. In any case, this leads to

$$\partial^* d_{X^{-r}}(x) \subset \text{Conv}(\partial^* d_X(\xi_{\neg X}(x)) \cup \mathbb{S}^{d-1}) \quad (3.11)$$

Case 6. $0 < d_X^{-r_i}(x_i) \leq r_i$ and $f_{r_i}(x_i) \geq c$

$\partial^* \phi_{r_i}(x_i) \subset \text{Conv}(\text{Nor}(X, \xi_{\neg X}(x)) \cap \mathbb{S}^{d-1}) + [0, 1] \cdot \nabla f_{r_i}(x_i)$. Now by compactness assume that $x_i \rightarrow x$. Then $x \in \partial X \cap f^{-1}(c)$ and thus

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \Delta(A_x) \geq \sigma > 0. \quad (3.12)$$

Case 7. $0 < d_X^{-r}(x_i) \leq r_i$ and $f_{r_i}(x_i) < c$

Then $\partial^* \phi_{r_i}(x_i) \subset \text{Conv}(\partial^* d_X(\xi_{\neg X}(x_i)) \cap \mathbb{S}^{d-1})$ which yields

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \mu > 0. \quad (3.13)$$

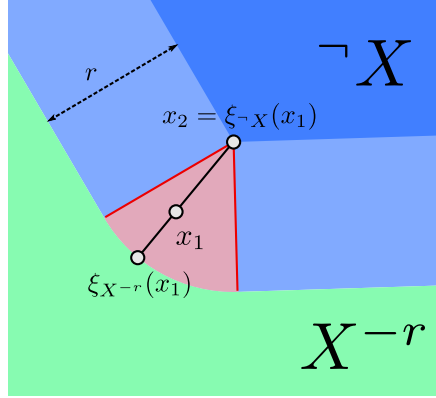


FIGURE 7: Visualisation of the inclusion $\partial^* d_{X^{-r}}(x) \subset \partial^* d_X(\xi_{\neg X}(x))$ for two points x_1 and x_2 , with $0 < r < \text{reach}(\neg X, x)$. The translated unit cone $x_2 + \text{Nor}(\neg X, x_2) \cap B(x_2, r)$ is depicted in red.

□

We are now able to build homotopies in neighborhoods of fixed size of both X_c and X_c^{-r} when r is small enough.

Lemma 3.9 (Deformation retractions around X_c and X_c^{-r}). *Let c be a regular value of $f|_X$. Using the notations of Definition 3.6, there exists $K > 0, M \geq 1$ as well as continuous, piecewise-smooth flows*

$$C : [0, 1] \times \phi^{-1}(-\infty, K] \rightarrow \phi^{-1}(-\infty, K]$$

$$C^r : [0, 1] \times \phi_r^{-1}(-\infty, K] \rightarrow \phi_r^{-1}(-\infty, K]$$

such that:

- $L := \sup\{\Delta(\partial^* \phi(y))^{-1} \mid y \in \phi^{-1}(0, K]\}$ is finite;
- For all $r > 0$ small enough, $(X_c)^{\frac{K}{M}} \subset \phi_r^{-1}(-\infty, K]$ and $(X_c^{-r})^{\frac{K}{M}} \subset \phi^{-1}(-\infty, K]$;
- $C(0, \cdot), C^r(0, \cdot)$ are the identity over their respective spaces of definition;
- $C(1, \phi^{-1}(-\infty, K]) = X_c$ and $C^r(1, \phi_r^{-1}(-\infty, K]) = X_c^{-r}$;
- For any $t \in [0, 1]$, $C(t, \cdot)|_{X_c}, C^r(t, \cdot)|_{X_c^{-r}}$ are the identity over X_c and X_c^{-r} ;
- $C(\cdot, \cdot)$ and $C^r(\cdot, \cdot)$ are $2KL$ -Lipschitz in the first variable when $r > 0$ is small enough.

Proof. Remark that $X_c = \phi^{-1}(0)$ and $X_c^{-r} = (\phi_r)^{-1}(0)$. We want to bound $\Delta \circ \partial^* \phi_r$ and $\Delta \circ \partial^* \phi$ from below to apply Proposition 2.10.

Let

$$\omega(s, K) := \inf_{\substack{r \in [0, s] \\ x \in \phi_r^{-1}(0, K)}} \Delta(\partial^* \phi_r(x)).$$

Lemma 3.8 states that

$$\liminf_{\substack{s \rightarrow 0^+ \\ K \rightarrow 0^+}} \omega(s, K) > 0. \quad (3.14)$$

When $K, s > 0$ are small enough, for all $r \in [0, s]$, $\Delta \partial^* \phi_r$ is uniformly bounded below by a positive number in $\phi_r^{-1}(0, K]$, allowing the offsets to be retracted by the approximate inverse flows C, C^r of respectively ϕ and ϕ_r by Proposition 2.10. For any positive ε , the flows can be chosen so that the gradients of the flows in the time parameter are bounded by $(1 + \frac{1}{2}\varepsilon)l_{r, K} = (1 + \frac{\varepsilon}{2})K \sup\{\Delta(\partial^* \phi_s(y))^{-1} \mid s \in [0, r], y \in \phi_r^{-1}(0, K]\}$ which is finite when r, K are taken small enough, and the supremum tends to a positive number L when r, K go to zero. When these numbers are small enough, the whole quantity is bounded by $(1 + \varepsilon)KL$.

Since the functions $(\phi_r)_{r \in [0, s]}$ are uniformly Lipschitz, we let $M := 1 + \sup\{\text{Lip}(\phi_r)_{r \in [0, s]}\}$. As the sets X_c^{-t} converge to X_c when t goes to 0 by Lemma 3.7, and $\|\phi - \phi_r\|_\infty = O(r)$, we have

$$(X_c^{-t})^{\frac{K}{M}} \subset \phi_r^{-1}(0, K]$$

for any t, r small enough. □

Corollary 3.10 (Homotopy Equivalence). *Let $X \subset \mathbb{R}^d$ be a complementary regular set and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function. Let c be a regular value of $f|_X$, let η be a smooth function with $\|\eta\|_\infty \leq 1$ and let $f_r : x \mapsto f(x + r\eta(x))$. Then for all $r > 0$ small enough, $X_c^{-r} = X^{-r} \cap f_r^{-1}(-\infty, c]$ and X_c have the same homotopy type.*

Proof. Since $\lim_{r \rightarrow 0} d_H(X_c^{-r}, X_c) = 0 < K/M$, the flows C, C^r are respectively well defined on X_c^{-r}, X_c for r small enough thanks to Lemma 3.9. Letting $\psi := C(1, \cdot)|_{X_c^{-r}} : X_c^{-r} \rightarrow X_c$ and $\psi^r := C^r(1, \cdot)|_{X_c} : X_c \rightarrow X_c^{-r}$, their composition $\psi \circ \psi^r$ is homotopic to Id_{X_c} via the map

$$\begin{cases} X_c \times [0, 1] & \rightarrow X_c \\ (x, t) & \mapsto C(1, C(t, C^r(t, x))). \end{cases}$$

In the same fashion, $\psi^r \circ \psi$ is homotopic to $\text{Id}_{X_c^{-r}}$ via $(t, x) \mapsto C^r(1, C^r(t, C(t, x)))$. □

3.3 Constant homotopy type lemma

In this section, we prove that the topology of the sublevel sets of a smooth map restricted to a complementary regular set does not evolve between critical values.

Theorem 3.11 (Constant homotopy type between critical values). *Let $X \subset \mathbb{R}^d$ be a complementary regular set. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth map and $a < b \in \mathbb{R}$ be such that $[a, b]$ contains only regular values of $f|_X$. Then X_a is a deformation retract of X_b .*

This theorem is a direct consequence of the compactness of $[a, b]$ and Lemma 3.13, which we will prove using the following technical lemma.

Lemma 3.12 (Regular values of the family $(\phi^c)_{c \in \mathbb{R}}$ are open.). *Let c be a regular value of $f|_X$ and let $\phi^s := d_X + \max(f - s, 0)$ for any $s \in \mathbb{R}$. Then we have:*

$$\lim_{\substack{\varepsilon \rightarrow 0^+ \\ K \rightarrow 0^+}} \inf \left\{ \Delta(\partial^* \phi^{c+a}(x)) \mid x \in (\phi^{c+a})^{-1}(0, K], a \in [-\varepsilon, \varepsilon] \right\} > 0.$$

Proof. We proceed by contradiction. Assuming the inequality is false, there exist two real sequences $a_i \rightarrow 0, K_i \rightarrow 0^+$, and $(x_i)_{i \in \mathbb{N}}$ a sequence in \mathbb{R}^d such that:

$$\forall i \in \mathbb{N}, 0 < \phi^{c+a_i}(x_i) \leq K_i \quad \text{and} \quad \lim_{i \rightarrow \infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) = 0.$$

We use the same distinction of sequences of $\phi_{c+a_i}^{-1}(0, K_i]$ into cases as in the proof of Lemma 3.8. Since $r = 0$, we distinguish 5 cases to compute $\partial^* \phi^{c+a_i}$.

Case 1. $f(x_i) < c + a_i$ and $d_X(x_i) > 0$.

Then $\partial^* \phi^{c+a_i}(x_i) = \partial^* d_X(x_i)$ and since $d_X(x_i) \leq K_i \rightarrow 0$, we have:

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) \geq \mu > 0.$$

Case 2. $x_i \in \text{int}(X)$ and $f(x_i) > c + a_i$.

Then $\partial^* \phi^{c+a_i}(x_i) = \{\nabla f(x_i)\}$ and thus

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) \geq \sigma > 0.$$

Cases 3, 4, 5.

$$\begin{cases} f(x_i) > c + a_i & \text{and} & d_X(x_i) > 0 \\ f(x_i) > c + a_i & \text{and} & x_i \in \partial X \\ f(x_i) = c + a_i & \text{and} & d_X(x_i) > 0. \end{cases}$$

In these 3 cases we have the inclusion $\partial^* \phi^{c+a_i}(x_i) \subset A_{x_i}$. As in the proof of Lemma 3.8, the map $y \mapsto A_y$ is semi-continuous. Now if (x_i) converges to a point x then this point belongs to $\partial X \cap f^{-1}(c)$. Since c is a regular value, we have:

$$\liminf_{i \rightarrow \infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) \geq \kappa > 0.$$

□

Lemma 3.13 (Local deformation retractions). *Let X be complementary regular, $f : \mathbb{R}^d \rightarrow \mathbb{R}$ smooth and let c be a regular value of $f|_X$. Then for all $\varepsilon > 0$ small enough and any $-\varepsilon \leq a \leq b \leq \varepsilon$, X_{c+a} is a deformation retract of X_{c+b} .*

Proof. By Lemma 3.12 there exist $\sigma, \varepsilon, K > 0$ such that for every $a \in [-\varepsilon, \varepsilon]$ we have

$$\Delta(\partial^* \phi^{c+a}(x)) \geq \sigma \text{ for all } x \text{ in } (\phi^{c+a})^{-1}(0, K]. \quad (3.15)$$

Thus by Proposition 2.10 for every $\alpha \in [-\varepsilon, \varepsilon]$ there exists a continuous $\frac{2K}{\sigma}$ -Lipschitz approximate flow of $\phi^{c+\alpha}$ on $(\phi^{c+\alpha})^{-1}(0, K]$ which we will denote $C_{c+\alpha}(\cdot, \cdot)$. By elementary computations one has for every $a < b \in [-\varepsilon, \varepsilon]$:

$$\phi^{c+a}(X_{c+b}) \subset [0, b-a] \subset [0, 2\varepsilon] \quad (3.16)$$

meaning that $X_{c+b} \subset (\phi^{c+a})^{-1}(0, K]$ when $\varepsilon > 0$ is small enough. The flow C_{c+a} makes ϕ^{c+a} decrease, leading to the following inclusions for $\varepsilon > 0$ small enough and any $t \in [0, 1]$:

$$C_{c+a}(t, X_{c+b}) \subset (\phi^{c+a})^{-1}[0, 2\varepsilon] \subset (\phi^{c+b})^{-1}[0, K]. \quad (3.17)$$

Consequently the composition $C_{c+b}(s, C_{c+a}(t, x))$ is well-defined for any $t, s \in [0, 1]$ and $x \in X_{c+b}$ and is continuous in every of these variables. Now letting i be the inclusion $X_{c+a} \rightarrow X_{c+b}$ and $\psi := C_{c+a}(1, \cdot) : X_{c+b} \rightarrow X_{c+a}$, one clearly has $\psi \circ i = \text{Id}_{X_{c+a}}$. The map $i \circ \psi$ is homotopic to $\text{Id}_{X_{c+b}}$ via the homotopy

$$\begin{cases} X_{c+b} \times [0, 1] & \rightarrow & X_{c+b} \\ (x, t) & \mapsto & C_{c+b}(1, C_{c+a}(t, x)). \end{cases} \quad (3.18)$$

□

3.4 Handle attachment around critical values

In this section, we study the properties of the critical points of a Morse function over a complementary regular set and obtain the handle attachment theorems around critical values. We fix some notations for the remainder of the section.

Definition 3.14 (Notations). *Let $X \subset \mathbb{R}^d$ be a complementary regular set and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be smooth. If $c \in \mathbb{R}$ is such that $f^{-1}(c)$ contains only one critical point x of $f|_X$ which is non-degenerate, we put for any $r > 0$:*

$$\gamma_r^c := y \mapsto y - r \frac{\nabla f(x)}{\|\nabla f(x)\|} \qquad f_{r,c} := f \circ \gamma_r^c$$

When the value c is clear from the context, we write γ_r and f_r instead to ease notations.

Remark. The results proved in this section also hold when $X \subset \mathbb{R}^d$ has positive reach, essentially because there is a correspondence between the critical points of $f|_X$ and those of $(-f)|_{\neg X}$ since $\text{Nor}(\neg X, x) = -\text{Nor}(X, x)$. The proofs have to be adapted by taking $\gamma_r^c : y \mapsto y + r \frac{\nabla f(x)}{\|\nabla f(x)\|}$ instead.

We begin by showing that non-degenerate critical points are isolated.

Proposition 3.15 (Critical points of a Morse function are isolated). *Let $X \subset \mathbb{R}^d$ be a set with positive reach or a complementary regular set and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function. Then in the set of critical points of $f|_X$, the non-degenerate critical points are isolated.*

Proof. Let x be a non-degenerate critical point that is an accumulation point of critical points of $f|_X$. There is a sequence x_i in ∂X of critical points all distinct from x converging to x . This means that for every $i \in \mathbb{N}$, the unit vector $n_i := -\frac{\nabla f(x_i)}{\|\nabla f(x_i)\|}$ lies in $\text{Nor}(X, x_i)$. The sequence (x_i, n_i) lies in $\text{Nor}(X)$ and converges to (x, n) where $n := -\frac{\nabla f(x)}{\|\nabla f(x)\|}$. Extracting a subsequence we can assume that $\frac{(x_i - x, n_i - n)}{\|(x_i - x, n_i - n)\|}$ converges to $(u, v) \in \text{Tan}(\text{Nor}(X), (x, n))$. Since x is non-degenerate, Proposition 2.4 implies that $\langle u, n \rangle = 0$. Moreover, the second fundamental form of X at (x, n) in the direction u is given by:

$$\mathbb{I}_{x,n}(u, u) = \langle u, v \rangle. \tag{3.19}$$

Since $\nu := -\frac{\nabla f}{\|\nabla f\|}$ is smooth around x , we have $\|n_i - n\| = \|\nu(x_i) - \nu(x)\| = O(\|x_i - x\|)$ ensuring that $u \neq 0$. By extracting a further subsequence, we can

assume that $\frac{x_i - x}{\|x_i - x\|}$ converges to $\frac{u}{\|u\|}$. The first order expansion of $n_i = \nu(x_i)$ gives

$$n_i - n = \|x_i - x\| T_x \nu \left(\frac{u}{\|u\|} \right) + o(\|x_i - x\|) \quad (3.20)$$

where $T_x \nu$ is the derivative of ν at x .

If $T_x \nu(u) = 0$, we have $\|n_i - n\| = o(\|x_i - x\|)$ meaning that $v = 0 = T_x \nu(u)$ and $\mathbb{I}_x(u, u) = 0$. Otherwise $\|n_i - n\| \sim C \|x_i - x\|$ for some $C > 0$. By elementary computations this also yields $T_x \nu(u) = v$ and we thus have in any case

$$T_x \nu(u) = v. \quad (3.21)$$

Now we can write the first order expansion of $\nabla f(x_i) + \|\nabla f(x_i)\| n_i$:

$$\begin{aligned} 0 &= \nabla f(x_i) + \|\nabla f(x_i)\| n_i \\ &= \|x_i - x\| (H_x f(u) + \|\nabla f(x)\| T_x \nu(u) - n \langle n, H_x f(u) \rangle) + o(\|x_i - x\|). \end{aligned}$$

Taking the scalar product of this vector with u yields:

$$H_x f|_X(u, u) = H_x f(u, u) + \|\nabla f(x)\| \langle u, v \rangle = 0. \quad (3.22)$$

This contradicts the non-degeneracy of $H_x f|_X$ in the direction u which belongs to $\pi_0(\text{Tan}(\text{Nor}(X)), (x, n)) \setminus \{0\}$. \square

We now describe how a cell is glued around the unique critical point when the level sets of a Morse function reach its associated critical value.

Lemma 3.16 (Local correspondence between critical points of $f|_X$ and $f_r|_{X^{-r}}$). *Let X be a complementary regular subset of \mathbb{R}^d . Assume x is a non-degenerate critical point of $f|_X$ and let ind_x be the index of the Hessian of $f|_X$ at x . Then $x^r = \gamma_r(x)$ is a critical point of $f_r|_{X^{-r}}$ such that $f_r(x^r) = f(x)$ for all $0 < r < \text{reach}(\neg X)$. When r is small enough, x^r is a non-degenerate critical point of $f_r|_{X^{-r}}$, whose Hessian at point x_r has index*

$$\text{ind}_x^r := \text{ind}_x + \text{number of infinite curvatures at } \left(x, \frac{\nabla f(x)}{\|\nabla f(x)\|} \right).$$

Proof. Let $n = \frac{\nabla f(x)}{\|\nabla f(x)\|} \in \text{Nor}(\neg X, x)$ the normalized gradient of f at this point. Keep in mind that $f_r : x \mapsto f(x - rn)$ is f translated in the direction n with magnitude r .

The pair $(x, n) \in \text{Nor}(\neg X)$ is regular by non-degeneracy of f at x . Denote by $(\kappa'_i)_{1 \leq i \leq d-1}$ the principal curvatures (defined in Proposition 2.4) of $\neg X$ at (x, n) sorted in ascending order and put $m := \max\{i \mid \kappa'_i < \infty\}$. From there we follow

the reasoning of Fu [5]. When $0 < r < \text{reach}(\neg X)$, X^{-r} is as $C^{1,1}$ -domain and the regularity of the pair (x, n) in X guarantees that the Gauss map $x \in \partial^- X^{-r} \mapsto n(x) \in \mathbb{S}^{d-1}$ is differentiable at $x+rn$. We have the following linear correspondance between tangent spaces:

$$\text{Tan}(\text{Nor}(\neg(X^{-r})), (x+rn, n)) = \{(\tau + r\sigma, \sigma) \mid (\tau, \sigma) \in \text{Tan}(\text{Nor}(\neg X), (x, n))\}.$$

Since $\text{Nor}(X^{-r}) = \{(z, -n) \mid (z, n) \in \text{Nor}(\neg(X^{-r}))\}$ we have:

$$\pi_0(\text{Tan}(\text{Nor}(X^{-r}), (x+rn, n))) = \{\tau - r\sigma \mid (\tau, \sigma) \in \text{Tan}(\text{Nor}(X), (x, n))\}.$$

This vector space is identifiable with the classical tangent space of differential geometry since and thus has dimension d . Proceeding exactly in the same fashion as the proof of [5, 4.6], we can write, for any $\tau - r\sigma, \tau' - r\sigma'$ in $\pi_0(\text{Tan}(\text{Nor}(X^{-r}), (x+rn, n)))$:

$$\begin{aligned} H_{x+rn} f_r|_{X^{-r}}(\tau - r\sigma, \tau' - r\sigma') \\ &= H_{x+rn} f_r(\tau - r\sigma, \tau' - r\sigma') + \|\nabla f_r(x^r)\| \mathbb{I}_{x+rn}(\tau - r\sigma, \tau - r\sigma') \\ &= H_x f(\tau - r\sigma, \tau' - r\sigma') + \|\nabla f(x)\| \langle \tau - r\sigma, \sigma' \rangle. \end{aligned}$$

We can decompose $\pi_0(\text{Tan}(\text{Nor}(X^{-r}), (x+rn, n)))$ as the direct sum of $F := \{\sigma \mid (0, \sigma) \in \text{Tan}(\text{Nor}(X), (x, n))\}$ and a supplementary subspace E . E has dimension m and F dimension $d-m$. By the structure theorem of tangent spaces, E and F are orthogonal. From the previous computation, identifying coefficients in front of the r -monomials, there are square matrices A_1, A_2, A_3 of size m , a square matrix B of size $d-m$ and a rectangular matrix C such that the Hessian $H_{x+rn} f_r|_{X^{-r}}$ has the form

$$\begin{pmatrix} A_1 + rA_2 + r^2A_3 & rC \\ rC^t & -r\|\nabla f(p)\| Id + r^2B \end{pmatrix}$$

where A_1 is similar to the matrix of $H_x f|_X$. It is the same computation as [5] except that we end up with a minus sign in front of the identity in the lower right corner. When $r > 0$ is small enough, this matrix is non-degenerate and its index is that of A_1 plus the dimension of the identity matrix in the lower right corner. \square

Lemma 3.17 (Critical points of $f_r|_{X^r}$ when r is small enough). *Let $X \subset \mathbb{R}^d$ be a complementary regular set. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function such that $f|_X$ is Morse. Assume x is the only critical point in $X \cap f^{-1}(c)$. Then for $\varepsilon, r > 0$ small enough, $x^r = x + r \frac{\nabla f(x)}{\|\nabla f(x)\|}$ is the only critical point of $f_r|_{X^{-r}}$ inside $f_r^{-1|X^{-r}}(c - \varepsilon, c + \varepsilon)$, and $f_r(x^r) = c$.*

Proof. First remark that $f_r(x^r) = f(x)$ and $\nabla f_r(x^r) = \nabla f(x)$. For $r > 0$ small enough, the normal consists in a sole line, in the same fashion as x_1 in Figure 7. More precisely, we have $\text{Nor}(X^{-r}, x^r) = -\text{Cone}(\nabla f(x))$ and x^r is a critical point of $f_r|_{X^{-r}}$. Assuming the claim of Lemma 3.17 is false, there are sequences $\varepsilon_i, r_i > 0$ converging to 0, and y_i a sequence in ∂X such that:

- $d_{-X}(y_i) = r_i$
- $y_i \neq x^{r_i}$
- $c - \varepsilon_i \leq f_{r_i}(y_i) \leq c + \varepsilon_i$
- $n_i := -\frac{\nabla f_{r_i}(y_i)}{\|\nabla f_{r_i}(y_i)\|} \in \text{Nor}(X, y_i)$.

By semi-continuity of the normal cones as functions of ∂X , which is a consequence of the identity $\text{Nor}(X, x) = \text{Cone}(\partial^* d_X(x))$, any accumulation point \bar{x} of the sequence $(y_i)_{i \in \mathbb{N}}$ is a critical point of $f|_X$ with $f(\bar{x}) = c$, thus showing that y_i converges to x . Now put $x_i := \xi_{-X}(y_i)$. If we assume that $x_i = x$ for all i , then $y_i = x + r_i n_i$. Since $y_i \neq x^{r_i}$, n_i and n are not equal, and we can also assume that $\frac{n_i - n}{\|n_i - n\|}$ converges to some unit vector $v' \in \mathbb{R}^d$ by extracting a subsequence. Then we would have

$$\begin{aligned} n_i - n &= -\frac{\nabla f(x + r_i(n_i - n))}{\|\nabla f(x + r_i(n_i - n))\|} + \frac{\nabla f(x)}{\|\nabla f(x)\|} \\ &= -r_i \|n_i - n\| (T_x \nu)(v) + o(r_i \|n_i - n\|) \\ &= o(\|n_i - n\|) \end{aligned}$$

which is absurd. We can thus assume without loss of generality that x_i is different from x for all $i \in \mathbb{N}$. Reasoning exactly as in the proof of Proposition 3.15, we can extract a subsequence such that the sequence $\frac{(x_i - x, n_i - n)}{\|(x_i - x, n_i - n)\|}$ converges to $(u, v) \in \text{Tan}(\text{Nor}(X), (x, n))$. The same computations yield that the restricted Hessian $H_x f|_X = H_x f + \|\nabla f(x)\| \mathbb{I}_{x, n}$ is degenerate in the direction $u \in \pi_0(\text{Tan}(\text{Nor}(X)), (x, n)) \setminus \{0\}$. \square

Theorem 3.18 (Handle attachment around unique critical values). *Let X be complementary regular and $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Assume $f|_X$ has only one critical point x in $f^{-1}(c)$ which is non degenerate. Then for any $\varepsilon > 0$ small enough, $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ with a λ_x -cell attached, where*

$$\begin{aligned} \lambda_x &:= \text{index of the Hessian of } f|_X \text{ at } x \\ &+ \text{number of infinite curvatures at } \left(x, \frac{\nabla f(x)}{\|\nabla f(x)\|} \right). \end{aligned}$$

Proof. By Lemma 3.17, when $\varepsilon, r > 0$ are small enough, there is only one critical point x_r in $f_{r|_{X^{-r}}}^{-1}((c - \varepsilon, c + \varepsilon))$. By $C^{1,1}$ Morse theory, $X_{c+\varepsilon}^{-r}$ has the homotopy

type of $X_{c-\varepsilon}^{-r}$ with a cell added around x^r . The dimension of the cell is λ_x for all $r > 0$ small enough by Lemma 3.16. Now by Corollary 3.10, when $r > 0$ is small enough, $X_{c+\varepsilon}^{-r}$ and $X_{c+\varepsilon}$ are homotopy equivalent, and so are $X_{c-\varepsilon}^{-r}$ and $X_{c-\varepsilon}$. This is summarized by Figure 8.

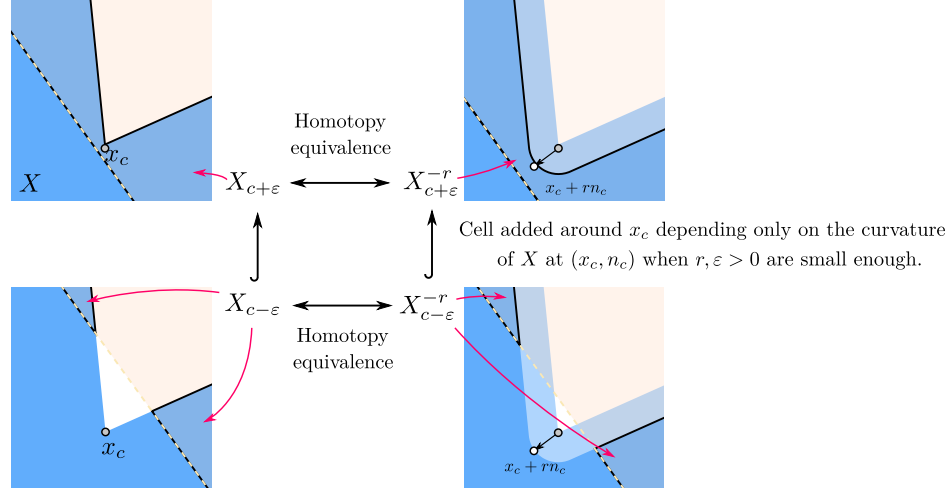


FIGURE 8: Commutative diagram in the proof of Theorem 3.18.

□

Theorem 3.19 (Morse Theory for complementary regular sets). *Let $X \subset \mathbb{R}^d$ be a complementary regular set. Suppose $f|_X$ has a finite number of critical points, which are all non-degenerate. Each critical level set $X \cap f^{-1}(\{c\})$ has a finite number p_c of critical points, whose indices (defined in Theorem 3.18) we denote by $\lambda_1^c, \dots, \lambda_{p_c}^c$. Then:*

- If $[a, b]$ does not contain any critical value, X_a is a deformation retract of X_b .
- If c is a critical value, $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ with exactly p_c cells attached around the critical points in $f^{-1}(c) \cap X$, of respective dimension $\lambda_{p_1}^c, \dots, \lambda_{p_c}^c$ for all $\varepsilon > 0$ small enough.

Proof. The first point is Theorem 3.11. We turn our attention to the second point.

Let c be a critical value of $f|_X$. Put x_1, \dots, x_p the critical points of $f|_X$ inside $f^{-1}(c)$. Put $n_i := -\frac{\nabla f(x_i)}{\|\nabla f(x_i)\|}$ and $x_i^r = x_i - rn_i$. Let $n(x)$ be the function mapping x to the n_i associated to the closest critical point x_i of x . This map is piecewise constant and defined almost everywhere. Let $U_i \subset V_i$ be respectively closed and

open balls containing x_i such that $\overline{V_i} \cap \overline{V_j} = \emptyset$ when $j \neq i$. Let η_c be a smooth function on \mathbb{R}^d with values in $[0, 1]$ such that η_c is constant of value 1 inside each U_i and 0 outside of $\bigcup V_i$. The map $n_c : y \mapsto \eta_c(y)n(y)$ is well-defined and smooth when the U_i are small enough. When r is small enough, the map $\gamma_r : y \mapsto y + rn_c(y)$ is a diffeomorphism. Now define f_r to be f locally translated around the critical points:

$$f_r = f \circ \gamma_r : y \mapsto f(y + rn_c(y)).$$

From Lemma 3.16 we know that the $(x_i^r)_{1 \leq i \leq p}$ are non-degenerate critical points of X^{-r} for $f_r|_{X^{-r}}$ with corresponding index $(\lambda_i^c)_{1 \leq i \leq p}$. From Lemma 3.17, we know that x_i^r is the only critical point of $f_r|_{X^{-r}}$ inside $\gamma_r(U_i)$ when r is small enough.

Now we prove that there are no critical points outside of $\bigcup_i \gamma_r(U_i)$ when r is small enough. On the one hand, outside of this set, the sets $\text{Nor}(X, x) \cap \mathbb{S}^{d-1}$ and $\frac{\nabla f(x)}{\|\nabla f(x)\|}$ have a fixed distance separating them. On the other hand, when r goes to 0, the sets $\text{Nor}(X^{-r}, x) \cap \mathbb{S}^{d-1}$ (resp. $\left\{ \left(x, \frac{\nabla f_r(x)}{\|\nabla f_r(x)\|} \right) \right\}$) converge uniformly in x (as will soon be precised) in the Hausdorff distance to $\text{Nor}(X, x) \cap \mathbb{S}^{d-1}$ (resp. $\frac{\nabla f(x)}{\|\nabla f(x)\|}$) meaning by semi-continuity that for r small enough, the two still cannot intersect.

More quantitatively, by the inverse function theorem X^{-r} has a $C^{1,1}$ boundary. Since ∇f does not vanish in a neighborhood of $f^{-1}(c) \cap X$, we know that $x \in X^{-r}$ is a critical point of $f_r|_{X^{-r}}$ if and only if $x \in \partial X^{-r}$, $\{\nu\} = \text{Nor}(X^{-r}, x) \cap \mathbb{S}^{d-1}$ (i.e. ν is the normal at x) and $\left\| \frac{\nabla f_r(x)}{\|\nabla f_r(x)\|} - \nu \right\| = 0$.

Remark that we have both

$$\text{Nor}(X^{-r}) = \{(x + r\nu, -\nu) \mid (x, \nu) \in \text{Nor}(X)\}$$

and

$$\sup_{(x, \nu) \in \text{Nor}(X)} \|\nabla f(x) - \nabla f_r(x + r\nu)\| = O(r)$$

leading to

$$\liminf_{r \rightarrow 0} \inf_{\substack{(x, \nu) \in \text{Nor}(X^{-r}) \\ x \notin \bigcup_i \gamma_r(U_i) \\ f_r(x) = c}} \left\| \frac{\nabla f_r(x)}{\|\nabla f_r(x)\|} - \nu \right\| \geq \inf_{\substack{(x, \nu) \in \text{Nor}(X) \\ x \notin \bigcup_i U_i \\ f(x) = c}} \left\| \frac{\nabla f(x)}{\|\nabla f(x)\|} - \nu \right\| > 0. \quad (3.23)$$

This shows that $\{x_1^r, \dots, x_p^r\}$ is exactly the set of critical points of $f_r|_{X^{-r}}$ with value c . We obtain $X_{c+\varepsilon}^{-r}$ from $X_{c-\varepsilon}^{-r}$ by gluing cells locally around each critical point as in classical Morse theory. \square

Remark. A similar argument holds assuming X has positive reach by taking $\eta_c(x)$ to be -1 near critical points instead of 1 . This shows that the Morse theorems holds when X has positive reach and f is a Morse function with several non-degenerate critical points sharing the same critical value.

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