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The Benjamin-Ono Initial-Value Problem for Rational Data

Elliot Blackstone^{*}, Louise Gassot[†], Patrick Gérard[‡], and Peter D. Miller[§]

Abstract. We show that the initial-value problem for the Benjamin-Ono equation on \mathbb{R} with $L^2(\mathbb{R})$ rational initial data with only simple poles can be solved in closed form via a determinant formula involving contour integrals. The dimension of the determinant depends on the number of simple poles of the rational initial data only and the matrix elements depend explicitly on the independent variables (t, x) and the dispersion coefficient ϵ . This allows for various interesting asymptotic limits to be resolved quite efficiently.

Key words. Benjamin-Ono equation, Cauchy problem, exact solutions

AMS subject classifications. 35C05, 35Q51, 37K10

1. Introduction.

The Benjamin-Ono equation

$$(1.1) \quad \partial_t u + \partial_x(u^2) = \epsilon \partial_x |D_x| u, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad \epsilon > 0,$$

where $\epsilon > 0$ measures the strength of the dispersion, is a well-known asymptotic model for internal waves in stratified fluids that was first proposed more than 50 years ago [1, 2]. The dispersive term is either defined by the Fourier multiplier $\widehat{|D_x|u}(\xi) = |\xi|\widehat{u}(\xi)$ or by the singular integral

$$(1.2) \quad |D_x|f(x) := -\frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{f'(y) dy}{y-x}.$$

Solutions $u = u(t, x)$ represent the vertical displacement of an interface (the *pycnocline*) between fluid layers of differing density, with the lower, denser layer being assumed to be infinitely deep.

It has been known since the late 1970's and early 1980's that the Benjamin-Ono equation has many of the features of a completely integrable dispersive nonlinear wave equation, the prototype of which is the Korteweg-de Vries equation. Indeed, the Benjamin-Ono equation has a Bäcklund transformation and infinitely-many conservation laws [27], can be represented as the compatibility condition of a Lax pair [7], is part of an infinite hierarchy of commuting Hamiltonian flows [12], has a variety of exact solutions including multi-solitons [23] and multi-phase waves [10], and has been solved by a formal inverse-scattering transform method [11, 17]. A generalization of the latter approach, the method of commuting flows, has recently been used to prove a sharp well-posedness [19] result for the Benjamin-Ono equation.

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While the inverse-scattering transform method in its original form has yet to be made fully rigorous (see [21, 30, 31]), there have been some developments in its use to study the initial-value problem on the line $x \in \mathbb{R}$ with initial data u_0 in a suitable space of functions with decay at infinity. Generalizing from an illustrative example in the paper [22], it was shown in [26] that all of the scattering data in the Fokas-Ablowitz inverse-scattering transform can be obtained essentially in explicit form when u_0 is a rational function. For instance, the discrete eigenvalues of the Lax operator are the zeros of an Evans function that is given explicitly as a determinant, the elements of which are analytic functions defined by contour integrals involving u_0 , and the dimension of the determinant is precisely the number of poles of u_0 in the upper complex half-plane. The reflection coefficient also has an explicit expression in terms of the solution of a linear system of the same dimension with coefficients that are similar contour integrals. Another remarkable recent result [14] is that the formal inverse-scattering algorithm can be “closed” to yield a formula for the solution $u(t, x)$ of the initial-value problem with completely general initial data $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ in terms of the resolvent of a linear operator depending explicitly on u_0 (see (2.3) below). This result has been extended to initial data $u_0 \in L^2(\mathbb{R})$ in [8] by using dispersive properties of the free Schrödinger evolution. Although the steps of the inverse-scattering transform themselves have not been fully justified, the resulting closed formula for $u(t, x)$ has been proved to provide the solution of the initial-value problem nonetheless.

The aim of this paper is to apply the exact solution formula in [14] to the special case that $u_0 \in L^2(\mathbb{R})$ is a real-valued rational function with simple poles:

$$(1.3) \quad u(0, x) = u_0(x) := \sum_{n=1}^N \left(\frac{c_n}{x - p_n} + \frac{c_n^*}{x - p_n^*} \right), \quad \text{Im}(p_n) > 0.$$

Clearly $u_0 \in L^2(\mathbb{R})$ for all pole locations $\{p_n\}$ in the upper half-plane and for all complex values of the coefficients $\{c_n\}$. We will show how completely explicit the formula from [14] becomes for such rational data. In particular, the solution $u(t, x)$ of (1.1) for initial data $u(0, x) = u_0(x)$ given by (1.3) turns out to be given by a ratio of $(N + 1) \times (N + 1)$ determinants, the entries of which are contour integrals involving the initial data. This result is therefore not unlike the computation in [26] of the Fokas-Ablowitz scattering data for u_0 of the form (1.3), but more to the point it also completely solves the corresponding inverse problem and returns explicitly the solution $u(t, x)$ instead of just its data in the transform domain. Obviously the availability of such a formula (see Theorem 1.3 below) makes it immediately tractable to analyze the solution $u(t, x)$ in various interesting asymptotic limits, and we have successfully used it to study in particular the small-dispersion limit $\epsilon \rightarrow 0$ for fixed rational initial data u_0 to exhibit multi-phase wave asymptotics away from wave caustic curves in the (t, x) -plane [3] as well as to prove universality theorems [6] in the vicinity of such curves and their intersection points, and to analyze the long-time limit [4] and prove the soliton resolution conjecture, in which we expect that the radiation term exhibits scattering if $xu_0 \in L^2$, and modified scattering otherwise.

Now we explain our main result. Choosing \log to be the principal branch such that

$|\operatorname{Im}(\log(\cdot))| < \pi$, we first define for $y \in \mathbb{R}$ negative and sufficiently large in absolute value,

$$(1.4) \quad h(y) := \frac{1}{4t}(y-x)^2 + \sum_{n=1}^N [c_n \log(y-p_n) + c_n^* \log(y-p_n^*)],$$

so that comparing with (1.3),

$$(1.5) \quad h'(y) = \frac{y-x}{2t} + u_0(y).$$

Note that when $y < 0$ with $|y|$ sufficiently large, $h(y) \in \mathbb{R}$. We want to analytically continue $y \mapsto h(y)$ to a maximal domain that is generally more complicated than implied by using the principal branch of the logarithm in (1.4) and taking $y = z$ to be complex. For this purpose, we start by allowing for general branch cuts $\{\Gamma_n, \bar{\Gamma}_n\}_{n=1}^N$ assumed only to have the following properties.

Definition 1.1 (Branch cuts of h). *The branch cuts $\Gamma_1, \dots, \Gamma_N$ are pairwise disjoint piecewise-smooth curves each emanating from exactly one of the poles $\{p_n\}_{n=1}^N$ and tending to $z = \infty$ in the direction asymptotic to the ray $\arg(z) = 3\pi/4$. All of these branch cuts are assumed to lie in a half-plane $\operatorname{Im}(z) > -\delta$ for some $\delta > 0$ sufficiently small (in particular, we assume $\delta < \min_n \{\operatorname{Im}(p_n)\}$).*

The branch cuts $\bar{\Gamma}_1, \dots, \bar{\Gamma}_N$ are straight horizontal rays each emanating from exactly one of the conjugate poles $\{p_n^\}_{n=1}^N$ and extending to $z = \infty$ in the left half-plane.*

Given the branch cuts, we index the poles $\{p_n\}_{n=1}^N$ such that in the vicinity of $z = \infty$ in the upper half-plane, the branch cuts $\Gamma_1, \dots, \Gamma_N$ are ordered left-to-right. See Figure 1, left-hand panel. We hence obtain a well-defined function $z \mapsto h(z)$ by analytic continuation from large negative real values of z where $h(z)$ is given by (1.4) to the domain

$$(1.6) \quad z \in \mathbb{C} \setminus (\Gamma_1 \cup \dots \cup \Gamma_N \cup \bar{\Gamma}_1 \cup \dots \cup \bar{\Gamma}_N).$$

We next define some relevant integration contours in the z -plane, see Figure 1, right-hand panel.

Definition 1.2 (Contours). *Let C_n , $n = 1, \dots, N$ denote the contour defined by one of the following alternatives:*

- *If ic_n/ϵ is a strictly negative integer, then C_n originates at $z = \infty$ in the direction $\arg(z) = 3\pi/4$ to the left of all branch cuts of h , lies in the domain of analyticity of h , and terminates at $z = p_n$. We call such an index n exceptional.*
- *Otherwise, C_n originates and terminates at $z = \infty$ in the direction $\arg(z) = 3\pi/4$, lies in the domain of analyticity of $h(z)$, and encircles with counterclockwise orientation precisely the branch cuts of $h(z)$ emanating from each of the points $z = p_m$, $1 \leq m \leq n$. Such an index n will be called non-exceptional.*

Finally, we let C_0 denote a path in the domain of analyticity of $h(z)$ originating at $z = \infty$ in the direction $\arg(z) = 3\pi/4$ to the left of all branch cuts $\Gamma_1, \dots, \Gamma_N$ of $h(z)$ and terminating at $z = \infty$ in the direction $\arg(z) = -\pi/4$.

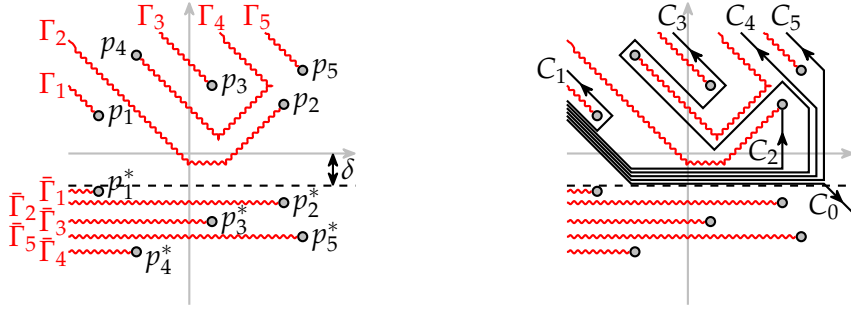


Figure 1. Left: admissible branch cuts of $h(z)$ in the z -plane for a rational initial condition with $N = 5$. Right: corresponding contours C_1, C_2, C_3, C_4, C_5 and C_0 for a situation where 2 is the only exceptional index.

Let $\mathbf{A}(t, x), \mathbf{B}(t, x)$ be two $(N + 1) \times (N + 1)$ matrices defined for $\epsilon > 0$ and $t > 0$ by

$$(1.7) \quad A_{j1}(t, x) := \int_{C_{j-1}} u_0(z) e^{-ih(z)/\epsilon} dz, \quad A_{jk}(t, x) := \int_{C_{j-1}} \frac{e^{-ih(z)/\epsilon} dz}{z - p_{k-1}},$$

$$(1.8) \quad B_{j1}(t, x) := \int_{C_{j-1}} e^{-ih(z)/\epsilon} dz, \quad B_{jk}(t, x) := \int_{C_{j-1}} \frac{e^{-ih(z)/\epsilon} dz}{z - p_{k-1}} = A_{jk}(t, x),$$

where the indices satisfy $1 \leq j \leq N + 1$ and $2 \leq k \leq N + 1$. Note that if j is an exceptional index, then $e^{-ih(z)/\epsilon}$ is actually a single-valued analytic function of z on a neighborhood of the branch cut Γ_j of $h(z)$, and this function vanishes to at least first order at the branch point p_j . Finally, we introduce a related matrix $\tilde{\mathbf{B}}(t, x)$, the first column of which is the same as that of $\mathbf{B}(t, x)$ while

$$(1.9) \quad \tilde{B}_{jk} = e^{i(x-p_{k-1})^2/(4t\epsilon)} B_{jk} = e^{i(x-p_{k-1})^2/(4t\epsilon)} \int_{C_{j-1}} \frac{e^{-ih(z)/\epsilon}}{z - p_{k-1}} dz, \quad k = 2, \dots, N + 1.$$

Theorem 1.3 (Solution of Benjamin-Ono for rational initial data). *Let $\epsilon > 0$. The solution of the Cauchy initial-value problem for the Benjamin-Ono equation (1.1) with rational initial condition $u(x, 0) = u_0(x)$ of the form (1.3) is*

$$(1.10) \quad u(t, x) = \Pi u(t, x) + \Pi u(t, x)^* = 2\text{Re}(\Pi u(t, x)), \quad t > 0,$$

$$(1.11) \quad \Pi u(t, x) = \frac{\det(\mathbf{A}(t, x))}{\det(\mathbf{B}(t, x))}.$$

An equivalent formula is

$$(1.12) \quad \Pi u(t, x) = i\epsilon \frac{\partial}{\partial x} \log(\det(\tilde{\mathbf{B}}(t, x))).$$

Also, we have $\det(\mathbf{B}(t, x)) \neq 0$ and $\det(\tilde{\mathbf{B}}(t, x)) \neq 0$ for all $(t, x) \in \mathbb{R}^2$ with $t > 0$ and all $\epsilon > 0$.

The expression Πu denotes the action of a Cauchy-Szegő projection with respect to the argument x , which is explained in Section 2.

Remark 1.4 (The limit $t \downarrow 0$ and negative times). The elements of the matrices $\mathbf{A}(t, x)$ and $\mathbf{B}(t, x)$, and hence also $\Pi u(t, x)$ given in (1.11), are undefined for $t = 0$ due to the factor of t^{-1} in the expression (1.4) for $h(z)$. However, since $h(z)$ appears in the exponent of each integrand, a steepest-descent analysis shows that as $t \downarrow 0$, we have $\Pi u(t, x) \rightarrow \Pi u_0(x)$ for each $x \in \mathbb{R}$. In this way, the solution can be defined by continuity for $t = 0$. Furthermore, to continue the solution to $t < 0$ we may use the fact that if u is a solution to (1.1), then $(t, x) \mapsto u(-t, -x)$ is also a solution to (1.1). Note that the map $u_0(x) \mapsto u_0(-x)$ preserves the class of rational functions in $L^2(\mathbb{R})$ and amounts to the change of rational data $\{p_n\}_{n=1}^N \mapsto \{-p_n^*\}_{n=1}^N$ and $\{c_n\}_{n=1}^N \mapsto \{-c_n^*\}_{n=1}^N$ in (1.3).

Remark 1.5 (Choice of branch cuts). Since it is known that the Cauchy problem with data $u_0 \in H^2(\mathbb{R})$ has a unique solution [28], the real part of the ratio of determinants in (1.11) is insensitive to the choice of admissible branch cuts and integration contours consistent with Definitions 1.1 and 1.2. For some applications it is sufficient to choose straight-ray cuts for which Γ_n is simply $p_n + e^{3\pi i/4}\mathbb{R}_+$. Furthermore, the branch cuts $\bar{\Gamma}_j$ in the lower half-plane can be chosen to be quite arbitrary as long as they do not intersect any of the contours of integration C_n . The flexibility of choosing general contours is especially useful in asymptotic problems such as those considered in [3] and [6].

Remark 1.6 (Two equivalent formulæ). Of the two equivalent formulæ for $u(t, x)$, (1.11) is preferable in situations where one is interested in the analysis of $u(t, x)$ in some asymptotic limit such as $\epsilon \rightarrow 0$ or $t \rightarrow \infty$, because one does not need to determine the effect of differentiation on error terms arising by expanding the determinants of $\mathbf{A}(t, x)$ and $\mathbf{B}(t, x)$. On the other hand, the formula (1.12) is more compact, better for exact computations such as those in Appendix A below, and also is reminiscent of formulæ for more traditional exact solutions as described in Remark 1.8.

Remark 1.7 (Comparison with the Calogero-Moser equation). Regarding the Calogero-Moser derivative NLS equation studied in [13], a similar formula as (1.11) for rational initial data can be derived from the general solution formula established in [18]. However, no formulation comparable to (1.12) is known. This could be explained by the different nature between the two equations. In fact, the solutions of the Benjamin-Ono equation stay bounded with respect to every Sobolev norm [27, 7]. On the contrary, for the Calogero-Moser derivative NLS equation, solutions with initial data of mass greater than the mass of the ground state can exhibit turbulent behavior [16] or even blow-up [20].

Remark 1.8 (Comparison with the N -soliton formula). The exact solution formula asserted in Theorem 1.3 for the Benjamin-Ono initial-value problem with rational initial data $u_0(x)$ resembles the N -soliton formula, which also can be written in the form (1.12) with a matrix of the form $\tilde{\mathbf{B}} = \mathbb{I} - i\mathbf{H}(t, x)/\epsilon$; here $\mathbf{H}(t, x)$ is a Hermitian matrix having simple structure (see [24, Section 3.1] and [5, Eqn. 1.8]). It is also true that the N -soliton formula is a rational function of x when $t = 0$ (see for instance [29, Eqn. 1.19]). However, the N -soliton solutions remain rational in x for all time $t > 0$, while in general this is not true of the solution $u(t, x)$ of the Cauchy problem with rational initial data. We wish to emphasize that the exact

formulae for $u(t, x)$ given in Theorem 1.3 represent also solutions that in the setting of the Fokas-Ablowitz inverse-scattering transform [11, 17] correspond to nonzero reflection coefficients. See [26] where the scattering data are computed explicitly for general rational initial conditions of the form (1.3); generally these solutions consist of both reflectionless/solitonic and reflective/dispersive components. This makes the explicit formula (1.11) especially remarkable.

Remark 1.9 (An explicit invariant space for $u(t, x)$). As pointed out in Remark 1.8 above, the linear subspace $L^2(\mathbb{R})$ consisting of rational functions with simple poles is not invariant under the Benjamin-Ono flow. However, one may define explicitly a *nonlinear* subset of $L^2(\mathbb{R})$ forward-invariant under (1.1) simply as the set of functions of x given by (1.10)–(1.11) for any fixed $t > 0$ and arbitrary u_0 of the form (1.3). According to Remark 1.4, this set clearly includes all rational functions of the form (1.3) (by taking $t \downarrow 0$) and can be further made backward-invariant by replacing t with $-t$ as indicated.

Remark 1.10 (Implementation). Since the elements of the matrices $\mathbf{A}(t, x)$ and $\mathbf{B}(t, x)$ involve the specific poles $\{p_n\}_{n=1}^N$ of u_0 in the upper half-plane, if one is given u_0 as a rational function in the form $u_0(x) = P(x)/Q(x)$ with $\deg(P) < \deg(Q)$ to guarantee $u_0 \in L^2(\mathbb{R})$, one must first have access to the factorization of $Q(x)$ into simple factors in order to be able to use Theorem 1.3.

The rest of the paper consists of the proof of Theorem 1.3. The proof of Theorem 1.3 relies on a more generally-applicable solution formula for the Benjamin-Ono equation found by one of the authors in [14]. We recall this formula in Section 2. Then in Section 3, we invert the operator involved in the solution formula when the initial data is rational, and we find conditions for the inverse to belong to the Hardy space in Section 4. We conclude that formulae (1.11) and (1.12) hold in Section 5, and finally we show the non-vanishing of $\det(\mathbf{B})$ and $\det(\bar{\mathbf{B}})$ in Section 6. In Appendix A we show how Theorem 1.3, when applied to the initial datum $u_0(x) = 2/(x^2 + 1)$ (essentially the general case for $N = 1$ with negative imaginary c_1), reproduces the well-known soliton solution in the case that also $\epsilon = 1$, but yields nontrivial results for other values of $\epsilon > 0$. In particular our method reveals what appears to be a new explicit formula for the M -soliton solution of the Benjamin-Ono equation in terms of just two contour integrals (see (A.11)).

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2. General solution formula. In this section, we recall the solution formula for equation (1.1) derived in [14].

The operator Π denotes the orthogonal projection from $L^2(\mathbb{R})$ to the Hardy space $L^2_+(\mathbb{R})$ of $L^2(\mathbb{R})$ functions with Fourier transform vanishing for frequencies $\xi < 0$. Hence in particular since $u(t, x)$ is real-valued, $u(t, x) = 2\text{Re}(\Pi u(t, x))$, and for the rational initial condition (1.3),

$$(2.1) \quad \Pi u_0(y) = \sum_{n=1}^N \frac{c_n^*}{y - p_n^*}.$$

The Hardy space $L^2_+(\mathbb{R})$ identifies with the holomorphic functions of the complex upper half-plane \mathbb{C}_+ satisfying

$$(2.2) \quad \sup_{v>0} \int_{\mathbb{R}} |f(u + iv)|^2 du < \infty,$$

in which case the Szegő projection Π identifies with the orthogonal projector onto this set of holomorphic functions.

For the initial-value problem written in the form (1.1) with initial data $u_0 \in L^2(\mathbb{R})$, the solution formula from [14] reads: for every $x \in \mathbb{C}_+$,

$$(2.3) \quad \Pi u(t, x) = \frac{1}{2i\pi} I_+ [(X^* + 2it\epsilon\partial_y + 2tT_{u_0} - x\text{Id})^{-1}(\Pi u_0)].$$

Here the ingredients are the following:

- I_+ is the linear functional defined on the subspace of $L^2(\mathbb{R})$ consisting of functions f with Fourier transforms \hat{f} right-continuous at zero frequency by $I_+[f] := \hat{f}(0+)$.
- $x\text{Id}$ is a constant multiple of the identity because on the right-hand side of this formula $x \in \mathbb{C}_+$ is simply a parameter. (On the left-hand side however, x denotes the $L^2_+(\mathbb{R})$ independent variable with respect to which we understand the projection Π .)
- T_{u_0} denotes the Toeplitz operator on $L^2_+(\mathbb{R})$ associated with multiplication by u_0 : $T_{u_0}f := \Pi(u_0f)$ for $u_0 \in L^\infty(\mathbb{R})$ and $f \in L^2_+(\mathbb{R})$.
- ∂_y denotes the differential operator with respect to the independent variable y of functions in the Hardy space $L^2_+(\mathbb{R})$.
- X^* denotes the $L^2_+(\mathbb{R})$ adjoint of the multiplication X of a Hardy function $f(y)$ by y . The functional I_+ is defined on the domain of X^* , and the action of X^* is explicitly given by [15, page 7]

$$(2.4) \quad X^*f(y) := yf(y) + \frac{1}{2i\pi} I_+[f].$$

The second term here is a constant function; the assertion that $f \in L^2_+(\mathbb{R})$ is in the domain of X^* is equivalent to the assertion that there exists a constant $\lambda \in \mathbb{C}$ such that $yf(y) - \lambda$ is in the Hardy space $L^2_+(\mathbb{R})$.

Note that the operator $X^* + 2it\epsilon\partial_y + 2tT_{u_0} - x\text{Id}$ can be written as $X^* - x\text{Id} - 2tL_{u_0}$ where $L_{u_0} = -i\epsilon\partial_y - T_{u_0}$ denotes the Lax operator for the Benjamin-Ono equation.

3. The action of the resolvent of $X^* + 2it\epsilon\partial_y + 2tT_{u_0}$ for rational u_0 . The main task in implementing the formula (2.3) is to invert the operator $X^* + 2it\epsilon\partial_y + 2tT_{u_0} - x \text{Id}$ on the Hardy function $\Pi u_0(y)$. In other words, we seek a function $y \mapsto f(y; t, x)$ in the Hardy space $L^2_+(\mathbb{R})$ and also in the domain of ∂_y and X^* such that

$$(3.1) \quad (X^* + 2it\epsilon\partial_y + 2tT_{u_0} - x \text{Id})f = \Pi u_0.$$

We will frequently omit the parameters t, x and simply write $f = f(y)$.

In this section, we prove the following lemma.

Lemma 3.1. *The solution f to (3.1) has the following integral representation:*

$$(3.2) \quad f(y) = -\frac{i}{2t\epsilon} e^{ih(y)/\epsilon} \int_{e^{3i\pi/4}\infty}^y \left(u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} \right) e^{-ih(z)/\epsilon} dz$$

with

$$(3.3) \quad \lambda = -\frac{1}{2i\pi} I_+[f], \quad V_n = 2tc_n f(p_n) - c_n, \quad n = 1, \dots, N.$$

Here, the path of integration originates at infinity in the indicated direction to the left of the branch cuts $\Gamma_1, \dots, \Gamma_N$ of h , and the path lies in the domain of analyticity of $h(z)$. Then f has the same domain of analyticity as h , namely $y \in \mathbb{C} \setminus (\Gamma_1 \cup \dots \cup \Gamma_N \cup \bar{\Gamma}_1 \cup \dots \cup \bar{\Gamma}_N)$, as shown in the left-hand panel of Figure 1.

Proof. If $f(\diamond) \in L^2_+(\mathbb{R})$, one can see that for $p \in \mathbb{C}_+$ and $y \in \mathbb{R}$, we have

$$(3.4) \quad \frac{f(y)}{y - p} = \frac{f(y) - f(p)}{y - p} + \frac{f(p)}{y - p},$$

where the first term on the right-hand side has a holomorphic extension to the upper half-plane \mathbb{C}_+ given by $z \in \mathbb{C}_+ \mapsto (f(z) - f(p))/(z - p)$, and the second term in the right-hand side has a holomorphic extension on \mathbb{C}_- given by $z \in \mathbb{C}_- \mapsto f(p)/(z - p)$. Hence the orthogonal projection of $y \in \mathbb{R} \mapsto f(y)/(y - p)$ on the Hardy space $L^2_+(\mathbb{R})$ is simply the first term:

$$(3.5) \quad \Pi \left(\frac{f(\diamond)}{\diamond - p} \right) (y) = \frac{f(y) - f(p)}{y - p}.$$

By linearity, we sum over the partial-fraction expansion (1.3) of $u_0(y)$ and hence infer that

$$(3.6) \quad T_{u_0} f(y) = u_0(y) f(y) - \sum_{n=1}^N \frac{c_n f(p_n)}{y - p_n}.$$

Using this along with (2.1) and (2.4) in (3.1), and writing $\Pi u_0(y) = u_0(y) - (\text{Id} - \Pi)u_0(y)$ on the right-hand side gives: for $y \in \mathbb{C}_+$,

$$(3.7) \quad 2it\epsilon f'(y) + [y - x + 2tu_0(y)] f(y) = u_0(y) + \lambda + \sum_{n=1}^N \frac{V_n}{y - p_n},$$

in which the constants λ and V_1, \dots, V_N are given by (3.3).

Now we observe that λ and V_1, \dots, V_N depend on the unknown function f , but they are constants and hence the form of the right-hand side of (3.7) as a function of y is explicit. This allows us to view (3.7) as a first-order linear differential equation for $f(y)$, the solution of which will depend on the constants λ and V_1, \dots, V_N . We will show in Section 4 that the constants are then uniquely determined so that when $\text{Im}(x) \geq 0$, $f(y)$ is a function in $L_+^2(\mathbb{R})$ that is also in the domain of X^* and of ∂_y . In particular, since (2.3) can be written as

$$(3.8) \quad \Pi u(t, x) = \frac{I_+[f]}{2i\pi} = -\lambda,$$

the value of the constant λ is exactly what is needed to solve the Cauchy problem (1.1)–(1.3):

$$(3.9) \quad u(t, x) = 2\text{Re}(\Pi u(t, x)) = -2\text{Re}(\lambda).$$

An integrating factor for (3.7) is $e^{-ih(z)/\epsilon}$, where h is defined for $z \in \mathbb{C} \setminus (\Gamma_1 \cup \dots \cup \Gamma_N \cup \bar{\Gamma}_1 \cup \dots \cup \bar{\Gamma}_N)$ as explained in the introduction. We assume that $\epsilon > 0$ and also that $t > 0$. Then, the dominant quadratic term in $h(z)$ behaves such that the function $e^{-ih(z)/\epsilon}$ decays exponentially to zero in the directions $\arg(z) = 3\pi/4$ and $\arg(z) = -\pi/4$, so the use of the integrating factor leads to the integral representation of a particular solution in the form (3.2). ■

4. Conditions for f to lie in the Hardy space. Now, assuming that $\text{Im}(x) \geq 0$, we seek conditions on λ and V_1, \dots, V_N such that f is a Hardy function of y . This requires two things:

- (i) that $f(y)$ is analytic at each of the points $y = p_1, \dots, p_N$ (guaranteeing that $f(y)$ is analytic except on the lower half-plane branch cuts $\bar{\Gamma}_1, \dots, \bar{\Gamma}_N$, and in particular for $\text{Im}(y) > -\delta$ for some $\delta > 0$);
- (ii) that the $L_+^2(\mathbb{R})$ -norm (2.2) of $f(y)$ is finite.

For condition (ii) to hold given (i), it will be enough to prove that $yf(y)$ has a limit as $y \rightarrow \infty$ in a horizontal strip $|\text{Im}(y)| < \delta$ centered on the real line. By confirming that this limit is the same in the left and right directions, we will further conclude that $f(y)$ is in the domain of X^* . The equation (3.1) then implies that $f(y)$ is also in the domain of ∂_y .

For condition (i), we show the following lemma. Recall the contours C_m for $m = 1, \dots, N$ defined in Definition 1.2.

Lemma 4.1. *Assume the conditions*

$$(4.1) \quad I_m := \int_{C_m} \left(u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} \right) e^{-ih(z)/\epsilon} dz = 0, \quad m = 1, \dots, N.$$

Then the formula (3.2) defines $f(y)$ as a function analytic for $\text{Im}(y) > -\delta$.

For the proof, we apply the reasoning behind [26, Corollary 1]. Before getting into the details, let us first give an idea of the proof in the simplest situation: $N = 1$. There are four cases to consider.

The exceptional case: $ic_1/\epsilon \in \mathbb{N}_{<0}$. In this case, the contour C_1 goes from $e^{3i\pi/4}\infty$ to p_1 and the integrand is integrable at $z = p_1$, so that by the condition $I_1 = 0$ one can change the starting point of the integration path in (3.2):

$$(4.2) \quad f(y) = -\frac{i}{2t\epsilon} e^{ih(y)/\epsilon} \int_{p_1}^y \left(u_0(z) + \lambda + \frac{V_1}{z - p_1} \right) e^{-ih(z)/\epsilon} dz.$$

This defines a holomorphic function at $z = p_1$. Indeed, since the phase has the expression $e^{-ih(z)/\epsilon} = (z - p_1)^{-ic_m/\epsilon} (z - p_1^*)^{-ic_m^*/\epsilon} e^{-i(z-x)^2/(4t\epsilon)}$, we can write the integrand in the form $H_1(z)(z - p_1)^{-ic_m/\epsilon-1}$ with H_1 being holomorphic at p_1 . Then by Taylor expansion of H_1 around $z = p_1$ and term-by-term integration, we see that $f(y)$ also has a convergent Taylor series around $y = p_1$.

The non-exceptional case with $\operatorname{Re}(ic_1/\epsilon) < 0$. In this case, the integrand is also integrable at $z = p_1$. Hence we can choose the contour C_1 to be the concatenation of C_1^{\searrow} from $e^{3i\pi/4}\infty$ to p_1 and C_1^{\swarrow} from p_1 to $e^{3i\pi/4}\infty$ along opposite sides of the branch cut Γ_1 . Correspondingly, the integral I_1 decomposes into $I_1 = I_1^{\searrow} + I_1^{\swarrow}$. The logarithmic jump being constant along Γ_1 , there holds $I_1 = (1 - e^{2\pi i(-ic_m/\epsilon-1)})I_1^{\searrow}$ so that I_1^{\searrow} vanishes. The proof of the exceptional case then applies and we conclude that f is holomorphic at $z = p_1$.

The non-exceptional case with $ic_1/\epsilon = M \in \mathbb{N}$. In this case, the integrand in (3.2) has a pole of order at most $M + 1$ at $z = p_1$. The condition $I_1 = 0$ then implies the vanishing of the residue of this integrand at $z = p_1$ so that the integral in the definition (3.2) of $f(y)$ defines a function analytic in the upper half-plane except for a pole of order at most M at $z = p_1$, which is compensated by zero of order M of $e^{ih(y)/\epsilon}$. Hence again f is holomorphic at $z = p_1$.

The non-exceptional case with $\operatorname{Re}(ic_1/\epsilon) \geq 0$ and $ic_1/\epsilon \notin \mathbb{N}$. In this case, we can integrate by parts a sufficient number of times to replace the factor $(z - p_1)^{-ic_m/\epsilon-1}$ coming from $e^{-ih(z)/\epsilon}/(z - p_1)$ in the integrand by $(z - p_1)^{-ic_m/\epsilon+M-1}$ for M large enough, after which the proof is the same as in the non-exceptional case with $\operatorname{Re}(ic_1/\epsilon) < 0$ with the same conclusion that $I_1 = 0$ implies that f is analytic at $y = p_1$.

Analyticity of f at just the one point $y = p_1$ is enough to guarantee analyticity for all $y \in \mathbb{C}_+$ in the $N = 1$ case. With the idea of the proof clear for $N = 1$, we now proceed to the general case.

Proof of Lemma 4.1. It is sufficient to show that the formula (3.2) defines an analytic function in the neighborhood of each of the points p_1, \dots, p_N .

Selecting an index m , first suppose that $\operatorname{Re}(ic_m/\epsilon) < 0$. Then, the integrand in (4.1) is integrable at $z = p_m$, and we claim that for y near p_m , $f(y)$ can be written in the equivalent form

$$(4.3) \quad f(y) = -\frac{i}{2t\epsilon} e^{ih(y)/\epsilon} \int_{p_m}^y \left(u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} \right) e^{-ih(z)/\epsilon} dz,$$

where the path of integration avoids the branch cut Γ_m of $h(z)$ emanating from p_m .

Indeed, in the special case that m is an exceptional index, deforming the path in (3.2) to pass through $z = p_m$ and using the condition $I_m = 0$ in (4.1) leads directly to (4.3).

If $\operatorname{Re}(ic_m/\epsilon) < 0$ but m is a non-exceptional index, we reason as follows. If there exists a non-exceptional index $j < m$, let j be the largest such index and observe that the difference $I_m - I_j$ from (4.1) is an integral \tilde{I}_m of the same integrand about a loop encircling in the counterclockwise sense only the branch cut Γ_m of $h(z)$ and no other singularities of the integrand (the integrand is analytic at $z = p_k$ for $j < k < m$ even though $h(z)$ has singularities because these are all exceptional indices). Otherwise, $\tilde{I}_m = I_m$ is itself such an integral. Since on opposite sides of the branch cut Γ_m the integrand differs by the non-unit factor $e^{2\pi i(-ic_m/\epsilon-1)}$

and since the integrand is integrable at $z = p_m$, the resulting integral condition $\tilde{I}_m = 0$ implies that the integral \tilde{I}_m^{\searrow} inwards from $z = \infty$ along just the one side of Γ_m vanishes (writing $\tilde{I}_m = \tilde{I}_m^{\searrow} + \tilde{I}_m^{\swarrow}$ we have $\tilde{I}_m^{\swarrow} = -e^{2\pi i(-ic_m/\epsilon-1)}\tilde{I}_m^{\searrow}$, and $1 - e^{2\pi i(-ic_m/\epsilon-1)} \neq 0$). Taking the sum of this “one-sided” integral \tilde{I}_m^{\searrow} and I_j from (4.1), or zero if there is no exceptional index $j < m$, one obtains an integral condition on a contour terminating at $z = p_m$, just as in the case that m is exceptional, that combines with the definition (3.2) to yield (4.3).

To make use of the formula for $f(y)$ rewritten in the form (4.3), we note that the integrand can be written in the form $H_m(z)(z - p_m)^{-ic_m/\epsilon-1}$ where $H_m(z)$ is holomorphic in a neighborhood of $z = p_m$ and the power function is cut along Γ_m , and that $e^{ih(y)/\epsilon}$ can be written in a similar form $G_m(y)(y - p_m)^{ic_m/\epsilon}$ with $G_m(y)$ analytic at p_m . Using these results in (4.3), expanding $H_m(z)$ in a Taylor series about $z = p_m$ and integrating term-by-term, the desired analyticity of $f(y)$ at $y = p_m$ is easily established under the operating assumption that $\text{Re}(ic_m/\epsilon) < 0$.

Next, suppose that ic_m/ϵ is a nonnegative integer M . Then the integrand in the conditions (4.1) has a pole of order at most $M + 1$ at $z = p_m$, while the factor $e^{ih(y)/\epsilon}$ is analytic with a zero of order M at $z = p_m$. Again letting j denote the largest non-exceptional index with $j < m$ as before, the difference $\tilde{I}_m = I_m - I_j$ (or just $\tilde{I}_m = I_m$ if there is no such index j) then yields an integral condition $\tilde{I}_m = 0$ on a contour that can be deformed to a loop surrounding the point $z = p_m$, which is the only singularity of the integrand therein. Therefore, the residue of the meromorphic integrand vanishes at $z = p_m$ and hence the integral factor in the definition (3.2) is a single-valued meromorphic function of y near p_m with a pole of order M , and again analyticity of $f(y)$ at $y = p_m$ follows.

Finally, we suppose that $\text{Re}(ic_m/\epsilon) \geq 0$ but that ic_m/ϵ is not an integer. Then, both in the definition (3.2) and the integrals I_k from (4.1) with non-exceptional indices k , we integrate by parts M times using the representation of the integrand

$$(4.4) \quad H_m(z)(z - p_m)^{-ic_m/\epsilon-1} = \frac{H_m(z)}{(-ic_m/\epsilon)_M} \frac{d^M}{dz^M} (z - p_m)^{-ic_m/\epsilon+M-1}$$

where $H_m(z)$ is analytic at $z = p_m$, $(a)_M := \Gamma(a + M)/\Gamma(a)$ denotes Pochhammer’s symbol, and M is sufficiently large that $\text{Re}(-ic_m/\epsilon) + M > 0$. No boundary terms are produced for the integrals I_k because under the indicated condition on k , the contour C_k has no finite endpoints. In the formula (3.2) however, each integration by parts will produce a contribution from the finite endpoint $z = y$. In this case, one uses the representation $e^{ih(y)/\epsilon} = G_m(y)(y - p_m)^{ic_m/\epsilon}$ with $G_m(y)$ analytic at $y = p_m$ to conclude that the contribution of each of these terms to $f(y)$ is holomorphic at $y = p_m$. For the remaining integral contribution to $f(y)$, the integrand is integrable at $z = p_m$ and one uses exactly the same argument as in the case $\text{Re}(ic_m/\epsilon) < 0$ with the rewritten integrals I_k to conclude finally that $f(y)$ is analytic at $y = p_m$. ■

Remark 4.2. Since Lemma 4.1 already guarantees that f is analytic on the potentially-complicated branch cuts $\Gamma_1, \dots, \Gamma_N$, for the rest of the argument it is sufficient to take the branch cuts of $h(z)$ to be straight rays from each of the points p_j to ∞ with $\arg(z) = 3\pi/4$ and to define h in the lower half-plane by Schwarz symmetry $h(z^*) = h(z)^*$, and we shall do so.

To deal with condition (ii), we let C_0 be as in Definition 1.2. See Figure 1, right-hand panel. Then we have the following.

Lemma 4.3. *Let $\text{Im}(x) \geq 0$. Suppose that the conditions (4.1) hold, and also that*

$$(4.5) \quad I_0 := \int_{C_0} \left(u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} \right) e^{-ih(z)/\epsilon} dz = 0.$$

Then $f(y)$ defined by (3.2) lies in the Hardy space $L_+^2(\mathbb{R})$, and also in the domain of X^* .

Before getting into the details, we first describe the main ideas of the proof. We first show that $f(y) \sim \lambda/y$ uniformly as $y \rightarrow \infty$ in a strip of the form $0 \leq \text{Im}(y) < \delta$ for some $\delta > 0$. We prove that this asymptotic expansion always holds as $\text{Re}(y) \rightarrow -\infty$, but we will need the condition (4.5) for the expansion to hold as $\text{Re}(y) \rightarrow +\infty$.

We start from expression (3.2) written in the form

$$(4.6) \quad f(y) = -\frac{i}{2t\epsilon} \int_{e^{3i\pi/4}\infty}^y \left(u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} \right) e^{-i(h(z)-h(y))/\epsilon} dz.$$

Then, we show that the dominant contribution in the phase $-i(h(z) - h(y))/\epsilon$ comes from the term $-i((z-x)^2 - (y-x)^2)/(4t\epsilon)$, whose real part is made maximal at $z = y$ on a well-chosen path from $e^{3i\pi/4}\infty$ to y . More precisely, to establish an asymptotic expansion, we use a real parameterization of this path (see (4.13) below), and write f in the form

$$(4.7) \quad f(y) = \lambda(y-x) \int_0^{+\infty} e^{-|y-x|^2 s} \phi(s) ds,$$

with $\phi(0) = 1$. Applying Watson's lemma (Laplace's method for exponential integrals on \mathbb{R}_+ with exponential factor e^{-as} in the integrand for large $a > 0$), we conclude that $f(y) \sim \lambda/y$ as $\text{Re}(y) \rightarrow -\infty$ and $0 \leq \text{Im}(y) < \delta$.

The condition (4.5) implies that there also holds

$$(4.8) \quad f(y) = -\frac{i}{2t\epsilon} \int_{e^{-i\pi/4}\infty}^y \left(u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} \right) e^{-i(h(z)-h(y))/\epsilon} dz.$$

Consequently, a similar study implies that f has the same asymptotic expansion $f(y) \sim \lambda/y$ as $\text{Re}(y) \rightarrow +\infty$ and $0 \leq \text{Im}(y) < \delta$.

This decay property of f at infinity implies that $f \in L_+^2(\mathbb{R})$, and the common limit of $yf(y)$ at $\pm\infty$ obtained from (4.5) also implies that f belongs to the domain of X^* . With this summary of the argument in hand, we proceed to the details.

Proof of Lemma 4.3. Applying Lemma 4.1, to prove that $f \in L_+^2(\mathbb{R})$, it is enough to show that $f(y) = O(y^{-1})$ uniformly as $y \rightarrow \infty$ in a horizontal strip of the form $0 \leq \text{Im}(y) < \delta$ for some $\delta > 0$. Indeed, in taking the supremum over $v > 0$ in (2.2), one can assume also that $0 < v < \delta$, and then by analyticity, and hence also continuity, of f on the closed strip $0 \leq \text{Im}(y) \leq \delta$, for every $L > 0$ there is a constant $K_L^{(1)} > 0$ such that $|f(u+iv)|^2 \leq K_L^{(1)}$

holds for $|u| \leq L$ and $0 \leq v \leq \delta$. Therefore, if $f(u + iv) = O(u^{-1})$ holds uniformly for $0 \leq v \leq \delta$ as $u \rightarrow \infty$, then for every $L > 0$ sufficiently large, there is a constant $K^{(2)} > 0$ such that $|f(u + iv)|^2 \leq K^{(2)}u^{-2}$ holds for $|u| \geq L$ and $0 \leq v \leq \delta$. Consequently,

$$\int_{\mathbb{R}} |f(u + iv)|^2 du \leq 2LK_L^{(1)} + 2K^{(2)} \int_L^\infty \frac{du}{u^2} = 2LK_L^{(1)} + 2K^{(2)}L^{-1} < \infty$$

holds for $0 \leq v < \delta$ and hence $f \in L^2_{\uparrow}(\mathbb{R})$.

To show that $f(u + iv) = O(u^{-1})$ as $u \rightarrow \infty$ holds uniformly for v bounded, we first introduce a large positive parameter by $r := |y - x| > 0$ and make the substitutions $y = x + rY$ and, in the integrand in (3.2), $z = x + rZ$. Here, Y and Z are variables corresponding to y and z respectively, and $|Y| = 1$. Then, in light of Remark 4.2,

$$(4.9) \quad \begin{aligned} e^{i\frac{h(y)}{\epsilon}} &= e^{i\frac{(y-x)^2}{4t\epsilon}} \prod_{j=1}^N (e^{i\frac{\pi}{4}}(y - p_j))^{i\frac{c_j}{\epsilon}} (e^{-i\frac{\pi}{4}}(y - p_j^*))^{i\frac{c_j^*}{\epsilon}} \\ &= r^{iP} e^{i\frac{r^2Y^2}{4t\epsilon}} \prod_{j=1}^N \left(e^{i\frac{\pi}{4}} \left(Y + \frac{x - p_j}{r} \right) \right)^{i\frac{c_j}{\epsilon}} \left(e^{-i\frac{\pi}{4}} \left(Y + \frac{x - p_j^*}{r} \right) \right)^{i\frac{c_j^*}{\epsilon}}, \end{aligned}$$

where

$$P := \frac{1}{\epsilon} \sum_{j=1}^N (c_j + c_j^*).$$

Likewise, the factors in the integrand in (3.2) are

$$(4.10) \quad e^{-i\frac{h(z)}{\epsilon}} = r^{-iP} e^{-i\frac{r^2Z^2}{4t\epsilon}} \prod_{j=1}^N \left(e^{i\frac{\pi}{4}} \left(Z + \frac{x - p_j}{r} \right) \right)^{-i\frac{c_j}{\epsilon}} \left(e^{-i\frac{\pi}{4}} \left(Z + \frac{x - p_j^*}{r} \right) \right)^{-i\frac{c_j^*}{\epsilon}}$$

and

$$(4.11) \quad \begin{aligned} u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} &= \lambda + \sum_{n=1}^N \frac{c_n + V_n}{z - p_n} + \sum_{n=1}^N \frac{c_n^*}{z - p_n^*} \\ &= \lambda + \frac{1}{r} \sum_{n=1}^N \left[\frac{c_n + V_n}{Z + (x - p_n)/r} + \frac{c_n^*}{Z + (x - p_n^*)/r} \right]. \end{aligned}$$

Then, from (3.2), we have

$$(4.12) \quad \begin{aligned} f(y) &= -\frac{ir}{2t\epsilon} \int_{e^{3i\frac{\pi}{4}}\infty}^Y e^{ir^2\frac{(Y^2-Z^2)}{4t\epsilon}} \cdot \prod_{j=1}^N \frac{(e^{i\frac{\pi}{4}}(Y + \frac{x-p_j}{r}))^{i\frac{c_j}{\epsilon}} (e^{-i\frac{\pi}{4}}(Y + \frac{x-p_j^*}{r}))^{i\frac{c_j^*}{\epsilon}}}{(e^{i\frac{\pi}{4}}(Z + \frac{x-p_j}{r}))^{i\frac{c_j}{\epsilon}} (e^{-i\frac{\pi}{4}}(Z + \frac{x-p_j^*}{r}))^{i\frac{c_j^*}{\epsilon}}} \\ &\quad \cdot \left(\lambda + \frac{1}{r} \sum_{n=1}^N \left[\frac{c_n + V_n}{Z + \frac{x-p_n}{r}} + \frac{c_n^*}{Z + \frac{x-p_n^*}{r}} \right] \right) dZ. \end{aligned}$$

Here the path of integration is that obtained from the original path in the z -plane under the affine mapping $Z = (z - x)/r = (z - x)/|y - x|$. The branch cuts of the factors in the denominator on the second line of (4.12) are also mapped into the Z -plane from those of $h(z)$ by this scaling (a similar remark applies to the factors in the numerator as functions of $Y = (y - x)/r = (y - x)/|y - x|$).

Now, we justify a change of variable of the form $i(Y^2 - Z^2)/(4t\epsilon) = -s$ for $s \in \mathbb{R}_+$ by deforming the path of integration. Taking y with a large negative real part and bounded imaginary part, we find that $r \gg 1$ while $Y \approx -1$, and we wish to deform the path of integration to lie along the branch of the hyperbola $\operatorname{Re}(Z^2) = \operatorname{Re}(Y^2) \approx 1$ that asymptotes to the ray $\arg(Z) = 3\pi/4$. Depending on the value of x with $\operatorname{Im}(x) \geq 0$ relative to the points p_1, \dots, p_N , the target hyperbola may cross one or more of the images in the Z -plane of the straight-line branch cuts of $h(z)$ emanating from the points p_1, \dots, p_N , in which case we can slightly shift them to the right far from the origin in the z -plane to ensure analyticity of the integrand as $Z \rightarrow \infty$ along the hyperbola in the indicated direction. Having selected the steepest descent path for which the exponential factor $e^{ir^2(Y^2 - Z^2)/(4t\epsilon)}$ is positive for Z along the path and maximized at the finite endpoint $Z = Y$, Laplace's method applies to give a complete asymptotic expansion of $f(y)$ in descending integer powers of r . Indeed, the substitution $Z = Z(s) := Y\sqrt{1 - 4ites/Y^2}$, where the square root is positive when $s = 0$ and analytic for $s \in \mathbb{R}$ when $Y^2 \approx 1$, parameterizes the desired branch of the hyperbola by $s > 0$. Using this explicit parameterization in (4.12) gives

$$(4.13) \quad f(y) = \frac{ir}{2t\epsilon} \int_0^{+\infty} e^{-r^2s} \cdot \prod_{j=1}^N \frac{(e^{i\frac{\pi}{4}}(Y + \frac{x-p_j}{r}))^{c_j^*} (e^{-i\frac{\pi}{4}}(Y + \frac{x-p_j^*}{r}))^{c_j^*}}{(e^{i\frac{\pi}{4}}(Z(s) + \frac{x-p_j}{r}))^{c_j^*} (e^{-i\frac{\pi}{4}}(Z(s) + \frac{x-p_j^*}{r}))^{c_j^*}} \cdot \left(\lambda + \frac{1}{r} \sum_{n=1}^N \left[\frac{c_n + V_n}{Z(s) + \frac{x-p_n}{r}} + \frac{c_n^*}{Z(s) + \frac{x-p_n^*}{r}} \right] \right) Z'(s) ds.$$

Applying Watson's Lemma and using $Z'(0) = -2it\epsilon/Y$ yields

$$(4.14) \quad f(y) = \frac{\lambda}{rY} + O(r^{-2}) = \frac{\lambda}{y-x} + O(|y-x|^{-2}) = \frac{\lambda}{y} + O(y^{-2}), \quad y \rightarrow \infty$$

with the expansion being valid for large negative $\operatorname{Re}(y)$ and bounded $\operatorname{Im}(y)$. This result does not require the condition (4.5).

To approximate $f(y)$ as $\operatorname{Re}(y) \rightarrow +\infty$ for $\operatorname{Im}(y)$ bounded, we first use (4.5) to write (3.2) in the form

$$(4.15) \quad f(y) = -\frac{i}{2t\epsilon} e^{ih(y)/\epsilon} \int_{e^{-i\pi/4}\infty}^y \left(u_0(z) + \lambda + \sum_{n=1}^N \frac{V_n}{z - p_n} \right) e^{-ih(z)/\epsilon} dz,$$

where the path of integration originates from $z = \infty$ in the indicated direction to the right of all branch cuts of $h(z)$ in the lower half-plane and lies in the domain of analyticity of the integrand. Again setting $r := |y - x|$, rescaling by $y = x + rY$, $z = x + rZ$, taking a hyperbolic path of integration parameterized by the same function $Z = Z(s) := Y\sqrt{1 - 4ites/Y^2}$ for

$s > 0$ (although in the present situation that $\operatorname{Re}(y)$ is large and positive, $Y \approx +1$ instead of -1) and possibly shifting some branch cuts of $h(z)$ in the lower half-plane to the left to maintain analyticity of the integrand along the integration path, we arrive at the same result: the expansion (4.14) is also valid for large positive $\operatorname{Re}(y)$ and bounded $\operatorname{Im}(y)$.

The validity of (4.14) for large $|\operatorname{Re}(y)|$ and bounded $\operatorname{Im}(y)$ implies that $f(u + iv) = O(u^{-1})$ as $u \rightarrow \pm\infty$ uniformly for bounded v , and hence that $f \in L^2_+(\mathbb{R})$. However from (4.14) we learn more, namely that $yf(y) - \lambda \in L^2_+(\mathbb{R})$, which shows that f lies in the domain of X^* as well. ■

5. Derivation of the solution formulæ (1.11) and (1.12).

5.1. Derivation of formula (1.11). Note that equations (4.1) and (4.5) amount to a linear system of $N + 1$ equations on the $N + 1$ unknowns λ, V_1, \dots, V_N :

$$(5.1) \quad \mathbf{B}\mathbf{x} = -\mathbf{b},$$

in which

$$(5.2) \quad \begin{aligned} \mathbf{x} &:= (\lambda, V_1, \dots, V_N)^\top, \\ \mathbf{b} &:= (A_{1,1}, \dots, A_{N+1,1})^\top \end{aligned}$$

and \mathbf{A} and \mathbf{B} are defined in (1.7) and (1.8). In particular, by Cramer's rule, the quantity $\lambda = \lambda(t, x)$ is given by

$$(5.3) \quad \lambda(t, x) = -\frac{\det(\mathbf{A}(t, x))}{\det(\mathbf{B}(t, x))}$$

provided the denominator is nonzero, where $\mathbf{A}(t, x)$ is the same as $\mathbf{B}(t, x)$ but with the first column replaced by $\mathbf{b}(t, x)$. Note that formula (5.3) extends by continuity to the case $\operatorname{Im}(x) = 0$. That the formula (1.11) solves the Cauchy problem for the Benjamin-Ono equation with rational initial data u_0 provided that $\det(\mathbf{B}(t, x)) \neq 0$ then follows from the fact that $\Pi u = -\lambda$ according to (3.8).

5.2. Derivation of formula (1.12). Next we prove the alternate formula (1.12). Let $j = 1, \dots, N + 1$. Note that using (1.4)–(1.5) and (1.7)–(1.8) we can obtain

$$(5.4) \quad \begin{aligned} \epsilon \frac{\partial B_{j1}}{\partial x} &= \int_{C_{j-1}} \left[-i \frac{x-z}{2t} \right] e^{-ih(z)/\epsilon} dz \\ &= \int_{C_{j-1}} [-i(u_0(z) - h'(z))] e^{-ih(z)/\epsilon} dz \\ &= -i \int_{C_{j-1}} u_0(z) e^{-ih(z)/\epsilon} dz \\ &= -iA_{j1}, \end{aligned}$$

because $e^{-ih(z)/\epsilon}$ vanishes at each endpoint (finite or infinite) of each contour C_{j-1} . An even simpler calculation shows that for $k = 2, \dots, N+1$,

$$(5.5) \quad \begin{aligned} \epsilon \frac{\partial B_{jk}}{\partial x} &= \frac{i}{2t} \int_{C_{j-1}} \frac{z-x}{z-p_{k-1}} e^{-ih(z)/\epsilon} dz \\ &= \frac{i}{2t} B_{j1} + \frac{i(p_{k-1}-x)}{2t} B_{jk}. \end{aligned}$$

Therefore, by writing the derivative of the determinant as a sum over $k = 1, \dots, N+1$ of the determinant obtained from \mathbf{B} by replacing column k by its derivative, and using the fact that all columns of \mathbf{A} and \mathbf{B} are the same except the first, we get

$$(5.6) \quad \epsilon \frac{\partial}{\partial x} \det(\mathbf{B}) = -i \det(\mathbf{A}) + \frac{i}{2t} \left(\sum_{k=2}^{N+1} (p_{k-1} - x) \right) \det(\mathbf{B}),$$

from which it follows that for any choice of the complex logarithm,

$$(5.7) \quad \begin{aligned} \frac{\det(\mathbf{A})}{\det(\mathbf{B})} &= i\epsilon \frac{\partial}{\partial x} \log(\det(\mathbf{B})) + \frac{1}{2t} \sum_{k=2}^{N+1} (p_{k-1} - x) \\ &= i\epsilon \frac{\partial}{\partial x} \log(\det(\mathbf{B})) + i\epsilon \frac{\partial}{\partial x} \log \left(\prod_{k=2}^{N+1} e^{i(x-p_{k-1})^2/(4t\epsilon)} \right). \end{aligned}$$

Hence, we obtain

$$(5.8) \quad \frac{\det(\mathbf{A})}{\det(\mathbf{B})} = i\epsilon \frac{\partial}{\partial x} \log(\det(\tilde{\mathbf{B}})),$$

where the first columns of \mathbf{B} and $\tilde{\mathbf{B}}$ agree while the remaining columns of $\tilde{\mathbf{B}}(t, x)$ are given by (1.9). Therefore, from (1.11) we arrive at the representation (1.12).

6. Nonvanishing of the denominators. The last step is to prove that $\det(\mathbf{B}(t, x))$ and $\det(\tilde{\mathbf{B}}(t, x))$ are nonvanishing. Since $\det(\mathbf{B}(t, x))$ and $\det(\tilde{\mathbf{B}}(t, x))$ are explicitly related, it is enough to consider $\det(\mathbf{B}(t, x))$. We first observe the following.

Lemma 6.1. *Let $t, \epsilon > 0$ be fixed. Then the function $x \mapsto \det(\mathbf{B}(t, x))$ is entire and not identically vanishing.*

Proof. The contour integrals in the matrix elements of $\mathbf{B}(t, x)$ are all absolutely convergent for $t, \epsilon > 0$ and arbitrary $x \in \mathbb{C}$, and the integrands are all entire functions of x , so it follows that $x \mapsto \det(\mathbf{B}(t, x))$ is entire. To show that this function does not vanish identically in x , we write $x = Re^{-i\pi/4}$ and consider the asymptotic behavior of the matrix elements of $\mathbf{B}(t, x)$ as $R \rightarrow +\infty$. For the first row of $\mathbf{B}(t, x)$, we observe that the factor $e^{-i(z-x)^2/(4t\epsilon)}$ in $e^{-ih(z)/\epsilon}$ has a simple critical point at $z = x = e^{-i\pi/4}R$ that may be taken to lie on the contour C_0 in the distant fourth quadrant of the complex z -plane. After suitable rescaling of $z - x$ and noting that the logarithms in $h(z)$ agree with the principal branch when $z \approx x$, the method of steepest descent applies to show that the first row of $\mathbf{B}(t, x)$ has the form

$$(6.1) \quad \mathbf{e}_1^\top \mathbf{B}(t, e^{-i\pi/4}R) = 2\sqrt{\pi t \epsilon} e^{-\pi\alpha/(4\epsilon)} R^{-i\alpha/\epsilon} (1, O(R^{-2}), \dots, O(R^{-2})), \quad R \rightarrow +\infty,$$

where $\alpha := 2\operatorname{Re}(c_1 + \cdots + c_N)$. To analyze the remaining rows of $\mathbf{B}(t, e^{-i\pi/4}R)$ in the same limit, we first observe that by invertible row operations and redefinition of the branch cuts $B_j, j = 1, \dots, N$ to diagonal rays $\tilde{B}_j := p_j + e^{3\pi i/4}\mathbb{R}_+$, at the cost of a nonzero constant factor in $\det(\mathbf{B}(t, x))$ we may assume that the contour C_{j-1} in row j is replaced with a ‘‘U-shaped’’ path surrounding with positive orientation only the new branch cut \tilde{B}_{j-1} (unless the index $j-1$ is exceptional, in which case we can simply take C_{j-1} itself to be the ray $p_{j-1} + e^{3\pi i/4}\mathbb{R}_+$ oriented away from p_{j-1}). We observe that all the dependence on $x = \operatorname{Re}^{-i\pi/4}$ in $e^{-ih(z)/\epsilon}$ enters via the exponential factor

$$(6.2) \quad \exp\left(-i\frac{(z - \operatorname{Re}^{-i\pi/4})^2}{4t\epsilon}\right) = \exp\left(-\frac{R^2}{4t\epsilon}\right) \exp\left(R\frac{e^{i\pi/4}z}{2t\epsilon}\right) \exp\left(-\frac{iz^2}{4t\epsilon}\right)$$

and aside from the first factor on the right-hand side that does not depend on z , the only dependence on $R > 0$ in $e^{-ih(z)/\epsilon}$ arises from the second factor. This factor (having an exponent that is proportional to R) decays exponentially in the direction $\arg(z) = 3\pi/4$ of the modified branch cuts and it is maximized at the corresponding branch point. The relevant steepest-descent procedure for such situations is based on Watson’s Lemma combined with repeated integration by parts to obtain integrability at the branch point (see [25, Section 4.8]). Using this method, one finds that row $j = 2, 3, \dots, N+1$ of $\mathbf{B}(t, e^{-i\pi/4}R)$ can be written in the form

$$(6.3) \quad \mathbf{e}_j^\top \mathbf{B}(t, e^{-i\pi/4}R) = C_j(t, \epsilon) \exp\left(-\frac{R^2}{4t\epsilon}\right) \exp\left(R\frac{e^{i\pi/4}p_{j-1}}{2t\epsilon}\right) R^{-ic_{j-1}/\epsilon} \\ \cdot \left(O(R^{-1}), \dots, O(R^{-1}), 1, O(R^{-1}), \dots, O(R^{-1})\right), \quad R \rightarrow +\infty,$$

where $C_j(t) \neq 0$ is a constant, and the largest entry is in the j^{th} column.

The determinant of $\mathbf{B}(t, x)$ is therefore dominated in this limit by the product of the diagonal entries, which is nonzero for sufficiently large $R > 0$. ■

With this result, we can now prove

Proposition 6.2. *We have $\det(\mathbf{B}(t, x)) \neq 0$ for every $x \in \mathbb{R}$ and $t, \epsilon > 0$.*

Proof. Given $t, \epsilon > 0$, it follows from Lemma 6.1 that $x \mapsto \det(\mathbf{B}(t, x))$ is an entire function having only isolated zeros. Given the form of the related determinant $\det(\tilde{\mathbf{B}}(t, x))$ in (1.9), we deduce that for $t, \epsilon > 0$ fixed, $x \mapsto \det(\tilde{\mathbf{B}}(t, x))$ is analytic on \mathbb{R} with only isolated zeros, exactly the same zeros as for $x \mapsto \det(\mathbf{B}(t, x))$. Suppose that $x_0 \in \mathbb{R}$ is such that $\det(\mathbf{B}(t, x_0)) = 0$. Then, $x = x_0$ is also an isolated zero of the analytic function $x \mapsto \det(\tilde{\mathbf{B}}(t, x))$, so there exists an analytic function $x \mapsto \mu(x)$ depending parametrically on t with $\mu(x_0) \neq 0$ and $k \in \mathbb{N}^*$ such that for x in the vicinity of x_0 ,

$$(6.4) \quad \det(\tilde{\mathbf{B}}(t, x)) = \mu(x)(x - x_0)^k.$$

Therefore, given (5.8), this means that for $x \neq x_0$ in the vicinity of x_0 , there holds

$$(6.5) \quad \Pi u(t, x) = i\epsilon \partial_x \log(\det(\tilde{\mathbf{B}}(t, x))) = \frac{i\epsilon k}{x - x_0} + i\epsilon \frac{\partial_x \mu(x)}{\mu(x)}.$$

The right-hand side is clearly not in $L^2_{\text{loc}}(\mathbb{R})$ in the vicinity of x_0 . However, global well-posedness of the Cauchy problem associated to (1.1) in $H^2(\mathbb{R})$ was proven in [28], ensuring that $x \mapsto \Pi u(t, x)$ is a well-defined function in $H^2(\mathbb{R})$. This is therefore a contradiction with the assumption that $\det(\mathbf{B}(t, x_0)) = 0$. \blacksquare

This completes the proof of Theorem 1.3.

Appendix A. The initial data $u_0(x) = 2/(x^2 + 1)$. To illustrate the use of Theorem 1.3, we consider perhaps the simplest nontrivial case by choosing the initial condition to be

$$(A.1) \quad u_0(x) = \frac{2}{x^2 + 1} = \frac{-i}{x - i} + \frac{i}{x + i}.$$

Here $N = 1$, $p_1 = i$, and $c_1 = -i$. Hence ic_1/ϵ is a strictly positive integer M whenever $\epsilon = 1/M$. Also, in this case it is known [22, 26] that the reflection coefficient in the Fokas-Ablovitz inverse-scattering transform vanishes, and hence the corresponding solution of the Benjamin-Ono equation (1.1) is an ensemble of M solitons (the M Lax eigenvalues are simply related to the roots of the Laguerre polynomial of degree M , as was first shown in [22]). For $M = 1$, the exact solution is known to be the Benjamin-Ono soliton:

$$(A.2) \quad u(t, x) = \frac{2}{(x - t)^2 + 1}, \quad \epsilon = M^{-1} = 1.$$

For larger integers M a formula is known in the literature [23] for the corresponding solution. However that formula is in terms of $M \times M$ determinants. We will now give a formula for the same solution as a 2×2 determinant (because $N = 1$ in (1.11)), that is moreover immediately reducible to a scalar expression. Then we will set $M = 1$ and show that the solution obtained is exactly (A.2).

A.1. General $M \in \mathbb{N}$. Letting $\epsilon = M^{-1}$ and taking $N = 1$, $p_1 = i$, and $c_1 = -i$, the exponential factor appearing in the integrands of the matrix elements of $\mathbf{A}(t, x)$ and $\mathbf{B}(t, x)$ is single-valued because $M \in \mathbb{N}$:

$$(A.3) \quad e^{-ih(z)/\epsilon} = e^{-iM(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^M},$$

according to (1.4). Hence, from (1.8) and (1.9), we have

$$(A.4) \quad \det(\tilde{\mathbf{B}}(t, x)) = e^{iM(x-i)^2/(4t)} \det(\mathbf{B}(t, x))$$

where the matrix $\mathbf{B}(t, x)$ is

$$(A.5) \quad \mathbf{B}(t, x) = \begin{bmatrix} \int_{C_0} e^{-iM(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^M} dz & \int_{C_0} e^{-iM(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^{M+1}} dz \\ \int_{C_1} e^{-iM(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^M} dz & \int_{C_1} e^{-iM(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^{M+1}} dz \end{bmatrix}.$$

Note that since we are in the non-exceptional case for the contour C_1 , it may be taken as a U-shaped contour surrounding only the pole at $z = i$ and tending to infinity in both directions

with $\arg(z) = 3\pi/4$. Because the integrand is single-valued, it follows that the integrals over C_1 may be evaluated by residues. It is convenient to first make the substitution $z = x + e^{-i\pi/4}w/\sigma$, with $\sigma := \sqrt{M/(4t)}$ so that for $k = 1, 2$

$$\begin{aligned}
(A.6) \quad B_{2k}(t, x) &= \int_{C_1} e^{-iM(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^{M+k-1}} dz \\
&= \left(e^{i\pi/4}\sigma \right)^{k-2} 2\pi i \operatorname{Res}_{w=e^{-3\pi i/4}\sigma(x-i)} e^{-w^2} \frac{(w - e^{-3\pi i/4}\sigma(x+i))^M}{(w - e^{-3\pi i/4}\sigma(x-i))^{M+k-1}} \\
&= \frac{2\pi i \left(e^{i\pi/4}\sigma \right)^{k-2}}{(M+k-2)!} \frac{d^{M+k-2}}{dw^{M+k-2}} \left[e^{-w^2} \left(w - e^{-3\pi i/4}\sigma(x+i) \right)^M \right] \Bigg|_{w=e^{-3\pi i/4}\sigma(x-i)}.
\end{aligned}$$

Expanding the derivative of the product,

$$\begin{aligned}
(A.7) \quad &\frac{d^{M+k-2}}{dw^{M+k-2}} \left[e^{-w^2} \left(w - e^{-3\pi i/4}\sigma(x+i) \right)^M \right] \\
&= \sum_{n=0}^{M+k-2} \frac{(M+k-2)!}{n!(M+k-2-n)!} \frac{d^n}{dw^n} e^{-w^2} \cdot \frac{d^{M+k-2-n}}{dw^{M+k-2-n}} \left(w - e^{-3\pi i/4}\sigma(x+i) \right)^M \\
&= \sum_{n=0}^{M+k-2} \frac{(M+k-2)!M!}{n!(M+k-2-n)!(n+2-k)!} \left(w - e^{-3\pi i/4}\sigma(x+i) \right)^{n+2-k} \frac{d^n}{dw^n} e^{-w^2}.
\end{aligned}$$

The remaining derivative can be expressed in terms of Hermite polynomials $H_n(\diamond)$ by Rodrigues' formula [9, Eqn. 18.5.5]:

$$\begin{aligned}
(A.8) \quad &\frac{d^{M+k-2}}{dw^{M+k-2}} \left[e^{-w^2} \left(w - e^{-3\pi i/4}\sigma(x+i) \right)^M \right] \\
&= e^{-w^2} \sum_{n=0}^{M+k-2} \frac{(M+k-2)!M!}{n!(M+k-2-n)!(n+2-k)!} \left(w - e^{-3\pi i/4}\sigma(x+i) \right)^{n+2-k} H_n(w).
\end{aligned}$$

Evaluating at $w = e^{-3\pi i/4}\sigma(x-i)$ then yields, for $k = 1, 2$,

$$(A.9) \quad B_{2k}(t, x) = 2\pi i M! (2i)^{2-k} e^{-iM(x-i)^2/(4t)} \sum_{n=0}^{M+k-2} \frac{(2ie^{-3\pi i/4}\sigma)^n H_n(e^{-3\pi i/4}\sigma(x-i))}{n!(M+k-2-n)!(n+2-k)!}.$$

Using (A.4) and expanding $\det(\mathbf{B}(t, x))$ then cancels the exponential factor $e^{-iM(x-i)^2/(4t)}$ and leaves an expression for $\det(\tilde{\mathbf{B}}(t, x))$ as a linear combination of two contour integrals:

$$(A.10) \quad \det(\tilde{\mathbf{B}}(t, x)) = P_2(t, x) \int_{C_0} e^{-i(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^M} dz - P_1(t, x) \int_{C_0} e^{-i(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^{M+1}} dz,$$

where $P_k(t, x) := e^{iM(x-i)^2/(4t)} B_{2k}(t, x)$ are polynomials in x obtained from (A.9). According to (1.12), the corresponding M -soliton solution of the Benjamin-Ono equation (1.1) is then

computed as the logarithmic derivative of this scalar expression:

$$(A.11) \quad u(t, x) = -2M^{-1} \operatorname{Im} \left(\frac{\partial}{\partial x} \log \left(P_2(t, x) \int_{C_0} e^{-i(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^M} dz \right. \right. \\ \left. \left. - P_1(t, x) \int_{C_0} e^{-i(z-x)^2/(4t)} \frac{(z+i)^M}{(z-i)^{M+1}} dz \right) \right).$$

Here C_0 may be taken as any contour originating at $z = \infty e^{3\pi i/4}$ and terminating at $z = \infty e^{-i\pi/4}$ that passes below the pole $z = i$. One can also replace C_0 with the real interval $[-R, R]$ and take the limit $R \rightarrow +\infty$ to define each integral as a convergent improper integral. Equation (A.11) appears to be a previously-unknown explicit formula for the M -soliton solution with the initial data (A.1) and $\epsilon = M^{-1}$.

A.2. The case $M = 1$. In the case $M = 1$, using $H_0(w) = 1$ and $H_1(w) = 2w$, we easily calculate

$$(A.12) \quad B_{21}(t, x) = -4\pi e^{-i(i-x)^2/(4t)}, \quad B_{22}(t, x) = 2\pi i e^{-i(i-x)^2/(4t)} \frac{t-x+i}{t}, \quad M = 1.$$

It then follows from (A.9) and (A.10) for $M = 1$ that

$$(A.13) \quad \det(\tilde{\mathbf{B}}(t, x)) = \frac{2\pi i}{t} \int_{C_0} e^{-i(z-x)^2/(4t)} \left[t-x+i + \frac{2i(i-x)}{z-i} + \frac{4t}{(z-i)^2} \right] dz \\ = \frac{2\pi i}{t} (t - (x-i)) I_0(t, x) + \frac{4\pi}{t} (x-i) I_1(t, x) + 8\pi i I_2(t, x),$$

where, by the substitution $z = w + x$,

$$(A.14) \quad I_P(t, x) := \int_{C_0} e^{-i(z-x)^2/(4t)} \frac{dz}{(z-i)^P} = \int_{C'_0} e^{-iw^2/(4t)} \frac{dw}{(w+(x-i))^P}, \quad P = 0, 1, 2, \dots$$

A useful identity comes from integration by parts:

$$(A.15) \quad I_P(t, x) = -\frac{1}{P-1} \int_{C'_0} e^{-iw^2/(4t)} \frac{d}{dw} \frac{1}{(w+(x-i))^{P-1}} dw \\ = -\frac{i}{2(P-1)t} \int_{C'_0} e^{-iw^2/(4t)} \frac{(w+(x-i)) - (x-i)}{(w+(x-i))^{P-1}} dw \\ = \frac{i(x-i)}{2(P-1)t} I_{P-1}(t, x) - \frac{i}{2(P-1)t} I_{P-2}(t, x), \quad P = 2, 3, 4, \dots$$

Using (A.15) for $P = 2$ allows (A.13) to be expressed in a simpler form as

$$(A.16) \quad \det(\tilde{\mathbf{B}}(t, x)) = \frac{2\pi i}{t} (t - (x+i)) I_0(t, x).$$

Since the shifted contour C'_0 can be taken to be locally independent of x by Cauchy's theorem, we see that

$$(A.17) \quad \frac{\partial I_0}{\partial x}(t, x) = 0.$$

Then, using (A.17), we get

$$(A.18) \quad \frac{\partial}{\partial x} \det(\tilde{\mathbf{B}}(t, x)) = -\frac{2\pi i}{t} I_0(t, x).$$

Consequently, according to (1.12), the solution of the Benjamin-Ono equation in this case is (taking $\epsilon = M^{-1}$ for $M = 1$)

$$(A.19) \quad u(t, x) = -2\text{Im} \left(\frac{\partial}{\partial x} \log(\det(\tilde{\mathbf{B}}(t, x))) \right) = -2\text{Im} \left(\frac{1}{x - t + i} \right) = \frac{2}{(x - t)^2 + 1}$$

in agreement with (A.2).

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