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► **To cite this version:**

Benoit Oriol. Asymptotic non-linear shrinkage formulas for weighted sample covariance. 2024. hal-04746757

HAL Id: hal-04746757

<https://hal.science/hal-04746757v1>

Preprint submitted on 21 Oct 2024

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Asymptotic non-linear shrinkage formulas for weighted sample covariance

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Abstract

We compute asymptotic non-linear shrinkage formulas for covariance and precision matrix estimators for weighted sample covariances, in the spirit of Ledoit and Pécché. We detail explicitly the formulas for exponentially-weighted sample covariances. Those new tools pave a way for applying non-linear shrinkage methods on weighted sample covariance. We show experimentally the performance of the asymptotic shrinkage formulas. Finally, we test the robustness of the theory to a heavy-tailed distributions.

1 Introduction and related work

Covariance estimation is a central topic in multivariate analysis. In high dimension and proportional number of samples - known as Kolmogorov asymptotics -, the sample covariance behaves badly due to an inherent spectrum deformation. This phenomenon is explained by Random Matrix Theory - RMT -, originally by the Marcenko-Pastur theorem [1].

One method to cope with this phenomenon and unbiased the sample covariance spectrum is called shrinkage [2, 3]: the idea is to shrink each sample covariance eigenvalue by a specific factor to counter the deformation induced by high dimension. The idea goes back to Stein, 1956 [4].

The applications of those estimators are diverse, and an extensive list was made by Ledoit and Wolf [5] in their literature review. We can include new works in it, as climatology [6], neuroscience [7], or sensor monitoring [8].

Linear shrinkage was studied, [9, 10, 11] to cite some. Then, RMT gave the mathematical tools to understand asymptotically the non-linear shrinkage. Ledoit and Pécché [12] found asymptotic equations for optimal non-linear shrinkage of the sample covariance in the class of rotation-invariant covariance and precision matrix estimators.

This work opened a way to several methods to estimate those complex non-linear optimal formulas: estimation through cross-validation [13], discrete Marcenko-Pastur inversion with QuEST [14, 5], or kernel density estimation [15, 16] thanks to the theoretical tools on kernel sample covariance spectrum density estimation studied by Jing in 2010 [17].

These methods focus on the standard sample covariance. However, applications in time series such as neurosciences [18], finance [19], suggest that the sample covariance suffers from the non-stationarity of the data.

Weighting schemes, such as the exponential weighted moving average (EWMA), are a model-free approach, and represent a transparent candidate for complex and hard-to-model dynamics. The **weighted** sample covariance was used in a shrinkage setup under Gaussian setting and EWMA weights in [20] for portfolio management. This was further studied for covariance filtering [21]. The spectrum of the EWMA sample covariance in a uncorrelated setting was studied in [22], and Tan et

al. in 2023 [23] developed a shrinkage algorithm of EWMA sample covariance, *à la* NERCOME, with applications in dynamic brain connectivity.

Theoretically, the asymptotic spectrum of the weighted sample covariance has been recently studied [24, 25]. However, to the best of our knowledge, no formulas for asymptotic non-linear shrinkage of the weighted sample covariance have been derived yet.

This work addresses this gap in the literature. In the context of weighted sample covariance, we generalize the asymptotic equation of Oriol [24] in the spirit of Ledoit and Péché [12] and use it to derive asymptotic non-linear shrinkage formulas. Those formulas give asymptotic optimal rotation-invariant covariance and precision matrix estimators. We detail the case of the exponentially-weighted sample covariance.

Finally, the experimental part numerically highlights the performance of the exact non-linear shrinkage. Moreover, we look at how they behave when the theoretical assumptions are not met, specifically when the underlying noise distribution has heavy tails.

2 Notation, definitions and hypotheses

Notations is not constant across major works on the spectrum of sample covariances and their use for shrinkage. In our work, we follow mostly Silverstein ones [26].

Notation 1 (The data matrix). *There are N samples of dimension n . We have:*

- $c_n = \frac{n}{N}$ the concentration ratio,
- Z_n is the noise $n \times N$ matrix composed of i.i.d centered complex entries of variance 1,
- T_n is the true covariance, a non-negative definite Hermitian matrix of size $n \times n$,
- W_n is the weight matrix, a $N \times N$ diagonal non-negative real matrix,
- $Y_n = T_n^{1/2} Z_n$ is the observed data matrix.

The object of interest is the **weighted** sample covariance B_n and its spectrum.

Notation 2 (Weighted sample covariance). *For $n \in \mathbb{N}^*$, the weighted sample covariance is defined by:*

$$B_n := \frac{1}{N} Y_n W_n Y_n^*.$$

Moreover, we denote by $(\tau_1^{(n)}, \dots, \tau_n^{(n)})$ the eigenvalues of T_n in decreasing order, and $(\lambda_1^{(n)}, \dots, \lambda_n^{(n)})$ the eigenvalues of B_n in decreasing order.

Example 1 (Standard weighting). $W_n = I_N$ is the standard weighting. In this situation, B_n is the standard sample covariance, and its asymptotic spectrum is described by Marcenko and Pastur [1].

Example 2 (Exponentially weighted scheme). *In time series analysis, a common choice is the exponentially weighted - EWMA - scheme. Parametrized by some $\alpha \in \mathbb{R}_+$, we define the weights as:*

$$\forall i \in \llbracket 1, N \rrbracket, (W_n)_{ii} = \beta e^{-\alpha i/N},$$

$$\beta = e^{-\alpha/N} \frac{1 - e^{-\alpha/N}}{1 - e^{-\alpha}}.$$

The spectrum of B_n is studied through its empirical spectrum distribution (e.s.d.).

Notation 3 (Empirical spectrum distribution). *We consider a Hermitian matrix A of size $n \times n$ with real eigenvalues (μ_1, \dots, μ_n) . We define the empirical spectrum distribution of A , denoted F^A , as:*

$$F^A := \frac{1}{n} \sum_{i=1}^n 1_{[\mu_i, +\infty[}$$

We describe now several assumptions, mostly the same used in Ledoit and Péché [12]. These assumptions fix the framework of what we call "high dimensional setting": the dimension and number samples grow linearly together, and the empirical spectrum distribution converges in this setting.

The assumption on bounded 12^{th} moments to be finite may appear quite strong for practitioner, however our experimental work lead to think that only bounded 4^{th} moments, or even less, are enough for the results to be applicable.

Assumption 1. We assume the following hypotheses.

H1: Z_n is a (n, N) matrix of real or complex iid random variables with zero mean, unit variance and bounded 12^{th} moments by a constant $B \in \mathbb{R}$ independent of n .

H2: T_n is a random Hermitian positive definite matrix, W_n is a diagonal random positive definite matrix, and Z_n, W_n and T_n are mutually independent.

H3: $c_n = \frac{n}{N} \rightarrow c \in \mathbb{R}_+^*$ as $n \rightarrow \infty$.

H4: $F^{T_n} \xrightarrow[n \rightarrow \infty]{\implies} H$ almost surely, where \implies denotes weak convergence. H defines a probability distribution function (p.d.f.), whose support $S_{F^{T_n}}$ is included in the compact interval $[h_1, h_2]$ with $0 < h_1 \leq h_2 < \infty$.

H5: $F^{W_n} \xrightarrow[n \rightarrow \infty]{\implies} D$ a.s.. D defines a probability distribution function, whose support $S_{F^{W_n}}$ is included in the compact interval $[d_1, d_2]$ with $0 < d_1 \leq d_2 < \infty$.

Remark that H5 naturally implies that a.s., $\int x dF^{W_n}(x) \rightarrow \int x dD(x)$.

3 Θ^g characterization, spectrum retrieval and non-linear shrinkage oracle formulas

This section aims at finding asymptotic optimal formulas to correct the spectrum of weighted sample covaraince and precision matrices. The previous section gave us the theoretical tools to understand the asymptotic spectrum, but when it comes to minimize the loss of a covariance or precision matrix estimator, the behavior of the eigenvectors plays also a role in the loss.

Notation 4 (Eigenvectors of B_n and T_n). We denote $(u_1^{(n)}, \dots, u_n^{(n)})$ a set of eigenvectors of B_n associated to the eigenvalues $(\lambda_1^{(n)}, \dots, \lambda_n^{(n)})$, and $(v_1^{(n)}, \dots, v_n^{(n)})$ a set of eigenvectors of T_n associated to the eigenvalues $(\tau_1^{(n)}, \dots, \tau_n^{(n)})$.

In this section, we look at functionals $\Theta_n^{(g)}$ introduced in a work of Ledoit and P  ch   [12] that carry the information needed to shrink properly the weighted sample eigenvalues for covariance and precision matrix estimation, taking in account the non-ideal projection of u onto v . For that, we need a notation to apply real functions to \mathbb{R} -diagonalizable matrices.

Notation 5 (Extension of real functions to \mathbb{R} -diagonalizable matrices). With $g : \mathbb{R} \rightarrow \mathbb{R}$ and $M = P\Lambda P^{-1}$, M a \mathbb{R} -diagonalizable $p \times p$ matrix, $p \in \mathbb{N}^*$, Λ a real diagonalization of M , we denote:

$$g(M) := P\tilde{\Lambda}P^{-1},$$

where $\forall i \neq j \in \llbracket 1, p \rrbracket, \tilde{\Lambda}_{ii} = g(\Lambda_{ii})$ and $\tilde{\Lambda}_{ij} = 0$.

We can now define the functionals $\Theta_n^{(g)}$ which are the objects of interest of this section.

Definition 1 (Θ_n^g). For $g : \mathbb{R} \rightarrow \mathbb{R}$ a bounded function with a finite points of discontinuity, we define:

$$\begin{aligned} \forall z \in \mathbb{C}_+, \Theta_n^g(z) &= \frac{1}{n} \sum_{i=1}^n \frac{1}{\lambda_i - z} \sum_{j=1}^n |u_i^* v_j|^2 g(\tau_j) \\ &= \frac{1}{n} \text{tr} \left((B_n - zI)^{-1} g(T_n) \right). \end{aligned}$$

For $k \in \mathbb{Z}$, we denote $\Theta_n^{(k)} := \Theta_n^g$ for $g : t \mapsto t^k$ and $m_n := \Theta_n^{(0)}$.

3.1 Asymptotic equation on Θ^g

In the spirit of Theorem 2 by Ledoit and Péché [12] for equally weighted sample covariance, the following result characterizes the limit of the functionals Θ_n^g defined above. The equation that arises is very similar to the one obtained in the Theorem 2 [24], except g at the numerator.

Theorem 1. *Assume the conditions H1-H5. For $g : \mathbb{R} \rightarrow \mathbb{R}$ a bounded function with a finite number of discontinuity, for all $z \in \mathbb{C}_+$, $\Theta_n^g(z) \rightarrow \Theta^g(z)$ almost surely and:*

$$\Theta^g(z) = -\frac{1}{z} \int \frac{g(\tau)}{\tau X(z) + 1} dH(\tau),$$

where for all $z \in \mathbb{C}_+$, $X(z)$ is the unique solution in \mathbb{C}_+ of the following equation:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

For $k \in \mathbb{Z}$, we denote $\Theta^{(k)} := \Theta^g$ for $g : t \mapsto t^k$ and $m := \Theta^{(0)}$.

As in Ledoit and Péché [12], for all $z \in \mathbb{C}_+$, the kernel of integration $\kappa_z(\tau) = \frac{1}{\tau X(z) + 1}$ is conserved disregarding the choice of g .

The next result controls the behavior of X and Θ^g near the real line, and is essential to derive oracle shrinkage formulas for rotation-invariant precision and covariance matrix estimators.

Theorem 2 (Continuity on \mathbb{R}^* , [24]). *Assume the conditions H1-H5. Then, for $\lambda \in \mathbb{R}^*$, we have that $\lim_{\eta \rightarrow 0^+} X(\lambda + i\eta)$, $\lim_{\eta \rightarrow 0^+} \Theta^{(0)}(\lambda + i\eta)$ and $\lim_{\eta \rightarrow 0^+} \Theta^{(1)}(\lambda + i\eta)$ exist. We denote them $\tilde{X}(\lambda)$, $\tilde{m}(\lambda)$ and $\tilde{\Theta}^{(1)}(\lambda)$ respectively.*

Additionally, for $g : \mathbb{R} \rightarrow \mathbb{R}$ with a finite number of discontinuity $\lim_{\eta \rightarrow 0^+} \text{Im}[\Theta^g(\lambda + i\eta)]$ exist, and we denote it $\text{Im}[\tilde{\Theta}^g(\lambda)]$.

The next part aims at using this result in the setting of rotation-invariant covariance and precision matrix estimation. By properly choosing g , with $x \mapsto x$ and $x \mapsto 1/x$, we will derive oracle shrinkage formulas for weighted estimators.

3.2 Oracle shrinkage formulas for rotation-invariant covariance and precision matrix estimators

A direct application of the Theorem 1 is the possibility to derive formulas for the optimal shrinkage of weighted sample eigenvalues in the class of rotation-invariant estimators, as it was done in the non-weighted case in Theorem 4 and 5 by Ledoit and Péché [12]. The minimization is asymptotically equivalent under Frobenius, Inverse Stein or Minimum Variance loss [16].

Definition 2 (Class of rotation-invariant estimator). *Given a weighted sample covariance S_n of size (n, n) , $n \in \mathbb{N}^*$, of the diagonalized form $U_n \Lambda_n U_n^*$ with Λ diagonal, real, we define the class of rotation-invariant estimators as:*

$$\mathcal{C}_n = \{U_n D_n U_n^* | D_n = \text{Diag}(d_1, \dots, d_n), (d_i)_{i=1}^n \in \mathbb{R}^n\}.$$

This class of estimators allows to modify - shrink - the eigenvalues of the weighted sample covariance estimator, but not the eigenvectors. The performance of an estimator in \mathcal{C}_n is thus directly linked to its capacity to correct the weighted sample covariance spectrum. Remark that the class of rotation invariant estimators suits to covariance matrix estimation as well as precision matrix estimation.

3.2.1 Asymptotic shrinkage for covariance matrix estimation

We firstly focus on the problem covariance estimation. It is very similar regarding the method to precision matrix estimation in our framework, which will be exposed after this part.

Under Frobenius norm, we can minimize the norm to the true covariance Σ_n and find the optimal covariance estimator in \mathcal{C}_n . This is an "oracle" estimator because the optimal eigenvalues \tilde{d}_i still depends on Σ_n .

Proposition 1 (Oracle rotation-invariant covariance estimator, [12] p.9). *With $\|\cdot\|$ the Frobenius loss, the minimization problem $\min_{D_n} \|U_n D_n U_n^* - \Sigma_n\|$ has the following minimizer:*

$$\tilde{D}_n = \text{Diag}(\tilde{d}_1, \dots, \tilde{d}_n),$$

where $\forall i \in \llbracket 1, n \rrbracket, \tilde{d}_i = u_i^* \Sigma_n u_i$. $(\tilde{d}_1, \dots, \tilde{d}_n)$ are denoted as "the oracle shrunk covariance eigenvalues".

The key object to study in order to derive asymptotic shrinkage formulas, introduced originally by Ledoit and P ech e [12], is the following Δ_n .

Δ_n contains the necessary information to find each \tilde{d}_i , which is what we need to shrink our sample eigenvalues. Using this object has a direct interest: it can be expressed using $\Theta_n^{(1)}$, which asymptotic behavior is totally characterized in Theorem 1.

To sum up, through Δ_n , we can describe the asymptotic behavior of the \tilde{d}_i , and thus find asymptotic shrinkage formulas.

Definition 3 (Δ_n , from [12]). *We define for all $x \in \mathbb{R}$:*

$$\Delta_n(x) = \frac{1}{n} \sum_{i=1}^n \tilde{d}_i 1_{[\lambda_i, +\infty[}(x).$$

Δ_n contains the necessary information to retrieve each \tilde{d}_i when the sample eigenvalues are all distinct:

$$\forall i \in \llbracket 1, n \rrbracket, \tilde{d}_i = \lim_{\epsilon \rightarrow 0^+} \frac{\Delta_n(\lambda_i + \epsilon) - \Delta_n(\lambda_i - \epsilon)}{F_n(\lambda_i + \epsilon) - F_n(\lambda_i - \epsilon)}.$$

Finally, for all $x \in \mathbb{R}$ such that Δ_n is continuous at x :

$$\Delta_n(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \text{Im}[\Theta_n^{(1)}(\xi + i\eta)] d\xi.$$

We can now describe the asymptotic behavior of Δ_n , which naturally leads to asymptotic shrinkage formulas.

Theorem 3 (Oracle covariance shrinkage formula). *Assume the conditions H1-H5 and that H and D are a finite mixture of diracs and p.d.f. that have a density. There exists a nonrandom function $\Delta : \mathbb{R} \rightarrow \mathbb{R}$ such that a.s, $\forall x \in \mathbb{R}_+, \Delta_n(x) \rightarrow \Delta(x)$. Moreover, we have by Stieltjes-Perron formula:*

$$\Delta(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \text{Im} \left[\Theta^{(1)}(\lambda + i\eta) \right] d\lambda.$$

Suppose $c < 1$. Then:

$$\Delta(x) = \int_{-\infty}^x h(\lambda) dF(\lambda),$$

where for $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$:

$$h(\lambda) = \frac{\int \frac{\tau^2}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}{\int \frac{\tau}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}.$$

Practically, this theorem means that replacing each weighted sample covariance eigenvalue λ_i by $h(\lambda)$, provided we know $\check{X}(\lambda)$ and H , will asymptotically minimize the Frobenius loss, in the class of rotation-invariant estimators. The issue of estimating H and computing \check{X} is addressed in [25] through the WeSpeR algorithm.

For $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$, we denote $\frac{\lambda}{h(\lambda)}$ as the *shrinkage intensity associated to λ* .

An example of shrinkage intensities in function of λ for $c = 0.2$ is given in figure 1. The true eigenvalue distribution used in the figure is $H = 0.2 \times 1_{[1, \infty[} + 0.4 \times 1_{[3, \infty[} + 0.4 \times 1_{[10, \infty[}$ as introduced by Bai and Silverstein [27]. The weight distribution is of the form $D(x) = \alpha \log(x) + \text{cst}$ and mimics an exponentially-weighted moving average (EWMA). The proper definition of this weight distribution is given in Section 4.

We can follow the same methodology to find oracle asymptotic shrinkage formulas for the **precision matrix**, the details and formulas are given in the Appendix.

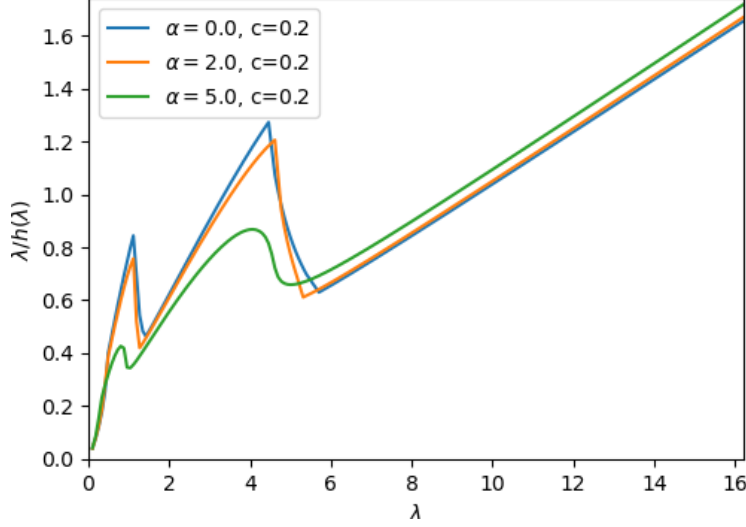


Figure 1: Shrinkage intensities $\lambda/h(\lambda)$ in function of λ , for $c = 0.2$, $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ and D exponentially-weighted with parameter $\alpha \in \{0, 2, 5\}$ as introduced in Section 4.

4 Details with exponential weight distribution

In this section, we want to give an explicit equation for $\Theta^{(1)}$ in function of m with weight distribution that mimics the exponentially weighted moving average (exponentially-weighted). We begin by describing mathematically the weight distribution.

Definition 4 (α -exponential weight law). We fix $\alpha \in \mathbb{R}_+^*$.

We define this law such as its cdf D follows: $D(\beta e^{-\alpha t}) = 1 - t$ for $t \in [0, 1]$.

Moreover, we impose that $\int \delta dD(\delta) = 1$.

We finally have: $\forall x \in [\beta e^{-\alpha}, \beta]$, $D(x) = 1 + \frac{1}{\alpha} \log\left(\frac{x}{\beta}\right)$, with $\beta = \frac{\alpha}{1-e^{-\alpha}}$.

We fix the parameter $\alpha \in \mathbb{R}_+^*$ for the remaining of the section. We can now give a more explicit equation to compute $\Theta^{(1)}$ in function of m .

Theorem 4 ($\check{\Theta}^{(1)}$ for exponential weight law). With an α -exponential weight law, and $\beta = \frac{\alpha}{1-e^{-\alpha}}$, we have:

$$\forall \lambda \in \mathbb{R}, \check{\Theta}^{(1)}(\lambda) = \frac{1}{\beta c} \frac{e^{\alpha c(1+\lambda\check{m}(\lambda))} - 1}{1 - e^{-\alpha + \alpha c(1+\lambda\check{m}(\lambda))}}.$$

As for all $\lambda \in (F')^{-1}(\mathbb{R}_+^*)$, $h(\lambda) = \text{Im}[\check{\Theta}^{(1)}(\lambda)] / \text{Im}[\check{m}(\lambda)]$, this theorem states that the statistical challenge for exponentially-weighted shrinkage is the same as it is for equally weighted sample covariance: estimate \check{m} . Indeed, as the oracle shrinkage formulas only depends on λ , \check{m} and $\check{\Theta}^{(1)}$, the relation above narrows down the estimation to \check{m} only, which can be done through kernel estimation for example [17].

5 Experimental results

5.1 Performance of the asymptotic oracle shrinkage formulas

To show the performance of the oracle asymptotic shrinkage formulas in finite sample, we consider the experiment used by Ledoit and Péché [12]. We are looking at the PRIAL of the rotation-invariant covariance estimator using the corrected eigenvalues with the asymptotic oracle formulas of Theorem 3. We approximate the shrinkage intensities by numerically solving equations from Theorem 2 [24] on a grid made of the sample eigenvalues $(\lambda_i)_i$. The reference distribution of true eigenvalues is $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$, introduced by Bai and Silverstein [27] and used by Ledoit and Péché [12].

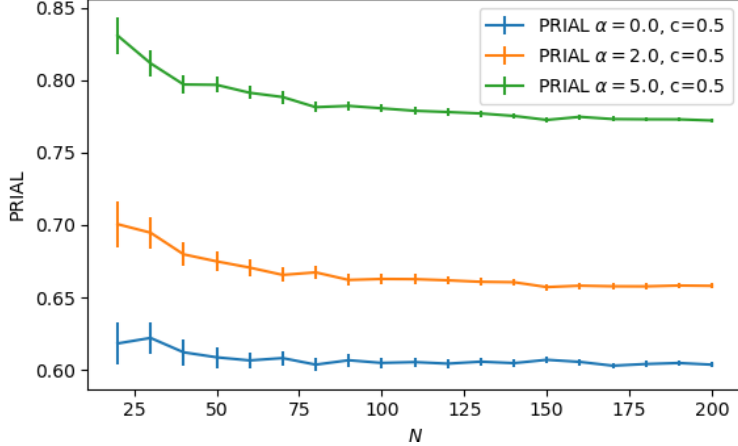


Figure 2: PRIAL of the oracle weighted shrunk estimator in function of the number of samples N , for $c = 0.5$, with $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ and exponentially-weighted distribution of parameter α , Monte-Carlo of $n_{MC} = 50$ draws for each point.

We remind that the PRIAL of an estimator $\hat{\Sigma}$ of the true covariance Σ compared to the sample covariance S is defined as:

$$\text{PRIAL}(\hat{\Sigma}) = \frac{\mathbb{E}[\|S - \Sigma\|^2] - \mathbb{E}[\|\hat{\Sigma} - \Sigma\|^2]}{\mathbb{E}[\|S - \Sigma\|^2]}$$

We approximate the expectations by Monte-Carlo. The result of the experiment is shown in figure 2

5.2 Convergence with heavy tails

This last experimental part aims at empirically relaxing the strong hypothesis we made in the theoretical part: we assumed bounded 12^{th} moments for the underlying distribution of X . Indeed, our experiments lead to think that only bounded 4^{th} moments is really necessary to see the convergence to the asymptotic density we described in Theorem 1. Even for heavier tails, the estimator seems to still show strong performance.

To visualize this point, we are plotting the histogram of Δ_n along with the asymptotic theoretical density $\frac{1}{\pi} \text{Im}[\tilde{\Theta}^{(1)}(\cdot)]$. We draw the (X_{ij}) from a normalized t-distribution with varying degree of freedom $\nu \in \{12, 3, 2.5\}$. At $\nu = 12$, we are at the edge of the theoretical requirement, but we see visible issues with the convergence only with $\nu < 3$, as a non negligible part of values are outside the theoretical support.

As previously, we chose $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ and an exponentially-weighted distribution of parameter $\alpha = 1$ for the weights. The concentration ratio is set at $c = 0.25$ and the dimension at $n = 2000$. The convergence is meant to be almost sure, so we did only one draw of spectrum per plot.

The results are shown in figure 3.

Regarding the PRIAL of the asymptotic oracle estimator, the experiment shows a higher robustness of the method, as even with very heavy-tails, such as $\nu = 3$, no particular artifact is appearing in our setting. We used the same parameters as in the previous experimental part concerning the PRIAL, just replacing the underlying Gaussian distribution with a t-distribution of various degrees of freedom $\nu \in \{4, 3, 2.5\}$. The result is shown in figure 4, and the Gaussian setting is in figure 2.

To conclude this experimental part, according to the previous experiments, the assumption requiring a bounded 12^{th} moments of the underlying distribution seems more of a technical interest rather than a real limitation. For Θ^g convergence, bounded 4^{th} moments seems sufficient, as it was noted by Ledoit and Wolf [28] in the equally-weighted setting.

Regarding the performance of the asymptotic estimator, an even higher robustness seems to be at stake. Indeed, the convergence of the PRIAL is observed until bounded 4^{th} moments. For heavier

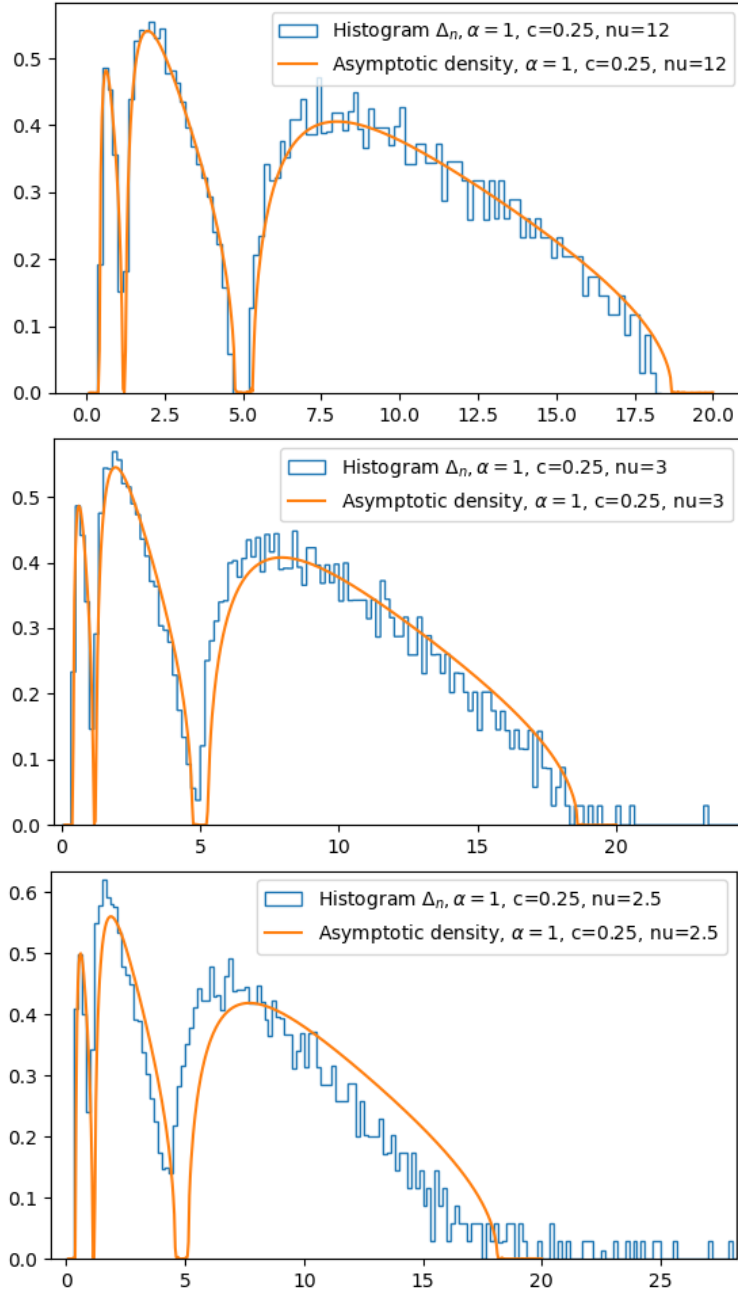


Figure 3: Histogram of Δ_n and its asymptotic theoretical density with underlying noise using t-distribution with $\nu \in \{12, 3, 2.5\}$ from top to bottom, for $n = 2000$, $c = 0.25$, $H = \frac{1}{5}\mathbf{1}_{[1, \infty[} + \frac{2}{5}\mathbf{1}_{[3, \infty[} + \frac{2}{5}\mathbf{1}_{[10, \infty[}$ and exponentially-weighted distribution of parameter $\alpha = 1$.

tails, the PRIAL of the non-linear shrinkage estimator has more variance but takes higher values than expected: the non-linear shrinkage estimator is less impacted by heavy tails than the weighted sample covariance.

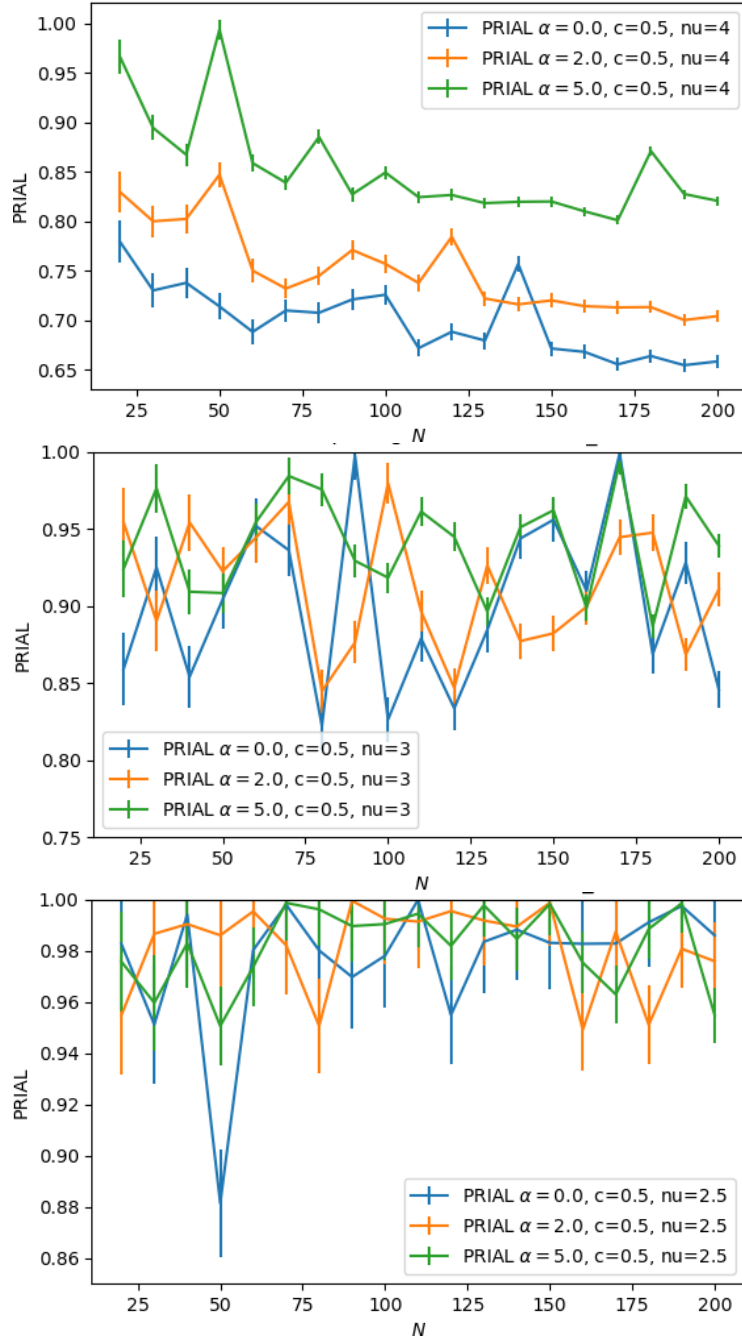


Figure 4: PRIAL of the oracle weighted shrinked estimator from a t-distribution with $\nu \in \{4, 3, 2.5\}$ from top to bottom, in function of N , for $c = 0.5$, with $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ and D the exponentially-weighted distribution of parameter $\alpha \in \{0, 2, 5\}$, Monte-Carlo of $n_{MC} = 400$ draws.

6 Conclusion

In this work, we compute asymptotic optimal formulas for non-linear shrinkage of weighted sample covariance in the class of rotation-invariant covariance and precision matrix estimators.

Experimentally, we underline the performance in PRIAL of the optimal shrinkage formulas. Finally, we confirm the robustness of the theory when the underlying distribution has only bounded 3^{rd} moments instead of 12 required in the theoretical part.

This theory paves a way for further works and extensions. As it was done recently with different approaches [15], [16], analytical estimators of the oracle shrinkage intensities could be derived using kernel density estimation of the weighted empirical spectrum.

Acknowledgment

We would like to thank Alexandre Miot and Gabriel Turinici for their insights and advices all along the work.

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7 Appendix A - Asymptotic shrinkage formulas for precision matrix estimation

For the precision matrix $\Gamma_n = \Sigma_n^{-1}$, we can use a very similar approach and extract an oracle estimator and asymptotic shrinkage formulas for the eigenvalues. We use the same method as with the covariance:

- find the oracle rotation-invariant estimator,
- define an object containing the useful information and that we can study asymptotically,
- and describe its asymptotic behavior.

Proposition 2 (Oracle rotation-invariant precision estimator, [12] p.9). *With $\|\cdot\|$ the Frobenius loss, the minimization problem $\min_{D_n} \|U_n D_n U_n^* - \Gamma_n\|$ has the following minimizer:*

$$\tilde{D}_n = \text{Diag}(\tilde{\gamma}_1, \dots, \tilde{\gamma}_n),$$

where $\forall i \in \llbracket 1, n \rrbracket, \tilde{\gamma}_i = u_i^* \Gamma_n u_i$. ($\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$) are denoted as "the oracle shrunk precision eigenvalues".

The key object Ψ_n is very similar to Δ_n in its conception.

Definition 5 (Ψ_n from Ledoit and Péché [12]). *We define for all $x \in \mathbb{R}$:*

$$\Psi_n(x) = \frac{1}{n} \sum_{i=1}^n \tilde{\gamma}_i 1_{[\lambda_i, +\infty[}(x).$$

Ψ_n contains the necessary information to retrieve each $\tilde{\gamma}_i$ when the sample eigenvalues are all distinct:

$$\forall i \in \llbracket 1, n \rrbracket, \tilde{\gamma}_i = \lim_{\epsilon \rightarrow 0^+} \frac{\Psi_n(\lambda_i + \epsilon) - \Psi_n(\lambda_i - \epsilon)}{F_n(\lambda_i + \epsilon) - F_n(\lambda_i - \epsilon)}.$$

Finally, for all $x \in \mathbb{R}$ such that Ψ_n is continuous at x :

$$\Psi_n(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \text{Im}[\Theta_n^{(-1)}(\xi + i\eta)] d\xi.$$

We can describe the asymptotic behavior of Ψ_n and deduce asymptotic shrinkage formulas.

Theorem 5 (Oracle precision shrinkage formula). *Assume the conditions H1-H5 and that H and D are a finite mixture of diracs and p.d.f. that have a density. There exists a nonrandom function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ such that a.s., $\forall x \in \mathbb{R}_+, \Psi_n(x) \rightarrow \Psi(x)$. Moreover, we have by Stieltjes-Perron formula:*

$$\Psi(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \text{Im} \left[\Theta^{(-1)}(\lambda + i\eta) \right] d\lambda.$$

Suppose $c < 1$. Then:

$$\Psi(x) = \int_{-\infty}^x t(\lambda) dF(\lambda)$$

where for $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$:

$$t(\lambda) = \frac{\int \frac{1}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}{\int \frac{\tau}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}.$$

Practically, similarly to Theorem 3, this theorem means that replacing each weighted sample precision eigenvalue $\frac{1}{\lambda_i}$ by $t(\lambda_i)$, provided you know H , will asymptotically minimize the Frobenius loss, in the class of rotation-invariant estimators. The estimation of H and computation of $\check{X}(\lambda_i)$ from it is discussed in [25].

8 Appendix B - Additional experiments on covariance and precision shrinkage

In this section, we illustrate how non-linear shrinkage formulas affect the spectrum of the weighted sample covariance and precision matrices.

We investigate firstly the case where the weight distribution follows an α -exponential weight law, for different α and c , for the same distribution H we used for illustration in the corpus: $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$, Z_{ij} . In Figures 5, 6 and 7, we show the histograms of the weighted sample eigenvalues (λ_i), the oracle non-linear shrinkage $u_i^* \Sigma u_i$ and the asymptotic non-linear shrinkage we derived ($h(\lambda_i)$). Through this experiment, we empirically assess the convergence of $\Delta_n(x)$ to $\Delta(x)$ for $x \in \mathbb{R}^*$.

The three last figures give the idea of what does non-linear shrinkage on the weighted sample spectrum estimation. We give one more figure, Figure 8 to show an example of **precision** matrix non-linear shrinkage. Through this experiment, we empirically assess the convergence of $\Psi_n(x)$ to $\Psi(x)$ for $x \in \mathbb{R}^*$. The behavior is very similar to the covariance shrinkage, so we will not dive more into details.

We finish this section with a more exotic weight distributions: a mixture of 5 diracs, used in [25] and create an additional gap in the sample spectrum for low c :

$$D = 0.593 \times \mathbf{1}_{[0.337, +\infty[} + 0.297 \times \mathbf{1}_{[0.674, +\infty[} + 0.074 \times \mathbf{1}_{[2.696, +\infty[} + 0.030 \times \mathbf{1}_{[6.74, +\infty[} + 0.006 \times \mathbf{1}_{[33.7, +\infty[}.$$

Results are shown in Figures 9 and 10 for two different c .

9 Appendix C - Proofs of theoretical results

9.1 Proof of Theorem 1

For this proof of Theorem 1, we adapt the proof of Ledoit and Péché for Theorem 1.2 [12]. It mostly relies on the following lemma.

Lemma 1 (Lemma 2.1 [12]). *Let $Y = (y_1, \dots, y_n)$ be a random vector with i.i.d entries satisfying:*

$$\mathbb{E}[y_1] = 0, \mathbb{E}[|y_1|^2] = 1, \mathbb{E}[|y_1|^{12}] \leq B, \quad (1)$$

where the constant B does not depend on n . Let also A be a given $n \times n$ matrix. Then there exists a constant $K > 0$ independent of n , A and Y such that:

$$\mathbb{E} \left[|YAY^* - \text{Tr}(A)|^6 \right] \leq K \|A\|^6 n^3. \quad (2)$$

In order to prove Theorem 1, we will firstly prove it for $g = 1$, for $g = Id$, then for g polynomial by recurrence, and for g continuous by density, and finally for g with a finite number of discontinuities also by recurrence.

We introduce new objects of interest in the analysis, namely q_j, r_j , and $B_{(j)}$.

Notation 6. *For $j \in \llbracket 1, N \rrbracket$, we denote:*

- $q_j = \frac{1}{\sqrt{n}} Z_{\cdot j}$,
- $r_j = \frac{1}{\sqrt{N}} T^{1/2} Z_{\cdot j} W_{jj}^{1/2}$,
- $B_{(j)} = B_n - r_j r_j^*$,

9.1.1 Case $g = 1$

Lemma 2. *Assume H1-H5. For all $z \in \mathbb{C}_+$, $m_n(z) := \Theta_n^{(0)}(z) \rightarrow \Theta^{(0)}(z)$ almost surely and:*

$$m(z) := \Theta^{(0)}(z) = -\frac{1}{z} \int \frac{1}{\tau X(z) + 1} dH(\tau),$$

where for all $z \in \mathbb{C}_+$, $X(z)$ is the unique solution in \mathbb{C}_+ of the following equation:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

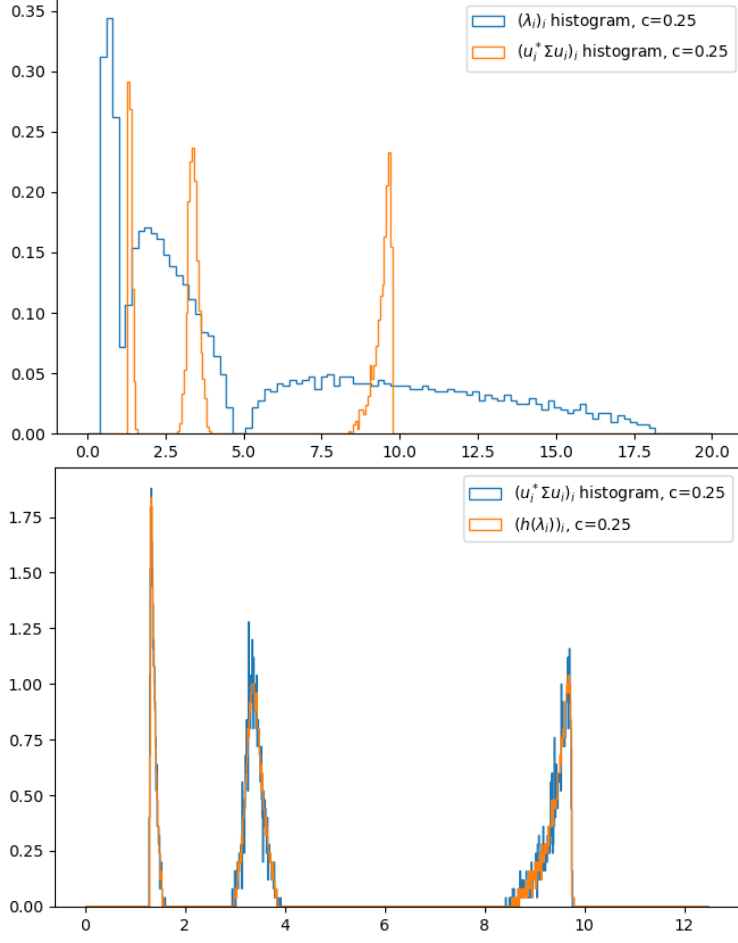


Figure 5: Histograms of the weighted sample eigenvalues λ_i , the optimal shrinkage $u_i^* \Sigma u_i$ (rescaled in top figure for readability) and the asymptotic formula $h(\lambda_i)$, weight distribution D α -exponential with $\alpha = 1$, $H = \frac{1}{5} \mathbf{1}_{[1, \infty[} + \frac{2}{5} \mathbf{1}_{[3, \infty[} + \frac{2}{5} \mathbf{1}_{[10, \infty[}$, Z_{ij} Gaussian, $c = 0.25$, $n = 2000$.

Proof. This result follows from Theorem 2 [24]. It is immediate that the assumptions of this theorem are fulfilled under H1-H5. So, we have that for all $z \in \mathbb{C}_+$, $\Theta_n^{(0)}(z) \rightarrow \Theta^{(0)}(z)$ almost surely and:

$$\Theta^{(0)}(z) = \int \frac{1}{\tau \tilde{m}(z) - z} dH(\tau),$$

where for all $z \in \mathbb{C}_+$, $\tilde{m}(z)$ is the unique solution in $\mathbb{C} \setminus \mathbb{C}_+$ of the following equation:

$$\tilde{m}(z) = \int \frac{\delta}{1 + \delta c \int \frac{\tau}{\tau \tilde{m}(z) - z} dH(\tau)} dD(\delta).$$

We then define:

$$\forall z \in \mathbb{C}_+, X(z) := -\frac{\tilde{m}(z)}{z}.$$

It is easy to check that:

$$X(z) = -\int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

Let $z \in \mathbb{C}_+$. Using the notation of the proof of Theorem 2 [24], it is proved that almost surely:

$$\tilde{m}_n(z) := \frac{1}{N} \sum_{j=1}^N \frac{W_{jj}}{1 + r_j^* (B_{(j)} - zI)^{-1} r_j} \xrightarrow{n \rightarrow +\infty} \tilde{m}(z). \quad (3)$$

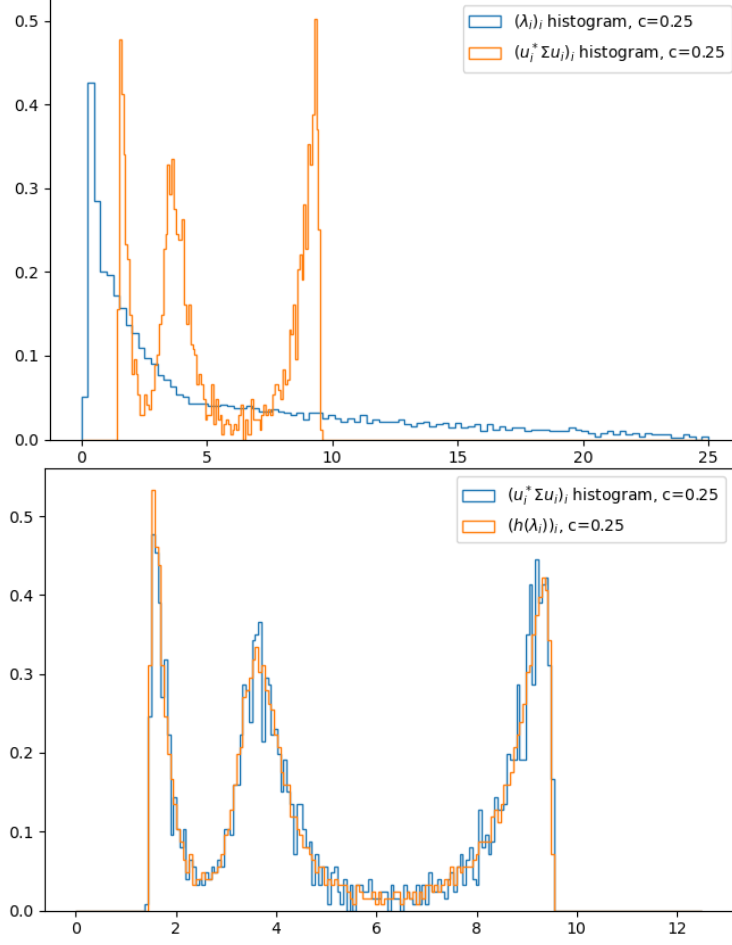


Figure 6: Histograms of the weighted sample eigenvalues λ_i , the optimal shrinkage $u_i^* \Sigma u_i$ and the asymptotic formula $h(\lambda_i)$, weight distribution D α -exponential with $\alpha = 5$, $H = \frac{1}{5} \mathbf{1}_{[1, \infty[} + \frac{2}{5} \mathbf{1}_{[3, \infty[} + \frac{2}{5} \mathbf{1}_{[10, \infty[}$, Z_{ij} Gaussian, $c = 0.25$, $n = 2000$.

So, using that $X(z) := -\frac{\tilde{m}(z)}{z}$, we have immediately that, almost surely:

$$X_n(z) := -\frac{1}{N} \sum_{j=1}^N \frac{W_{jj}}{z + r_j^*(B_{(j)}/z - I)^{-1} r_j} \xrightarrow{n \rightarrow +\infty} X(z). \quad (4)$$

Using the proof of Lemma 3 [24], we have that $\text{Im}[r_j^*(B_{(j)}/z - I)^{-1} r_j] \geq 0$, so $\forall n \in \mathbb{N}, X_n(z) \in \mathbb{C}_+$. So, $\text{Im}[X(z)] \geq 0$.

Moreover, using the definition $X(z) = -\frac{\tilde{m}(z)}{z}$, it is easy to check that:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

It is also immediate that Equation (9.1.1) does not admit any real solution because $z \in \mathbb{C}_+$. As we proved that $\text{Im}[X(z)] \geq 0$, we deduce now that $X(z) \in \mathbb{C}_+$.

The last point to check is the unicity in \mathbb{C}_+ of the solution of (9.1.1). Let $X_1, X_2 \in \mathbb{C}_+$, $X_1 \neq X_2$ solving Equation (9.1.1). Then, in the spirit of the proof of unicity for $\tilde{m}(z)$ in [24], we have with

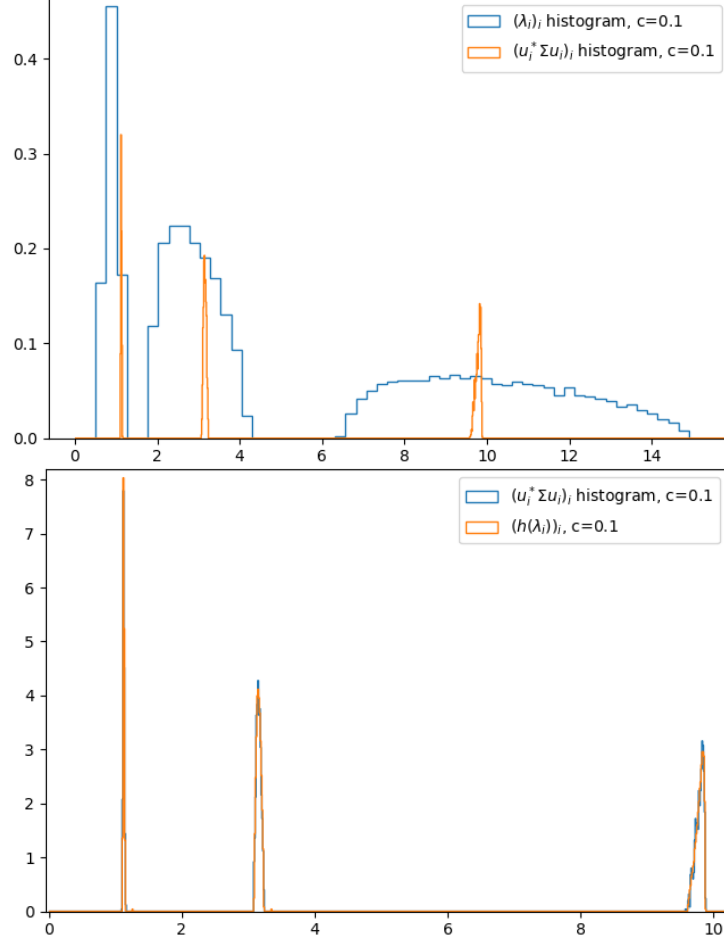


Figure 7: Histograms of the weighted sample eigenvalues λ_i , the optimal shrinkage $u_i^* \Sigma u_i$ (rescaled in top figure for readability) and the asymptotic formula $h(\lambda_i)$, weight distribution D α -exponential with $\alpha = 1$, $H = \frac{1}{5} \mathbf{1}_{[1, \infty[} + \frac{2}{5} \mathbf{1}_{[3, \infty[} + \frac{2}{5} \mathbf{1}_{[10, \infty[}$, Z_{ij} Gaussian, $c = 0.1$, $n = 2000$.

Hölder inequality:

$$\begin{aligned}
|X_1 - X_2|^2 &= |X_1 - X_2|^2 \times \left| \int \frac{\delta^2 c \int \frac{\tau^2}{(\tau X_1 + 1)(\tau X_2 + 1)} dH(\tau)}{\left(z - \delta c \int \frac{\tau}{\tau X_1 + 1} dH(\tau) \right) \left(z - \delta c \int \frac{\tau}{\tau X_2 + 1} dH(\tau) \right)} dD(\delta) \right|^2 \\
&\leq |X_1 - X_2|^2 \times \left| \int \frac{\delta^2 c \int \frac{\tau^2}{|\tau X_1 + 1|^2} dH(\tau)}{\left| z - \delta c \int \frac{\tau}{\tau X_1 + 1} dH(\tau) \right|^2} dD(\delta) \right| \times \left| \int \frac{\delta^2 c \int \frac{\tau^2}{|\tau X_2 + 1|^2} dH(\tau)}{\left| z - \delta c \int \frac{\tau}{\tau X_2 + 1} dH(\tau) \right|^2} dD(\delta) \right| \\
&< |X_1 - X_2|^2 \times \frac{|\operatorname{Im}[X_1]|}{|\operatorname{Im}[X_1]|} \times \frac{|\operatorname{Im}[X_2]|}{|\operatorname{Im}[X_2]|} \\
|X_1 - X_2|^2 &< |X_1 - X_2|^2
\end{aligned} \tag{5}$$

The inequality is strict because $H(]0, \infty[) > 0$ and $D(]0, \infty[) > 0$ according to Assumption H3-H4. The result is absurd, hence the unicity in \mathbb{C}_+ , which concludes the proof of the lemma. \square

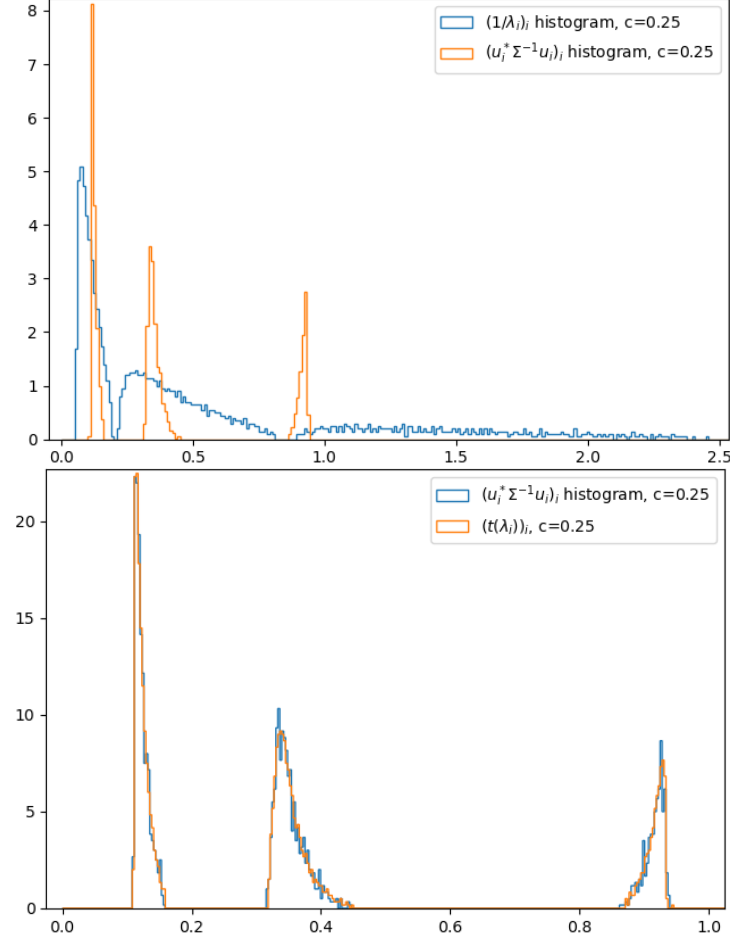


Figure 8: Histograms of the weighted sample precision eigenvalues $1/\lambda_i$, the optimal shrinkage $u_i^* \Sigma^{-1} u_i$ (rescaled in top figure for readability) and the asymptotic formula $t(\lambda_i)$, weight distribution D α -exponential with $\alpha = 1$, $H = \frac{1}{5} \mathbf{1}_{[1, \infty[} + \frac{2}{5} \mathbf{1}_{[3, \infty[} + \frac{2}{5} \mathbf{1}_{[10, \infty[}$, Z_{ij} Gaussian, $c = 0.25$, $n = 2000$.

9.1.2 Case $g = Id$

Lemma 3. Assume H1-H5. For all $z \in \mathbb{C}_+$, $\Theta_n^{(1)}(z) \rightarrow \Theta^{(1)}(z)$ almost surely and:

$$\Theta^{(1)}(z) = -\frac{1}{z} \int \frac{\tau}{\tau X(z) + 1} dH(\tau),$$

where for all $z \in \mathbb{C}_+$, $X(z)$ is the unique solution in \mathbb{C}_+ of the following equation:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

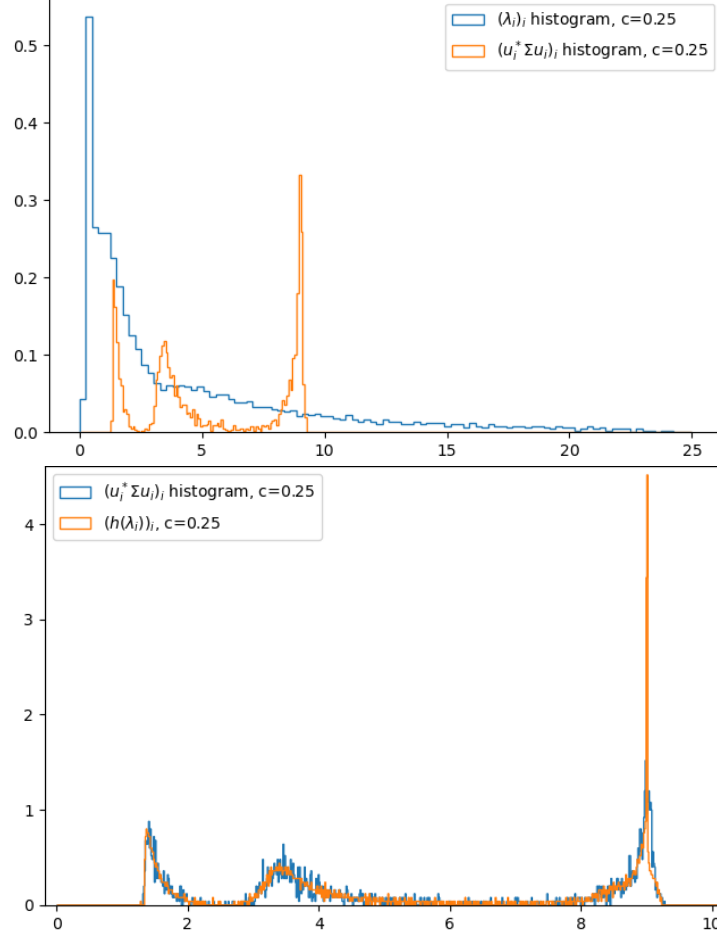


Figure 9: Histograms of the weighted sample covariance eigenvalues λ_i , the optimal shrinkage $u_i^* \Sigma u_i$ (rescaled in top figure for readability) and the asymptotic formula $h(\lambda_i)$, weight distribution D mixture of 5 diracs, $H = \frac{1}{5} \mathbf{1}_{[1, \infty[} + \frac{2}{5} \mathbf{1}_{[3, \infty[} + \frac{2}{5} \mathbf{1}_{[10, \infty[}$, Z_{ij} Gaussian, $c = 0.25$, $n = 2000$.

Proof. Let $z \in \mathbb{C}_+$. We note $v := \text{Im}[z]$. Using derivations detailed in section 2.3.1 [24], we have in our setting:

$$\begin{aligned} \left| \tilde{m}_n(z) \Theta_n^{(1)}(z) - (1 + z m_n(z)) \right| &= \left| \tilde{m}_n(z) \Theta_n^{(1)}(z) - \frac{1}{n} \sum_{j=1}^N \frac{r_j^* (B_{(j)} - zI)^{-1} r_j}{1 + r_j^* (B_{(j)} - zI)^{-1} r_j} \right| \\ &= \left| \frac{1}{N} \sum_{j=1}^N \frac{W_{jj} \left(\Theta_n^{(1)}(z) - q_j^* (B_{(j)} - zI)^{-1} q_j \right)}{1 + r_j^* (B_{(j)} - zI)^{-1} r_j} \right| \end{aligned} \quad (6)$$

$$\left| \tilde{m}_n(z) \Theta_n^{(1)}(z) - (1 + z m_n(z)) \right| \leq \frac{|z| d_2}{v} \max_{j \leq N} \left| \Theta_n^{(1)}(z) - q_j^* T^{1/2} (B_{(j)} - zI)^{-1} T^{1/2} q_j \right|.$$

Using Lemma 1, we have that:

$$\mathbb{E} \left[\left| \Theta_n^{(1)}(z) - q_j^* T^{1/2} (B_{(j)} - zI)^{-1} T^{1/2} q_j \right|^6 \right] \leq K \frac{h_2^6}{n^3 v^6}. \quad (7)$$

So,

$$\max_{j \leq N} \left| \Theta_n^{(1)}(z) - q_j^* T^{1/2} (B_{(j)} - zI)^{-1} T^{1/2} q_j \right| \xrightarrow[n \rightarrow \infty]{} 0 \text{ a.s.} \quad (8)$$

We can conclude:

$$\text{a.s., } \left| \tilde{m}_n(z) \Theta_n^{(1)}(z) - (1 + z m_n(z)) \right| \xrightarrow[n \rightarrow \infty]{} 0. \quad (9)$$

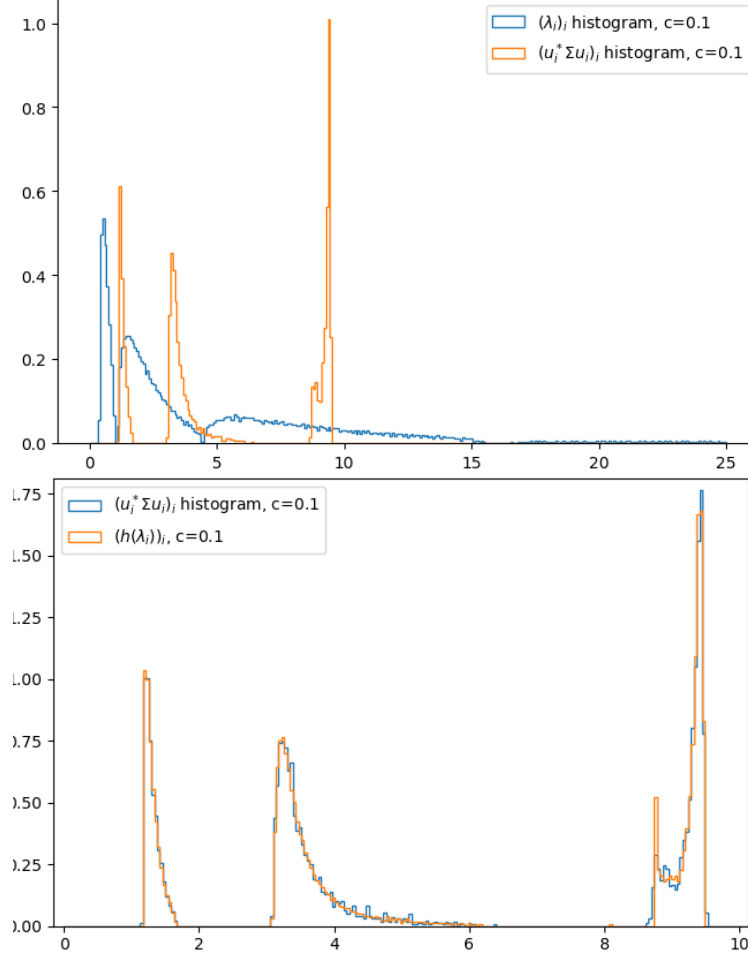


Figure 10: Histograms of the weighted sample covariance eigenvalues λ_i , the optimal shrinkage $u_i^* \Sigma u_i$ (rescaled in top figure for readability) and the asymptotic formula $h(\lambda_i)$, weight distribution D mixture of 5 diracs, $H = \frac{1}{5} \mathbf{1}_{[1, \infty[} + \frac{2}{5} \mathbf{1}_{[3, \infty[} + \frac{2}{5} \mathbf{1}_{[10, \infty[}$, Z_{ij} Gaussian, $c = 0.1$, $n = 2000$.

We also proved that a.s., $\tilde{m}_n(z) \xrightarrow{n \rightarrow \infty} \tilde{m}(z) = \int \frac{\delta}{1 + \delta c \int \frac{\tau}{\tau \tilde{m}(z) - z} dH(\tau)} dD(\delta)$. This equation ensures that $\tilde{m}(z) \neq 0$. To conclude, a.s.,

$$\Theta_n^{(1)}(z) \xrightarrow{n \rightarrow \infty} \frac{1 + zm(z)}{\tilde{m}(z)} = -\frac{1 + zm(z)}{zX(z)} = -\frac{1}{z} \int \frac{\tau}{\tau X(z) + 1} dH(\tau) =: \Theta^{(1)}(z). \quad (10)$$

□

9.1.3 Case g polynomial

Lemma 4. For $q \in \mathbb{N}$ and $z \in \mathbb{C}_+$, $\Theta_n^{(q+1)}(z) \rightarrow \Theta^{(q+1)}(z)$ almost surely and:

$$\Theta^{(q+1)}(z) = -\frac{1}{zX(z)} \left[z\Theta^{(q)}(z) + \int \tau^q dH(\tau) \right]. \quad (11)$$

Proof. Let $z \in \mathbb{C}_+$ and $q \in \mathbb{N}$. We note $v := \text{Im}[z]$. By recurrence, suppose a.s. $\Theta_n^{(q)}(z) \rightarrow \Theta^{(q)}(z)$. We have:

$$\frac{1}{n} \text{tr} (T^q + zT^q(B_n - zI)^{-1}) = \frac{1}{n} \sum_{j=1}^N \frac{r_j^* T^q (B_{(j)} - zI)^{-1} r_j}{1 + r_j^* (B_{(j)} - zI)^{-1} r_j}. \quad (12)$$

Using Lemma 1 and elementary matrix calculus, we prove that:

- $\mathbb{E} \left[\left| r_j^* (B_{(j)} - zI)^{-1} r_j - \frac{W_{jj}}{N} \operatorname{tr} (T(B_{(j)} - zI)^{-1}) \right|^6 \right] \leq \frac{d_2^6 h_2^6 c_n^3}{v^6 N^3},$
- $\mathbb{E} \left[\left| r_j^* T^q (B_{(j)} - zI)^{-1} r_j - \frac{W_{jj}}{N} \operatorname{tr} (T^{q+1}(B_{(j)} - zI)^{-1}) \right|^6 \right] \leq \frac{d_2^6 h_2^{6(q+1)} c_n^3}{v^6 N^3},$
- for all $k \in \mathbb{N}$, $\max_{j \leq N} \left| \frac{1}{N} \operatorname{tr} (T^k(B_{(j)} - zI)^{-1} - T^k(B_n - zI)^{-1}) \right| \leq \max_{j \leq N} \frac{|z| h_2^k d_2 c_n \|q_j\|^2}{N v^3} \rightarrow 0$ a.s.

Using these three points, we prove that:

$$\frac{1}{n} \operatorname{tr} (T^q + zT^q(B_n - zI)^{-1}) - \frac{1}{n} \sum_{j=1}^N \frac{\frac{W_{jj}}{N} \operatorname{tr} (T^{q+1}(B_n - zI)^{-1})}{1 + \frac{W_{jj}}{N} \operatorname{tr} (T(B_n - zI)^{-1})} \xrightarrow{n \rightarrow +\infty} 0 \text{ a.s.} \quad (13)$$

Finally, we remark that, by assumption, a.s.:

$$\frac{1}{n} \operatorname{tr} (T^q + zT^q(B_n - zI)^{-1}) \xrightarrow{n \rightarrow +\infty} \int \tau^q dH(\tau) + z\Theta^{(q)}(z). \quad (14)$$

Moreover,

$$\frac{1}{n} \sum_{j=1}^N \frac{\frac{W_{jj}}{N} \operatorname{tr} (T^{q+1}(B_n - zI)^{-1})}{1 + \frac{W_{jj}}{N} \operatorname{tr} (T(B_n - zI)^{-1})} = \Theta_n^{q+1}(z) \int \frac{\delta}{1 + \delta c_n \Theta_n^{(1)}(z)} dF^{W_n}(\delta). \quad (15)$$

As $\left| \frac{\delta}{1 + \delta c_n \Theta_n^{(1)}(z)} \right| \leq \frac{2}{\operatorname{Im}[c\Theta^{(1)}(z)]}$ for n large enough, we have a.s.:

$$\int \frac{\delta}{1 + \delta c_n \Theta_n^{(1)}(z)} dF^{W_n}(\delta) \xrightarrow{n \rightarrow +\infty} \int \frac{\delta}{1 + \delta c \Theta(1)(z)} dD(\delta) = -zX(z). \quad (16)$$

To sum up, we have a.s.:

$$\Theta_n^{(q+1)}(z) \xrightarrow{n \rightarrow +\infty} -\frac{1}{zX(z)} \left[z\Theta^{(q)}(z) + \int \tau^q dH(\tau) \right]. \quad (17)$$

□

Lemma 5. For $k \in \mathbb{N}$ and $z \in \mathbb{C}_+$, a.s. $\Theta_n^{(k)}(z) \xrightarrow{n \rightarrow +\infty} \Theta^{(k)}(z)$ and:

$$\Theta^{(k)}(z) = -\frac{1}{z} \int \frac{\tau^k}{\tau X(z) + 1} dH(\tau) \quad (18)$$

Moreover, let g a real polynomial function. Then, for $z \in \mathbb{C}_+$, a.s. $\Theta_n^g(z) \xrightarrow{n \rightarrow +\infty} \Theta^g(z)$ and:

$$\Theta^g(z) = -\frac{1}{z} \int \frac{g(\tau)}{\tau X(z) + 1} dH(\tau) \quad (19)$$

Proof. The proof is immediate by recurrence on k , using the previous lemmas. The linearity in g proves the second part. □

9.1.4 Case g continuous

Lemma 6. Assume H1-H5. For all $z \in \mathbb{C}_+$, g continuous on $[h_1, h_2]$, $\Theta_n^g(z) \rightarrow \Theta^g(z)$ almost surely and:

$$\Theta^g(z) = -\frac{1}{z} \int \frac{g(\tau)}{\tau X(z) + 1} dH(\tau),$$

where for all $z \in \mathbb{C}_+$, $X(z)$ is the unique solution in \mathbb{C}_+ of the following equation:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

Proof. By Weierstrass approximation theorem, there exists a sequence of polynomials that converge to g uniformly on $[h_1, h_2]$. By Lemma 5, the results holds for every polynomial in the sequence, therefore it also holds for the limit g . □

9.1.5 General case

By recurrence on the number of discontinuity points, we prove Theorem 1 in the general case.

Proof. We prove the result by recurrence. Let $k \in \mathbb{N}$. Suppose Theorem 1 holds for bounded functions with at most k discontinuity points. Let $g : [h_1, h_2] \rightarrow \mathbb{R}$ bounded with $k + 1$ discontinuity points, one of them is denoted by $\nu \in [h_1, h_2]$. We define $\rho : x \in [h_1, h_2] \mapsto g(x) \times (x - \nu)$. By assumption, $\forall z \in \mathbb{C}_+$, $\Theta_n^\rho \rightarrow \Theta^\rho(z) = -\frac{1}{z} \int \frac{\rho(\tau)}{\tau X(z) + 1} dH(\tau)$ a.s. Using the same proof as Lemma 4, we prove that, for $z \in \mathbb{C}_+$, a.s.:

$$\Theta_n^\rho(z) = -\frac{1}{zX(z)} \left[\int g(\tau) dH(\tau) + z\Theta_n^g(z) \right] - \nu\Theta_n^g(z) + o(1). \quad (20)$$

We deduce then that, a.s. for $z \in \mathbb{C}_+$:

$$\Theta_n^\rho(z) \xrightarrow{n \rightarrow +\infty} -\frac{zX(z)\Theta^\rho(z) + \int g(\tau) dH(\tau)}{z(1 + \nu X(z))} =: \Theta^g(z). \quad (21)$$

Using the assumption on Θ^ρ , we have for $z \in \mathbb{C}_+$:

$$\begin{aligned} \Theta^g(z) &= -\frac{1}{z(1 + \nu X(z))} \left[-X(z) \int \frac{g(\tau) \times (\tau - \nu)}{\tau X(z) + 1} dH(\tau) + \int g(\tau) dH(\tau) \right] \\ \Theta^g(z) &= -\frac{1}{z} \int \frac{g(\tau)}{\tau X(z) + 1} dH(\tau), \end{aligned} \quad (22)$$

which concludes the proof. \square

9.2 Proof of Theorems 3

Theorem 1 shows that:

$$\forall z \in \mathbb{C}_+, \Theta_n^{(1)}(z) \rightarrow \Theta^{(1)}(z) = -\frac{1}{z} \int \frac{\tau}{\tau X(z) + 1} dH(\tau) \text{ a.s.} \quad (23)$$

Moreover, we have that, from Equation (12) in [12], for every x where Δ_n is continuous:

$$\Delta_n(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \text{Im} \left[\Theta_n^{(1)}(\lambda + i\eta) \right] d\lambda. \quad (24)$$

The following Lemma generalizes to the weighted setting the Lemma 3.1 [12].

Lemma 7. *Let g a real bounded function on $[h_1, h_2]$ with finitely many points of discontinuity. Consider the function Ω_n^g defined by:*

$$\forall x \in \mathbb{R}, \Omega_n^g(x) = \frac{1}{n} \sum_{i=1}^n 1_{[\lambda_i, +\infty[}(x) \sum_{j=1}^n |u_i^* v_j|^2 \times g(\tau_j). \quad (25)$$

Then, there exists a nonrandom function Ω^g defined on \mathbb{R} such that $\Omega_n^g(x) \rightarrow \Omega^g(x)$ a.s. at all points of continuity of Ω^g . Furthermore,

$$\Omega^g(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \text{Im} [\Theta^g(\lambda + i\eta)] d\lambda, \quad (26)$$

for all x where Ω^g is continuous.

Proof. The proof is very similar to the proof of Lemma 3.1 [12]. The Cauchy-Stieltjes transform of Ω_n^g is Θ_n^g . From Theorem 1, $\forall z \in \mathbb{C}_+$, $\Theta_n^g(z) \rightarrow \Theta^g(z)$ a.s. Therefore, Equation (2.5) [29] implies that $\lim_{n \rightarrow +\infty} \Omega_n^g(x) =: \Omega^g(x)$ exists for all x where Ω^g is continuous. Furthermore, the Cauchy-Stieltjes transform of Ω^g is Θ^g . Then, (26) is simply the Stieltjes-Perron formula, inversion of the Stieltjes transform. \square

By Lemma 7, $\Delta(x) := \lim_{n \rightarrow +\infty} \Delta_n(x)$ exists a.s. at all point of continuity of Δ and is equal to:

$$\Delta(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \operatorname{Im} \left[\Theta^{(1)}(\lambda + i\eta) \right] d\lambda. \quad (27)$$

We need the following lemma to invert limit and integral.

Lemma 8. *Under the assumptions of Lemma 7 and that $c < 1$, we have for $(x_1, x_2) \in \mathbb{R}^2$:*

$$\Omega^g(x_2) - \Omega^g(x_1) = \frac{1}{\pi} \int_{x_1}^{x_2} \lim_{\eta \rightarrow 0^+} \operatorname{Im}[\Theta^g(\lambda + i\eta)] d\lambda. \quad (28)$$

Proof. We know from Theorem 2 that $\forall x \in \mathbb{R}^*$, $\lim_{z \in \mathbb{C}_+ \rightarrow x} \operatorname{Im}[\Theta^g(z)] =: \operatorname{Im}[\check{\Theta}^g(x)]$ exists. Let us prove it for $x = 0$. From Lemma 3.3 [12] and Marcenko-Pastur theorem for $c < 1$, we have that F is constant over the interval $]-\infty, (1 - \sqrt{c})^2 h_1 d_1[$. So $\lim_{z \in \mathbb{C}_+ \rightarrow 0} \operatorname{Im}[m(z)] \rightarrow 0$. So, $\lim_{z \in \mathbb{C}_+ \rightarrow 0} \operatorname{Im}[\Theta^g(z)] \rightarrow 0$.

Θ^g is the Cauchy-Stieltjes transform Ω^g . Theorem 2.1 [30] implies that Ω^g is differentiable at $x \in \mathbb{R}$ and its derivative is $\pi^{-1} \operatorname{Im}[\check{\Theta}^g(\cdot)]$. We get the result by integration. \square

We can now complete the proof of Theorem 3 with the following lemma. Due to Theorem 2 [25], for $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$, $\check{X}(\lambda) := \lim_{\eta \rightarrow 0^+} X(\lambda + i\eta)$ exists.

Lemma 9. *Suppose $c < 1$. Let $x \in \mathbb{R}$.*

$$\Delta(x) = \int_{-\infty}^x h(\lambda) dF(\lambda),$$

where for $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$:

$$h(\lambda) = \frac{\int \frac{\tau^2}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}{\int \frac{\tau}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}.$$

Proof. We have, using Lemma 8:

$$\begin{aligned} \Delta(x) &= \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \operatorname{Im} \left[\Theta_n^{(1)}(\lambda + i\eta) \right] d\lambda \\ \Delta(x) &= \int_{-\infty}^x \frac{1}{\pi} \operatorname{Im}[\check{\Theta}^{(1)}(\lambda)] d\lambda. \end{aligned} \quad (29)$$

Remark that $\forall \lambda \in \mathbb{R}$, $\operatorname{Im}[\check{\Theta}^{(1)}(\lambda)] = 0 \iff \operatorname{Im}[\check{m}(\lambda)] = 0$. So, as F has a density $F' := \pi^{-1} \operatorname{Im}[\check{m}(\cdot)]$ (see Corollary 1 [25]):

$$\begin{aligned} \Delta(x) &= \int_{]-\infty, x[\cap (F')^{-1}(\mathbb{R}_+^*)} \frac{1}{\pi} \operatorname{Im}[\check{\Theta}^{(1)}(\lambda)] d\lambda \\ &= \int_{]-\infty, x[\cap (F')^{-1}(\mathbb{R}_+^*)} \frac{\operatorname{Im}[\check{\Theta}^{(1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} \times \frac{1}{\pi} \operatorname{Im}[\check{m}(\lambda)] d\lambda \\ \Delta(x) &= \int_{]-\infty, x[\cap (F')^{-1}(\mathbb{R}_+^*)} \frac{\operatorname{Im}[\check{\Theta}^{(1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} dF(\lambda). \end{aligned} \quad (30)$$

Let $\lambda \in (F')^{-1}(\mathbb{R}_+^*)$. We proved that $F'(0) = \pi^{-1} \operatorname{Im}[\check{m}(0)] = 0$, so $\lambda \neq 0$. Then, using Corollary 1 [25]:

$$\begin{aligned} \frac{\operatorname{Im}[\check{\Theta}^{(1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} &= \frac{\operatorname{Im} \left[\int \frac{\tau}{\tau \check{X}(\lambda) + 1} dH(\tau) \right]}{\operatorname{Im} \left[\int \frac{1}{\tau \check{X}(\lambda) + 1} dH(\tau) \right]} \\ &= \frac{\int \frac{\tau^2}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}{\int \frac{\tau}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)} \\ \frac{\operatorname{Im}[\check{\Theta}^{(1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} &= h(\lambda). \end{aligned} \quad (31)$$

$h(\cdot)$ is well-defined on $\mathbb{R}^* \setminus \{x \in S_F | F'(x) = 0\}$, due to Corollary 1 [25]. We extend h on \mathbb{R} with any constant value $\lambda \in \{x \in S_F | F'(x) = 0\} \cup \{0\}$, for example, for $h(\lambda) := 0$. The value does not matter as $dF((F')^{-1}(\{0\})) = 0$ and $dF(\{0\}) = 0$ as $c < 1$. Finally:

$$\begin{aligned}\Delta(x) &= \int_{]-\infty, x[\cap(F')^{-1}(\mathbb{R}_+^*)} h(\lambda) dF(\lambda) \\ \Delta(x) &= \int_{-\infty}^x h(\lambda) dF(\lambda),\end{aligned}\tag{32}$$

which concludes the proof. \square

9.3 Proof of Theorem 5

Proof. By Lemma 7, $\Psi(x) := \lim_{n \rightarrow +\infty} \Psi_n(x)$ exists a.s. at all points of continuity of Ψ and is equal to:

$$\Psi(x) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \operatorname{Im} \left[\Theta^{(-1)}(\lambda + i\eta) \right] d\lambda.\tag{33}$$

We have, using Lemma 8:

$$\begin{aligned}\Psi(x) &= \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^x \operatorname{Im} \left[\Theta_n^{(-1)}(\lambda + i\eta) \right] d\lambda \\ \Psi(x) &= \int_{-\infty}^x \frac{1}{\pi} \operatorname{Im}[\check{\Theta}^{(-1)}(\lambda)] d\lambda.\end{aligned}\tag{34}$$

Remark that $\forall \lambda \in \mathbb{R}, \operatorname{Im}[\check{\Theta}^{(-1)}(\lambda)] = 0 \iff \operatorname{Im}[\check{m}(\lambda)] = 0$. So, as F has a density $F' := \pi^{-1} \operatorname{Im}[\check{m}(\cdot)]$ (see Corollary 1 [25]):

$$\begin{aligned}\Psi(x) &= \int_{]-\infty, x[\cap(F')^{-1}(\mathbb{R}_+^*)} \frac{1}{\pi} \operatorname{Im}[\check{\Theta}^{(-1)}(\lambda)] d\lambda \\ &= \int_{]-\infty, x[\cap(F')^{-1}(\mathbb{R}_+^*)} \frac{\operatorname{Im}[\check{\Theta}^{(-1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} \times \frac{1}{\pi} \operatorname{Im}[\check{m}(\lambda)] d\lambda \\ \Psi(x) &= \int_{]-\infty, x[\cap(F')^{-1}(\mathbb{R}_+^*)} \frac{\operatorname{Im}[\check{\Theta}^{(-1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} dF(\lambda).\end{aligned}\tag{35}$$

Let $\lambda \in (F')^{-1}(\mathbb{R}_+^*)$. We proved that $F'(0) = \pi^{-1} \operatorname{Im}[\check{m}(0)] = 0$, so $\lambda \neq 0$. Then, using Corollary 1 [25]:

$$\begin{aligned}\frac{\operatorname{Im}[\check{\Theta}^{(-1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} &= \frac{\operatorname{Im} \left[\int \frac{\tau^{-1}}{\tau \check{X}(\lambda) + 1} dH(\tau) \right]}{\operatorname{Im} \left[\int \frac{1}{\tau \check{X}(\lambda) + 1} dH(\tau) \right]} \\ &= \frac{\int \frac{1}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)}{\int \frac{\tau}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau)} \\ \frac{\operatorname{Im}[\check{\Theta}^{(-1)}(\lambda)]}{\operatorname{Im}[\check{m}(\lambda)]} &= t(\lambda).\end{aligned}\tag{36}$$

$t(\cdot)$ is well-defined on $\mathbb{R}^* \setminus \{x \in S_F | F'(x) = 0\}$, due to Corollary 1 [25]. We extend h on \mathbb{R} with any constant value $\lambda \in \{x \in S_F | F'(x) = 0\} \cup \{0\}$, for example, for $t(\lambda) := 0$. The value does not matter as $dF((F')^{-1}(\{0\})) = 0$ and $dF(\{0\}) = 0$ as $c < 1$. Finally:

$$\begin{aligned}\Psi(x) &= \int_{]-\infty, x[\cap(F')^{-1}(\mathbb{R}_+^*)} t(\lambda) dF(\lambda) \\ \Psi(x) &= \int_{-\infty}^x t(\lambda) dF(\lambda),\end{aligned}\tag{37}$$

which concludes the proof. \square

9.4 Proof of Theorem 4

Let $\alpha \in \mathbb{R}_+$, $\beta = \frac{\alpha}{1-e^{-\alpha}}$, and $z \in \mathbb{C}_+$. We remark that, noting $m := \Theta^{(0)}$:

$$1 + zm(z) = -zX(z)\Theta^{(1)}(z) \text{ and } -zX(z) = \int \frac{\delta}{1 + \delta c\Theta^{(1)}(z)} dD(\delta). \quad (38)$$

So, we deduce that:

$$1 + zm(z) = \Theta^{(1)}(z) \int \frac{\delta}{1 + \delta c\Theta^{(1)}(z)} dD(\delta). \quad (39)$$

So, in the case of α -exponential weight law, we have, assuming complex-valued logarithm:

$$1 + zm(z) = \frac{1}{\alpha c} \left(\log \left[1 + \beta c\Theta^{(1)}(z) \right] - \log \left[1 + \beta e^{-1} c\Theta^{(1)}(z) \right] \right). \quad (40)$$

So,

$$\exp(\alpha c(1 + zm(z))) = \frac{1 + \beta c\Theta^{(1)}(z)}{1 + \beta e^{-1} c\Theta^{(1)}(z)}. \quad (41)$$

And,

$$\Theta^{(1)}(z) = \frac{e^{\alpha c(1+zm(z))} - 1}{\beta c(1 - e^{-\alpha + \alpha c(1+zm(z))})} \quad (42)$$

Due to Theorem 2 [25] and Theorem 2.2 [30], $\check{\Theta}^{(1)}$ is continuous on \mathbb{R}^* . So, for all $x \in \mathbb{R}^*$, $c(1 + \lambda \check{m}(\lambda)) \neq 1$. So, by continuity, we have for all $\lambda \in \mathbb{R}^*$:

$$\check{\Theta}^{(1)}(\lambda) = \frac{e^{\alpha c(1+\lambda \check{m}(\lambda))} - 1}{\beta c(1 - e^{-\alpha + \alpha c(1+\lambda \check{m}(\lambda))})}, \quad (43)$$

which concludes the proof.

As final remark, we have for $\lambda \in (F')^{-1}(\mathbb{R}_+^*)$, through the method of Lemma 4:

$$\check{\Theta}^{(-1)}(\lambda) = -\check{X}(\lambda)\check{m}(\lambda) - \frac{1}{\lambda} \int \frac{1}{\tau} dH(\tau) = \frac{\check{m}(\lambda)(1 + \lambda \check{m}(\lambda))}{\lambda \check{\Theta}^{(1)}(\lambda)} - \frac{1}{\lambda} \int \frac{1}{\tau} dH(\tau), \quad (44)$$

so,

$$t(\lambda) = \frac{\text{Im} \left[\frac{\check{m}(\lambda)(1 + \lambda \check{m}(\lambda))}{\lambda \check{\Theta}^{(1)}(\lambda)} \right]}{\text{Im}[\check{m}(\lambda)]}. \quad (45)$$

As for all $\lambda \in (F')^{-1}(\mathbb{R}_+^*)$, $h(\lambda) = \text{Im}[\check{\Theta}^{(1)}(\lambda)] / \text{Im}[\check{m}(\lambda)]$ and $t(\lambda) = \text{Im} \left[\frac{\check{m}(\lambda)(1 + \lambda \check{m}(\lambda))}{\lambda \check{\Theta}^{(1)}(\lambda)} \right] / \text{Im}[\check{m}(\lambda)]$, Theorem 4 states that the statistical challenge for exponentially-weighted shrinkage is the same as it is for equally weighted sample covariance: estimate \check{m} , which can be done through kernel estimation for example [17].