



**HAL**  
open science

# WeSpeR: Population spectrum retrieval and spectral density estimation of weighted sample covariance

Benoit Oriol

► **To cite this version:**

Benoit Oriol. WeSpeR: Population spectrum retrieval and spectral density estimation of weighted sample covariance. 2024. hal-04746747

**HAL Id: hal-04746747**

**<https://hal.science/hal-04746747v1>**

Preprint submitted on 21 Oct 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# WeSpeR: Population spectrum retrieval and spectral density estimation of weighted sample covariance

---

**Benoît Oriol**

CEREMADE, Université Paris-Dauphine, PSL, Paris, France  
 Société Générale Corporate and Investment Banking, Puteaux, France  
 benoit.oriol@dauphine.eu

## Abstract

The spectrum of the weighted sample covariance shows a asymptotic non random behavior when the dimension grows with the number of samples. In this setting, we prove that the asymptotic spectral distribution  $F$  of the weighted sample covariance has a continuous density on  $\mathbb{R}^*$ . We address then the practical problem of numerically finding this density. We propose a procedure to compute it, to determine the support of  $F$  and define an efficient grid on it. We use this procedure to design the *WeSpeR* algorithm, which estimates the spectral density and retrieves the true spectral covariance spectrum. Empirical tests confirm the good properties of the *WeSpeR* algorithm.

## 1 Introduction and related work

In Random Matrix Theory - RMT -, the sample covariance spectrum has a non-random limit, denoted  $F$ , when the dimension grows linearly with the number of samples. In this regime, the sample spectral distribution does not converge to the population spectral distribution, but to a limit described by the Marcenko-Pastur equation [1]. The joint work of Silverstein and Choi [2] gives important results on the asymptotic distribution  $F$ :  $F$  has a density on  $\mathbb{R}^*$ , and its support can be computed with a simple procedure without any sampling of high dimensional sample covariance matrices.

Those results were directly used in the design of the algorithm QuEST by Ledoit and Wolf [3], aiming at retrieving the population covariance spectrum and the sample covariance asymptotic density  $F'$ .

Those works focus on the sample covariance matrix, however in practice we often face **weighted** sample covariance matrices, in particular in multivariate time series analysis. Indeed, weighting schemes, such as the exponential weighted moving average (EWMA), are a model-free approach, and represent a widely used method to estimate statistics. They were used in covariance estimation for portfolio management in [4], further studied for covariance filtering in [5], for financial spectrum estimation in [6], and recently Tan et al.[7] developed a NERCOME-like approach for EWMA sample covariance in a dynamic brain connectivity setting.

For the generalization to the weighted sample covariance of the asymptotic results on the spectral distribution, the work of Oriol [8] gives a Fundamental Equation similar to Marcenko-Pastur equation. The high dimensional spectrum also converges to a non-random distribution  $F$ .

This work aims at studying  $F$  for weighted sample covariances, in order to retrieve the population spectrum, and numerically compute the sample density.

We prove that the asymptotic sample spectrum has a density on  $\mathbb{R}^*$ . We then provide a procedure to compute it at any point  $x \in \mathbb{R}^*$ . Moreover, we design a method to find numerically the asymptotic support  $S_F$  of the spectrum  $F$ , so that we can numerically detect spectral gaps and determine a precise discretization grid on  $S_F$ . This is the approach used by Ledoit and Wolf [3] in the QuEST function

for the standard sample covariance: we present here *WeSpeR* that generalizes the idea to weighted sample covariance and makes use of automatic differentiation for a greatly simplified implementation.

Experimentally, we propose an algorithm based on these results. It retrieves the population spectrum from a weighted sample covariance, and computes the asymptotic density  $F'$  of the sample spectrum.

## 2 Notations, definitions and hypotheses

Notation is not constant across major works on the spectrum of sample covariances, we mostly follow Silverstein [9].

**Notation 1** (The data matrix). *There are  $N$  samples of dimension  $n$ . We have:*

- $c_n = \frac{n}{N}$  the concentration ratio,
- $Z_n$  is the noise  $n \times N$  matrix composed of i.i.d centered complex entries of variance 1,
- $T_n$  is the true covariance, a non-negative definite Hermitian matrix of size  $n \times n$ ,
- $W_n$  is the weight matrix, a  $N \times N$  diagonal non-negative real matrix,
- $Y_n = T_n^{1/2} Z_n$  is the observed data matrix.

The object of interest is the weighted sample covariance  $B_n$ , particularly its spectrum of eigenvalues, which is introduced below.

**Notation 2** (Weighted sample covariance). *For  $n \in \mathbb{N}^*$ , the weighted sample covariance is defined:*

$$B_n := \frac{1}{N} T_n^{1/2} Z_n W_n Z_n^* T_n^{1/2}.$$

Moreover, we denote  $(\tau_1^{(n)}, \dots, \tau_n^{(n)})$  the eigenvalues of  $T_n$  in decreasing order, and  $(\lambda_1^{(n)}, \dots, \lambda_n^{(n)})$  the eigenvalues of  $B_n$  in decreasing order.

**Example 1** (Standard weighting). *The most simple weighting is the constant weighting:  $W_n = I_N$ . In this situation,  $B_n$  is the standard sample covariance, and its asymptotic spectrum is described by the Marcenko-Pastur theorem [1].*

**Example 2** (Exponentially weighted scheme). *A common choice in time series analysis is the exponentially weighted scheme. Parametrized by some  $\alpha \in \mathbb{R}_+^*$ , we define the weights as:*

$$\begin{aligned} \forall i \in \llbracket 1, N \rrbracket, (W_n)_{ii} &= \beta e^{-\alpha i/N}, \\ \beta &= e^{-\alpha/N} \frac{1 - e^{-\alpha/N}}{1 - e^{-\alpha}}. \end{aligned}$$

**Notation 3** (Empirical spectrum distribution). *We consider a Hermitian matrix  $A$  of size  $n \times n$  with real eigenvalues  $(\mu_1, \dots, \mu_n)$ . We define the empirical spectrum distribution of  $A$ , denoted  $F^A$ , as:*

$$F^A := \frac{1}{n} \sum_{i=1}^n 1_{[\mu_i, +\infty[}.$$

We describe now several assumptions, the same used in Ledoit and Pécché [10], extended to the weighted situation. These assumptions define the framework of what we call "high dimensional setting": the dimension and number samples grow linearly together, and the empirical spectrum distribution converges.

**Assumption 1.** *We assume the following hypotheses.*

*H1:  $Z_n$  is a  $(n, N)$  matrix of real or complex iid random variables with zero mean, unit variance.*

*H2:  $T_n$  is a random Hermitian positive definite matrix,  $W_n$  is a diagonal random positive definite matrix, and  $Z_n$ ,  $W_n$  and  $T_n$  are mutually independent.*

*H3:  $c_n = \frac{n}{N} \rightarrow c \in \mathbb{R}_+^*$  as  $n \rightarrow \infty$ .*

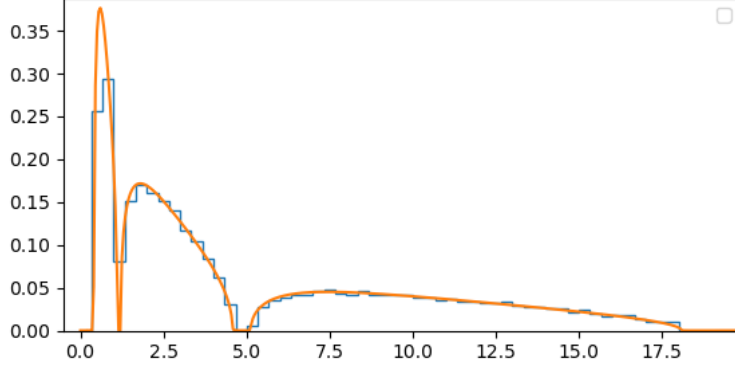


Figure 1: Theoretical  $F$ , and observed  $F^{B_n}$  using  $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ ,  $D$  uniform between  $1/2$  and  $3/2$ ,  $c = 0.25$ , and empirical eigenvalues sampled with  $n = 2000$ .

$H4$ :  $F^{T_n} \xrightarrow[n \rightarrow \infty]{} H$  almost surely, where  $\xrightarrow{} \implies$  denotes weak convergence.  $H$  defines a probability distribution function (p.d.f.), whose support  $S_H$  is included in the compact interval  $[h_1, h_2]$  with  $0 < h_1 \leq h_2 < \infty$ .

$H5$ :  $F^{W_n} \xrightarrow[n \rightarrow \infty]{} D$  a.s..  $D$  defines a probability distribution function, whose support  $S_D$  is included in the compact interval  $[d_1, d_2]$  with  $0 < d_1 \leq d_2 < \infty$ . Moreover, a.s.,  $\int x dF^{W_n}(x) \rightarrow \int x dD(x)$ .

### 3 The asymptotic spectrum $F$ has a density on $\mathbb{R}^*$

As stated in the Theorem 2 [8],  $F_n$  is asymptotically non random, we denote its limit  $F$ . We give here a weaker form of the Theorem, that needs fewer assumptions.

**Theorem 1** (Theorem 2 [8]). *Assume H1 to H5. Then, almost surely,  $F^{B_n}$  converges weakly to a nonrandom p.d.f  $F$ , whose Cauchy-Stieltjes transform  $m := m_F$  satisfies for all  $z \in \mathbb{C}_+$ :*

$$m(z) = -\frac{1}{z} \int \frac{1}{\tau X(z) + 1} dH(\tau), \quad (1)$$

where for all  $z \in \mathbb{C}_+$ ,  $X(z)$  is the unique solution in  $\mathbb{C}_+$  of the following equation:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

An example of asymptotic spectrum  $F$  is given in Figure 1. A key objective of this work is to derive properties of  $F$  through the analysis of  $X$ . This study leads to show that  $F$  has a density on  $\mathbb{R}^*$ , and even  $\mathbb{R}$  if  $c < 1$ . Moreover, we prove that the support of  $F$ , denoted  $S_F$  can be retrieved with a simple function analysis.

We define the functionals  $\Theta^g$  which are the objects of interest of this section. This functional play a central role in the asymptotic analysis of  $F$ , through  $m$  [8], and its optimal shrinkage, through  $\Theta^{(1)}$  and  $\Theta^{(-1)}$  particularly [10].

**Definition 1** ( $\Theta^g$ ). *For  $g : [h_1, h_2] \rightarrow \mathbb{R}$  a bounded function with a finite number of discontinuities, we define:*

$$\Theta^g(z) = -\frac{1}{z} \int \frac{g(\tau)}{\tau X(z) + 1} dH(\tau),$$

where for all  $z \in \mathbb{C}_+$ ,  $X(z)$  is the unique solution in  $\mathbb{C}_+$  of the following equation:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta).$$

For  $k \in \mathbb{Z}$ , we denote  $\Theta^{(k)} = \Theta^g$  for  $g : t \mapsto t^k$  and  $m := \Theta^{(0)}$ .

The next result controls the behavior of  $X$ ,  $m$  and  $\Theta^g$  near the real line. This theorem and its proof contain most of the theoretical ideas that lead to the rest of the article.

**Theorem 2** (Continuity on  $\mathbb{R}^*$ ). *Assume the conditions H1-H5. Then, for  $\lambda \in \mathbb{R}^*$ , we have that  $\check{X}(\lambda) := \lim_{\eta \rightarrow 0^+} X(\lambda + i\eta)$ ,  $\check{m}(\lambda) := \lim_{\eta \rightarrow 0^+} m(\lambda + i\eta)$  and  $\check{\Theta}^{(1)}(\lambda) := \lim_{\eta \rightarrow 0^+} \Theta^{(1)}(\lambda + i\eta)$  exist.*

Moreover, for  $g : [h_1, h_2] \rightarrow \mathbb{R}$  bounded with a finite number of discontinuity points,  $\text{Im}[\check{\Theta}^g(\lambda)] := \lim_{\eta \rightarrow 0^+} \text{Im}[\Theta^g(\lambda + i\eta)]$  exists.

One key implication of this theorem is the existence of a density for  $F$  on  $\mathbb{R}^*$ . More generally, most of the interesting information we use from  $\Theta^g$  or  $m$  comes from the limit of the imaginary part on the real line in the unweighted case: density, support retrieval [2], shrinkage [10] for example. So, the existence of such a limit on the real line is an essential result to extend the results of unweighted sample covariance to the weighted case.

**Corollary 1** (Density of  $F$ ). *Assume the conditions H1-H5. Then,  $F$  has a density on  $\mathbb{R}^*$ . If  $c < 1$ ,  $F$  has a density on  $\mathbb{R}$ . Its density is  $F' = \frac{1}{\pi} \text{Im}[\check{m}(\cdot)]$ .*

Moreover, for  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$  and  $g : [h_1, h_2] \rightarrow \mathbb{R}$  bounded with a finite number of discontinuity points:

$$\begin{aligned}\check{m}(\lambda) &= -\frac{1}{\lambda} \int \frac{1}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \check{\Theta}^{(1)}(\lambda) &= -\frac{1}{\lambda} \int \frac{\tau}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \text{Im}[\check{\Theta}^g(\lambda)] &= \frac{1}{\lambda} \int \frac{g(\tau) \tau \text{Im}[\check{X}(\lambda)]}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau).\end{aligned}$$

One interesting point that emerges directly from this corollary is that no matter the regularity of  $D$  and  $H$ , typically even if they are mixtures of Diracs, then  $F$  always has a density on  $\mathbb{R}^*$ . It was noticed in [2] for the unweighted scenario, and the result is generalized here.

#### 4 Numerical computation of $\check{X}$ , $\check{m}$ and $\text{Im}[\check{\Theta}^g]$

In this section, we adress the practical problem of numerically computing  $\check{X}$  at a point  $\lambda \in \mathbb{R}^*$  while knowing  $H$ ,  $D$  and  $c$ . We saw in Corollary 1 that computing  $\check{X}(\lambda)$  is enough to compute  $\check{m}(\lambda)$  or  $\text{Im}[\check{\Theta}^g](\lambda)$ .

Those quantities are in practice what we are interested in: for example  $\frac{1}{\pi} \check{m}(\lambda)$  is the density of  $F$  in  $\lambda$ ,  $\text{Im}[\check{\Theta}^{(1)}(\lambda)]$  and  $\text{Im}[\check{\Theta}^{(-1)}(\lambda)]$  are closely linked to non-linear shrinkage [10].

We define, for  $z \in \mathbb{C}$ :

$$f_z : X \mapsto X + \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X + 1} dH(\tau)} dD(\delta)$$

We assume that  $f_z$  is easy to evaluate at any  $X \in \mathbb{C}$  in its domain of definition. This is the practical case where  $H$  and  $D$  are finite mixtures of diracs for example, or when they can be efficiently sampled for Monte-Carlo evaluation.

In this situation,  $X(z)$  for  $z \in \mathbb{C}_+$  is the unique solution to  $f_z(X) = 0$  for  $X \in \mathbb{C} \setminus \mathbb{C}_-$  and can be solved as a classical minimization problem. In our experiments, a first-order minimization algorithm worked efficiently for this task.

However, computing the limit  $\check{X}(\lambda) = \lim_{z \in \mathbb{C}_+ \rightarrow \lambda} X(z)$  for  $\lambda \in \mathbb{R}^*$  is *a priori* a more difficult task. Fortunately, we prove that  $\check{X}(\lambda)$  is a solution to the equation  $f_\lambda(X) = 0$ , which is a first important step.

But, we cannot apply the same method as with  $z \in \mathbb{C}_+$  and directly solve  $f_\lambda(X) = 0$  for  $X \in \mathbb{C} \setminus \mathbb{C}_-$  because there can be many solutions to this equation. Indeed, considering  $D = 1_{[1, +\infty]}$  and  $H$  a mixture of  $N \in \mathbb{N}^*$  diracs,  $f_\lambda(X) = 0$  as up to  $N$  different solutions in  $\mathbb{C} \setminus \mathbb{C}_-$ .

We can split the problem into two different scenarios:

- if  $\check{X}(\lambda) \in \mathbb{C}_+$ , then  $\check{X}(\lambda)$  is the **unique** solution in  $\mathbb{C}_+$  (and not  $\mathbb{C} \setminus \mathbb{C}_-$  this time) of the equation  $f_\lambda(X) = 0$ ,
- otherwise, i.e.  $\check{X}(\lambda) \in \mathbb{R}$ , the equation (on  $X$ )  $f_\lambda(X) = 0$  has only **real** solutions and  $\check{X}(\lambda)$  is one of them.

This is formally stated in the following theorem.

**Theorem 3** ( $\check{X}$  computing). *Assume the conditions H1-H5. Then, for  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$ , we denote  $f_\lambda : X \in \mathbb{C}_+ \cup R \mapsto X + \int \frac{\delta}{\lambda - \delta c} \frac{dH(\tau)}{\tau X + 1} dD(\delta)$ , where  $R$  is its domain of definition included in  $\mathbb{R}$ . Then, we have that  $f_\lambda(\check{X}(\lambda)) = 0$ . Moreover,*

$$\begin{aligned} \check{X}(\lambda) \in \mathbb{C}_+ &\iff \\ f_\lambda(X) = 0 &\text{ has at least a solution in } X \in \mathbb{C}_+. \end{aligned}$$

Based on this result, we suggest the following procedure to compute  $\check{X}(\lambda)$ .

**Proposition 1** (Procedure for computing  $\check{X}(\lambda)$ ,  $\lambda \in \mathbb{R}^*$ ). *Let  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$ . To compute  $\check{X}(\lambda)$ :*

- try to solve  $f_\lambda(X) = 0$  on  $\mathbb{C}_+$  (or  $\mathbb{C}_{+\epsilon} := \{z \in \mathbb{C}, \text{Im}[z] \geq \epsilon\}$  for some  $\epsilon > 0$ ), if it succeeds with a solution  $X_0 \in \mathbb{C}_+$  (or in  $\mathbb{C}_{+\epsilon}$  for numerical reasons) then  $\check{X}(\lambda) = X_0$ ,
- otherwise,  $\check{X}(\lambda) \in \mathbb{R}$ , solve  $f_{\lambda+i\eta}(X)$  in  $\mathbb{C}_+$ , for some  $\eta > 0$  small, there is a unique solution  $X(\lambda + i\eta)$  and use  $\check{X}(\lambda) = \text{Re}[X(\lambda + i\eta)]$ .

Remark that as  $F'$  is continuous - due to Theorem 2.2 [9] -,  $\{x \in S_F, F'(x) = 0\}$  is of measure zero.

The other question we need to adress in order to compute for example the density of  $F$ , is: what is the support of  $F$ , denoted  $S_F$  ?

Without it, it can be hard to find a grid on  $\mathbb{R}^*$  to cover the "interesting" parts of the distribution. One simple idea is to draw a sample covariance and compute its spectrum. Only made of Diracs, this will not disclose the spectral gaps and the grid can easily miss them if not chosen appropriately. Moreover, based on a remark of Ledoit and Wolf [3], the density as a square-root behaviour near the border of  $S_F$ , and for numerical approximation we benefit from increasing the density of the grid near the border.

The following section propose a simple way to find  $S_F$  without drawing a sample covariance, generalizing an idea from Silverstein and Choi [2] for unweighted sample covariance.

## 5 Identification of $S_F$

The purpose of this section is to find the support of  $F$ , denoted  $S_F$ . The idea is to use one or several well-chosen real functions, easy to compute, and deduce the border of  $S_F$  from the zero's of their derivatives. This method do not rely on sampling any weighted sample matrix and can detect even very small spectral gaps.

Let us start where the support of the weight distribution, denoted  $S_D$ , is convex.

### 5.1 Identification of the support of $F$ in function of $H$ when $S_D$ is convex

This case is very similar to the unweighted scenario, studied in [2]. Indeed, the number of spectral gaps in  $S_F$  is bounded by the number of gaps in  $S_H$ , and one function is enough to detect all of them. The function we are interested in to determine  $S_F$  is  $x_F$ , defined below.

**Definition 2.** *Suppose  $S_D$  is convex, i.e.  $S_D$  is of the form  $[d_1, d_2]$ . We define:*

$$m_{LD} : x \in S_D^c \mapsto \int \frac{\delta}{\delta - x} dD(\delta) \in \mathbb{R}^*.$$

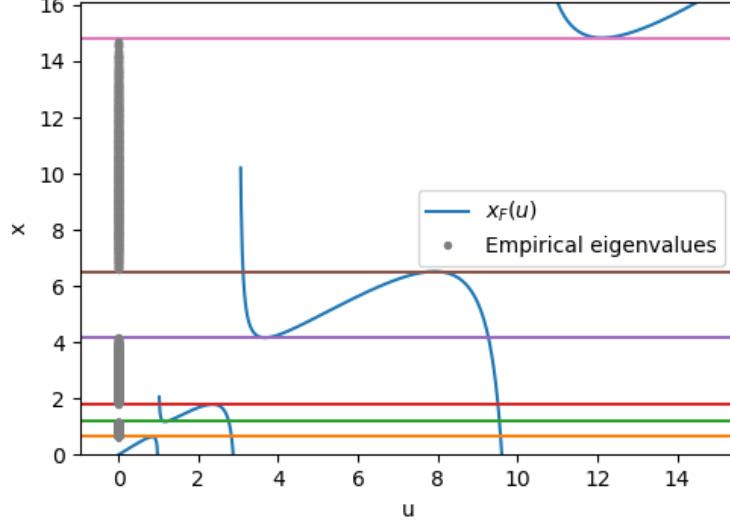


Figure 2:  $x_F(-1/u)$  for  $u \in S_H^c$ , using  $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ ,  $D$  uniform between  $1/2$  and  $3/2$ ,  $c = 0.1$ , and empirical eigenvalues sampled with  $n = 1000$ , Gaussian noise.

Notice that  $m_{LD}$  is invertible. Moreover, we define, with  $B = \{y \in \mathbb{R}, y \neq 0, -\frac{1}{y} \in S_H^c\}$ :

$$x_F : X \in B \mapsto \begin{cases} \frac{h(X)}{X} m_{LD}^{-1}(h(X)) & , \text{ if } h(X) \neq 0, \\ -\frac{1}{X} \int \delta dD(\delta) & , \text{ otherwise.} \end{cases}$$

$$\text{with } h : X \in B \mapsto cX \int \frac{\tau}{\tau X + 1} dH(\tau).$$

**Proposition 2.** Suppose  $S_D$  is convex. Then  $x_F \in \mathcal{C}^1(B, \mathbb{R})$ .

The practical result in the following theorem links  $S_F$  with the derivative of  $x_F$ .

**Theorem 4.** Assume H1-H5. Suppose  $S_D$  is convex. Then,  $x \in S_F^c \iff X \in B$  and  $x'_F(X) > 0$ , where  $X$  and  $x$  are linked respectively by  $X = \tilde{X}(x)$  and  $x = x_F(X)$ .

The procedure to use this theorem is simple:

- find the open intervals  $]a_i, b_i[_i$  of  $B$ ,  $a_i < b_i$ , where  $\forall i, \forall X \in ]a_i, b_i[, x'_F(X) > 0$ ,
- then,  $S_F^c = \bigcup_i ]x_F(a_i), x_F(b_i)[$ .

Numerically, we only need to find the zeros of  $x'_F$  and compute the value of  $x_F$  at those points.

An example is given in Figure 2, and illustrate the use of the theorem, and the precision of the prediction. Additional figures and experiments for diverse weight distributions are detailed in the Appendix and implementation is given in the supplementary material.

## 5.2 Identification of $S_F$ in function of $H$ when $S_D$ is a finite union of intervals

In the more general case where  $S_D$  is a finite union of intervals, the method can be extended. This case is useful in the case of  $D$  being a finite mixture of Diracs for example. This method requires as many functions  $x_F^{(k)}$  as there are disjoint intervals in  $S_D$ .

**Definition 3.** Suppose  $S_D$  finite union of  $M \in \mathbb{N}^*$  intervals, i.e. there exists  $\delta_1^{(1)} \leq \delta_2^{(1)} < \dots < \delta_1^{(M)} \leq \delta_2^{(M)}$  such that  $S_D = \bigcup_{k=1}^M ]\delta_1^{(k)}, \delta_2^{(k)}]$ . We define for  $k \in \llbracket 1, M-1 \rrbracket$ :

$$m_{LD}^{(k)} : x \in ]\delta_2^{(k)}, \delta_1^{(k+1)}[ \mapsto \int \frac{\delta}{\delta - x} dD(\delta),$$

and,

$$m_{LD}^{(M)} : x \in \mathbb{R} \setminus [\delta_1^{(1)}, \delta_2^{(M)}] \mapsto \int \frac{\delta}{\delta - x} dD(\delta).$$

For all  $k \in \llbracket 1, M \rrbracket$ ,  $m_{LD}^{(k)}$  is invertible. Moreover, we define for  $k \in \llbracket 1, M - 1 \rrbracket$ , with  $B = \{y \in \mathbb{R}, y \neq 0, -\frac{1}{y} \in S_H^c\}$ :

$$x_F^{(k)} : X \in B \mapsto \frac{h(X)}{X} \left(m_{LD}^{(k)}\right)^{-1} (h(X)),$$

and we define  $x_F^{(M)} : X \rightarrow \mathbb{R}$  as, for  $X \in B$ :

$$x_F^{(M)}(X) = \begin{cases} \frac{h(X)}{X} \left(m_{LD}^{(M)}\right)^{-1} (h(X)) & , \text{ if } h(X) \neq 0, \\ -\frac{1}{X} \int \delta dD(\delta) & , \text{ otherwise.} \end{cases}$$

As Proposition 2, we have the following result.

**Proposition 3.** Suppose  $S_D$  finite union of  $M \in \mathbb{N}^*$  intervals, then  $\forall k \in \llbracket 1, M \rrbracket, x_F^{(k)} \in \mathcal{C}^1(B, \mathbb{R})$ .

In this scenario, the theorem linking  $S_F$  to the  $(x_F^{(k)})_k$  is similar to Theorem 4: if any of the  $x_F^{(k)}$  is increasing on an interval  $I \subset \mathbb{R}$ , then  $x_F^{(k)}(I) \subset S_F^c$ . This is formally stated in the following theorem.

**Theorem 5.** Suppose H1-H5 and  $S_D$  finite union of  $M \in \mathbb{N}^*$  intervals. Let  $x \in \mathbb{R}^*$ . Then,  $x \in S_F^c \iff \exists k \in \llbracket 1, M \rrbracket, \exists X \in B, \left(x_F^{(k)}\right)'(X) > 0$  and  $\left(x_F^{(k)}\right)(X) = x$ .

This result highlights a phenomom of higher spectral separation than the previous section where  $S_D$  was made of only one interval. Let us look at the same example of distribution  $H$  we studied previously:  $H$  being a mixture of 3 diracs in 1, 3, and 10 with respectively weights 0.2, 0.4 and 0.4. With  $S_D$  made of one interval,  $S_F$  is a union of at most 3 distinct intervals. Now, the situation is different, each separation in the support of the **weights** can lead to a spectral separation in the empirical spectrum.

We show this specific behaviour with weights following a mixture of two diracs in Figure 10 where **3** spectral gaps are visible in  $S_F$  while  $S_H$  has only **2** gaps. The case with  $N$  diracs is discussed in the Appendix, along with a way to compute it efficiently.

In practice, it can be computationally demanding to compute each of the  $x_F^{(k)'}$  and find its zeros if  $M$  is large. A general heuristic we observed experimentally is that studying  $x_F^{(M)'}$  is enough to find out most of the gaps if  $S_D$  has no "large" gaps.

## 6 WeSpeR: retrieving $H$ and computing $F'$

In this section, we propose an application of the previous theoretical results for retrieving the true spectrum distribution  $H$  as a mixture of diracs from observed weighted sample eigenvalues  $(\lambda_i)_{i=1}^n$ , and using this distribution  $H$  to compute the empirical spectrum density  $F'$ .

The following algorithm, denoted *WeSpeR* for *Weighted sample covariance Spectrum Retrieval* algorithm, generalizes to weighted sample covariance the idea of the QuEST algorithm [3], with the help of auto-differentiation to greatly simplify its implementation.

- 1- As input, we take the observed sample spectrum distribution  $F_n = \frac{1}{n} \sum_{i=1}^n 1_{[\lambda_i, \infty[}$  and the weight matrix  $W$ .
- 2- Find the estimated true spectrum  $\hat{H}(\tau) = \frac{1}{n} \sum_{i=1}^n 1_{[\tau_i, +\infty[}$  where  $\tau$  solves:

$$\min_{\tau \in \mathbb{R}^n} \mathbb{E}_Z \left[ \left\| \tilde{F}_n(Z) - F_n \right\|_{\mathcal{W}, 2}^2 \right] \quad (2)$$

where  $\|\cdot\|_{\mathcal{W}, 2}$  is the 2-Wasserstein norm and  $\tilde{F}_n(Z) := F^{\frac{1}{N}} \sqrt{T} Z W Z^* \sqrt{T}$  with:

- $T = \text{Diag}((\tau_i)_{i=1}^n)$ ,
- $Z$  of size  $(n, N)$  with iid  $Z_{ij} \sim \mathcal{N}(0, 1)$ .



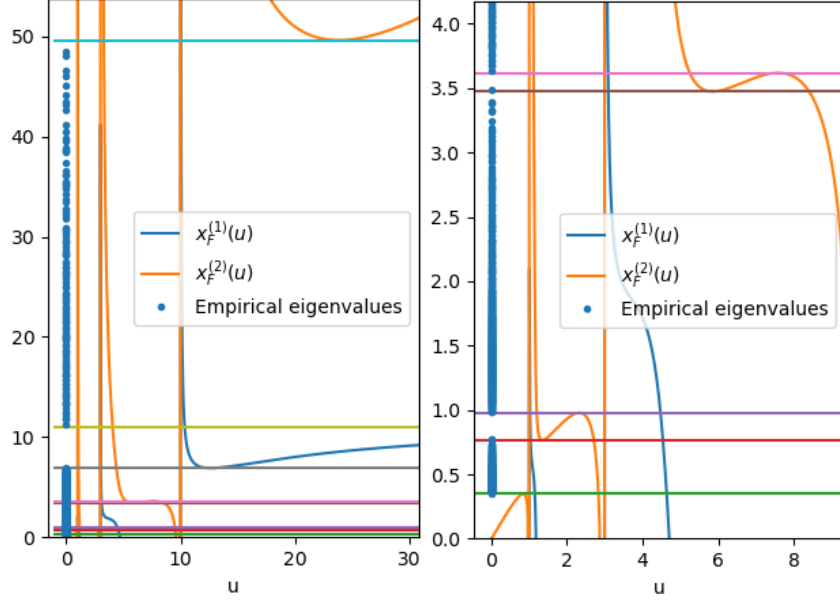


Figure 3:  $u \mapsto x_F^{(k)}(-1/u)$  for  $u \in S_H^c$ , using  $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ ,  $D = (1 - \frac{1}{80})\mathbf{1}_{[\frac{1}{2},\infty[} + \frac{1}{80}\mathbf{1}_{[\frac{79}{160},\infty[}$ ,  $c = 0.1$ , and empirical eigenvalues sampled with  $n = 1000$ . (Left) Whole spectrum. (Right) Zoom for small eigenvalues.

We use automatic differentiation to solve it.

- 3- Find the support  $S_F$  of  $F$  using Theorems 4 or 5 for  $\hat{H}$ ,  $c$  and the considered weight distribution in input. For all, or a subset of,  $k \in \llbracket 1, M \rrbracket$ , find the zeros of  $(x_F^{(k)})'$ .
- 4- Define a grid  $(\xi_i)_{i=1}^n$  on  $S_F$  with higher density near  $\partial S_F$ , as defined in [3], compute  $\tilde{m}$  on the grid with Proposition 1 and deduce  $F' = \frac{1}{\pi} \text{Im}[\tilde{m}]$ .

The algorithm is fully detailed in the Appendix, notably points (3) and (4), and the implementation given in the supplementary material.

**Remark 1** (Noise sampling). *We note that the expectation in (2) can be computed under any centered and standardized distribution, irrespectively of the noise of the observed phenomenon due to the universality of Theorem 2 [8].*

An experimental result is shown in Figure 4 to illustrate how *WeSpeR* works. We used  $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ ,  $D$  exponentially weighted with  $\alpha = 1$ ,  $c = 0.1$ , and  $Z_{ij} \sim \mathcal{N}(0, 1)$ . More experiments are available in the Appendix.

## 7 Conclusion

This work aims at studying the inherent properties of the asymptotic spectrum  $F$  of weighted sample covariance matrices. We prove that  $F$  has a density on  $\mathbb{R}^*$  and find numerical procedure to compute it.

We propose a method to retrieve the support  $S_F$  of  $F$  through a study of simple real functions, enlighting a new phenomenon of spectral gaps that does not exist with the standard sample covariance.

We use these theoretical results and procedure to design *WeSpeR*: an algorithm addressing the problem of numerically estimating the true covariance spectrum  $H$  and the asymptotic sample density  $F'$  of a weighted sample covariance, two essential but non-observable objects. Empirical tests confirm the performance of the *WeSpeR* algorithm.

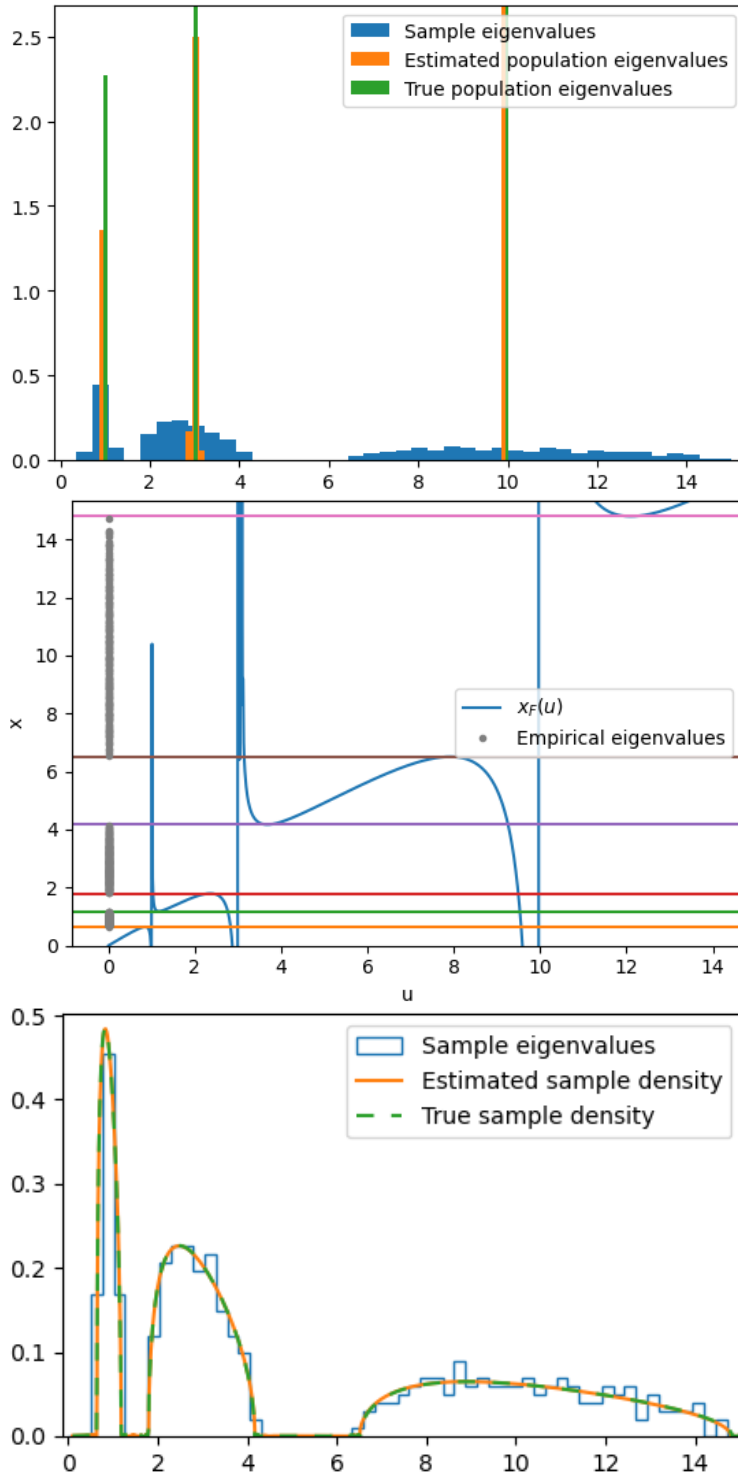


Figure 4: (Top) Histograms of sample eigenvalues, estimated population eigenvalues  $\hat{H}$  with *WeSpeR*, and true population eigenvalues  $H$ . (Middle)  $u \mapsto x_F(-1/u)$  using the estimated  $\hat{H}$  to detect the estimated support  $S_F$ . (Bottom) Estimated and true sample density computed on  $S_F$  and sample eigenvalues histogram.

## Acknowledgment

We would like to thank Alexandre Miot and Gabriel Turinici for their insights and advices all along the work.

## References

- [1] V A Marčenko and L A Pastur. Distribution of eigenvalues for some sets of random matrices. *Mathematics of the USSR-Sbornik*, 1(4):457, apr 1967.
- [2] Jack W. Silverstein and Sang Il Choi. Analysis of the limiting spectral distribution of large dimensional random matrices. *Journal of Multivariate Analysis*, 54:295–309, 1995.
- [3] Olivier Ledoit and Michael Wolf. Spectrum estimation: A unified framework for covariance matrix estimation and pca in large dimensions. *Journal of Multivariate Analysis*, 139:360–384, 2015.
- [4] Szilard Pafka, Marc Potters, and Imre Kondor. Exponential weighting and random-matrix-theory-based filtering of financial covariance matrices for portfolio optimization, 2004.
- [5] J Daly, M Crane, and H J Ruskin. Random matrix theory filters and currency portfolio optimisation. *Journal of Physics: Conference Series*, 221(1):012003, apr 2010.
- [6] Jens Svensson. The asymptotic spectrum of the ewma covariance estimator. *Physica A: Statistical Mechanics and its Applications*, 385(2):621–630, 2007.
- [7] Vincent Tan and Stefan Zohren. Estimation of large financial covariances: A cross-validation approach, 2023.
- [8] Benoit Oriol. Asymptotic spectrum of weighted sample covariance: a Marcenko-Pastur generalization, 2024.
- [9] J. W. Silverstein. Strong convergence of the empirical distribution of eigenvalues of large dimensional random matrices. *Journal of Multivariate Analysis*, 55(2):331–339, 1995.
- [10] Olivier Ledoit and Sandrine Péché. Eigenvectors of some large sample covariance matrix ensembles, 2009.
- [11] Olivier Ledoit and Michael Wolf. Numerical implementation of the quest function, 2016.
- [12] John Hagood and Brian Thomson. Recovering a function from a dini derivative. *The American Mathematical Monthly*, 113, 01 2006.
- [13] Z. D. Bai and Jack W. Silverstein. CLT for linear spectral statistics of large-dimensional sample covariance matrices. *The Annals of Probability*, 32(1A):553 – 605, 2004.
- [14] Manfred Möller and Vyacheslav Pivovarchik. *Spectral Theory of Operator Pencils, Hermite-Biehler Functions, and their Applications*. Birkhäuser Cham, 2015.

## 8 Appendix A - Additional experiments, implementation details

### 8.1 Support identification for $S_D$ convex

#### 8.1.1 Examples of weight distribution: EWMA distribution

In time series analysis, the Exponentially-Weighted Moving Average - EWMA - is a widely used weighting scheme from neuroscience to finance as detailed in the introduction (see [4, 5, 6, 7]). The asymptotic distribution p.d.f.  $D_\alpha$  of the weights in a EWMA setting is defined in the following definition. The decay of the EWMA is controlled through the parameter  $\alpha \in \mathbb{R}_+$ : the larger  $\alpha$ , the steeper the decay. Some examples of densities for different values of  $\alpha$  are shown in figure 5.

**Definition 4** (EWMA distribution). *For  $\alpha \in \mathbb{R}_+$ , we consider the following weight p.d.f.:*

$$D_\alpha : x \in [\beta e^{-\alpha}, \beta] \mapsto 1 + \frac{1}{\alpha} \log\left(\frac{x}{\beta}\right) \text{ with } \beta = \frac{\alpha}{1 - e^{-\alpha}}. \quad (3)$$

Then, we have the following closed-form formulas for  $m_{LD}$  and  $m_{LD}^{-1}$ :

$$\begin{aligned} m_{LD_\alpha} : x \in \mathbb{R} \setminus [\beta e^{-\alpha}, \beta] &\mapsto \frac{1}{\alpha} \log\left(1 + \frac{\alpha}{\beta e^{-\alpha} - x}\right), \\ m_{LD_\alpha}^{-1} : y \in \mathbb{R}^* &\mapsto \beta e^{-\alpha} + \frac{\alpha}{1 - e^{\alpha y}}. \end{aligned} \quad (4)$$

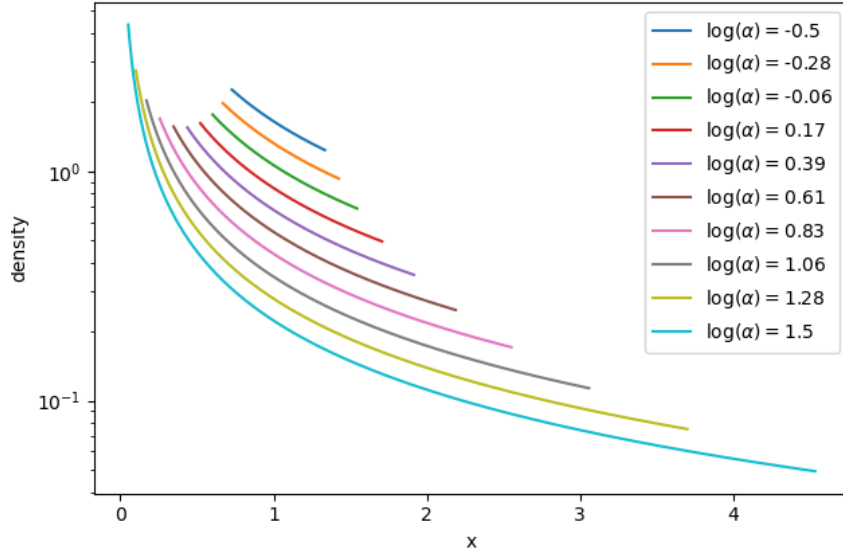


Figure 5: EWMA density  $D'_\alpha$  for different  $\alpha$ .

Two examples of identification of the support are shown in figure 6 when there is spectral separation, and in figure 7 when there is not.

#### 8.1.2 Examples of weight distribution: uniform distribution

**Proposition 4** (Uniform distribution). *For  $\alpha \in [0, 2[$ , we consider the following weight distribution:*

$$D_\alpha : x \in \mathbb{R} \mapsto \frac{x - 1 + \alpha/2}{\alpha} 1_{[1-\alpha/2, 1+\alpha/2]}(x) + 1_{[1+\alpha/2, +\infty[}(x). \quad (5)$$

Then, we have the following closed-form formulas for  $m_{LD}$ :

$$m_{LD_\alpha} : x \in \mathbb{R} \setminus [1 - \alpha/2, 1 + \alpha/2] \mapsto 1 + \frac{x}{\alpha} \log\left(1 + \frac{\alpha}{1 - \frac{\alpha}{2} - x}\right). \quad (6)$$

$m_{LD}^{-1}$  has no closed-form formulas and it can be retrieved through numerical optimization.

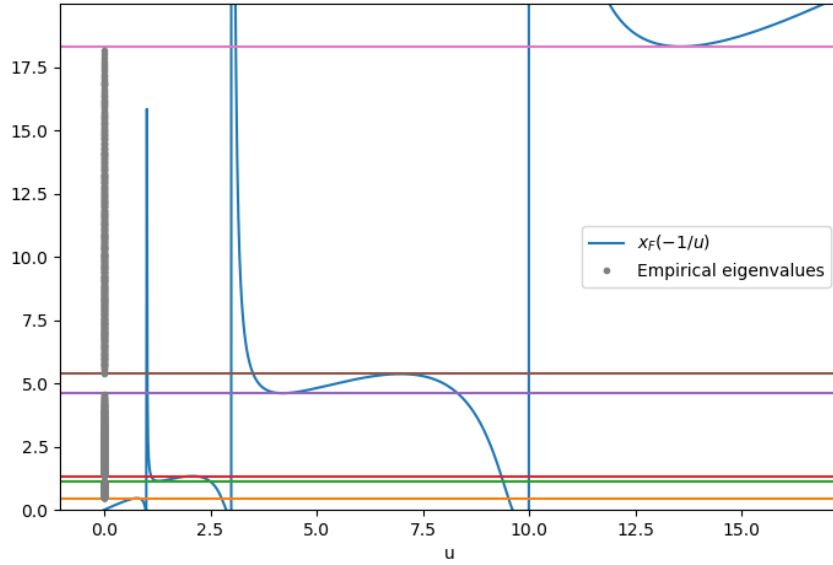


Figure 6:  $x_F(-1/u)$  for  $u \in S_H^c$ , using  $H$  being a mixture of 3 diracs in 1, 3, and 10 with respectively weights 0.2, 0.4 and 0.4 as in [2],  $D_\alpha$  EWMA distribution with  $\alpha = 5$ ,  $c = 0.1$ , and empirical eigenvalues sampled with  $p = 1000$ . Horizontal lines are plotted at the zeros of  $x'_F$ , they represent the theoretical borders of  $S_F$ .

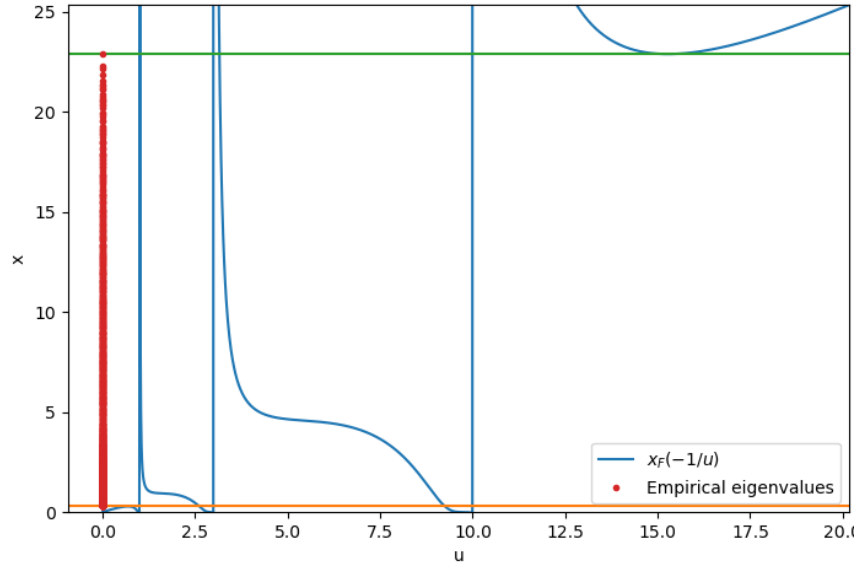


Figure 7:  $x_F(-1/u)$  for  $u \in S_H^c$ , using  $H$  being a mixture of 3 diracs in 1, 3, and 10 with respectively weights 0.2, 0.4 and 0.4 as in [2],  $D_\alpha$  EWMA distribution with  $\alpha = 10$ ,  $c = 0.1$ , and empirical eigenvalues sampled with  $p = 1000$ . Horizontal lines are plotted at the zeros of  $x'_F$ , they represent the theoretical borders of  $S_F$ .

Two examples of identification of the support are shown in figure 8 when there is spectral separation, and in figure 9 when there is not.

### 8.1.3 Numerical implementation of spectrum support identification

In all the implementation, we assume we can compute  $m_{LD}^{-1}$ ,  $t : u \in S_H^c \mapsto c \int \frac{\tau}{\tau-u} dH(\tau)$ , and the derivatives  $m'_{LD}$ ,  $m''_{LD}$ ,  $t'$  and  $t''$  at any point. Moreover, we assume we have access to a root-finding algorithm taking in input a real function  $f$  and two real values  $(x_0, x_1)$  such that  $f(x_0)$  and  $f(x_1)$  have different signs. The idea is inspired from the QuEST algorithm [3].

We implement the algorithm in the scenario where  $S_H$  can be written as a finite disjoint union of intervals, *i.e.* for some  $K \in \mathbb{N}^*$ ,  $S_H = \bigcup_{i=1}^K [\tau_i^l, \tau_i^r]$  where  $\tau_1^l \leq \tau_1^r < \dots < \tau_K^l \leq \tau_K^r$ .

For convenience, we use the increasing change of variable  $u \mapsto -1/X$ , and consider the function  $y_F : u \in S_H^c \mapsto x_F(-1/u)$ . We have, for  $u \in S_H^c$  with  $t : u \in S_H^c \mapsto c \int \frac{\tau}{\tau-u} dH(\tau)$ :

$$\begin{aligned} y_F(u) &= -ut(u)m_{LD}^{-1}(t(u)), \\ y'_F(u) &= -t(u)m_{LD}^{-1}(t(u)) - ut'(u) \left( \frac{t(u)}{m'_{LD}(m_{LD}^{-1}(t(u)))} + m_{LD}^{-1}(t(u)) \right), \\ y''_F(u) &= -(2t'(u) + ut''(u))m_{LD}^{-1}(t(u)) - \frac{2(t(u) + ut'(u))t'(u) + ut(u)t''(u)}{m'_{LD}(m_{LD}^{-1}(t(u)))} - \frac{ut(u)t'(u)^2 m''_{LD}(m_{LD}^{-1}(t(u)))}{m'_{LD}(m_{LD}^{-1}(t(u)))^3}. \end{aligned} \quad (7)$$

We are going to construct iteratively  $S_F^c$ . At step 0, we consider  $(S_F^c)_0 = \emptyset$ . There are three different situations.

- Firstly, on the interval  $] -\infty, \tau_1^l[$ , we are looking for the unique zero of  $y'_F$ . As  $y'_F(u) \xrightarrow{u \rightarrow -\infty} \int \delta dD(\delta) > 0$  and  $y'_F(u) \xrightarrow{u \rightarrow \tau_1^l-} -\infty$ , we can find easily with line search two points  $(u_1, u_2) \in ] -\infty, \tau_1^l[$  such that  $y'_F(u_1) > 0$  and  $y'_F(u_2) < 0$ . We can use the root-finding algorithm of  $y'_F$  between  $u_1$  and  $u_2$ , giving us the solution  $u_i^*$ . In conclusion of this part, we update  $(S_F^c)_1 = (S_F^c)_0 \cup ] -\infty, y_F(u_i^*)[$ .
- Similarly, on the interval  $] \tau_K^r, +\infty[$ , we are looking for the unique zero of  $y'_F$ . As  $y'_F(u) \xrightarrow{u \rightarrow +\infty} \int \delta dD(\delta) > 0$  and  $y'_F(u) \xrightarrow{u \rightarrow \tau_K^r+} -\infty$ , we can apply the previous procedure in  $] \tau_K^r, +\infty[$  and find  $u_r^*$ , root of  $y'_F$  in  $] \tau_K^r, +\infty[$ . In conclusion of this part, we update  $(S_F^c)_2 = (S_F^c)_1 \cup ] y_F(u_r^*), +\infty[$ .
- For each  $i \in \llbracket 1, K-1 \rrbracket$ , we consider the interval  $] \tau_i^r, \tau_{i+1}^l[$ . This time, we have  $y''_F(u) \xrightarrow{u \rightarrow \tau_i^r+} +\infty$  and  $y''_F(u) \xrightarrow{u \rightarrow \tau_{i+1}^l-} -\infty$ . Still through line-search, we can use the root-finding algorithm on  $y''_F$ . We expect  $y''_F$  to have only one zero, denoted by  $u_0 \in ] \tau_i^r, \tau_{i+1}^l[$  on this interval.
  - If  $y'_F(u_0) \leq 0$ , there is no spectral gap to be found on this interval.
  - Otherwise, we are looking for two zeros of  $y'_F$ : one on  $] \tau_i^r, u_0[$  and one on  $] u_0, \tau_{i+1}^l[$ . As  $y'_F(u) \xrightarrow{u \rightarrow \tau_i^r+} -\infty$  and  $y'_F(u) \xrightarrow{u \rightarrow \tau_{i+1}^l-} -\infty$ , we can apply the line-search and use the root-finding algorithm on each interval, outputting two solutions:  $u_i^* \in ] \tau_i^r, u_0[$  and  $u_r^* \in ] u_0, \tau_{i+1}^l[$ . We update  $(S_F^c)_{i+2} = (S_F^c)_{i+1} \cup ] y_F(u_i^*), y_F(u_r^*)[$ .

In the end, we have  $S_F^c = (S_F^c)_{K+1}$ .

## 8.2 Support identification for $D$ mixture of diracs

We give examples of applications of the Theorem 5, when  $S_D$  is a finite union of intervals.

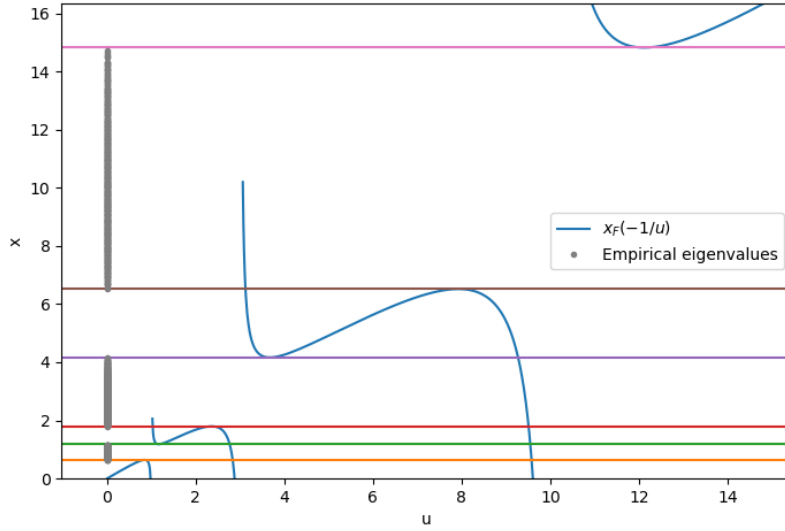


Figure 8:  $x_F(-1/u)$  for  $u \in S_H^c$ , using  $H$  being a mixture of 3 diracs in 1, 3, and 10 with respectively weights 0.2, 0.4 and 0.4 as in [2],  $D_\alpha$  uniform with  $\alpha = 1$ ,  $c = 0.1$ , and empirical eigenvalues sampled with  $p = 1000$ . Horizontal lines are plotted at the zeros of  $x'_F$ , they represent the theoretical borders of  $S_F$ .

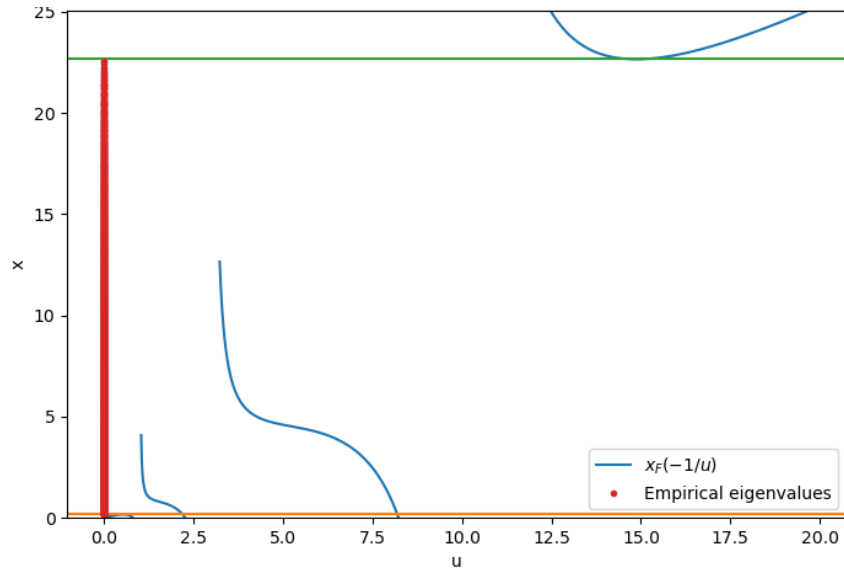


Figure 9:  $x_F(-1/u)$  for  $u \in S_H^c$ , using  $H$  being a mixture of 3 diracs in 1, 3, and 10 with respectively weights 0.2, 0.4 and 0.4 as in [2],  $D_\alpha$  uniform with  $\alpha = 1$ ,  $c = 0.5$ , and empirical eigenvalues sampled with  $p = 5000$ . Horizontal lines are plotted at the zeros of  $x'_F$ , they represent the theoretical borders of  $S_F$ .

### 8.2.1 Examples of weight distribution: mixture of 2 diracs

We detail the simpler case of  $S_D$  as union of several intervals:  $S_D$  as union of two points, *i.e.*  $D$  being a mixture of two diracs p.d.f.

**Proposition 5** (Mixture of two diracs). *For  $\alpha \in ]0, 1[$ ,  $w \in ]0, 1[$ , we consider the following weight distribution, with  $\beta = \frac{\alpha w}{1-w}$ :*

$$D_{\alpha,w} = w1_{[1-\alpha,\infty[} + (1-w)1_{[1+\beta,\infty[}. \quad (8)$$

Then, we have the following closed-form formulas for  $m_{LD}$  and  $m_{LD}^{-1}$ :

$$\begin{aligned} m_{LD_{\alpha,w}}^{(1)} : x \in ]1-\alpha, 1+\alpha[ &\mapsto \frac{w(1-\alpha)}{1-\alpha-x} + \frac{(1-w)(1+\beta)}{1+\beta-x}, \\ m_{LD_{\alpha,w}}^{(2)} : x \in \mathbb{R} \setminus [1-\alpha, 1+\alpha] &\mapsto \frac{w(1-\alpha)}{1-\alpha-x} + \frac{(1-w)(1+\beta)}{1+\beta-x}, \\ \left(m_{LD_{\alpha,w}}^{(1)}\right)^{-1} : y \in \mathbb{R}^* &\mapsto \frac{-b(y)^2 + \sqrt{b(y)^2 - 4a(y)c(y)}}{2a(y)}, \\ \left(m_{LD_{\alpha,w}}^{(2)}\right)^{-1} : y \in \mathbb{R}^* &\mapsto \frac{-b(y)^2 - \sqrt{b(y)^2 - 4a(y)c(y)}}{2a(y)}. \end{aligned} \quad (9)$$

with for  $y \in \mathbb{R}^*$ :

$$\begin{aligned} a(y) &= y(1-w), \\ b(y) &= 1-w - (1-2w(1-\alpha) + 1-\alpha)y, \\ c(y) &= (1-\alpha)(1-w(1-\alpha)) - (1-\alpha)(1-w(a-\alpha)). \end{aligned} \quad (10)$$

An example of identification of the support is shown in figure 10 where a new spectral separation for the empirical spectrum is induced by the weight distribution - itself having large gaps. Thus,  $S_F$  is made of 4 intervals, while  $S_H$  is only made of 3. This is a new behavior due to the weight distribution  $D$  that we do not observe in the classical setting with equal weights, where  $S_F$  could only be made of at most 3 intervals with this type of true spectrum  $H$ .

### 8.2.2 Examples of weight distribution: mixture of $N$ diracs

We detail the computation and the result for the  $D$  mixture of  $N$  diracs. The efficient implementation of this problem is discussed after.

**Proposition 6** (Mixture of  $N$  diracs). *For  $M \in \mathbb{N}^*$ ,  $w \in ]0, 1]^M$ ,  $\delta \in (\mathbb{R}_+^*)^M$  such that  $\sum_{i=1}^M w_i = 1$ ,  $\sum_{i=1}^M w_i \delta_i = 1$ , we consider the following weight distribution:*

$$D_\alpha = \sum_{i=1}^M w_i 1_{[\delta_i, \infty[}. \quad (11)$$

Then, we have the following closed-form formulas for  $m_{LD}$ :

$$\begin{aligned} \forall k \in [1, M-1], m_{LD}^{(k)} : x \in ]\delta_2^{(k)}, \delta_1^{(k+1)}[ &\mapsto \sum_{i=1}^M w_i \frac{\delta_i}{\delta_i - x}, \\ m_{LD}^{(M)} : x \in ]-\infty, \delta_1^{(1)}[ \cup ]\delta_2^{(M)}, +\infty[ &\mapsto \sum_{i=1}^M w_i \frac{\delta_i}{\delta_i - x}. \end{aligned} \quad (12)$$

$\left(m_{LD}^{(k)}\right)^{-1}$  has no general closed-form formulas when  $M \geq 5$  due to Abel–Ruffini theorem and it can be retrieved through numerical optimization.

An example of identification of the support are shown in figure 11 when an important separation of the weight diracs implies a new spectral separation for the empirical spectrum, in order to show the role of each  $\left(m_{LD}^{(k)}\right)^{-1}$  in the determination of the support.



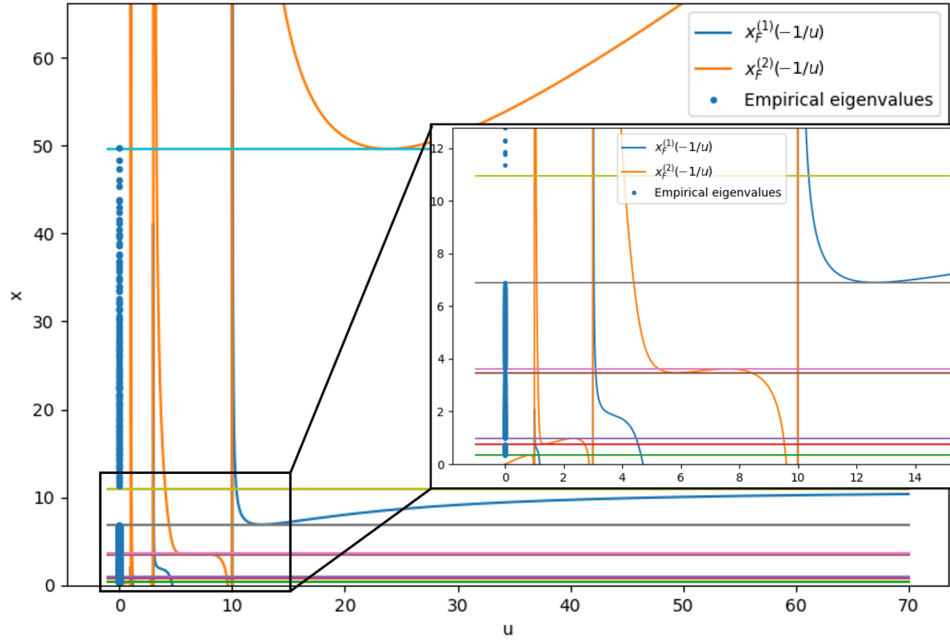


Figure 10:  $x_F^{(k)}(-1/u)$  for  $u \in S_H^c$ , using  $H$  being a mixture of 3 diracs in 1, 3, and 10 with respectively weights 0.2, 0.4 and 0.4 as in [2],  $D_{\alpha,w}$  mixture of 2 diracs with  $\alpha = 0.5$ ,  $w = 1 - \frac{1}{80}$ ,  $c = 0.1$ , and empirical eigenvalues sampled with  $p = 4000$ . Horizontal lines are plotted at the zeros of  $x_F^{(k) \prime}$ , they represent the theoretical borders of  $S_F$ .

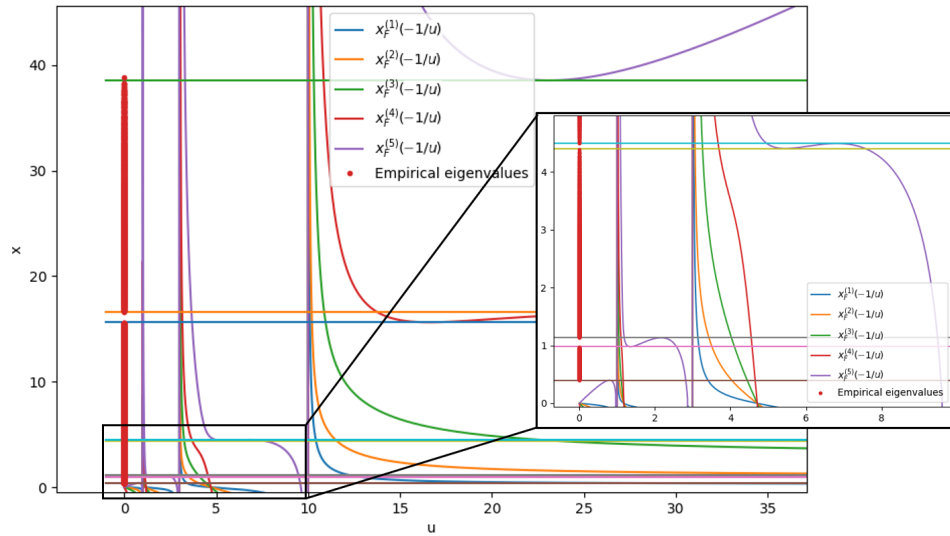


Figure 11:  $x_F^{(k)}(-1/u)$  for  $u \in S_H^c$ , using  $H$  being a mixture of 3 diracs in 1, 3, and 10 with respectively weights 0.2, 0.4 and 0.4 as in [2],  $D$  mixture of 5 diracs with in 0.34, 0.67, 2.7, 6.74, and 34, with respective weights 0.59, 0.30, 0.074, 0.03, and 0.006,  $c = 0.1$ , and empirical eigenvalues sampled with  $p = 5000$ . Horizontal lines are plotted at the zeros of  $x_F^{(k) \prime}$ , they represent the theoretical borders of  $S_F$ .

### 8.2.3 Numerical implementation of spectrum support for $D$ finite mixture of Diracs

We mostly use the same implementation scheme provided for the convex setting in Section 8.1.3, applied to each  $y_F^{(k)} : u \in S_F^c \mapsto x_F^{(k)}(-1/u)$  (extended in 0 by 0).

Using the same notation, the only difference in the implementation is about the handling of  $y_F^{(k)}$ ,  $k \in \llbracket 1, M-1 \rrbracket$  on the outside interval  $]\tau_K^r, +\infty[$ . For numerical reasons, we apply back the change of variable  $X = -1/u$ , as it is easier to study  $x_F^{(k)}$  on  $] -1/\tau_K^r, 0[$  than  $y_F^{(k)}$  on the initial interval. On each of these intervals, we use the the same idea we used on the bounded intervals: find the zero  $X_0$  of  $x_F^{(k)''}$ , and if  $x_F^{(k)'}(X_0) > 0$ , find the zero  $X_l^*$  of  $x_F^{(k)'}$  on  $] -1/\tau_K^r, X_0[$  and the zero  $X_r^*$  of  $x_F^{(k)'}$  on  $]X_0, 0[$ . Then add  $] -1/x_F(X_l^*), -1/x_F(X_r^*)[$  to the current  $(S_F^c)_i$ .

The central numerical problem remaining is to compute the function  $v^{(k)} : u \mapsto m_{LD}^{(k)-1} \left( c \int \frac{\tau}{\tau-u} dH(\tau) \right)$  for the desired  $u \in S_H^c$ .

In this section, the term  $t : u \in S_H^c \mapsto c \int \frac{\tau}{\tau-u} dH(\tau)$  is supposed to be easy to compute. In the case of  $H$  being a finite mixture of Diracs, it is a rational function.

The more complex part resides in computing  $v^{(k)}$ . We suppose that  $D$  is a finite mixture of Diracs, i.e. there exists  $M \in \mathbb{N}^*$ ,  $(w_i)_{i=1}^M \in \mathbb{R}_+^*$ ,  $(\delta_i)_{i=1}^M \in \mathbb{R}_+^*$ , such that  $\sum_{i=1}^M w_i = 1$  and  $D = \sum_{i=1}^M w_i 1_{[\delta_i, +\infty[}$ .

Let  $t \in \mathbb{R}$ . Computing  $v^{(k)}(t)$  for all  $k \in \llbracket 1, M \rrbracket$  is equivalent to finding the  $M$  distinct roots of the rational function  $x \in S_D^c \mapsto m_{LD}(x) - t$ . And this is equivalent to finding the  $M$  distinct roots of the polynomial  $P - tQ$  where:

$$\begin{aligned} P(X) &= \sum_{i=1}^M w_i \delta_i \prod_{j=1, j \neq i}^M (\delta_j - X), \\ Q(X) &= \prod_{i=1}^M (\delta_i - X). \end{aligned} \tag{13}$$

We suggest to use MPSolve (useable in Python, Matlab, C, Octave...) to find efficiently and simultaneously the  $M$  roots of the resulting polynomial routinely for large  $M$ . Otherwise, for moderately large  $M$ , using the eigenvalues of the companion matrix of  $P - tQ$  is possible and makes the implementation slightly easier.

Once we computed through this method all the functions  $u \mapsto m_{LD}^{(k)-1} \left( c \int \frac{\tau}{\tau-u} dH(\tau) \right)$  on the desired grid of  $u \in S_H^c$ , we can easily deduce the  $y_F^{(k)}$  and the its derivative  $y_F^{(k)'}$ ,  $y_F^{(k)''}$  with the following formulas:

$$\begin{aligned} y_F^{(k)}(u) &= -ut(u)v^{(k)}(t(u)), \\ y_F^{(k)'}(u) &= -t(u)v^{(k)}(t(u)) - ut'(u) \left( \frac{t(u)}{m'_{LD}(v^{(k)}(t(u)))} + v^{(k)}(t(u)) \right), \\ y_F^{(k)''}(u) &= -(2t'(u) + ut''(u))v^{(k)}(t(u)) - \frac{2(t(u) + ut'(u))t'(u) + ut(u)t''(u)}{m'_{LD}(v^{(k)}(t(u)))} - \frac{ut(u)t'(u)^2 m''_{LD}(v^{(k)}(t(u)))}{m'_{LD}(v^{(k)}(t(u)))^3}. \end{aligned} \tag{14}$$

### 8.3 WeSpeR: implementation details and additional experiments

In this section we discuss the details, the implementation and experiments of the proposed algorithm to retrieve  $H$  and  $F'$ .

### 8.3.1 Implementation details

**Step (2)** We used PyTorch to implement this step. The Wasserstein distance of real distributions is a  $\mathcal{L}^2$ -norm on the cumulative distribution functions, which makes it easy to compute for mixture of Diracs. Auto-differentiation makes it even easier to implement. We chose Adam optimizer by default.

**Step (3)** For finding the support, we use the implementation for  $N$ -diracs given in Section 8.2.3. If the product  $M \times n$  is too large compared to the desired execution time, it is possible to compute only a subset of  $S_F^c$ . For that, analyze only  $x_F^M$  and  $x_F^k$  for  $k$  in a (small, even empty) subset of  $\llbracket 1, M-1 \rrbracket$ .

**Step (4)** From the previous step, we have determined  $S_F$  of the form  $S_F = \cup_{i=1}^{\nu} [u_i^l, u_i^r]$ ,  $u_1^l < u_1^r < \dots < u_{\nu}^l < u_{\nu}^r$ . As proposed in [11], a suitable choice for a grid follows the arcsine distribution on each sub-interval  $[u_i^l, u_i^r]$ . Formally, for each  $i \in \llbracket 1, \nu \rrbracket$ , we define the sub-grid  $\{\xi_i^j\}_{j=0}^{\omega_i+1}$  with  $\omega_i \in \mathbb{N}$  points:

$$\forall j \in \llbracket 0, \omega_i + 1 \rrbracket, \xi_i^j = u_i^l + (u_i^r - u_i^l) \sin \left( \frac{\pi j}{2(\omega_i + 1)} \right)^2. \quad (15)$$

The choice of  $\omega_i \in \mathbb{N}$  is up to the practitioner. We propose three strategies for a total number of points of  $\Omega + 2\nu \in \mathbb{N}^*$ :

- the uniform weighting:  $\forall i \in \llbracket 1, \nu \rrbracket, \omega_i^{(u)} = \Omega/\nu$  (up to integer discretization),
- the frequentist weighting, inspired from [11]:  $\forall i \in \llbracket 1, \nu \rrbracket, \omega_i^{(f)} = \frac{\nu}{p} |\{k \in \llbracket 1, p \rrbracket | \lambda_k \in [u_i^l, u_i^r]\}|$  (up to integer discretization), where more points are used in intervals where we find a lot of empirical eigenvalues,
- a mix of both strategies with parameter  $\mu \in [0, 1]$ :  $\forall i \in \llbracket 1, \nu \rrbracket, \omega_i^{(m)} = \mu \omega_i^{(u)} + (1 - \mu) \omega_i^{(f)}$ . With  $\mu = 0.1$  for example, this strategy ensures that a minimal number of discretization points will be used even if no empirical eigenvalues were found in the interval.

### 8.3.2 Additional experiments

We explore the behavior of the algorithm with a different concentration ratio  $c = 0.5$  to see the effect of fewer samples on the estimation. Results are shown in Figure 12. Fewer samples compared to the dimension makes it hard to estimate accurately  $H$ . We clearly see that the estimated  $H$  is more spread around the true diracs than it were with  $c = 0.1$  in the experiment shown in the main corpus.

We also experiment the effect of heavy tails in the estimation. In [8], it is remarked that even if the convergence of  $F_n$  to  $F$  is almost sure as long as we have bounded  $2^{nd}$  moments, this convergence is slower as the tail is heavier.

In this experiment, we fix the dimension  $n = 400$ ,  $c = 0.1$ , and we study the impact of  $\nu > 2$  when the underlying noise  $Z_{ij}$  follows a Student distribution  $t_{\nu}(0, 1)$ . In this scenario, nothing particular happens while  $\nu \geq 4$  roughly,  $F_n$  is consistent, and the estimation remains barely affected. However, for very low  $\nu$ , around  $2 < \nu < 4$ , the sample spectrum  $F_n$  tends to have some very high eigenvalues outside of the asymptotic support  $S_F$ . Everything else fixed, decreasing  $\nu$  increases the amount of outlier eigenvalues in  $F_n$ . Of course their frequency vanishes while  $n \rightarrow +\infty$  but they exist in moderate dimension.

In order to study the impact of these outliers in the estimation of  $H$ , we consider an extreme setting  $\nu = 3$  where we consistently draw high outliers when sampling  $F_n$ . This experiment brings the algorithm far from the theoretical requirements H1-H5 that assume  $\nu > 12$ .

As shown in the Figure 13, these outliers in the sample spectrum affect badly the estimation of  $H$ , skewing it towards high values. Fortunately, as their frequency is quite low, only the higher values in  $H$  are deteriorated and the estimated densited  $F'$  is still accurate in the core of the distribution. Some artefacts appear in the tail of the estimated  $F'$  and  $H$ , with unwanted and isolated high values that fit the observed outliers. These outliers create small intervals in the estimated support  $S_F$  around the observed outliers.

In this extreme setting with  $\nu = 3$ , far from the theoretical requirement, if one uses heavy tails, we recommend to transform the observed eigenvalues in order to reject the highest quantiles.

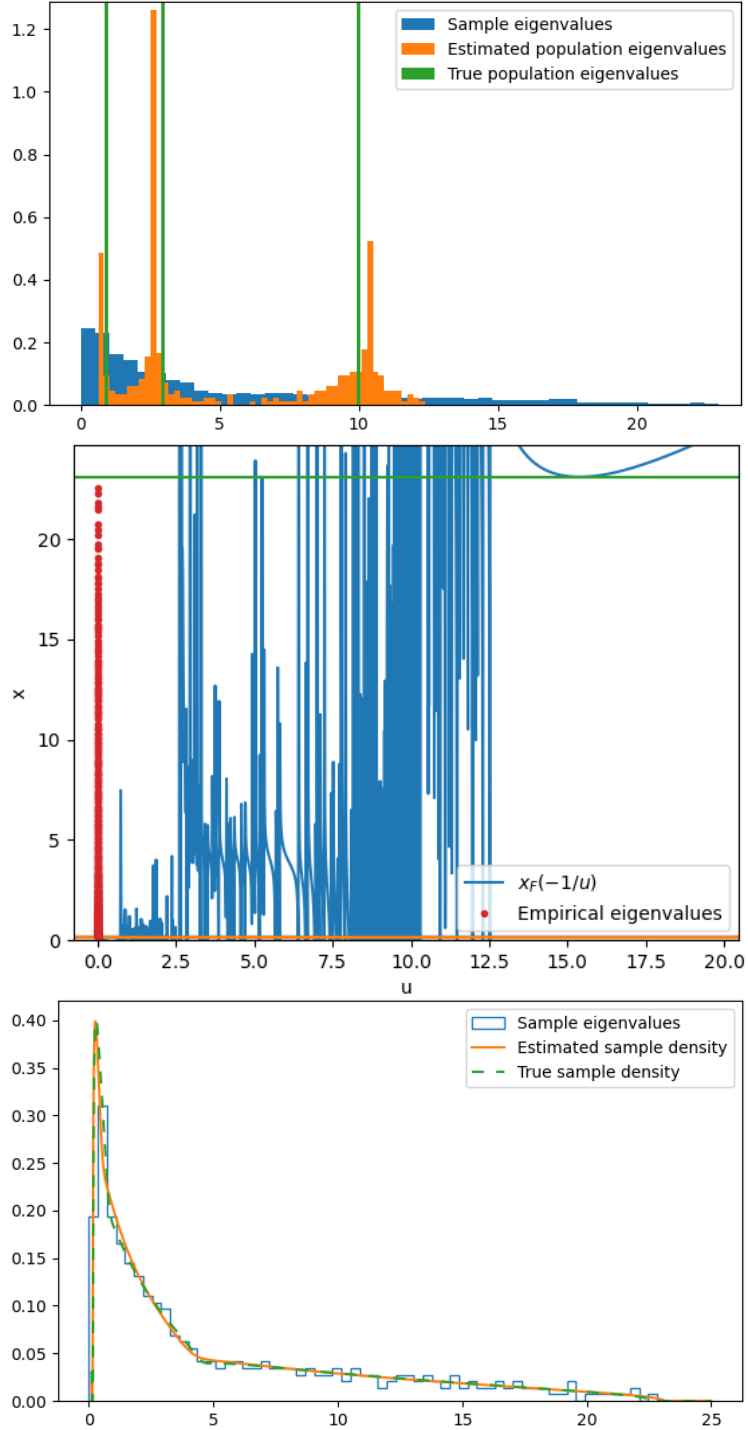


Figure 12: (Top) Histograms of sample eigenvalues, estimated population eigenvalues  $\hat{H}$  with *WeSpeR*, and true population eigenvalues  $H$ . (Middle)  $u \mapsto x_F(-1/u)$  using the estimated  $\hat{H}$  to detect the estimated support  $S_F$ . (Bottom) Estimated and true sample density computed on  $S_F$  and sample eigenvalues histogram.  $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ ,  $D$  exponentially weighted with  $\alpha = 1$ ,  $c = 0.5$ , and  $Z_{ij} \sim \mathcal{N}(0, 1)$ .

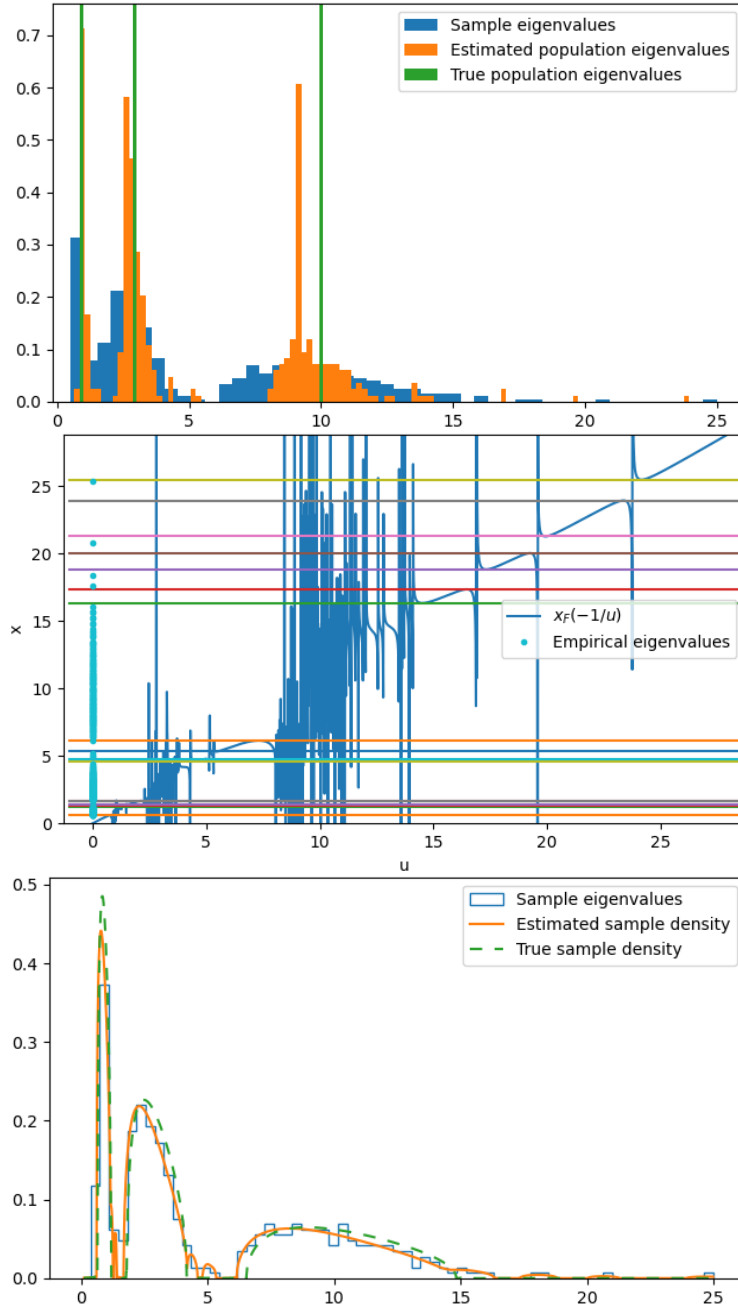


Figure 13: (Top) Histograms of sample eigenvalues, estimated population eigenvalues  $\hat{H}$  with *WeSpeR*, and true population eigenvalues  $H$ . (Bottom) Estimated and true sample density computed on  $S_F$  and sample eigenvalues histogram.  $H = \frac{1}{5}\mathbf{1}_{[1,\infty[} + \frac{2}{5}\mathbf{1}_{[3,\infty[} + \frac{2}{5}\mathbf{1}_{[10,\infty[}$ ,  $D$  exponentially weighted with  $\alpha = 1$ ,  $c = 0.1$ , and heavy tails  $Z_{ij} \sim t_3(0, 1)$ .

## 9 Appendix B - Proofs

### 9.1 Remark on Theorem 1

In the main corpus, we used Theorem 2 [8] with the following form.

**Theorem 6** (Theorem 2 [8]). *Assume H1 to H5. Then, almost surely,  $F^{B_n}$  converges weakly to a nonrandom p.d.f  $F$ , whose Cauchy-Stieltjes transform  $m := m_F$  satisfies for all  $z \in \mathbb{C}_+$ :*

$$m(z) = -\frac{1}{z} \int \frac{1}{\tau X(z) + 1} dH(\tau), \quad (16)$$

where for all  $z \in \mathbb{C}_+$ ,  $X(z)$  is the unique solution in  $\mathbb{C}_+$  of the following equation:

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta). \quad (17)$$

One note the use of  $X(z)$  is this formulation, instead of  $\tilde{m}(z) = -zX(z)$  in the original work [8]. We clarify the only non-trivial point of the equivalence between those two formulations (see [8] Theorem 2 for the original formulation and its notation): for  $z \in \mathbb{C}_+$ ,  $X(z) := -\frac{\tilde{m}(z)}{z} \in \mathbb{C}_+$  and the solution of Equation (17) is unique in  $\mathbb{C}_+$ .

Let  $z \in \mathbb{C}_+$ . Using the notation of the proof of Theorem 2 [8], it is proved that almost surely:

$$\frac{1}{N} \sum_{j=1}^N \frac{W_{jj}}{1 + r_j^*(B_{(j)} - zI)^{-1}r_j} \xrightarrow{N \rightarrow +\infty} \tilde{m}(z). \quad (18)$$

So, using that  $X(z) := -\frac{\tilde{m}(z)}{z}$ , we have immediatly that, almost surely:

$$-\frac{1}{N} \sum_{j=1}^N \frac{W_{jj}}{z + r_j^*(B_{(j)}/z - I)^{-1}r_j} \xrightarrow{N \rightarrow +\infty} X(z). \quad (19)$$

Using the proof of Lemma 3 [8], we have that  $\text{Im}[r_j^*(B_{(j)}/z - I)^{-1}r_j]$ , so  $\forall N, -\frac{1}{N} \sum_{j=1}^N \frac{W_{jj}}{z + r_j^*(B_{(j)}/z - I)^{-1}r_j} \in \mathbb{C}_+$ . So,  $\text{Im}[X(z)] \geq 0$ .

Moreover, using the definition  $X(z) = -\frac{\tilde{m}(z)}{z}$ , it is immediate from Theorem 2 [8] that  $X(z)$  is a solution of Equation (17). It is also immediate that Equation (17) does not admit any real solution because  $z \in \mathbb{C}_+$ . As we proved that  $\text{Im}[X(z)] \geq 0$ , we deduce now that  $X(z) \in \mathbb{C}_+$ .

The last point is the unicity in  $\mathbb{C}_+$  of the solution of (17). Let  $X_1, X_2 \in \mathbb{C}_+$ ,  $X_1 \neq X_2$  solving Equation (17). Then, in the spirit of the proof of unicity for  $\tilde{m}(z)$  in [8], we have with Hölder inequality:

$$\begin{aligned} |X_1 - X_2|^2 &= |X_1 - X_2|^2 \times \left| \int \frac{\delta^2 c \int \frac{\tau^2}{(\tau X_1 + 1)(\tau X_2 + 1)} dH(\tau)}{\left(z - \delta c \int \frac{\tau}{\tau X_1 + 1} dH(\tau)\right) \left(z - \delta c \int \frac{\tau}{\tau X_2 + 1} dH(\tau)\right)} dD(\delta) \right|^2 \\ &\leq |X_1 - X_2|^2 \times \left| \int \frac{\delta^2 c \int \frac{\tau^2}{|\tau X_1 + 1|^2} dH(\tau)}{\left|z - \delta c \int \frac{\tau}{\tau X_1 + 1} dH(\tau)\right|^2} dD(\delta) \right| \times \left| \int \frac{\delta^2 c \int \frac{\tau^2}{|\tau X_2 + 1|^2} dH(\tau)}{\left|z - \delta c \int \frac{\tau}{\tau X_2 + 1} dH(\tau)\right|^2} dD(\delta) \right| \\ &< |X_1 - X_2|^2 \times \frac{|\text{Im}[X_1]|}{|\text{Im}[X_1]|} \times \frac{|\text{Im}[X_2]|}{|\text{Im}[X_2]|} \\ |X_1 - X_2|^2 &< |X_1 - X_2|^2 \end{aligned} \quad (20)$$

The inequality is strict because  $H(]0, \infty[) > 0$  and  $D(]0, \infty[) > 0$  according to Assumption H3-H4. The result is absurd, hence the unicity in  $\mathbb{C}_+$ .

## 9.2 Proof of Theorem 2 and Corollary 1

In the whole following section, we assume H1-H5 to hold. We denote  $\tilde{m}(z) := -zX(z)$ .

**Lemma 1.**  $\tilde{m}$  is bounded on every bounded region of  $\mathbb{C}_+$  bounded away from  $\{0\}$ .

*Proof.* Suppose there exists  $(z_n)_n \in \mathbb{C}_+^{\mathbb{N}}$  bounded and bounded away from  $\{0\}$ , i.e. there exists  $\eta > 0$  such that  $|z_n| \geq \eta$ , such that  $|X(z_n)| \rightarrow +\infty$ . Moreover, for  $z \in \mathbb{C}_+$ ,

$$\left| \frac{\tau}{\tau X(z) + 1} \right| \geq \frac{h_2}{h_1 |X(z)| - 1}. \quad (21)$$

So,

$$\left| \frac{\delta}{z + \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} \right| \leq \frac{d_2}{|z| - cd_2 \frac{h_2}{h_1 |X(z)| - 1}}. \quad (22)$$

By hypothesis, for  $n$  large enough,  $|X(z_n)| \geq \max\left(\frac{1}{h_1} \left(1 + \frac{2cd_2 h_2}{\eta}\right), 2\frac{d_2}{\eta}\right) + 1$ . Then, using the fact that for all  $z \in \mathbb{C}_+$ :

$$X(z) = - \int \frac{\delta}{z - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} dD(\delta), \quad (23)$$

we have that:

$$\begin{aligned} |X(z_n)| &\leq \int \left| \frac{\delta}{z_n - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} \right| dD(\delta) \\ &\leq \frac{d_2}{|z_n| - cd_2 \frac{h_2}{h_1 |X(z_n)| - 1}} \\ &\leq 2 \frac{d_n}{|z_n|} \end{aligned} \quad (24)$$

$$|X(z_n)| < |X(z_n)|,$$

which is absurd, and concludes the proof.  $\square$

From Theorem 2 [8],  $m := \Theta^{(0)}$  is the Cauchy-Stieltjes transform of a probability distribution function - p.d.f. - that we denote  $F$ .

**Lemma 2.** Let  $x_0 \in S_F^c, x_0 \neq 0$ . Then,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \tilde{m}(z)$  exists.

*Proof.* Let  $x_0 \in S_F^c, x_0 \neq 0$ . By property of the Cauchy-Stieltjes transform of a p.d.f.,  $m$  is analytic on  $\mathbb{C} \setminus S_F$ . Thus,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} m(z) := \tilde{m}(z) \in \mathbb{R}$  exists. Then,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \text{Im}[\tilde{m}(z)] = 0$ . Now, suppose there exists  $z_{1,n} \in \mathbb{C}_+^{\mathbb{N}}$  such that  $z_{1,n} \rightarrow x_0$  and  $\tilde{m}(z_{1,n}) \rightarrow \tilde{m}_1$ , and similarly  $z_{2,n} \in \mathbb{C}_+^{\mathbb{N}}$  such that  $z_{2,n} \rightarrow x_0$  and  $\tilde{m}(z_{2,n}) \rightarrow \tilde{m}_2$ . As  $m(z_{1,n}) \rightarrow \tilde{m}(z) \in \mathbb{R}$  and  $m(z_{2,n}) \rightarrow \tilde{m}(z) \in \mathbb{R}$ , we have that  $\text{Re}[m(z_{1,n}) - m(z_{2,n})] \rightarrow 0$ . And,

$$\text{Re}[m(z_{1,n}) - m(z_{2,n})] = \int \frac{\tau \text{Re}[\tilde{m}(z_{1,n}) - \tilde{m}(z_{2,n})] - \text{Re}[z_{1,n} - z_{2,n}]}{|\tau \tilde{m}_{1,n} - z_{1,n}|^2 \times |\tau \tilde{m}_{2,n} - z_{2,n}|^2} dH(\tau). \quad (25)$$

So,

$$\begin{aligned} |\text{Re}[\tilde{m}(z_{1,n}) - \tilde{m}(z_{2,n})]| &= \left| \frac{\text{Re}[m(z_{1,n}) - m(z_{2,n})] + \text{Re}[z_{1,n} - z_{2,n}] \int \frac{1}{|\tau \tilde{m}_{1,n} - z_{1,n}|^2 \times |\tau \tilde{m}_{2,n} - z_{2,n}|^2} dH(\tau)}{\int \frac{\tau}{|\tau \tilde{m}_{1,n} - z_{1,n}|^2 \times |\tau \tilde{m}_{2,n} - z_{2,n}|^2} dH(\tau)} \right| \\ |\text{Re}[\tilde{m}(z_{1,n}) - \tilde{m}(z_{2,n})]| &\leq \left| \frac{\text{Re}[m(z_{1,n}) - m(z_{2,n})]}{\int \frac{\tau}{|\tau \tilde{m}_{1,n} - z_{1,n}|^2 \times |\tau \tilde{m}_{2,n} - z_{2,n}|^2} dH(\tau)} \right| + \frac{1}{h_1} |\text{Re}[z_{1,n} - z_{2,n}]|. \end{aligned} \quad (26)$$

Suppose  $\text{Re}[\tilde{m}_1] \neq \text{Re}[\tilde{m}_2]$ . Then, from the previous inequation, we deduce that  $\int \frac{\tau}{|\tau \tilde{m}_{1,n} - z_{1,n}|^2 \times |\tau \tilde{m}_{2,n} - z_{2,n}|^2} dH(\tau) \rightarrow 0$ . So  $|\tilde{m}_{1,n}| \rightarrow +\infty$  and  $|\tilde{m}_{2,n}| \rightarrow +\infty$ , which is absurd. So  $\text{Re}[\tilde{m}_1] = \text{Re}[\tilde{m}_2]$ . We proved at the beginning of the proof that  $\text{Im}[\tilde{m}_1] = \text{Im}[\tilde{m}_2]$ . As  $\tilde{m}$  is bounded on every bounded region of  $\mathbb{C}_+$  bounded away from  $\{0\}$ , we deduce that  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \tilde{m}(z)$  exists (and is equal to  $\tilde{m}_1$ ).  $\square$

We define:

$$m_{LD} : z \in \mathbb{C} \setminus S_D \mapsto \int \frac{\delta}{\delta - z} dD(\delta). \quad (27)$$

Remark that  $m_{LD}$  is analytic on  $\mathbb{C} \setminus S_D$ .

**Lemma 3.** *Let  $x_0 \in S_F, x_0 \neq 0$ . We denote  $X(z) = -\frac{\tilde{m}(z)}{z}$ . Suppose there exists  $z_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $X(z_n) \rightarrow X_0 \in \mathbb{C}_+$  and  $\hat{z}_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $X(\hat{z}_n) \rightarrow \hat{X}_0 \in \mathbb{C} \setminus \mathbb{C}_-, \hat{X}_0 \neq X_0$ . Then there exists  $\bar{z}_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $X(\bar{z}_n) \rightarrow \bar{X}_0 \in \mathbb{C}_+$  and  $m'_{LD}(\check{Y}) \neq 0$  with  $\check{Y} := \frac{x_0}{c \int \frac{\tau}{\tau \bar{X} + 1} dH(\tau)}$ .*

*Proof.* Consider the following procedures:

- (i) Suppose  $\text{Im}[X_0] \neq \text{Im}[\hat{X}_0]$ . Let  $n \in \mathbb{N}$ . By continuity of  $X$  on  $\mathbb{C}_+$ , there exists  $u_n \in \mathbb{C}_+$  in the complex segment  $[z_n, \hat{z}_n]_{\mathbb{C}}$  such that  $\text{Im}[X(u_n)] = \frac{1}{2} (\text{Im}[X(z_n)] + \text{Im}[X(\hat{z}_n)])$ .  $u_n \rightarrow x_0 \neq 0$  so  $X(u_n)$  is bounded, so there exists an extraction  $\bar{z}_n^{(1)}$  of  $u_n$  such that  $\bar{z}_n^{(1)} \rightarrow x_0$  and  $X(\bar{z}_n^{(1)}) \rightarrow \bar{X}^{(1)} \in \mathbb{C}_+, \text{Im}[\bar{X}^{(1)}] = \frac{1}{2} (\text{Im}[X_0] + \text{Im}[\hat{X}_0])$ . Do the procedure (i) again with  $z_n, \bar{z}_n^{(1)}$  instead of  $z_n, \hat{z}_n$  in order to construct  $\bar{z}_n^{(2)}$  and  $\bar{X}^{(2)}$ .
- (ii) Suppose  $\text{Im}[X_0] = \text{Im}[\hat{X}_0] > 0$ , then  $\text{Re}[X_0] \neq \text{Re}[\hat{X}_0]$ . Similarly, there exists  $u_n \in \mathbb{C}_+$  in the complex segment  $[z_n, \hat{z}_n]_{\mathbb{C}}$  such that  $\text{Re}[X(u_n)] = \frac{1}{2} (\text{Re}[X(z_n)] + \text{Re}[X(\hat{z}_n)])$ .  $u_n \rightarrow x_0 \neq 0$  so  $X(u_n)$  is bounded, so there exists an extraction  $\bar{v}_n$  of  $u_n$  such that  $\bar{v}_n \rightarrow x_0$  and  $X(\bar{v}_n) \rightarrow \bar{X} \in \mathbb{C} \setminus \mathbb{C}_-, \text{Re}[\bar{X}] = \frac{1}{2} (\text{Re}[X_0] + \text{Re}[\hat{X}_0])$ .
  - If  $\text{Im}[\bar{X}] = 0$ , do the procedure (i) with  $z_n, \bar{v}_n$  instead of  $z_n, \hat{z}_n$  in order to construct  $\bar{z}_n^{(1)}$  and  $\bar{X}^{(1)} \in \mathbb{C}_+$ .
  - If  $\text{Im}[\bar{X}] = \text{Im}[X_0]$ , define  $z_n^{(1)} := \bar{v}_n$  and  $\bar{X}^{(1)} := \bar{X} \in \mathbb{C}_+$ , and do the procedure (ii) again with  $z_n, \bar{z}_n^{(1)}$  instead of  $z_n, \hat{z}_n$  in order to construct  $\bar{z}_n^{(2)}, \dots$  and  $\bar{X}^{(2)} \in \mathbb{C}_+$ .
  - Otherwise, also define  $z_n^{(1)} := \bar{v}_n$  and  $\bar{X}^{(1)} := \bar{X} \in \mathbb{C}_+$ , and go to procedure (i) with  $z_n, \bar{z}_n^{(1)}$  instead of  $z_n, \hat{z}_n$  in order to construct  $\bar{z}_n^{(2)}$  and  $\bar{X}^{(2)} \in \mathbb{C}_+$ .

Using this procedure, we construct iteratively  $\{\bar{X}^{(k)}, k \in \mathbb{N}^*\}$ , where for all  $k \in \mathbb{N}^*, \bar{X}^{(k)} \in \mathbb{C}_+$  is an adherence point of  $X$  as  $z \in \mathbb{C}_+ \rightarrow x_0$ , and by construction,  $\forall k \neq k' \in \mathbb{N}^*, \bar{X}^{(k)} \neq \bar{X}^{(k')}$ .

Suppose that  $\forall k \in \mathbb{N}^*, m'_{LD} \left( \frac{x_0}{c \int \frac{\tau}{\tau \bar{X}^{(k)} + 1} dH(\tau)} \right) = 0$ .  $D \neq 0$  by hypothesis so  $m_{LD}$  is not constant over  $\mathbb{C}_+$ , and as  $m_{LD}$  is analytic on  $\mathbb{C}_+$  it implies that  $\left\{ \int \frac{\tau}{\tau \bar{X}^{(k)} + 1} dH(\tau), k \in \mathbb{N}^* \right\}$  is finite while  $\{\bar{X}^{(k)}, k \in \mathbb{N}^*\}$  is countable. So, as  $\phi : X \in \mathbb{C}_+ \mapsto \int \frac{\tau}{\tau X + 1} dH(\tau)$  is analytic, we have that  $\phi$  is constant over  $\mathbb{C}_+$ . But  $H \neq 0$  by hypothesis, this is absurd and it concludes the proof.  $\square$

**Lemma 4.** *Let  $x_0 \in S_F, x_0 \neq 0$ . Suppose there exists  $z_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $X(z_n) \rightarrow \check{X} \in \mathbb{C}_+$  and  $m'_{LD}(\check{Y}) \neq 0$  with  $\check{Y} := \frac{x_0}{c \int \frac{\tau}{\tau \check{X} + 1} dH(\tau)}$ . Then  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} X(z)$  exists and  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} X(z) = \check{X}$ .*

*Proof.* We have for all  $z \in \mathbb{C}_+$ , with  $X(z) = -\frac{\tilde{m}(z)}{z}$ :

$$m_{LD} \left( \frac{z}{c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} \right) = c - c \int \frac{1}{\tau X(z) + 1} dH(\tau). \quad (28)$$

Suppose  $z_n \in \mathbb{C}_+ \rightarrow x_0, X(z_n) \rightarrow \check{X} \in \mathbb{C}_+$  and, with  $\check{Y} := \frac{x_0}{c \int \frac{\tau}{\tau \check{X} + 1} dH(\tau)} \in \mathbb{C}_+$  we have  $m'_{LD}(\check{Y}) \neq 0$ . So by the holomorphic inverse function theorem,  $m_{LD}$  is locally invertible at  $\check{Y}$  in an open set  $B$  containing  $\check{Y}$ . We denote its local inverse  $g_{LD} : m_{LD}(B) \rightarrow B$ , which is also analytic.



As  $Y_n := \frac{z_n}{c \int \frac{\tau}{\tau X(z_n)+1} dH(\tau)} \rightarrow \tilde{Y}$ , for  $n$  large enough,  $Y_n \in B$ . We have then, for  $n$  large enough:

$$z_n = c \int \frac{\tau}{\tau X(z_n)+1} dH(\tau) g_{LD} \left( c - c \int \frac{1}{\tau X(z_n)+1} dH(\tau) \right). \quad (29)$$

So,

$$x_0 = c \int \frac{\tau}{\tau \tilde{X}+1} dH(\tau) g_{LD} \left( c - c \int \frac{1}{\tau \tilde{X}+1} dH(\tau) \right). \quad (30)$$

Let  $\epsilon > 0$  and  $B' = B(\tilde{X}, \epsilon)$  so that  $\left\{ c - c \int \frac{1}{\tau X+1} dH(\tau), X \in B' \right\} \subset m_{LD}(B)$ . We can choose  $\epsilon > 0$  arbitrarily small because  $m_{LD}(B)$  is open due to the open mapping theorem,  $c - c \int \frac{1}{\tau X+1} dH(\tau) \in m_{LD}(B)$  and  $X \mapsto \int \frac{1}{\tau X+1} dH(\tau)$  is analytic on  $\mathbb{C} \setminus \{x \in \mathbb{R}, -x^{-1} \in S_H\}$ . We define:

$$z_F : X \in B' \mapsto c \int \frac{\tau}{\tau X+1} dH(\tau) g_{LD} \left( c - c \int \frac{1}{\tau X+1} dH(\tau) \right). \quad (31)$$

By the open mapping theorem, as  $z_F$  is analytic and non-constant over  $B'$ ,  $z_F(B')$  is open and contains  $x_0$ . Thus, for any  $(\tilde{z}_n)_n \in \mathbb{C}_+^{\mathbb{N}}$  such that  $\tilde{z}_n \rightarrow x_0$ , we have that  $\tilde{z}_n \in z_F(B')$  for  $n$  large enough. So, for these  $\tilde{z}_n$ , there exists  $X_n \in B'$  such that  $z_F(X_n) = \tilde{z}_n$ . So,  $X_n = X(\tilde{z}_n) \in B'$ . As we can choose  $\epsilon > 0$  arbitrarily small, we have that  $X(\tilde{z}_n) \rightarrow \tilde{X}$ , which concludes the proof.  $\square$

At this stage, we know that, for  $x_0 \in S_F^c \setminus \{0\}$ ,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \tilde{m}(z)$  exists, and for  $x_0 \in S_F \setminus \{0\}$ , either  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \text{Im}[\tilde{m}(z)] = 0$  or  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \tilde{m}(z)$  exists.

**Lemma 5.** Assume  $0 \notin ]m_a, m_b[ \subset \mathbb{R}$ . If, for all  $m_0 \in ]m_a, m_b[$ , there exists  $(h_n)_n \in (\mathbb{R}_+^*)^{\mathbb{N}}$  such that  $h_n \rightarrow 0$  and  $\int \frac{h_n \tau^2 dH(\tau)}{(1+\tau m_0)^2 + \tau^2 h_n^2} \rightarrow 0$ , then  $] -m_a^{-1}, -m_b^{-1}[ \subset S_H^c$ .

*Proof.* Let  $m_0 \in ]m_a, m_b[$  and  $n \in \mathbb{N}$ . Let  $g(\tau) = \frac{h_n \tau^2}{(1+\tau m_0)^2 + \tau^2 h_n^2}$ .  $g$  is increasing on  $] -m_0^{-1} - h_n, -m_0^{-1}[$  and decreasing on  $] -m_0^{-1}, -m_0^{-1} + h_n[$ . Then,

$$\begin{aligned} \int g(\tau) dH(\tau) &\geq \int_{-m_0^{-1}-h_n}^{-m_0^{-1}} \frac{h_n(m_n^{-1} + h_n)^2 dH(\tau)}{(h_n m_0)^2 + h_n^2(m_0^{-1} + h_n)^2} \\ \int g(\tau) dH(\tau) &\geq \frac{(m_0^{-1} + h_n)^2}{m_0^2 + (m_0^{-1} + h_n)^2} \frac{H(-m_0^{-1}) - H(-m_0^{-1} - h_n)}{h_n}. \end{aligned} \quad (32)$$

Thus,  $\frac{H(-m_0^{-1}) - H(-m_0^{-1} - h_n)}{h_n} \rightarrow 0$ . Then, the lower-left Dini derivative (see [12] for a reference on the subject) is null, i.e.  $D_- H(-m_0^{-1}) := \liminf_{h \rightarrow 0^+} \frac{H(-m_0^{-1}) - H(-m_0^{-1} - h)}{h} = 0$ . As it is null,  $D_- H$  is continuous on  $] -m_a^{-1}, -m_b^{-1}[$ . Moreover,  $H$  is càdlàg, nondecreasing and  $D_- H = 0$  on  $] -m_a^{-1}, -m_b^{-1}[$  so  $H$  is continuous on  $] -m_a^{-1}, -m_b^{-1}[$ . The two latter points imply that the three other Dini derivatives -  $D^- H$ ,  $D_+ H$  and  $D^+ H$  - are continuous on  $] -m_a^{-1}, -m_b^{-1}[$ . As  $H$  is monotone on  $] -m_a^{-1}, -m_b^{-1}[$ , it is almost everywhere differentiable due to Lebesgue's theorem. So, almost everywhere on  $] -m_a^{-1}, -m_b^{-1}[$ , all four Dini derivatives are equal, and their value is 0 due to  $D_-$ . And from continuity, everywhere on  $]m_a, m_b[$ , all four Dini derivatives are equal to 0. So  $H$  is differentiable on  $] -m_a^{-1}, -m_b^{-1}[$  and  $H' = 0$ . So  $H$  is constant on  $] -m_a^{-1}, -m_b^{-1}[$ , which finally implies that  $] -m_a^{-1}, -m_b^{-1}[ \subset S_H^c$ .  $\square$

**Lemma 6.** Assume  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \text{Im}[X(z)] = 0$ . Let  $z_n \in \mathbb{C}_+ \rightarrow x_0$  and  $\hat{z}_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $X(z_n) \rightarrow X_0 \in \mathbb{R}$  and  $X(\hat{z}_n) \rightarrow \hat{X}_0 \in \mathbb{R}$ ,  $X_0 < \hat{X}_0$ . Then,  $\forall \tilde{X} \in ]X_0, \hat{X}_0[$ , there exists  $(\tilde{z}_n)_n \in \mathbb{C}_+^{\mathbb{N}}$  such that  $\tilde{z}_n \rightarrow x_0$  and  $X(\tilde{z}_n) \rightarrow \tilde{X}$ .  $(\tilde{z}_n)_n$  can be chosen so that  $\text{Re}[X(\tilde{z}_n)] = \tilde{X}$ .

*Proof.* The same proof as Lemma 3.6 [9].  $\square$

**Lemma 7.** Let  $x_0 \in S_F$ ,  $x_0 \neq 0$ . Suppose  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \text{Im}[X(z)] = 0$ , then  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} X(z)$  exists.

*Proof.* Suppose  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \text{Im}[X(z)] = 0$  but  $\text{Re}[X(z)]$  does not converge as  $z \in \mathbb{C}_+ \rightarrow x_0$ . Using Lemma 6, we can find  $X_0 \in \mathbb{R}$  and  $\hat{X}_0 \in \mathbb{R}$  adherence points of  $X$  at  $x_0$ ,  $X_0 < \hat{X}_0$  such that  $0 \notin ]X_0, \hat{X}_0[$ . Let  $\bar{X} \in ]X_0, \hat{X}_0[$ . From Lemma 6, there exists  $(\bar{z}_n)_n \in \mathbb{C}_+^{\mathbb{N}}$  such that  $\bar{z}_n \rightarrow x_0$  and  $X(\bar{z}_n) \rightarrow \bar{X}$  and  $\text{Re}[X(\bar{z}_n)] = \bar{X}$ . So,  $\text{Im}[X(\bar{z}_n)] \rightarrow 0$ . By the function equation on  $X$ , we have:

$$\text{Im}[X(\bar{z}_n)] = \int \frac{\delta \left( \text{Im}[\bar{z}_n] + \delta c \int \frac{\tau^2 \text{Im}[X(\bar{z}_n)]}{|\tau X(\bar{z}_n) + 1|^2} dH(\tau) \right)}{\left| \bar{z}_n - \delta c \int \frac{\tau}{\tau X(\bar{z}_n) + 1} dH(\tau) \right|^2}. \quad (33)$$

$\left| \bar{z}_n - \delta c \int \frac{\tau}{\tau X(\bar{z}_n) + 1} dH(\tau) \right|^2$  is bounded, so  $\int \frac{\tau^2 \text{Im}[X(\bar{z}_n)]}{|\tau X(\bar{z}_n) + 1|^2} dH(\tau) \rightarrow 0$ . Using Lemma 5 with  $m_a := X_0$ ,  $m_b := \hat{X}_0$  and  $h_n := \text{Im}[X(\bar{z}_n)]$ , we deduce that  $]-X_0^{-1}, -\hat{X}_0^{-1}[ \in S_H^c$ . Without loss of generality, suppose  $\left] c - c \int \frac{1}{\tau X_0 + 1} dH(\tau), c - c \int \frac{1}{\tau \hat{X}_0 + 1} dH(\tau) \right[ \subset \mathbb{R}^*$ . For  $\bar{z}_n \rightarrow x_0$  such that  $X(\bar{z}_n) \rightarrow \bar{X} \in ]X_0, \hat{X}_0[$ , we have that  $m_{LD} \left( \frac{\bar{z}_n}{c \int \frac{\tau}{\tau X(\bar{z}_n) + 1} dH(\tau)} \right) = c - c \int \frac{1}{\tau X(\bar{z}_n) + 1} dH(\tau) \rightarrow c - c \int \frac{1}{\tau \bar{X} + 1} dH(\tau) \in \mathbb{R}^*$ . Thus,  $\int \frac{\tau}{\tau X(\bar{z}_n) + 1} dH(\tau) \rightarrow \int \frac{\tau}{\tau \bar{X} + 1} dH(\tau) \neq 0$  (otherwise  $m_{LD} \left( \frac{\bar{z}_n}{c \int \frac{\tau}{\tau X(\bar{z}_n) + 1} dH(\tau)} \right) \rightarrow 0$ ), and  $\frac{\bar{z}_n}{c \int \frac{\tau}{\tau X(\bar{z}_n) + 1} dH(\tau)} \rightarrow \frac{x_0}{c \int \frac{\tau}{\tau \bar{X} + 1} dH(\tau)} \in \mathbb{R}$ . Then,  $\left] \frac{x_0}{c \int \frac{\tau}{\tau X_0 + 1} dH(\tau)}, \frac{x_0}{c \int \frac{\tau}{\tau \hat{X}_0 + 1} dH(\tau)} \right[ \subset S_D^c$ .

So, for  $\bar{X} \in ]X_0, \hat{X}_0[$ ,  $m_{LD}$  is locally invertible at  $\bar{Y} = \frac{x_0}{c \int \frac{\tau}{\tau \bar{X} + 1} dH(\tau)}$ , on an open  $B$  containing this point, we denote its inverse  $g_{LD}$ . Indeed,  $m'_{LD}(\bar{Y}) = \int \frac{\delta}{(\delta - \bar{Y})^2} dD(\delta) > 0$ .

We define  $B' := \left\{ X \in \mathbb{C}^* \mid -X^{-1} \notin S_H \text{ and } c - c \int \frac{1}{\tau X + 1} dH(\tau) \in m_{LD}(B) \right\}$ . Remark that  $B'$  is open as  $m_{LD}(B)$  open and  $X \in \{X \in \mathbb{C}^* \mid -X^{-1} \notin S_H\} \mapsto c - c \int \frac{1}{\tau X + 1} dH(\tau)$  analytic. We define  $z_F : X \in B' \mapsto c \int \frac{\tau}{\tau X + 1} dH(\tau) g_{LD} \left( c - c \int \frac{1}{\tau X + 1} dH(\tau) \right)$ . For all  $\bar{X} \in B'$ , we have  $x_0 = z_F(\bar{X})$ , so  $z_F$  is constant on  $B'$ , which is absurd. So  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} X(z)$  exists.  $\square$

**Lemma 8.**  $X$  - extended on  $\mathbb{R}^*$  with  $\check{X}$  - is a continuous function on  $\mathbb{C}_+ \cup \mathbb{R}^*$ .

*Proof.* Immediate consequence of Theorem 2.2 [9].  $\square$

The following Lemma concludes the proof of the first part of Theorem 2.

**Lemma 9.** For  $x_0 \neq 0$ ,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} m(z)$  and  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \Theta^{(1)}(z)$  exist.

*Proof.* We recall that  $\forall z \in \mathbb{C}_+$ ,  $\Theta^g(z) = -\frac{1}{z} \int \frac{g(\tau)}{\tau X(z) + 1} dH(\tau) = \frac{1}{X(z)} \int \frac{g(\tau)}{\tau + X(z)^{-1}} dH(\tau)$ . Let  $x_0 \neq 0$ . The case where  $\text{Im}[\check{X}(x_0)] \neq 0$  is trivial, with dominated convergence theorem we have that  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \Theta^g(z)$  exists and  $\check{\Theta}^g(x_0) := \lim_{z \in \mathbb{C}_+ \rightarrow x_0} \Theta^g(z) = -\frac{1}{x_0} \int \frac{g(\tau)}{\tau \check{X}(x_0) + 1} dH(\tau)$ .

Now suppose  $\text{Im}[\check{X}(x_0)] = 0$ . Suppose  $\check{X}(x_0) = 0$ . Then,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \Theta^g(z) = -\frac{1}{x_0} \int g(\tau) dH(\tau)$  by D.C.T. Now suppose  $\text{Re}[\check{X}(x_0)] \neq 0$ .

Suppose there exists  $z_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $\left| \int \frac{\tau}{\tau X(z_n) + 1} dH(\tau) \right| \rightarrow +\infty$ . As  $|X(z_n)| \leq \int \left| \frac{\delta}{z_n - \delta c \int \frac{\tau}{\tau X(z) + 1} dH(\tau)} \right| dD(\delta)$  and  $S_D \subset [d_1, d_2]$ , we have that  $\check{X}(x_0) = 0$ , which is absurd. So  $\int \frac{\tau}{\tau X(z) + 1} dH(\tau)$  remains bounded as  $z \in \mathbb{C}_+ \rightarrow x_0$ .

Suppose now there exists  $z_n \in \mathbb{C}_+ \rightarrow x_0$  and  $\hat{z}_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $\int \frac{\tau}{\tau X(z_n) + 1} dH(\tau) \rightarrow m$  and  $\int \frac{\tau}{\tau X(\hat{z}_n) + 1} dH(\tau) \rightarrow \hat{m}$ . Remark that  $m, \hat{m} \in \mathbb{R}$ , and say  $m < \hat{m}$ . Using the proof of Lemma

6, for all  $\bar{m} \in ]m, \hat{m}[$ , there exists  $\bar{z}_n \in \mathbb{C}_+ \rightarrow x_0$  such that  $\operatorname{Re} \left[ \int \frac{\tau}{\tau X(\bar{z}_n)+1} dH(\tau) \right] = \bar{m}$ . And immediately from the functional equation on  $X$  we have that  $\operatorname{Im} \left[ \int \frac{\tau}{\tau X(\bar{z}_n)+1} dH(\tau) \right] \rightarrow 0$ .

Now, following the proof of Lemma 5, we have that  $]x_0/cm, x_0/cm[ \in S_D^c$ . It is in direct contradiction to the fact that  $X(z)$  converges as  $z \in \mathbb{C}_+ \rightarrow x_0$ .

So,  $\check{\Theta}^{(1)}(x_0) := \lim_{z \in \mathbb{C}_+ \rightarrow x_0} \int \frac{\tau}{\tau X(z)+1} dH(\tau)$  exists.

As  $\forall z \in \mathbb{C}_+$ ,  $\Theta^{(1)}(z) = \frac{1}{X(z)} \int \frac{\tau}{\tau + X(z)-1} dH(\tau) = \frac{1}{X(z)} \left( 1 - \int \frac{1}{\tau X(z)+1} dH(\tau) \right)$ , we deduce that  $\check{m}(x_0) := \lim_{z \in \mathbb{C}_+ \rightarrow x_0} \int \frac{1}{\tau X(z)+1} dH(\tau)$  exists.  $\square$

**Lemma 10.** *Let  $g$  a bounded real function on  $[h_1, h_2]$  with a finite number of discontinuities. Then, for  $x_0 \neq 0$ ,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \operatorname{Im}[\Theta^g(z)]$  exists.*

*Proof.* For  $z \in \mathbb{C}_+$ , we have that  $\operatorname{Im}[\Theta^g(z)] = \int \frac{g(\tau)(\tau \operatorname{Im}[-\bar{m}(z)] + \operatorname{Im}[z])}{|\tau \bar{m}(z) - z|^2} dH(\tau)$ . So, if  $\operatorname{Im}[\check{m}(x_0)] = 0$ , then, as  $g$  bounded on  $[h_1, h_2]$ ,  $\operatorname{Im}[\Theta^g(z)] \rightarrow 0$ .

Suppose  $\operatorname{Im}[\check{m}(x_0)] \neq 0$ , then  $\operatorname{Im}[\check{\Theta}^{(1)}(x_0)] \neq 0$ , so by the functional equation on  $X$ ,  $\operatorname{Im}[\check{X}(x_0)] \neq 0$ . With dominated convergence theorem we have that  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} \Theta^g(z)$  exists and  $\check{\Theta}^g(x_0) := \lim_{z \in \mathbb{C}_+ \rightarrow x_0} \Theta^g(z) = -\frac{1}{x_0} \int \frac{g(\tau)}{\tau \check{X}(x_0)+1} dH(\tau)$ .  $\square$

We now extend the result to  $x_0 = 0$  in the case  $c < 1$ .

**Lemma 11.** *Suppose  $c < 1$ , for  $g$  bounded with a finite number of discontinuities,  $\lim_{z \in \mathbb{C}_+ \rightarrow 0} \operatorname{Im}[\Theta^g(z)]$  exists.*

*Proof.* Using Equation (1.9b) [13] for  $c < 1$ , we have for  $\eta = (1 - \sqrt{c})^2 h_1 d_1$  and any  $l > 0$ ,  $\mathbb{P}(\lambda_{\min}^{B_n} \leq \eta) = o(n^{-l})$ . So,  $F_n(\eta) \xrightarrow{n \rightarrow +\infty} 0$ . So, for any  $x < \eta$ ,  $F(x) = 0$ , in particular  $F(0) = 0$  and  $F$  continuous in 0. So,  $\lim_{z \in \mathbb{C}_+ \rightarrow 0} \operatorname{Im}[m(z)] \rightarrow 0$ . So,  $\lim_{z \in \mathbb{C}_+ \rightarrow 0} \operatorname{Im}[\Theta^g(z)] \rightarrow 0$ .  $\square$

With the last Lemma, we proved the second part of Theorem 2.

The following corollary proves the first part of Corollary 1 on  $\mathbb{R}^*$ , or  $\mathbb{R}$  if  $c < 1$ .

**Lemma 12.**  *$F$  has a density on  $\mathbb{R}^*$  or  $\mathbb{R}$  if  $c < 1$ .*

*Proof.* Applying Theorem 2.2 [9] on  $m$ , we have that  $\check{m}$  is continuous on  $\mathbb{R}^*$ . Due to Theorem 2.1 [9],  $F$  is differentiable on  $\mathbb{R}^*$ , and  $F' = \frac{1}{\pi} \operatorname{Im}[m(\cdot)]$ . As  $\operatorname{Im}[m(\cdot)]$ ,  $F'$  is continuous on  $\mathbb{R}^*$ .

Suppose  $c < 1$ . The previous proof stated that for any  $x < \eta = (1 - \sqrt{c})^2 h_1 d_1$ ,  $F(x) = 0$ . Moreover,  $F'$  exists and is continuous on  $\mathbb{R}^*$ . So  $F'$  exists and is continuous density on  $R$  when  $c < 1$ .  $\square$

The second part of Corollary 1 on  $\check{m}$ ,  $\check{\Theta}^{(1)}$  and  $\operatorname{Im}[\check{\Theta}^g]$  uses the following lemma.

**Lemma 13.** *We have:  $\forall \lambda \in S_F^c, \lambda \neq 0, -\check{X}(\lambda)^{-1} \in S_H^c$  and if  $\check{\Theta}^{(1)}(\lambda) \neq 0$ , then  $\frac{\lambda}{c\check{\Theta}^{(1)}(\lambda)} \in S_D^c$ .*

*Proof.* Let  $x_0 \in S_F^c, x_0 \neq 0$ . Suppose  $\check{X}(x_0) = 0$ . Then, by the functional equation,  $\left| \int \frac{\tau}{\tau X(z)+1} dH(\tau) \right|_{z \in \mathbb{C}_+ \rightarrow x_0} \xrightarrow{z \in \mathbb{C}_+ \rightarrow x_0} +\infty$ , but we also proved that  $\left| \int \frac{\tau}{\tau X(z)+1} dH(\tau) \right|_{z \in \mathbb{C}_+ \rightarrow x_0} \xrightarrow{z \in \mathbb{C}_+ \rightarrow x_0} \check{\Theta}^{(1)}(x_0) \in \mathbb{C}$ , which is absurd. So  $\forall x_0 \in S_F^c, x_0 \neq 0, \check{X}(x_0) \in \mathbb{R}^*$ . Then,  $\forall x_0 \in S_F^c, x_0 \neq 0, D_-H(-\check{X}(x_0)^{-1}) = 0$ .  $S_F^c$  is open, and  $\check{X}$  is continuous and non-constant, so  $\{-\check{X}(x_0)^{-1}, x_0 \in S_F^c\}$  is open. So,  $\forall x_0 \in S_F^c, x_0 \neq 0, H'(-\check{X}(x_0)^{-1})$  is defined and  $H'(-\check{X}(x_0)^{-1}) = 0$ . So,  $\{-\check{X}(x_0)^{-1}, x_0 \in S_F^c\} \subset S_H^c$ . In particular,  $-\check{X}(\lambda)^{-1} \in S_H^c$ .

Moreover, from the following formula:

$$\forall z \in \mathbb{C}_+, m_{LD} \left( \frac{z}{c\check{\Theta}^{(1)}(z)} \right) = c - c \int \frac{1}{\tau X(z) + 1} dH(\tau), \quad (34)$$

we prove that, as  $z \in \mathbb{C}_+ \rightarrow x_0 \in \mathbb{R}^*$ , the right side converges to a real number  $c - c \int \frac{1}{\tau \check{X}(x_0) + 1} dH(\tau)$ , because we have just proved that  $-\check{X}(x_0)^{-1} \in S_H^c$  and D.C.T finishes it.

Suppose that  $\check{\Theta}^{(1)}(x_0) \neq 0$ . This implies that, as  $\frac{z}{c\check{\Theta}^{(1)}(z)} \rightarrow \frac{x_0}{c\check{\Theta}^{(1)}(x_0)} \in \mathbb{R}^*$ ,  $m_{LD} \left( \frac{z}{c\check{\Theta}^{(1)}(z)} \right)$  converges to a real number. So, similarly to what we did in the proof of Lemma 7, we have that  $D_- D \left( \frac{x_0}{c\check{\Theta}^{(1)}(x_0)} \right) = 0$ .  $S_F^c$  is open, and  $x_0 \mapsto \frac{x_0}{c\check{\Theta}^{(1)}(x_0)}$  is continuous and non-constant, so  $\left\{ \frac{x_0}{c\check{\Theta}^{(1)}(x_0)}, x_0 \in S_F^c \right\}$  is open. So,  $\left\{ \frac{x_0}{c\check{\Theta}^{(1)}(x_0)} \mid x_0 \in S_F^c, \check{\Theta}^{(1)}(x_0) \neq 0 \right\} \subset S_D^c$ .  $\square$

The following lemma finishes the proof of Corollary 1.

**Lemma 14.** For  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$  and  $g : [h_1, h_2] \rightarrow \mathbb{R}$  bounded with a finite number of discontinuity points:

$$\begin{aligned} \check{m}(\lambda) &= -\frac{1}{\lambda} \int \frac{1}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \check{\Theta}^{(1)}(\lambda) &= -\frac{1}{\lambda} \int \frac{\tau}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \text{Im}[\check{\Theta}^g(\lambda)] &= \frac{1}{\lambda} \int \frac{g(\tau)\tau \text{Im}[\check{X}(\lambda)]}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau). \end{aligned}$$

*Proof.* Let  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$ . If  $F'(\lambda) > 0$ , then  $\check{m}(\lambda) \in \mathbb{C}_+$ , so  $\check{X}(\lambda) \in \mathbb{C}_+$ . By D.C.T, we immediatly conclude that:

$$\begin{aligned} \check{m}(\lambda) &= -\frac{1}{\lambda} \int \frac{1}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \check{\Theta}^{(1)}(\lambda) &= -\frac{1}{\lambda} \int \frac{\tau}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \text{Im}[\check{\Theta}^g(\lambda)] &= \frac{1}{\lambda} \int \frac{g(\tau)\tau \text{Im}[\check{X}(\lambda)]}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau). \end{aligned}$$

Otherwise,  $F'(\lambda) = 0$ . As  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$ , we have that  $\lambda \in S_F^c$  and  $\check{X}(\lambda) \in \mathbb{R}$ . Using Lemma 13, we have that  $-\check{X}(\lambda)^{-1} \in S_H^c$ . Then, by D.C.T, we deduce immediatly:

$$\begin{aligned} \check{m}(\lambda) &= -\frac{1}{\lambda} \int \frac{1}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \check{\Theta}^{(1)}(\lambda) &= -\frac{1}{\lambda} \int \frac{\tau}{\tau \check{X}(\lambda) + 1} dH(\tau), \\ \text{Im}[\check{\Theta}^g(\lambda)] &= \frac{1}{\lambda} \int \frac{g(\tau)\tau \text{Im}[\check{X}(\lambda)]}{|\tau \check{X}(\lambda) + 1|^2} dH(\tau). \end{aligned}$$

$\square$

### 9.3 Proof of Theorem 3

Let us proof the equivalence in two steps. Assume H1-H5 and let  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$ .

We prove firstly that  $f_\lambda(\check{X}(\lambda)) = 0$ . If  $\check{X}(\lambda) \in \mathbb{C}_+$ , this is immediate by D.C.T. Otherwise, if  $\check{X}(\lambda) \in \mathbb{R}$ , then  $\lambda \in S_F^c$  because  $\lambda \in \mathbb{R}^* \setminus \{x \in S_F, F'(x) = 0\}$ . Using Lemma 13, we have that  $\check{X}(\lambda) \neq 0$ ,  $-\check{X}(\lambda)^{-1} \in S_H^c$ . If  $\check{\Theta}^{(1)} \neq 0$ , then  $\frac{\lambda}{c\check{\Theta}^{(1)}(\lambda)} \in S_D^c$ . So, by D.C.T., we have that  $f_\lambda(\check{X}(\lambda)) = 0$ . Otherwise, if  $\check{\Theta}^{(1)} = 0$ , we also have directly by D.C.T. that  $f_\lambda(\check{X}(\lambda)) = 0$ , which concludes the first part of the proof.

For the direct way, suppose that  $\check{X}(\lambda) \in \mathbb{C}_+$ . Then, by D.C.T, we have immediatly that  $f_\lambda(\check{X}(\lambda)) = 0$ . So  $f_\lambda(X) = 0$  has indeed at least one solution in  $\mathbb{C}_+$ .

For the other way, suppose now that there exists  $X_0 \in \mathbb{C}_+$  such that  $f_\lambda(X_0) = 0$ . We are using the Theorem 9.1.1 [14] to conclude. In order to apply it, we have that:

- $f_\lambda(X_0) = 0$ ,
- $(x, X) \mapsto f_x(X)$  is analytic on an open of  $\mathbb{C}^2$  containing  $(\lambda, X_0)$ ,
- $f_\lambda(\cdot)$  is not identically 0 in a neighborhood of  $X_0$  (because of the imaginary part for example).

Then, there exists  $\delta > 0, \epsilon > 0, m \in \mathbb{N}^*$  such that for all  $z \in \mathbb{C}, |z - x_0| < \epsilon, f_z(\cdot)$  has exactly  $m$  zeros in  $\{X \in \mathbb{C}, |X - X_0| < \delta\}$  and  $X_{k_j}(z) = X_0 + \sum_{n=1}^{\infty} a_{k_n} \left( ((z - \lambda)^{1/p_k})_j \right)^n$ . We refer to Section 9 [14] for the notation.

So, for  $|z - \lambda|$  small enough,  $X_{k_j}(z) \in \mathbb{C}_+$  as  $X_0 \in \mathbb{C}_+$  and  $X_{k_j}$  continuous. Moreover,  $f_z(X_{k_j}(z)) = 0$  for all  $z \in \mathbb{C}_+$  and  $|z - \lambda|$  small enough. By unicity of the solution of  $f_z(\cdot) = 0$  when  $z \in \mathbb{C}_+$ , we deduce that  $\forall k_j \in \llbracket 1, m \rrbracket, X_{k_j}(z) = X(z)$ .

In conclusion, that  $\forall k_j \in \llbracket 1, m \rrbracket, X_{k_j}(z) \xrightarrow{z \rightarrow x_0} X_0$  by continuity of  $X_{k_j}$ , and  $\forall k_j \in \llbracket 1, m \rrbracket, X_{k_j}(z) \xrightarrow{z \rightarrow x_0} \check{X}(\lambda)$  by continuity of  $X(\cdot)$  on  $\mathbb{C}^*$ . So,  $\check{X}(\lambda) = X_0 \in \mathbb{C}_+$ , which concludes the proof.

## 9.4 Proof of Proposition 2

*Proof.* Firstly, we have that  $h : X \in B \mapsto cX \int \frac{\tau}{\tau X + 1} dH(\tau)$  is analytic so is in  $\mathcal{C}^\infty(B, \mathbb{R})$ . We can write:

$$x_F : X \in B \mapsto \begin{cases} \frac{h(X)}{X} m_{LD}^{-1}(h(X)) & , \text{if } h(X) \neq 0, \\ -\frac{1}{X} \int \delta dD(\delta) & , \text{otherwise.} \end{cases} \quad (35)$$

Let  $y \in \mathbb{R}^*$  and  $x := m_{LD}^{-1}(y)$ . Then, by Taylor extension as  $y \rightarrow 0$ , possible as  $S_D = [d_1, d_2]$ , we have:

$$\begin{aligned} yx &\sim - \int \delta dD(\delta), \\ \frac{yx + \int \delta dD(\delta)}{y} &\sim \frac{\int \delta^2 dD(\delta)}{\int \delta dD(\delta)}. \end{aligned} \quad (36)$$

So,

$$y \in \mathbb{R} \mapsto \begin{cases} y m_{LD}^{-1}(y) & , \text{if } y \neq 0, \\ -y \int \delta dD(\delta) & , \text{otherwise.} \end{cases} \quad (37)$$

is in  $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$ . So, by composition,  $x_F \in \mathcal{C}^1(B, \mathbb{R})$ .  $\square$

## 9.5 Proof of Theorems 4 and 5

### 9.5.1 Proof of Theorem 4

Suppose the assumptions of Theorem 4.

**Lemma 15.** *For any  $x_0 \in S_F^c, x_0 \neq 0$ , let  $X_0 = \check{X}(x_0)$ . Then  $X_0 \in B := \{X \in \mathbb{R} : X \neq 0, -X^{-1} \in S_H^c\}$ ,  $x_0 = x_F(X_0)$  and  $x_F'(X_0) > 0$ .*

*Proof.* For  $z \in \mathbb{C}_+$ ,  $X$  is differentiable - because analytic -, and we have from the functional equation on  $X$ :

$$X'(z) = \frac{\int \frac{\delta}{(z - \delta c_{\check{\Theta}^{(1)}}(z))^2} dD(\delta)}{1 + \int \frac{\tau^2}{(\tau X(z) + 1)^2} dH(\tau) \times \int \frac{\delta^2 c}{(z - \delta c_{\check{\Theta}^{(1)}}(z))^2} dD(\delta)} \quad (38)$$

Let  $x_0 \in S_F^c, x_0 \neq 0$ . Then,  $\text{Im}[\check{X}(x_0)] = 0$ . Using Lemma 13, we have that  $-\check{X}(x_0)^{-1} \in S_H^c$  and if  $\check{\Theta}^{(1)}(x_0) \neq 0$ , then  $\frac{x_0}{c_{\check{\Theta}^{(1)}}(x_0)} \in S_D^c$ .

We can prove now by D.C.T, using the previous result if  $\Theta^{(1)}(x_0) \neq 0$  or using directly D.C.T if  $\check{\Theta}^{(1)}(x_0) = 0$  that:

$$X'(z) \xrightarrow{z \rightarrow x_0} \check{X}'(x_0) := \frac{\int \frac{\delta}{(x_0 - \delta c \check{\Theta}^{(1)}(x_0))^2} dD(\delta)}{1 + \int \frac{\tau^2}{(\tau \check{X}(x_0) + 1)^2} dH(\tau) \times \int \frac{\delta^2 c}{(x_0 - \delta c \check{\Theta}^{(1)}(x_0))^2} dD(\delta)} \in \mathbb{R}_+^*. \quad (39)$$

As  $X$  is continuous on  $\mathbb{C}_+ \cup S_F^c \setminus \{0\}$ , differentiable on  $\mathbb{C}_+$  and  $X'$  admits a limit on  $S_F^c \setminus \{0\}$ , then  $X$  is differentiable on  $\mathbb{C}_+ \cup S_F^c \setminus \{0\}$  and is derivative on  $S_F^c \setminus \{0\}$  is indeed  $\check{X}'$ .

Similarly, for  $x_0 \in S_F^c \setminus \{0\}$  and  $X_0 := \check{X}(x_0)$  and supposing that  $\check{\Theta}^{(1)}(x_0) \neq 0$ , we also prove by D.C.T, for the same reasons, that:

$$m_{LD} \left( \frac{x_0}{c \int \frac{\tau}{\tau X_0 + 1} dH(\tau)} \right) = c X_0 \int \frac{\tau}{\tau X_0 + 1} dH(\tau). \quad (40)$$

As  $\frac{x_0}{c \int \frac{\tau}{\tau X_0 + 1} dH(\tau)} \in S_D^c$ ,  $m_{LD}$  is invertible at this point, so  $x_0 = x_F(\check{X}(x_0))$ .

Now supposing that  $\check{\Theta}^{(1)}(x_0) = 0$ , we have by D.C.T on (23) that:

$$X_0 = \check{X}(x_0) = -\frac{1}{x_0} \int \delta dD(\delta). \quad (41)$$

So, in this case we also have that  $x_0 = x_F(\check{X}(x_0))$ .

So,  $x_F$  is indeed the inverse of  $\check{X}$  on  $S_F^c \setminus \{0\}$ , and as  $\check{X}'(x_0) > 0$ , we conclude that  $x'_F(X_0) > 0$ .  $\square$

**Lemma 16.** *Suppose that  $X_0 \in B$  and  $x'_F(X_0) > 0$ . Let  $x_0 = x_F(X_0)$ . Then,  $x_0 \in S_F^c$ ,  $\lim_{z \in \mathbb{C}_+ \rightarrow x_0} X(z) =: \check{X}(x_0)$  exists, and  $\check{X}(x_0) = X_0$ .*

*Proof.* Let  $X_0 \in B$  such that  $x'_F(X_0) > 0$ , and let  $x_0 = x_F(X_0)$ . Firstly, remark that  $m_{LD}$  - seen as an analytic function on  $\mathbb{C} \setminus S_D$  - is locally invertible at any  $y \in S_D^c$ ,  $y$  real. Indeed, for such  $y$ ,  $m'_{LD}(y) = \int \frac{\delta}{(\delta - y)^2} dD(\delta) > 0$ .

We prove the result for  $X_0 \in B$  such that  $\int \frac{\tau}{\tau X_0 + 1} dH(\tau) \neq 0$  firstly, and we will conclude using the continuity of  $F'$ .

Assuming  $\int \frac{\tau}{\tau X_0 + 1} dH(\tau) \neq 0$ , we define:  $y_0 := \frac{x_0}{c \int \frac{\tau}{\tau X_0 + 1} dH(\tau)}$ . Remark that  $y_0 \in S_D^c$  because  $y_0 = m_{LD}^{-1} \left( c \int \frac{\tau X_0}{\tau X_0 + 1} dH(\tau) \right)$  by definition. Then,  $m_{LD}$  is locally invertible in a neighborhood  $N$  of  $\mathbb{C}$  containing  $y_0$ , we denote its local inverse by  $g_{LD}$ .  $g_{LD}$  is analytic as  $m_{LD}$  is analytic. Obviously, on  $m_{LD}(N \cap \mathbb{R})$ , which is open in  $\mathbb{R}$ , we have  $g_{LD} = m_{LD}^{-1}$ . So, on  $m_{LD}(N)$  which is open in  $\mathbb{C}$ ,  $g_{LD}$  extends analytically  $m_{LD}^{-1}$ .

We define now an extension of  $x_F$ , denoted  $z_F$ , defined on an open  $N' := \left\{ X \in \mathbb{C} \mid X \neq 0, X \notin S_H, c \int \frac{\tau X}{\tau X + 1} dH(\tau) \in m_{LD}(N) \right\}$  of  $\mathbb{C}$  containing  $X_0$ :

$$z_F : X \in N' \mapsto c \int \frac{\tau}{\tau X + 1} dH(\tau) g_{LD} \left( c \int \frac{\tau X}{\tau X + 1} dH(\tau) \right). \quad (42)$$

$N'$  is open because  $X \mapsto c \int \frac{\tau X}{\tau X + 1} dH(\tau)$  is analytic on  $\mathbb{C} \setminus S_H$  so continuous, and non constant.  $X_0 \in N'$  by construction.

As  $z'_F(X_0) = x'_F(X_0)$  because both functions coincides on the real line and  $z_F$  is differentiable because analytic, we have that  $z'_F(X_0) > 0$ , so  $z_F$  is locally invertible at  $X_0$  on an open ball  $B(X_0, \epsilon)$  for some  $\epsilon > 0$ , and we denote its local inverse  $\check{X}$ .

We prove that there exists  $\epsilon' \in ]0, \epsilon]$  such that  $\forall X \in B(X_0, \epsilon')$ ,  $z_F(X) \in \mathbb{R} \iff X \in \mathbb{R}$ . For that, suppose there exists a sequence  $(X_n)_{n \geq 1}$  such that  $\forall n \in \mathbb{N}^*$ ,  $X_n \in B(X_0, \epsilon) \setminus \mathbb{R}$ ,  $X_n \rightarrow X_0$  and

$z_F(X_n) \in \mathbb{R}$ .  $z_F$  and  $x_F$  coincides on the real line  $z_F(\cdot|X_0 - \epsilon, X_0 + \epsilon) = x_F(\cdot|X_0 - \epsilon, X_0 + \epsilon)$ . Moreover, as  $x'_F(X_0) > 0$ ,  $x_F$  is locally invertible (in the  $\mathbb{R}$ -topology) at  $X_0$ , thus there exists  $\eta \in ]0, \epsilon]$  such that  $x_F(\cdot|X_0 - \eta, X_0 + \eta)$  is an open interval and  $x_F(X_0) \in x_F(\cdot|X_0 - \eta, X_0 + \eta)$ . So  $z_F(\cdot|X_0 - \eta, X_0 + \eta)$  is an open interval of  $\mathbb{R}$  and  $z_F(X_0) \in z_F(\cdot|X_0 - \eta, X_0 + \eta)$ . As  $X_n \rightarrow X_0$ , by continuity, we have  $z_F(X_n) \rightarrow z_F(X_0)$ . So, as  $z_F(X_n) \in \mathbb{R}$ , for  $n$  large enough, we have that  $z_F(X_n) \in z_F(\cdot|X_0 - \eta, X_0 + \eta)$ . However,  $X_n \notin ]X_0 - \eta, X_0 + \eta[$  by assumption. So,  $z_F$  is not injective on  $B(X_0, \epsilon)$ , which is contradictory with the fact that it is invertible. So there exists  $\epsilon' \in ]0, \epsilon]$  such that  $\forall X \in B(X_0, \epsilon')$ ,  $z_F(X) \in \mathbb{R} \iff X \in \mathbb{R}$ .

We deduce then by continuity that either  $z_F(B(X_0, \epsilon') \cap \mathbb{C}_+) \subset \mathbb{C}_+$ , or  $z_F(B(X_0, \epsilon') \cap \mathbb{C}_+) \subset \mathbb{C}_-$ . Because  $z'_F(X_0) > 0$ , we conclude that  $z_F(B(X_0, \epsilon') \cap \mathbb{C}_+) \subset \mathbb{C}_+$ . So, for all  $X \in B(X_0, \epsilon') \cap \mathbb{C}_+$ , we have that  $z_F(X) \in \mathbb{C}_+$  and  $X = -\int \frac{\delta}{z_F(X) - \delta c \int \frac{\tau}{\tau X + 1} dH(\tau)} dD(\delta)$ . In other words,  $X$  solve (23) at  $z_F(X) \in \mathbb{C}_+$ . By unicity of the solution of (23) in  $\mathbb{C}_+$ , we have that  $X = X(z_F(X))$ . Moreover, we also have by local invertibility that  $X = \hat{X}(z_F(X))$ . Then, by continuity of  $\hat{X}$ , for all  $x \in z_F(\cdot|X_0 - \epsilon', X_0 + \epsilon')$ ,  $X(z) \xrightarrow{z \in \mathbb{C}_+ \rightarrow x} \hat{X}(x) \in \mathbb{R}$  (it is real

because we proved  $z_F(X) \in \mathbb{R} \iff X \in \mathbb{R}$ ). In particular, we have that  $\check{X}(x_0)$  exists and  $\check{X}(x_0) = \hat{X}(x_0) = X_0$ . We remind that  $z_F(\cdot|X_0 - \epsilon', X_0 + \epsilon')$  contains an open (of  $\mathbb{R}$ ) containing  $x_0$  because  $z'_F(X_0) > 0$  and  $z_F(X_0) = x_0$ . Then, as  $\check{X}(x_0) = X_0 \in B$ ,  $B$  is an open of  $\mathbb{R}$  and  $\check{X}$  is continuous, there is an open interval  $I \subset z_F(\cdot|X_0 - \epsilon', X_0 + \epsilon')$  such that  $x_0 \in I$  and  $\forall x \in I$ ,  $\hat{X}(x) \in B$ .

As a consequence, by D.C.T. from the fact that  $\forall x \in I$ ,  $\hat{X}(x) \in B$ , we have that  $\forall x \in I$ ,  $m(z) \xrightarrow{z \in \mathbb{C}_+ \rightarrow x} \frac{1}{x} \int \frac{1}{\tau \hat{X}(x) + 1} dH(\tau) \in \mathbb{R}$ . Therefore, by Theorem 2.1 [9],  $\forall x \in I$ ,  $F'(x) = 0$ .

So  $I \subset S_F^c$ , and so  $x_0 \in S_F^c$ , which concludes the proof in the case  $\int \frac{\tau}{\tau X_0 + 1} dH(\tau) \neq 0$ .

Finally, suppose  $\int \frac{\tau}{\tau X_0 + 1} dH(\tau) = 0$ . Then, as  $X \mapsto \int \frac{\tau}{\tau X + 1} dH(\tau)$  is analytic non-constant on a neighborhood of  $X_0$ , there exists a real neighborhood  $I' \subset B$  of  $X_0$  such that  $\forall X \in I' \setminus \{X_0\}$ ,  $\int \frac{\tau}{\tau X + 1} dH(\tau) \neq 0$ . By the previous proof we know that  $\forall X \in I' \setminus \{X_0\}$ ,  $\check{X}(x_F(X)) = X$ . Moreover, as  $\int \frac{\tau}{\tau X_0 + 1} dH(\tau) = 0$ , we have that  $x_0 := x_F(X_0) = -\frac{1}{X_0} \int \delta dD(\delta) \in \mathbb{R}^*$ . As  $\check{X}$  is well-defined and continuous on  $\mathbb{R}^*$  from Lemma 9, we have that  $\check{X}(x_0)$  is well-defined and  $\check{X}$  continuous at  $x_0$ . By continuity of  $x_F$  on  $I' \cap B$  from Proposition 2, we deduce that  $\check{X}(x_0) = X_0 \in B$ . So,  $\forall X \in I'$ ,  $\check{X}(x_F(X)) = X$ . As  $x'_F(X_0) > 0$ ,  $x_F(I')$  contains an open interval  $J$  containing  $x_0$ . For all  $x \in J$ , we prove by D.C.T. that  $m(z) \xrightarrow{z \in \mathbb{C}_+ \rightarrow x} \frac{1}{x} \int \frac{1}{\tau \hat{X}(x) + 1} dH(\tau) \in \mathbb{R}$ , which implies that  $F'(x) = 0$ . So,  $x_0 \in S_F^c$  which concludes the proof.  $\square$

### 9.5.2 Proof of Theorem 5

Suppose the assumptions of Theorem 5.

**Lemma 17.** *For any  $x_0 \in S_F^c$ ,  $x_0 \neq 0$ , let  $X_0 = \check{X}(x_0)$ . Then  $X_0 \in B := \{X \in \mathbb{R} : X \neq 0, -X^{-1} \in S_H^c\}$ , and there exists  $k \in \llbracket 1, M \rrbracket$  such that  $x_0 = x_F^{(k)}(X_0)$  and  $x_F^{(k)'}(X_0) > 0$ .*

*Proof.* We apply the same method as the proof of Lemma 15. We develop here only the parts that differ. We consider firstly the case where  $\check{\Theta}^{(1)}(x_0) \neq 0$ . Then, we prove that  $y_0 := \frac{x_0}{c \int \frac{\tau}{\tau X_0 + 1} dH(\tau)} \in S_D^c$ .

So by assumption on  $S_D$ , either there exists  $k \in \llbracket 1, M - 1 \rrbracket$  such that  $y_0 \in I^{(k)} := ]\delta_2^{(k)}, \delta_1^{(k+1)}[$  or for  $k = M$ ,  $y_0 \in I^{(M)} = ]-\infty, \delta_1^{(1)}[ \cup ]\delta_2^{(M)}, +\infty[$ . As  $y_0 \in S_D^c$ ,  $m_{LD}$  is invertible at this point and  $m_{LD} = m_{LD}^{(k)}$  in a neighborhood of  $y_0$ , so we deduce that  $x_0 = x_F^{(k)}(X_0)$ .

By continuity, we also have in a neighborhood  $N$  of  $x_0$  that  $\forall x \in N$ ,  $\frac{x}{c \int \frac{\tau}{\tau \hat{X}(x) + 1} dH(\tau)} \in I^{(k)}$  and  $x = x_F^{(k)}(\hat{X}(x))$ .

If  $\check{\Theta}^{(1)}(x_0) = 0$ , then by D.C.T:

$$X_0 = \check{X}(x_0) = -\frac{1}{x_0} \int \delta dD(\delta). \quad (43)$$

So in this case we have that  $x_0 = x_F^{(M)}(\check{X}(x_0))$ . By continuity of  $\Theta^{(1)}$ , we also have in a neighborhood  $N$  of  $x_0$  that  $\forall x \in N, \frac{\tau}{\tau\check{X}(x)+1}dH(\tau) = 0$  or  $\frac{x}{c \int \frac{\tau}{\tau\check{X}(x)+1}dH(\tau)} \in I^{(M)}$ . So,  $\forall x \in N, x = x_F^{(M)}(\check{X}(x))$ .

So,  $x_F^{(k)}$ , with the appropriate  $k$  chosen above, is indeed the inverse of  $\check{X}$  in a neighborhood of  $x_0$ . As  $\check{X}'(x_0) > 0$  (see the proof of Lemma 15), we conclude that  $x_F^{(k)'}(X_0) > 0$  and it completes the proof. □

**Lemma 18.** *Suppose that  $X_0 \in B$  and there exists  $k \in \llbracket 1, M \rrbracket$  such that  $x_F(k)'(X_0) > 0$ . Let  $x_0 = x_F^{(k)}(X_0)$ . Then,  $x_0 \in S_F^c$  and  $\check{X}(x_0) = X_0$ .*

*Proof.* We use the same proof as Lemma 16, replacing  $x_F$  by  $x_F^{(k)}$ , and the result follows. □