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# ON LOCAL COMPACTNESS OF SPACES OF CONTINUOUS VALUATIONS

JEAN GOUBAULT-LARRECQ

ABSTRACT. We show that the spaces of continuous valuations, resp. subprobability valuations on a locally compact space is locally compact; similarly with probability valuations on locally compact, compact spaces. Continuous valuations are close cousins of measures. No separation property is assumed.

## 1. Introduction

A *continuous valuation* on a topological space  $X$  is a close cousin of a Borel measure. Continuous valuations on  $X$  form a space  $\mathbf{V}X$ , with a topology known as the weak topology, and similarly for the subspaces  $\mathbf{V}_{\leq 1}X$  of subprobability valuations and  $\mathbf{V}_1X$  of probability valuations. We will define these notions more precisely below. It is known that  $\mathbf{V}_{\leq 1}$  preserves various properties: stable compactness [2, Theorem 39], being a continuous dcpo [16, Theorem 5.2], being a quasi-continuous dcpo [14, Theorem 5.1], for example. Some of these preservation theorems extend over to  $\mathbf{V}_1$  or to  $\mathbf{V}$ , but a conspicuously absent property in the list is local compactness. This is what we address in this paper.

The proof relies on credibilities, as already used in [14], on the Smyth powerdomain construction [23], on Jones' celebrated theorem that the probabilistic powerdomain preserves continuous dcpos [16, Theorem 5.2], and on a few things about certain functionals called superlinear previsions [12]. As a whole, the proof is therefore rather elaborate; but it also results from little more than an assemblage of previously known results, and as a consequence, is pretty short.

## 2. Preliminaries

For background on topology, we refer the reader to [11].

A *compact* subset  $A$  of a space  $X$  is one such that one can extract a finite subcover from any of its open covers. No separation property is

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assumed. A subset  $A$  of  $X$  is *saturated* if and only if it is equal to the intersection of its open neighborhoods, or equivalently if and only if it is upwards-closed in the specialization preordering of  $X$ . The latter is defined by  $x \leq y$  if and only if every open neighborhood of  $x$  contains  $y$ .

A space  $X$  is *locally compact* if and only if every point has a base of compact neighborhoods, or equivalently of compact saturated neighborhoods, since for any compact subset  $K$  of  $X$ , the upward closure  $\uparrow K$  of  $K$  with respect to the specialization preordering of  $X$  is compact saturated. In non-Hausdorff spaces, a compact space may fail to be locally compact.

A space is  $T_0$  if and only if its specialization preordering is antisymmetric. An *irreducible* closed subset  $C$  of  $X$  is a non-empty closed subset such that, for any two closed subsets  $C_1$  and  $C_2$  of  $X$  such that  $C \subseteq C_1 \cup C_2$ ,  $C$  is included in  $C_1$  or in  $C_2$  already; equivalently, if  $C$  intersects two open sets, it must intersect their intersection. A space  $X$  is *sober* if and only if it is  $T_0$  and every irreducible closed subset is of the form  $\downarrow x$  for some point  $x \in X$ . Every Hausdorff space, for example, is sober. The notation  $\downarrow x$  stands for the *downward closure* of  $x$  in  $X$ , namely the set of points  $y$  below  $x$ ; that is also the closure of  $\{x\}$ . Symmetrically,  $\uparrow x$  stands for the *upward closure* of  $x$ , namely the set of points  $y$  above  $x$ . This notation extends to  $\uparrow A$ , for any subset  $A$ , denoting  $\bigcup_{x \in A} \uparrow x$ .

A function  $f: P \rightarrow Q$  between posets is *monotonic* if and only if for all  $x, x' \in P$ ,  $x \leq x'$  implies  $f(x) \leq f(x')$ . It is *Scott-continuous* if and only if  $f$  is monotonic and for every directed family  $(x_i)_{i \in I}$  with a supremum  $x$  in  $P$ , the (necessarily directed) family of elements  $f(x_i)$  has  $f(x)$  as supremum. A *directed* family is a non-empty family  $D$  such that any two elements of  $D$  have an upper bound in  $D$ . We write  $\sup^\uparrow D$  for the supremum of a directed family  $D$ . Scott-continuity is equivalent to continuity with the respective Scott topologies on  $P$  and  $Q$ . The *Scott topology* on a poset  $P$  consists of those subsets  $U$  that are upwards closed ( $x \in U$  and  $x \leq x'$  implies  $x' \in U$ ) and such that every directed family  $D$  that has a supremum in  $U$  intersects  $U$ .

A *dcpo* (short for *directed-complete partial order*) is a poset  $P$  in which every directed subset has a supremum. In a dcpo  $P$ , let  $x \ll y$  (“ $x$  is *way below*  $y$ ”) if and only if every directed family  $D$  such that  $y \leq \sup^\uparrow D$  contains an element  $d \in D$  such that  $x \leq d$ . A *basis* for  $P$  is a subset  $B$  such that, for every  $x \in P$ ,  $\downarrow_B x \stackrel{\text{def}}{=} \{b \in B \mid b \ll x\}$  is directed and has  $x$  as its supremum. A dcpo is *continuous* if and only

if it has a basis. If so, the sets  $\uparrow b \stackrel{\text{def}}{=} \{x \in P \mid b \ll x\}$  form a base of the Scott topology.

Let  $\overline{\mathbb{R}}_+$  be the set of extended non-negative real numbers  $\mathbb{R}_+ \cup \{\infty\}$ , with its usual ordering. When needed, we will consider it with its Scott topology, whose open sets are the intervals  $]t, \infty]$ ,  $t \in \mathbb{R}_+$ , plus  $\emptyset$  and  $\overline{\mathbb{R}}_+$  itself. In passing,  $\overline{\mathbb{R}}_+$  is an example of a continuous dcpo, with  $s \ll t$  if and only if  $s = 0$  or  $s < t$ .

We write  $\mathcal{O}X$  for the lattice of open subsets of a space  $X$ . A *continuous valuation* on  $X$  is a map  $\nu: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$  that is *strict* ( $\nu(\emptyset) = 0$ ), *modular* (for all  $U, V \in \mathcal{O}X$ ,  $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$ ) and Scott-continuous. We say that  $\nu$  is *bounded* if  $\nu(X) < \infty$ , a *probability valuation* if and only if  $\nu(X) = 1$ , and a *subprobability valuation* if and only if  $\nu(X) \leq 1$ .

Continuous valuations are an alternative to measures that have become popular in domain theory [17, 16]. The first results that connected continuous valuations and measures are due to Saheb-Djahromi [21] and Lawson [19]. The current state of the art on this matter is the following. In one direction, every measure on the Borel  $\sigma$ -algebra of  $X$  induces a continuous valuation on  $X$  by restriction to the open sets, if  $X$  is hereditarily Lindelöf (namely, if every directed family of open sets contains a cofinal monotone sequence). This is an easy observation, and one half of Adamski's theorem [1, Theorem 3.1], which states that a space is hereditary Lindelöf if and only if every measure on its Borel  $\sigma$ -algebra restricts to a continuous valuation on its open sets. In the other direction, every continuous valuation on a space  $X$  extends to a measure on the Borel sets provided that  $X$  is an LCS-complete space [6, Theorem 1]. An LCS-complete space is a  $G_\delta$  subspace of a locally compact sober space, and the class of LCS-complete spaces contains all locally compact sober spaces, in particular all continuous dcpos from domain theory, all of M. de Brecht's quasi-Polish spaces [13] and therefore all Polish spaces.

The extension of a continuous valuation  $\nu$  to a measure on the Borel  $\sigma$ -algebra is unique if  $\nu$  is bounded. This is a consequence of the  $\pi$ - $\lambda$  theorem [3, Theorem 3.2], usually applied to probability measures [3, Theorem 3.3], but the same proof works for bounded measures.

Hence the above maps from Borel measures to continuous valuations and conversely, defined on hereditary Lindelöf LCS-complete spaces (in particular, quasi-Polish spaces), are inverse of each other when restricted to bounded measures and bounded continuous valuations.

Let  $\mathbf{V}X$  denote the space of continuous valuations on a space  $X$ , with the *weak topology*, defined by subbasic open sets  $[U > r] \stackrel{\text{def}}{=} \{\nu \in$

$\mathbf{V}X \mid \nu(U) > r\}$ , where  $U \in \mathcal{O}X$  and  $r \in \mathbb{R}_+$ . We define its subspace  $\mathbf{V}_1X$  of probability valuations and  $\mathbf{V}_{\leq 1}X$  (subprobability) similarly. In general, we write  $\mathbf{V}_\bullet X$ , where  $\bullet$  can be nothing, “ $\leq 1$ ”, or “1”. The specialization ordering of each is the *stochastic ordering*  $\leq$  given by  $\nu \leq \nu'$  if and only if  $\nu(U) \leq \nu'(U)$  for every  $U \in \mathcal{O}X$ ; indeed,  $\nu \leq \nu'$  if and only if for every  $U \in \mathcal{O}X$ , for every  $r \in \mathbb{R}_+$ ,  $\nu \in [U > r]$  implies  $\nu' \in [U > r]$ .

There is a related topology on spaces of probability measures. A sequence of probability measures  $P_n$ ,  $n \in \mathbb{N}$ , on a metric space *converges weakly* to a probability measure  $P$  if and only if  $\liminf_{n \in \mathbb{N}} P_n(U) \geq P(U)$  for every open subset  $U$  [4, Portmanteau Theorem 2.1]. It is easy to see that this is equivalent to: for every open subset  $U$  and for every  $r \in \mathbb{R}_+$ ,  $P(U) > r$  implies that  $P_n(U) > r$  for  $n$  large enough. Hence we may define the *weak topology* on spaces of measures (not just probability measures) on a topological space  $X$  (not just a metric space) as given via a subbasis of sets  $\{P \mid P(U) > r\}$ ,  $U \in \mathcal{O}X$ ,  $r \in \mathbb{R}_+$ . We obtain the following.

**Proposition 2.1.** *The weak topology makes the spaces  $\mathcal{M}_{\leq 1}X$  and  $\mathcal{M}_1X$  of subprobability, resp. probability measures homeomorphic to  $\mathbf{V}_{\leq 1}X$  and  $\mathbf{V}_1X$ , respectively, for every hereditarily Lindelöf LCS-complete space  $X$ .*

$\mathbf{V}_\bullet$  is the object part of an endofunctor on the category of topological spaces, whose morphism part is defined as follows: for every continuous map  $f: X \rightarrow Y$ ,  $\mathbf{V}_\bullet f$  maps every  $\nu \in \mathbf{V}_\bullet X$  to its *image valuation*  $f[\nu]$ , where  $f[\nu](V) \stackrel{\text{def}}{=} \nu(f^{-1}(V))$  for every  $V \in \mathcal{O}Y$ .

We will prove slightly more than the fact that  $\mathbf{V}_\bullet$  preserves local compactness, where  $\bullet$  is nothing, “ $\leq 1$ ”, or “1”. A space  $X$  is *core-compact* if and only if  $\mathcal{O}X$  is a continuous dcpo; every locally compact space is core-compact [11, Theorem 5.2.9]. The connection between the two notions can be made more precise as follows. Every topological space  $X$  has a *sobrification*  $\mathcal{S}(X)$  (or  $X^s$ ), which is the free sober space over  $X$  [11, Theorem 8.2.44]; then  $X$  is core-compact if and only if  $\mathcal{S}(X)$  is locally compact [11, Proposition 8.3.11].  $\mathcal{S}(X)$  can be built as the collection of irreducible closed subsets, with the topology whose open sets (all of them, not just a base) are  $\diamond U \stackrel{\text{def}}{=} \{F \in \mathcal{S}(X) \mid F \cap U \neq \emptyset\}$ ,  $U \in \mathcal{O}X$ . In particular,  $\diamond: U \mapsto \diamond U$  is an order-isomorphism between  $\mathcal{O}X$  and  $\mathcal{O}\mathcal{S}(X)$ . This induces a homeomorphism between  $\mathbf{V}_\bullet X$  and  $\mathbf{V}_\bullet \mathcal{S}(X)$ .

### 3. Credibilities, previsions and the Smyth powerdomain

The proof of our main theorem will rely on reducing the question to spaces of valuations on Smyth powerdomains, and on using notions of credibilities [10] and of previsions [9, 12]. We need to give a quick tour of the part of that theory that we need.

Let  $\mathcal{L}X$  denote the dcpo of all lower semicontinuous maps from  $X$  to  $\overline{\mathbb{R}}_+$ ; a map  $f: X \rightarrow \overline{\mathbb{R}}_+$  is lower semicontinuous if and only if it is continuous, where  $\overline{\mathbb{R}}_+$  is given its Scott topology. The ordering on  $\mathcal{L}X$  is the pointwise ordering. For every monotone map  $\nu: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ , for every  $h \in \mathcal{L}X$ , there is a *Choquet integral*  $\int_{x \in X} h(x) d\nu$ , defined as the indefinite Riemann integral  $\int_0^\infty \nu(h^{-1}(]t, \infty])) dt$ . We need to reprove in part some of the results of [10] on Choquet integrals, since that paper only considers integrals of maps from  $X$  to  $\mathbb{R}_+$ ; we will do this in Lemma 3.1 below.

A *prevision* on a space  $X$  is a Scott-continuous map  $F: \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$  that is *positively homogeneous* in the sense that  $F(ah) = aF(h)$  for all  $a \in \mathbb{R}_+$  and  $h \in \mathcal{L}X$ . There is a space  $\mathbb{P}X$  of previsions on  $X$ , whose topology is generated by sets  $[h > r] \stackrel{\text{def}}{=} \{F \mid F(h) > r\}$ ,  $h \in \mathcal{L}X$ ,  $r \in \mathbb{R}_+$ .

For example, any continuous valuation  $\nu$  on  $X$  gives rise to a prevision  $G: h \mapsto \int_{x \in X} h(x) d\nu$ . Such a prevision is *linear*, in the sense that  $G(h + h') = G(h) + G(h')$  for all  $h, h' \in \mathcal{L}X$ . Let  $\mathbb{P}_p X$  be the subspace of  $\mathbb{P}X$  of linear previsions. In the reverse direction, every linear prevision  $G \in \mathbb{P}_p X$  gives rise to a continuous valuation  $U \mapsto G(\chi_U)$ , where  $\chi_U$  is the characteristic map of the open set  $U$ , and the two constructions are inverse of each other. Additionally, those two constructions define continuous maps between  $\mathbf{V}X$  and  $\mathbb{P}_p X$  [24, Satz 4.16]. We will therefore equate continuous valuations with linear previsions.

A prevision is *superlinear* if and only if  $G(h + h') \geq G(h) + G(h')$  for all  $h, h' \in \mathcal{L}X$ . As in [12], we write  $\mathbb{P}_{\text{dp}} X$  for the subspace of  $\mathbb{P}X$  consisting of superlinear previsions.

Among the continuous valuations, there are the probability valuations and the subprobability valuations. Similarly, we say that a prevision  $F$  is *subnormalized* (resp., *normalized*) iff  $F(\mathbf{1} + h) \leq \mathbf{1} + F(h)$  (resp.,  $=$ ) for every  $h \in \mathcal{L}X$ , where  $\mathbf{1}$  is the constant function with value 1. The homeomorphism between  $\mathbf{V}X$  and  $\mathbb{P}_p X$  restricts to homeomorphisms between  $\mathbf{V}_{\leq 1} X$  (resp.,  $\mathbf{V}_1 X$ ) and the subspace  $\mathbb{P}_p^{\leq 1} X$  (resp.,  $\mathbb{P}_p^1 X$ ) of subnormalized (resp., normalized) linear previsions on  $X$ . We

write  $\mathbb{P}_{\text{DP}}^{\leq 1} X$ ,  $\mathbb{P}_{\text{DP}}^1 X$  for the corresponding spaces of (sub)normalized, superlinear previsions. In general, we write  $\mathbb{P}_{\text{DP}}^\bullet X$ , where  $\bullet$  can be nothing, “ $\leq 1$ ”, or “1”.

For every topological space  $X$ , let  $\mathcal{Q}X$  be the collection of non-empty compact saturated subsets of  $X$ , ordered by reverse inclusion  $\supseteq$ . This is the *Smyth powerdomain* of  $X$  [23]; see also [22, Section 7]. Let  $\mathcal{Q}_V X$  be the same set, but with the *upper Vietoris topology*, whose basic open subsets are  $\square U \stackrel{\text{def}}{=} \{Q \in \mathcal{Q}X \mid Q \subseteq U\}$ ,  $U \in \mathcal{O}X$ . We will first use this in item (6) of Lemma 3.1 below. The functions  $\sum_{j=1}^m a_j \mathbf{u}_{Q_j}$  of item (7) of the same lemma are the (*simple*) *credibilities* of [10, Theorem 6.18].

**Lemma 3.1.** *The following properties hold, where  $h \in \mathcal{L}X$  and  $\nu$  is a monotonic map from  $\mathcal{O}X$  to  $\overline{\mathbb{R}}_+$ .*

- (1) *The Choquet integral  $\int_{x \in X} h(x) d\nu$  is linear in  $\nu$ , monotonic and even Scott-continuous in  $\nu$ .*
- (2) *If  $\nu$  is Scott-continuous, then the Choquet integral is Scott-continuous in  $h$ .*
- (3) *If  $\nu$  is a continuous valuation, then the Choquet integral is also linear in  $h$ .*
- (4) *For every  $U \in \mathcal{O}X$ ,  $\int_{x \in X} \chi_U(x) d\nu = \nu(U)$ .*
- (5) *Given any continuous valuation  $\mu$  on  $\mathcal{Q}_V X$ , there is a Scott-continuous map  $\mu^\square: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$  defined by  $\mu^\square(U) \stackrel{\text{def}}{=} \mu(\square U)$  for every  $U \in \mathcal{O}X$ . Then, for every  $h \in \mathcal{L}X$ , the map  $h^*: Q \mapsto \min_{x \in Q} h(x)$  is in  $\mathcal{L}\mathcal{Q}_V X$  and  $\int_{x \in X} h(x) d\mu^\square = \int_{Q \in \mathcal{Q}_V X} h^*(Q) d\mu$ .*
- (6) *For every compact saturated subset  $Q$  of  $X$ , the unanimity game  $\mathbf{u}_Q: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ , which maps every  $U \in \mathcal{O}X$  to 1 if  $Q \subseteq U$  and to 0 otherwise, is a Scott-continuous map from  $\mathcal{O}X$  to  $\overline{\mathbb{R}}_+$ .*
- (7) *Letting  $\nu \stackrel{\text{def}}{=} \sum_{j=1}^m a_j \mathbf{u}_{Q_j}$ , where each  $Q_j$  is compact saturated and  $a_j \in \mathbb{R}_+$ , the map  $F: \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$  defined by:*

$$F(h) \stackrel{\text{def}}{=} \int_{x \in X} h(x) d\nu = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$$

*for every  $h \in \mathcal{L}X$ , is a superlinear prevision.*

*Proof.* (1) The fact that the Choquet integral is linear in  $\nu$ , namely that it commutes with scalar products by non-negative real numbers and with addition of continuous valuations, follows from the linearity of indefinite Riemann integration. It is also monotonic in  $\nu$ . In order to show Scott-continuity, we consider a directed family  $(\nu_i)_{i \in I}$ , with (pointwise) supremum  $\nu$ , and we observe that  $\int_{x \in X} h(x) d\nu = \int_0^\infty \sup_{i \in I}^\uparrow \nu_i(h^{-1}([t, \infty])) dt$ . The key is that the maps  $t \mapsto \nu_i(h^{-1}([t, \infty]))$

are antitone (all antitone maps are Riemann-integrable), and that indefinite Riemann integration of antitone maps  $f$  is Scott-continuous in  $f$ , see [24, Lemma 4.2]. Therefore  $\int_{x \in X} h(x) d\nu = \sup_{i \in I}^{\uparrow} \int_0^{\infty} \nu_i(h^{-1}(]t, \infty])) dt = \sup_{i \in I}^{\uparrow} \int_{x \in X} h(x) d\nu_i$ .

(2) The proof works as Tix's original proof of the same result in the special case where  $\nu$  is a continuous valuation [24, Satz 4.4], and relies on [24, Lemma 4.2], just like item (1). Explicitly, let  $(h_i)_{i \in I}$  be a directed family in  $\mathcal{L}X$ , with (pointwise) supremum  $h$ . For every  $t \in \mathbb{R}_+$ ,  $h^{-1}(]t, \infty]) = \{x \in X \mid \sup_{i \in I}^{\uparrow} h_i(x) > t\} = \bigcup_{i \in I}^{\uparrow} h_i^{-1}(]t, \infty])$ . Therefore,

$$\begin{aligned} \int_{x \in X} h(x) d\nu &= \int_0^{\infty} \nu\left(\bigcup_{i \in I}^{\uparrow} h_i^{-1}(]t, \infty])\right) dt \\ &= \int_0^{\infty} \sup_{i \in I}^{\uparrow} \nu(h_i^{-1}(]t, \infty])) dt \\ &= \sup_{i \in I}^{\uparrow} \int_0^{\infty} \nu(h_i^{-1}(]t, \infty])) dt = \sup_{i \in I}^{\uparrow} \int_{x \in X} h_i(x) d\nu, \end{aligned}$$

using the Scott-continuity of  $\nu$  and the Scott-continuity of indefinite Riemann integration of antitone maps.

(3) This is a result of Tix [24, Satz 4.4].

(4)  $\int_{x \in X} \chi_U(x) d\nu = \int_0^{\infty} \nu(\chi_U^{-1}(]t, \infty])) dt = \int_0^1 \nu(U) dt + \int_1^{\infty} 0 dt = \nu(U)$ .

(5) This is as with [10, Lemma 7.5]. The fact that  $\mu^{\square}$  is Scott-continuous follows from the fact that  $\mu$  is, and that the  $\square$  operator is, too. For the latter, observe that for every directed family  $(U_i)_{i \in I}$  of open subsets of  $X$ , for every  $Q \in \mathcal{Q}_V X$ ,  $Q \in \square \bigcup_{i \in I}^{\uparrow} U_i$  if and only if  $Q \subseteq \bigcup_{i \in I}^{\uparrow} U_i$ , which is equivalent to  $Q \subseteq U_i$  (namely,  $Q \in \square U_i$ ) for some  $i \in I$ , because  $Q$  is compact.

For every  $Q \in \mathcal{Q}_V X$ , the minimum of  $h(x)$  when  $x$  ranges over  $Q$  is reached, since  $Q$  is compact and non-empty. For every  $t \in \mathbb{R}_+$ ,  $Q \in h^{*-1}(]t, \infty])$  if and only if  $h^*(Q) > t$ . The latter certainly implies that  $h(x) > t$  for every  $x \in Q$ , hence that  $Q \in \square h^{-1}(]t, \infty])$ . Conversely, if  $Q \in \square h^{-1}(]t, \infty])$ , then let us pick  $x \in Q$  such that  $h(x)$  is the least value reached by  $h$  on  $Q$ ; then  $h^*(Q) = h(x) > t$ , so  $Q \in h^{*-1}(]t, \infty])$ . Hence  $h^{*-1}(]t, \infty]) = \square h^{-1}(]t, \infty])$ . This implies that  $h^*$  is in  $\mathcal{L}\mathcal{Q}_V X$ , in particular.



Now:

$$\begin{aligned} \int_{Q \in \mathcal{Q}_V X} h^*(Q) d\mu &= \int_0^\infty \mu(h^{*-1}([t, \infty])) dt \\ &= \int_0^\infty \mu(\square h^{-1}([t, \infty])) dt \\ &= \int_0^\infty \mu^\square(h^{-1}([t, \infty])) dt = \int_{x \in X} h(x) d\mu^\square. \end{aligned}$$

(6) Monotonicity is clear. For every directed family  $(U_i)_{i \in I}$  of open subsets of  $X$ ,  $\mathbf{u}_Q(\bigcup_{i \in I}^\uparrow U_i) = 1$  if and only if  $Q \in \square \bigcup_{i \in I}^\uparrow U_i$ , which is equivalent to the existence of an  $i \in I$  such that  $Q \in \square U_i$  (equivalently,  $\mathbf{u}_Q(U_i) = 1$ ), as we have seen at the beginning of the proof of item (5).

(7) That would be a consequence of [10, Propositions 7.2, 7.6], except for the fact that our functions  $h$  take their values in  $\overline{\mathbb{R}}_+$ . We verify that  $\int_{x \in X} h(x) d\nu = \sum_{j=1}^m a_j \int_{x \in X} h(x) d\mathbf{u}_{Q_j} = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$ : the first equality is by item (1), and the second one is because  $\int_{x \in X} h(x) d\mathbf{u}_{Q_i}$  is equal to  $\int_0^\infty \mathbf{u}_{Q_i}(h^{-1}([t, \infty])) dt = \int_0^{\min_{x \in Q_i} h(x)} 1 dt + \int_{\min_{x \in Q_i} h(x)}^\infty 0 dt = \min_{x \in Q_i} h(x)$ .

It is easy to see that  $F(h)$  is superlinear, because  $\min_{x \in Q_i} ah(x) = a \min_{x \in Q_i} h(x)$  and  $\min_{x \in Q_i} (h(x) + h'(x)) \geq \min_{x \in Q_i} h(x) + \min_{x \in Q_i} h'(x)$ , for all  $a \in \mathbb{R}_+$  and  $h, h' \in \mathcal{L}X$ . Scott-continuity comes from the fact that  $F(h) = \int_{x \in X} h(x) d\nu$ , where  $\nu \stackrel{\text{def}}{=} \sum_{j=1}^m a_j \mathbf{u}_{Q_j}$ , that  $\nu$  is Scott-continuous (by item (6)), and by using item (2).  $\square$

Let us turn to  $\mathcal{Q}_V$ . The specialization preordering of  $\mathcal{Q}_V X$  is  $\supseteq$ , and in particular  $\mathcal{Q}_V X$  is  $T_0$ . The Scott topology on  $\mathcal{Q}X$  is finer than the upper Vietoris topology when  $X$  is sober, and coincides with it when  $X$  is locally compact and sober [11, Lemma 8.3.26], and then  $\mathcal{Q}X$  itself is a continuous dcpo [11, Proposition 8.3.25].

$\mathcal{Q}_V$  is an endofunctor on the category of topological spaces, whose action on continuous maps  $f: X \rightarrow Y$  is given by  $\mathcal{Q}_V f(Q) \stackrel{\text{def}}{=} \uparrow f[Q]$ , where  $f[Q]$  denotes the image of  $Q$  under  $f$ . That functor is part of a monad [22, Section 7], of which we need to know the unit  $\eta^\mathcal{Q}$ : for every space  $X$ , for every  $x \in X$ ,  $\eta_X^\mathcal{Q}(x) = \uparrow x$ . We also note that  $(\eta_X^\mathcal{Q})^{-1}(\square U) = U$  for every  $U \in \mathcal{O}X$ .

By [12, Lemma 3.20], for every space  $X$ , there is a continuous map  $s_{\text{DP}}^\bullet: \mathbb{P}_{\text{DP}}^\bullet X \rightarrow \mathcal{Q}_V \mathbb{P}_P^\bullet X$  defined by  $s_{\text{DP}}^\bullet(F) \stackrel{\text{def}}{=} \{G \in \mathbb{P}_P^\bullet X \mid G \geq F\}$ . The ordering  $\leq$  between previsions is the specialization ordering, which is pointwise, and  $\geq$  is the opposite ordering. Up to the homeomorphism between  $\mathbb{P}_P^\bullet X$  and  $\mathbf{V}_\bullet X$ , we restate the latter as follows.

**Lemma 3.2.** *For every topological space  $X$ , there is a continuous map  $s_{\text{DP}}^\bullet: \mathbb{P}_{\text{DP}}^\bullet X \rightarrow \mathcal{Q}_V \mathbf{V}_\bullet X$ , defined by:*

$$s_{\text{DP}}^\bullet(F) \stackrel{\text{def}}{=} \left\{ \nu' \in \mathbf{V}_\bullet X \mid \forall h \in \mathcal{L}X, \int_{x \in X} h(x) d\nu' \geq F(h) \right\}.$$

In order to show that  $\mathbf{V}_\bullet X$  is locally compact (if  $X$  is), we will need to find enough compact subsets. They will be provided to us by the images of certain superlinear previsions under  $s_{\text{DP}}^\bullet$ .

#### 4. The main theorem

We spend most of this section proving the following theorem. We remember that all locally compact spaces are core-compact.

**Theorem 4.1.** *For every core-compact space  $X$ ,  $\mathbf{V}X$  and  $\mathbf{V}_{\leq 1}X$  are locally compact. If  $X$  is also compact, then  $\mathbf{V}_1X$  is locally compact.*

*Proof.* Replacing  $X$  by  $\mathcal{S}(X)$  if necessary, we may assume that  $X$  is locally compact and sober.

A fundamental theorem due to Jones [16, Theorem 5.2] states that for every continuous dcpo  $P$ ,  $\mathbf{V}_{\leq 1}P$  is a continuous dcpo under the stochastic ordering, and that a basis is given by the *simple valuations*, namely those of the form  $\sum_{i=1}^n a_i \delta_{x_i}$ , where each  $a_i$  is in  $\mathbb{R}_+$  and  $x_i \in P$ . The notation  $\delta_x$  refers to the *Dirac valuation* at  $x$ , defined by  $\delta_x(U) \stackrel{\text{def}}{=} 1$  if  $x \in U$ , 0 otherwise, for every  $U \in \mathcal{O}X$ .

A similar result holds for  $\mathbf{V}P$  [8, Theorem IV-9.16], and for  $\mathbf{V}_1P$  provided that  $P$  is also pointed [7, Corollary 3.3]. (A poset is *pointed* if and only if it has a least element.) We will apply those results to  $P \stackrel{\text{def}}{=} \mathcal{Q}X$ , and we notice that if  $X$  is compact, then  $P$  is pointed, as  $X$  itself will be the least element of  $P$  in that case.

Let  $\nu \in \mathbf{V}_\bullet X$ , and let  $\mathcal{U}$  be any open neighborhood of  $\nu$ . Then  $\nu$  is in some finite intersection of subbasic open sets  $\bigcap_{i=1}^n [U_i > r_i]$  that is included in  $\mathcal{U}$ , where each  $U_i$  is open in  $X$  and  $r_i \in \mathbb{R}_+$ . We consider  $\mu \stackrel{\text{def}}{=} \mathbf{V}\eta_X^\mathcal{Q}(\nu) \in \mathbf{V}_\bullet \mathcal{Q}_V X$ . We recall that  $\eta_X^\mathcal{Q}$  is the unit of the  $\mathcal{Q}_V$  monad, and maps every point  $x \in X$  to  $\uparrow x \in \mathcal{Q}_V X$ . For every open subset  $U$  of  $X$ ,  $\mu(\square U) = \eta_X^\mathcal{Q}[\nu](\square U) = \nu((\eta_X^\mathcal{Q})^{-1}(\square U)) = \nu(U)$ . It follows that  $\mu$  is in  $\bigcap_{i=1}^n [\square U_i > r_i]$ . The latter is open in the upper Vietoris topology on  $\mathbf{V}_\bullet \mathcal{Q}_V X = \mathbf{V}_\bullet P$ , hence in the Scott topology of the stochastic ordering. Since  $X$  is locally compact and sober,  $P$  is a continuous dcpo, so  $\mathbf{V}_\bullet P$  is a continuous dcpo with a basis of simple valuations, and therefore there is a simple valuation  $\xi \stackrel{\text{def}}{=} \sum_{j=1}^m a_j \delta_{Q_j}$  in  $\mathbf{V}_\bullet P$  that is way below  $\mu$  and in  $\bigcap_{i=1}^n [\square U_i > r_i]$ .

We may form  $\xi^\square$ , as in Lemma 3.1, item (5): for every  $U \in \mathcal{O}X$ ,  $\xi^\square(U) = \xi(\square U)$ . We check easily that  $\xi^\square$  is equal to  $\sum_{j=1}^m a_j \mathbf{u}_{Q_j}$ . Using Lemma 3.1, item (7) allows us to build a superlinear prevision  $F$  on  $X$  by letting  $F(h) \stackrel{\text{def}}{=} \int_{x \in X} h(x) d\xi^\square = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$  for every  $h \in \mathcal{L}X$ .

Now  $s_{\text{DP}}^\bullet(F)$  is a compact saturated subset of  $\mathbf{V}_\bullet X$ , by Lemma 3.2. We claim that  $\nu$  is in the interior of  $s_{\text{DP}}^\bullet(F)$ , and that  $s_{\text{DP}}^\bullet(F)$  is included in  $\mathcal{U}$ ; this will end our proof.

We start by showing that  $s_{\text{DP}}^\bullet(F) \subseteq \mathcal{U}$ . Let  $\nu'$  be any element of  $s_{\text{DP}}^\bullet(F)$ . In other words, for every  $h \in \mathcal{L}X$ ,  $\int_{x \in X} h(x) d\nu' \geq F(h) = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$ . For each  $i \in \{1, \dots, n\}$ , we apply this to  $h \stackrel{\text{def}}{=} \chi_{U_i}$ ; then  $\min_{x \in Q_j} h(x) = \mathbf{u}_{Q_j}(U_i)$ , so  $F(\chi_{U_i}) = \sum_{j=1}^m a_j \mathbf{u}_{Q_j}(U_i) = \xi^\square(U_i) = \xi(\square U_i)$ , and therefore  $\nu'(U_i) = \int_{x \in X} \chi_{U_i}(x) d\nu' \geq F(\chi_{U_i}) = \xi(\square U_i) > r_i$ , since  $\xi \in [\square U_i > r_i]$ . Hence  $\nu' \in \bigcap_{i=1}^n [U_i > r_i] \subseteq \mathcal{U}$ .

Next, we verify that  $\nu$  is in the interior of  $s_{\text{DP}}^\bullet(F)$ . We use the fact that  $\xi$  is way below  $\mu$ , equivalently that  $\mu$  is in the open set  $\uparrow \xi$ . Since  $\mu = \mathbf{V}\eta_X^\mathcal{Q}(\nu)$ ,  $\nu$  is in  $(\mathbf{V}\eta_X^\mathcal{Q})^{-1}(\uparrow \xi)$ , which is open since  $\mathbf{V}\eta_X^\mathcal{Q}$  is continuous. It remains to show that  $(\mathbf{V}\eta_X^\mathcal{Q})^{-1}(\uparrow \xi)$  is included in  $s_{\text{DP}}^\bullet(F)$ .

For every  $\nu' \in (\mathbf{V}\eta_X^\mathcal{Q})^{-1}(\uparrow \xi)$ , by definition  $\xi$  is way below, in particular less than or equal to  $\mu' \stackrel{\text{def}}{=} \mathbf{V}\eta_X^\mathcal{Q}(\nu')$ . The latter is such that  $\mu'(\square U) = \eta_X^\mathcal{Q}[\nu'](\square U) = \nu'((\eta_X^\mathcal{Q})^{-1}(\square U)) = \nu'(U)$  for every  $U \in \mathcal{O}X$ . Hence  $\nu' = \mu'^{\square}$ . By Lemma 3.1, item (5),  $\int_{x \in X} h(x) d\nu' = \int_{Q \in \mathcal{Q}_{\mathbf{V}X}} h^*(Q) d\mu'$ . Since  $\xi \leq \mu'$ , the latter is larger than or equal to  $\int_{Q \in \mathcal{Q}_{\mathbf{V}X}} h^*(Q) d\xi = \int_{x \in X} h(x) d\xi^\square = F(h)$ , using Lemma 3.1, item (5) once again. We have shown that  $\int_{x \in X} h(x) d\nu' \geq F(h)$  for every  $h \in \mathcal{L}X$ , so  $\nu' \in s_{\text{DP}}^\bullet(F)$ , as promised.  $\square$

When  $X$  is compact,  $\mathbf{V}_1 X = s_{\text{DP}}^\bullet(F)$ , where  $F$  is the normalized superlinear prevision  $h \in \mathcal{L}X \mapsto \int_{x \in X} h(x) d\mathbf{u}_X = \min_{y \in X} h(y)$ . Indeed, for every probability valuation  $\nu$  on  $X$ , for every  $h \in \mathcal{L}X$ ,  $\int_{x \in X} h(x) d\nu \geq \int_{x \in X} \min_{y \in X} h(y) d\nu = \min_{y \in X} h(y)$ . We therefore obtain the following.

**Fact 4.2.** *If  $X$  is compact, then  $\mathbf{V}_1 X$  is compact.*

**Lemma 4.3.** *Let  $\bullet$  be nothing, “ $\leq 1$ ”, or “1”. For every space  $X$ ,  $\mathbf{V}_\bullet X$  is sober.*

*Proof.* For every space  $X$ ,  $\mathbf{V}X$  is sober. The argument is due to R. Tix [24, Satz 5.4], following ideas by R. Heckmann (see [15, Section 2.3]). The sober subspaces of a sober space  $Z$  coincide with the subsets that are closed in the strong topology on  $Z$  [18, Corollary 3.5]. The latter

is also known as the Skula topology, and is the smallest one generated by the original topology on  $Z$  and all the downwards-closed subsets. In particular, any closed subspace of a sober space is sober, any saturated subspace of a sober space is sober. Hence  $\mathbf{V}_{\leq 1}Y$  is sober, being equal to the closed subspace  $\mathbf{V}Y \setminus [Y > 1]$  of  $\mathbf{V}Y$ , and  $\mathbf{V}_1Y$  is sober, being upwards-closed in  $\mathbf{V}_{\leq 1}Y$ .  $\square$

**Corollary 4.4.**  *$\mathbf{V}$  and  $\mathbf{V}_{\leq 1}$  restrict to endofunctors on the category of locally compact (resp., locally compact sober) spaces and continuous maps.  $\mathbf{V}_1$  restricts to an endofunctor on the category of compact, locally compact (resp., compact, locally compact sober) spaces.*

We finish with the mandatory application to spaces of measures, which follows directly, using the considerations developed in Section 2 on the relationship between measures and continuous valuations.

**Corollary 4.5.** *For every hereditarily Lindelöf, LCS-complete space  $X$ —in particular, for every quasi-Polish space  $X$ —if  $X$  is locally compact, then  $\mathcal{M}_{\leq 1}X \cong \mathbf{V}_{\leq 1}X$  are locally compact; if  $X$  is locally compact and compact, then  $\mathcal{M}_1X \cong \mathbf{V}_1X$  are locally compact and compact.*

The locally compact, quasi-Polish spaces are exactly the locally compact, sober, second-countable spaces [5, Theorem 44]. It is easy to see that the  $\mathbf{V}_\bullet$  functor preserves second countability. Using Lemma 4.3, we obtain our final corollary.

**Corollary 4.6.** *For every locally compact quasi-Polish space  $X$ ,  $\mathcal{M}_{\leq 1}X \cong \mathbf{V}_{\leq 1}X$  are locally compact and quasi-Polish.*

## 5. CONCLUSION

When trying to show that a space  $Z$  is locally compact, the direct approach—which is how we proceeded—consists in picking an arbitrary point  $z \in Z$ , an arbitrary open neighborhood  $V$  of  $z$ , and to find a compact (saturated) neighborhood  $Q$  of  $z$  included in  $V$ . The main difficulty is in finding enough compact saturated sets  $Q$ .

Previous work has shown that  $\mathbf{V}_{\leq 1}X$  is stably compact for every stably compact space  $X$ , and gives a hint at what compact saturated subsets of  $\mathbf{V}_{\leq 1}X$  we may need for the task [2, Theorem 39]. (A stably compact space is a compact, locally compact sober space that is *coherent*, in the sense that the intersection of any two compact saturated subsets is compact.) Indeed, in that case, the subsets  $[Q \geq r]^{\leq 1} \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_{\leq 1}X \mid \forall U \in \mathcal{O}X, Q \subseteq U \Rightarrow \nu(U) \geq r\}$  form the required family of compact saturated sets, where  $Q$  ranges over the compact saturated subsets of  $X$  and  $r \in \mathbb{R}_+$ . We might therefore assume that analogous

sets  $[Q \geq r]^\bullet \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_\bullet X \mid \forall U \in \mathcal{O}X, Q \subseteq U \Rightarrow \nu(U) \geq r\}$  would be compact saturated in  $\mathbf{V}_\bullet X$ , and could be used to form enough compact saturated neighborhoods of points  $\nu \in \mathbf{V}_\bullet X$ , when  $X$  is locally compact sober—but without coherence (or compactness).

Without using the detour through superlinear prevision, however, it is not completely clear how one might prove that  $[Q \geq r]^\bullet$  is compact, to start with. It *is* true that  $[Q \geq r]^\bullet$  is compact saturated in  $\mathbf{V}_\bullet X$ , for every compact saturated subset  $Q$  of any topological space  $X$ , and for every  $r \in \mathbb{R}_+$ . Indeed,  $[Q \geq r]^\bullet = s_{\text{DP}}^\bullet(F)$ , where  $F$  is the superlinear prevision  $h \in \mathcal{L}X \mapsto r \int_{x \in X} h(x) d\mathbf{u}_Q$ , as one may check by using Lemma 3.1.

However, the proof we have given needs many more compact saturated subsets, and we have obtained them as  $s_{\text{DP}}^\bullet(F)$  for superlinear prevision of the form  $h \in \mathcal{L}X \mapsto \int_{x \in X} h(x) d \sum_{i=1}^n a_i \mathbf{u}_{Q_i}$ . In general,  $s_{\text{DP}}^\bullet(F)$  is compact saturated in  $\mathbf{V}_\bullet X$ , for any superlinear prevision  $F$  on  $X$ . This naturally leads to the following question.

- (1) What are the compact saturated subsets of  $\mathbf{V}_\bullet X$ , for an arbitrary topological space  $X$ ?

In investigating (1), one may restrict  $X$  to some interesting subclasses. The answer to (1) is already known for stably compact spaces: the compact saturated subsets of  $\mathbf{V}_{\leq 1} X$  are exactly the intersections of finite unions of sets of the form  $[Q \geq r]^{\leq 1}$  given as above [2]. One should also compare this with Prokhorov's characterization of compact sets of probability measures as those sets that are uniformly tight [20], see also the fundamental paper by Topsøe [25]. In the meantime, one should observe that for every topological space  $X$ , there is a continuous map  $r_{\text{DP}}: \mathcal{Q}_V(\mathbf{V}_\bullet X) \rightarrow \mathbb{P}_{\text{DP}}^\bullet X$  such that  $r_{\text{DP}} \circ s_{\text{DP}}^\bullet = \text{id}_{\mathbb{P}_{\text{DP}}^\bullet X}$  and  $s_{\text{DP}}^\bullet \circ r_{\text{DP}} \leq \text{id}_{\mathcal{Q}_V(\mathbf{V}_\bullet X)}$  [12, Proposition 3.22], and that those maps define a homeomorphism between  $\mathbb{P}_{\text{DP}}^\bullet X$  and the subspace  $\mathcal{Q}_V^{cvx}(\mathbf{V}_\bullet X)$  of convex elements of  $\mathcal{Q}_V \mathbf{V}_\bullet X$  [12, Theorem 4.15]. Hence the *convex* compact saturated subsets of  $\mathbf{V}_\bullet X$  are known: they are exactly the sets  $s_{\text{DP}}^\bullet(F)$ , where  $F$  is a superlinear prevision on  $X$ .

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