



HAL
open science

On Local Compactness of Spaces of Continuous Valuations

Jean Goubault-Larrecq

► **To cite this version:**

Jean Goubault-Larrecq. On Local Compactness of Spaces of Continuous Valuations. 2024. hal-04746726v1

HAL Id: hal-04746726

<https://hal.science/hal-04746726v1>

Preprint submitted on 23 Oct 2024 (v1), last revised 18 Nov 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Submitted to
Topology Proceedings

ON LOCAL COMPACTNESS OF SPACES OF CONTINUOUS VALUATIONS

JEAN GOUBAULT-LARRECQ

ABSTRACT. We show that the spaces of continuous valuations, resp. subprobability valuations on a locally compact space is locally compact; similarly with probability valuations on locally compact, compact spaces. Continuous valuations are close cousins of measures. No separation property is assumed.

1. INTRODUCTION

A *continuous valuation* on a topological space X is a close cousin of a Borel measure. Continuous valuations on X form a space $\mathbf{V}X$, with a topology known as the weak topology, and similarly for the subspaces $\mathbf{V}_{\leq 1}X$ of subprobability valuations and \mathbf{V}_1X of probability valuations. We will define these notions more precisely below. It is known that $\mathbf{V}_{\leq 1}$ preserves various properties: stable compactness [2, Theorem 39], being a continuous dcpo [16, Theorem 5.2], being a quasi-continuous dcpo [14, Theorem 5.1], for example. Some of these preservation theorems extend over to \mathbf{V}_1 or to \mathbf{V} , but a conspicuously absent property in the list is local compactness. This is what we address in this paper.

The proof relies on credibilities, as already used in [14], on the Smyth powerdomain construction [23], on Jones' celebrated theorem that the probabilistic powerdomain preserves continuous dcpos [16, Theorem 5.2], and on a few things about certain functionals called superlinear previsions

2010 *Mathematics Subject Classification*. Primary 54G99; secondary 28A33, 68Q55.

Key words and phrases. Continuous valuations, spaces of measures, local compactness, previsions.

[12]. As a whole, the proof is therefore rather elaborate; but it also results from little more than an assemblage of previously known results, and as a consequence, is pretty short.

2. PRELIMINARIES

For background on topology, we refer the reader to [11].

A *compact* subset A of a space X is one such that one can extract a finite subcover from any of its open covers. No separation property is assumed. A subset A of X is *saturated* if and only if it is equal to the intersection of its open neighborhoods, or equivalently if and only if it is upwards-closed in the specialization preordering of X . The latter is defined by $x \leq y$ if and only if every open neighborhood of x contains y .

A space X is *locally compact* if and only if every point has a base of compact neighborhoods, or equivalently of compact saturated neighborhoods, since for any compact subset K of X , the upward closure $\uparrow K$ of K with respect to the specialization preordering of X is compact saturated. In non-Hausdorff spaces, a compact space may fail to be locally compact.

A space is T_0 if and only if its specialization preordering is antisymmetric. An *irreducible* closed subset C of X is a non-empty closed subset such that, for any two closed subsets C_1 and C_2 of X such that $C \subseteq C_1 \cup C_2$, C is included in C_1 or in C_2 already; equivalently, if C intersects two open sets, it must intersect their intersection. A space X is *sober* if and only if it is T_0 and every irreducible closed subset is of the form $\downarrow x$ for some point $x \in X$. Every Hausdorff space, for example, is sober. The notation $\downarrow x$ stands for the *downward closure* of x in X , namely the set of points y below x ; that is also the closure of $\{x\}$. Symmetrically, $\uparrow x$ stands for the *upward closure* of x , namely the set of points y above x . This notation extends to $\uparrow A$, for any subset A , denoting $\bigcup_{x \in A} \uparrow x$.

A function $f: P \rightarrow Q$ between posets is *monotonic* if and only if for all $x, x' \in P$, $x \leq x'$ implies $f(x) \leq f(x')$. It is *Scott-continuous* if and only if f is monotonic and for every directed family $(x_i)_{i \in I}$ with a supremum x in P , the (necessarily directed) family of elements $f(x_i)$ has $f(x)$ as supremum. A *directed* family is a non-empty family D such that any two elements of D have an upper bound in D . We write $\sup^\uparrow D$ for the supremum of a directed family D . Scott-continuity is equivalent to continuity with the respective Scott topologies on P and Q . The *Scott topology* on a poset P consists of those subsets U that are upwards closed ($x \in U$ and $x \leq x'$ implies $x' \in U$) and such that every directed family D that has a supremum in U intersects U .

A *dcpo* (short for *directed-complete partial order*) is a poset P in which every directed subset has a supremum. In a dcpo P , let $x \ll y$ (“ x is way below y ”) if and only if every directed family D such that $y \leq \sup^\uparrow D$

contains an element $d \in D$ such that $x \leq d$. A *basis* for P is a subset B such that, for every $x \in P$, $\downarrow_B x \stackrel{\text{def}}{=} \{b \in B \mid b \ll x\}$ is directed and has x as its supremum. A dcpo is *continuous* if and only if it has a basis. If so, the sets $\uparrow b \stackrel{\text{def}}{=} \{x \in P \mid b \ll x\}$ form a base of the Scott topology.

Let $\overline{\mathbb{R}}_+$ be the set of extended non-negative real numbers $\mathbb{R}_+ \cup \{\infty\}$, with its usual ordering. When needed, we will consider it with its Scott topology, whose open sets are the intervals $]t, \infty]$, $t \in \mathbb{R}_+$, plus \emptyset and $\overline{\mathbb{R}}_+$ itself. In passing, $\overline{\mathbb{R}}_+$ is an example of a continuous dcpo, with $s \ll t$ if and only if $s = 0$ or $s < t$.

We write $\mathcal{O}X$ for the lattice of open subsets of a space X . A *continuous valuation* on X is a map $\nu: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ that is *strict* ($\nu(\emptyset) = 0$), *modular* (for all $U, V \in \mathcal{O}X$, $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$) and Scott-continuous. We say that ν is *bounded* if $\nu(X) < \infty$, a *probability valuation* if and only if $\nu(X) = 1$, and a *subprobability valuation* if and only if $\nu(X) \leq 1$.

Continuous valuations are an alternative to measures that have become popular in domain theory [17, 16]. The first results that connected continuous valuations and measures are due to Saheb-Djahromi [21] and Lawson [19]. The current state of the art on this matter is the following. In one direction, every measure on the Borel σ -algebra of X induces a continuous valuation on X by restriction to the open sets, if X is hereditarily Lindelöf (namely, if every directed family of open sets contains a cofinal monotone sequence). This is an easy observation, and one half of Adamski's theorem [1, Theorem 3.1], which states that a space is hereditarily Lindelöf if and only if every measure on its Borel σ -algebra restricts to a continuous valuation on its open sets. In the other direction, every continuous valuation on a space X extends to a measure on the Borel sets provided that X is an LCS-complete space [6, Theorem 1]. An LCS-complete space is a G_δ subspace of a locally compact sober space, and the class of LCS-complete spaces contains all locally compact sober spaces, in particular all continuous dcpos from domain theory, all of M. de Brecht's quasi-Polish spaces [13] and therefore all Polish spaces.

The extension of a continuous valuation ν to a measure on the Borel σ -algebra is unique if ν is bounded. This is a consequence of the π - λ theorem [3, Theorem 3.2], usually applied to probability measures [3, Theorem 3.3], but the same proof works for bounded measures.

Hence the above maps from Borel measures to continuous valuations and conversely, defined on hereditary Lindelöf LCS-complete spaces (in particular, quasi-Polish spaces), are inverse of each other when restricted to bounded measures and bounded continuous valuations.

Let $\mathbf{V}X$ denote the space of continuous valuations on a space X , with the *weak topology*, defined by subbasic open sets $[U > r] \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}X \mid \nu(U) > r\}$, where $U \in \mathcal{O}X$ and $r \in \mathbb{R}_+$. We define its subspace \mathbf{V}_1X of probability valuations and $\mathbf{V}_{\leq 1}X$ (subprobability) similarly. In general, we write $\mathbf{V}_\bullet X$, where \bullet can be nothing, “ ≤ 1 ”, or “1”. The specialization ordering of each is the *stochastic ordering* \leq given by $\nu \leq \nu'$ if and only if $\nu(U) \leq \nu'(U)$ for every $U \in \mathcal{O}X$; indeed, $\nu \leq \nu'$ if and only if for every $U \in \mathcal{O}X$, for every $r \in \mathbb{R}_+$, $\nu \in [U > r]$ implies $\nu' \in [U > r]$.

There is a related topology on spaces of probability measures. A sequence of probability measures P_n , $n \in \mathbb{N}$, on a metric space *converges weakly* to a probability measure P if and only if $\liminf_{n \in \mathbb{N}} P_n(U) \geq P(U)$ for every open subset U [4, Portmanteau Theorem 2.1]. It is easy to see that this is equivalent to: for every open subset U and for every $r \in \mathbb{R}_+$, $P(U) > r$ implies that $P_n(U) > r$ for n large enough. Hence we may define the *weak topology* on spaces of measures (not just probability measures) on a topological space X (not just a metric space) as given via a subbasis of sets $\{P \mid P(U) > r\}$, $U \in \mathcal{O}X$, $r \in \mathbb{R}_+$. We obtain the following.

Proposition 2.1. *The weak topology makes the spaces $\mathcal{M}_{\leq 1}X$ and \mathcal{M}_1X of subprobability, resp. probability measures homeomorphic to $\mathbf{V}_{\leq 1}X$ and \mathbf{V}_1X , respectively, for every hereditarily Lindelöf LCS-complete space X .*

\mathbf{V}_\bullet is the object part of an endofunctor on the category of topological spaces, whose morphism part is defined as follows: for every continuous map $f: X \rightarrow Y$, $\mathbf{V}_\bullet f$ maps every $\nu \in \mathbf{V}_\bullet X$ to its *image valuation* $f[\nu]$, where $f[\nu](V) \stackrel{\text{def}}{=} \nu(f^{-1}(V))$ for every $V \in \mathcal{O}Y$.

We will prove slightly more than the fact that \mathbf{V}_\bullet preserves local compactness, where \bullet is nothing, “ ≤ 1 ”, or “1”. A space X is *core-compact* if and only if $\mathcal{O}X$ is a continuous dcpo; every locally compact space is core-compact [11, Theorem 5.2.9]. The connection between the two notions can be made more precise as follows. Every topological space X has a *sobrification* $\mathcal{S}(X)$ (or X^s), which is the free sober space over X [11, Theorem 8.2.44]; then X is core-compact if and only if $\mathcal{S}(X)$ is locally compact [11, Proposition 8.3.11]. $\mathcal{S}(X)$ can be built as the collection of irreducible closed subsets, with the topology whose open sets (all of them, not just a base) are $\diamond U \stackrel{\text{def}}{=} \{F \in \mathcal{S}(X) \mid F \cap U \neq \emptyset\}$, $U \in \mathcal{O}X$. In particular, $\diamond: U \mapsto \diamond U$ is an order-isomorphism between $\mathcal{O}X$ and $\mathcal{O}\mathcal{S}(X)$. This induces a homeomorphism between $\mathbf{V}_\bullet X$ and $\mathbf{V}_\bullet \mathcal{S}(X)$.

3. CREDIBILITIES, PREVISIONS AND THE SMYTH POWERDOMAIN

The proof of our main theorem will rely on reducing the question to spaces of valuations on Smyth powerdomains, and on using notions of credibilities [10] and of previsions [9, 12]. We need to give a quick tour of the part of that theory that we need.

Let $\mathcal{L}X$ denote the dcpo of all lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$; a map $f: X \rightarrow \overline{\mathbb{R}}_+$ is lower semicontinuous if and only if it is continuous, where $\overline{\mathbb{R}}_+$ is given its Scott topology. The ordering on $\mathcal{L}X$ is the pointwise ordering. For every monotone map $\nu: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$, for every $h \in \mathcal{L}X$, there is a *Choquet integral* $\int_{x \in X} h(x) d\nu$, defined as the indefinite Riemann integral $\int_0^\infty \nu(h^{-1}([t, \infty])) dt$. We need to reprove in part some of the results of [10] on Choquet integrals, since that paper only considers integrals of maps from X to \mathbb{R}_+ ; we will do this in Lemma 3.1 below.

A *prevision* on a space X is a Scott-continuous map $F: \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$ that is *positively homogeneous* in the sense that $F(ah) = aF(h)$ for all $a \in \mathbb{R}_+$ and $h \in \mathcal{L}X$. There is a space $\mathbb{P}X$ of previsions on X , whose topology is generated by sets $[h > r] \stackrel{\text{def}}{=} \{F \mid F(h) > r\}$, $h \in \mathcal{L}X$, $r \in \mathbb{R}_+$.

For example, any continuous valuation ν on X gives rise to a prevision $G: h \mapsto \int_{x \in X} h(x) d\nu$. Such a prevision is *linear*, in the sense that $G(h + h') = G(h) + G(h')$ for all $h, h' \in \mathcal{L}X$. Let $\mathbb{P}_p X$ be the subspace of $\mathbb{P}X$ of linear previsions. In the reverse direction, every linear prevision $G \in \mathbb{P}_p X$ gives rise to a continuous valuation $U \mapsto G(\chi_U)$, where χ_U is the characteristic map of the open set U , and the two constructions are inverse of each other. Additionally, those two constructions define continuous maps between $\mathbf{V}X$ and $\mathbb{P}_p X$ [24, Satz 4.16]. We will therefore equate continuous valuations with linear previsions.

A prevision is *superlinear* if and only if $G(h + h') \geq G(h) + G(h')$ for all $h, h' \in \mathcal{L}X$. As in [12], we write $\mathbb{P}_{dp} X$ for the subspace of $\mathbb{P}X$ consisting of superlinear previsions.

Among the continuous valuations, there are the probability valuations and the subprobability valuations. Similarly, we say that a prevision F is *subnormalized* (resp., *normalized*) iff $F(\mathbf{1} + h) \leq \mathbf{1} + F(h)$ (resp., $=$) for every $h \in \mathcal{L}X$, where $\mathbf{1}$ is the constant function with value 1. The homeomorphism between $\mathbf{V}X$ and $\mathbb{P}_p X$ restricts to homeomorphisms between $\mathbf{V}_{\leq 1} X$ (resp., $\mathbf{V}_1 X$) and the subspace $\mathbb{P}_p^{\leq 1} X$ (resp., $\mathbb{P}_p^1 X$) of subnormalized (resp., normalized) linear previsions on X . We write $\mathbb{P}_{dp}^{\leq 1} X$, $\mathbb{P}_{dp}^1 X$ for the corresponding spaces of (sub)normalized, superlinear previsions. In general, we write $\mathbb{P}_{dp}^\bullet X$, where \bullet can be nothing, “ ≤ 1 ”, or “1”.

For every topological space X , let $\mathcal{Q}X$ be the collection of non-empty compact saturated subsets of X , ordered by reverse inclusion \supseteq . This

is the *Smyth powerdomain* of X [23]; see also [22, Section 7]. Let $\mathcal{Q}_V X$ be the same set, but with the *upper Vietoris topology*, whose basic open subsets are $\square U \stackrel{\text{def}}{=} \{Q \in \mathcal{Q}X \mid Q \subseteq U\}$, $U \in \mathcal{O}X$. We will first use this in item (6) of Lemma 3.1 below. The functions $\sum_{j=1}^m a_j \mathbf{u}_{Q_j}$ of item (7) of the same lemma are the (*simple*) *credibilities* of [10, Theorem 6.18].

Lemma 3.1. *The following properties hold, where $h \in \mathcal{L}X$ and ν is a monotonic map from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$.*

- (1) *The Choquet integral $\int_{x \in X} h(x) d\nu$ is linear in ν , monotonic and even Scott-continuous in ν .*
- (2) *If ν is Scott-continuous, then the Choquet integral is Scott-continuous in h .*
- (3) *If ν is a continuous valuation, then the Choquet integral is also linear in h .*
- (4) *For every $U \in \mathcal{O}X$, $\int_{x \in X} \chi_U(x) d\nu = \nu(U)$.*
- (5) *Given any continuous valuation ν on $\mathcal{Q}_V X$, there is a Scott-continuous map $\nu^\square: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ defined by $\nu^\square(U) \stackrel{\text{def}}{=} \nu(\square U)$ for every $U \in \mathcal{O}X$. Then, for every $h \in \mathcal{L}X$, the map $h^*: Q \mapsto \min_{x \in Q} h(x)$ is in $\mathcal{L}\mathcal{Q}_V X$ and $\int_{x \in X} h(x) d\nu^\square = \int_{Q \in \mathcal{Q}_V X} h^*(Q) d\nu$.*
- (6) *For every compact saturated subset Q of X , the unanimity game $\mathbf{u}_Q: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$, which maps every $U \in \mathcal{O}X$ to 1 if $Q \subseteq U$ and to 0 otherwise, is a Scott-continuous map from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$.*
- (7) *Letting $\nu \stackrel{\text{def}}{=} \sum_{j=1}^m a_j \mathbf{u}_{Q_j}$, where each Q_j is compact saturated and $a_j \in \mathbb{R}_+$, the map $F: \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$ defined by:*

$$F(h) \stackrel{\text{def}}{=} \int_{x \in X} h(x) d\nu = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$$

for every $h \in \mathcal{L}X$, is a superlinear prevision.

Proof. (1) The fact that the Choquet integral is linear in ν , namely that it commutes with scalar products by non-negative real numbers and with addition of continuous valuations, follows from the linearity of indefinite Riemann integration. It is also monotonic in ν . In order to show Scott-continuity, we consider a directed family $(\nu_i)_{i \in I}$, with (pointwise) supremum ν , and we observe that $\int_{x \in X} h(x) d\nu = \int_0^\infty \sup_{i \in I}^\uparrow \nu_i(h^{-1}([t, \infty])) dt$. The key is that the maps $t \mapsto \nu_i(h^{-1}([t, \infty]))$ are antitone (all antitone maps are Riemann-integrable), and that indefinite Riemann integration of antitone maps f is Scott-continuous in f , see [24, Lemma 4.2]. Therefore $\int_{x \in X} h(x) d\nu = \sup_{i \in I}^\uparrow \int_0^\infty \nu_i(h^{-1}([t, \infty])) dt = \sup_{i \in I}^\uparrow \int_{x \in X} h(x) d\nu_i$.

(2) The proof works as Tix's original proof of the same result in the special case where ν is a continuous valuation [24, Satz 4.4], and relies

on [24, Lemma 4.2], just like item (1). Explicitly, let $(h_i)_{i \in I}$ be a directed family in $\mathcal{L}X$, with (pointwise) supremum h . For every $t \in \mathbb{R}_+$, $h^{-1}(]t, \infty]) = \{x \in X \mid \sup_{i \in I}^\uparrow h_i(x) > t\} = \bigcup_{i \in I}^\uparrow h_i^{-1}(]t, \infty])$. Therefore,

$$\begin{aligned} \int_{x \in X} h(x) d\nu &= \int_0^\infty \nu(\bigcup_{i \in I}^\uparrow h_i^{-1}(]t, \infty])) dt \\ &= \int_0^\infty \sup_{i \in I}^\uparrow \nu(h_i^{-1}(]t, \infty])) dt \\ &= \sup_{i \in I}^\uparrow \int_0^\infty \nu(h_i^{-1}(]t, \infty])) dt = \sup_{i \in I}^\uparrow \int_{x \in X} h_i(x) d\nu, \end{aligned}$$

using the Scott-continuity of ν and the Scott-continuity of indefinite Riemann integration of antitone maps.

(3) This is a result of Tix [24, Satz 4.4].

(4) $\int_{x \in X} \chi_U(x) d\nu = \int_0^\infty \nu(\chi_U^{-1}(]t, \infty])) dt = \int_0^1 \nu(U) dt + \int_1^\infty 0 dt = \nu(U)$.

(5) This is as with [10, Lemma 7.5]. The fact that ν^\square is Scott-continuous follows from the fact that ν is, and that the \square operator is, too. For the latter, observe that for every directed family $(U_i)_{i \in I}$ of open subsets of X , for every $Q \in \mathcal{Q}_V X$, $Q \in \square \bigcup_{i \in I}^\uparrow U_i$ if and only if $Q \subseteq \bigcup_{i \in I}^\uparrow U_i$, which is equivalent to $Q \subseteq U_i$ (namely, $Q \in \square U_i$) for some $i \in I$, because Q is compact.

For every $Q \in \mathcal{Q}_V X$, the minimum of $h(x)$ when x ranges over Q is reached, since Q is compact and non-empty. For every $t \in \mathbb{R}_+$, $Q \in h^{*-1}(]t, \infty])$ if and only if $h^*(Q) > t$. The latter certainly implies that $h(x) > t$ for every $x \in Q$, hence that $Q \in \square h^{-1}(]t, \infty])$. Conversely, if $Q \in \square h^{-1}(]t, \infty])$, then let us pick $x \in Q$ such that $h(x)$ is the least value reached by h on Q ; then $h^*(Q) = h(x) > t$, so $Q \in h^{*-1}(]t, \infty])$. Hence $h^{*-1}(]t, \infty]) = \square h^{-1}(]t, \infty])$. This implies that h^* is in $\mathcal{L}\mathcal{Q}_V X$, in particular.

Now:

$$\begin{aligned} \int_{Q \in \mathcal{Q}_V X} h^*(Q) d\nu &= \int_0^\infty \nu(h^{*-1}(]t, \infty])) dt \\ &= \int_0^\infty \nu(\square h^{-1}(]t, \infty])) dt \\ &= \int_0^\infty \nu^\square(h^{-1}(]t, \infty])) dt = \int_{x \in X} h(x) d\nu^\square. \end{aligned}$$

(6) Monotonicity is clear. For every directed family $(U_i)_{i \in I}$ of open subsets of X , $u_Q(\bigcup_{i \in I}^\uparrow U_i) = 1$ if and only if $Q \in \square \bigcup_{i \in I}^\uparrow U_i$, which is equivalent to the existence of an $i \in I$ such that $Q \in \square U_i$ (equivalently, $u_Q(U_i) = 1$), as we have seen at the beginning of the proof of item (5).

(7) That would be a consequence of [10, Propositions 7.2, 7.6], except for the fact that our functions h take their values in $\overline{\mathbb{R}}_+$. We verify that $\int_{x \in X} h(x) d\nu = \sum_{j=1}^m a_j \int_{x \in X} h(x) du_{Q_j} = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$: the first equality is by item (1), and the second one is because $\int_{x \in X} h(x) du_{Q_i}$ is equal to $\int_0^\infty u_{Q_i}(h^{-1}([t, \infty])) dt = \int_0^{\min_{x \in Q_i} h(x)} 1 dt + \int_{\min_{x \in Q_i} h(x)}^\infty 0 dt = \min_{x \in Q_i} h(x)$.

It is easy to see that $F(h)$ is superlinear, because $\min_{x \in Q_i} ah(x) = a \min_{x \in Q_i} h(x)$ and $\min_{x \in Q_i} (h(x) + h'(x)) \geq \min_{x \in Q_i} h(x) + \min_{x \in Q_i} h'(x)$, for all $a \in \mathbb{R}_+$ and $h, h' \in \mathcal{L}X$. Scott-continuity comes from the fact that $F(h) = \int_{x \in X} h(x) d\nu$, where $\nu \stackrel{\text{def}}{=} \sum_{j=1}^m a_j u_{Q_j}$, that ν is Scott-continuous (by item (6)), and by using item (2). \square

Let us turn to \mathcal{Q}_V . The specialization preordering of $\mathcal{Q}_V X$ is \supseteq , and in particular $\mathcal{Q}_V X$ is T_0 . The Scott topology on $\mathcal{Q}X$ is finer than the upper Vietoris topology, and coincides with it when X is locally compact and sober [11, Lemma 8.3.26], and then $\mathcal{Q}X$ itself is a continuous depo [11, Proposition 8.3.25].

\mathcal{Q}_V is an endofunctor on the category of topological spaces, whose action on continuous maps $f: X \rightarrow Y$ is given by $\mathcal{Q}_V f(Q) \stackrel{\text{def}}{=} \uparrow f[Q]$, where $f[Q]$ denotes the image of Q under f . That functor is part of a monad [22, Section 7], of which we need to know the unit $\eta^{\mathcal{Q}}$: for every space X , for every $x \in X$, $\eta_X^{\mathcal{Q}}(x) = \uparrow x$. We also note that $(\eta_X^{\mathcal{Q}})^{-1}(\square U) = U$ for every $U \in \mathcal{O}X$.

By [12, Lemma 3.20], for every space X , there is a continuous map $s_{\text{DP}}^\bullet: \mathbb{P}_{\text{DP}}^\bullet X \rightarrow \mathcal{Q}_V \mathbb{P}_{\text{DP}}^\bullet X$ defined by $s_{\text{DP}}^\bullet(F) \stackrel{\text{def}}{=} \{G \in \mathbb{P}_{\text{DP}}^\bullet X \mid G \geq F\}$. The ordering \leq between previsions is the specialization ordering, which is pointwise, and \geq is the opposite ordering. Up to the homeomorphism between $\mathbb{P}_{\text{DP}}^\bullet X$ and $\mathbf{V}_\bullet X$, we restate the latter as follows.

Lemma 3.2. *For every topological space X , there is a continuous map $s_{\text{DP}}^\bullet: \mathbb{P}_{\text{DP}}^\bullet X \rightarrow \mathcal{Q}_V \mathbf{V}_\bullet X$, defined by:*

$$s_{\text{DP}}^\bullet(F) \stackrel{\text{def}}{=} \left\{ \nu' \in \mathbf{V}_\bullet X \mid \forall h \in \mathcal{L}X, \int_{x \in X} h(x) d\nu' \geq F(h) \right\}.$$

In order to show that $\mathbf{V}_\bullet X$ is locally compact (if X is), we will need to find enough compact subsets. They will be provided to us by the images of certain superlinear previsions under s_{DP}^\bullet .

4. THE MAIN THEOREM

We spend most of this section proving the following theorem. We remember that all locally compact spaces are core-compact.

Theorem 4.1. *For every core-compact space X , $\mathbf{V}X$ and $\mathbf{V}_{\leq 1}X$ are locally compact. If X is also compact, then \mathbf{V}_1X is locally compact.*

Proof. Replacing X by $\mathcal{S}(X)$ if necessary, we may assume that X is locally compact and sober.

A fundamental theorem due to Jones [16, Theorem 5.2] states that for every continuous depo P , $\mathbf{V}_{\leq 1}P$ is a continuous depo under the stochastic ordering, and that a basis is given by the *simple valuations*, namely those of the form $\sum_{i=1}^n a_i \delta_{x_i}$, where each a_i is in \mathbb{R}_+ and $x_i \in P$. The notation δ_x refers to the *Dirac valuation* at x , defined by $\delta_x(U) \stackrel{\text{def}}{=} 1$ if $x \in U$, 0 otherwise, for every $U \in \mathcal{O}X$.

A similar result holds for $\mathbf{V}P$ [8, Theorem IV-9.16], and for \mathbf{V}_1P provided that P is also pointed [7, Corollary 3.3]. (A poset is *pointed* if and only if it has a least element.) We will apply those results to $P \stackrel{\text{def}}{=} \mathcal{Q}X$, and we notice that if X is compact, then P is pointed, as X itself will be the least element of P in that case.

Let $\nu \in \mathbf{V}_{\bullet}X$, and let \mathcal{U} be any open neighborhood of ν . Then ν is in some finite intersection of subbasic open sets $\bigcap_{i=1}^n [U_i > r_i]$ that is included in \mathcal{U} , where each U_i is open in X and $r_i \in \mathbb{R}_+$. We consider $\mu \stackrel{\text{def}}{=} \mathbf{V}\eta_X^{\mathcal{Q}}(\nu) \in \mathbf{V}_{\bullet}\mathcal{Q}X$. We recall that $\eta_X^{\mathcal{Q}}$ is the unit of the $\mathcal{Q}_{\mathbf{V}}$ monad, and maps every point $x \in X$ to $\uparrow x \in \mathcal{Q}X$. For every open subset U of X , $\mu(\square U) = \eta_X^{\mathcal{Q}}[\nu](\square U) = \nu((\eta_X^{\mathcal{Q}})^{-1}(\square U)) = \nu(U)$. It follows that μ is in $\bigcap_{i=1}^n [\square U_i > r_i]$. The latter is open in the upper Vietoris topology on $\mathbf{V}_{\bullet}\mathcal{Q}X = \mathbf{V}_{\bullet}P$, hence in the Scott topology of the stochastic ordering. Since X is locally compact and sober, P is a continuous depo, so $\mathbf{V}_{\bullet}P$ is a continuous depo with a basis of simple valuations, and therefore there is a simple valuation $\xi \stackrel{\text{def}}{=} \sum_{j=1}^m a_j \delta_{Q_j}$ in $\mathbf{V}_{\bullet}P$ that is way below μ and in $\bigcap_{i=1}^n [\square U_i > r_i]$.

We may form ξ^{\square} , as in Lemma 3.1, item (5): for every $U \in \mathcal{O}X$, $\xi^{\square}(U) = \xi(\square U)$. We check easily that ξ^{\square} is equal to $\sum_{j=1}^m a_j \mathbf{u}_{Q_j}$. Using Lemma 3.1, item (7) allows us to build a superlinear prevision F on X by letting $F(h) \stackrel{\text{def}}{=} \int_{x \in X} h(x) d\xi^{\square} = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$ for every $h \in \mathcal{L}X$.

Now $s_{\text{dp}}^{\bullet}(F)$ is a compact saturated subset of $\mathbf{V}_{\bullet}X$, by Lemma 3.2. We claim that ν is in the interior of $s_{\text{dp}}^{\bullet}(F)$, and that $s_{\text{dp}}^{\bullet}(F)$ is included in \mathcal{U} ; this will end our proof.

We start by showing that $s_{\text{dp}}^{\bullet}(F) \subseteq \mathcal{U}$. Let ν' be any element of $s_{\text{dp}}^{\bullet}(F)$. In other words, for every $h \in \mathcal{L}X$, $\int_{x \in X} h(x) d\nu' \geq F(h) =$

$\sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$. For each $i \in \{1, \dots, n\}$, we apply this to $h \stackrel{\text{def}}{=} \chi_{U_i}$; then $\min_{x \in Q_j} h(x) = \mathbf{u}_{Q_j}(U_i)$, so $F(\chi_{U_i}) = \sum_{j=1}^m a_j \mathbf{u}_{Q_j}(U_i) = \xi^\square(U_i) = \xi(\square U_i)$, and therefore $\nu'(U_i) = \int_{x \in X} \chi_{U_i}(x) d\nu' \geq F(\chi_{U_i}) = \xi(\square U_i) > r_i$, since $\xi \in [\square U_i > r_i]$. Hence $\nu' \in \bigcap_{i=1}^n [U_i > r_i] \subseteq \mathcal{U}$.

Next, we verify that ν is in the interior of $s_{\text{dp}}^\bullet(F)$. We use the fact that ξ is way below μ , equivalently that μ is in the open set $\uparrow \xi$. Since $\mu = \mathbf{V}\eta_X^\square(\nu)$, ν is in $(\mathbf{V}\eta_X^\square)^{-1}(\uparrow \xi)$, which is open since $\mathbf{V}\eta_X^\square$ is continuous. It remains to show that $(\mathbf{V}\eta_X^\square)^{-1}(\uparrow \xi)$ is included in $s_{\text{dp}}^\bullet(F)$.

For every $\nu' \in (\mathbf{V}\eta_X^\square)^{-1}(\uparrow \xi)$, by definition ξ is way below, in particular less than or equal to $\mu' \stackrel{\text{def}}{=} \mathbf{V}\eta_X^\square(\nu')$. The latter is such that $\mu'(\square U) = \eta_X^\square[\nu'](\square U) = \nu'((\eta_X^\square)^{-1}(\square U)) = \nu'(U)$ for every $U \in \mathcal{O}X$. Hence $\nu' = \mu'^\square$. By Lemma 3.1, item (5), $\int_{x \in X} h(x) d\nu' = \int_{Q \in \mathcal{Q}_{\mathbf{V}X}} h^*(Q) d\mu'$. Since $\xi \leq \mu'$, the latter is larger than or equal to $\int_{Q \in \mathcal{Q}_{\mathbf{V}X}} h^*(Q) d\xi = \int_{x \in X} h(x) d\xi^\square = F(h)$, using Lemma 3.1, item (5) once again. We have shown that $\int_{x \in X} h(x) d\nu' \geq F(h)$ for every $h \in \mathcal{L}X$, so $\nu' \in s_{\text{dp}}^\bullet(F)$, as promised. \square

When X is compact, $\mathbf{V}_1 X = s_{\text{dp}}^\bullet(F)$, where F is the normalized superlinear prevision $h \in \mathcal{L}X \mapsto \int_{x \in X} h(x) du_X = \min_{y \in X} h(y)$. Indeed, for every probability valuation ν on X , for every $h \in \mathcal{L}X$, $\int_{x \in X} h(x) d\nu \geq \int_{x \in X} \min_{y \in X} h(y) d\nu = \min_{y \in X} h(y)$. We therefore obtain the following.

Fact 4.2. *If X is compact, then $\mathbf{V}_1 X$ is compact.*

Lemma 4.3. *Let \bullet be nothing, “ ≤ 1 ”, or “1”. For every space X , $\mathbf{V}_\bullet X$ is sober.*

Proof. For every space X , $\mathbf{V}X$ is sober. The argument is due to R. Tix [24, Satz 5.4], following ideas by R. Heckmann (see [15, Section 2.3]). The sober subspaces of a sober space Z coincide with the subsets that are closed in the strong topology on Z [18, Corollary 3.5]. The latter is also known as the Skula topology, and is the smallest one generated by the original topology on Z and all the downwards-closed subsets. In particular, any closed subspace of a sober space is sober, any saturated subspace of a sober space is sober. Hence $\mathbf{V}_{\leq 1} Y$ is sober, being equal to the closed subspace $\mathbf{V}Y \setminus [Y > 1]$ of $\mathbf{V}Y$, and $\mathbf{V}_1 Y$ is sober, being upwards-closed in $\mathbf{V}_{\leq 1} Y$. \square

Corollary 4.4. *\mathbf{V} and $\mathbf{V}_{\leq 1}$ restrict to endofunctors on the category of locally compact (resp., locally compact sober) spaces and continuous maps. \mathbf{V}_1 restricts to an endofunctor on the category of compact, locally compact (resp., compact, locally compact sober) spaces.*

We finish with the mandatory application to spaces of measures, which follows directly, using the considerations developed in Section 2 on the relationship between measures and continuous valuations.

Corollary 4.5. *For every hereditarily Lindelöf, LCS-complete space X —in particular, for every quasi-Polish space X —if X is locally compact, then $\mathcal{M}_{\leq 1}X \cong \mathbf{V}_{\leq 1}X$ are locally compact; if X is locally compact and compact, then $\mathcal{M}_1X \cong \mathbf{V}_1X$ are locally compact and compact.*

The locally compact, quasi-Polish spaces are exactly the locally compact, sober, second-countable spaces [5, Theorem 44]. It is easy to see that the \mathbf{V}_\bullet functor preserves second countability. Using Lemma 4.3, we obtain our final corollary.

Corollary 4.6. *For every locally compact quasi-Polish space X , $\mathcal{M}_{\leq 1}X \cong \mathbf{V}_{\leq 1}X$ are locally compact and quasi-Polish.*

5. CONCLUSION

When trying to show that a space Z is locally compact, the direct approach—which is how we proceeded—consists in picking an arbitrary point $z \in Z$, an arbitrary open neighborhood V of z , and to find a compact (saturated) neighborhood Q of z included in V . The main difficulty is in finding enough compact saturated sets Q .

Previous work has shown that $\mathbf{V}_{\leq 1}X$ is stably compact for every stably compact space X , and gives a hint at what compact saturated subsets of $\mathbf{V}_{\leq 1}X$ we may need for the task [2, Theorem 39]. (A stably compact space is a compact, locally compact sober space that is *coherent*, in the sense that the intersection of any two compact saturated subsets is compact.)

Indeed, in that case, the subsets $[Q \geq r]^{\leq 1} \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_{\leq 1}X \mid \forall U \in \mathcal{O}X, Q \subseteq U \Rightarrow \nu(U) \geq r\}$ form the required family of compact saturated sets, where Q ranges over the compact saturated subsets of X and $r \in \mathbb{R}_+$.

We might therefore assume that analogous sets $[Q \geq r]^\bullet \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_\bullet X \mid \forall U \in \mathcal{O}X, Q \subseteq U \Rightarrow \nu(U) \geq r\}$ would be compact saturated in $\mathbf{V}_\bullet X$, and could be used to form enough compact saturated neighborhoods of points $\nu \in \mathbf{V}_\bullet X$, when X is locally compact sober—but without coherence (or compactness).

Without using the detour through superlinear previsions, however, it is not completely clear how one might prove that $[Q \geq r]^\bullet$ is compact, to start with. It *is* true that $[Q \geq r]^\bullet$ is compact saturated in $\mathbf{V}_\bullet X$, for every compact saturated subset Q of any topological space X , and for every $r \in \mathbb{R}_+$. Indeed, $[Q \geq r]^\bullet = s_{\text{DP}}^\bullet(F)$, where F is the superlinear prevision $h \in \mathcal{L}X \mapsto r \int_{x \in X} h(x) du_Q$, as one may check by using Lemma 3.1.

However, the proof we have given needs many more compact saturated subsets, and we have obtained them as $s_{\text{DP}}^{\bullet}(F)$ for superlinear previsions of the form $h \in \mathcal{L}X \mapsto \int_{x \in X} h(x) d \sum_{i=1}^n a_i u_{Q_i}$. In general, $s_{\text{DP}}^{\bullet}(F)$ is compact saturated in $\mathbf{V}_{\bullet}X$, for any superlinear prevision F on X . This naturally leads to the following question.

- (1) What are the compact saturated subsets of $\mathbf{V}_{\bullet}X$, for an arbitrary topological space X ?

In investigating (1), one may restrict X to some interesting subclasses. The answer to (1) is already known for stably compact spaces: the compact saturated subsets of $\mathbf{V}_{\leq 1}X$ are exactly the intersections of finite unions of sets of the form $[Q \geq r]^{\leq 1}$ given as above [2]. One should also compare this with Prokhorov's characterization of compact sets of probability measures as those sets that are uniformly tight [20], see also the fundamental paper by Topsøe [25]. In the meantime, one should observe that for every topological space X , there is a continuous map $r_{\text{DP}}: \mathcal{Q}_{\mathbf{V}}(\mathbf{V}_{\bullet}X) \rightarrow \mathbb{P}_{\text{DP}}^{\bullet}X$ such that $r_{\text{DP}} \circ s_{\text{DP}}^{\bullet} = \text{id}_{\mathbb{P}_{\text{DP}}^{\bullet}X}$ and $s_{\text{DP}}^{\bullet} \circ r_{\text{DP}} \leq \text{id}_{\mathcal{Q}_{\mathbf{V}}(\mathbf{V}_{\bullet}X)}$ [12, Proposition 3.22], and that those maps define a homeomorphism between $\mathbb{P}_{\text{DP}}^{\bullet}X$ and the subspace $\mathcal{Q}_{\mathbf{V}}^{\text{conv}}(\mathbf{V}_{\bullet}X)$ of convex elements of $\mathcal{Q}_{\mathbf{V}}\mathbf{V}_{\bullet}X$ [12, Theorem 4.15]. Hence the *convex* compact saturated subsets of $\mathbf{V}_{\bullet}X$ are known: they are exactly the sets $s_{\text{DP}}(F)$, where F is a superlinear prevision on X .

ACKNOWLEDGMENTS

I thank the anonymous referee for carefully rereading this paper.

REFERENCES

- [1] Wolfgang Adamski. τ -smooth Borel measures on topological spaces. *Mathematische Nachrichten*, 78:97–107, 1977.
- [2] Mauricio Alvarez-Manilla, Achim Jung, and Klaus Keimel. The probabilistic powerdomain for stably compact spaces. *Theoretical Computer Science*, 328(3):221–244, 2004.
- [3] Patrick Billingsley. *Probability and Measure*. Wiley series in probability and mathematical statistics. John Wiley and Sons, 3rd edition, 1995.
- [4] Patrick Billingsley. *Convergence of Probability Measures*. Wiley series in probability and mathematical statistics. John Wiley and Sons, 2nd edition, 1999.
- [5] Matthew de Brecht. Quasi-Polish spaces. *Annals of Pure and Applied Logic*, 164(3):356–381, 2013.
- [6] Matthew de Brecht, Jean Goubault-Larrecq, Xiaodong Jia, and Zhenchao Lyu. Domain-complete and LCS-complete spaces. *Electronic Notes in Theoretical Computer Science*, 2019. Presented at the 8th International Symposium of Domain Theory and its Applications (ISDT'19). Available on arXiv:1902.11142 [math.GN].
- [7] Abbas Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.

- [8] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie Don Lawson, Michael Mislove, and Dana Stewart Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003.
- [9] Jean Goubault-Larrecq. Continuous previsions. In Jacques Duparc and Thomas A. Henzinger, editors, *Proceedings of the 16th Annual EACSL Conference on Computer Science Logic (CSL'07)*, volume 4646 of *Lecture Notes in Computer Science*, pages 542–557, Lausanne, Switzerland, September 2007. Springer.
- [10] Jean Goubault-Larrecq. De Groot duality and models of choice: Angels, demons and nature. *Mathematical Structures in Computer Science*, 20(2):169–237, 2010.
- [11] Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology*, volume 22 of *New Mathematical Monographs*. Cambridge University Press, 2013.
- [12] Jean Goubault-Larrecq. Isomorphism theorems between models of mixed choice. *Mathematical Structures in Computer Science*, 27(6):1032–1067, September 2017.
- [13] Jean Goubault-Larrecq. Products and projective limits of continuous valuations on T_0 spaces. *Mathematical Structures in Computer Science*, 31(2):234–254, 2021.
- [14] Jean Goubault-Larrecq. Probabilistic powerdomains and quasi-continuous domains. *Topology Proceedings*, 60:1–16, 2022.
- [15] Reinhold Heckmann. Spaces of valuations. In S. Andima, R. C. Flagg, G. Itzkowitz, Y. Kong, R. Kopperman, and P. Misra, editors, *Papers on General Topology and its Applications*, volume 806, pages 174–200. Annals of the New York Academy of Science, Dec 1996.
- [16] Claire Jones. *Probabilistic Non-Determinism*. PhD thesis, University of Edinburgh, 1990. Technical Report ECS-LFCS-90-105.
- [17] Claire Jones and Gordon Plotkin. A probabilistic powerdomain of evaluations. In *Proceedings of the 4th Annual Symposium on Logic in Computer Science*, pages 186–195. IEEE Computer Society Press, 1989.
- [18] Klaus Keimel and Jimmie Lawson. d -completions and the d -topology. *Annals of Pure and Applied Logic*, 159:292–306, 2009.
- [19] Jimmie D. Lawson. Valuations on continuous lattices. In R.-E. Hoffmann, editor, *Mathematische Arbeitspapiere*, volume 27, pages 204–225, Universität Bremen, 1982.
- [20] Yuri Vasilyevich Prokhorov. Convergence of random processes and limit theorems in probability theory. *Teoriya Veroyatnostoni i ee Primeneniya*, 1(2):177–238, 1956. English translation in *Theory of Probabilities and its Applications*, 1(2), 1956, pages 157–214.
- [21] Nait Saheb-Djahromi. Cpo's of measures for nondeterminism. *Theoretical Computer Science*, 12:19–37, 1980.
- [22] Andrea Schalk. *Algebras for Generalized Power Constructions*. PhD thesis, Technische Hochschule Darmstadt, 1993. Available from <http://www.cs.man.ac.uk/~schalk/publ/diss.ps.gz>.
- [23] Michael B. Smyth. Powerdomains and predicate transformers: A topological view. In Josep Diaz, editor, *Automata, Languages and Programming*, pages 662–675, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.
- [24] Regina Tix. Stetige Bewertungen auf topologischen Räumen. Diplomarbeit, Technische Hochschule Darmstadt, 1995.
- [25] Flemming Topsøe. Compactness and tightness in spaces of measures with the topology of weak convergence. *Mathematica Scandinavica*, 34:187–210, 1974.

UNIVERSITÉ PARIS-SACLAY, CNRS, ENS PARIS-SACLAY, LABORATOIRE MÉTH-
ODES FORMELLES, 91190, GIF-SUR-YVETTE, FRANCE.
E-mail address: `jgl@mf.cnrs.fr`