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ON LOCAL COMPACTNESS OF SPACES OF CONTINUOUS VALUATIONS

JEAN GOUBAULT-LARRECQ

ABSTRACT. We show that the spaces of continuous valuations, resp. subprobability valuations on a locally compact space is locally compact; similarly with probability valuations on locally compact, compact spaces. Continuous valuations are close cousins of measures. No separation property is assumed.

1. INTRODUCTION

A *continuous valuation* on a topological space X is a close cousin of a Borel measure. Continuous valuations on X form a space $\mathbf{V}X$, with a topology known as the weak topology, and similarly for the subspaces $\mathbf{V}_{\leq 1}X$ of subprobability valuations and \mathbf{V}_1X of probability valuations. We will define these notions more precisely below. It is known that $\mathbf{V}_{\leq 1}$ preserves various properties: stable compactness [2, Theorem 39], being a continuous dcpo [16, Theorem 5.2], being a quasi-continuous dcpo [14, Theorem 5.1], for example. Some of these preservation theorems extend over to \mathbf{V}_1 or to \mathbf{V} , but a conspicuously absent property in the list is local compactness. This is what we address in this paper.

The proof relies on credibilities, as already used in [14], on the Smyth powerdomain construction [23], on Jones' celebrated theorem that the probabilistic powerdomain preserves continuous dcpos [16, Theorem 5.2], and on a few things about certain functionals called superlinear previsions

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[12]. As a whole, the proof is therefore rather elaborate; but it also results from little more than an assemblage of previously known results, and as a consequence, is pretty short.

2. PRELIMINARIES

For background on topology, we refer the reader to [11].

A *compact* subset A of a space X is one such that one can extract a finite subcover from any of its open covers. No separation property is assumed. A subset A of X is *saturated* if and only if it is equal to the intersection of its open neighborhoods, or equivalently if and only if it is upwards-closed in the specialization preordering of X . The latter is defined by $x \leq y$ if and only if every open neighborhood of x contains y .

A space X is *locally compact* if and only if every point has a base of compact neighborhoods, or equivalently of compact saturated neighborhoods, since for any compact subset K of X , the upward closure $\uparrow K$ of K with respect to the specialization preordering of X is compact saturated. In non-Hausdorff spaces, a compact space may fail to be locally compact.

A space is T_0 if and only if its specialization preordering is antisymmetric. An *irreducible* closed subset C of X is a non-empty closed subset such that, for any two closed subsets C_1 and C_2 of X such that $C \subseteq C_1 \cup C_2$, C is included in C_1 or in C_2 already; equivalently, if C intersects two open sets, it must intersect their intersection. A space X is *sober* if and only if it is T_0 and every irreducible closed subset is of the form $\downarrow x$ for some point $x \in X$. Every Hausdorff space, for example, is sober. The notation $\downarrow x$ stands for the *downward closure* of x in X , namely the set of points y below x ; that is also the closure of $\{x\}$. Symmetrically, $\uparrow x$ stands for the *upward closure* of x , namely the set of points y above x . This notation extends to $\uparrow A$, for any subset A , denoting $\bigcup_{x \in A} \uparrow x$.

A function $f: P \rightarrow Q$ between posets is *monotonic* if and only if for all $x, x' \in P$, $x \leq x'$ implies $f(x) \leq f(x')$. It is *Scott-continuous* if and only if f is monotonic and for every directed family $(x_i)_{i \in I}$ with a supremum x in P , the (necessarily directed) family of elements $f(x_i)$ has $f(x)$ as supremum. A *directed* family is a non-empty family D such that any two elements of D have an upper bound in D . We write $\sup^\uparrow D$ for the supremum of a directed family D . Scott-continuity is equivalent to continuity with the respective Scott topologies on P and Q . The *Scott topology* on a poset P consists of those subsets U that are upwards closed ($x \in U$ and $x \leq x'$ implies $x' \in U$) and such that every directed family D that has a supremum in U intersects U .

A *dcpo* (short for *directed-complete partial order*) is a poset P in which every directed subset has a supremum. In a dcpo P , let $x \ll y$ (“ x is way below y ”) if and only if every directed family D such that $y \leq \sup^\uparrow D$

contains an element $d \in D$ such that $x \leq d$. A *basis* for P is a subset B such that, for every $x \in P$, $\downarrow_B x \stackrel{\text{def}}{=} \{b \in B \mid b \ll x\}$ is directed and has x as its supremum. A dcpo is *continuous* if and only if it has a basis. If so, the sets $\uparrow b \stackrel{\text{def}}{=} \{x \in P \mid b \ll x\}$ form a base of the Scott topology.

Let $\overline{\mathbb{R}}_+$ be the set of extended non-negative real numbers $\mathbb{R}_+ \cup \{\infty\}$, with its usual ordering. When needed, we will consider it with its Scott topology, whose open sets are the intervals $]t, \infty]$, $t \in \mathbb{R}_+$, plus \emptyset and $\overline{\mathbb{R}}_+$ itself. In passing, $\overline{\mathbb{R}}_+$ is an example of a continuous dcpo, with $s \ll t$ if and only if $s = 0$ or $s < t$.

We write $\mathcal{O}X$ for the lattice of open subsets of a space X . A *continuous valuation* on X is a map $\nu: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ that is *strict* ($\nu(\emptyset) = 0$), *modular* (for all $U, V \in \mathcal{O}X$, $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$) and Scott-continuous. We say that ν is *bounded* if $\nu(X) < \infty$, a *probability valuation* if and only if $\nu(X) = 1$, and a *subprobability valuation* if and only if $\nu(X) \leq 1$.

Continuous valuations are an alternative to measures that have become popular in domain theory [17, 16]. The first results that connected continuous valuations and measures are due to Saheb-Djahromi [21] and Lawson [19]. The current state of the art on this matter is the following. In one direction, every measure on the Borel σ -algebra of X induces a continuous valuation on X by restriction to the open sets, if X is hereditarily Lindelöf (namely, if every directed family of open sets contains a cofinal monotone sequence). This is an easy observation, and one half of Adamski's theorem [1, Theorem 3.1], which states that a space is hereditarily Lindelöf if and only if every measure on its Borel σ -algebra restricts to a continuous valuation on its open sets. In the other direction, every continuous valuation on a space X extends to a measure on the Borel sets provided that X is an LCS-complete space [6, Theorem 1]. An LCS-complete space is a G_δ subspace of a locally compact sober space, and the class of LCS-complete spaces contains all locally compact sober spaces, in particular all continuous dcpos from domain theory, all of M. de Brecht's quasi-Polish spaces [13] and therefore all Polish spaces.

The extension of a continuous valuation ν to a measure on the Borel σ -algebra is unique if ν is bounded. This is a consequence of the π - λ theorem [3, Theorem 3.2], usually applied to probability measures [3, Theorem 3.3], but the same proof works for bounded measures.

Hence the above maps from Borel measures to continuous valuations and conversely, defined on hereditary Lindelöf LCS-complete spaces (in particular, quasi-Polish spaces), are inverse of each other when restricted to bounded measures and bounded continuous valuations.

Let $\mathbf{V}X$ denote the space of continuous valuations on a space X , with the *weak topology*, defined by subbasic open sets $[U > r] \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}X \mid \nu(U) > r\}$, where $U \in \mathcal{O}X$ and $r \in \mathbb{R}_+$. We define its subspace \mathbf{V}_1X of probability valuations and $\mathbf{V}_{\leq 1}X$ (subprobability) similarly. In general, we write $\mathbf{V}_\bullet X$, where \bullet can be nothing, “ ≤ 1 ”, or “1”. The specialization ordering of each is the *stochastic ordering* \leq given by $\nu \leq \nu'$ if and only if $\nu(U) \leq \nu'(U)$ for every $U \in \mathcal{O}X$; indeed, $\nu \leq \nu'$ if and only if for every $U \in \mathcal{O}X$, for every $r \in \mathbb{R}_+$, $\nu \in [U > r]$ implies $\nu' \in [U > r]$.

There is a related topology on spaces of probability measures. A sequence of probability measures P_n , $n \in \mathbb{N}$, on a metric space *converges weakly* to a probability measure P if and only if $\liminf_{n \in \mathbb{N}} P_n(U) \geq P(U)$ for every open subset U [4, Portmanteau Theorem 2.1]. It is easy to see that this is equivalent to: for every open subset U and for every $r \in \mathbb{R}_+$, $P(U) > r$ implies that $P_n(U) > r$ for n large enough. Hence we may define the *weak topology* on spaces of measures (not just probability measures) on a topological space X (not just a metric space) as given via a subbasis of sets $\{P \mid P(U) > r\}$, $U \in \mathcal{O}X$, $r \in \mathbb{R}_+$. We obtain the following.

Proposition 2.1. *The weak topology makes the spaces $\mathcal{M}_{\leq 1}X$ and \mathcal{M}_1X of subprobability, resp. probability measures homeomorphic to $\mathbf{V}_{\leq 1}X$ and \mathbf{V}_1X , respectively, for every hereditarily Lindelöf LCS-complete space X .*

\mathbf{V}_\bullet is the object part of an endofunctor on the category of topological spaces, whose morphism part is defined as follows: for every continuous map $f: X \rightarrow Y$, $\mathbf{V}_\bullet f$ maps every $\nu \in \mathbf{V}_\bullet X$ to its *image valuation* $f[\nu]$, where $f[\nu](V) \stackrel{\text{def}}{=} \nu(f^{-1}(V))$ for every $V \in \mathcal{O}Y$.

We will prove slightly more than the fact that \mathbf{V}_\bullet preserves local compactness, where \bullet is nothing, “ ≤ 1 ”, or “1”. A space X is *core-compact* if and only if $\mathcal{O}X$ is a continuous dcpo; every locally compact space is core-compact [11, Theorem 5.2.9]. The connection between the two notions can be made more precise as follows. Every topological space X has a *sobrification* $\mathcal{S}(X)$ (or X^s), which is the free sober space over X [11, Theorem 8.2.44]; then X is core-compact if and only if $\mathcal{S}(X)$ is locally compact [11, Proposition 8.3.11]. $\mathcal{S}(X)$ can be built as the collection of irreducible closed subsets, with the topology whose open sets (all of them, not just a base) are $\diamond U \stackrel{\text{def}}{=} \{F \in \mathcal{S}(X) \mid F \cap U \neq \emptyset\}$, $U \in \mathcal{O}X$. In particular, $\diamond: U \mapsto \diamond U$ is an order-isomorphism between $\mathcal{O}X$ and $\mathcal{O}\mathcal{S}(X)$. This induces a homeomorphism between $\mathbf{V}_\bullet X$ and $\mathbf{V}_\bullet \mathcal{S}(X)$.

3. CREDIBILITIES, PREVISIONS AND THE SMYTH POWERDOMAIN

The proof of our main theorem will rely on reducing the question to spaces of valuations on Smyth powerdomains, and on using notions of credibilities [10] and of previsions [9, 12]. We need to give a quick tour of the part of that theory that we need.

Let $\mathcal{L}X$ denote the dcpo of all lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$; a map $f: X \rightarrow \overline{\mathbb{R}}_+$ is lower semicontinuous if and only if it is continuous, where $\overline{\mathbb{R}}_+$ is given its Scott topology. The ordering on $\mathcal{L}X$ is the pointwise ordering. For every monotone map $\nu: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$, for every $h \in \mathcal{L}X$, there is a *Choquet integral* $\int_{x \in X} h(x) d\nu$, defined as the indefinite Riemann integral $\int_0^\infty \nu(h^{-1}([t, \infty])) dt$. We need to reprove in part some of the results of [10] on Choquet integrals, since that paper only considers integrals of maps from X to \mathbb{R}_+ ; we will do this in Lemma 3.1 below.

A *prevision* on a space X is a Scott-continuous map $F: \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$ that is *positively homogeneous* in the sense that $F(ah) = aF(h)$ for all $a \in \mathbb{R}_+$ and $h \in \mathcal{L}X$. There is a space $\mathbb{P}X$ of previsions on X , whose topology is generated by sets $[h > r] \stackrel{\text{def}}{=} \{F \mid F(h) > r\}$, $h \in \mathcal{L}X$, $r \in \mathbb{R}_+$.

For example, any continuous valuation ν on X gives rise to a prevision $G: h \mapsto \int_{x \in X} h(x) d\nu$. Such a prevision is *linear*, in the sense that $G(h + h') = G(h) + G(h')$ for all $h, h' \in \mathcal{L}X$. Let $\mathbb{P}_p X$ be the subspace of $\mathbb{P}X$ of linear previsions. In the reverse direction, every linear prevision $G \in \mathbb{P}_p X$ gives rise to a continuous valuation $U \mapsto G(\chi_U)$, where χ_U is the characteristic map of the open set U , and the two constructions are inverse of each other. Additionally, those two constructions define continuous maps between $\mathbf{V}X$ and $\mathbb{P}_p X$ [24, Satz 4.16]. We will therefore equate continuous valuations with linear previsions.

A prevision is *superlinear* if and only if $G(h + h') \geq G(h) + G(h')$ for all $h, h' \in \mathcal{L}X$. As in [12], we write $\mathbb{P}_{\text{dp}} X$ for the subspace of $\mathbb{P}X$ consisting of superlinear previsions.

Among the continuous valuations, there are the probability valuations and the subprobability valuations. Similarly, we say that a prevision F is *subnormalized* (resp., *normalized*) iff $F(\mathbf{1} + h) \leq \mathbf{1} + F(h)$ (resp., $=$) for every $h \in \mathcal{L}X$, where $\mathbf{1}$ is the constant function with value 1. The homeomorphism between $\mathbf{V}X$ and $\mathbb{P}_p X$ restricts to homeomorphisms between $\mathbf{V}_{\leq 1} X$ (resp., $\mathbf{V}_1 X$) and the subspace $\mathbb{P}_p^{\leq 1} X$ (resp., $\mathbb{P}_p^1 X$) of subnormalized (resp., normalized) linear previsions on X . We write $\mathbb{P}_{\text{dp}}^{\leq 1} X$, $\mathbb{P}_{\text{dp}}^1 X$ for the corresponding spaces of (sub)normalized, superlinear previsions. In general, we write $\mathbb{P}_{\text{dp}}^\bullet X$, where \bullet can be nothing, “ ≤ 1 ”, or “1”.

For every topological space X , let $\mathcal{Q}X$ be the collection of non-empty compact saturated subsets of X , ordered by reverse inclusion \supseteq . This

is the *Smyth powerdomain* of X [23]; see also [22, Section 7]. Let $\mathcal{Q}_V X$ be the same set, but with the *upper Vietoris topology*, whose basic open subsets are $\square U \stackrel{\text{def}}{=} \{Q \in \mathcal{Q}X \mid Q \subseteq U\}$, $U \in \mathcal{O}X$. We will first use this in item (6) of Lemma 3.1 below. The functions $\sum_{j=1}^m a_j \mathbf{u}_{Q_j}$ of item (7) of the same lemma are the (*simple*) *credibilities* of [10, Theorem 6.18].

Lemma 3.1. *The following properties hold, where $h \in \mathcal{L}X$ and ν is a monotonic map from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$.*

- (1) *The Choquet integral $\int_{x \in X} h(x) d\nu$ is linear in ν , monotonic and even Scott-continuous in ν .*
- (2) *If ν is Scott-continuous, then the Choquet integral is Scott-continuous in h .*
- (3) *If ν is a continuous valuation, then the Choquet integral is also linear in h .*
- (4) *For every $U \in \mathcal{O}X$, $\int_{x \in X} \chi_U(x) d\nu = \nu(U)$.*
- (5) *Given any continuous valuation ν on $\mathcal{Q}_V X$, there is a Scott-continuous map $\nu^\square: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$ defined by $\nu^\square(U) \stackrel{\text{def}}{=} \nu(\square U)$ for every $U \in \mathcal{O}X$. Then, for every $h \in \mathcal{L}X$, the map $h^*: Q \mapsto \min_{x \in Q} h(x)$ is in $\mathcal{L}\mathcal{Q}_V X$ and $\int_{x \in X} h(x) d\nu^\square = \int_{Q \in \mathcal{Q}_V X} h^*(Q) d\nu$.*
- (6) *For every compact saturated subset Q of X , the unanimity game $\mathbf{u}_Q: \mathcal{O}X \rightarrow \overline{\mathbb{R}}_+$, which maps every $U \in \mathcal{O}X$ to 1 if $Q \subseteq U$ and to 0 otherwise, is a Scott-continuous map from $\mathcal{O}X$ to $\overline{\mathbb{R}}_+$.*
- (7) *Letting $\nu \stackrel{\text{def}}{=} \sum_{j=1}^m a_j \mathbf{u}_{Q_j}$, where each Q_j is compact saturated and $a_j \in \mathbb{R}_+$, the map $F: \mathcal{L}X \rightarrow \overline{\mathbb{R}}_+$ defined by:*

$$F(h) \stackrel{\text{def}}{=} \int_{x \in X} h(x) d\nu = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$$

for every $h \in \mathcal{L}X$, is a superlinear prevision.

Proof. (1) The fact that the Choquet integral is linear in ν , namely that it commutes with scalar products by non-negative real numbers and with addition of continuous valuations, follows from the linearity of indefinite Riemann integration. It is also monotonic in ν . In order to show Scott-continuity, we consider a directed family $(\nu_i)_{i \in I}$, with (pointwise) supremum ν , and we observe that $\int_{x \in X} h(x) d\nu = \int_0^\infty \sup_{i \in I}^\uparrow \nu_i(h^{-1}([t, \infty])) dt$. The key is that the maps $t \mapsto \nu_i(h^{-1}([t, \infty]))$ are antitone (all antitone maps are Riemann-integrable), and that indefinite Riemann integration of antitone maps f is Scott-continuous in f , see [24, Lemma 4.2]. Therefore $\int_{x \in X} h(x) d\nu = \sup_{i \in I}^\uparrow \int_0^\infty \nu_i(h^{-1}([t, \infty])) dt = \sup_{i \in I}^\uparrow \int_{x \in X} h(x) d\nu_i$.

(2) The proof works as Tix's original proof of the same result in the special case where ν is a continuous valuation [24, Satz 4.4], and relies

on [24, Lemma 4.2], just like item (1). Explicitly, let $(h_i)_{i \in I}$ be a directed family in $\mathcal{L}X$, with (pointwise) supremum h . For every $t \in \mathbb{R}_+$, $h^{-1}(]t, \infty]) = \{x \in X \mid \sup_{i \in I}^\uparrow h_i(x) > t\} = \bigcup_{i \in I}^\uparrow h_i^{-1}(]t, \infty])$. Therefore,

$$\begin{aligned} \int_{x \in X} h(x) d\nu &= \int_0^\infty \nu(\bigcup_{i \in I}^\uparrow h_i^{-1}(]t, \infty])) dt \\ &= \int_0^\infty \sup_{i \in I}^\uparrow \nu(h_i^{-1}(]t, \infty])) dt \\ &= \sup_{i \in I}^\uparrow \int_0^\infty \nu(h_i^{-1}(]t, \infty])) dt = \sup_{i \in I}^\uparrow \int_{x \in X} h_i(x) d\nu, \end{aligned}$$

using the Scott-continuity of ν and the Scott-continuity of indefinite Riemann integration of antitone maps.

(3) This is a result of Tix [24, Satz 4.4].

(4) $\int_{x \in X} \chi_U(x) d\nu = \int_0^\infty \nu(\chi_U^{-1}(]t, \infty])) dt = \int_0^1 \nu(U) dt + \int_1^\infty 0 dt = \nu(U)$.

(5) This is as with [10, Lemma 7.5]. The fact that ν^\square is Scott-continuous follows from the fact that ν is, and that the \square operator is, too. For the latter, observe that for every directed family $(U_i)_{i \in I}$ of open subsets of X , for every $Q \in \mathcal{Q}_V X$, $Q \in \square \bigcup_{i \in I}^\uparrow U_i$ if and only if $Q \subseteq \bigcup_{i \in I}^\uparrow U_i$, which is equivalent to $Q \subseteq U_i$ (namely, $Q \in \square U_i$) for some $i \in I$, because Q is compact.

For every $Q \in \mathcal{Q}_V X$, the minimum of $h(x)$ when x ranges over Q is reached, since Q is compact and non-empty. For every $t \in \mathbb{R}_+$, $Q \in h^{*-1}(]t, \infty])$ if and only if $h^*(Q) > t$. The latter certainly implies that $h(x) > t$ for every $x \in Q$, hence that $Q \in \square h^{-1}(]t, \infty])$. Conversely, if $Q \in \square h^{-1}(]t, \infty])$, then let us pick $x \in Q$ such that $h(x)$ is the least value reached by h on Q ; then $h^*(Q) = h(x) > t$, so $Q \in h^{*-1}(]t, \infty])$. Hence $h^{*-1}(]t, \infty]) = \square h^{-1}(]t, \infty])$. This implies that h^* is in $\mathcal{L}\mathcal{Q}_V X$, in particular.

Now:

$$\begin{aligned} \int_{Q \in \mathcal{Q}_V X} h^*(Q) d\nu &= \int_0^\infty \nu(h^{*-1}(]t, \infty])) dt \\ &= \int_0^\infty \nu(\square h^{-1}(]t, \infty])) dt \\ &= \int_0^\infty \nu^\square(h^{-1}(]t, \infty])) dt = \int_{x \in X} h(x) d\nu^\square. \end{aligned}$$

(6) Monotonicity is clear. For every directed family $(U_i)_{i \in I}$ of open subsets of X , $u_Q(\bigcup_{i \in I}^\uparrow U_i) = 1$ if and only if $Q \in \square \bigcup_{i \in I}^\uparrow U_i$, which is equivalent to the existence of an $i \in I$ such that $Q \in \square U_i$ (equivalently, $u_Q(U_i) = 1$), as we have seen at the beginning of the proof of item (5).

(7) That would be a consequence of [10, Propositions 7.2, 7.6], except for the fact that our functions h take their values in $\overline{\mathbb{R}}_+$. We verify that $\int_{x \in X} h(x) d\nu = \sum_{j=1}^m a_j \int_{x \in X} h(x) du_{Q_j} = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$: the first equality is by item (1), and the second one is because $\int_{x \in X} h(x) du_{Q_i}$ is equal to $\int_0^\infty u_{Q_i}(h^{-1}([t, \infty])) dt = \int_0^{\min_{x \in Q_i} h(x)} 1 dt + \int_{\min_{x \in Q_i} h(x)}^\infty 0 dt = \min_{x \in Q_i} h(x)$.

It is easy to see that $F(h)$ is superlinear, because $\min_{x \in Q_i} ah(x) = a \min_{x \in Q_i} h(x)$ and $\min_{x \in Q_i} (h(x) + h'(x)) \geq \min_{x \in Q_i} h(x) + \min_{x \in Q_i} h'(x)$, for all $a \in \mathbb{R}_+$ and $h, h' \in \mathcal{L}X$. Scott-continuity comes from the fact that $F(h) = \int_{x \in X} h(x) d\nu$, where $\nu \stackrel{\text{def}}{=} \sum_{j=1}^m a_j u_{Q_j}$, that ν is Scott-continuous (by item (6)), and by using item (2). \square

Let us turn to \mathcal{Q}_V . The specialization preordering of $\mathcal{Q}_V X$ is \supseteq , and in particular $\mathcal{Q}_V X$ is T_0 . The Scott topology on $\mathcal{Q}X$ is finer than the upper Vietoris topology, and coincides with it when X is locally compact and sober [11, Lemma 8.3.26], and then $\mathcal{Q}X$ itself is a continuous depo [11, Proposition 8.3.25].

\mathcal{Q}_V is an endofunctor on the category of topological spaces, whose action on continuous maps $f: X \rightarrow Y$ is given by $\mathcal{Q}_V f(Q) \stackrel{\text{def}}{=} \uparrow f[Q]$, where $f[Q]$ denotes the image of Q under f . That functor is part of a monad [22, Section 7], of which we need to know the unit $\eta^{\mathcal{Q}}$: for every space X , for every $x \in X$, $\eta_X^{\mathcal{Q}}(x) = \uparrow x$. We also note that $(\eta_X^{\mathcal{Q}})^{-1}(\square U) = U$ for every $U \in \mathcal{O}X$.

By [12, Lemma 3.20], for every space X , there is a continuous map $s_{\text{DP}}^\bullet: \mathbb{P}_{\text{DP}}^\bullet X \rightarrow \mathcal{Q}_V \mathbb{P}_{\text{DP}}^\bullet X$ defined by $s_{\text{DP}}^\bullet(F) \stackrel{\text{def}}{=} \{G \in \mathbb{P}_{\text{DP}}^\bullet X \mid G \geq F\}$. The ordering \leq between previsions is the specialization ordering, which is pointwise, and \geq is the opposite ordering. Up to the homeomorphism between $\mathbb{P}_{\text{DP}}^\bullet X$ and $\mathbf{V}_\bullet X$, we restate the latter as follows.

Lemma 3.2. *For every topological space X , there is a continuous map $s_{\text{DP}}^\bullet: \mathbb{P}_{\text{DP}}^\bullet X \rightarrow \mathcal{Q}_V \mathbf{V}_\bullet X$, defined by:*

$$s_{\text{DP}}^\bullet(F) \stackrel{\text{def}}{=} \left\{ \nu' \in \mathbf{V}_\bullet X \mid \forall h \in \mathcal{L}X, \int_{x \in X} h(x) d\nu' \geq F(h) \right\}.$$

In order to show that $\mathbf{V}_\bullet X$ is locally compact (if X is), we will need to find enough compact subsets. They will be provided to us by the images of certain superlinear previsions under s_{DP}^\bullet .

4. THE MAIN THEOREM

We spend most of this section proving the following theorem. We remember that all locally compact spaces are core-compact.

Theorem 4.1. *For every core-compact space X , $\mathbf{V}X$ and $\mathbf{V}_{\leq 1}X$ are locally compact. If X is also compact, then \mathbf{V}_1X is locally compact.*

Proof. Replacing X by $\mathcal{S}(X)$ if necessary, we may assume that X is locally compact and sober.

A fundamental theorem due to Jones [16, Theorem 5.2] states that for every continuous depo P , $\mathbf{V}_{\leq 1}P$ is a continuous depo under the stochastic ordering, and that a basis is given by the *simple valuations*, namely those of the form $\sum_{i=1}^n a_i \delta_{x_i}$, where each a_i is in \mathbb{R}_+ and $x_i \in P$. The notation δ_x refers to the *Dirac valuation* at x , defined by $\delta_x(U) \stackrel{\text{def}}{=} 1$ if $x \in U$, 0 otherwise, for every $U \in \mathcal{O}X$.

A similar result holds for $\mathbf{V}P$ [8, Theorem IV-9.16], and for \mathbf{V}_1P provided that P is also pointed [7, Corollary 3.3]. (A poset is *pointed* if and only if it has a least element.) We will apply those results to $P \stackrel{\text{def}}{=} \mathcal{Q}X$, and we notice that if X is compact, then P is pointed, as X itself will be the least element of P in that case.

Let $\nu \in \mathbf{V}_{\bullet}X$, and let \mathcal{U} be any open neighborhood of ν . Then ν is in some finite intersection of subbasic open sets $\bigcap_{i=1}^n [U_i > r_i]$ that is included in \mathcal{U} , where each U_i is open in X and $r_i \in \mathbb{R}_+$. We consider $\mu \stackrel{\text{def}}{=} \mathbf{V}\eta_X^{\mathcal{Q}}(\nu) \in \mathbf{V}_{\bullet}\mathcal{Q}X$. We recall that $\eta_X^{\mathcal{Q}}$ is the unit of the $\mathcal{Q}_{\mathbf{V}}$ monad, and maps every point $x \in X$ to $\uparrow x \in \mathcal{Q}_{\mathbf{V}}X$. For every open subset U of X , $\mu(\square U) = \eta_X^{\mathcal{Q}}[\nu](\square U) = \nu((\eta_X^{\mathcal{Q}})^{-1}(\square U)) = \nu(U)$. It follows that μ is in $\bigcap_{i=1}^n [\square U_i > r_i]$. The latter is open in the upper Vietoris topology on $\mathbf{V}_{\bullet}\mathcal{Q}_{\mathbf{V}}X = \mathbf{V}_{\bullet}P$, hence in the Scott topology of the stochastic ordering. Since X is locally compact and sober, P is a continuous depo, so $\mathbf{V}_{\bullet}P$ is a continuous depo with a basis of simple valuations, and therefore there is a simple valuation $\xi \stackrel{\text{def}}{=} \sum_{j=1}^m a_j \delta_{Q_j}$ in $\mathbf{V}_{\bullet}P$ that is way below μ and in $\bigcap_{i=1}^n [\square U_i > r_i]$.

We may form ξ^{\square} , as in Lemma 3.1, item (5): for every $U \in \mathcal{O}X$, $\xi^{\square}(U) = \xi(\square U)$. We check easily that ξ^{\square} is equal to $\sum_{j=1}^m a_j \mathbf{u}_{Q_j}$. Using Lemma 3.1, item (7) allows us to build a superlinear prevision F on X by letting $F(h) \stackrel{\text{def}}{=} \int_{x \in X} h(x) d\xi^{\square} = \sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$ for every $h \in \mathcal{L}X$.

Now $s_{\text{dp}}^{\bullet}(F)$ is a compact saturated subset of $\mathbf{V}_{\bullet}X$, by Lemma 3.2. We claim that ν is in the interior of $s_{\text{dp}}^{\bullet}(F)$, and that $s_{\text{dp}}^{\bullet}(F)$ is included in \mathcal{U} ; this will end our proof.

We start by showing that $s_{\text{dp}}^{\bullet}(F) \subseteq \mathcal{U}$. Let ν' be any element of $s_{\text{dp}}^{\bullet}(F)$. In other words, for every $h \in \mathcal{L}X$, $\int_{x \in X} h(x) d\nu' \geq F(h) =$

$\sum_{j=1}^m a_j \min_{x \in Q_j} h(x)$. For each $i \in \{1, \dots, n\}$, we apply this to $h \stackrel{\text{def}}{=} \chi_{U_i}$; then $\min_{x \in Q_j} h(x) = \mathbf{u}_{Q_j}(U_i)$, so $F(\chi_{U_i}) = \sum_{j=1}^m a_j \mathbf{u}_{Q_j}(U_i) = \xi^\square(U_i) = \xi(\square U_i)$, and therefore $\nu'(U_i) = \int_{x \in X} \chi_{U_i}(x) d\nu' \geq F(\chi_{U_i}) = \xi(\square U_i) > r_i$, since $\xi \in [\square U_i > r_i]$. Hence $\nu' \in \bigcap_{i=1}^n [U_i > r_i] \subseteq \mathcal{U}$.

Next, we verify that ν is in the interior of $s_{\text{dp}}^\bullet(F)$. We use the fact that ξ is way below μ , equivalently that μ is in the open set $\uparrow \xi$. Since $\mu = \mathbf{V}\eta_X^\square(\nu)$, ν is in $(\mathbf{V}\eta_X^\square)^{-1}(\uparrow \xi)$, which is open since $\mathbf{V}\eta_X^\square$ is continuous. It remains to show that $(\mathbf{V}\eta_X^\square)^{-1}(\uparrow \xi)$ is included in $s_{\text{dp}}^\bullet(F)$.

For every $\nu' \in (\mathbf{V}\eta_X^\square)^{-1}(\uparrow \xi)$, by definition ξ is way below, in particular less than or equal to $\mu' \stackrel{\text{def}}{=} \mathbf{V}\eta_X^\square(\nu')$. The latter is such that $\mu'(\square U) = \eta_X^\square[\nu'](\square U) = \nu'((\eta_X^\square)^{-1}(\square U)) = \nu'(U)$ for every $U \in \mathcal{O}X$. Hence $\nu' = \mu'^\square$. By Lemma 3.1, item (5), $\int_{x \in X} h(x) d\nu' = \int_{Q \in \mathcal{Q}_{\mathbf{V}X}} h^*(Q) d\mu'$. Since $\xi \leq \mu'$, the latter is larger than or equal to $\int_{Q \in \mathcal{Q}_{\mathbf{V}X}} h^*(Q) d\xi = \int_{x \in X} h(x) d\xi^\square = F(h)$, using Lemma 3.1, item (5) once again. We have shown that $\int_{x \in X} h(x) d\nu' \geq F(h)$ for every $h \in \mathcal{L}X$, so $\nu' \in s_{\text{dp}}^\bullet(F)$, as promised. \square

When X is compact, $\mathbf{V}_1 X = s_{\text{dp}}^\bullet(F)$, where F is the normalized superlinear prevision $h \in \mathcal{L}X \mapsto \int_{x \in X} h(x) du_X = \min_{y \in X} h(y)$. Indeed, for every probability valuation ν on X , for every $h \in \mathcal{L}X$, $\int_{x \in X} h(x) d\nu \geq \int_{x \in X} \min_{y \in X} h(y) d\nu = \min_{y \in X} h(y)$. We therefore obtain the following.

Fact 4.2. *If X is compact, then $\mathbf{V}_1 X$ is compact.*

Lemma 4.3. *Let \bullet be nothing, “ ≤ 1 ”, or “1”. For every space X , $\mathbf{V}_\bullet X$ is sober.*

Proof. For every space X , $\mathbf{V}X$ is sober. The argument is due to R. Tix [24, Satz 5.4], following ideas by R. Heckmann (see [15, Section 2.3]). The sober subspaces of a sober space Z coincide with the subsets that are closed in the strong topology on Z [18, Corollary 3.5]. The latter is also known as the Skula topology, and is the smallest one generated by the original topology on Z and all the downwards-closed subsets. In particular, any closed subspace of a sober space is sober, any saturated subspace of a sober space is sober. Hence $\mathbf{V}_{\leq 1} Y$ is sober, being equal to the closed subspace $\mathbf{V}Y \setminus [Y > 1]$ of $\mathbf{V}Y$, and $\mathbf{V}_1 Y$ is sober, being upwards-closed in $\mathbf{V}_{\leq 1} Y$. \square

Corollary 4.4. *\mathbf{V} and $\mathbf{V}_{\leq 1}$ restrict to endofunctors on the category of locally compact (resp., locally compact sober) spaces and continuous maps. \mathbf{V}_1 restricts to an endofunctor on the category of compact, locally compact (resp., compact, locally compact sober) spaces.*

We finish with the mandatory application to spaces of measures, which follows directly, using the considerations developed in Section 2 on the relationship between measures and continuous valuations.

Corollary 4.5. *For every hereditarily Lindelöf, LCS-complete space X —in particular, for every quasi-Polish space X —if X is locally compact, then $\mathcal{M}_{\leq 1}X \cong \mathbf{V}_{\leq 1}X$ are locally compact; if X is locally compact and compact, then $\mathcal{M}_1X \cong \mathbf{V}_1X$ are locally compact and compact.*

The locally compact, quasi-Polish spaces are exactly the locally compact, sober, second-countable spaces [5, Theorem 44]. It is easy to see that the \mathbf{V}_\bullet functor preserves second countability. Using Lemma 4.3, we obtain our final corollary.

Corollary 4.6. *For every locally compact quasi-Polish space X , $\mathcal{M}_{\leq 1}X \cong \mathbf{V}_{\leq 1}X$ are locally compact and quasi-Polish.*

5. CONCLUSION

When trying to show that a space Z is locally compact, the direct approach—which is how we proceeded—consists in picking an arbitrary point $z \in Z$, an arbitrary open neighborhood V of z , and to find a compact (saturated) neighborhood Q of z included in V . The main difficulty is in finding enough compact saturated sets Q .

Previous work has shown that $\mathbf{V}_{\leq 1}X$ is stably compact for every stably compact space X , and gives a hint at what compact saturated subsets of $\mathbf{V}_{\leq 1}X$ we may need for the task [2, Theorem 39]. (A stably compact space is a compact, locally compact sober space that is *coherent*, in the sense that the intersection of any two compact saturated subsets is compact.) Indeed, in that case, the subsets $[Q \geq r]^{\leq 1} \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_{\leq 1}X \mid \forall U \in \mathcal{O}X, Q \subseteq U \Rightarrow \nu(U) \geq r\}$ form the required family of compact saturated sets, where Q ranges over the compact saturated subsets of X and $r \in \mathbb{R}_+$. We might therefore assume that analogous sets $[Q \geq r]^\bullet \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_\bullet X \mid \forall U \in \mathcal{O}X, Q \subseteq U \Rightarrow \nu(U) \geq r\}$ would be compact saturated in $\mathbf{V}_\bullet X$, and could be used to form enough compact saturated neighborhoods of points $\nu \in \mathbf{V}_\bullet X$, when X is locally compact sober—but without coherence (or compactness).

Without using the detour through superlinear previsions, however, it is not completely clear how one might prove that $[Q \geq r]^\bullet$ is compact, to start with. It *is* true that $[Q \geq r]^\bullet$ is compact saturated in $\mathbf{V}_\bullet X$, for every compact saturated subset Q of any topological space X , and for every $r \in \mathbb{R}_+$. Indeed, $[Q \geq r]^\bullet = s_{\text{DP}}^\bullet(F)$, where F is the superlinear prevision $h \in \mathcal{L}X \mapsto r \int_{x \in X} h(x) du_Q$, as one may check by using Lemma 3.1.

However, the proof we have given needs many more compact saturated subsets, and we have obtained them as $s_{\text{DP}}^{\bullet}(F)$ for superlinear previsions of the form $h \in \mathcal{L}X \mapsto \int_{x \in X} h(x) d \sum_{i=1}^n a_i u_{Q_i}$. In general, $s_{\text{DP}}^{\bullet}(F)$ is compact saturated in $\mathbf{V}_{\bullet}X$, for any superlinear prevision F on X . This naturally leads to the following question.

- (1) What are the compact saturated subsets of $\mathbf{V}_{\bullet}X$, for an arbitrary topological space X ?

In investigating (1), one may restrict X to some interesting subclasses. The answer to (1) is already known for stably compact spaces: the compact saturated subsets of $\mathbf{V}_{\leq 1}X$ are exactly the intersections of finite unions of sets of the form $[Q \geq r]^{\leq 1}$ given as above [2]. One should also compare this with Prokhorov's characterization of compact sets of probability measures as those sets that are uniformly tight [20], see also the fundamental paper by Topsøe [25]. In the meantime, one should observe that for every topological space X , there is a continuous map $r_{\text{DP}}: \mathcal{Q}_{\mathbf{V}}(\mathbf{V}_{\bullet}X) \rightarrow \mathbb{P}_{\text{DP}}^{\bullet}X$ such that $r_{\text{DP}} \circ s_{\text{DP}}^{\bullet} = \text{id}_{\mathbb{P}_{\text{DP}}^{\bullet}X}$ and $s_{\text{DP}}^{\bullet} \circ r_{\text{DP}} \leq \text{id}_{\mathcal{Q}_{\mathbf{V}}(\mathbf{V}_{\bullet}X)}$ [12, Proposition 3.22], and that those maps define a homeomorphism between $\mathbb{P}_{\text{DP}}^{\bullet}X$ and the subspace $\mathcal{Q}_{\mathbf{V}}^{\text{conv}}(\mathbf{V}_{\bullet}X)$ of convex elements of $\mathcal{Q}_{\mathbf{V}}\mathbf{V}_{\bullet}X$ [12, Theorem 4.15]. Hence the *convex* compact saturated subsets of $\mathbf{V}_{\bullet}X$ are known: they are exactly the sets $s_{\text{DP}}(F)$, where F is a superlinear prevision on X .

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UNIVERSITÉ PARIS-SACLAY, CNRS, ENS PARIS-SACLAY, LABORATOIRE MÉTH-
ODES FORMELLES, 91190, GIF-SUR-YVETTE, FRANCE.
E-mail address: `jgl@mf.cnrs.fr`