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# Tight bounds on adjacency labels for monotone graph classes

Édouard Bonnet\*    Julien Duron†    John Sylvester‡    Viktor Zamaraev§  
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## Abstract

A class of graphs admits an adjacency labeling scheme of size  $f(n)$ , if the vertices of any  $n$ -vertex graph  $G$  in the class can be assigned binary strings (aka labels) of length  $f(n)$  so that the adjacency between each pair of vertices in  $G$  can be determined only from their labels. The Implicit Graph Conjecture claimed that any graph class which is *hereditary* (i.e. closed under taking induced subgraphs) and *factorial* (i.e. containing  $2^{\Theta(n \log n)}$  graphs on  $n$  vertices) admits an adjacency labeling scheme of order optimal size  $\mathcal{O}(\log n)$ . This 30-year old conjecture was recently disproved by Hatami and Hatami who exposed hereditary factorial classes that do not admit an adjacency labeling scheme of size  $n^{1/2-\delta}$  for any fixed  $\delta > 0$ .


In this work we show that the Implicit Graph Conjecture does not hold even for graph classes that are *monotone*, i.e. closed under taking subgraphs. More specifically, we show that there are monotone factorial graph classes for which the size of any adjacency labeling scheme is  $\Omega(\log^2 n)$ . In contrast to the general case of hereditary classes, this bound deviates only a factor of  $\log n$  from the order optimal one. In fact this deviation is best possible, as any monotone factorial class admits an adjacency labeling scheme of size  $\mathcal{O}(\log^2 n)$ .


This is a consequence of our general result that establishes tight bounds on the size of adjacency labeling schemes for monotone graph classes: for any function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  with  $\log x \leq f(x) \leq x^{1-\delta}$  for some constant  $\delta > 0$ , that satisfies some natural condition, there exist monotone graph classes, in which the number of  $n$ -vertex graphs grows as  $2^{\mathcal{O}(nf(n))}$  and that do not admit adjacency labels of size at most  $f(n) \log n$ . On the other hand any such class admits adjacency labels of size  $\mathcal{O}(f(n) \log n)$ , which is a factor of  $\log n$  away from the order optimal bound  $\mathcal{O}(f(n))$ . This is the first example of tight bounds on adjacency labels for graph classes that do not admit order optimal adjacency labeling schemes.


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# 1 Introduction

A *class* of graphs is a set of graphs which is closed under isomorphism. For a class of graphs  $\mathcal{X}$  we denote by  $\mathcal{X}_n$  the set of graphs in  $\mathcal{X}$  with vertex set  $[n]$ . A *coding* of graphs is representation of graphs by words in a finite alphabet. In this paper we will always assume the binary alphabet  $\{0, 1\}$ . One of the main considerations with graph representations is their succinctness; clearly, any representation of  $n$ -vertex graphs in a class  $\mathcal{X}$  would require at least  $\lceil \log |\mathcal{X}_n| \rceil$  bits for some graphs in  $\mathcal{X}_n$ . Another consideration is whether the representation is global or local.

Standard graph representations, such as adjacency matrix or adjacency lists, are examples of *global* representations, where a graph is stored in a single data structure that needs to be accessed in order to query some information about the graph, e.g., adjacency between a pair of vertices. By contrast, in *local* graph representations, the encoding of a graph is distributed over its vertices in such a way that the queries can be answered by looking only into the local information associated with the vertices involved in the query. In this work we are concerned with local graph representations for adjacency queries, i.e., queries that given two vertices answer whether they are adjacent or not.

Let  $\mathcal{X}$  be a class of graphs and  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a function. An  $f(n)$ -bit adjacency labeling scheme (or simply  $f(n)$ -bit labeling scheme) for  $\mathcal{X}$  is a pair (encoder, decoder) of algorithms where for any  $n$ -vertex graph  $G \in \mathcal{X}_n$  the encoder assigns to the vertices of  $G$  binary strings of length  $f(n)$ , called *labels*, such that the adjacency between any pair of vertices can be inferred by the decoder only from their labels. We note that the decoder depends on the class  $\mathcal{X}$ , but not on the graph  $G$ . The function  $f$  is the *size* of the labeling scheme. Adjacency labeling schemes were introduced by Kannan, Naor, and Rudich [KNR88, KNR92], and independently by Muller [Mul88] in the late 1980's and have been actively studied since then.

Adjacency labeling schemes are closely related to induced universal graphs, which we will refer to simply as universal graphs, as this is the only type of universal graph that we will see in this paper. For a function  $u : \mathbb{N} \rightarrow \mathbb{N}$ , a *universal graph sequence* or simply *universal graph* of size  $u(n)$  is a sequence of graphs  $(U_n)_{n \in \mathbb{N}}$  such that for every  $n \in \mathbb{N}$  the graph  $U_n$  has at most  $u(n)$  vertices and every  $n$ -vertex graph in  $\mathcal{X}$  is an induced subgraph of  $U_n$ . It was observed in [KNR92] that for a class of graphs the existence of an  $f(n)$ -bit labeling scheme is equivalent to the existence of a universal graph of size  $2^{f(n)}$ .

The binary word, obtained by concatenating labels of the vertices of a graph  $G \in \mathcal{X}_n$  assigned by an adjacency labeling scheme, uniquely determines graph  $G$ . Thus, an  $f(n)$ -bit labeling scheme cannot represent more than  $2^{nf(n)}$  graphs on  $n$  vertices, and therefore, if  $\mathcal{X}$  admits an  $f(n)$ -bit labeling scheme, then  $|\mathcal{X}_n| \leq 2^{nf(n)}$ . This implies a lower bound of  $\log |\mathcal{X}_n|/n$  on the size  $f(n)$  of any adjacency labeling scheme for  $\mathcal{X}$ . A natural and important question is: which classes admit an adjacency labeling scheme of size that matches this information-theoretic lower bound?

We say that a graph class  $\mathcal{X}$  admits an *implicit representation*, if it admits an *order optimal* adjacency labeling scheme, i.e., if  $\mathcal{X}$  has an  $f(n)$ -bit labeling scheme, where  $f(n) = \mathcal{O}(\log |\mathcal{X}_n|/n)$ . Equivalently,  $\mathcal{X}$  admits an implicit representation if  $\mathcal{X}$  has a universal graph of size  $2^{\mathcal{O}(\log |\mathcal{X}_n|/n)}$ . For example, the class  $\mathcal{A}$  of all graphs admits an implicit representation, because  $|\mathcal{A}_n| = 2^{\binom{n}{2}} = 2^{\Theta(n^2)}$  and  $f(n) = \mathcal{O}(\log |\mathcal{A}_n|/n) = \mathcal{O}(n)$ , and one can easily design an  $\mathcal{O}(n)$ -bit labeling scheme for  $\mathcal{A}$ , e.g., by assigning to each vertex of a graph an  $n$ -bit label which is the corresponding row in an adjacency matrix of the graph.

However, not every class admits an implicit representation. The following example is due to Muller [Mul88] (see also [Spi03]). Let  $\mathcal{Y}$  be the class of graphs in which the number of edges does not exceed the number of vertices. It is easy to estimate that  $|\mathcal{Y}_n| = 2^{\mathcal{O}(n \log n)}$ . To show that this class does not admit an implicit representation, consider an arbitrary  $n$ -vertex graph  $G$ . Obviously,  $G$  does not necessarily belong to  $\mathcal{Y}$ , but after adding  $n^2 - n$  isolated vertices to  $G$ , we obtain a

graph  $H$  on  $n^2$  vertices that belongs to  $\mathcal{Y}$ . Now, if an  $\mathcal{O}(\log n)$ -bit labeling scheme for  $\mathcal{Y}$  existed, then the  $\mathcal{O}(\log n^2)$ -bit adjacency labels for  $H$  could be used as  $\mathcal{O}(\log n)$ -bit adjacency labels for  $G$ . Since,  $G$  was chosen arbitrarily, this is in contradiction with the lower bound of  $\log |\mathcal{A}_n|/n = \Omega(n)$  on the size of any labeling scheme for the class  $\mathcal{A}$  of all graphs.

The crucial property used in the above example is that by adding isolated vertices to a graph not in  $\mathcal{Y}$  one can obtain a graph in  $\mathcal{Y}$ . Using more familiar terminology, one would say that class  $\mathcal{Y}$  is not *hereditary*, i.e., it is not closed under vertex removal or, equivalently, under taking induced subgraphs. Many natural graph classes (e.g., forests, planar graphs, disk graphs) are hereditary. It turns out that finding a hereditary graph class that does not admit an implicit representation is a non-trivial question. For factorial graph classes, this question was asked by Kannan, Naor, and Rudich [KNR88], which was later stated by Spinrad [Spi03] in the form of a conjecture, that became known as the *Implicit Graph Conjecture*.

**Conjecture 1.1** (Implicit Graph Conjecture [KNR88, Spi03]). *Any hereditary class of at most factorial speed admits an implicit representation, i.e., an  $\mathcal{O}(\log n)$ -bit labeling scheme.*

This question remained open for over 30 years until a recent breakthrough by Hatami and Hatami [HH22]. They showed that, for any  $\delta > 0$ , there exists a hereditary factorial class that does not admit a labeling scheme of size  $n^{1/2-\delta}$ , which is very far from the information-theoretic lower bound of  $\Omega(\log n)$ . This result leaves wide open the question of characterizing factorial hereditary graph classes that admit an implicit representation (see [HWZ22] for more discussion).

Factorial classes form an important family of hereditary classes, as many classes of theoretical or practical interest are factorial. However, as was noted by Spinrad [Spi03], there is nothing that prevents one from considering implicit representability of other hereditary graph classes. Reformulating the question of Spinrad [Spi03], which he dubbed the *Generalized Implicit Graph Question*, we state the following

**Question 1** ([Spi03]). *Which hereditary graph classes admit implicit representations?*

The answer to this question is known for classes with  $|\mathcal{X}_n| = 2^{\Omega(n^2)}$ , and for *subfactorial* graph classes, i.e., classes  $\mathcal{X}$  with  $|\mathcal{X}_n| = 2^{o(n \log n)}$ . Indeed, for the latter classes, it is known that they have at most exponential speed, i.e.,  $|\mathcal{X}_n| = 2^{\mathcal{O}(n)}$  [Ale97, SZ94], and also admit  $\mathcal{O}(1)$ -bit labeling schemes [Sch99]. For the former classes, the  $\mathcal{O}(n)$ -bit labeling scheme mentioned above for the class  $\mathcal{A}$  of all graphs is an order optimal labeling scheme. In fact, in this regime, *asymptotically optimal* (up to the second-order term) labelling schemes are available. For the class of all graphs, such results (in the language of universal graphs) were available since 1965 [Moo65, AKTZ15, Alo17]. For proper hereditary graph classes  $\mathcal{X}$  with the speed  $2^{\Omega(n^2)}$ , by the Alekseev–Bollobás–Thomason theorem [Ale92, BT95], their speed is  $|\mathcal{X}_n| = 2^{(1-1/k(\mathcal{X}))n^2/2+o(n^2)}$ , where  $k(\mathcal{X})$  is an integer greater than 1. Recently, Bonamy, Esperet, Groenland, and Scott showed [BEGS21] that all such classes have asymptotically optimal adjacency labeling schemes of size  $(1 - 1/k(\mathcal{X}))n/2 + o(n)$ .

For the classes in the intermediate range, i.e., the classes with the speed between  $2^{\Omega(n \log n)}$  and  $2^{o(n^2)}$  the picture is much less understood (see Figure 1). Most known information is concentrated around the lower extreme of the range, i.e., around factorial speed, which was promoted by the Implicit Graph Conjecture. Factorial graph classes from certain families are known to admit implicit representations: proper minor-closed graph classes [GL07], graph classes of bounded degeneracy (equivalently, of bounded arboricity) [KNR88], clique-width [CV03, Spi03] (see also [Ban22]), and twin-width [BGK<sup>+</sup>22] all admit implicit representations. The only lower bound witnessing (non-constructively) factorial classes that do not admit an implicit representation is the above-mentioned result by Hatami and Hatami [HH22]. A notable family of hereditary graph classes where Question 1

remains open is the *small* graph classes, i.e., classes  $\mathcal{X}$  with  $|\mathcal{X}_n| \leq c^n n!$  for some constant  $c$ . These classes encompass only the bottom part of the factorial layer and include proper minor-closed classes [NSTW06], and more generally, classes of bounded twin-width [BGK<sup>+</sup>22]. However, it is still unknown if all such classes admit an implicit representation (see [BDS<sup>+</sup>23] for more details on implicit representation of small classes). Alon showed [Alo23] that every hereditary graph class  $\mathcal{X}$  with  $|\mathcal{X}_n| = 2^{o(n^2)}$  admits an  $n^{1-\delta}$ -bit labeling scheme for some  $\delta > 0$ .

## 1.1 Our contribution

In this paper, we study Question 1 for *monotone* graph classes, i.e., graph classes that are closed under taking subgraphs. Monotone graph classes form a subfamily of hereditary graph classes. The following result shows that any monotone class with non-decreasing speed admits a labeling scheme of size at most  $\mathcal{O}(\log n)$  away from the information-theoretic lower bound.

**Proposition 1.2.** *Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a non-decreasing function. Then, any monotone class of graphs  $\mathcal{X}$  with the speed  $|\mathcal{X}_n| = 2^{\mathcal{O}(nf(n))}$  admits an adjacency labeling scheme of size  $\mathcal{O}(f(n) \log n)$ .*

This upper bound is an easy consequence of an estimation of the number of edges in graphs from monotone classes combined with a standard labeling scheme for  $c$ -degenerate graphs [KNR88], i.e., graphs in which every induced subgraph contains a vertex of degree at most  $c$ .

Our main result shows that this upper bound is attained by some monotone classes. Before stating the result formally we must briefly introduce a family of non-decreasing functions we call “decent”. Roughly speaking, on some domain  $[s, \infty)$ , decent functions are sub-multiplicative, i.e.,  $f(xy) \leq f(x)f(y)$ , and slow-growing, that is  $\log x \leq f(x) \leq x^{1-\delta}$  for some constant  $\delta \in (0, 1)$ .

**Theorem 1.3.** *Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a decent function. Then, there exists a monotone graph class  $\mathcal{X}$  with speed  $|\mathcal{X}_n| = 2^{\mathcal{O}(nf(n))}$  that does not admit a universal graph of size at most  $2^{f(n) \log n}$ . Equivalently,  $\mathcal{X}$  admits no adjacency labeling scheme of size at most  $f(n) \log n$ .*

Theorem 1.3 is the main contribution of the paper, and it gives the existence of monotone classes requiring labels whose size is a  $\log n$ -factor above the information-theoretic lower bound. In particular this shows that Proposition 1.2 is tight.

A special case of Theorem 1.3 (when  $f(x) = \log x$ ) implies that the Implicit Graph Conjecture (Conjecture 1.1) does not hold even for monotone graph classes. Combining this observation with Proposition 1.2 gives the following result.

**Corollary 1.4.** *For any constant  $c > 0$ , there are factorial monotone classes that do not admit a  $(c \log^2 n)$ -bit labeling scheme, while any factorial monotone class admits an  $\mathcal{O}(\log^2 n)$ -bit labeling scheme.*

## 1.2 Outline

The rest of the paper is organized as follows. In Sections 2.1 and 2.2, we cover some common notation, definitions and lemmas. In Section 2.3 we introduce two key concepts used in our proofs. Firstly, we give the notion of *f-good* graphs, which are the building blocks for the monotone classes used to prove our main result. Secondly, we formally define *decent* functions which describe the speeds of these monotone graph classes, before concluding Section 2.3 with some natural examples of decent functions. In Section 3, we prove a result about random graphs which is the main technical ingredient of our lower bound. In Section 4, we establish the lower and upper bounds on labeling schemes for monotone classes. We conclude the paper in Section 5 with a discussion and some open problems.

Speed $ \mathcal{X}_n $	Hereditary Classes		
Dense $2^{\Theta(n^2)}$	✓ [BEGS21]	Monotone	
Super-Factorial $2^{o(n^2)}$	UB: $n^{1-\delta}$ [Alo23]	✗ LB/UB: $\frac{\log  \mathcal{X}_n }{n} \cdot \log n$ [Theorem 1.3 & Proposition 1.2]	
Factorial $2^{\mathcal{O}(n \log n)}$	LB: $n^{1/2-\delta}$ [HH22]	✗ LB/UB: $\log^2 n$ [Corollary 1.4]	Bdd Degeneracy ✓ [KNR92]
Small-Factorial $c^n \cdot n!$	?	?	
	Bdd Twinwidth ✓ [BGK+22]		Minor-Closed [GL07]
Sub-Factorial $2^{o(n \log n)}$	✓ [Sch99]		

Figure 1: The status of implicit representability and best known upper and lower bounds on the size of adjacency labelling schemes for some well-known hereditary classes. The presence of a ✓ denotes the existence of an implicit representation for any class of that type, likewise ✗ indicates that there is a class of the given type that does not admit an implicit representation, and ? shows this question is open. A ✓ is inherited by every sub-region, a ✗ is inherited to the left of the marked region, and a ? only holds in that region. The upper and lower bounds (denoted UB and LB respectively) are stated up to constants which may depend on the class.

## 2 Preliminaries

### 2.1 Standard definitions and notation

For two real numbers  $i, j$ , we let  $[i, j] := \{\lceil i \rceil, \lceil i \rceil + 1, \dots, \lfloor j \rfloor - 1, \lfloor j \rfloor\}$ . Note that if  $j < i$ , then  $[i, j]$  is the empty set. We may use  $[i]$  as a short-hand for  $[1, [i]]$ , and  $\ln^c x$  as a short-hand for  $(\ln x)^c$ . We use  $\mathbb{R}_{\geq 0}$  to denote the set of non-negative real numbers.

We use  $X \sim \mathcal{D}$  to denote that the random variable  $X$  has distribution  $\mathcal{D}$ . We say that a sequence of events  $(A_n)$  holds *with high probability (w.h.p.)* if  $\mathbb{P}[A_n] \rightarrow 1$  as  $n \rightarrow \infty$ .

**Graphs.** We consider finite undirected graphs, without loops or multiple edges. Given a graph  $G$ , we write  $V(G)$  for its vertex set, and  $E(G)$  for its edge set. A graph  $H$  is a *subgraph* of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . Thus,  $H$  can be obtained from  $G$  by vertex and edge deletions. The graph  $H$  is an *induced subgraph* of  $G$  if  $V(H) \subseteq V(G)$ , and  $E(H)$  consists exactly of the edges in  $E(G)$  with both endpoints in  $V(H)$ . In that case,  $H$  can be obtained from  $G$  by vertex deletions only. In the usual way, for a set of vertices  $U \subseteq V(G)$ , we denote by  $G[U]$  the induced subgraph of  $G$  with the set of vertices  $U$ . We denote by  $e(G)$  the number of edges in  $G$ .

When we refer to an  $n$ -vertex graph  $G$  as *labeled*, we mean that the vertex set of  $G$  is  $[n]$  and we distinguish two different labeled graphs even if they are isomorphic. In contrast, if we refer to  $G$  as *unlabeled* graph, its vertices are indistinguishable and two isomorphic graphs correspond to the same unlabeled graph.

**Graph classes.** A graph class is *hereditary* if it is closed under taking induced subgraphs, and it is *monotone* if it closed under taking subgraphs. For a set  $\mathcal{X}$  of graphs we let  $\text{Her}(\mathcal{X})$  denote the hereditary closure of  $\mathcal{X}$ , i.e., the inclusion-wise minimal hereditary class that contains  $\mathcal{X}$ ; and  $\text{Mon}(\mathcal{X})$  denote the monotone closure of  $\mathcal{X}$ , i.e., the minimal monotone class that contains  $\mathcal{X}$ .

### 2.2 Useful lemmas

We use standard notation  $G(n, p)$  to denote the distribution on  $n$ -vertex graphs where each edge is included independently with probability  $p$ , and  $G(n, m)$  to denote the uniform distribution on  $n$ -vertex graphs with  $m$  edges, see (for example) [FK23]. The following lemma allows us to transfer results from one graph model to another.

**Lemma 2.1.** *Let  $\mathcal{P}$  be any graph property (i.e., graph class) and  $0 \leq p \leq 1$  satisfy  $p \binom{n}{2} \rightarrow \infty$  and  $\binom{n}{2} - p \binom{n}{2} \rightarrow \infty$  and  $m = \lceil p \binom{n}{2} \rceil$ . Then, for  $G_n \sim G(n, m)$  and  $G'_n \sim G(n, p)$ , we have*

$$\mathbb{P}[G_n \in \mathcal{P}] \leq 10\sqrt{m} \cdot \mathbb{P}[G'_n \in \mathcal{P}].$$

Lemma 2.1 follows by a very minor adaption of [FK23, Lemma 3.2], the only difference is a ceiling in the number of edges, which makes no difference in the proof.

We will make use of the following version of the Chernoff bound (see [AS08, Theorem A.1.15]), where  $\text{Bin}(N, p)$  denotes the binomial distribution with parameters  $N$  and  $p$ .

**Lemma 2.2** (Chernoff bound). *Let  $\xi \sim \text{Bin}(N, p)$ ,  $\mu = Np$ , and  $a, t > 0$ . Then,*

$$\mathbb{P}(\xi > (1 + a)\mu) \leq \left( \frac{e^a}{(1 + a)^{1+a}} \right)^\mu \leq \exp \left( -(1 + a)\mu \cdot \ln \frac{1 + a}{e} \right).$$

### 2.3 Good graphs and decent functions

**Definition 2.3** (*f-good*). Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a function. An  $n$ -vertex graph  $G$  is *f-good* if the number of edges in any subgraph on  $k$  vertices is bounded from above by

$$\begin{cases} \frac{k \cdot f(k)}{\log k} & \text{if } 2 \leq k \leq \sqrt{n} \\ k \cdot f(k) & \text{if } \sqrt{n} < k \leq n \end{cases}.$$

We observe that *f-goodness* is a monotone property, i.e., if a graph  $G$  is *f-good*, then so is any of its subgraphs. Indeed moving the threshold (between the first and the second, more relaxed, upper bound) from  $\sqrt{n}$  down to a smaller value may only help in satisfying these bounds.

**Definition 2.4** ( $(\delta, C, s)$ -*decent*). For constants  $\delta \in (0, 1)$ ,  $C \geq 1$  and  $s \geq 2$ , we say that a non-decreasing function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is  $(\delta, C, s)$ -*decent* if the following properties hold

**(Moderate-growth):**  $\log x \leq f(x) \leq C \cdot x^{1-\delta}$  holds for every  $x \in [s, \infty)$ ,

**(Sub-multiplicativity):**  $f(xy) \leq C \cdot f(x) \cdot f(y)$  holds for any  $x, y \in [s, \infty)$ .

We say that a function  $f$  is *decent* if there exist some constants  $\delta \in (0, 1)$ ,  $C \geq 1$ , and  $s \geq 2$  such that  $f$  is  $(\delta, C, s)$ -decent. We now give some natural examples of decent functions.

**Lemma 2.5.** *For any constants  $\alpha > 0, \beta \geq 1$  and  $d \in (0, 1)$ , the following functions are decent:*

(i)  $f(x) = \alpha x^d,$

(ii)  $f(x) = \exp(\alpha(\ln x)^d),$

(iii)  $f(x) = \exp((\alpha + 1) \cdot \ln^\beta(\log x)),$

(iv)  $f(x) = (\log x)^\beta$

(v)  $f(x) = \beta \cdot g(x),$  where  $g(x)$  is decent.

(vi)  $f(x) = g(x) \cdot h(x),$  where  $g(x), h(x)$  are decent and  $g(x) \cdot h(x)$  is *moderately-growing*.

*Proof.* For (i), if we set  $s_1 := \left(\frac{2}{d \max\{\alpha, 1\}}\right)^{2/d}$  then we have

$$f(s) = \alpha \cdot \left(\frac{2}{d \max\{\alpha, 1\}}\right)^2 \geq \frac{2}{d} \cdot \frac{2}{d \max\{\alpha, 1\}} \geq \frac{2}{d} \cdot \log \frac{2}{d \max\{\alpha, 1\}} = \log s.$$

Furthermore, there exists some constant  $s_2 := s_2(\alpha, d)$  such that  $\frac{\alpha x^d}{\log x}$  is increasing for all  $x \geq s_2$ , establishing *moderate-growth* on  $[\max\{s_1, s_2\}, \infty)$  with  $C = \alpha$  and  $\delta = 1 - d$ . Observe also that  $f(xy) = \frac{1}{\alpha} \cdot f(x)f(y)$ , and thus  $f(x) = \alpha x^d$  is  $\left(1 - d, \max\{\alpha, \frac{1}{\alpha}\}, \max\{s_1, s_2\}\right)$ -decent.

For (ii), *moderate-growth* holds for  $C = 1$ , any fixed  $\delta \in (0, 1)$ , and sufficiently large  $s \geq 2$ . For  $x, y \geq 0$  let  $g_x(y) = (x+y)^d - x^d - y^d$  and observe that  $g_x(0) = 0$  and  $g'_x(y) = d(x+y)^{d-1} - dy^{d-1} \leq 0$ . Consequently,  $g_x(y) \leq 0$  for all  $x, y \geq 0$ , or equivalently  $(x+y)^d \leq x^d + y^d$ . This implies that  $f$  is *sub-multiplicative* as

$$f(xy) = \exp\left(\alpha(\ln x + \ln y)^d\right) \leq \exp\left(\alpha((\ln x)^d + (\ln y)^d)\right) = f(x) \cdot f(y).$$

For (iii), it will be useful to show that

$$\ln^\beta(x+y) \leq \ln^\beta x + \ln^\beta y, \quad \text{for all } x, y \in [e^\beta, \infty). \quad (1)$$

To prove (1), we first observe that the function  $g(x) = \frac{\ln^\beta x}{x}$  is non-increasing for  $x \in [e^\beta, \infty)$ . This follows since  $g$  is differentiable when  $x \neq 0$  and  $g'(x) = \frac{(\beta - \ln x) \ln^{\beta-1} x}{x^2} < 0$  for all  $x > e^\beta > 1$ . Thus (1) follows from this observation, since for any  $x, y \in [e^\beta, \infty)$  we have

$$\ln^\beta(x+y) = x \cdot \frac{\ln^\beta(x+y)}{x+y} + y \cdot \frac{\ln^\beta(x+y)}{x+y} \leq x \cdot \frac{\ln^\beta x}{x} + y \cdot \frac{\ln^\beta y}{y} = \ln^\beta x + \ln^\beta y.$$

We now see that  $f$  is **sub-multiplicative** for any  $x, y \in [2^{e^\beta}, \infty)$  as by (1) we have

$$f(xy) = \exp\left((\alpha+1) \cdot \ln^\beta(\log x + \log y)\right) \leq \exp\left((\alpha+1) \cdot (\ln^\beta(\log x) + \ln^\beta(\log y))\right) = f(x) \cdot f(y).$$

Since  $\beta \geq 1$  and  $\alpha > 0$ ,  $f$  is also **moderately-growing** for a sufficiently large  $s$ .

For (iv), we see that  $f(x) = \log^\beta x$  is **moderately-growing** for large  $s \geq 2$  as  $\beta \geq 1$ . For **sub-multiplicativity**, for any  $x, y \in [s, \infty)$

$$f(x) = \log^\beta(xy) = (\log x + \log y)^\beta \leq (2 \log(x) \log(y))^\beta \leq 2^\beta f(x) f(y).$$

For (v), if  $g$  is  $(\delta_g, C_g, s_g)$ -decent, then it is easy to check that  $\beta g$  is  $(\delta_g, \beta C_g, s_g)$ -decent.

For (vi), let  $g$  be  $(\delta_g, C_g, s_g)$ -decent and  $h$  be  $(\delta_h, C_h, s_h)$ -decent, and  $f(x) := g(x) \cdot h(x)$ . As  $\log x \leq f(x) \leq C' x^{1-\delta'}$  for some  $\delta' \in (0, 1)$ ,  $C' > 0$ , and  $s' \geq 2$ , by assumption, it remains to show **sub-multiplicativity**. For any  $x, y \in [s, \infty)$ , where  $s = \max\{s', s_g, s_h\}$ , we have

$$f(xy) = g(xy) \cdot h(xy) \leq C_g g(x) g(y) \cdot C_h h(x) h(y) \leq C_g \cdot C_h \cdot f(x) f(y),$$

and thus  $f$  is  $(\delta', \max\{C', C_g \cdot C_h\}, s)$ -decent.  $\square$

### 3 Growth of the number of edges in subgraphs of $G(n, p)$

**Theorem 3.1.** *For any constants  $\delta \in (0, 1)$ ,  $C \geq 1$ , and  $s \geq 2$ , let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be  $(\delta, C, s)$ -decent. Then, for any fixed  $\gamma > 1$ , there exists  $c := c(\delta, C, s, \gamma) > 0$  such that, for large  $n$ ,*

$$\mathbb{P}[G(n, \gamma f(n)/n) \text{ is not } (cf)\text{-good}] \leq n^{-2}.$$

*Proof.* Let  $p := p(n) = \gamma f(n)/n$ , and let  $c_1, c_2$  be sufficiently large constants (depending on  $\gamma$ ) fixed later. Let  $\mathcal{E}_{1,k}$  (respectively  $\mathcal{E}_{2,k}$ ) be the event that there are no subgraphs of size  $k$  with more than  $c_1 k f(k)/\log k$  edges (respectively  $c_2 k f(k)$  edges). Observe that if  $c = \max\{c_1, c_2, \binom{s}{2}\}$ , then

$$\{G(n, p) \text{ is not } (cf)\text{-good}\} \subseteq \left( \bigcup_{k=s}^{\lfloor \sqrt{n} \rfloor} \neg \mathcal{E}_{1,k} \right) \cup \left( \bigcup_{k=\lfloor \sqrt{n} \rfloor + 1}^n \neg \mathcal{E}_{2,k} \right). \quad (2)$$

Let  $k$  denote the number of vertices in a subgraph, and thus  $\xi \sim \text{Bin}\left(\binom{k}{2}, p\right)$  denotes the number

of edges in a given  $k$ -vertex subgraph. The expectation of  $\xi$  is

$$\mu := \binom{k}{2} p = \frac{\gamma}{2} \cdot \frac{k(k-1)f(n)}{n}.$$

On the other hand, the number of ways to choose a  $k$ -vertex subgraph is

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k = \exp\left(k \ln \frac{n}{k} + k\right) \leq \exp(2k \ln n). \quad (3)$$

Our strategy will be to bound the probability of the events on the right hand side of (2) using the union and Chernoff bounds.

We begin by considering events of the form  $\mathcal{E}_{2,k}$  and thus can assume that  $\lfloor \sqrt{n} \rfloor + 1 \leq k \leq n$ . Observe that since  $f$  is **sub-multiplicative**, non-decreasing, and **moderately-growing**, we have

$$\frac{f(k)}{f(n)} = \frac{f(k)}{f\left(\frac{n}{k} \cdot k\right)} \geq \frac{f(k)}{C \cdot f\left(\frac{n}{k}\right) \cdot f(k)} \geq \frac{f(k)}{C \cdot f\left(\frac{sn}{k}\right) \cdot f(k)} \geq \frac{f(k)}{C^2 \cdot \left(\frac{sn}{k}\right)^{1-\delta} \cdot f(k)} \geq \frac{k}{C^2 s \cdot n}. \quad (4)$$

If we now fix

$$c_2 = C^2 s \cdot e^2 \cdot \gamma > 6, \quad (5)$$

then by (4) we have

$$\frac{2c_2 n f(k)}{e\gamma(k-1)f(n)} = \frac{2C^2 s e \cdot n f(k)}{(k-1)f(n)} \geq \frac{2ek}{k-1} > e. \quad (6)$$

So, applying Chernoff bound (Lemma 2.2) with  $1+a = \frac{c_2 k f(k)}{\mu} = \frac{2c_2 n f(k)}{\gamma(k-1)f(n)}$  gives

$$\begin{aligned} \mathbb{P}(\xi > c_2 k f(k)) &\leq \exp\left(- (1+a)\mu \cdot \ln \frac{1+a}{e}\right) \\ &= \exp\left(-c_2 k f(k) \cdot \ln \frac{2c_2 n f(k)}{e\gamma(k-1)f(n)}\right) \\ &\stackrel{(6)}{\leq} \exp(-c_2 k f(k)) \\ &\stackrel{(5)}{\leq} \exp(-6k f(k)). \end{aligned} \quad (7)$$

Thus, by (3), (7), the union bound, and as  $f(k) \geq \log k > \ln k$ , we have

$$\mathbb{P}\left(\bigcup_{k=\lfloor \sqrt{n} \rfloor + 1}^n -\mathcal{E}_{2,k}\right) \leq \sum_{k=\lfloor \sqrt{n} \rfloor + 1}^n \exp(2k \ln n) \cdot \exp(-6k f(k)) \leq \sum_{k=\lfloor \sqrt{n} \rfloor + 1}^n k^{-k} \leq \exp(-\sqrt{n}). \quad (8)$$

We now treat events of the form  $\mathcal{E}_{1,k}$  and thus we can assume that  $s \leq k \leq \lfloor \sqrt{n} \rfloor$ . Observe that for any fixed constant  $d > 0$  and sufficiently large  $n$  we have  $\frac{n^{2/3}}{k(\log k)^d} \geq s$  as  $k \leq \sqrt{n}$ . Thus, by **sub-multiplicativity**, and **moderate-growth** we have

$$\begin{aligned} f\left(\frac{n^{2/3}}{(\log k)^d}\right) &= f\left(\frac{n^{2/3}}{k(\log k)^d} \cdot k\right) \\ &\leq C \cdot f\left(\frac{n^{2/3}}{k(\log k)^d}\right) \cdot f(k) \end{aligned}$$

$$\begin{aligned}
&\leq C^2 \cdot \left( \frac{n^{2/3}}{k(\log k)^d} \right)^{1-\delta} \cdot f(k) \\
&\leq C^2 \cdot \frac{n^{2/3}}{k(\log k)^d} \cdot f(k).
\end{aligned}$$

Similarly, by [sub-multiplicativity](#) and [moderate-growth](#), we have

$$\begin{aligned}
f(n) &= f \left( \frac{n^{2/3}}{(\log k)^d} \cdot n^{1/3}(\log k)^d \right) \\
&\leq C \cdot f \left( \frac{n^{2/3}}{(\log k)^d} \right) \cdot f \left( n^{1/3}(\log k)^d \right) \\
&\leq C^2 \cdot f \left( \frac{n^{2/3}}{(\log k)^d} \right) \cdot n^{(1-\delta)/3}(\log k)^{(1-\delta) \cdot d}.
\end{aligned}$$

If we set  $d = 1/\delta > 0$  then the two bounds above give

$$\frac{f(k)}{f(n)} \geq \frac{f \left( \frac{n^{2/3}}{(\log k)^d} \right) \cdot \frac{k(\log k)^d}{2C^2 n^{2/3}}}{C^2 \cdot f \left( \frac{n^{2/3}}{(\log k)^d} \right) \cdot n^{(1-\delta)/3}(\log k)^{(1-\delta) \cdot d}} = \frac{k(\log k)^{\delta d}}{2C^4 n^{1-\delta/3}} = \frac{k \log k}{2C^4 n} \cdot n^{\delta/3}. \quad (9)$$

Foreseeing the need for the constant 15 later on, we now set

$$c_1 = e \cdot 15 \cdot C^4 \gamma / \delta. \quad (10)$$

We now set  $1 + a := \frac{c_1 k f(k)}{\mu \cdot \log k}$ , which by (9) satisfies

$$1 + a = \frac{c_1 k f(k)}{\mu \cdot \log k} = \frac{2c_1 n f(k)}{\gamma(k-1)f(n) \log k} \geq \frac{c_1 k}{\gamma(k-1)C^4} \cdot n^{\delta/3} > e \cdot n^{\delta/3}. \quad (11)$$

As before, Chernoff bound (Lemma 2.2) with this  $1 + a$  gives

$$\begin{aligned}
\mathbb{P} \left( \xi > \frac{c_1 k f(k)}{\log k} \right) &\leq \exp \left( -\frac{c_1 k f(k)}{\log k} \cdot \ln \frac{1+a}{e} \right) \\
&\stackrel{(11)}{\leq} \exp \left( -\frac{c_1 k f(k)}{\log k} \cdot \frac{\delta}{3} \ln n \right) \\
&\stackrel{(*)}{\leq} \exp \left( -c_1 k \cdot \frac{\delta}{3} \ln n \right) \\
&\stackrel{(10)}{\leq} \exp(-5k \ln n), \quad (12)
\end{aligned}$$

where (\*) follows as  $f(k) \geq \log k$  by [moderate-growth](#). Thus, by (3), (12), and the union bound,

$$\mathbb{P} \left( \bigcup_{k=s}^{\lfloor \sqrt{n} \rfloor} \neg \mathcal{E}_{1,k} \right) \leq \sum_{k=s}^{\lfloor \sqrt{n} \rfloor} \exp(2k \ln n) \cdot \exp(-5k \ln n) \leq \sqrt{n} \cdot n^{-3s} \leq n^{-5}. \quad (13)$$

The result follows by taking  $c = \max\{c_1, c_2, \binom{s}{2}\}$ , (2), and the union bound over (8) and (13).  $\square$

We can now use this result to bound the number of  $cf$ -good graphs from below.

**Lemma 3.2.** *Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be  $(\delta, C, s)$ -decent for some constants  $\delta \in (0, 1)$ ,  $C \geq 1$ , and  $s \geq 2$ . Then, for any fixed  $\gamma > 1$ , there exists some  $c := c(\gamma, \delta, C, s) > 0$  such that for every  $n \in \mathbb{N}$  there are at least  $2^{(\delta\gamma/2 - o(1)) \cdot nf(n) \log n}$  many unlabeled  $(cf)$ -good graphs.*

*Proof.* Let  $m := \lceil \frac{\gamma(n-1)f(n)}{2} \rceil$  and  $G_n \sim G(n, m)$ . Observe that by Theorem 3.1 and Lemma 2.1, there exists some fixed  $c > 0$  such that for sufficiently large  $n$

$$\mathbb{P}[G_n \text{ is } (cf)\text{-good}] \geq 1 - 10\sqrt{\lceil \frac{\gamma(n-1)f(n)}{2} \rceil} \cdot n^{-2} = 1 - o(1). \quad (14)$$

The number of labeled graphs in the support of  $G(n, m)$  is

$$\binom{\binom{n}{2}}{m} = \binom{\binom{n}{2}}{\lceil \frac{\gamma(n-1)f(n)}{2} \rceil} \geq \left( \frac{n}{\gamma f(n)} \right)^{\frac{\gamma(n-1)f(n)}{2}} = 2^{\frac{\gamma}{2} \cdot (n-1)f(n) \cdot (\log n - \log(\gamma f(n)))}.$$

By (14), a  $1 - o(1)$  fraction of these labeled graphs are  $(cf)$ -good. Furthermore, there are at most  $n! \leq n^n$  labelings of a given unlabeled graph. Thus, the number of unlabeled  $n$ -vertex  $(cf)$ -good graphs is bounded from below by

$$\begin{aligned} (1 - o(1)) \cdot \frac{1}{n^n} \cdot 2^{\frac{\gamma}{2} \cdot (n-1)f(n) \cdot (\log n - \log(\gamma f(n)))} &= 2^{\frac{\gamma}{2} \cdot nf(n) \cdot (\log n - \log(f(n)) - \mathcal{O}(1))} \\ &\geq 2^{\frac{\gamma}{2} \cdot nf(n) \cdot (\log n - (1-\delta) \log(n) - \mathcal{O}(1))} \\ &= 2^{(\delta\gamma/2 - o(1)) \cdot nf(n) \log n}, \end{aligned}$$

as claimed, since  $\log n \leq f(n) \leq Cn^{1-\delta}$  by [moderate-growth](#). □

## 4 Tight bounds on labeling schemes for monotone factorial classes

We begin in Section 4.1 with a lemma which is useful for bounding the speed when constructing monotone classes with no implicit representation. This is then used to prove our lower bound in Section 4.2. Finally, in Section 4.3 we give a matching upper bound on labeling schemes for monotone classes, this follows from [\[KNR92\]](#) and included mainly for completeness.

### 4.1 Construction of monotone tiny classes

We begin with a lemma showing, for a decent function  $f$ , we can create monotone classes from the union of many  $f$ -good graphs and still maintain control over the speed. The proof follows the broad idea of [\[HH22, Claim 3.1\]](#).

**Lemma 4.1.** *Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be  $(\delta, C, s)$ -decent for some constants  $\delta \in (0, 1)$ ,  $C \geq 1$ , and  $s \geq 2$ . Let  $c > 0$  be a constant, and, for every  $n \in \mathbb{N}$ , let  $M_n$  be any set of  $(cf)$ -good unlabeled  $n$ -vertex graphs satisfying  $|M_n| \leq \lceil 2^{\sqrt{nf(n)}} \rceil$ . Then the speed of  $\mathcal{X} := \text{Mon}(\cup_{n \in \mathbb{N}} M_n)$  is  $2^{\mathcal{O}(nf(n))}$ .*

*Proof.* Let  $\mathcal{Y} := \text{Her}(\cup_{n \in \mathbb{N}} M_n)$ . Note that  $\mathcal{X} = \text{Mon}(\mathcal{Y})$ . We first estimate the speed of  $\mathcal{Y}$ . For an  $n$ -vertex graph  $G \in \mathcal{Y}$ , let  $N$  be the smallest integer such that  $G$  is an induced subgraph of a graph  $H \in M_N$ . We split the proof over two cases: (i):  $N \geq n^2$ , and (ii):  $N < n^2$ .

Case (i): Since  $H$  is a  $(cf)$ -good  $N$ -vertex graph and  $G$  is its  $n$ -vertex induced subgraph, where  $n \leq \sqrt{N}$ , it follows from Definition 2.3 that  $G$  must have at most  $g(n) := cnf(n)/\log n$  many edges. The number of such graphs is at most

$$\binom{\binom{n}{2}}{g(n)} \leq \left(\frac{n^2 e}{g(n)}\right)^{g(n)} = 2^{g(n) \cdot \log \frac{n^2 e}{g(n)}} = 2^{c \frac{nf(n)}{\log n} \cdot \log \frac{n^2 e}{g(n)}} = 2^{\mathcal{O}(nf(n))},$$

and so  $\mathcal{Y}$  contains  $2^{\mathcal{O}(nf(n))}$  many  $n$ -vertex labeled graphs each of which is an induced subgraph of a graph in  $M_N$  for some  $N$  with  $n \leq \sqrt{N}$ .

Case (ii): For this case, we simply use the fact that any  $H \in M_N$  has at most  $N^n$  many  $n$ -vertex induced subgraphs.

Thus, the number of  $n$ -vertex labeled graphs in  $\mathcal{Y}$  each of which is an induced subgraph of a graph in  $M_N$  for some  $N$  with  $N < n^2$  is bounded from above by

$$\begin{aligned} n! \cdot \sum_{N=n}^{n^2} N^n \cdot |M_N| &\leq n! \cdot \sum_{N=n}^{n^2} N^n \cdot \left\lceil 2^{\sqrt{Nf(N)}} \right\rceil \\ &\leq n! \cdot n^2 \cdot (n^2)^n \cdot \left\lceil 2^{\sqrt{n^2 f(n^2)}} \right\rceil \\ &\leq 2^{\mathcal{O}(n \log n)} \cdot \left\lceil 2^{\sqrt{C}nf(n)} \right\rceil \\ &= 2^{\mathcal{O}(nf(n))}, \end{aligned}$$

where in the last inequality we used [sub-multiplicativity](#) of  $f$ , and in the final equality we used the fact that  $f(x) \geq \log x$ .

Thus,  $|\mathcal{Y}_n| = 2^{\mathcal{O}(nf(n))}$ . Now, since every  $n$ -vertex labeled graph in  $\mathcal{X}$  is a subgraph of an  $n$ -vertex labeled graph in  $\mathcal{Y}$ , and, due to  $(cf)$ -goodness, every graph in  $\mathcal{Y}_n$  has at most  $2^{cnf(n)}$   $n$ -vertex subgraphs, we conclude that  $|\mathcal{X}_n| \leq |\mathcal{Y}_n| \cdot 2^{cnf(n)} = 2^{\mathcal{O}(nf(n))}$ .  $\square$

## 4.2 Lower bound

We can now show the main result of the paper, which we recall for convenience.

**Theorem 1.3.** *Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a decent function. Then, there exists a monotone graph class  $\mathcal{X}$  with speed  $|\mathcal{X}_n| = 2^{\mathcal{O}(nf(n))}$  that does not admit a universal graph of size at most  $2^{f(n) \log n}$ . Equivalently,  $\mathcal{X}$  admits no adjacency labeling scheme of size at most  $f(n) \log n$ .*

*Proof.* By assumption  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is  $(\delta, C, s)$ -decent for some constants  $\delta \in (0, 1)$ ,  $C \geq 1$ , and  $s \geq 2$ . We will construct a monotone class (via the probabilistic method) with the speed  $2^{\mathcal{O}(nf(n))}$  that does not admit a universal graph of size  $u_n := 2^{f(n) \log n}$ . Fix  $\gamma := 4/\delta > 1$  and let  $c := c(\gamma, \delta, C, s) > 0$  be the satisfying constant from Lemma 3.2 corresponding to this choice of  $\gamma$ . Let  $k_n := \left\lceil 2^{\sqrt{nf(n)}} \right\rceil$ .

The number of distinct  $u_n$ -vertex graphs is at most  $2^{u_n^2}$  and the number of  $n$ -vertex induced subgraphs of a fixed  $u_n$ -vertex graph is at most  $\binom{u_n}{n}$ . Hence the number of collections of  $k_n$  graphs on  $n$  vertices that are induced subgraphs of a  $u_n$ -vertex (universal) graph is at most

$$2^{u_n^2} \cdot \binom{\binom{u_n}{n}}{k_n} \leq 2^{u_n^2} \cdot u_n^{k_n \cdot n}. \tag{15}$$

On the other hand, from Lemma 3.2, the number of different collections of  $n$ -vertex  $(cf)$ -good graphs of cardinality  $k_n$  is at least

$$\binom{2^{(\gamma\delta/2-o(1))\cdot nf(n)\log n}}{k_n} \geq \left( \frac{2^{(\gamma\delta/2-o(1))\cdot nf(n)\log n}}{k_n} \right)^{k_n} = 2^{k_n \cdot (\gamma\delta/2-o(1))\cdot nf(n)\log n}, \quad (16)$$

as  $\log k_n = \mathcal{O}(\sqrt{nf(n)}) = o(nf(n)\log n)$ . By taking logarithms, we can see that for sufficiently large  $n$  the upper bound (15) is smaller than the lower bound (16). In particular, taking the logarithm of (15) gives

$$\begin{aligned} \log \left( 2^{u_n^2} \cdot u_n^{k_n \cdot n} \right) &= u_n^2 + k_n \cdot n \log u_n \\ &= 2^{2f(n)\log n} + k_n \cdot nf(n) \log n \\ &= (1 + o(1)) \cdot k_n \cdot nf(n) \log n, \end{aligned}$$

as  $k_n := \left\lceil 2\sqrt{nf(n)} \right\rceil = \omega(2^{2f(n)\log n})$ . However, since  $\gamma = 4/\delta$ , the logarithm of (16) is

$$\begin{aligned} \log \left( 2^{k_n \cdot (\gamma\delta/2-o(1))\cdot nf(n)\log n} \right) &= k_n \cdot (\gamma\delta/2 - o(1)) \cdot nf(n) \log n \\ &= (2 - o(1)) \cdot k_n \cdot nf(n) \log n. \end{aligned}$$

Thus, for any sufficiently large  $n$ , there exists a collection  $M_n$  of  $k_n$   $(cf)$ -good  $n$ -vertex graphs that are not representable by any universal graph of size at most  $u_n = 2^{2f(n)\log n}$ . Consequently, by Lemma 4.1, the speed of  $\mathcal{X} := \text{Mon}(\cup_n M_n)$  is  $|\mathcal{X}_n| = 2^{\mathcal{O}(nf(n))}$  and  $\mathcal{X}$  does not admit a universal graph of size at most  $2^{f(n)\log n}$ .  $\square$

### 4.3 Upper bound

**Proposition 4.2.** *Let  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a non-decreasing function. Then, any monotone class of graphs  $\mathcal{X}$  with the speed  $|\mathcal{X}_n| = 2^{\mathcal{O}(nf(n))}$  admits an adjacency labeling scheme of size  $\mathcal{O}(f(n)\log n)$ .*

*Proof.* Let  $\mathcal{X}$  be a monotone class with at most  $2^{Cnf(n)}$  labeled  $n$ -vertex graphs for every  $n$ . If an  $n$ -vertex graph  $G \in \mathcal{X}$  has  $m$  edges, then  $\mathcal{X}$  contains at least  $2^m$  labeled  $n$ -vertex graphs, as every subgraph of  $G$  also belongs to  $\mathcal{X}$  due to monotonicity.

This implies that every  $n$ -vertex graph  $G$  in  $\mathcal{X}$  contains at most  $Cnf(n)$  edges, and hence, has a vertex of degree at most  $2Cf(n)$ . Due to monotonicity of  $f$ , the same is true for every subgraph of  $G$ . Indeed, if  $H$  is a  $k$ -vertex subgraph of  $G$ , then, since  $H$  belongs to  $\mathcal{X}$ , the number of edges in  $H$  is at most  $Ckf(k) \leq Ckf(n)$ , and therefore  $H$  has a vertex of degree at most  $2Cf(n)$ .

Thus, every  $n$ -vertex graph in  $\mathcal{X}$  is  $2Cf(n)$ -degenerate, and the standard adjacency labeling scheme for  $c$ -degenerate graphs [KNR92] implies that  $\mathcal{X}$  admits a  $2Cf(n)\log n$ -bit labeling scheme.  $\square$

## 5 Conclusions

Our main result shows that for any ‘decent’ function  $f$  we can find a monotone class of graphs  $\mathcal{X}$  with speed  $|\mathcal{X}_n| = 2^{\mathcal{O}(nf(n))}$  for which any labeling scheme requires a multiplicative factor  $\log n$  more bits than the information-theoretic lower bound. Furthermore, we gave an upper bound on

the size of labeling schemes for any monotone class with non-decreasing speed which matches our lower bound, up-to a constant.

A natural question arising from our work is to characterize monotone classes that admit an implicit representation, i.e., an adjacency labeling scheme of order optimal size. Motivated by the Implicit Graph Conjecture, of particular interest is the case of factorial classes.

**Question 2.** *Which monotone factorial graph classes admit an  $\mathcal{O}(\log n)$ -bit labeling scheme?*

An analogous question is completely understood for constant-size *adjacency sketches* (a probabilistic version of adjacency labeling schemes) that were studied in [FK09, Har20, HWZ22]. The importance of constant-size adjacency sketches is that they can be derandomized to  $\mathcal{O}(\log n)$ -bit adjacency labels [Har20, HWZ22]. Thus, if a class admits constant-size adjacency sketches, then it admits an  $\mathcal{O}(\log n)$ -bit labeling scheme. Though, the converse is not always true. Esperet, Harms, and Kupavskii showed [EHK22] that a monotone class admits constant-size adjacency sketches if and only if it has bounded degeneracy. This result may suggest that bounded degeneracy also characterises monotone classes that admit  $\mathcal{O}(\log n)$ -bit labeling schemes. This, however, is not the case, as the class of subgraphs of hypercubes is monotone, has unbounded degeneracy, and admits an  $\mathcal{O}(\log n)$ -bit labeling scheme [EHZ23].

Recall that Question 1 (first raised in [Spi03]), asks which hereditary graph classes admit implicit representations. A prominent instance of Question 1 is whether every small class (i.e., class  $\mathcal{X}$  with  $|\mathcal{X}_n| \leq c^n \cdot n!$  for some constant  $c > 0$ ), or even every monotone small class, admits an implicit representation. It is known that for any  $\kappa > 0$  there is a small monotone class which does not admit a  $(\kappa \log n)$ -bit labeling scheme [BDS<sup>+</sup>23]. In particular, some small classes admit no asymptotically optimal labeling scheme. However, this result does not rule out the existence of an order optimal labeling scheme for each small class of graphs.

Finally, a more technical (yet natural) question is whether the conditions ([moderate-growth](#) and [sub-multiplicativity](#)) of ‘decent’ can be relaxed. Due to the discussion in the introduction, the [moderate-growth](#) condition is essentially necessary. It is not so clear to what extent the [sub-multiplicativity](#) condition is necessary, however, if one is to follow our method, some notion of global “smoothness” is required to prove Theorem 3.1. To see this, consider a function  $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$  such that  $f(n) = \log n$ , if  $n$  is odd, and  $f(n) = \sqrt{n}$ , if  $n$  is even. Then, for any  $c > 0$ , and large enough even  $n$ ,  $G(n, f(n)/n)$  will not be  $cf$ -good as the restriction on the subgraphs with odd number of vertices is far too stringent.

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