Symmetric-Difference (Degeneracy) and Signed Tree Models

Édouard Bonnet, Julien Duron, John Sylvester, Viktor Zamaraev

Symmetric-Difference (Degeneracy) and Signed Tree Models

Édouard Bonnet* Julien Duron† John Sylvester‡ Viktor Zamaraev§

Abstract

We introduce a dense counterpart of graph degeneracy, which extends the recently-proposed invariant symmetric difference. We say that a graph has *sd-degeneracy* (for symmetric-difference degeneracy) at most d if it admits an elimination order of its vertices where a vertex u can be removed whenever it has a d-twin, i.e., another vertex v such that at most d vertices outside $\{u, v\}$ are neighbors of exactly one of u, v. The family of graph classes of bounded sd-degeneracy is a superset of that of graph classes of bounded degeneracy or of bounded flip-width, and more generally, of bounded symmetric difference. Unlike most graph parameters, sd-degeneracy is not hereditary: it may be strictly smaller on a graph than on some of its induced subgraphs. In particular, every n-vertex graph is an induced subgraph of some $O(n^2)$-vertex graph of sd-degeneracy 1. In spite of this and the breadth of classes of bounded sd-degeneracy, we devise $\tilde{O}(\sqrt{n})$-bit adjacency labeling schemes for them, which are optimal up to the hidden polylogarithmic factor. This is attained on some even more general classes, consisting of graphs G whose vertices bijectively map to the leaves of a tree T, where transversal edges and anti-edges added to T define the edge set of G. We call such graph representations *signed tree models* as they extend the so-called tree models (or twin-decompositions) developed in the context of twin-width, by adding transversal anti-edges.

While computing the degeneracy of a graph takes linear time, we show that determining its symmetric difference is para-co-NP-complete. This may seem surprising as symmetric difference can serve as a short-sighted first approximation of twin-width, whose computation is para-NP-complete. Indeed, we show that deciding if the symmetric difference of an input graph is at most 8 is co-NP-complete. We also show that deciding if the sd-degeneracy is at most 1 is NP-complete, contrasting with the symmetric difference.

1 Introduction

There are two theories of sparse graphs: the so-called Sparsity Theory pioneered by Nešetřil and Ossona de Mendez [NO12], and the theory behind the equivalent notions of bounded degeneracy, maximum average degree, subgraph density, and arboricity. One of the many merits of the former theory is to capture efficient first-order model checking within subgraph-closed classes, with the so-called *nowhere dense* classes [GKS17]. *Monadic stability* constitutes a dense analogue of nowhere denseness with similar algorithmic properties [DEM+23]. The second theory has, as we will see, simple but useful connections with the chromatic number and adjacency labeling schemes. One of our two main motivations is to introduce and explore dense analogues of it.

The degeneracy of a graph G is the minimum integer d such that every induced subgraph of G has a vertex of degree at most d. As removing vertices may only decrease the degree of the

*Univ. Lyon, ENS de Lyon, UCBL, CNRS, LIP, France, edouard.bonnet@ens-lyon.fr
†Univ. Lyon, ENS de Lyon, UCBL, CNRS, LIP, France, julien.duron@ens-lyon.fr
‡Department of Computer Science, University of Liverpool, UK, john.sylvester@liverpool.ac.uk
§Department of Computer Science, University of Liverpool, UK, viktor.zamaraev@liverpool.ac.uk

remaining ones, checking that the degeneracy is at most d can be done greedily. This prompts the following equivalent definition, equal to the *coloring number*[1] minus one [EH66, Die17]. A graph has degeneracy at most d if there is a total order, called *degeneracy ordering*, on its vertices such that every vertex v has at most d neighbors following v in the order. The degeneracy is then the least integer d such that an ordering witnessing degeneracy at most d exists. Given such an ordering, a graph can be properly $(d+1)$-colored by a greedy strategy: use the smallest available color looping through vertices in the reverse order. Another advantage of the definition via degeneracy ordering is that it yields a polynomial-time algorithm to compute the degeneracy. While the graph is nonempty, find a vertex of minimum degree, append it to the order, and remove it from the graph. Degeneracy is frequently used to bound the chromatic number from above. For instance, until recently [NPS23] the Kostochka–Thomason degeneracy bound of graphs without K_t minor [Kos84, Tho84] was the best way we knew of coloring these graphs.

Another application of bounding the degeneracy is to obtain implicit representations. Indeed, graphs of bounded degeneracy admit $f(n)$-bit *adjacency labeling schemes* with $f(n) = O(\log n)$.[2] In other words, given a class of graphs of degeneracy at most d, there exists an algorithm, called *decoder*, such that the vertices of any n-vertex graph G from the class can be assigned *labels* (which are binary strings) of length $f(n)$ in such a way that the decoder can infer the adjacency of any two vertices u, v in G from their mere labels. An $O(\log n)$-bit labeling scheme is easy to design for any class $\mathcal{C}$ of bounded degeneracy. From an ordering of $G \in \mathcal{C}$ witnessing degeneracy d, the label of each vertex stores its own index in the ordering and the indices of its at most d neighbors that follow it in the order. Then the decoder just checks whether the index of one of u, v is among the indices of the neighbors of the other vertex. Note that each label has size at most $(d+1)\lceil \log n \rceil$. For example, this was recently used to show that every subgraph-closed class with single-exponential speed of growth admits such a labeling scheme [BDS+23].

Adjacency labeling schemes of size $O(\log n)$ are at the heart of the recently-refuted Implicit Graph Conjecture (IGC) [KNR88, Mul88]. The IGC speculated that the information-theoretic necessary condition for a hereditary graph class to have an $O(\log n)$-bit labeling scheme is also sufficient. This necessary condition comes from the observation that a string of length $O(n \log n)$ obtained by concatenating all vertex labels is an encoding of the graph. Therefore a class of graphs that admits an adjacency labeling scheme of size $O(\log n)$ contains at most $2^{O(n \log n)}$ (un)labeled n-vertex graphs. Graph classes with such a bound on the number of (un)labeled n-vertex graphs are called *factorial*. In this terminology, the IGC can be stated as follows: any hereditary factorial graph class admits an $O(\log n)$-bit adjacency labeling scheme.

The IGC has been refuted by a wide margin; in a breakthrough work, Hatami and Hatami [HH22] showed that there are factorial hereditary graph classes for which any adjacency labeling scheme requires labels of length $\Omega(\sqrt{n})$. However, the refutation is based on a counting argument and does not pinpoint an explicit counterexample. There are a number of explicit factorial graph classes that could refute the IGC, but the conjecture is still open for these classes. Let us call EIGC (for Explicit Implicit Graph Conjecture) this very challenge. For instance, whether the IGC holds within intersection graphs of segments, unit disks, or disks in the plane, and more generally semi-algebraic graph classes, is unsettled. Despite the workable definitions of these classes, the geometric representations alone cannot lead to $O(\log n)$-bit labeling schemes [MM13]. If such labeling schemes exist, they are likely to utilize some non-trivial structural properties of these graphs.

The graph parameter *symmetric difference* was introduced to design a candidate to explicitly refute the IGC [ACLZ15]. A graph G has symmetric difference at most d if in every induced subgraph

[1] not to be confused with the *chromatic* number

[2] Throughout the paper, log denotes the logarithm function in base 2.

of G there is a pair of vertices u, v such that there are at most d vertices different from u and v that are adjacent to exactly one of u, v. In other words, u and v are *d-twins*, i.e., they become twins after removing at most d vertices from the graph. One can construe symmetric difference as a dense analogue of the first definition of degeneracy given above. Symmetric difference is a hereditary graph parameter: it can only decrease when taking induced subgraphs. Like classes of bounded degeneracy, classes of bounded symmetric difference are factorial [ACLZ15]. Symmetric difference generalizes degeneracy in the sense that any class of graphs of bounded degeneracy has bounded symmetric difference. Indeed, if a graph has degeneracy at most d, then it has symmetric difference as most $2d$: for any graph with an ordering witnessing degeneracy d, the first two vertices in the order are $2d$-twins. Notice that, on the other hand, complete graphs have unbounded degeneracy, but their symmetric difference is 0. The existence of an $O(\log n)$-bit adjacency labeling scheme for graphs of bounded symmetric difference remains open.

Our contribution. We introduce another dense analogue of degeneracy based on the second given definition. The *sd-degeneracy* (for *symmetric-difference degeneracy*) of a graph G is the least integer d for which there is an ordering of the vertices of G such that every vertex v but the last one admits a d-twin in the subgraph of G induced by v and all the vertices following it in the order. It follows from the definitions that graphs with sd-degeneracy at most d form a superset of graphs with symmetric difference at most d. Contrary to what happens in the sparse setting with degeneracy, this superset is strict. In fact, there are classes with sd-degeneracy 1 and unbounded symmetric difference.

Proposition 1.1. *For any n-vertex graph G, there exists a graph of sd-degeneracy 1 with less than n^2 vertices containing G as an induced subgraph.*

Proof. Take any n-vertex graph G, and fix an arbitrary ordering $v_1, v_2, \dots, v_n$ of its vertices. There is a graph G' with at most $(n-1)(n-3)$ vertices that both contains G as an induced subgraph, and has sd-degeneracy at most 1. The graph G' can be built by adding to G, for every $i \in [n-1]$, up to $n-3$ vertices $v_{i,1}, \dots, v_{i,h_i}$ gradually "interpolating" between the neighborhood of v_i and that of v_{i+1} (in G). For instance, $v_{i,1}, \dots, v_{i,h'_i}$ remove one-by-one the neighbors of v_i that are not neighbors of v_{i+1}, and $v_{i,h'_i+1}, \dots, v_{i,h_i}$ add one-by-one the neighbors of v_{i+1} that are not neighbors of v_i. (We put no edge between a pair of added vertices $v_{i,p}, v_{i',p'}$.) Then an ordering witnessing sd-degeneracy at most 1 is $v_1, v_{1,1}, v_{1,2}, \dots, v_{1,h_1}, v_2, v_{2,1}, v_{2,2}, \dots, v_{2,h_2}, \dots, v_{n-1}, v_{n-1,1}, v_{n-1,2}, \dots, v_{n-1,h_{n-1}}, v_n$, for which the 1-twin of a vertex is its successor. □

By an aforementioned counting argument, the class of all graphs requires labeling schemes of size $\Theta(n)$. Therefore, by Proposition 1.1, the (non-hereditary) class of graphs with sd-degeneracy at most 1 requires adjacency labels of size $\Omega(\sqrt{n})$. Surprisingly, we match this lower bound with a labeling scheme, tight up to a polylogarithmic factor, for any class of bounded sd-degeneracy.

Theorem 1.2. *The class of all graphs with sd-degeneracy at most d admits an $O(\sqrt{dn}\log^3 n)$-bit adjacency labeling scheme.*

The tool behind the proof of Theorem 1.2 is the second motivation of the paper. We wish to unify and extend twin-decompositions of low width (also called tree models) [BGK+21, BNO+21] developed in the context of twin-width, and spanning paths (or Welzl orders) of low crossing number (or low alternation number) [Wel88], which are useful orders in answering geometric range queries; these orders also appear in the context of implicit graph representations [Alo23, AAA+23], and were utilized as part of efficient first-order model checking algorithms [DEM+23]. We thus introduce *signed tree models*. A signed tree model of a graph G is a tree whose leaves are in one-to-one

correspondence with the vertices of G, together with extra transversal edges and anti-edges, which fully determine (see the exact rules in Section 3) the edges of G. The novelty compared to the existing tree models is the presence of transversal anti-edges. We show that graphs with signed tree models of degeneracy at most d admit a labeling scheme as in Theorem 1.2. The latter theorem is then obtained by building such a signed tree model for any graph of sd-degeneracy d.

When given the vertex ordering witnessing sd-degeneracy d, the labeling scheme can be effectively computed. However, we show that computing the sd-degeneracy of a graph (hence, in particular a witnessing order) is NP-complete, even when the sd-degeneracy is guaranteed to be below a fixed constant. In the language of parameterized complexity, sd-degeneracy is para-NP-complete.

Theorem 1.3. *Deciding if a graph has sd-degeneracy at most 1 is NP-complete.*

This is in stark contrast with the existing linear-time cograph recognition [CPS85], which is equivalent to determining if the sd-degeneracy is (at most) 0. We show that, surprisingly, the other dense analogue of degeneracy, symmetric difference, is co-NP-complete. Again, the associate parameterized problem is para-co-NP-complete.

Theorem 1.4. *Deciding if a graph has symmetric difference at most 8 is co-NP-complete.*

This is curious because sd-degeneracy and symmetric difference similarly extend to the dense world two equivalent definitions of degeneracy. Nevertheless, one can explain the apparent tension between Theorems 1.3 and 1.4: a vertex ordering witnesses an upper bound in the sd-degeneracy, whereas an induced subgraph witnesses a lower bound in the symmetric difference. We leave as an open question whether classes of bounded symmetric difference have labeling schemes of (poly)logarithmic size. This is excluded for bounded sd-degeneracy, for which we now know essentially optimal labeling schemes. While not an absolute barrier, the likely absence of polynomial certificates tightly upper bounding the symmetric difference complicates matters in settling this open question.

Beside compact labeling schemes, we mentioned *graph coloring* as another motivation for classes of bounded degeneracy. A natural dense analogue for bounded chromatic number is the notion of χ-boundedness, that is, the property of having chromatic number bounded by a function of the clique number. However, even classes of bounded symmetric difference may not be χ-bounded, as witnessed by shift graphs on pairs. These graphs have vertex sets $\{(i,j) \ : \ i < j \in [n]\}$ (for increasing values of n) and (i,j) is adjacent to (k,ℓ) whenever $j = k$ or $i = \ell$. This defines a class of triangle-free graphs with unbounded chromatic number [EH68], hence not χ-bounded. We claim that shift graphs have symmetric difference at most 2. Let H be any induced subgraph of a shift graph. Consider the smallest integer i such that there two vertices (a,i) and (b,i) in H (or symmetrically the largest i such that $(i,a),(i,b) \in V(H)$). One can see that (a,i) and (b,i) (or symmetrically $(i,a),(i,b)$) can have at most one private neighbor each. If no such integer i exists, H is a disjoint union of paths, which has symmetric difference at most 1.

Organization. Section 2 gives definitions and notation. In Section 3, we introduce signed tree models, and prove that graphs of bounded sd-degeneracy admit signed tree models of bounded width. In Section 4, we show how to balance these signed tree models, and complete the proof of Theorem 1.2. In Section 5, we prove Theorem 1.4, and in Section 6, Theorem 1.3.

Acknowledgments. That classes of bounded symmetric difference need not be χ-bounded was observed by Romain Bourneuf, Colin Geniet, Stéphan Thomassé, and the first author. We thank them for letting us include this remark in the introduction.

2 Preliminaries

We denote by $[i, j]$ the set of integers that are at least i and at most j, and $[i]$ is a shorthand for $[1, i]$. We follow standard asymptotic notation throughout, and additionally by $f(n) = \tilde{O}(g(n))$ we mean that there exist constants $c, n_0 > 0$ such that for any $n \geqslant n_0$ we have $f(n) \leqslant g(n) \log^c n$.

We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively. Given a vertex u of a graph G, we denote by $N_G(u)$ the set of neighbors of u in G (*open neighborhood*) and by $N_G[u]$ the set $N_G(u) \cup \{u\}$ (*closed neighborhood*). For a set $S \subseteq V(G)$, we denote by $N_G(S)$ the set of vertices in $V(G) \setminus S$ that have a neighbor in S in graph G. When H, G are two graphs, we may denote by $H \subseteq_i G$ (resp. $H \subseteq G$) the fact that H is an induced subgraph (resp. subgraph) of G, i.e., can be obtained by removing vertices of G (resp. by removing vertices and edges of G). We denote by $G[S]$ the subgraph of G induced by S, formed by removing every vertex of $V(G) \setminus S$. We use $G - S$ as a shorthand for $G[V(G) \setminus S]$, and $G - v$, for $G - \{v\}$.

Given two sets A and B, we denote by $A \triangle B$ their symmetric difference, that is, $(A \setminus B) \cup (B \setminus A)$. Given a graph G, and two distinct vertices $u, v \in V(G)$, we set

$$\mathrm{sd}_G(u, v) := |(N_G(u) \setminus \{v\}) \triangle (N_G(v) \setminus \{u\})|.$$

The *symmetric difference* of G, $\mathrm{sd}(G)$, is defined as $\max_{H \subseteq_i G} \min_{u \neq v \in V(H)} \mathrm{sd}_H(u, v)$. Symmetric difference was implicitly introduced in [ACLZ15] and later explicitly defined in [AAL21]. We call *sd-degeneracy* of G, denoted by $\mathrm{sdd}(G)$, the smallest non-negative integer d such that $|V(G)| = 1$ or there is a pair $u \neq v \in V(G)$ satisfying $\mathrm{sd}_G(u, v) \leqslant d$ and $G - v$ has sd-degeneracy at most d. We say that an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G witnesses that the *sd-degeneracy* of G is at most d if for every $i \in [n-1]$, there is a $j > i$ such that $\mathrm{sd}_{G - \{v_k \,:\, k \in [i-1]\}}(v_i, v_j) \leqslant d$. It thus holds that for any graph G, $\mathrm{sdd}(G) \leqslant \mathrm{sd}(G)$, since for every $i \in [n]$, $G - \{v_k \,:\, k \in [i-1]\}$ is an induced subgraph of G. But, as shown by Proposition 1.1, there are some graphs with sd-degeneracy 1 and unbounded symmetric difference.

Two vertices u, v are said to be *d-twins* in a graph G if they are distinct and $|(N_G(u) \setminus N_G[v]) \cup (N_G(v) \setminus N_G[u])| \leqslant d$. The $a \times b$ *rook graph* has vertex set $\{(i, j) \,:\, i \in [a], j \in [b]\}$ and edge set $\{(i, j)(k, \ell) \,:\, (i, j) \neq (k, \ell),\ i = k \text{ or } j = \ell\}$. Equivalently it is the line graph of the bipartite complete graph $K_{a,b}$. For every $a, b \geqslant 3$, the symmetric difference of the $a \times b$ *rook graph* is $2(\min(a, b) - 1)$.

We will extensively use *tree orders*, i.e., partial orders defined by ancestor–descendant relationships in a rooted tree. We denote by $\prec_T$ the corresponding relation in rooted tree T. That is, $u \prec_T u'$ means that u is a *strict ancestor* of u' in T, and $u \preceq_T u'$ means that u is an *ancestor* of u', i.e., $u = u'$ or $u \prec_T u'$. We extend this partial order to elements of $\binom{V(T)}{2}$. An unordered pair uv is an *ancestor* of $u'v'$ in T, denoted by $uv \preceq_T u'v'$, whenever either $u \preceq_T u'$ and $v \preceq_T v'$, or $v \preceq_T u'$ and $u \preceq_T v'$ holds. We write $uv \prec_T u'v'$ when $uv \preceq_T u'v'$ and $\{u, v\} \neq \{u', v'\}$. A rooted binary tree is *full* if all its *internal nodes*, i.e., non-leaf nodes, have exactly two children. A rooted binary tree is *complete* if all its levels are completely filled, except possibly the last one, wherein leaves are left-aligned. The *depth* of a rooted tree is the maximum number of nodes in a root-to-leaf path.

3 Signed tree models

A wide range of structural graph invariants, called width parameters, can be expressed via so-called *tree layouts* (or at least parameters functionally equivalent to them can). A *tree layout* of an n-vertex graph G is a full binary tree T such that the leaves of T, that we may denote by $L(T)$, are in one-to-one correspondence with $V(G)$. Width parameters are typically defined through evaluating

a particular function on bipartitions of $V(G)$ made by the two connected components of T when removing one edge of T. The width is then the minimum over tree layouts of the maximum over all such evaluations. We depart from this viewpoint, and instead augment T with a sparse structure encoding the graph G.

An unordered pair of vertices in T that is not in an ancestor–descendant relationship is called a *transversal pair of* T. Two transversal pairs $uv, u'v'$ of T *cross* if either $u \prec_T u'$ and $v' \prec_T v$, or $u' \prec_T u$ and $v \prec_T v'$, or $u \prec_T v'$ and $u' \prec_T v$, or $v' \prec_T u$ and $v \prec_T u'$. A *signed tree model* $\mathcal{T}$ is a triple $(T, A(T), B(T))$, where T is a full binary tree, $A(T)$ (for Android green, or Anti) is a set of transversal pairs of T, called *transversal anti-edges*, and $B(T)$ (for Blue, or Biclique) is a set of transversal pairs of T, called *transversal edges*, such that $A(T) \cap B(T) \neq \emptyset$ and no $uv, u'v' \in A(T) \cup B(T)$ cross. We may refer to the transversal anti-edges as *green edges*, and to the transversal edges as *blue edges*.

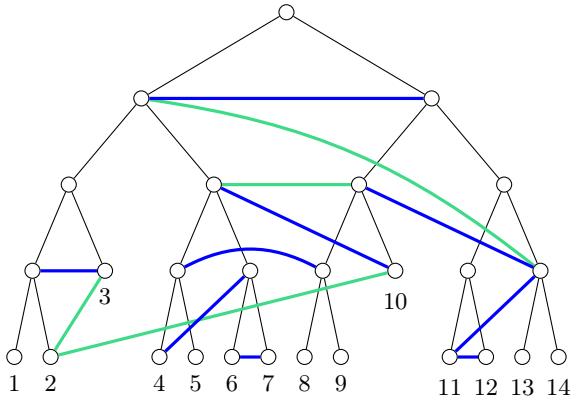

Figure 1: A signed tree model of a 14-vertex graph.

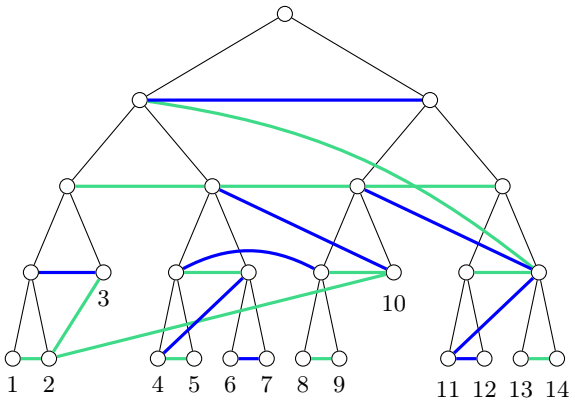

Figure 2: The signed tree model of Figure 1 made clean.

The *width* of the signed tree model $(T, A(T), B(T))$ is the degeneracy of the graph $(V(T), A(T) \cup B(T))$. Note that if $(V(T), A(T) \cup B(T))$ is d-degenerate, then $(V(T), A(T) \cup B(T) \cup E(T))$ is $(d+2)$-degenerate. The signed tree model is *d-sparse* if $|A(T) \cup B(T)| \leqslant d|V(T)|$. We observe that a signed tree model of width d is d-sparse, but an $O(1)$-sparse signed tree model can have width $\Omega(\sqrt{|V(T)|})$.

The signed tree model $\mathcal{T} := (T, A(T), B(T))$ defines a graph $G := G_{\mathcal{T}}$ with vertex set $L(T)$. Two leaves $u, v \in L(T)$ are adjacent in G if there is $u'v' \in B(T)$ such that $u'v' \preceq_T uv$, and there is no $u''v'' \in A(T)$ with $u'v' \prec_T u''v'' \preceq_T uv$. For example, in the representation of Figure 1, vertices 4 and 8 are adjacent in G because of the blue edge between their parents (below the green edge between their grandparents), but vertices 7 and 8 are non-adjacent because of the green edge between their grandparents (below the blue edge between their great-grand-parents). We may say that a graph G admits (or has) a signed tree model of width d if there is a signed tree model of this width that defines G. Every graph G admits a signed tree model as one can simply set $A(T) := \emptyset$, $B(T) := E(G)$ on an arbitrary full binary tree T with $L(T) = V(G)$. However this representation may have large width, while a more subtle one (linking nodes higher up in the tree) may have a lower width.

Every graph of twin-width d admits a signed tree model with $A(T) = \emptyset$ and width at most $d+1$. *Tree models* or *twin-decompositions* are signed tree models with $A(T) = \emptyset$, and further technical requirements. We observe that similar objects to signed tree models were utilized in [BGOT23] to

attain a fast matrix multiplication on matrices of low twin-width. We will not need a definition of twin-width, and refer the interested reader to [BKTW22]. In Section 1 we also mentioned Welzl orders with low alternation number [Wel88], let us now elaborate on that.

A *Welzl order* of *alternation number* d for a graph G is a total order $<$ on $V(G)$ such that the neighborhood of every vertex is the union of at most d intervals along $<$. We claim that bipartite graphs $G = (X \uplus Y, E(G))$ with a Welzl order $<$ of alternation number d admit a signed tree model of width $2d$. Note that we can assume that for every $x \in X$ and $y \in Y$, $x < y$. We build a signed tree model $(T, A(T), B(T))$ of G as follows. Let us call *left binary comb* a full binary tree whose internal nodes induce a path, rooted at an endpoint of this path, and every right child is a leaf. We make the root of T adjacent to the roots of two left binary combs with $|X|$ and $|Y|$ leaves, respectively. The leaves are labeled from left to right with the vertices of G in the order $<$. To simplify the notations, assume that these labels describe $[n]$ in the natural order. To represent that vertex $i \in X$ has $[j, k] \subseteq Y$ in the partition of its open neighborhood into maximal intervals, we add a blue edge between leaf i and the parent of k, and a green edge between i and the sibling of j (to stop the interval). Finally observe that $(V(T), A(T) \cup B(T))$ has degeneracy at most $2d$. (The subtree whose leaves are the vertices of X need not be a binary comb.)

A similar construction would work for graphs G of chromatic number q, and would yield a signed tree model of width $2(q-1)d$. A more permissive definition of signed tree models, allowing leaf-to-ancestor transversal edges and relaxing the notion of crossing, would give models of width $2d$ for any graph with a Welzl order of alternation number d. However, with this alternative definition, the consequences of the next section would not follow. Hence we stick to the given definition of signed tree models.

A signed tree model is said to be *clean* if every pair of siblings are linked by a green or blue edge. It is easy to turn a signed tree model into a clean one representing the same graph: simply add green edges between every pair of siblings that were previously not linked (by a blue or green edge). This operation may only increase the width of the signed tree model by 1. The advantage of working with a clean signed tree model is that for every pair of leaves u, v with least common ancestor w, there is at least one transversal edge or anti-edge connecting the paths (in T) between w and u and between w and v. Clean tree models will be useful in Section 4 when we balance the trees associated with the tree models.

Given a clean signed tree model $(T, A(T), B(T))$ and $u, v \in L(T)$, we denote by $e_T(u, v)$ the unique green or blue edge $u'v'$ such that $u'v' \preceq_T uv$ and no green or blue edge $u''v''$ satisfies $u'v' \prec_T u''v'' \preceq_T uv$. The edge $e_T(u, v)$ exists because the signed tree model is clean, and is unique because no green or blue edges may cross (or be equal). Then, u, v are adjacent in G if and only if $e_T(u, v) \in B(T)$, i.e., $e_T(u, v)$ is a blue edge. We first show that graphs of bounded sd-degeneracy (and in particular, of bounded symmetric difference) admit clean signed tree models of bounded width.

Lemma 3.1. *Any graph of sd-degeneracy d admits a clean signed tree model of width $d + 1$.*

Proof. Let $v_1, \ldots, v_n$ be a vertex ordering that witnesses sd-degeneracy d for an n-vertex graph G. For $i \in [n]$, let $G_i := G - \{v_j \ : \ 1 \leqslant j \leqslant i-1\}$. In particular, $G_1 = G$. Let u_i be a d-twin of v_i in G_i. Initially we consider a forest of n distinct 1-vertex rooted trees, each root labeled by a distinct vertex of G. We will build T (and in parallel, the transversal anti-edges and edges) by iteratively giving a common parent to two roots of this forest of n singletons. Note that different nodes of T may have the same label, as the labels will range in $V(G)$ whereas T has $2n-1$ nodes.

For i ranging from 1 to $n-1$:

- add a blue (resp. green) edge between v_i and u_i if $u_iv_i \in E(G)$ (resp. $u_iv_i \notin E(G)$),
- add a blue edge between v_i and the roots labeled by w for $w \in N_{G_i}(v_i) \setminus N_{G_i}[u_i]$,

- add a green edge between v_i and the roots labeled by w for $w \in N_{G_i}(u_i) \setminus N_{G_i}[v_i]$, and
- create a common parent, labeled by u_i, for the roots labeled u_i (left child) and v_i (right child).

This defines a full binary tree T such that $L(T) = V(G)$. In $(V(T), A(T) \cup B(T))$, the leaves labeled by v_1 and u_1 have degree at most $d+1$ and 1, respectively. Hence an immediate induction on $(T, A(T), B(T))$ (after removing these two leaves, and following the order $v_2, \ldots, v_n$) shows that $(V(T), A(T) \cup B(T))$ is $(d+1)$-degenerate. As we only add transversal anti-edges and edges between pairs of roots, no pair in $A(T) \cup B(T)$ can cross. Indeed, if x, y are two nodes of T that are both roots in some G_i, then it cannot happen that x', y' are also both roots of some $G_{i'}$ with $x \prec_T x'$ and $y' \prec_T y$. The first item further ensures that the signed tree model $(T, A(T), B(T))$ of width $d+1$ is clean.

Let us finally check that for every $u, v \in L(T)$, $e_T(u, v)$ is a blue edge if and only if $uv \in E(G)$. This is a consequence of the following property.

Claim 3.2. Let x, y be two nodes of T labeled by u, v respectively. Let x' be a child of x, labeled by u', such that $x'y$ is neither a blue nor a green edge. Further assume that y was a root when the parent of x' (i.e., x) was created. Then, $uv \in E(G)$ if and only if $u'v \in E(G)$.

Proof of Claim. If x' is the left child of x, the conclusion holds since $u = u'$. We can thus assume that x' is the right child of x, and not the sibling of y since it would contradict that $x'y$ is neither a blue nor a green edge. Node x' was not linked to y by a blue or a green edge, so v cannot be a neighbor of exactly one of u, u'. ◊

Consider the moment $e_T(u, v)$ was added to the signed tree model, say between the then-roots x and y, labeled by u' and v', respectively. By the way blue and green edges are introduced, xy is a blue edge if $u'v' \in E(G)$, and xy is green if $u'v' \notin E(G)$. Thus we conclude by iteratively applying Claim 3.2. □

4 Balancing Signed Tree Models

For any signed tree model of width d of an n-vertex graph, we get an adjacency labeling scheme with labels of size $O(dh \log n)$, where h is the depth of T. Indeed, one can label a leaf v of T (i.e., vertex of G) by the identifiers (each of $\log(2n)$ bits) of all the nodes of the path from v to the root of T, followed by the identifiers of the outneighbors of these at most h nodes in a fixed orientation of $(V(T), A(T) \cup B(T))$ with maximum outdegree at most d, allocating an extra bit for the color of each corresponding edge. One can then decode the adjacency of any pair $u, v \in V(G)$ by looking at the color of $e_T(u, v)$. The latter is easy to single out, based on the labels of u and v.

Proposition 4.1. *Let G be an n-vertex graph with a signed tree model of width d and depth h. Then, G admits an $O(dh \log n)$-bit adjacency labeling scheme.*

Unfortunately, the depth of the tree T of a signed tree model of low width obtained for an n-vertex graph of low sd-degeneracy could be as large as n. This makes a direct application of Proposition 4.1 inadequate. Instead, we first decrease the depth of the signed tree model, while controlling its sparsity. We start with some notation and auxiliary tools. Throughout this section we fix n and denote by R a full, complete, rooted binary tree whose leaves are natural numbers $1, 2, \ldots, n$ read from left to right.

Lemma 4.2. *Any interval $[i, j]$ with $i, j \in [n]$ is the disjoint union of the leaves of at most $2 \log n$ rooted subtrees of R.*

Proof. Let $X \subseteq V(R)$ be such that the leaves of the subtrees rooted at a node of X partition $[i, j]$, and X is of minimum cardinality among node subsets with this property. Let k be the first level of R intersected by X (with the root being at level 1). At most two nodes x, y of X are at level k (and exactly one node when $k = 2$), with $x = y$ or x to the left of y. Observe that if $x \neq y$, then x, y have to be consecutive along the left-to-right ordering of level k, but cannot be siblings (otherwise they can be substituted by their parent). At level $k + 1$, at most two nodes can be part of X: the node just to the left of the leftmost child of x, and the node just to the right of the rightmost child of y. This property propagates to the last level. Thus $|X| \leqslant \max(2(\lceil \log n \rceil - 1), \lceil \log n \rceil) \leqslant 2 \log n$. □

From the previous proof it can also be seen that there is a unique minimum-cardinality set X representing an interval $I = [i, j]$, which we denote by X_I. Furthermore, we observe that the minimality of X_I implies that the elements of X_I are incomparable in R, i.e., for any distinct $x, y \in X_I$, neither $x \prec_R y$, nor $y \prec_R x$.

Let T be any full binary tree with n leaves that are natural numbers $1, 2, \ldots, n$ read from left to right. For any node x of T the leaves of the subtree of T rooted at x form an interval in $[n]$, which we denote by $I_T(x)$. We proceed with further auxiliary statements.

Observation 4.3. *For every $x, y \in V(T)$, if $I_T(x)$ and $I_T(y)$ intersect, then one is included in the other, and thus $x \preceq_T y$ or $y \preceq_T x$.*

The observation above helps us establish the following lemma.

Lemma 4.4. *Let p and s be two distinct nodes of T, and a and b be two nodes of R such that $a \in X_{I_T(p)}$ and $b \in X_{I_T(s)}$. If $a \preceq_R b$, then $p \preceq_T s$ or $s \preceq_T p$. Furthermore, if $a \prec_R b$, then $p \prec_T s$.*

Proof. Since $a \in X_{I_T(p)}$ and $b \in X_{I_T(s)}$, we have $I_R(a) \subseteq I_T(p)$ and $I_R(b) \subseteq I_T(s)$. Thus, if $a \preceq_R b$, then $I_R(b) \subseteq I_R(a)$, and therefore $I_T(p)$ and $I_T(s)$ intersect. By Observation 4.3, this implies that $I_T(p) \subseteq I_T(s)$ or $I_T(s) \subseteq I_T(p)$, and thus $p \preceq_T s$ or $s \preceq_T p$.

Suppose now that $a \prec_R b$. Since the elements of $X_{I_T(s)}$ are incomparable in R and b belongs to this set, we conclude that $a \notin X_{I_T(s)}$. Thus, again by minimality of $X_{I_T(s)}$, there exists a leaf $x \in I_R(a)$ that does not belong to $I_T(s)$, and since $I_R(a) \subseteq I_T(p)$, we conclude that $I_T(p) \not\subseteq I_T(s)$. Therefore, we must have $I_T(s) \subsetneq I_T(p)$, where the strictness of the inclusion is due to the fact that $x \in I_T(p) \setminus I_T(s)$. Consequently, we conclude that $p \prec_T s$, as desired. □

We are now ready to prove the main lemma of this section. Recall that R a full, complete, rooted binary tree with n leaves, and T is a full binary tree with n leaves, where in both trees the leaves are natural numbers $1, 2, \ldots, n$ read from left to right.

Lemma 4.5. *Let $(T, A(T), B(T))$ be a clean d-sparse signed tree model of an n-vertex graph G. Then, G admits a $4d \log^2 n$-sparse signed tree model $(R, A(R), B(R))$ of depth $\lceil \log n \rceil + 1$.*

Proof. We start by describing the construction of $A(R) \cup B(R)$. For every transversal anti-edge (resp. edge) $xy \in A(T)$ (resp. $xy \in B(T)$), we add to $A(R)$ (resp. $B(R)$) all the unordered pairs ab with $a \in X_{I_T(x)}$ and $b \in X_{I_T(y)}$; we say that each such pair ab *originated* from xy. It may happen that some ab is added to both $A(R)$ and $B(R)$. In which case, ab originated from both $x_0y_0 \in A(T)$ and $x_1y_1 \in B(T)$ such that $x_0y_0 \prec_T x_1y_1$ or $x_1y_1 \prec_T x_0y_0$. In the former case, we remove ab from $A(R)$ (and only keep it in $B(R)$), and in the latter, we remove ab from $B(R)$ (and only keep it in $A(R)$). This finishes the construction of $\mathcal{R} := (R, A(R), B(R))$.

Let us first argue that no two transversal pairs in $A(R) \cup B(R)$ cross. Assume for the sake of contradiction that $a_1b_1, a_2b_2 \in A(R) \cup B(R)$ satisfy $a_1 \prec_R a_2$ and $b_2 \prec_R b_1$. Let $p_1s_1, p_2s_2 \in A(T) \cup B(T)$ be transversal pairs from which a_1b_1, a_2b_2, respectively, originated. By Lemma 4.4,

we have $p_1 \prec_T p_2$ and $s_2 \prec_T s_1$, implying that p_1s_1 and p_2s_2 cross in T, which is not possible. This contradiction proves that $\mathcal{R}$ is a signed tree model.

Next, let us show that $\mathcal{R}$ represents G. Fix $u, v \in V(G)$ and let $xy := e_T(u, v)$. Without loss of generality, assume that xy belongs to $A(T)$. Let ab be the transversal pair originating from xy such that $u \in I_R(a)$ and $v \in I_R(b)$. By construction ab belongs to $A(R) \cup B(R)$. We claim that $ab \in A(R)$ and $ab = e_R(u, v)$. First, suppose that $ab \notin A(R)$, and thus $ab \in B(R)$. By construction, this is possible only if ab has also originated from some $x_0y_0 \in B(T)$ and $xy \prec_T x_0y_0$. Note that as $u \in I_R(a)$ and $v \in I_R(b)$, we have that $a \preceq_R u$ and $b \preceq_R v$. Thus, by Lemma 4.4, $x_0 \preceq_T u$ or $u \preceq_T x_0$, and $y_0 \preceq_T v$ or $v \preceq_T y_0$, and since u and v are leaves in T, we conclude that $x_0y_0 \preceq_T uv$. Consequently, $xy \prec_T x_0y_0 \preceq_T uv$, which contradicts the fact that $xy = e_T(u, v)$. Now, suppose that $ab \neq e_R(u, v)$, i.e., there exists a_1b_1 such that $ab \prec_R a_1b_1 \preceq_R uv$. Again by Lemma 4.4, this implies $xy \prec_T x_1y_1 \preceq_T uv$, where x_1y_1 is the transversal pair of T from which a_1b_1 originated. This contradicts the assumption that $xy = e_T(u, v)$.

Finally, we establish the claimed sparseness of $\mathcal{R}$. By design, the depth of R is $\lceil \log n \rceil + 1$. As $\mathcal{T} := (T, A(T), B(T))$ is d-sparse, it has at most $(2n - 1)d$ transversal (anti-)edges. Each blue or green edge of $\mathcal{T}$ gives rise to at most $(2 \log n)^2$ blue or green edges of $\mathcal{T}'$, by Lemma 4.2. Hence $\mathcal{R}$ is $4d \log^2 n$-sparse. □

We finally need this folklore observation.

Observation 4.6. *Every m-edge graph has degeneracy at most $\lceil \sqrt{2m} \rceil - 1$.*

Proof. It is enough to show that any m-edge graph G has a vertex of degree at most $\lceil \sqrt{2m} \rceil - 1$. If all the vertices of G have degree at least $\lceil \sqrt{2m} \rceil$, then $m \geqslant \frac{1}{2}n\lceil \sqrt{2m} \rceil$. But also $n \geqslant \lceil \sqrt{2m} \rceil + 1$ for a vertex to possibly have $\lceil \sqrt{2m} \rceil$ neighbors. Thus $m \geqslant \frac{1}{2}\sqrt{2m}(\sqrt{2m} + 1) > m$, a contradiction. □

Combining Lemmas 3.1 and 4.5, Observation 4.6, and Proposition 4.1 yields Theorem 1.2.

Proof of Theorem 1.2. Let G be an n-vertex graph of sd-degeneracy d. By Lemma 3.1, G admits a clean signed tree model of width at most $d + 1$, hence $(d + 1)$-sparse. Thus by Lemma 4.5, G has a $4(d + 1) \log^2 n$-sparse signed tree model $\mathcal{T}$ of depth $\lceil \log n \rceil + 1$. By Observation 4.6, $\mathcal{T}$ has width at most

$$\sqrt{16(d + 1)n \log^2 n} = 4\sqrt{(d + 1)n} \log n.$$

Therefore, by Proposition 4.1, G has a $O(\sqrt{dn} \log^3 n)$-bit labeling scheme. □

5 Symmetric Difference is para-co-NP-complete

For any fixed even integer $d \geqslant 8$, we show that the following problem is NP-complete: Does the input graph G have an induced subgraph with at least two vertices and no pair of d-twins? We call such an induced subgraph a $(d + 1)$-*diverse graph*. The membership of this problem to NP is straightforward, as a $(d + 1)$-diverse induced subgraph H of G is a polynomial-size witness. One can indeed check in polynomial-time that H has at least two vertices, and that for every pair u, v of vertices of H, at least $d + 1$ other vertices of H are neighbors of exactly one of u, v.

The *d-twin graph* $T_d(G)$ of a graph G is a graph with vertex set $V(G)$ and edges between every pair of d-twins.

Observation 5.1. *The vertices of a $(d+1)$-diverse induced subgraph of G form an independent set of $T_d(G)$.*

Given any 3-SAT formula φ with at most three occurrences of each variable, clauses of size two or three, and at least three clauses, we build a graph $G := G(\varphi)$ such that G has a $(d+1)$-diverse induced subgraph if and only if φ is satisfiable. Such a restriction of 3-SAT is known to be NP-complete [Tov84].

5.1 Bubble gadget

A *bubble gadget* B (or *bubble* for short) is a $w \times w$ rook graph, with $w := \frac{d}{2} + 2$, deprived of the two rightmost vertices of its top row. Let the gadget B be an induced subgraph of a graph G and let S be the set of neighbors in G of the bubble outside of B. We say that B is *properly attached* to S in G if each vertex of the top row (of width $\frac{d}{2}$) and of the rightmost column (of height $\frac{d}{2}+1$) has one or two neighbors outside the gadget, whereas the other vertices of B have no neighbors outside $V(B)$.

We say that B is *neatly attached to* S if it is properly attached to S, and further, vertices of the top row and rightmost column have *exactly* one outside neighbor, and at most one vertex of S has neighbors in both the top row and rightmost column. The *neat* attachments that we will use all satisfy $3 \leqslant |S| \leqslant 5$. Hence they can be described by a tuple of size between 3 and 5, listing the number of neighbors of vertices in S among $V(B)$, starting with the top row and ending with the rightmost column. For instance, Figure 3 depicts a neat $(2,2,2,7)$-attachment. A bubble properly

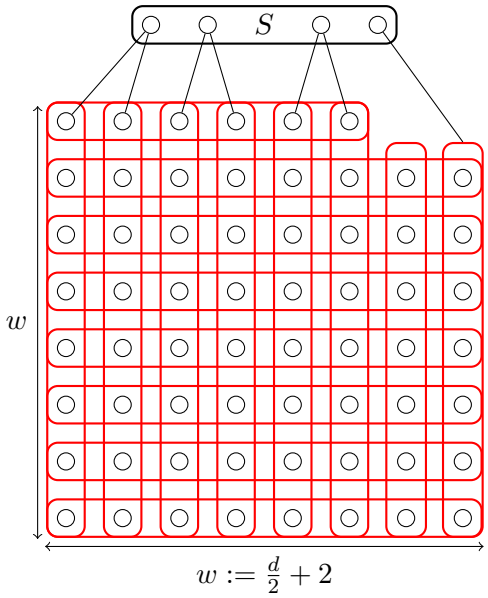

Figure 3: A neatly $(2,2,2,7)$-attached bubble gadget, with $d = 12$. The vertical and horizontal red boxes are cliques.

attached to S is in a delicate state. It may entirely survive in a $(d+1)$-diverse induced subgraph of G.

Observation 5.2. *Let B be a bubble gadget properly attached to S in G. No pair of vertices of B are d-twins in $G[V(B) \cup S]$.*

Proof. In B the only pairs with symmetric difference at most d, in fact exactly d, consist of a vertex in the top row and another vertex in its column, or two vertices of the same row in the two rightmost columns. In both cases, these pairs have symmetric difference at least $d+1$ in $G[V(B) \cup S]$ since

the vertices of the top row or rightmost column have at least one neighbor in S, while all other vertices of B have no neighbor in S. □

However, deletions that cause one vertex of the top row or two vertices of the rightmost column to no longer have outside neighbors cause the bubble to completely collapse.

Lemma 5.3. *Let B be a bubble gadget properly attached to S in G. Let H be any $(d+1)$-diverse induced subgraph of G, such that at least one vertex of the top row or at least two vertices of the rightmost column has no neighbor in $V(H) \setminus V(B)$. Then, H contains at most one vertex of B.*

Proof. We first deal with the case when a vertex v of the top row (in the entire B) has no neighbor in $V(H) \setminus V(B)$. By symmetry, assume that v is the topmost vertex of the first column. Vertex v is thus d-twin with all the other vertices of the first column. Hence by Observation 5.1, either v is not in H, or none of the $d/2+1$ vertices below v are in H.

If the latter holds, then any two vertices in the same column, outside the top row and rightmost column, are now d-twins in H. By Observation 5.1, within these vertices, H can only contain at most one vertex per column. In turn, the kept vertices are pairwise d-twins in H, so at most one can be kept overall. We conclude since the vertices of $N_H(S) \cap V(B)$ have at most two neighbors in S.

We now suppose that v is not in H. Then, in each row but the topmost, the vertices in the first and penultimate columns are d-twins in H. Thus, within each pair, at most one vertex can be in H. This implies that any two vertices in the same column, outside the top row and rightmost column, are now d-twins in H. Thus we conclude as in the previous paragraph.

We now deal with the case when two vertices x, y of the right most column have no neighbor in $V(H) \setminus V(B)$. By symmetry, we can assume that x is in the second row, and y is in the third row. Then, in H, x (resp. y) is d-twin with the vertex just to its left. After one vertex is removed in each pair, in each column but the last two, the vertices in the second and third rows have become d-twins in H. Therefore, H can only contain at most one vertex from all these pairs. We reach the state that any two vertices in the same row, outside the top row and rightmost column, are d-twins in H, and we can conclude as previously. □

In the incoming construction, all the bubble gadgets will be neatly attached. Furthermore, every vertex a bubble is attached to will have at least one neighbor on the top row, or at least two neighbors in the rightmost column. Thus the deletion of *any* vertex a bubble B is attached to will result, by Lemma 5.3, in deleting all the vertices of B but at most one.

5.2 Variable and clause gadgets

The *variable gadget* of variable x used in φ is simply two vertices $x, \neg x$ adjacent to a set N_x of $t := \frac{d}{2}+1$ shared neighbors. Vertices x and $\neg x$ have one or two other neighbors in G corresponding

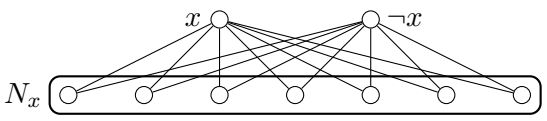

Figure 4: The variable gadget of x with $d = 12$.

to the clause they belong to, as we will soon see. Some vertices of the first two sets N_x (N_{x_1} and N_{x_2}) will have neighbors in these two sets. Apart from N_{x_1}, each set N_x will form an independent set. All other neighbors of the vertices of any N_x will belong to at most four bubble gadgets. This

will be described in detail shortly. The *clause gadget* of clause c consists of a pair of adjacent vertices v_c, d_c. We make v_c (but not d_c) adjacent to the two or three vertices corresponding to the literals of c.

5.3 Construction of $G(\varphi)$

Unsurprisingly, we add one variable gadget per variable, and one clause gadget per clause of φ. Let $x_1, \dots, x_n$ be a numbering of the variables, and $c_1, \dots, c_m$, of the clauses. We neatly attach a bubble gadget to S_j made of the five vertices $z_j, v_{c_j}, d_{c_j}, v_{c_{j+1}}, d_{c_{j+1}}$ for every $j \in [m-1]$, with (in this order) a $(1, \lfloor d/4 \rfloor, \lfloor d/4 \rfloor, \lceil d/4 \rceil, \lceil d/4 \rceil)$-attachment, where z_j is a vertex of some N_x. The choice of z_j is irrelevant, but we take all the vertices z_j pairwise distinct. This is possible since there are at most $3n/2$ clauses, and more than $dn/2$ vertices contained in the union of the sets N_x. For every $j \in [m-1]$, we make $\{v_{c_j}, d_{c_j}\}$ fully adjacent to $\{v_{c_{j+1}}, d_{c_{j+1}}\}$. The construction of G is almost complete; see Figure 5 for an illustration.

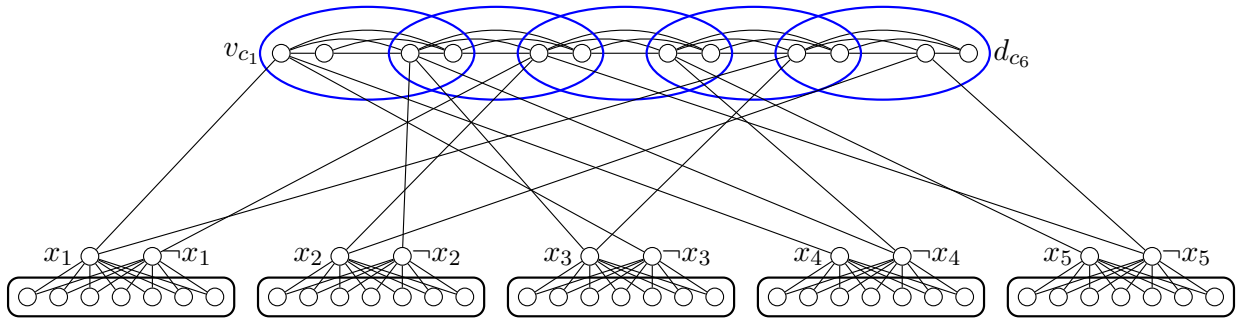

Figure 5: The essential part of G built so far, for a 3-CNF formula φ whose first two clauses are $x_1 \vee \neg x_3 \vee x_4$ and $\neg x_2 \vee x_3 \vee \neg x_4$. The blue ellipses represent the bubbles attached to the four enclosed vertices (recall that the bubble is attached to a fifth vertex among the sets N_x).

At this point, all the vertices v_{c_j}, d_{c_j} such that $j \in [2, m-1]$ have exactly $\lfloor d/4 \rfloor + \lceil d/4 \rceil = d/2$ neighbors in (two) bubble gadgets. Let $y_1, \dots, y_{nt}$ be the ordering of the vertices in $\bigcup_x N_x$ from left to right in how they appear in Figure 5. We neatly attach

1. a bubble gadget to (v_{c_1}, d_{c_1}, y_1) by a $(\lceil d/4 \rceil, \lceil d/4 \rceil, d + 1 - 2\lceil d/4 \rceil)$-attachment;
2. a bubble gadget to $(v_{c_m}, d_{c_m}, y_{nt})$ by a $(\lfloor d/4 \rfloor, \lfloor d/4 \rfloor, d + 1 - 2\lfloor d/4 \rfloor)$-attachment;
3. for every $i \in [nt-2]$, a bubble gadget to S'_i made of the three vertices y_i, y_{i+1}, y_{i+2} with a $(1, d/2, d/2)$-attachment.

Finally, we make a clique on the first $\lceil d/4 \rceil$ vertices of N_{x_1} (i.e., on the vertices $y_1, y_2, \dots, y_{\lceil d/4 \rceil}$) and fully connect them to the first $\lceil d/4 \rceil$ vertices of N_{x_2} (i.e., to the vertices $y_{t+1}, y_{t+2}, \dots, y_{t+\lceil d/4 \rceil}$). This finishes the construction.

We make some observations. As all the bubble gadgets are neatly attached, no two vertices outside a bubble gadget B can share a neighbor in B.

Observation 5.4. *For every $j \in [m]$, v_{c_j}, d_{c_j} each have exactly $d/2$ neighbors in bubble gadgets (all of which are non-adjacent to any other vertex outside their respective bubble).*

Vertices in $\bigcup_x N_x$ have more neighbors in bubbles.

Observation 5.5. *For every $i \in [nt]$, y_i has at least $d/2+1$ neighbors in bubble gadgets (all of which are non-adjacent to any other vertex outside the respective bubble). Furthermore, if $i \geqslant 3$, then y_i has at least d neighbors in bubble gadgets.*

5.4 Correctness

We can now show the following strengthening of Theorem 1.4.

Theorem 5.6. *For every fixed even integer $d \geqslant 8$, deciding if an input graph has symmetric difference at most d is co-NP-complete.*

As the graph $G := G(\varphi)$ presented in Section 5.3 can be constructed in polynomial time, to prove Theorem 5.6, we shall simply check the equivalence between the satisfiability of φ and the existence of a $(d+1)$-diverse induced subgraph of $G(\varphi)$. We do this in the next two lemmas. Recall that, by definition, G does *not* have symmetric difference at most d if and only if it has a $(d+1)$-diverse induced subgraph.

Lemma 5.7. *If φ is satisfiable, then G admits a $(d+1)$-diverse induced subgraph.*

Proof. Let $\mathcal{A}$ be a satisfying assignment of φ. For each variable x of φ, we delete vertex $\neg x$ if $\mathcal{A}$ sets x to true, and we delete vertex x otherwise (if $\mathcal{A}$ sets x to false). Let us call H the obtained induced subgraph of G (with at least two vertices). We claim that H has no pair of d-twins, and successively rule out such pairs

(i) within the same bubble,
(ii) between a vertex in a bubble B and a vertex outside B (but possibly in another bubble),
(iii) between two vertices both outside every bubble gadget.

(i) As H contains all the vertices of G on which bubble gadgets are attached, by Observation 5.2, no two distinct vertices in the same bubble are d-twins.

(ii) Let us fix a bubble gadget B attached to S, and two vertices $u \in V(B)$ and $v \in V(H) \setminus V(B)$. First observe that u has at least $d/2+1$ neighbors in $V(B)$ (hence in H) that are not neighbors of v. All the vertices $v \in V(H) \setminus (V(B) \cup S)$ have at least $d/2$ neighbors in H that are not neighbors of u. For these vertices v, $\mathrm{sd}_H(u,v) > d$. We thus focus on the case when $v \in S$. Recall that bubble gadgets are attached only to vertices in $\{y_1, \ldots, y_{nt}\} \cup \bigcup_{j\in[m]}\{v_{c_j}, d_{c_j}\}$. First, assume $v \in \{y_3, y_4, \ldots, y_{nt}\}$. Each such vertex has at least $d/2$ neighbors in each of two distinct bubbles, and thus it has $d/2$ neighbors outside B implying $\mathrm{sd}_H(u,v) > d$. Next, assume $v \in \{y_1, y_2\}$. Then v has one neighbor in some bubble gadget, say B_1, and at least $d/2$ neighbors in some other gadget, say B_2. If $B = B_1$, then clearly $\mathrm{sd}_H(u,v) > d$. If $B = B_2$, then v has at least $(\lceil d/4 \rceil - 2) + \lceil d/4 \rceil + 1 + 1 \geqslant d/2$ neighbors outside B that are not neighbors of u, where $\lceil d/4 \rceil - 2$ neighbors are coming from the clique $\{y_1, y_2, \ldots, y_{\lceil d/4 \rceil}\}$ (note that one of those vertices is v, and at most one of them is a neighbor of u, as u can have at most one neighbor outside B, hence minus 2), $\lceil d/4 \rceil$ are coming from $\{y_{t+1}, y_{t+2}, \ldots, y_{t+\lceil d/4 \rceil}\}$, one neighbor is coming from $\{x_1, \neg x_1\}$, and one more is coming from B_1. Thus, in this case, we also have $\mathrm{sd}_H(u,v) > d$.

Finally, assume that v is v_{c_j} or d_{c_j} for some $j \in [m]$. We can further assume that u is in the top row or rightmost column of B, otherwise it has d neighbors that are not neighbors of v (and v has at least one private neighbor). Now we observe that

$$\mathrm{sd}_H(u,v) \geqslant |N_H(u) \setminus N_H[v]| + |N_H(v) \setminus N_H[u]| \geqslant d/2 + 1 + d/2 - 1 - \lceil d/4 \rceil + \lfloor d/4 \rfloor + 2 \geqslant d+1,$$

where $d/2+1$ lower bounds the number of neighbors of u whose neighborhood is included in $V(B)$, $d/2 - 1 - \lceil d/4 \rceil$ lower bounds the number of neighbors of u in the top row or rightmost column

of B that are not adjacent to v, $\lfloor d/4 \rfloor$ lower bounds the number of neighbors of v in another bubble than B, and 2 accounts for the at least two neighbors of v that are not neighbors of u among $\{v_{c_{j-1}}, d_{c_{j-1}}, v_{c_j}, d_{c_j}, v_{c_{j+1}}, d_{c_{j+1}}\}$, whichever exist. (Here we use the fact that φ has at least two clauses.)

(*iii*) Let u, v be two distinct vertices outside every bubble gadget. First, recall that, by Observations 5.4 and 5.5, every vertex $v_{c_j}, d_{c_j}, j \in [m]$ has $d/2$ neighbors in bubble gadgets, and every vertex in $\bigcup_x N_x$ has at least $d/2+1$ neighbors in bubble gadgets. Since no two vertices outside bubble gadgets can share a neighbor in a bubble gadget, we conclude that if $u \in \bigcup_x N_x \cup \bigcup_{j\in[m]} \{v_{c_j}, d_{c_j}\}$ and $v \in \bigcup_x N_x$, then u and v are not d-twins.

Furthermore, if u is x_i or $\neg x_i$ for some $i \in [n]$ and $v \in \{y_3, y_4, \ldots, y_{nt}\}$, then, by Observation 5.5 and the fact that u has at least one neighbor v_{c_j} (for some $j \in [m]$), the vertices u and v are also not d-twins. Next, since each x_i and $\neg x_i$ has $d/2+1$ neighbors in N_{x_i}, and no neighbors in $N_{x_{i'}}$ for $i' \neq i$, we conclude that if u is x_i or $\neg x_i$ for some $i \in [n]$ and v is $x_{i'}$ or $\neg x_{i'}$ for some $i' \in [n]$, or v_{c_j} or d_{c_j} for some $j \in [m]$, then u and v are not d-twins.

Now, assume that u is x_i or $\neg x_i$ for some $i \in [n]$, and $v \in \{y_1, y_2\}$. Note that none of y_1 and y_2 has neighbors in N_{x_i} for $i \geqslant 3$, and therefore a non-trivial argument is only required for $i \leqslant 2$. Then v has at least $\lceil d/4 \rceil - 1$ neighbors in $N_{x_1} \cup N_{x_2}$ that are not neighbors of u, and by Observation 5.5, v has at least $d/2+1$ additional private neighbors in bubble gadgets. On the other hand, u has at least $d/2+1-\lceil d/4 \rceil$ private neighbors in N_{x_2}. This implies that u and v are not d-twins in H.

This leaves us with the cases when u and v are two vertices in clause gadgets. By Observation 5.4, each of these vertices has $d/2$ private neighbors in bubble gadgets. As there are at least three clauses in φ, two vertices u, v from distinct clause gadgets have at least two additional private neighbors. Thus, we can assume that $u = v_{c_j}$ and $v = d_{c_j}$ for some $j \in [m]$. As $\mathcal{A}$ is a satisfying assignment, at least one vertex x or $\neg x$ adjacent to v_{c_j} has survived in H. Hence $\mathrm{sd}_H(u, v) \geqslant d/2 + d/2 + 1 = d+1$. □

Lemma 5.8. *If φ is not satisfiable, then G has no $(d+1)$-diverse induced subgraph.*

Proof. Suppose, toward a contradiction, that φ is not satisfiable, but there exists a set $S \subseteq V(G)$ such that $G[S]$ is a $(d+1)$-diverse graph. Our strategy is to start with $V' = V(G)$ and iteratively reduce V' while ensuring that it still contains S. To do this, we will repeatedly apply Lemma 5.3 and the fact that at most one vertex in any pair of d-twins in $G[V']$ can belong to S. Eventually, we will reduce V' to a single vertex, and thus arrive at a contradiction that $G[S]$ is a $(d+1)$-diverse graph.

For each variable x, the vertices $x, \neg x$ are 3-twins, thus at least one of them has to be removed in a $(d+1)$-diverse induced subgraph. The kept literals (if any) define a (partial) truth assignment. By assumption, this assignment does not satisfy at least one clause c_j. This implies that v_{c_j}, d_{c_j} are d-twins in the corresponding induced subgraph. Indeed, they each have exactly $d/2$ private neighbors in bubble gadgets, and no other private neighbors.

By Lemma 5.3, the bubble attached to S_j is reduced to at most one vertex, say w_j (if any). In turn, this makes the pairs $v_{c_{j-1}}, d_{c_{j-1}}$ and $v_{c_{j+1}}, d_{c_{j+1}}$ d-twins (when they exist). Indeed their symmetric difference is at most $3+2\lceil d/4 \rceil + 1 \leqslant d$ (as d is even and at least 8), where 3 accounts for the three literals of the clause, and 1 for vertex w_j. This iteratively collapses every bubble attached to some $S_{j'}$ to a single vertex, as well as the two bubble gadgets attached to $\{v_{c_1}, d_{c_1}, y_1\}$ and $\{v_{c_m}, d_{c_m}, y_{nt}\}$, to say, w_0 and w_m. Now all the vertices w_j (for $j \in [0, m]$) have degree at most 1, hence are pairwise 2-twins. So at most one of them can be kept. We recall that at most one vertex per clause gadget could be kept. For j going from 1 to $m-1$, the vertex kept (if any) from the clause gadget of c_j is an 8-twin of the vertex kept in the next surviving clause gadget. This implies that

from all the clause gadgets and all the bubble gadgets attached to them, one can only keep at most one vertex overall, say z. This vertex has degree at most 3 in the resulting induced subgraph.

Vertices y_1, y_2 are now d-twins, so the bubble gadget attached to S'_1 collapses to at most one vertex. Vertices y_1 and z are now d-twins, so at most one can survive, which we keep calling z. Even if y_2 is kept, it is now a d-twin of y_3, thus at most one of y_2, y_3 can be kept. This implies the collapse of the bubble gadget attached to S'_2 to at most one vertex, absorbed by z. In turn, y_2 and z collapse to a single vertex. This process progressively eats up all the vertices y_j, and all the bubble gadgets attached to them. As soon as a vertex x or $\neg x$ has three remaining neighbors, it becomes a 6-twin of z, and is absorbed by it. We end up with the single vertex z. □

6 Deciding sd-degeneracy at most 1 is NP-complete

The membership to NP of computing the sd-degeneracy is readily given by the witnessing order, which is a polynomial certificate. Indeed, for any integer d, testing if a vertex has a d-twin can be done in polynomial time. In this section, we reduce 3-SAT where every variable occurs in two or three clauses and every clause has three literals to deciding if the sd-degeneracy of a graph is at most 1. As mentioned in the previous section, such a restriction of 3-SAT is known to be NP-complete [Tov84]. More precisely, the reduction is from determining if a variable assignment satisfies all but at most one clause. Obviously this problem remains NP-hard: simply duplicate an instance of 3-SAT on two disjoint sets of variables. One can satisfy all but at most one clause of the resulting instance if and only if the original instance is satisfiable.

For a given formula φ, we define a graph $G := G(\varphi)$, which is of sd-degeneracy at most 1 if and only if all clauses of φ but at most one are satisfied by some assignment. The graph G is made of variable gadgets, clause gadgets, and two extra vertices: a vertex of large degree γ and an isolated vertex ι.

6.1 Construction of $G(\varphi)$

For each variable v, we define the *variable gadget* $\mathrm{Var}(v)$. Let p be the number of clauses v occurs in (note that p is either 2 or 3). The gadget $\mathrm{Var}(v)$ is made of $2p+1$ vertices. Two special vertices v_0 and v_1 named respectively *representatives of* $\neg v$ and v, and $2p-1$ vertices named the *transition vertices from* v_0 *to* v_1. The $2p+1$ vertices of $\mathrm{Var}(v)$ form an independent set. For a literal l of variable v, we will write $\mathrm{Var}(l)$ to refer to gadget $\mathrm{Var}(v)$.

For each clause c appearing in φ, we define the *clause gadget* $\mathrm{Clause}(c)$. Let l_1, l_2 and l_3 be the literals of the clause c. The gadget $\mathrm{Clause}(c)$ is made of five vertices: $c_\uparrow, c_{l_1}, c_{l_2}, c_{l_3}$ and $c_\downarrow$. The five vertices of $\mathrm{Clause}(c)$ form a clique. The graph G is obtained from the disjoint union of the variable gadgets, clause gadgets, and two extra vertices γ and ι, on which we add the following edges.

1. For each clause $c = l_1 \vee l_2 \vee l_3$ and $i \in [3]$, we link both c_{l_i} and $c_\uparrow$ to the representative of l_i; see Figure 6.
2. For each clause c, we add an edge between $c_\downarrow$ and γ.
3. At this point, for each variable gadget $\mathrm{Var}(v)$, one can describe the neighborhood of the representatives v_0 and v_1 using two integers a and b, where $a+b$ is the number of occurrences of variable v in φ. Recall from Item 1 that the neighborhood of v_0 is a set of $2a$ vertices in clause gadgets (two vertices per clause containing $\neg v$), say $\{x_1, \ldots, x_{2a}\}$, where we order them such that the vertices x_i for $i \in [a]$ are of the form $c_\uparrow$ for some clause c, and the vertices x_i for $i \in [a+1, 2a]$ are of the form c_{l_j}. In the neighborhood of v_1, which is a set $\{y_1, \ldots, y_{2b}\}$ of $2b$ vertices, the order is reversed. Namely, the vertices y_i for $i \in [b]$ are of the form c_{l_j}, and the vertices y_i for $i \in [b+1, 2b]$ are of the form $c_\uparrow$ for some clause c. Let $\{t_1, \ldots, t_{2(a+b)-1}\}$ be

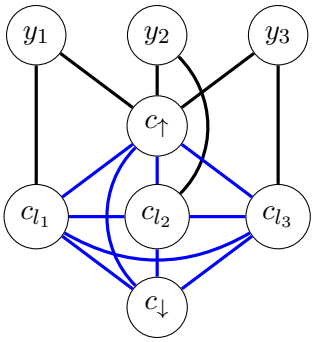

Figure 6: Adjacencies between the clause gadget of $c = l_1 \vee l_2 \vee l_3$ and y_1, y_2 and y_3 the representatives of l_1, l_2, and l_3, respectively. The blue edges represent the internal edges of the clause gadget.

the transition vertices from v_0 to v_1. We set $N(t_1)$ to $N(v_0) \cup \{y_1\}$, and for $i \in [2, 2b]$, we set $N(t_i)$ to $N(t_{i-1}) \cup \{y_i\}$. Note that $N(t_{2b}) = N(v_0) \cup N(v_1)$. Then for $i \in [2b+1, 2a+2b-1]$, we set $N(t_i)$ to $N(t_{i-1}) \setminus \{x_{i-2b}\}$.

We finally add ι, an isolated vertex. This finishes the construction; see Figure 7 for the overall picture.

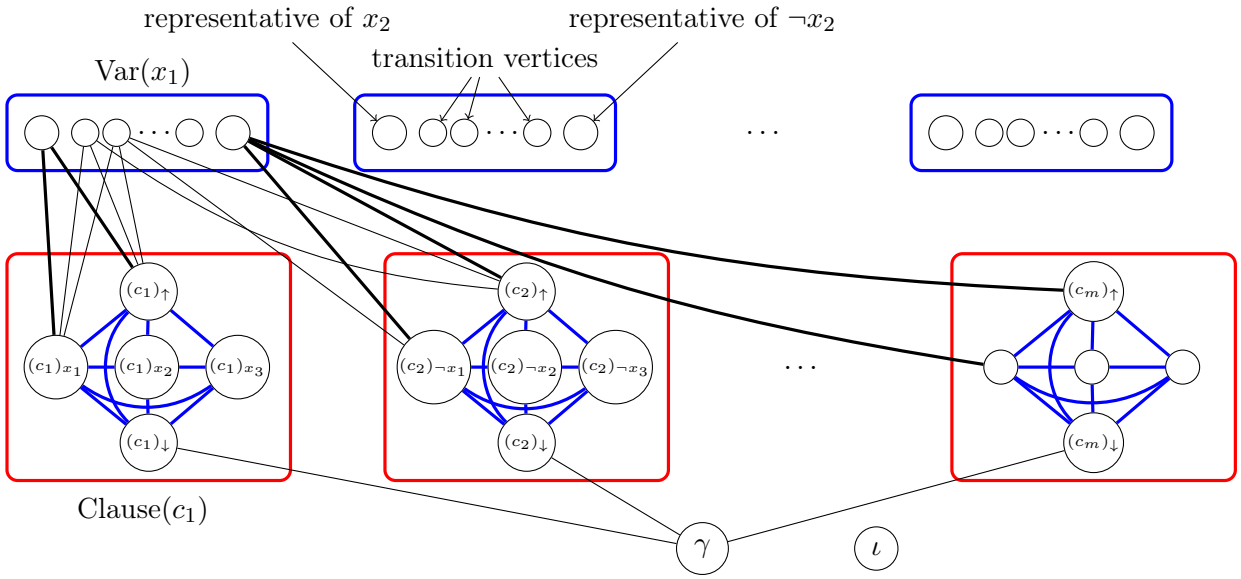

Figure 7: Depiction of $G(\varphi)$ when the first two clauses of φ are $c_1 = x_1 \vee x_2 \vee x_3$ and $c_2 = \neg x_1 \vee \neg x_2 \vee \neg x_3$, literal x_1 appears only once, and $\neg x_1$ is the first literal of the second and last clauses, c_2 and c_m respectively. The blue boxes represent the variable gadgets and the red boxes represent the clause gadgets. For legibility, only few edges between the variable gadgets and the clause gadgets were drawn.

6.2 Correctness

Henceforth we say that a vertex v can be *eliminated* in an induced subgraph H of G, if v has a 1-twin in H, and an elimination order of G is an ordering $u_1, u_2, \ldots$ of $V(G)$, equivalently represented by a total order $\prec$ on $V(G)$ such that u_i can be eliminated in $G - \{u_j \ : \ 1 \leqslant j \leqslant i-1\}$. Given an

elimination order $\prec$ of G and a vertex v, we denote the graph $G - \{u \in V(G) \ : \ u \prec v\}$ by G_v.

Let us first give an overview of the proof. In one direction, (Lemma 6.1) we turn an assignment satisfying all but at most one clause into an elimination order. This is relatively straightforward. We start by eliminating in each variable gadget $\mathrm{Var}(v)$ every vertex but v_0 when v is set to true, and every vertex but v_1 when v is set to false. After which, we can remove every vertex of $\mathrm{Clause}(c)$ for every satisfied clause c. From there one can easily finish the elimination order of G.

Let us now consider the other direction (Lemma 6.5). If G admits an elimination order $\prec$, an assignment is obtained by considering the last vertex eliminated in each variable gadget. If the neighborhood of this last vertex contains $N(v_0)$, then v is set to true, otherwise v is set to false. We denote this assignment by $\mathcal{A}(\prec)$. We need to argue that $\mathcal{A}(\prec)$ satisfies all but at most one clause of φ.

First, we show that since γ is adjacent to all the vertices $c_\downarrow$, for γ to be eliminated one needs to have at least started the elimination of all but at most one clause gadget. Second, we prove that for a clause gadget $\mathrm{Clause}(c)$ to start its elimination before γ is eliminated, one needs, for at least one of its literals l, that all the vertices of $\mathrm{Var}(l)$ that are fully adjacent to the neighborhood of its representative are already eliminated. Finally, we show that for each variable v, the last eliminated vertex of $\mathrm{Var}(v)$ is eliminated after each clause gadget adjacent to $\mathrm{Var}(v)$ has started to be eliminated. This implies that $\mathcal{A}(\prec)$ satisfies all but at most one clause of φ.

Lemma 6.1. *If all but at most one clause of φ can be satisfied, then G has sd-degeneracy at most 1.*

Proof. Let $\mathcal{A}$ be a variable assignment satisfying all but at most one clause of φ; say, c_0 if an unsatisfied clause indeed exists. We give an elimination order of G. For each variable v, we eliminate in $\mathrm{Var}(v)$ every vertex but v_0 if v is set to true, and every vertex but v_1, if instead v is set to false. This is possible since $v_0, t_1, \ldots, t_h, v_1$ is a *path* of 1-twins, where $t_1, \ldots, t_h$ are the transition vertices of $\mathrm{Var}(v)$.

Now we eliminate the clause gadgets. Let $c = l_1 \vee l_2 \vee l_3$ be a clause satisfied by $\mathcal{A}$. Assume without loss of generality (since $\mathrm{Clause}(c)$ is symmetric) that l_1 is satisfied by $\mathcal{A}$. So the only remaining vertex in $\mathrm{Var}(l_1)$ is the representative of $\neg l_1$. In particular, $\mathrm{Var}(l_1)$ is no longer adjacent to $\mathrm{Clause}(c)$. Thus $c_\uparrow$ and c_{l_2} are 1-twins: they may only differ on the last remaining vertex of $\mathrm{Var}(l_3)$. We eliminate $c_\uparrow$. In turn, c_{l_1} and c_{l_2} are 1-twins: they may only differ on the last remaining vertex of $\mathrm{Var}(l_2)$. We eliminate c_{l_2}. The same holds for c_{l_1} and c_{l_3}, and we eliminate c_{l_3}. At this point, $\mathrm{Clause}(c)$ contains exactly two vertices: c_{l_1} and $c_\downarrow$. The neighborhood of c_{l_1} is reduced to $\{c_\downarrow\}$, and the neighborhood of $c_\downarrow$ is $\{c_{l_1}, \gamma\}$. We eliminate c_{l_1} followed by $c_\downarrow$; the latter is now a 1-twin of ι, since it has degree 1.

All the vertices of all the clause gadgets except possibly $\mathrm{Clause}(c_0)$ are now eliminated. Hence γ is of degree at most 1, thereby is a 1-twin of ι. We eliminate γ. It remains to eliminate $\mathrm{Clause}(c_0)$ (if it exists). Let l be a literal of c_0. Now $c_\downarrow$ and $(c_0)_l$ are 1-twins. We eliminate $(c_0)_l$ for each literal l of c_0. The vertex $(c_0)_\downarrow$ and all remaining vertices of the variable gadgets are now of degree at most 1. We eliminate them, as 1-twin of ι. We finally eliminate $(c_0)_\uparrow$, and remain with a single vertex, ι. □

We now proceed with some lemmas which will establish the other direction of the reduction. Given two sets $X, Y \subseteq V(G)$, we write $X \prec Y$ if for every $x \in X$ and $y \in Y$ it holds that $x \prec y$, and we write $x \prec Y$ or $Y \prec x$ when $X = \{x\}$.

Lemma 6.2. *Let $\prec$ be an elimination order for G. Then, there is at most one clause gadget $\mathit{Clause}(c)$ such that $\gamma \prec V(\mathit{Clause}(c))$.*

Proof. The vertex γ is adjacent to all the vertices $c_\downarrow$. By construction, no vertex of G apart from γ has more than one $c_\downarrow$ in its closed neighborhood. Let w be a 1-twin of γ in G_γ. Assume for the sake of contradiction that c_1 and c_2 are two clauses of φ with $\gamma \prec V(\text{Clause}(c_1)) \cup V(\text{Clause}(c_2))$. As γ is adjacent to both $(c_1)_\downarrow$ and $(c_2)_\downarrow$, w must be either a vertex of Clause(c_1) or a vertex of Clause(c_2). In both cases, it still has four neighbors in G_γ inside its clause gadget, thus w cannot be a 1-twin of γ; a contradiction. □

Next we formalize the dependence between eliminating the variable gadgets and eliminating the clause gadgets.

Lemma 6.3. *Let $\prec$ be an elimination order for G. Let $c = l_1 \vee l_2 \vee l_3$ be a clause of φ. Let v be the first eliminated vertex of Clause(c). Then either $\gamma \prec v$ or there is an $l \in \{l_1, l_2, l_3\}$ such that for any vertex y in Var(l) such that $N(y)$ contains the (open) neighborhood of the representative of l, we have $y \prec v$.*

Proof. Let w be a 1-twin of v in G_v. Note that G_v still contains the whole gadget Clause(c), and so $N_{G_v}[v]$ contains $V(\text{Clause}(c))$. As all the vertices of G apart from those of Clause(c) have at most two neighbors in Clause(c), w has to be a vertex of Clause(c). Assume, for the sake of contradiction, that $v \prec \gamma$ and for each $l \in \{l_1, l_2, l_3\}$, there is a vertex y_l of Var(l) with $v \prec y_l$ that is adjacent to the whole open neighborhood of the representative of l. We consider three cases.

Case 1: v is $c_\uparrow$. Then, by construction of G, v is adjacent to y_{l_1}, y_{l_2}, and y_{l_3}, and so w must be adjacent to at least two vertices among y_{l_1}, y_{l_2} and y_{l_3}. But no vertex in $w \in V(\text{Clause}(c)) \setminus \{c_\uparrow\}$ has this property.

Case 2: v is of the form c_l for $l \in \{l_1, l_2, l_3\}$. Without loss of generality, assume $v = c_{l_1}$. Then vertex v is adjacent to y_{l_1} and not adjacent to y_{l_2} and y_{l_3}. Thus w cannot be $c_\uparrow$, nor c_{l_2}, nor c_{l_3}. Vertex w cannot be $c_\downarrow$ either, since it is still adjacent to γ.

Case 3: $v = c_\downarrow$. By the previous cases $c_\downarrow$ has no 1-twins in G_v. □

Lemma 6.4. *Let $\prec$ be an elimination order for G. Let l be a literal appearing in clauses $c_1, \ldots, c_p$, and let v_l be its representative. Assume that the last eliminated vertex from Var(l), say v, is fully adjacent to $N_G(v_l)$. Then, for each $i \in [p]$, there is a vertex x_i of the gadget Clause(c_i) with $x_i \prec v$.*

Proof. Assume, for the sake of contradiction, that there is a clause c among $c_1, \ldots, c_p$ with all its vertices in G_v. Let w be a 1-twin of v in G_v. Then v has exactly two neighbors in Clause(c): $c_\uparrow$ and c_l. This implies that w cannot be a vertex of Clause(c), as w still has four neighbors in $V(\text{Clause}(c))$. Vertex w cannot be in another clause gadget, since it would then have no neighbors in $V(\text{Clause}(c))$. Vertex w is not γ nor ι since they are not adjacent to $c_\uparrow$ and c_l.

Therefore, w is a vertex of another variable gadget, and it is adjacent to $c_\uparrow$. Then w is adjacent to a vertex $c_{l'}$ of Clause(c), with $l' \neq l$. Thus v and w are not 1-twins; a contradiction. □

We can now turn an elimination order into a variable assignment that satisfies all the clauses but at most one, which together with Lemma 6.1, proves Theorem 1.3.

Lemma 6.5. *If G admits an elimination order, then all clauses of φ but at most one can be satisfied.*

Proof. Let $\prec$ be an elimination order for G. For each variable x, let v_x be the last vertex eliminated from the variable gadget Var(x). If $N_G(v_x)$ contains $N_G(x_0)$ then we set x to true, otherwise we set x to false. This gives a truth assignment $\mathcal{A}(\prec)$.

Assume for the sake of contradiction that there are two clauses c_1, c_2 not satisfied by $\mathcal{A}(\prec)$. Assume that $v \in$ Clause(c_1) is the first eliminated vertex among Clause$(c_1) \cup$ Clause(c_2). By Lemma 6.2, the vertex γ is in G_v. Thus by Lemma 6.3, there is a literal l appearing in c_1 such that

any vertex in $\mathrm{Var}(l)$ adjacent to the neighborhood of the representative of l is eliminated before v. By definition of $\mathcal{A}(\prec)$, this implies that $\mathrm{Var}(l)$ is eliminated before any vertex of $\mathrm{Clause}(c_1)$ has been eliminated, which is a contradiction by Lemma 6.4. □

References

[AAA+23] Bogdan Alecu, Vladimir E. Alekseev, Aistis Atminas, Vadim Lozin, and Viktor Zamaraev. Graph parameters, implicit representations and factorial properties. *Discrete Mathematics*, 346(10):113573, 2023.

[AAL21] Bogdan Alecu, Aistis Atminas, and Vadim V. Lozin. Graph functionality. *J. Comb. Theory, Ser. B*, 147:139–158, 2021.

[ACLZ15] Aistis Atminas, Andrew Collins, Vadim V. Lozin, and Victor Zamaraev. Implicit representations and factorial properties of graphs. *Discret. Math.*, 338(2):164–179, 2015.

[Alo23] Noga Alon. Implicit representation of sparse hereditary families. *Discret. Comput. Geom.*, to appear, 2023.

[BDS+23] Édouard Bonnet, Julien Duron, John Sylvester, Viktor Zamaraev, and Maksim Zhukovskii. Tight bounds on adjacency labels for monotone graph classes. *arXiv preprint arXiv:2310.20522*, 2023.

[BGK+21] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width III: max independent set, min dominating set, and coloring. In *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021*, volume 198 of *LIPIcs*, pages 35:1–35:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[BGOT23] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, and Stéphan Thomassé. Twin-width V: linear minors, modular counting, and matrix multiplication. In *40th International Symposium on Theoretical Aspects of Computer Science, STACS 2023*, volume 254 of *LIPIcs*, pages 15:1–15:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023.

[BKTW22] Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. *J. ACM*, 69(1):3:1–3:46, 2022.

[BNO+21] Édouard Bonnet, Jaroslav Nesetril, Patrice Ossona de Mendez, Sebastian Siebertz, and Stéphan Thomassé. Twin-width and permutations. *arXiv preprint arXiv:2102.06880*, 2021.

[CPS85] Derek G. Corneil, Yehoshua Perl, and Lorna K. Stewart. A linear recognition algorithm for cographs. *SIAM J. Comput.*, 14(4):926–934, 1985.

[DEM+23] Jan Dreier, Ioannis Eleftheriadis, Nikolas Mählmann, Rose McCarty, Michal Pilipczuk, and Szymon Torunczyk. First-order model checking on monadically stable graph classes. *CoRR*, abs/2311.18740, 2023.

[Die17] Reinhard Diestel. *Graph Theory*. Springer Berlin Heidelberg, 2017.

[EH66] Paul Erdős and András Hajnal. On chromatic number of graphs and set-systems. *Acta Mathematica Hungarica*, 17(1-2):61–99, 1966.

[EH68] Paul Erdos and András Hajnal. On chromatic number of infinite graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 83–98, 1968.

[GKS17] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):17:1–17:32, 2017.

[HH22] Hamed Hatami and Pooya Hatami. The implicit graph conjecture is false. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS 2022)*, pages 1134–1137. IEEE, 2022.

[KNR88] Sampath Kannan, Moni Naor, and Steven Rudich. Implicit representation of graphs. In *Proceedings of the twentieth annual ACM symposium on Theory of computing (STOC 1988)*, pages 334–343, 1988.

[Kos84] Alexandr V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.

[MM13] Colin McDiarmid and Tobias Müller. Integer realizations of disk and segment graphs. *Journal of Combinatorial Theory, Series B*, 103(1):114–143, 2013.

[Mul88] John H. Muller. *Local Structure in Graph Classes*. PhD thesis, 1988.

[NO12] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012.

[NPS23] Sergey Norin, Luke Postle, and Zi-Xia Song. Breaking the degeneracy barrier for coloring graphs with no K_t minor. *Advances in Mathematics*, 422:109020, 2023.

[Tho84] Andrew Thomason. An extremal function for contractions of graphs. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 95, pages 261–265. Cambridge University Press, 1984.

[Tov84] Craig A. Tovey. A simplified NP-complete satisfiability problem. *Discret. Appl. Math.*, 8(1):85–89, 1984.

[Wel88] Emo Welzl. Partition trees for triangle counting and other range searching problems. In Herbert Edelsbrunner, editor, *Proceedings of the Fourth Annual Symposium on Computational Geometry (SoCG 1988)*, pages 23–33. ACM, 1988.