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# Modelling and inference for a degradation process with partial maintenance effects

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## Abstract

This paper proposes a new way of modelling imperfect maintenance in degradation models, by assuming that maintenance affects only a part of the degradation process. More precisely, the global degradation process is the sum of two dependent Wiener processes with drift. Maintenance has an effect of the  $ARD_1$ -type on only one of these processes: it reduces the degradation level of a quantity which is proportional to the amount of degradation of this process accumulated since previous maintenance. Two particular cases of the model are considered: perturbed  $ARD_1$  and partial replacement models. The usual  $ARD_1$  model is also a specific case of this new model. The system is regularly inspected in order to measure the global degradation level. Two observation schemes are considered. In the complete scheme, the degradation levels are measured both between maintenance actions and at maintenance times (just before and just after). In the general scheme, the degradation levels are measured only between maintenance actions. The maximum likelihood estimation of the model parameters is studied for both observation schemes in both particular models. The quality of the estimators is assessed through a simulation study.

*Keywords:* Degradation modelling, Imperfect maintenance, Observation schemes, Statistical inference, Wiener process,  $ARD_1$  model.

## 1 Introduction

Industrial or technological devices can be subject to degradation during their lifetime because of the intrinsic wear, environmental factors or operating conditions. To reduce degradation and extend their lifetime, maintenance actions are regularly performed. Since maintenances generally do not renew the system, they are said to be imperfect. The devices

reliability depends heavily on the kinetics of the degradation process and the effects of maintenance actions. Therefore, to assess reliability and make efficient maintenance decisions, it is necessary to have accurate models of the degradation and maintenance processes.

There is a huge literature on degradation modelling. Review papers have been proposed by Ye and Xie [31], Shahraki et al [24] and Kang et al [9]. The evolution of the degradation level over time is often modeled by stochastic processes such as Gamma processes [1, 12, 26], Wiener processes [3, 29, 32, 34], inverse Gaussian processes [30] or variants such as Gamma-Wiener [15], trend-Gamma [28], variance-Gamma [22] or transformed Gamma processes [4]. When the degrading systems are subjected to maintenance actions, the basic assumptions are that maintenance has no effect on the degradation level (minimal maintenance, or As-Bad-As-Old situation) or that maintenance renews the system (perfect maintenance, or As-Good-As-New situation).

Recently, the inclusion of imperfect maintenance effects in degradation models has attracted a lot of attention. In Zhang et al [33], Zhao et al [36] and Ma et al [17], the impact of maintenance is expressed by a random improvement factor affecting the trend of the degradation process. Most of imperfect maintenance models assume that maintenance effect is a reduction of the degradation level. This kind of model has been studied by Mercier and Castro [19], Huynh and Grall [6], Kamrafar et al [8], Corset et al [2] when the underlying degradation process is a Gamma process, while Li et al [14] and Leroy et al [13] considered an underlying Wiener process. Other features in these models such as multivariate degradation (Wang et al [27]), multi-phase maintenance activities (Ma et al [17]) or maintenance optimization (Omsi and Grall [21]) have been studied. Finally, Mercier and Castro [19] and Wang et al [27] also considered the possibility that maintenance effect is a reduction of the system age. An important point is that, in all these studies, maintenance effect is on the whole degradation process.

However, in practice, maintenance actions are often performed only on a part of the system, and therefore, they have an effect only on a part of the degradation process. For instance, within the framework of railway infrastructure management and track geometry maintenance, longitudinal leveling by tamping (ie compacting the ballast to correct the track geometry) can be considered as a partial repair : they indeed restore in part the track geometry, but since the ballast itself is not replaced and rail fastenings are not tightened, this operation does not results in a as-good-as new state for the track level [18]. Another example is a vehicle's braking system, where maintenance actions are generally performed only on the break pads and not on the whole disc. More generally, when a system is

made of several components, maintenance often consists in replacing only a smaller group of components [25, 11]. This is called partial replacement. Maintenance is perfect for the replaced components, but is imperfect for the whole system. This situation has often been evoked to justify the interest of imperfect maintenance models, but these models are generally not built using directly the partial replacement assumption. An exception is the partial repair model based on superimposed renewal processes proposed in [16]. Therefore, to model degradation, it seems relevant to consider that maintenance can affect only a part of the degradation process. In this paper a degradation model with imperfect maintenance of that kind is proposed.

The model assumes that the degradation process has two parts  $X^U$  and  $X^M$ , which are dependent Wiener processes with drift. Between two successive maintenance actions, the degradation process is the sum of  $X^U$  and  $X^M$ . Only  $X^M$  is affected by maintenance actions and maintenance effect is of the  $ARD_1$ -type (Arithmetic Reduction of Degradation), introduced by Mercier and Castro [19]. It means that maintenance reduces the degradation level by a quantity which is proportional to the amount of degradation of  $X^M$  accumulated since the previous maintenance action. Two particular cases of the model are considered: the perturbed  $ARD_1$  and partial replacement models.

The principle of building a degradation process from underlying degradation processes has also been used in a few other papers, but with a different point of view than ours. In Mercier et al [20], the trivariate reduction method is used to build a bivariate gamma process in order to provide a model for the deterioration of railway tracks. In Salem et al [22], a variance gamma process is used to provide a model for the leakage rate of centrifugal pump. This process can be seen as the difference of two gamma processes.

The model parameters have to be estimated from field data. For this purpose, the system is regularly inspected in order to measure the global degradation level. The estimation depends on the way data are collected, i.e. on the observation scheme. Several observation schemes have been considered in the papers on degradation modelling with imperfect maintenance. In Zhao et al [35], degradation levels are measured just before and just after every maintenance action. In Zhang et al [33] and Giorgio and Pulcini [5] degradation levels are only observed between maintenance actions. In Salles et al [23] and Corset et al [2], degradation levels are only observed just before the maintenance actions. In Salem et al [22], degradation levels are regularly measured between and just before maintenance actions. In Leroy et al [13], four observation schemes are considered, including all the cases presented in previous papers.

In the present paper, two observation schemes are considered. In the so-called complete scheme, the degradation levels are measured both between maintenance actions and at maintenance times (just before and just after). In the so-called general scheme, the degradation levels are measured only between maintenance actions. At first sight, the complete observation scheme seems the most natural and reasonable. However, it may not be possible to implement this ideal scheme in practice, for various technological and engineering reasons. This is for example the case if the measurement devices and technical crew for inspection are not the same as the technical devices and crew for maintenance, such as for railway assets monitoring and maintenance, where inspections and maintenance actions cannot be performed at the same time because of an incompressible maintenance intervention delay [18]. In such cases, it is necessary to model the observation procedures in a more realistic way (accounting for delays between the inspections and the maintenance actions, or non-periodic inspections). This can be done in the general observation scheme. Besides, a measurement at a maintenance time is a limit case of a measurement between maintenance times, when the measurement (inspection) time tends to the maintenance time. Therefore, all the possible observations schemes (as the 4 schemes considered in [13]) are limit cases of the general scheme. That is why it is called general. For both schemes, the parameters are estimated by the maximum likelihood method. In the full model, an identifiability problem appears, so the estimation is made only for the perturbed  $ARD_1$  and partial replacement models.

Finally, a simulation study is carried out in order to assess the quality of the estimators. The impact of different features of the model and data (values of the parameters, number of observations, number of maintenances) is analyzed. Through a study of the impact on observation locations, it is also possible to compare the complete and general observation schemes.

The paper is organized as follows. The notations used throughout the paper are given in Section 2. The new degradation model is presented in Section 3. The statistical inference is studied in Section 4 for the perturbed  $ARD_1$  and partial replacement models, under both observation schemes. The results of the simulation study are presented in Section 5. Finally, concluding comments are given and some prospects are raised in Section 6.

## 2 Notations

- $\tau$ : end of observation time.

- $k$  : number of maintenance actions.
- $\tau_j$  :  $j^{\text{th}}$  maintenance time.  $\tau_0 = 0$ ,  $\tau_{k+1} = \tau$ .
- $n_j$  : number of observations of degradation levels on  $(\tau_{j-1}, \tau_j)$ .
- $n$  : total number of observations on  $[0, \tau]$ .
- $t_{j,i}$  : time of the  $i^{\text{th}}$  observation of the degradation level in  $[\tau_{j-1}, \tau_j]$ .  $t_{j,n_j+1} = t_{j+1,0} = \tau_j$ .
- $\Delta t_{j,i} = t_{j,i} - t_{j,i-1}$ : time interval between the  $(i-1)^{\text{th}}$  and  $i^{\text{th}}$  observations in  $[\tau_{j-1}, \tau_j]$ .
- $X^U = \{X^U(t)\}_{t \geq 0}$  : part of the underlying degradation process not affected by maintenance actions ( $U$  stands for Unmaintained).
- $X^M = \{X^M(t)\}_{t \geq 0}$  : part of the underlying degradation process affected by maintenance actions ( $M$  stands for Maintained).
- $X^S = \{X^S(t)\}_{t \geq 0}$  : global underlying degradation process of the unmaintained system,  $X^S = X^U + X^M$ .
- $\Delta X_j^\ell = X^\ell(\tau_j) - X^\ell(\tau_{j-1})$  : increment of the degradation process  $\ell$  ( $\ell \in \{S, M\}$ ) between the  $(j-1)^{\text{th}}$  and the  $j^{\text{th}}$  maintenance action.
- $Y = \{Y(t)\}_{t \geq 0}$  : global degradation process of the maintained system.
- $Y(\tau_j^-) = Y(t_{j,n_j+1})$  : degradation level just before the  $j^{\text{th}}$  maintenance action.
- $Y(\tau_j^+) = Y(t_{j+1,0})$  : degradation level just after the  $j^{\text{th}}$  maintenance action.
- $\Delta Y_{j,i}^c = Y(t_{j,i}) - Y(t_{j,i-1})$  :  $i^{\text{th}}$  degradation increment in  $[\tau_{j-1}, \tau_j]$ .
- $\Delta Y_j^c = Y(\tau_j^-) - Y(\tau_{j-1}^+) = \Delta X_j^S$  : observed degradation increment on  $[\tau_{j-1}, \tau_j]$  under the complete observation scheme.
- $\Delta Y_j^g = Y(t_{j,n_j}) - Y(t_{j,1_{j>1}})$  : observed degradation increment on  $(\tau_{j-1}, \tau_j)$  under the general observation scheme. As the first degradation increment  $\Delta y_{1,1}^c$  is always observed,  $\Delta Y_1^g = Y(t_{1,n_1})$  and  $\forall j \in \{2, \dots, k+1\}$ ,  $\Delta Y_j^g = Y(t_{j,n_j}) - Y(t_{j,1})$ .
- $\Delta \tau_j = \tau_j - \tau_{j-1}$  : elapsed time between the  $(j-1)^{\text{th}}$  and the  $j^{\text{th}}$  maintenance action.

- $\Delta t_j^g$  : elapsed time between the first and last observation on  $(\tau_{j-1}, \tau_j)$  under the general observation scheme. For  $j \geq 2$ ,  $\Delta t_j^g = t_{j,n_j} - t_{j,1}$ . For  $j = 1$ ,  $\Delta t_1^g = t_{1,n_1}$ .
- $Z_j^c = Y(\tau_j^+) - Y(\tau_j^-)$  : observed degradation jump at the  $j^{\text{th}}$  maintenance time for the complete observation scheme.
- $Z_j^g = \Delta Y_{j+1,1}^c + Z_j^c + \Delta Y_{j,n_j+1}^c$  : observed degradation jump around the  $j^{\text{th}}$  maintenance time for the general observation scheme.
- $\mu_U, \mu_M, \mu_S$  : drift parameters of the underlying degradation processes.
- $\sigma_U^2, \sigma_M^2, \sigma_S^2$  : variance parameters of the underlying degradation processes.
- $c_{UM}, c_{SM}$  : covariance parameters of the underlying degradation processes.
- $r_{UM}, r_{SM}$  : coefficient of correlation parameters of the underlying degradation processes.
- $\rho$  : maintenance efficiency parameter.
- $\Theta$  : set of all parameters in the second parametrization,  $\Theta = (\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, \rho, r_{SM})$ 
  - $\Theta_1 = (\mu, \sigma_S^2, \sigma_M^2, \rho, r_{SM})$  : set of parameters for the perturbed  $ARD_1$  model.
  - $\Theta_2 = (\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, r_{SM})$  : set of parameters for the partial replacement model.
- $f_X$  : density of  $X$ .
- $f_{X|Y}$  : conditional density of  $X$  given  $Y$ .
- $\mathcal{O}_t^h$  : set of observed data before time  $t$  for scheme  $h$ .  $h \in \{c, g\}$  where  $c$  stands for “complete” and  $g$  stands for “general”.
- The random quantities are denoted by uppercase letters and their realizations by lowercase letters. For instance,  $\Delta y_{j,i}^c$  is the observed value of  $\Delta Y_{j,i}^c$ .

A part of the notations presented in this section is illustrated in [Figure 1](#). In this figure, a trajectory of the degradation process  $Y$  is presented according to the two observation schemes used in this paper, which will be described in [Section 4](#). The black dots depict the observed degradation levels. The left figure represents the complete observation scheme

where degradation levels are observed just before and just after every maintenance action. The general scheme is represented in the right figure, where degradation levels are only observed between maintenance actions. This observation scheme is called general because the complete scheme is a limit case of this scheme, obtained when  $t_{j,n_j}$  and  $t_{j+1,1}$  get closer to  $\tau_j$ .

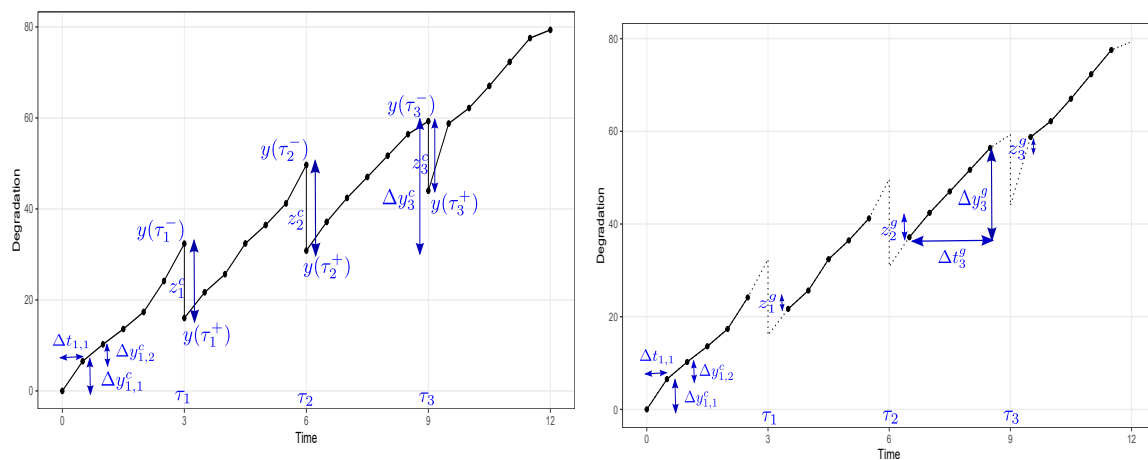


Figure 1: Detailed notations for respectively the complete (left) and general (right) observation schemes

### 3 A degradation model with partial maintenance effects

This section presents the proposed degradation model, for which maintenance affects only a part of the degradation process. The model is described in [Section 3.1](#). An alternative parameterization is proposed in [Section 3.2](#). Some particular cases are listed in [Section 3.3](#) and an identifiability issue is raised in [Section 3.4](#).

#### 3.1 Model description

The assumptions of the proposed model are given below.

- The degradation process is made of two parts  $X^U = \{X^U(t)\}_{t \geq 0}$  and  $X^M = \{X^M(t)\}_{t \geq 0}$ .



- Between two maintenances, the degradation process evolves as the sum of the underlying degradation processes  $X^U$  and  $X^M$ .
- Maintenance only affects  $X^M$ .
- Maintenance effect is of the  $ARD_1$ -type [19]: at maintenance times, the degradation level is reduced of a quantity which is proportional to the amount of degradation of  $X^M$  accumulated since the last maintenance action.
- $X^U$  and  $X^M$  are dependent Wiener processes with drift, or  $(X^U, X^M)^T$  is a bivariate Wiener process with drift. Therefore, their increments are independent and  $(X^U(t), X^M(t))^T$  is a Gaussian vector:

$$\begin{pmatrix} X^U(t) \\ X^M(t) \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mu_U \\ \mu_M \end{pmatrix} t, \Sigma_{UM} t \right) \quad (1)$$

where  $\Sigma_{UM} = \begin{pmatrix} \sigma_U^2 & c_{UM} \\ c_{UM} & \sigma_M^2 \end{pmatrix}$ .

$\mu_U, \mu_M$  are the drift parameters,  $\sigma_U^2, \sigma_M^2$  are the variance parameters, and  $c_{UM}$  is a covariance parameter. Note that the coefficient of correlation between  $X_U(t)$  and  $X_M(t)$  is

$$\text{Corr}(X_U(t), X_M(t)) = \frac{c_{UM}t}{\sqrt{\sigma_U^2 t \sigma_M^2 t}} = \frac{c_{UM}}{\sigma_U \sigma_M} = r_{UM} \in [-1, 1].$$

Therefore, the coefficient of correlation does not depend on time.

The system is maintained at times  $(\tau_1, \tau_2, \dots)$ . Let  $Y(t)$  be the degradation level at time  $t$  of a maintained system, with  $Y(0) = 0$ . Let  $Y(\tau_j^-)$  be the degradation level just before the  $j^{\text{th}}$  maintenance at time  $\tau_j$  and  $Y(\tau_j^+)$  be the degradation level just after  $\tau_j$ .

Before the first maintenance action, the degradation level  $Y$  of the system evolves as the sum of  $X^U$  and  $X^M$ .

$$\forall t \in [0, \tau_1), Y(t) = X^U(t) + X^M(t).$$

Just before the first maintenance,  $Y(\tau_1^-) = X^U(\tau_1) + X^M(\tau_1)$ .

At time  $\tau_1$ , an  $ARD_1$ -type maintenance is performed that only affects  $X^M$ . The maintenance effect is to reduce the degradation level of a quantity proportional to the amount of degradation of  $X^M$  accumulated since the origin. Let  $\rho$  be the coefficient of proportionality, which expresses the efficiency of maintenance.

$$Y(\tau_1^+) = Y(\tau_1^-) - \rho [X^M(\tau_1) - X^M(0)] = X^U(\tau_1) + (1 - \rho)X^M(\tau_1)$$

Between  $\tau_1$  and  $\tau_2$ , the system is degrading as a non-maintained system.

$$\begin{aligned} \forall t \in [\tau_1, \tau_2), \quad Y(t) &= Y(\tau_1^+) + X^U(t) + X^M(t) - X^U(\tau_1) - X^M(\tau_1) \\ &= X^U(t) + X^M(t) - \rho X^M(\tau_1) \end{aligned}$$

At  $\tau_2$ , an  $ARD_1$ -type maintenance is performed.

$$Y(\tau_2^+) = Y(\tau_2^-) - \rho [X^M(\tau_2) - X^M(\tau_1)] = X^U(\tau_2) + (1 - \rho)X^M(\tau_2)$$

By recurrence, it is easy to show that the degradation process of the maintained system is defined as follows.

$$\forall j \geq 1, \quad Y(\tau_j^-) = X^U(\tau_j) + X^M(\tau_j) - \rho X^M(\tau_{j-1}) \quad (2)$$

$$Y(\tau_j^+) = X^U(\tau_j) + (1 - \rho)X^M(\tau_j) \quad (3)$$

$$\forall t \in [\tau_{j-1}, \tau_j), \quad Y(t) = X^U(t) + X^M(t) - \rho X^M(\tau_{j-1}) \quad (4)$$

The effect of a maintenance action is represented by a jump of the degradation level. Let  $Z_j^c$  be the observed jump at the  $j^{\text{th}}$  maintenance time.

$$Z_j^c = Y(\tau_j^+) - Y(\tau_j^-) = -\rho [X^M(\tau_j) - X^M(\tau_{j-1})] \quad (5)$$

From the properties of the Wiener process  $X^M$ , the  $Z_j^c$  are independent random variables and their respective distributions are,  $\forall j \geq 1$ :

$$Z_j^c \sim \mathcal{N}(-\rho\mu_M\Delta\tau_j, \rho^2\sigma_M^2\Delta\tau_j). \quad (6)$$

where  $\Delta\tau_j = \tau_j - \tau_{j-1}$ . Note that the jumps are not independent of the previous degradation increments.

The distribution of the degradation increments inside each inter-maintenance interval

can also be easily derived from (4):

$$\forall j \geq 1, \forall \tau_{j-1} < s < t < \tau_j, Y(t) - Y(s) = X^U(t) - X^U(s) + X^M(t) - X^M(s).$$

Therefore, the distribution of the increments is given by,  $\forall j \geq 1, \forall \tau_{j-1} < s < t < \tau_j$ :

$$Y(t) - Y(s) \sim \mathcal{N}((\mu_U + \mu_M)(t - s), (\sigma_U^2 + \sigma_M^2 + 2c_{UM})(t - s)). \quad (7)$$

The independence of increments of the bivariate process  $(X^U, X^M)$  implies the independence of increments of  $Y$ .

### 3.2 An alternative parameterization

From a statistical perspective, the model parameters will have to be estimated from field data. The observed data are degradation increments and degradation jumps, whose distributions are given by (6) and (7). Therefore, in order to simplify the writings in statistical inference, it appears relevant to introduce an alternative parameterization of the model. Let

$$X^S(t) = X^U(t) + X^M(t) \quad (8)$$

$X^S$  is the global underlying degradation process of the unmaintained system and  $X^M$  is the part of this process which is affected by maintenance actions.  $(X^S(t), X^M(t))^T$  is a linear transform of the Gaussian vector  $(X^U(t), X^M(t))^T$ :

$$\begin{pmatrix} X^S(t) \\ X^M(t) \end{pmatrix} = A \begin{pmatrix} X^U(t) \\ X^M(t) \end{pmatrix}, \text{ where } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

So  $(X^S(t), X^M(t))^T$  is still a Gaussian vector. [From usual results on random vectors](#), its mean is

$$AE \begin{pmatrix} X^U(t) \\ X^M(t) \end{pmatrix} = \begin{pmatrix} \mu_U + \mu_M \\ \mu_M \end{pmatrix} t$$

and its covariance matrix is

$$A\Sigma_{UM}tA^T = \begin{pmatrix} \sigma_U^2 + \sigma_M^2 + 2c_{UM} & c_{UM} + \sigma_M^2 \\ c_{UM} + \sigma_M^2 & \sigma_M^2 \end{pmatrix} t.$$

Therefore, let us introduce new parameters  $\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, c_{SM}, r_{SM}$ , such that  $(X^S(t), X^M(t))^T$  is a Gaussian vector with mean  $(\mu_S, \mu_M)^T t$  and covariance matrix  $\Sigma_{SM}t$ , where  $\Sigma_{SM} = \begin{pmatrix} \sigma_S^2 & c_{SM} \\ c_{SM} & \sigma_M^2 \end{pmatrix}$  and  $r_{SM} = c_{SM}/(\sigma_S\sigma_M)$ . The links between the first parameterization  $(U, M)$  and the second parameterization  $(S, M)$  are given in Table 1.

Table 1: Links between the two model parameterizations

	$(X^U, X^M)$	$(X^S, X^M)$
Drift	$\mu_U = \mu_S - \mu_M$	$\mu_S = \mu_U + \mu_M$
Variance	$\sigma_U^2 = \sigma_S^2 + \sigma_M^2 - 2c_{SM}$	$\sigma_S^2 = \sigma_U^2 + \sigma_M^2 + 2c_{UM}$
Covariance	$c_{UM} = c_{SM} - \sigma_M^2$	$c_{SM} = c_{UM} + \sigma_M^2$
Correlation	$r_{UM} = \frac{r_{SM}\sigma_S - \sigma_M}{\sqrt{\sigma_S^2 + \sigma_M^2 - 2r_{SM}\sigma_S\sigma_M}}$	$r_{SM} = \frac{r_{UM}\sigma_U + \sigma_M}{\sqrt{\sigma_U^2 + \sigma_M^2 + 2r_{UM}\sigma_U\sigma_M}}$

To simplify writings, let  $\forall \ell \in \{S, M\}$ ,  $\Delta X_j^\ell = X^\ell(\tau_j) - X^\ell(\tau_{j-1})$  be the increment of the degradation process  $\ell$  ( $\ell \in \{S, M\}$ ) between the  $j-1^{th}$  and the  $j^{th}$  maintenance action.

In the new parameterization, the degradation level and jumps are defined as follows.

$$\forall j \geq 1, \forall t \in [\tau_{j-1}, \tau_j), Y(t) = X^S(t) - \rho X^M(\tau_{j-1}). \quad (9)$$

$$\forall j \geq 1, Z_j^c = -\rho [X^M(\tau_j) - X^M(\tau_{j-1})] = -\rho \Delta X_j^M. \quad (10)$$

The distributions of the degradation increments and jumps are:

$$\forall j \geq 1, \forall \tau_{j-1} \leq s < t \leq \tau_j, Y(t) - Y(s) \sim \mathcal{N}(\mu_S(t-s), \sigma_S^2(t-s)). \quad (11)$$

$$\forall j \geq 1, Z_j^c \sim \mathcal{N}(-\rho\mu_M\Delta\tau_j, \rho^2\sigma_M^2\Delta\tau_j). \quad (12)$$

From (11) and (12), it can be seen that parameters  $(\mu_S, \sigma_S^2)$  are closely linked to degradation increments between maintenance actions while parameters  $(\mu_M, \sigma_M^2, \rho)$  are closely linked to degradation jumps, so to maintenance efficiency. This remark will be useful in

Section 4 for the estimation of these parameters.

Finally, the distribution of the degradation level at time  $t$  can be easily computed from (9):

$$Y(t) \sim \mathcal{N}(\mu_S t - \rho \mu_M \tau_{j-1}, \sigma_S^2 t + \rho^2 \sigma_M^2 \tau_{j-1} - 2\rho r_{SM} \sigma_S \sigma_M \tau_{j-1})$$

The same degradation trajectory under both parameterizations is presented in Figure 2. The observed degradation process  $Y$  is represented with the underlying processes  $X^U$  and  $X^M$  on the left figure and with  $X^S$  and  $X^M$  on the right figure. This trajectory has been simulated using the following parameters :  $\mu_U = 4$ ,  $\mu_M = 2$ ,  $\sigma_U^2 = 10$ ,  $\sigma_M^2 = 7$ ,  $r_{UM} = 0.8$ ,  $\rho = 0.7$ , which is equivalent to  $\mu_S = 6$ ,  $\mu_M = 2$ ,  $\sigma_S^2 = 30.39$ ,  $\sigma_M^2 = 7$ ,  $r_{SM} = 0.94$ ,  $\rho = 0.7$ .

In this example, four maintenance actions are periodically performed at times  $\{3, 6, 9, 12\}$ . The underlying processes are simulated using consecutive degradation increments on a time increment equals to 0.001. In the first parameterization, between maintenance actions, the degradation process  $Y$  (in black) evolves as the sum of  $X^U$  (in green) and  $X^M$  (in blue). In the second parameterization,  $Y$  evolves as  $X^S$  (in red) between maintenances. The correlation between the underlying degradation processes clearly appears in this figure. The maintenance effect is also clearly visible for the first three maintenances. For the fourth one at  $\tau_4 = 12$ , the maintenance effect is not visible because the amount of degradation accumulated for process  $X^M$  between  $\tau_3 = 9$  and  $\tau_4 = 12$  is very small.

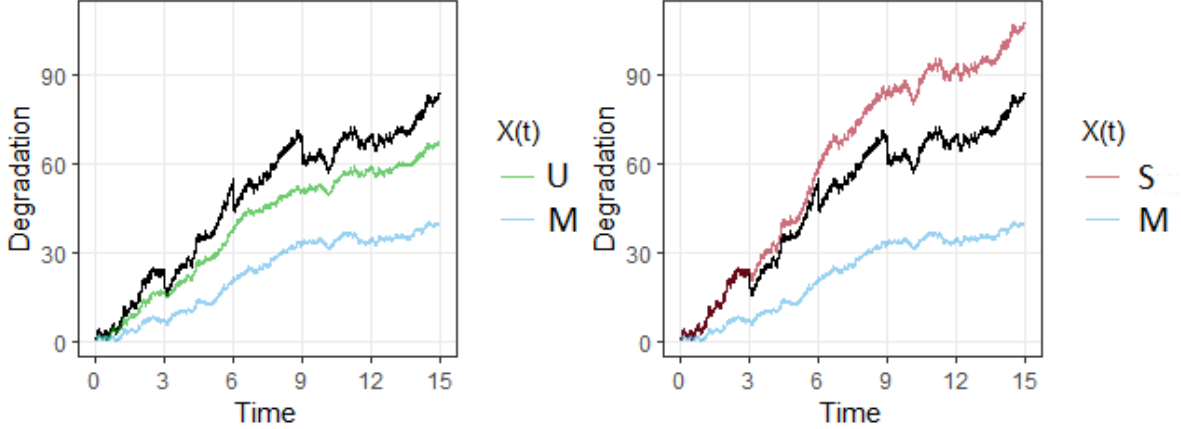


Figure 2: Degradation trajectories of  $Y$  (in black) with  $(X^U, X^M)$  (in green and blue) in the left figure and  $(X^S, X^M)$  (in red and blue) in the right figure.

### 3.3 Specific cases

The model defined by equation (9) to (12) will be called hereafter the full model. Some interesting specific cases of this full model can be considered.

- **ABAO.** When  $\rho = 0$ ,  $Y(t) = X^S(t) = X^U(t) + X^M(t)$ . Maintenance has no effect on the system, the system after maintenance is As Bad As Old (ABAO).
- **Usual  $ARD_1$  model.** The new model is equivalent to the usual Wiener-based  $ARD_1$  model [19, 13] when the whole degradation process is affected by maintenance actions. This is the case for  $X^U(t) = 0, \forall t \geq 0$ , i.e.  $\mu_U = \sigma_U^2 = c_{UM} = 0$ . In this case,  $X^S = X^M$  so  $\mu_S = \mu_M$ ,  $\sigma_S^2 = \sigma_M^2 = c_{SM}$  and  $r_{SM} = 1$ . In this situation,  $\rho = 1$  entails a perfect maintenance action.
- **Perturbed  $ARD_1$  model.** When  $\mu_S = \mu_M$  or  $\mu_U = 0$ ,  $X^U$ , which can depend on the degradation process, is a white noise. So the degradation process  $Y$  can be considered as a usual Wiener-based  $ARD_1$  model perturbed by this white noise.
- **Partial replacement.** In the usual  $ARD_1$  model,  $\rho = 1$  means that the system is renewed, or As Good As New (AGAN), after each maintenance. Here,  $\rho = 1$  means that maintenance deletes all the amount of degradation of  $X^M$  accumulated since the last maintenance. Globally, maintenance is not perfect ( $Y(\tau_j^+) = X^U(\tau_j) \neq 0$ ) but can be considered as optimal. This corresponds to the situation where maintenance consists in replacing a part of the system components, but not the whole system. This situation is particularly interesting in practice and will be studied in Section 4.
- When  $X^U$  and  $X^M$  are independent,  $r_{UM} = 0$  so  $r_{SM} = \frac{\sigma_M}{\sqrt{\sigma_U^2 + \sigma_M^2}}$ . Therefore  $X^S$  and  $X^M$  are dependent and positively correlated.
- When  $X^S$  and  $X^M$  are independent,  $r_{SM} = 0$  so  $r_{UM} = -\frac{\sigma_M}{\sqrt{\sigma_S^2 + \sigma_M^2}}$ . Therefore  $X^U$  and  $X^M$  are dependent and negatively correlated.
- $r_{UM} = 1 \Rightarrow r_{SM} = 1$ . A perfect positive correlation under the  $(U, M)$  parameterization entails a perfect positive correlation under the  $(S, M)$  parameterization.
- $r_{SM} = 1 \Rightarrow r_{UM} = \frac{\sigma_S - \sigma_M}{\sqrt{(\sigma_S - \sigma_M)^2}} \Rightarrow r_{UM} = \{-1, 1\}$ . A perfect positive correlation under the  $(S, M)$  parameterization entails a perfect correlation under the  $(U, M)$  parameterization, which can be either positive or negative.

### 3.4 Identifiability

The definition of the **full** model in (9) to (12) shows that two different sets of parameters  $(\mu_M, \sigma_M^2, \rho)$  and  $(\tilde{\mu}_M, \tilde{\sigma}_M^2, \tilde{\rho})$  such that  $\rho\mu_M = \tilde{\rho}\tilde{\mu}_M$  and  $\rho\sigma_M = \tilde{\rho}\tilde{\sigma}_2$  define the same model.

More precisely,  $X^S(t)$  and  $X^M(t)$  can be written  $X^S(t) = \mu_S t + \sigma_S B^S(t)$  and  $X^M(t) = \mu_M t + \sigma_M B^M(t)$ , where  $B^S$  and  $B^M$  are dependent standard Brownian motions. Let  $\tilde{X}^M(t) = \tilde{\mu}_M t + \tilde{\sigma}_2 B^M(t)$ , where  $\tilde{\mu}_M = \rho\mu_M/\tilde{\rho}$  and  $\tilde{\sigma}_2 = \rho\sigma_M/\tilde{\rho}$ . We have  $\tilde{X}^M(t) = \rho X^M(t)/\tilde{\rho}$  and  $\tilde{\rho}\tilde{X}^M(t) = \rho X^M(t)$ . Therefore, the degradation process  $\tilde{Y}(t) = X^S(t) - \tilde{\rho}\tilde{X}^M(t)$  is exactly the same as  $Y(t) = X^S(t) - \rho X^M(t)$ . Moreover, it is easy to show that  $Cov(X^S(t), \tilde{X}^M(t)) = \rho c_{SM} t / \tilde{\rho} = \tilde{c}_{SM} t$  and  $\tilde{r}_{SM} = \tilde{c}_{SM} / \sigma_S \tilde{\sigma}_M = r_{SM}$ .

So finally,  $\forall \tilde{\rho} \in (0, 1]$  the sets of parameters  $(\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, r_{SM}, \rho)$  and  $(\mu_S, \rho\mu_M/\tilde{\rho}, \sigma_S^2, \rho^2\sigma_M^2/\tilde{\rho}^2, r_{SM}, \tilde{\rho})$  define the same model. Although it is less visible, the same problem also appears using the  $(U, M)$  parameterization.

Hence, the **full** model is not identifiable. Therefore, in Section 4, constraints will be imposed in order to estimate the parameters of identifiable models: the perturbed  $ARD_1$  and partial replacement models.

In order to study statistical inference in the following section, the model's likelihood is first derived for the **full** model.

## 4 Statistical Inference

Let  $\Theta = (\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, r_{SM}, \rho)$  be the set of model parameters under the second parameterization. This section studies the statistical estimation of these parameters from field data. The system is assumed to be regularly inspected and degradation levels  $Y_{j,i}$  are measured. Two observation schemes (i.e. two ways of collecting data) are considered, the complete scheme and the general scheme. For both schemes, the parameters are estimated by the maximum likelihood method. Each scheme leads to a different writing of the likelihood and then to different estimations of the parameters. The observation schemes are described in Section 4.1. The general expressions of the likelihoods are determined in Section 4.2. Due to the identifiability issue raised in Section 3.4, the estimation is made in Section 4.3 only for the perturbed  $ARD_1$  and partial replacement models under both observation schemes.

## 4.1 Observation schemes

The considered system is observed in a time interval  $[0, \tau]$ . In this interval,  $k$  maintenances are performed at times  $\tau_1, \dots, \tau_k$ . Let us denote  $\tau_0 = 0$  and  $\tau_{k+1} = \tau$ .

$\forall j \in \{1, \dots, k+1\}$ , let  $n_j$  be the number of observations of the degradation level in  $(\tau_{j-1}, \tau_j)$ . The degradation levels are observed at times  $t_{j,i}$ , where  $t_{j,i}$  is the time of the  $i^{\text{th}}$  observation in  $(\tau_{j-1}, \tau_j) \forall i \in \{1, \dots, n_j\}$ . If observations are made at the maintenance times (just before and/or just after), the corresponding observation times are denoted  $t_{j,n_j+1} = \tau_j = t_{j+1,0}$ . Let  $\Delta t_{j,i} = t_{j,i} - t_{j,i-1}$  be the time elapsed between two successive observations.

### 4.1.1 Complete observation scheme

In the complete scheme, the degradation levels are measured both between maintenance actions and at maintenance times (just before and just after). In [Figure 3](#), a degradation trajectory is simulated for  $\mu_S = 10$ ,  $\mu_M = 5$ ,  $\sigma_S^2 = 30$ ,  $\sigma_M^2 = 20$ ,  $r_{SM} = 0.7$ ,  $\rho = 0.8$ . In this example, degradation levels are periodically observed and represented by black dots. Maintenance actions are periodically performed at times  $\{3, 6, 9, 12\}$ . The two underlying processes  $X^S$  and  $X^M$  are respectively depicted by the red and blue lines. The observed degradation increments of the system are represented by the black line.

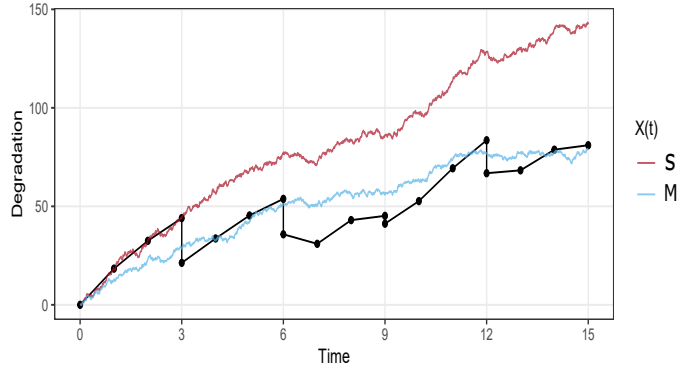


Figure 3: Observations of the degradation levels of the maintained system (in black) under the complete observation scheme

Due to the properties of the Wiener process, it is more convenient to consider that the degradation increments (instead of degradation levels) are observed. Therefore, in the complete scheme, the observations are made of:



- The degradation increments  $\Delta Y_{j,i}^c = Y(t_{j,i}) - Y(t_{j,i-1}) = X^S(t_{j,i}) - X^S(t_{j,i-1})$ , for  $1 \leq j \leq k+1$  and  $1 \leq i \leq n_j+1$ . They are independent and their respective distributions are  $\Delta Y_{j,i}^c \sim \mathcal{N}(\mu_S \Delta t_{j,i}, \sigma_S^2 \Delta t_{j,i})$ .
- The degradation jumps  $Z_j^c = Y(\tau_j^+) - Y(\tau_j^-) = -\rho [X^M(\tau_j) - X^M(\tau_{j-1})] = -\rho \Delta X_j^M$ , for  $1 \leq j \leq k$ . They are not independent of the observations prior to  $\tau_j$ . The set of all these observations is denoted:

$$\mathcal{O}_{\tau_j^-}^c = \{\Delta y_{1,1}^c, \dots, \Delta y_{1,n_1+1}^c, z_1^c, \Delta y_{2,1}^c, \dots, \Delta y_{j-1,n_{j-1}+1}^c, z_{j-1}^c, \Delta y_{j,1}^c, \dots, \Delta y_{j,n_j+1}^c\}$$

The likelihood function is the joint density of all observations. Therefore, the likelihood function in the complete scheme, denoted  $L_c(\Theta)$ , is given by

$$L_c(\Theta) = \prod_{j=1}^{k+1} \prod_{i=1}^{n_j+1} f_{\Delta Y_{j,i}^c}(\Delta y_{j,i}^c) \prod_{j=1}^k f_{Z_j^c | \mathcal{O}_{\tau_j^-}^c}(z_j^c) \quad (13)$$

where  $f_{Z_j^c | \mathcal{O}_{\tau_j^-}^c}$  is the conditional density of the  $j^{\text{th}}$  jump given all the observations before  $\tau_j$ .

The expression of this likelihood is developed in [Section 4.2.1](#).

In Leroy et al [13], this observation scheme has been considered for a usual  $ARD_1$  Wiener-based degradation model (first observation scheme in [13]). In this model, the value of a jump is proportional to the amount of degradation accumulated since the last maintenance action. Thus, under the complete observation scheme, the sizes of the jumps are deterministic given the previous observations. This is very unlikely in practice. Here,  $Z_j^c = -\rho \Delta X_j^M$  and the  $\Delta X_j^M$  are not observed, so the problem of the  $ARD_1$  model does not occur. Therefore, the proposed model is more realistic than the  $ARD_1$  model for the complete observation scheme.

#### 4.1.2 General observation scheme

In the general scheme, the degradation levels are measured only between maintenance actions. In [Figure 4](#), a degradation trajectory is simulated for the same parameters as in [Figure 3](#). In this example, contrary to [Figure 3](#), the observations are not periodic and the numbers of observations between two successive maintenances are not constant. As in [Figure 3](#), each black dot represents an observed degradation level. Dotted lines correspond

to the unobserved degradation increments around maintenance times. The two underlying processes  $X^S$  and  $X^M$  are respectively depicted by the red and blue lines. The observed degradation increments of the system are represented by the black line.

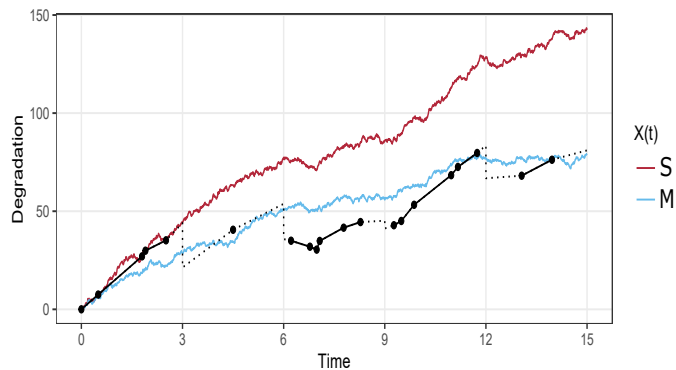


Figure 4: Observations of the degradation levels of the maintained system (in black) under the general observation scheme

In this case, neither  $Y(\tau_j^-)$  nor  $Y(\tau_j^+)$  are observed. Therefore, the true degradation jumps  $Z_j^c$  are also not observed. The observation around the  $j^{\text{th}}$  maintenance time  $\tau_j$  is the difference between the first observed degradation level after  $\tau_j$  and the last observed degradation level before  $\tau_j$ :

$$\forall j \geq 1, Z_j^g = Y(t_{j+1,1}) - Y(t_{j,n_j}) \quad (14)$$

Therefore, in the general scheme, the observations are made of:

- The degradation increments  $\Delta Y_{j,i}^c$ , for  $2 \leq j \leq k$  and  $2 \leq i \leq n_j$ , and  $1 \leq i \leq n_1$  for  $j = 1$ . Indeed, with respect to the complete scheme, the first and last degradation increments of each interval are no longer observed, except the very first one  $\Delta Y_{1,1}^c$ . As before, they are independent and their respective distributions are  $\Delta Y_{j,i}^c \sim \mathcal{N}(\mu_S \Delta t_{j,i}, \sigma_S^2 \Delta t_{j,i})$ .
- The new jumps  $Z_j^g = Y(t_{j+1,1}) - Y(t_{j,n_j})$ , for  $1 \leq j \leq k$ . They are not independent of the observations prior to  $t_{j,n_j}$ . The set of all these observations is denoted:

$$\mathcal{O}_{t_j, n_j}^g = \{\Delta y_{1,1}^c, \dots, \Delta y_{1,n_1}^c, Z_1^g, \Delta y_{2,2}^c, \dots, \Delta y_{j-1, n_{j-1}}^c, Z_{j-1}^g, \Delta y_{j,2}^c, \dots, \Delta y_{j, n_j}^c\}$$

Therefore, the likelihood function in the general scheme, denoted  $L_g(\Theta)$ , is given by

$$L_g(\Theta) = \prod_{j=1}^{k+1} \prod_{i=1+\mathbb{1}_{j>1}}^{n_j} f_{\Delta Y_{j,i}^c}(\Delta y_{j,i}^c) \prod_{j=1}^k f_{Z_j^g | \mathcal{O}_{t_j, n_j}^g}(z_j^g) \quad (15)$$

The expression of this likelihood is developed in [Section 4.2.2](#).

## 4.2 Derivation of the likelihood

### 4.2.1 Complete observation scheme

In order to write the likelihood (13), the only difficulty is to derive the conditional distributions of the jumps  $Z_j^c = -\rho \Delta X_j^M$  given  $\mathcal{O}_{\tau_j^-}^c$ .

We have  $\Delta X_j^M = X^M(\tau_j) - X^M(\tau_{j-1}) = \sum_{i=1}^{n_j+1} \Delta X_{j,i}^M$  and  $\Delta X_j^S = X^S(\tau_j) - X^S(\tau_{j-1}) = \sum_{i=1}^{n_j+1} \Delta X_{j,i}^S$ .  $\Delta X_{j,i}^M$  is independent of all observed degradation increments before  $\tau_j$ , except  $\Delta X_{j,i}^S = \Delta Y_{j,i}^c$ . So finally, the conditional distribution of  $Z_j$  given  $\mathcal{O}_{\tau_j^-}^c$  is simply the conditional distribution of  $-\rho \Delta X_j^M$  given  $\Delta X_j^S = \Delta y_j^c$ .

$(\Delta X_j^M, \Delta X_j^S)^T$  is a Gaussian vector:

$$\begin{pmatrix} \Delta X_j^M \\ \Delta X_j^S \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mu_M \\ \mu_S \end{pmatrix} \Delta \tau_j, \begin{pmatrix} \sigma_M^2 & c_{SM} \\ c_{SM} & \sigma_S^2 \end{pmatrix} \Delta \tau_j \right) \quad (16)$$

So the distribution of  $\Delta X_j^M$  given  $\Delta X_j^S$  can be derived thanks to the lemma given in [Appendix A](#), with  $A = \Delta X_j^M$ ,  $B = \Delta X_j^S$ ,  $\mu_A = \mu_M \Delta \tau_j$ ,  $\mu_B = \mu_S \Delta \tau_j$ ,  $\Sigma_A = \sigma_M^2 \Delta \tau_j$ ,  $\Sigma_B = \sigma_S^2 \Delta \tau_j$ , and  $\Sigma_{AB} = c_{SM} \Delta \tau_j$ .

Straightforward computations lead to the desired conditional distribution:

$$Z_j^c | \mathcal{O}_{\tau_j^-}^c \sim \mathcal{N} \left( -\rho \left( \mu_M \Delta \tau_j + r_{SM} \frac{\sigma_M}{\sigma_S} (\Delta y_j^c - \mu_S \Delta \tau_j) \right), \rho^2 \Delta \tau_j \sigma_M^2 (1 - r_{SM}^2) \right) \quad (17)$$

Hence, the log-likelihood for the complete observation scheme is:

$$\begin{aligned} \ln L_c(\Theta) = & -\frac{1}{2} \left( \sum_{j=1}^{k+1} \sum_{i=1}^{n_j+1} \ln(2\pi\sigma_S^2\Delta t_{j,i}) + \frac{(\Delta y_{j,i}^c - \mu_S\Delta t_{j,i})^2}{\sigma_S^2\Delta t_{j,i}} \right. \\ & \left. + \sum_{j=1}^k \ln(2\pi\rho^2\Delta\tau_j\sigma_M^2(1-r_{SM}^2)) + \frac{(z_j^c + \rho\mu_M\Delta\tau_j + \rho r_{SM}\frac{\sigma_M}{\sigma_S}(\Delta y_j^c - \mu_S\Delta\tau_j))^2}{\rho^2\Delta\tau_j\sigma_M^2(1-r_{SM}^2)} \right) \end{aligned} \quad (18)$$

#### 4.2.2 General observation scheme

In order to write the likelihood (15), the only difficulty is to derive the conditional distribution of the observed jump around the  $j^{\text{th}}$  maintenance  $Z_j^g$  given  $\mathcal{O}_{t_j, n_j}^g$ .  $\forall j \in \{1, \dots, k\}$ , we have:

$$\begin{aligned} Z_j^g &= Y(t_{j+1,1}) - Y(t_{j, n_j}) \\ &= Y(t_{j+1,1}) - Y(\tau_j^+) + Y(\tau_j^+) - Y(\tau_j^-) + Y(\tau_j^-) - Y(t_{j, n_j}) \\ &= \Delta Y_{j+1,1}^c + Z_j^c + \Delta Y_{j, n_j+1}^c \\ &= \Delta X_{j+1,1}^S - \rho \Delta X_j^M + \Delta X_{j, n_j+1}^S \\ &= \Delta X_{j+1,1}^S - \rho \Delta X_{j,1}^M \mathbf{1}_{j>1} - \rho \sum_{i=1+\mathbf{1}_{j>1}}^{n_j} \Delta X_{j,i}^M - \rho \Delta X_{j, n_j+1}^M + \Delta X_{j, n_j+1}^S \end{aligned} \quad (19)$$

In this sum:

- $\Delta X_{j+1,1}^S$  is not observed and is independent of  $\mathcal{O}_{t_j, n_j}^g$ .
- $\Delta X_{j,1}^M \mathbf{1}_{j>1}$  is not observed. It depends on  $\Delta X_{j,1}^S \mathbf{1}_{j>1}$ , which is also not observed, but involved in  $Z_j^g$ .
- $\sum_{i=1+\mathbf{1}_{j>1}}^{n_j} \Delta X_{j,i}^M$  is not observed but depends on  $\sum_{i=1+\mathbf{1}_{j>1}}^{n_j} \Delta X_{j,i}^S$ , which is observed.
- $\Delta X_{j, n_j+1}^M$  and  $\Delta X_{j, n_j+1}^S$  are not observed, are independent of  $\mathcal{O}_{t_j, n_j}^g$ , but are dependent one from each other.

Finally, the conditional distribution of  $Z_j^g$  given  $\mathcal{O}_{t_j, n_j}^g$  is the conditional distribution of

$Z_j^g$  given  $\left( \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta X_{j,i}^S, Z_{j-1}^g \right)$ . This distribution will be obtained by using again the

lemma of Appendix A, starting from the Gaussian vector  $\left( Z_j^g, \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta X_{j,i}^S, Z_{j-1}^g \right)$ .

To simplify writings, let us denote:

- $\Delta t_j^g = \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta t_{j,i}$  the elapsed time between the first and last observation on the  $j^{\text{th}}$  inter-maintenance interval. For  $j \geq 2$ ,  $\Delta t_j^g = t_{j,n_j} - t_{j,1}$ . For  $j = 1$ ,  $\Delta t_1^g = t_{1,n_1}$ .
- $\Delta Y_j^g = \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta X_{j,i}^S$  the total increment of degradation observed on the  $j^{\text{th}}$  inter-maintenance interval. The Gaussian vector of interest can now be written  $\left( Z_j^g, \Delta Y_j^g, Z_{j-1}^g \right)$ .

The expectation of the Gaussian vector is given by:

$$\begin{aligned} \mu_{Z_j^g} &= \mathbb{E}[Z_j^g] = \mathbb{E}[\Delta Y_{j+1,1}^c] + \mathbb{E}[Z_j^c] + \mathbb{E}[\Delta Y_{j,n_j+1}^c] \\ &= \mu_S(\Delta t_{j+1,1} + \Delta t_{j,n_j+1}) - \rho \mu_M \Delta \tau_j. \end{aligned} \quad (20)$$

$$\mathbb{E}[\Delta Y_j^g] = \mu_S \Delta t_j^g. \quad (21)$$

For the covariance matrix, one has to compute:

$$\begin{aligned} \sigma_{Z_j^g}^2 &= \text{Var}[Z_j^g] \\ &= \text{Var}[\Delta X_{j+1,1}^S - \rho \Delta X_{j,1}^M \mathbb{1}_{j>1} - \rho \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta X_{j,i}^M - \rho \Delta X_{j,n_j+1}^M + \Delta X_{j,n_j+1}^S] \\ &= \text{Var}[\Delta X_{j+1,1}^S] + \rho^2 \text{Var}[\Delta X_{j,1}^M \mathbb{1}_{j>1}] + \rho^2 \text{Var}\left[ \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta X_{j,i}^M \right] \\ &\quad + \rho^2 \text{Var}[\Delta X_{j,n_j+1}^M] + \text{Var}[\Delta X_{j,n_j+1}^S] - 2\rho \text{Cov}(\Delta X_{j,n_j+1}^S, \Delta X_{j,n_j+1}^M) \\ &= \sigma_S^2 \Delta t_{j+1,1} + \rho^2 \sigma_M^2 \Delta t_{j,1} \mathbb{1}_{j>1} + \rho^2 \sigma_M^2 \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta t_{j,i} \\ &\quad + \rho^2 \sigma_M^2 \Delta t_{j,n_j+1} + \sigma_S^2 \Delta t_{j,n_j+1} - 2\rho c_{SM} \Delta t_{j,n_j+1} \\ &= \sigma_S^2 (\Delta t_{j,n_j+1} + \Delta t_{j+1,1}) + \rho^2 \sigma_M^2 \Delta \tau_j - 2\rho r_{SM} \sigma_S \sigma_M \Delta t_{j,n_j+1}. \end{aligned} \quad (22)$$

$$\text{Var}[\Delta Y_j^g] = \sigma_S^2 \Delta t_j^g. \quad (23)$$

$$\text{Cov}(Z_j^g, Z_{j-1}^g) = \text{Cov}(-\rho \Delta X_{j,1}^M, \Delta X_{j,1}^S) = -\rho c_{SM} \Delta t_{j,1}. \quad (24)$$

$$\text{Cov}(Z_j^g, \Delta Y_j^g) = \text{Cov}\left(-\rho \sum_{i=1}^{n_j} \Delta X_{j,i}^M, \sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta X_{j,i}^S\right) = -\rho c_{SM} \Delta t_j^g. \quad (25)$$

$$\text{Cov}(\Delta Y_j^g, Z_{j-1}^g) = \text{Cov}\left(\sum_{i=1+\mathbb{1}_{j>1}}^{n_j} \Delta X_{j,i}^S, \Delta X_{j,1}^S\right) = 0. \quad (26)$$

Finally,  $\forall j \in \{2, \dots, k\}$ , the distribution of the Gaussian vector  $(Z_j^g, \Delta Y_j^g, Z_{j-1}^g)$  is given by:

$$\begin{pmatrix} Z_j^g \\ \Delta Y_j^g \\ Z_{j-1}^g \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{Z_j^g} \\ \mu_S \Delta t_j^g \\ \mu_{Z_{j-1}^g} \end{pmatrix}, \begin{pmatrix} \sigma_{Z_j^g}^2 & -\rho r_{SM} \sigma_S \sigma_M \Delta t_j^g & -\rho r_{SM} \sigma_S \sigma_M \Delta t_{j,1} \\ -\rho r_{SM} \sigma_S \sigma_M \Delta t_j^g & \sigma_S^2 \Delta t_j^g & 0 \\ -\rho r_{SM} \sigma_S \sigma_M \Delta t_{j,1} & 0 & \sigma_{Z_{j-1}^g}^2 \end{pmatrix}\right) \quad (27)$$

The distribution of  $Z_j^g$  given  $(\Delta Y_j^g, Z_{j-1}^g)$  can be derived thanks to the lemma given in Appendix A, with  $A = Z_j^g$ ,  $B = \begin{pmatrix} \Delta Y_j^g \\ Z_{j-1}^g \end{pmatrix}$ ,  $\mu_A = \mu_{Z_j^g}$ ,  $\mu_B = \begin{pmatrix} \mu_S \Delta t_j^g \\ \mu_{Z_{j-1}^g} \end{pmatrix}$ ,  $\Sigma_A = \sigma_{Z_j^g}^2$ ,  $\Sigma_{AB} = (-\rho r_{SM} \sigma_S \sigma_M \Delta t_j^g, -\rho r_{SM} \sigma_S \sigma_M \Delta t_{j,1})$ ,  $\Sigma_{BA} = \Sigma_{AB}^T$  and  $\Sigma_B = \begin{pmatrix} \sigma_S^2 \Delta t_j^g & 0 \\ 0 & \sigma_{Z_{j-1}^g}^2 \end{pmatrix}$ .

After computations, the desired conditional distribution is obtained:

$$\begin{aligned} Z_j^g \mid \mathcal{O}_{t_j, n_j}^g &\sim \mathcal{N}\left(\mu_{Z_j^g} - \frac{\rho r_{SM} \sigma_M}{\sigma_S} (\Delta y_j^g - \mu_S \Delta t_j^g) - \frac{\rho r_{SM} \sigma_S \sigma_M \Delta t_{j,1} \mathbb{1}_{j>1}}{\sigma_{Z_{j-1}^g}^2} (z_{j-1}^g - \mu_{Z_{j-1}^g}), \right. \\ &\quad \left. \sigma_{Z_j^g}^2 - \rho^2 r_{SM}^2 \sigma_S^2 \sigma_M^2 \left(\frac{\Delta t_j^g}{\sigma_S^2} + \frac{\Delta t_{j,1}^2 \mathbb{1}_{j>1}}{\sigma_{Z_{j-1}^g}^2}\right)\right) \end{aligned} \quad (28)$$

Finally, the log-likelihood for the general observation scheme is:

$$\begin{aligned}
\ln L_g(\Theta) = & -\frac{1}{2} \left( \sum_{j=1}^{k+1} \sum_{i=1+\mathbb{1}_{j>0}}^{n_j} \ln(2\pi\sigma_S^2\Delta t_{j,i}) + \frac{(\Delta y_{j,i}^c - \mu_S\Delta t_{j,i})^2}{\sigma_S^2\Delta t_{j,i}} \right. \\
& + \sum_{j=1}^k \ln \left( 2\pi \left( \sigma_{Z_j^g}^2 - \rho^2 r_{SM}^2 \sigma_S^2 \sigma_M^2 \left( \frac{\Delta t_j^g}{\sigma_S^2} + \frac{\Delta t_{j,1}^2 \mathbb{1}_{j>1}}{\sigma_{Z_{j-1}^g}^2} \right) \right) \right) \\
& \left. + \frac{\left( z_j^g - \mu_{Z_j^g} + \frac{\rho r_{SM} \sigma_M}{\sigma_S} (\Delta y_j^g - \mu_S \Delta t_j^g) + \frac{\rho r_{SM} \sigma_S \sigma_M \Delta t_{j,1} \mathbb{1}_{j>1}}{\sigma_{Z_{j-1}^g}^2} (z_{j-1}^g - \mu_{z_{j-1}^{(2)}}) \right)^2}{\sigma_{Z_j^g}^2 - \rho^2 r_{SM}^2 \sigma_S^2 \sigma_M^2 \left( \frac{\Delta t_j^g}{\sigma_S^2} + \frac{\Delta t_{j,1}^2 \mathbb{1}_{j>1}}{\sigma_{Z_{j-1}^g}^2} \right)} \right) \quad (29)
\end{aligned}$$

Let us recall that this second observation scheme has been called general because the complete scheme is a limit case of this scheme when  $t_{j,n_j}$  and  $t_{j+1,1}$  tend to  $\tau_j$ . In this case,  $\Delta t_{j+1,1}$  and  $\Delta t_{j,n_j+1}$  tend to 0,  $\Delta t_j^g$  tends to  $\Delta\tau_j$  and  $\Delta y_j^g$  tends to  $\Delta y_j^c$ . Under these assumptions, it is easy to check that the conditional distribution in the general case (28) converges to the conditional distribution in the complete case (17).

### 4.3 Estimation for the perturbed $ARD_1$ and partial replacement models

The identifiability issue raised in Section 3.4 leads that the parameters  $\Theta = (\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, \rho, r_{SM})$  of the full model cannot be estimated. But if some constraints are imposed, the model may become identifiable. Therefore, we will consider in the following the parameter estimation for two specific models introduced in Section 3.3, the perturbed  $ARD_1$  and partial replacement models.

- **Perturbed  $ARD_1$ .** The degradation process is an  $ARD_1$  model perturbed by a white noise. This corresponds to the case  $\mu_S = \mu_M = \mu$ . Therefore, the model parameters in this case are  $\Theta_1 = (\mu, \sigma_S^2, \sigma_M^2, \rho, r_{SM})$ . The log-likelihoods in the complete and general observation schemes are respectively  $\ln L_c^{PA}(\Theta_1) = \ln L_c(\mu, \mu, \sigma_S^2, \sigma_M^2, \rho, r_{SM})$  and  $\ln L_g^{PA}(\Theta_1) = \ln L_g(\mu, \mu, \sigma_S^2, \sigma_M^2, \rho, r_{SM})$ , which can be computed using (18) and (29).
- **Partial replacement.** The maintenance is optimal since it deletes all the amount of removable degradation accumulated since the last maintenance. This corresponds to

the case  $\rho = 1$ . Therefore, the model parameters in this case are  $\Theta_2 = (\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, r_{SM})$ . The log-likelihoods in the complete and general observation schemes are respectively  $\ln L_c^{PR}(\Theta_2) = \ln L_c(\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, 1, r_{SM})$  and  $\ln L_g^{PR}(\Theta_2) = \ln L_g(\mu_S, \mu_M, \sigma_S^2, \sigma_M^2, 1, r_{SM})$ , which can be computed using (18) and (29).

## 5 Simulation study

Deriving the theoretical properties of the estimators would require sophisticated mathematical derivations, which are out of the scope of this paper. Therefore, the quality of the estimators is assessed through a simulation study. In order to analyze the impact on the estimation quality of the parameters values, the number of maintenance actions and the number of observations, several situations are considered and compared to each other. Each situation corresponds to different levels of information and available data, so as to provide a set of convincing elements giving a good insight into the behavior and performance of the proposed estimation method in different situations representative of the reality. Through a study of the impact on observation locations, it is also possible to compare the complete and general observation schemes.

Only the Perturbed  $ARD_1$  and Partial replacement model are considered. Maintenance actions and degradation measurements are supposed to be periodically performed. A situation is defined by the values of the model parameters, the number of observations between maintenance actions ( $n_j$ ), the number of maintenance actions ( $k$ ), the total number of observations ( $n$ ), and the maintenance period. For each situation, 5000 degradation trajectories are simulated. The five model's parameters are estimated by maximizing the log-likelihoods (18) and (29) with the Nelder-Mead method. To obtain good estimation results, the initial parameter values of this algorithm have to be carefully chosen. The employed technique is described in Appendix B. The results are presented through box-plots of the empirical distribution of each estimator.

### 5.1 Quality of parameter estimation for the complete observation scheme

#### 5.1.1 Perturbed $ARD_1$ model

For the Perturbed  $ARD_1$  model, 8 situations are considered, described in Table 2. For each situation, 5000 degradation trajectories are simulated. For each trajectory, maintenances are performed every 3 time units. The boxplots of the estimators are presented in Figure



5.

Table 2: Simulation situations for the Perturbed  $ARD_1$  model

Situation	$\mu$	$\sigma_S^2$	$\sigma_M^2$	$\rho$	$r_{SM}$	$n_j$	$k$	$n$
1	5	10	7	0.5	0.7	2	4	20
2	5	10	7	0.5	0.7	0	9	20
3	5	10	7	0.5	0.7	5	2	21
4	5	10	7	0.5	0.7	2	49	200
5	5	10	7	0.5	0.1	2	4	20
6	5	10	20	0.5	0.7	2	4	20
7	5	10	7	0.2	0.7	2	4	20
8	5	10	7	0.8	0.7	2	4	20

- Situation 1 is the reference situation, with an average maintenance efficiency ( $\rho = 0.5$ ), a quite high correlation between the underlying degradation processes ( $r_{SM} = 0.7$ ),  $k = 4$  maintenance actions and  $n_j = 2$  observations of the degradation level between each maintenance.
- Situation 2 has the same parameters and the same amount of data, but with more maintenances and no observations between maintenances. Since we have more information on the jumps and less information between maintenance,  $\sigma_M^2$  and  $\rho$  are better estimated and  $\sigma_S^2$  is less well estimated.  $\mu$  and  $r_{SM}$  are also better estimated.
- Situation 3 is the dual of the previous one, with less maintenances and more observations between maintenances. Logically, all the estimations are degraded, except that of  $\sigma_S^2$ .
- Situation 4 is similar to situation 1, with much more data. As expected, the estimation quality of all estimators increases with the number of data.
- Situation 5 is similar to situation 1, except that the coefficient of correlation is much smaller ( $r_{SM} = 0.1$ ). The box-plot shows that it is difficult to estimate a so small coefficient of correlation. The estimations of other parameters are not affected except  $\rho$ , which is slightly less well estimated as in Situation 1.
- Situation 6 is similar to situation 1, except that the second variance is much larger ( $\sigma_M^2 = 20$ ). Unsurprisingly, only the estimation of  $\sigma_M^2$  is affected.

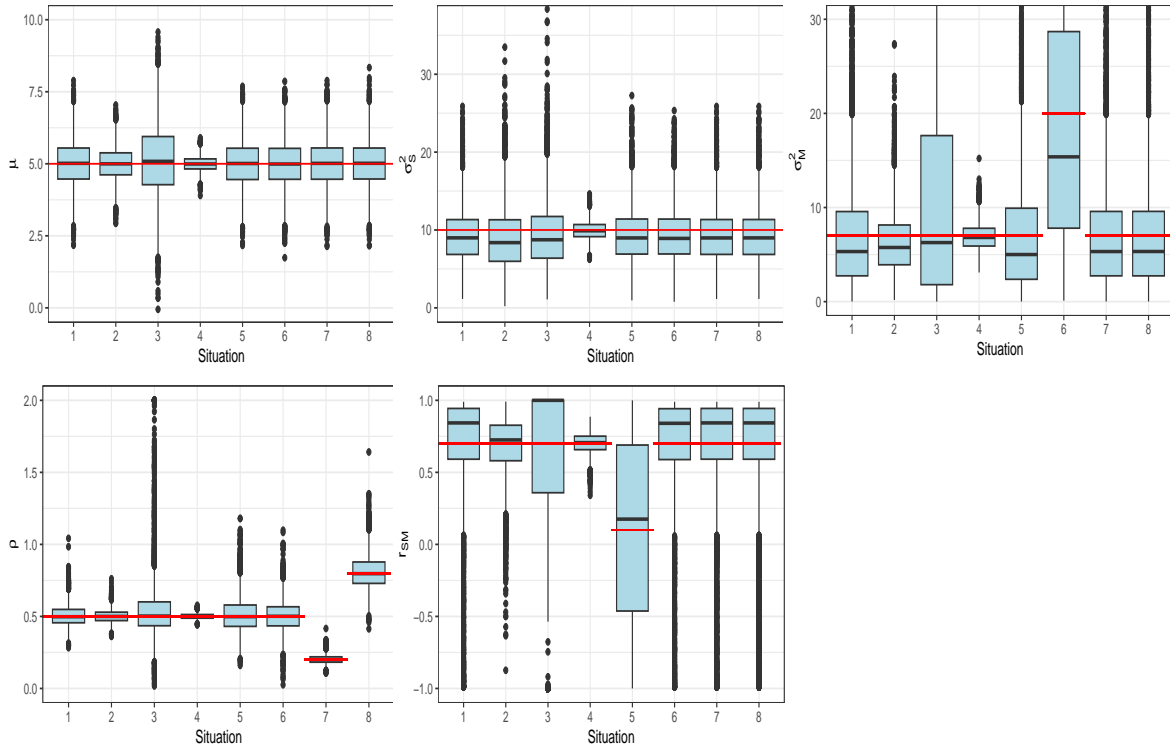


Figure 5: Estimations of the parameters of the Perturbed  $ARD_1$  model for the complete observation scheme

- Situation 7 is similar to situation 1, except that the maintenance efficiency is much smaller ( $\rho = 0.2$ ). It appears that the estimation of  $\rho$  is improved and the estimations of other parameters are not affected.
- Situation 8 is similar to situation 1, except that the maintenance efficiency is much larger ( $\rho = 0.8$ ). Only the estimation of  $\rho$  is slightly degraded.

These results and other simulations not reported here lead us to draw the following conclusions. The parameters are generally well estimated, even for a rather small amount of data.  $\sigma_S^2$  and  $\sigma_M^2$  are slightly underestimated and  $r_{SM}$  is overestimated. A higher number of maintenances improves the estimation of the parameters linked to the jumps  $\sigma_M^2$ ,  $\rho$  and  $r_{SM}$ . A higher number of observed degradation levels between maintenances improves the estimation of  $\sigma_S^2$ . It is difficult to estimate  $r_{SM}$  when its value is close to 0 or 1. The variances of the estimators of  $\sigma_M^2$  and  $\rho$  increase when the true parameter values increase.

### 5.1.2 Partial replacement model

For the Partial replacement model, 6 situations are considered, described in [Table 3](#). The boxplots of the estimators are presented in [Figure 6](#). The results are completely similar to those of the Perturbed  $ARD_1$  model. Whereas the true values of the drift parameters are different, the estimations quality of  $\mu_S$  and  $\mu_M$  remains comparable.

Table 3: Simulation situations for the Partial replacement model

Situation	$\mu_S$	$\mu_M$	$\sigma_S^2$	$\sigma_M^2$	$r_{SM}$	$n_j$	$k$	$n$
1	10	5	10	7	0.7	2	4	20
2	10	5	10	7	0.7	0	9	20
3	10	5	10	7	0.7	5	2	21
4	10	5	10	7	0.7	2	49	200
5	10	5	10	7	0.1	2	4	20
6	10	5	10	20	0.7	2	4	20

## 5.2 Impact of the observations locations and the observation schemes

The aim of this section is to look if the observations locations between two successive maintenance actions can have an impact on the estimation quality. To that aim, degradation levels are not necessarily periodically observed. Five situations are considered, described

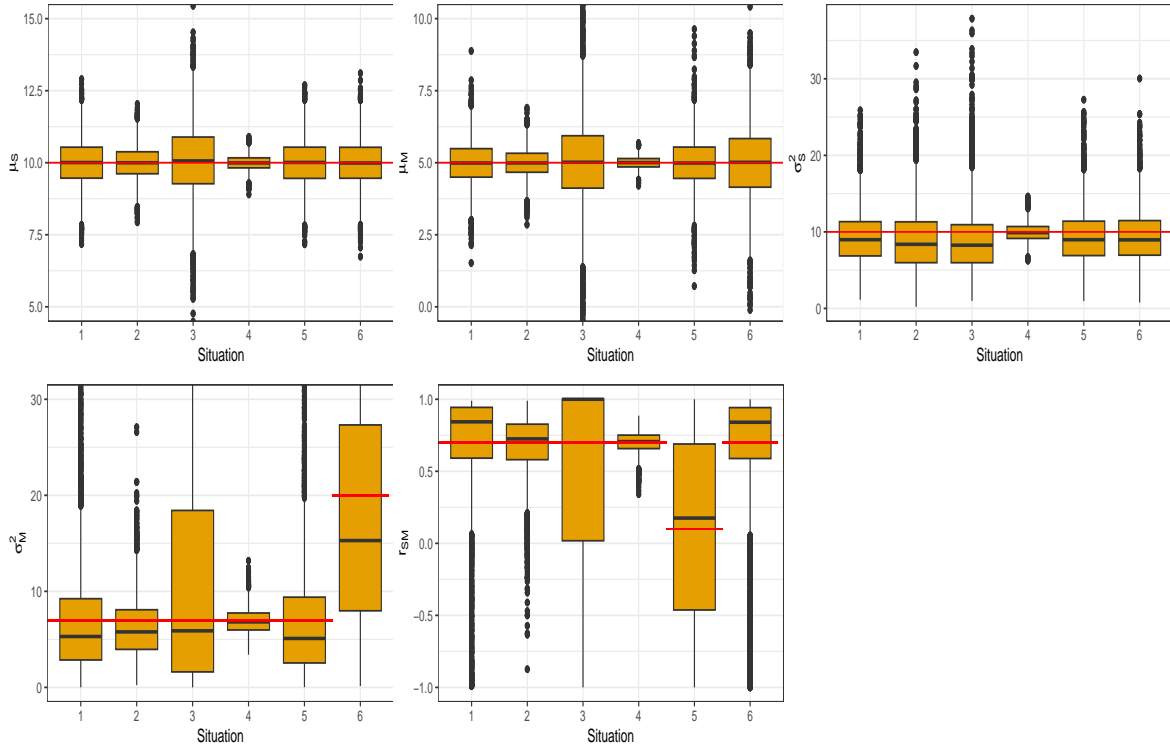


Figure 6: Estimations of the parameters of the Partial replacement model for the complete observation scheme

in Table 4 and illustrated in Figure 7. The total numbers of data are equivalent in all situations. The considered model is the Perturbed  $ARD_1$  model.

Table 4: Simulation situations for assessing the impact of observations locations

Observation scheme	Situation	$\mu$	$\sigma_S^2$	$\sigma_M^2$	$\rho$	$r_{SM}$	$n_j$	$k$	$n$
c	1	5	10	7	0.5	0.7	0	9	20
g	2	5	10	7	0.5	0.7	2	9	21
g	3	5	10	7	0.5	0.7	2	9	21
g	4	5	10	7	0.5	0.7	2	9	21
g	5	5	10	7	0.5	0.7	2	9	21

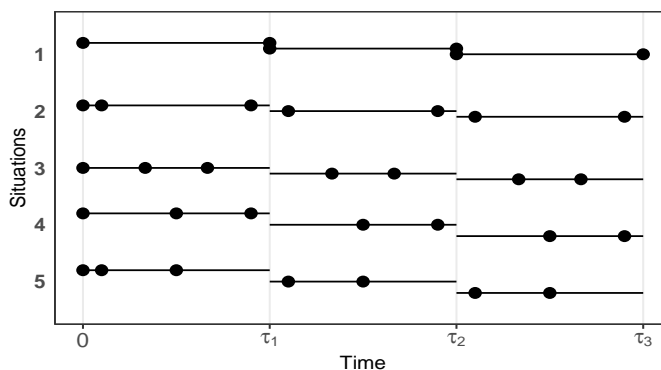


Figure 7: Observations location for the five considered situations and the first three maintenance times.

- In situation 1, the data are collected under the complete observation scheme, with no observed degradation levels between maintenances.
- In situation 2, the data are collected under the general observation scheme. Two degradation levels are measured in each interval, and the measurement times are close to the maintenance times. Here, the degradation measurements are not periodic:  $\forall j \in \{1, \dots, 9\}$ ,  $t_{j,n_j} = \tau_j - \frac{1}{5}$  and  $t_{j+1,1} = \tau_j + \frac{1}{5}$ .
- In situation 3, the data are collected under the general observation scheme. Two degradation levels are measured in each interval, and the measurement times are periodically spaced in the intervals between maintenances.

- In situation 4, the data are collected under the general observation scheme. Two degradation levels are measured in each interval, and the measurement times are just before the maintenance times. Here, the degradation measurements are not periodic:  $\forall j \in \{1, \dots, 9\}, t_{j,n_j} = \tau_j - \frac{1}{5}$  and  $t_{j+1,1} = \tau_j + \frac{1}{2}$ .
- In situation 5, the data are collected under the general observation scheme. Two degradation levels are measured in each interval, and the measurement times are just after the maintenance times. Here, the degradation measurements are not periodic:  $\forall j \in \{1, \dots, 9\}, t_{j,n_j} = \tau_j - \frac{1}{2}$  and  $t_{j+1,1} = \tau_j + \frac{1}{5}$ .

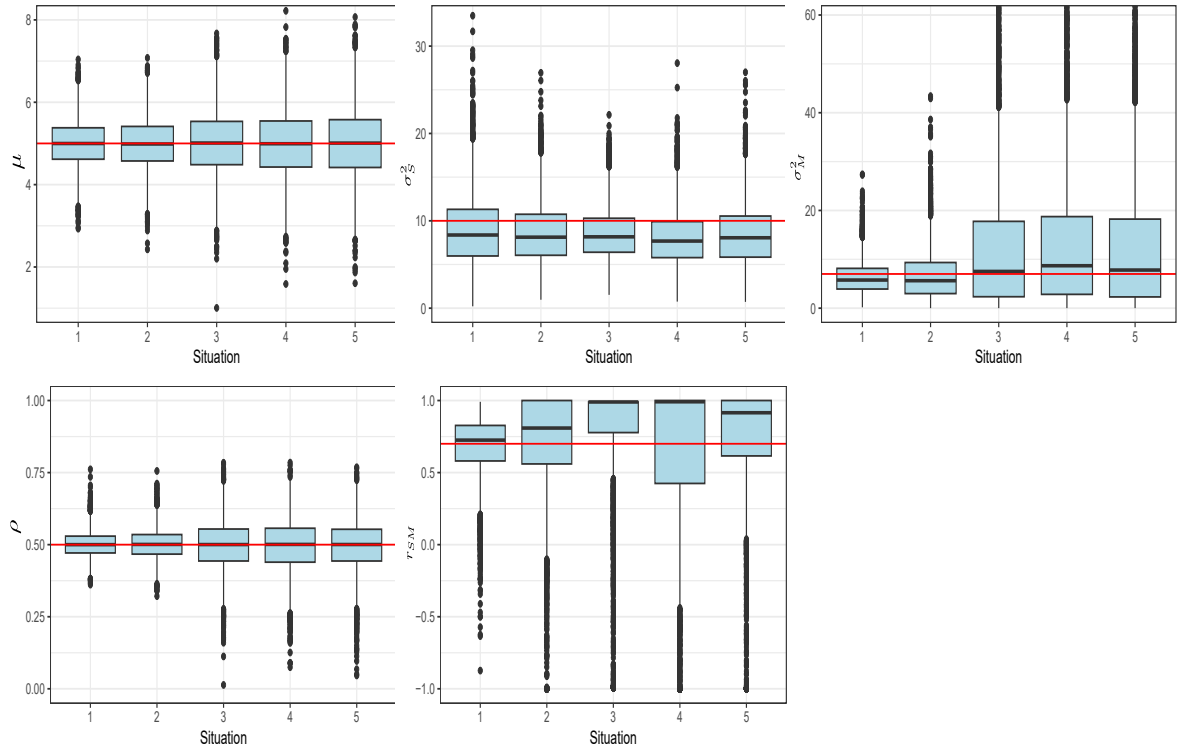


Figure 8: Estimations of the parameters of the Perturbed  $ARD_1$  model for the situations of [Table 4](#)

The boxplots of the estimators are presented in [Figure 8](#). This figure shows that the best estimations are obtained for situation 1, except for  $\sigma_S^2$ . It means that in order to estimate the parameters mainly linked to the maintenance effect, it is better to observe degradation

levels at maintenance times. Conversely, for parameter  $\sigma_S^2$  mainly linked to the intrinsic degradation, it is recommended to observe degradation levels between maintenances.

The results for situation 2 are closer to the results of situation 1 than those of situation 3. This illustrates the fact that Situation 1 is a limit case of situation 2, or that the complete observation scheme is a limit case of the general observation scheme. In this case, the observed jumps in the general scheme  $Z_j^g$  are close to the real jumps  $Z_j^c$ .

The estimations in situation 4 and 5 are comparable except for the correlation coefficient  $r_{SM}$ , which is better estimated when degradation levels are observed just after maintenance actions.

The estimation quality is slightly better in situation 5 compared to situation 4. Measuring degradation levels just after maintenance actions improves the estimations quality. In the usual  $ARD_1$  model, the best estimations are not obtained for the complete observation scheme [13]. On the contrary, for the proposed model, the complete observation scheme provides the best estimation.

## 6 Conclusion and perspectives

This article has introduced a new degradation model with imperfect maintenance, which assumes that maintenance effect only affects a part of the degradation process. Two particular models of interest have been studied, the Perturbed  $ARD_1$  and the Partial replacement models. The parameters of these models have been estimated under two different observation schemes. The quality of estimation has been assessed through a simulation study and the impact of different features of the model and data has been analyzed. [A first prospect of this study is to derive the theoretical properties of the estimators.](#)

In the future, decision making techniques could be carried out in order to establish optimal maintenance policies. Generally, it is assumed that a failure occurs when the degradation process hits for the first time a given degradation threshold. For usual Wiener processes, it is well known that the first hitting time is inverse Gaussian distributed [7]. For the new degradation model presented in this article, the first hitting time's distribution is not inverse Gaussian anymore but appears to be more complex as the degradation trajectory is discontinuous at maintenance times. So an interesting prospect of this work is the determination of this distribution.

In Appendix C, a preliminary step is given, through an empirical estimate of this distribution's density. Maintenance effects seem to cause the multimodal shape of the distri-

bution. Due to the maintenance effect, the risk of failure is higher before a maintenance than after. Further studies could be carried out in the future in order to determine the theoretical distribution of the first hitting time for this degradation model.

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# Appendices

## A Conditional distributions in Gaussian vectors

The following lemma gives the conditional distributions of the components of a Gaussian vector. It can be found for instance in Kotz et al [10].

Let  $\begin{pmatrix} A \\ B \end{pmatrix}$  be a Gaussian vector such that  $\begin{pmatrix} A \\ B \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma)$ , where  $\mu = \begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix}$  and  $\Sigma = \begin{pmatrix} \Sigma_A & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_B \end{pmatrix}$ . Then, the conditional distribution of  $A$  given  $[B = b]$  is multivariate normal  $\mathcal{N}(\tilde{\mu}, \tilde{\Sigma})$  where  $\tilde{\mu} = \mu_A + \Sigma_{AB}\Sigma_B^{-1}(b - \mu_B)$  and  $\tilde{\Sigma} = \Sigma_A - \Sigma_{AB}\Sigma_B^{-1}\Sigma_{BA}$ .

## B Parameters initialization for the maximum likelihood estimation algorithm

Model's parameters are estimated by maximizing the log-likelihood with the Nelder-Mead algorithm. In most situations, this method is not really sensitive to its initialization. However, a good initialization provides a more efficient algorithm and avoids potential local minima. In order to provide an efficient initialization, we will use the remark made in [Section 3.2](#) saying that  $(\mu_S, \sigma_S^2)$  are closely linked to degradation increments while parameters  $(\mu_M, \sigma_M^2, \rho)$  are closely linked to degradation jumps, so to maintenance efficiency. For the complete observation scheme with periodic maintenances of periodicity  $\Delta\tau$ , the initialization of the parameters is made as follows. The initial values of  $(\mu_S, \sigma_S^2, \mu_M, \sigma_M^2, r_{SM}, \rho)$  will be denoted  $(\mu_S^{(0)}, \sigma_S^{2(0)}, \mu_M^{(0)}, \sigma_M^{2(0)}, r_{SM}^{(0)}, \rho^{(0)})$ .

- **Initialization of  $\mu_S$  and  $\sigma_S^2$ .** A preliminary estimation of  $\mu_S$  and  $\sigma_S^2$  can be done by considering that only the degradation increments  $\Delta Y_{j,i}^c$  are observed. These random variables are independent and have respectively the  $\mathcal{N}(\mu_S \Delta t_{j,i}, \sigma_S^2 \Delta t_{j,i})$  distributions. So  $\mu_S^{(0)}$  and  $\sigma_S^{2(0)}$  will be the values of  $\mu_S$  and  $\sigma_S^2$  which maximize  $L(\mu_S, \sigma_S^2) = \prod_j \prod_i f_{\Delta Y_{j,i}^c}(\Delta y_{j,i}^c)$ . It is easy to show that:

$$\mu_S^{(0)} = \frac{\sum_{j=1}^{k+1} \sum_{i=1}^{n_j+1} \Delta y_{j,i}^c}{\sum_{j=1}^{k+1} \sum_{i=1}^{n_j+1} \Delta t_{j,i}} = \frac{y(\tau)}{\tau} \text{ and } \sigma_S^{2(0)} = \frac{1}{N+k+1} \sum_{j=1}^{k+1} \sum_{i=1}^{n_j+1} \frac{(\Delta y_{j,i}^c - \mu_S^{(0)} \Delta t_{j,i})^2}{\Delta t_{j,i}},$$

where  $N = \sum_{j=1}^{k+1} n_j$ .

Note that for the perturbed  $ARD_1$  model ( $\mu_S = \mu_M = \mu$ ),  $\mu_S^{(0)}$  is an estimator of  $\mu$ .

- **Initialization of  $\rho$  and  $\mu_M$ .** The jumps  $Z_j^c$  are independent and their respective distributions are  $\mathcal{N}(-\rho\mu_M\Delta\tau, \rho^2\sigma_M^2\Delta\tau)$ . So  $\forall j \in \{1, \dots, k\}$ ,  $\mathbb{E}\left[\frac{Z_j^c}{\mu_M\Delta\tau}\right] = -\rho$ . Therefore, for the Perturbed  $ARD_1$  model, one can choose  $\mu_M^{(0)} = \mu_S^{(0)}$  and  $\rho^{(0)} = -\frac{1}{k\mu_M^{(0)}\Delta\tau} \sum_{j=1}^k z_j^c$ . For the Partial replacement model,  $\rho = 1$  so  $\mu_M^{(0)} = -\frac{1}{k\Delta\tau} \sum_{j=1}^k z_j^c$ .
- **Initialization of  $\sigma_M^2$ .** If only the jumps  $Z_j^c$  are observed, the likelihood is  $L(\mu_M, \sigma_M^2, \rho) = \prod_{j=1}^k f_{Z_j^c}(z_j^c)$ . It is easy to show that the parameter values which maximize this function are such that  $\sigma_M^2 = \frac{1}{k\rho^2\Delta\tau} \sum_{j=1}^k (z_j^c + \rho\mu_M\Delta\tau)^2$ . Therefore, for the Perturbed  $ARD_1$  model,  $\sigma_M^2{}^{(0)} = \frac{1}{k\rho^{(0)2}\Delta\tau} \sum_{j=1}^k (z_j^c + \rho^{(0)}\mu_M^{(0)}\Delta\tau)^2$ . And for the Partial replacement model,  $\sigma_M^2{}^{(0)} = \frac{1}{k\Delta\tau} \sum_{j=1}^k (z_j^c + \mu_M^{(0)}\Delta\tau)^2$ .
- **Initialization of  $r_{SM}$ .** Since  $Z_j^c = -\rho\Delta X_j^M$  and  $\Delta Y_j = \Delta X_j^S$ ,  $\text{Cov}\left(\frac{Z_j^c}{-\rho}, \Delta Y_j\right) = \text{Cov}(\Delta X_j^M, \Delta X_j^S) = c_{SM}\Delta\tau$ . Therefore,  $c_{SM}$  can be estimated by the empirical covariance between the  $Z_j^c/(-\rho)$  and the  $\Delta Y_j^c/\Delta\tau$ :

$$c_{SM}^{(0)} = -\frac{1}{\Delta\tau\rho^{(0)}} \left[ \frac{1}{k} \sum_{j=1}^k Z_j^c \Delta Y_j - \frac{1}{k^2} \sum_{j=1}^k Z_j^c \sum_{i=1}^k \Delta Y_i \right], \text{ and } r_{SM}^{(0)} = \frac{c_{SM}^{(0)}}{\sigma_S^{(0)}\sigma_M^{(0)}}.$$

If the estimation of  $r_{SM}$  is greater than 1, then  $r_{SM}^{(0)}$  is set close to 1 (at 0.9). If the estimation of  $r_{SM}$  is lower than  $-1$ , then  $r_{SM}^{(0)}$  is set close to  $-1$  (at  $-0.9$ ).

## C First hitting time empirical distribution

Degradation trajectories are simulated for the proposed model and first hitting times are collected. An example of a simulated trajectory and observed hitting time is given in [Figure 9](#). An histogram of the obtained values is plotted in [Figure 10](#).

It appears that the first hitting time is multimodal distributed. In [Figure 10](#), the modes of the empirical density are clearly close to the maintenance times. In fact, the multimodal

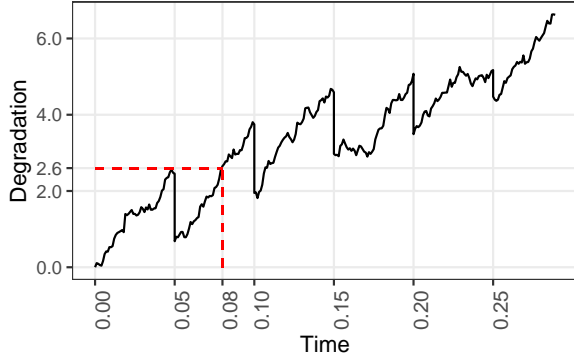


Figure 9: An example of a simulated degradation trajectory

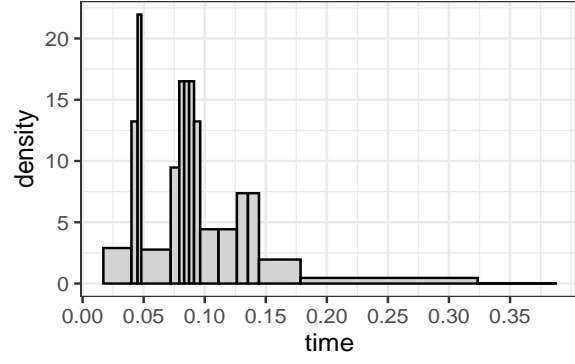


Figure 10: Empirical distribution of the first passage time

shape of the distribution is a consequence of the maintenance effect on the degradation.