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# The Beurling-Malliavin density, the Pólya density and their connection

Rita Giuliano \* Georges Grekos<sup>†</sup> Ladislav Mišik<sup>‡§</sup>

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## Abstract

In this paper we present a new formulation of the Beurling-Malliavin density (Proposition 1). Then we consider the upper Pólya density and show how its existence is connected with the concept of subadditivity; moreover, by means of some quantities introduced for proving Proposition 1, a theorem is presented that clarifies the connection between the upper Pólya and the Beurling-Malliavin densities. In the last section we discuss the classical definition of the upper Pólya density and we prove a result which seems to be new.

**Keywords:** Beurling–Malliavin density, upper Pólya density, increasing sequence, substantial interval, counting function, subadditivity,  $\eta$ -covering

**MSC 2020:** 11B25, 11B05

## 1 Introduction

Let  $\Lambda = (\lambda_n)_{n \in \mathbb{Z}}$  be an indexed family of real numbers. In order to solve the problem of finding the radius of completeness of  $\Lambda$  (denoted by  $\mathcal{R}(\Lambda)$ ), A. Beurling and P. Malliavin introduced for the first time in the paper [1] the quantity  $b(\Lambda)$ , defined as follows: if  $(I_n)_{n \in \mathbb{Z}}$  is a sequence of disjoint intervals on  $\mathbb{R}$ , call it *short* if

$$\sum_{n \in \mathbb{Z}} \frac{|I_n|^2}{1 + \text{dist}^2(0, I_n)} < \infty$$

(where  $|I_n|$  =length of  $I_n$  and  $\text{dist}(0, I_n) = \min_{x \in I_n} |x|$ ) and *long* otherwise; then define

$$b(\Lambda) := \sup\{d : \exists \text{ long } (I_n)_{n \in \mathbb{Z}} \text{ such that } \#(\Lambda \cap I_n) > d|I_n|, \forall n \in \mathbb{Z}\}$$

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(the particular formulation used above comes from [6] and [7]). The celebrated Theorem of [1] states that the radius of completeness of  $\Lambda$  is connected with  $b(\Lambda)$  by the formula

$$\mathcal{R}(\Lambda) = 2\pi b(\Lambda).$$

Later the same quantity  $b(\Lambda)$  has been studied by other authors, and various equivalent formulations have been found; see [7] and the recent [3] for exhaustive lists of references.

Another interesting type of density is introduced in [8] with the scope of studying gaps and singularities of power series. It is usually named as “upper Pólya density”.

In the present paper we first analyze the BM-density according to the definition of [5] and in particular we present a further formulation of this concept (see Proposition 1). Then we study the upper Pólya density and show how its existence is connected with the concept of subadditivity; moreover we prove a result (Theorem 10) that clarifies the connection between the upper Pólya and the BM-densities (by means of some quantities introduced for proving Proposition 1).

In the last section of the paper we prove that the inner limit appearing in the definition of the upper Pólya density, which in its original definition in [8] is calculated along the reals, can actually be calculated along the sequence of integers only. This fact seems to have not been noticed anywhere in the past.

The BM-density is defined in [1] for general families  $(\lambda_n)_{n \in \mathbb{Z}}$  of real numbers; on the other hand, the Pólya density concerns sequences of real numbers (i.e. indexed by  $n \in \mathbb{N}^*$ ), positive and strictly increasing ( $0 \leq \lambda_1 < \lambda_2 < \dots$ ) and such that  $\lim_{n \rightarrow \infty} \lambda_n = +\infty$ . Thus, in order to compare these two concepts, in what follows *we shall confine ourselves to sequences of the second (i.e. Pólya’s) kind*. We also emphasize that our study concerns sequences of real numbers and not only of integers. In the sequel, by the term *sequence* we always mean a sequence of Pólya’s kind, unless otherwise specified. Notice in particular that we are not dealing with finite sequences and sequences with repetitions.

The *counting function* of  $(\lambda_n)_{n \in \mathbb{N}^*}$  is the function

$$F_\Lambda(t) = \begin{cases} 0 & t = 0 \\ \#\{k \in \mathbb{N}^* : \lambda_k \leq t\} & t > 0. \end{cases}$$

For  $0 \leq a < b$  we have clearly

$$F_\Lambda(b) - F_\Lambda(a) = \#\{k \in \mathbb{N}^* : \lambda_k \in (a, b]\}.$$

The counting function of a sequence  $(\lambda_n)_{n \in \mathbb{N}^*}$  is non-decreasing, is identified by the sequence itself, and viceversa; since the definitions of the Pólya and B-M densities can be formulated in terms of  $F_\Lambda$ , in the sequel we shall adopt the point of view of counting functions in place of that of sequences.

## 2 The Beurling-Malliavin density

The BM-density, firstly defined in [1], has been studied in more recent times in [5]. This book employs a definition which is apparently completely different from the original one and proves the equivalence of the two concepts. In the present paper we shall follow [5], of course with an adaptation to our restricted framework.

Let  $\mathfrak{C}$  be the family of all sequences

$$\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*}$$

of intervals in  $(0, +\infty)$  such that  $a_n < b_n \leq a_{n+1}$  for all  $n \in \mathbb{N}^*$  and

$$\sum_{n=1}^{\infty} \left( \frac{b_n}{a_n} - 1 \right)^2 = +\infty.$$

In [5] these systems of intervals are called *substantial* (they are nothing but the *long* sequences of [6] and [7]). Then put

$$\mathfrak{R} = \left\{ R \geq 0 : \exists \mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C}, \frac{F_{\Lambda}(b_n) - F_{\Lambda}(a_n)}{b_n - a_n} \geq R \text{ for each sufficiently large } n \right\}.$$

Then the *Beurling-Malliavin density*  $b(\Lambda)$  is defined as the supremum of  $\mathfrak{R}$ ; in formula

$$b(\Lambda) = \sup \mathfrak{R}.$$

*Remark 1.* Despite the different notation, this definition coincides with the one of [6] and [7].

For every  $A \subseteq [1, +\infty]$  we shall be interested in the subset of  $\mathfrak{C}$  defined as

$$\mathfrak{C}_A = \left\{ \mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C} : \limsup_{n \rightarrow \infty} \frac{b_n}{a_n} \in A \right\}.$$

Accordingly, we shall denote

$$\mathfrak{R}_A = \left\{ R \geq 0 : \exists \mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C}_A, \frac{F_{\Lambda}(b_n) - F_{\Lambda}(a_n)}{b_n - a_n} \geq R \text{ for each sufficiently large } n \right\},$$

and finally

$$b_A(\Lambda) = \sup \mathfrak{R}_A. \tag{1}$$

In the case  $A = \{\alpha\}$  (i.e.  $A$  is a singleton) we shall simplify this set of notation to  $\mathfrak{C}_{\alpha}$ ,  $\mathfrak{R}_{\alpha}$  and  $b_{\alpha}(\Lambda)$  respectively.

For  $A \subseteq B \subseteq [1, +\infty]$  we point out the obvious relations

$$\mathfrak{C}_A \subseteq \mathfrak{C}_B, \quad \mathfrak{R}_A \subseteq \mathfrak{R}_B, \quad b_A(\Lambda) \leq b_B(\Lambda). \tag{2}$$

Since  $a_n < b_n$  for every substantial sequence of intervals, of course

$$\limsup_{n \rightarrow \infty} \frac{b_n}{a_n} \in [1, +\infty];$$

thus, according to the preceding notation, we have  $b(\Lambda) = b_{[1,+\infty]}(\Lambda)$ .

For every  $\mathcal{I} = \{(a_n, b_n]\}_{n \in \mathbb{N}^*} \in \mathfrak{C}$ , let

$$\ell_{\mathcal{I}} = \liminf_{n \rightarrow \infty} \frac{F_{\Lambda}(b_n) - F_{\Lambda}(a_n)}{b_n - a_n}.$$

Formula (3) in the following Proposition provides an alternative definition of  $b_A(\Lambda)$ , to be used in the following section. The result is almost self-evident, anyway we prove it in detail.

**Proposition 1.** *Let  $A \subseteq [1, +\infty]$  be fixed. Then*

$$b_A(\Lambda) = \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}_A\}. \quad (3)$$

*Proof.* Denote for simplicity  $\alpha_A(\Lambda) = \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}_A\}$ .

(i) We prove first that

$$\alpha_A(\Lambda) \leq b_A(\Lambda). \quad (4)$$

By the definition of supremum, for every  $\epsilon > 0$ , there exists  $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C}_A$  such that

$$\alpha_A(\Lambda) - \epsilon < \ell_{\mathcal{I}}.$$

By definition of liminf, ultimately we have

$$\frac{F_{\Lambda}(b_n) - F_{\Lambda}(a_n)}{b_n - a_n} \geq \alpha_A(\Lambda) - \epsilon.$$

The last inequality implies that  $\alpha_A(\Lambda) - \epsilon \in \mathfrak{R}_A$ , so that  $\alpha_A(\Lambda) - \epsilon \leq \sup \mathfrak{R}_A = b_A(\Lambda)$ , and (4) follows by the arbitrariness of  $\epsilon$ .

(ii) For every  $R \in \mathfrak{R}_A$ , there exists  $\mathcal{I}_R = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C}_A$  such that ultimately

$$\frac{F_{\Lambda}(b_n) - F_{\Lambda}(a_n)}{b_n - a_n} \geq R.$$

By passing to the liminf in the above inequality, we get for such  $\mathcal{I}_R$  that  $R \leq \ell_{\mathcal{I}_R}$ , hence

$$b_A(\Lambda) = \sup \mathfrak{R}_A \leq \sup_R \ell_{\mathcal{I}_R} \leq \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}_A\} = \alpha_A(\Lambda).$$

□

### 3 Analysis of the Beurling-Malliavin density

The aim of the present Section is to give a precise mathematical formulation and a rigorous proof of the intuitive feelings that, in defining the BM-density,

- (i) all the sequences of substantial intervals  $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*}$  with  $\limsup_{n \rightarrow \infty} \frac{b_n}{a_n} > 1$  have the same “status”, so to say (see Proposition 3);
- (ii) for identifying the value of  $b(\Lambda)$  the only important sequences of substantial intervals  $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*}$  are those with  $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = 1$  (see Theorem 2 here below).

**Theorem 2.**  $b(\Lambda) = b_1(\Lambda)$ .

For proving Theorem 2 the first step is the following Proposition, which says that in formula (3) we can restrict ourselves to taking the supremum of  $\ell_{\mathcal{I}}$  in particular classes  $\mathfrak{C}_A$  (i.e. for particular sets  $A$ ) in place of the whole class  $\mathfrak{C}$ .

**Proposition 3.** For  $k > 1$  consider the set  $A_k = [1, k]$ . Then, for every  $k > 1$ ,

$$b(\Lambda) = b_{A_k}(\Lambda).$$

*Proof.* The inequality  $\geq$  is obvious by the last relation in (2). Thus it is sufficient to show that

**Proposition 4.** For fixed  $k > 1$  consider the set  $A_k = [1, k]$ . Then

- (i) for every  $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C}$ , there exists  $\mathcal{J}_{\mathcal{I}} \in \mathfrak{C}_{A_k}$  such that  $\ell_{\mathcal{J}_{\mathcal{I}}} \geq \ell_{\mathcal{I}}$ . Hence  $b(\Lambda) = \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}\} \leq \sup\{\ell_{\mathcal{J}_{\mathcal{I}}}, \mathcal{I} \in \mathfrak{C}\} \leq \sup\{\ell_{\mathcal{J}}, \mathcal{J} \in \mathfrak{C}_{A_k}\} = b_{A_k}(\Lambda)$ .
- (ii)  $\mathcal{J}_{\mathcal{I}} = ((c_n, d_n])_{n \in \mathbb{N}^*}$  can be chosen in such a way that  $\lim_{n \rightarrow \infty} \frac{d_n}{c_n}$  exists (and belongs to  $A_k$ ).

For the proof of Proposition 4 we need a lemma.

**Lemma 5.** (i) Let  $k > 1$ . It is possible to split any interval  $(a, b]$  into some number  $r$  of disjoint subintervals

$$(a, b] = \bigcup_{i=1}^r (c_i, d_i],$$

where  $a = c_1 < d_1 = c_2 < \dots = c_r < d_r = b$ , in such a way that, for every  $i = 1, \dots, r$ ,

$$1 < \frac{d_i}{c_i} \leq k.$$

(ii) Let the family of intervals  $\{(c_i, d_i], i = 0, \dots, r\}$ , covers the interval  $(a, b]$ . Then there exists an integer  $j \in \{0, \dots, r\}$  such that

$$\frac{F_{\Lambda}(d_j) - F_{\Lambda}(c_j)}{d_j - c_j} \geq \frac{F_{\Lambda}(b) - F_{\Lambda}(a)}{\sum_{j=0}^r (d_i - c_i)}.$$

*Proof of Lemma 5.* (i) Take  $r = \lfloor \log_k \frac{b}{a} \rfloor + 1$ ,  $\alpha = (\frac{b}{a})^{\frac{1}{r}}$  and, for  $i = 1, 2, \dots, r$ , define

$$c_i = \alpha^{i-1}a, \quad d_i = \alpha^i a,$$

noticing that

$$c_1 = a, \quad d_r = b, \quad \frac{d_i}{c_i} = \alpha \leq k.$$

(ii) Assume the contrary, i.e.

$$\frac{F_\Lambda(d_i) - F_\Lambda(c_i)}{d_i - c_i} < \frac{F_\Lambda(b) - F_\Lambda(a)}{\sum_{j=0}^r (d_i - c_i)}, \quad \forall i = 0, 1, 2, \dots, r.$$

Then, as the family  $\{(c_i, d_i]\}$  covers  $(a, b]$ ,

$$F_\Lambda(b) - F_\Lambda(a) \leq \sum_{i=1}^r (F_\Lambda(d_i) - F_\Lambda(c_i)) < \frac{F_\Lambda(b) - F_\Lambda(a)}{\sum_{j=0}^r (d_i - c_i)} \sum_{i=1}^r (d_i - c_i) = F_\Lambda(b) - F_\Lambda(a),$$

a contradiction.  $\square$

*Proof of Proposition 4.* For any  $n \in \mathbb{N}^*$ , by Lemma 5 (i) it is possible to split the interval  $(a_n, b_n]$  into  $r_n$  subintervals  $(c_{n,i}, d_{n,i}]$  such that

$$1 \leq \frac{d_{n,i}}{c_{n,i}} \leq k, \quad i = 1, \dots, r_n. \quad (5)$$

By Lemma 5 (ii), there exists  $j_n \in \{1, 2, \dots, r_n\}$  such that

$$\frac{F_\Lambda(d_{n,j_n}) - F_\Lambda(c_{n,j_n})}{d_{n,j_n} - c_{n,j_n}} \geq \frac{F_\Lambda(b_n) - F_\Lambda(a_n)}{b_n - a_n}. \quad (6)$$

The sequence of numbers

$$\left( \frac{d_{n,j_n}}{c_{n,j_n}} \right)_{n \in \mathbb{N}^*}$$

is bounded by (5), hence, by possibly passing to a subsequence, we can assume that it is convergent. It is clear that the sequence of intervals

$$\mathcal{J} = ((c_{n,j_n}, d_{n,j_n}])_{n \in \mathbb{N}^*}$$

belongs to  $\mathfrak{C}_{A_k}$  and, by (6), verifies the relation  $\ell_{\mathcal{J}} \geq \ell_{\mathcal{I}}$ . The Proposition is proved.  $\square$

For proving Theorem 2 we shall use a further lemma.

**Lemma 6.** *Let  $\eta > 1$ . It is possible to split any interval  $(a, b]$  (with  $0 < a < b$ ) into some number  $r$  of disjoint subintervals*

$$(a, b] = \bigcup_{i=1}^r (c_i, d_i],$$

where  $a = c_1 < d_1 = c_2 < \dots = c_r < d_r = b$  in such a way that

$$\frac{d_i}{c_i} = \eta, \quad i = 1, \dots, r-1, \quad \frac{d_r}{c_r} \leq \eta.$$

*Proof.* Consider the (increasing divergent) sequence  $(\eta^{i-1}a)_{i \in \mathbb{N}^*}$  and denote

$$r = \max\{i \in \mathbb{N}^* : \eta^{i-1}a < b\},$$

which means that  $\eta^{r-1}a < b$  and  $\eta^r a \geq b$ . Then put

$$c_i = \eta^{i-1}a, \quad d_i = (\eta^i a) \wedge b, \quad i = 1, 2, \dots, r.$$

Then

$$\frac{d_i}{c_i} = \frac{\eta^i a}{\eta^{i-1}a} = \eta, \quad i = 1, \dots, r-1; \quad \frac{d_r}{c_r} = \frac{(\eta^r a) \wedge b}{\eta^{r-1}a} \leq \frac{\eta^r a}{\eta^{r-1}a} = \eta.$$

□

*Proof of Theorem 2.* The inequality  $\geq$  is obvious by the last relation in (2). Thus, by Proposition 3, it suffices to prove that for each sequence of intervals  $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C}_{A_2}$  there exists a sequence of intervals  $\mathcal{J}_{\mathcal{I}} \in \mathfrak{C}_1$  such that  $\ell_{\mathcal{J}_{\mathcal{I}}} \geq \ell_{\mathcal{I}}$ . This implies

$$b(\Lambda) = \sup\{\ell_{\mathcal{I}}, \mathcal{I} \in \mathfrak{C}\} \leq \sup\{\ell_{\mathcal{J}_{\mathcal{I}}}, \mathcal{I} \in \mathfrak{C}\} \leq \sup\{\ell_{\mathcal{J}}, \mathcal{J} \in \mathfrak{C}_1\} = b_1(\Lambda).$$

Let  $n \in \mathbb{N}^*$  be fixed and put  $\eta_n = (\frac{b_n}{a_n})^{\frac{1}{n}}$ . Applying Lemma 6 and afterwards Lemma 5 (ii), we can construct an interval

$$J_n = (c_n, d_n] \subseteq (a_n, b_n]$$

such that

$$1 \leq \frac{d_n}{c_n} \leq \eta_n \tag{7}$$

and

$$\frac{F_{\Lambda}(d_n) - F_{\Lambda}(c_n)}{d_n - c_n} \geq \frac{F_{\Lambda}(b_n) - F_{\Lambda}(a_n)}{b_n - a_n}.$$

Now notice that, since  $\limsup_{n \rightarrow \infty} \frac{b_n}{a_n} \leq 2$ , the sequence  $(\frac{b_n}{a_n})_{n \in \mathbb{N}^*}$  is bounded by some constant  $c$ , hence  $\lim_{n \rightarrow \infty} \eta_n = 1$  by the inequalities

$$1 \leq \liminf_{n \rightarrow \infty} \eta_n \leq \limsup_{n \rightarrow \infty} \eta_n \leq \lim_{n \rightarrow \infty} \sqrt[n]{c} = 1.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{d_n}{c_n} = 1$$

by (7), which concludes the proof. □

Let  $A$  be a subset of  $[1, +\infty]$  containing 1. Then, by the last relation in (2) and by Theorem 2,

$$b_1(\Lambda) \leq b_A(\Lambda) \leq b(\Lambda) = b_1(\Lambda).$$

This proves

**Corollary 7.**  $b_A(\Lambda) = b(\Lambda)$  if  $1 \in A$ .



## 4 The upper Pólya density $\bar{p}(\Lambda)$

The *upper Pólya density* of  $\Lambda$  is the number

$$\bar{p}(\Lambda) = \lim_{\xi \rightarrow 1^-} \limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(\xi x)}{x - x\xi}.$$

In [8] it is proved that the above limit exists; see also [4]. In this section we give a new proof of this fact (see Proposition 9). The aim is to show how the existence of  $\bar{p}(\Lambda)$  is connected with the concept of subadditivity.

The following result is a generalization of a famous lemma due to M. Fekete (see [2]).

**Lemma 8.** *Let  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a function such that there exists a continuous non-decreasing function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with the following property: for every  $a, b \geq 0$ ,*

$$g(a + b) \leq \frac{\phi(a)}{\phi(a + b)}g(a) + \left(1 - \frac{\phi(a)}{\phi(a + b)}\right)g(b). \quad (8)$$

Then  $\lim_{x \rightarrow +\infty} g(x)$  exists and

$$\lim_{x \rightarrow +\infty} g(x) = \sup_{x > 0} g(x).$$

*Proof.* First it is easily seen by a simple recursive argument that, for every  $a \geq 0$  and for every  $n \in \mathbb{N}^*$ ,

$$g(na) \leq g(a). \quad (9)$$

Let  $y > 0$  be fixed. For every  $x \in (0, y)$ , let  $n(x) = \lfloor \frac{y}{x} \rfloor - 1$ , put

$$z(x) = y - n(x) \cdot x \quad (10)$$

and observe that, by the inequalities  $a - 1 < \lfloor a \rfloor \leq a$ , we have

$$x \leq z(x) \leq 2x. \quad (11)$$

Since  $y = n(x) \cdot x + z(x)$ , we get from (8) and (9) that

$$g(y) \leq \frac{\phi(n(x) \cdot x)}{\phi(y)}g(x) + \left(1 - \frac{\phi(n(x) \cdot x)}{\phi(y)}\right)g(z(x)). \quad (12)$$

Now we distinguish the two cases (i)  $\sup_{x > 0} g(x) \in \mathbb{R}$ ; (ii)  $\sup_{x > 0} g(x) = +\infty$ .

(i) It follows from (12) that

$$g(y) \leq \frac{\phi(n(x) \cdot x)}{\phi(y)}g(x) + \left(1 - \frac{\phi(n(x) \cdot x)}{\phi(y)}\right)\left(\sup_{x > 0} g(x)\right).$$

Let  $x \rightarrow 0^+$ ; we have from (11) that  $z(x) \rightarrow 0$ , hence  $n(x) \cdot x \rightarrow y$  by (10); thus, by the continuity of  $\phi$ ,

$$\lim_{x \rightarrow 0^+} \frac{\phi(n(x) \cdot x)}{\phi(y)} = 1$$

and

$$g(y) \leq \left( \liminf_{x \rightarrow 0^+} g(x) \right) \lim_{x \rightarrow 0^+} \frac{\phi(n(x) \cdot x)}{\phi(y)} + \left( \sup_{x > 0} g(x) \right) \lim_{x \rightarrow 0^+} \left( 1 - \frac{\phi(n(x) \cdot x)}{\phi(y)} \right) = \liminf_{x \rightarrow 0^+} g(x).$$

Now we pass to the supremum in  $y$  and get

$$\sup_{y > 0} g(y) \leq \liminf_{x \rightarrow 0^+} g(x) \leq \limsup_{x \rightarrow 0^+} g(x) \leq \sup_{y > 0} g(y),$$

and we are done.

(ii) We get from (12)

$$g(y) - \left( 1 - \frac{\phi(n(x) \cdot x)}{\phi(y)} \right) g(z(x)) \leq \frac{\phi(n(x) \cdot x)}{\phi(y)} g(x).$$

Assume that  $\liminf_{x \rightarrow 0^+} g(x) < +\infty$ . Then, passing to the  $\liminf_{x \rightarrow 0^+}$ , the above relation gives

$$g(y) \leq \liminf_{x \rightarrow 0^+} g(x) < +\infty,$$

hence the absurdum by passing to the sup in  $y$ . □

**Proposition 9.** *The limit*

$$\bar{p}(\Lambda) = \lim_{\xi \rightarrow 1^-} \limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(\xi x)}{x - x\xi}$$

exists. Moreover

$$\lim_{\xi \rightarrow 1^-} \limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(\xi x)}{x - x\xi} = \sup_{\xi < 1} \limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(\xi x)}{x - x\xi}. \quad (13)$$

*Proof.* The aim is to apply the previous Lemma to the function

$$g(x) = \limsup_{y \rightarrow \infty} \frac{F_\Lambda(y) - F_\Lambda(ye^{-x})}{y - ye^{-x}}, \quad x > 0,$$

obtaining

$$\begin{aligned} \lim_{\xi \rightarrow 1^-} \limsup_{y \rightarrow \infty} \frac{F_\Lambda(y) - F_\Lambda(\xi y)}{y - y\xi} &= \lim_{x \rightarrow 0^+} \limsup_{y \rightarrow \infty} \frac{F_\Lambda(y) - F_\Lambda(ye^{-x})}{y - ye^{-x}} \\ &= \sup_{x > 0} \limsup_{y \rightarrow \infty} \frac{F_\Lambda(y) - F_\Lambda(ye^{-x})}{y - ye^{-x}} = \sup_{\xi < 1} \limsup_{y \rightarrow \infty} \frac{F_\Lambda(y) - F_\Lambda(\xi y)}{y - y\xi}. \end{aligned}$$

So we prove the subadditivity of  $g$ . We have

$$\begin{aligned} &\frac{F_\Lambda(y) - F_\Lambda(ye^{-(a+b)})}{y - ye^{-(a+b)}} \\ &= \frac{F_\Lambda(y) - F_\Lambda(ye^{-a})}{y - ye^{-a}} \cdot \frac{1 - e^{-a}}{1 - e^{-(a+b)}} + \frac{F_\Lambda(ye^{-a}) - F_\Lambda(ye^{-(a+b)})}{ye^{-a}(1 - e^{-b})} \cdot \left( 1 - \frac{1 - e^{-a}}{1 - e^{-(a+b)}} \right); \end{aligned}$$

passing to the limsup in  $y$  we obtain

$$g(a+b) \leq \frac{1 - e^{-a}}{1 - e^{-(a+b)}} g(a) + \left( 1 - \frac{1 - e^{-a}}{1 - e^{-(a+b)}} \right) g(b).$$

Since the function  $\phi(x) = 1 - e^{-x}$  is trivially non-decreasing and continuous, the proof is concluded. □

## 5 Comparison between $\bar{p}(\Lambda)$ and $b_{(1,+\infty]}(\Lambda)$

In this section we are concerned with  $\mathfrak{C}_{(1,+\infty]}$ ,  $\mathfrak{R}_{(1,+\infty]}$  and  $b_{(1,+\infty]}(\Lambda)$ , which we shall denote as  $\mathfrak{C}_{>1}$ ,  $\mathfrak{R}_{>1}$  and  $b_{>1}(\Lambda)$  respectively for easier writing.

It is known that in general  $\bar{p}(\Lambda) \leq b(\Lambda)$  (see [5]). The aim of this section is to prove a more precise relation. In fact

**Theorem 10.** *We have*

$$\bar{p}(\Lambda) = b_{>1}(\Lambda).$$

We split the proof into two parts, namely Propositions 11 and 12.

**Proposition 11.** *The following inequality holds true:*

$$\bar{p}(\Lambda) \leq b_{>1}(\Lambda).$$

*Proof.* It suffices to show that, for every  $R < \bar{p}(\Lambda)$  we have  $R \leq b_{>1}(\Lambda)$ . By definition of  $\bar{p}(\Lambda)$ , there exists  $\xi < 1$  such that

$$\limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(x\xi)}{x - x\xi} > R.$$

By definition of limsup, there exists a sequence  $(x_n)_{n \in \mathbb{N}^*}$  such that

$$\lim_{n \rightarrow \infty} x_n = +\infty, \quad \lim_{n \rightarrow \infty} \frac{F_\Lambda(x_n) - F_\Lambda(x_n\xi)}{x_n - x_n\xi} = \limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(x\xi)}{x - x\xi} > R. \quad (14)$$

Set  $n_1 = 1$ ,  $y_1 = x_{n_1} = x_1$  and  $n_2 = \min\{n > 1 : x_n \geq \frac{x_1}{\xi}\}$  ( $n_2$  exists since otherwise the sequence  $(x_n)_{n \in \mathbb{N}^*}$  would be bounded). Put  $y_2 = x_{n_2}$ ; then  $y_2 \geq \frac{y_1}{\xi}$ .

Assume we have constructed  $n_2, \dots, n_r$  and  $y_2, \dots, y_r$  such that  $y_k \geq \frac{y_{k-1}}{\xi}$  for each  $k = 2, \dots, r$ . Let  $n_{r+1} = \min\{n > n_r : x_n \geq \frac{x_{n_r}}{\xi}\}$  ( $n_{r+1}$  exists for the same reason as above) and let  $y_{r+1} = x_{n_{r+1}}$ . By this recursive construction we obtain a subsequence  $(y_n)_{n \in \mathbb{N}^*}$  of  $(x_n)_{n \in \mathbb{N}^*}$  with the property that

$$y_{n+1} \geq \frac{y_n}{\xi}, \quad n = 1, 2, \dots; \quad (15)$$

now set  $a_n = y_n\xi$  and  $b_n = y_n$ .

It is easy to see that the sequence of intervals  $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*}$  belongs to  $\mathfrak{C}_{>1}$ :

(i)  $a_n < b_n \leq a_{n+1}$  since this means  $y_n\xi < y_n \leq y_{n+1}\xi$ , which is true by (15);

(ii)

$$\sum_{n=1}^{\infty} \left( \frac{b_n}{a_n} - 1 \right)^2 = \sum_{n=1}^{\infty} \left( \frac{1}{\xi} - 1 \right)^2 = \infty;$$

(iii)

$$\limsup_{n \rightarrow \infty} \frac{b_n}{a_n} = \frac{1}{\xi} > 1.$$

Since ultimately

$$\frac{F_\Lambda(b_n) - F_\Lambda(a_n)}{b_n - a_n} > R$$

by (14), we deduce that  $R \in \mathfrak{R}_{>1}$ , hence  $R \leq \sup \mathfrak{R}_{>1} = b_{>1}(\Lambda)$ . □

Now we are concerned with the reverse inequality.

**Proposition 12.** *We have*

$$b_{>1}(\Lambda) \leq \bar{p}(\Lambda).$$

*Proof.* Let  $R \in \mathfrak{R}_{>1}$  and let  $\mathcal{I} = ((a_n, b_n])_{n \in \mathbb{N}^*} \in \mathfrak{C}_{>1}$  with  $\frac{\Lambda(b_n) - \Lambda(a_n)}{b_n - a_n} \geq R$ . Denote

$$\limsup_{n \geq 1} \frac{b_n}{a_n} = L \geq 1.$$

Fix  $\epsilon > 0$ ; we have ultimately

$$a_n > \frac{b_n}{L + \epsilon},$$

hence

$$\begin{aligned} R &\leq \frac{F_\Lambda(b_n) - F_\Lambda(a_n)}{b_n - a_n} \leq \frac{F_\Lambda(b_n) - F_\Lambda(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \cdot \frac{b_n(1 - \frac{1}{L+\epsilon})}{b_n - a_n} = \frac{F_\Lambda(b_n) - F_\Lambda(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \cdot \frac{\frac{b_n}{a_n}(1 - \frac{1}{L+\epsilon})}{\frac{b_n}{a_n} - 1} \\ &\leq \frac{F_\Lambda(b_n) - F_\Lambda(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \cdot \frac{L + \epsilon - 1}{\frac{b_n}{a_n} - 1}. \end{aligned}$$

We deduce

$$\frac{F_\Lambda(b_n) - F_\Lambda(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \geq R \cdot \frac{\frac{b_n}{a_n} - 1}{L + \epsilon - 1}$$

and, passing to the limsup in  $x$ ,

$$\limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(\frac{x}{L+\epsilon})}{x - \frac{x}{L+\epsilon}} \geq \limsup_{x \rightarrow \infty} \frac{F_\Lambda(b_n) - F_\Lambda(\frac{b_n}{L+\epsilon})}{b_n - \frac{b_n}{L+\epsilon}} \geq R \cdot \limsup_{n \rightarrow \infty} \frac{\frac{b_n}{a_n} - 1}{L + \epsilon - 1} = R \cdot \frac{L - 1}{L + \epsilon - 1}.$$

Thus, observing that  $\frac{1}{L+\epsilon} < 1$  and by Proposition 9 (see relation (13)), we get

$$\bar{p}(\Lambda) = \sup_{\xi < 1} \limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(x\xi)}{x - x\xi} \geq \limsup_{x \rightarrow \infty} \frac{F_\Lambda(x) - F_\Lambda(\frac{x}{L+\epsilon})}{x - \frac{x}{L+\epsilon}} \geq R \cdot \frac{L - 1}{L + \epsilon - 1},$$

for every  $\epsilon > 0$ . Now pass to the limit as  $\epsilon \rightarrow 0$  to obtain that

$$\bar{p}(\Lambda) \geq R, \quad \forall R \in \mathfrak{R}_{>1},$$

and optimizing

$$\bar{p}(\Lambda) \geq \sup \mathfrak{R}_{>1} = b_{>1}(\Lambda). \quad \square$$

## 6 On the limit in the definition of the upper Pólya density

As we have seen at the beginning of Section 4, the inner limit in the definition of  $\bar{p}(\Lambda)$  is calculated as  $x \rightarrow \infty$ , where  $x$  is a real variable. Actually Pólya in [8] uses the symbol  $r$  (instead of  $x$ ) without specifying where  $r$  varies, but there is no reason to suppose that he didn't have the real numbers in mind. Anyway, in this Section we prove the following result:

**Theorem 13.** *The limit*

$$\ell := \lim_{\eta \rightarrow 1^-} \limsup_n \frac{F(n) - F(\eta n)}{(1 - \eta)n}$$

exists and its value is  $\bar{p}(\Lambda)$ .

The proof is rather intricate and needs some preparation. In particular we need to construct a particular covering of the interval  $(\xi x, x]$ , i.e. a finite family of intervals  $\{(a_i, b_i]\}$  with right endpoints  $b_i$  belonging to  $\mathbb{N}$  and such that

$$(\xi x, x] \subseteq \bigcup_i (a_i, b_i]$$

### 6.1 Construction of an $\eta$ -covering of $(\xi x, x]$

For  $x > 0$  and  $\eta \in (0, 1)$  let  $\phi(x) = \lceil x \rceil$  and  $\psi(x) = \eta x$ ,  $f(x) = (\phi \circ \psi)(x) = \lceil \eta x \rceil$ . We denote by  $f^m$  the function obtained by composing  $f$  with itself  $m$  times, i.e.

$$f^0(x) = x; \quad f^{m+1}(x) = (f \circ f^m)(x).$$

Similarly for  $\psi^m$ .

**Lemma 14.** *We have the following facts:*

- (i)  $t \leq \phi(t)$ , for every  $t > 0$ ;
- (ii)  $\psi^m(t) \leq f^m(t)$  for every integer  $m \geq 0$ ;
- (iii)  $(f^m \circ \phi)(x) \leq \eta^m x + \sum_{k=0}^m \eta^k$  for every integer  $m \geq 0$ .

*Proof.* (i) is evident. We prove (ii) by induction: the case  $m = 0$  is obvious; the case  $m = 1$  follows from (i) with  $\psi(t)$  in place of  $t$ . For the case  $m + 1$  we have,

$$\psi^{m+1}(t) = \psi(\psi^m(t)) \leq \psi(f^m(t)) \leq f(f^m(t)) = f^{m+1}(t)$$

where, besides the inductive assumption, we have used the case  $m = 1$  and the fact that  $\psi$  is increasing.

Now we prove (iii), again by induction. The case  $m = 0$  is obvious (recall that  $f^0(x) = x$ ) and reads as  $\phi(x) \leq x + 1$ . The inductive step uses the inductive assumption and the fact that  $f$  is nondecreasing:

$$\begin{aligned} f^{m+1}(\phi(x)) &= f(f^m(\phi(x))) \leq f\left(\eta^m x + \sum_{k=0}^m \eta^k\right) = \left\lceil \eta(\eta^m x + \sum_{k=0}^m \eta^k) \right\rceil \\ &\leq \eta\left(\eta^m x + \sum_{k=0}^m \eta^k\right) + 1 = \eta^{m+1} x + \sum_{k=0}^{m+1} \eta^k. \end{aligned}$$

□

*Remark 2.* By Lemma 14 (iii),

$$\eta^m x \leq (f^m \circ \phi)(x) \leq \eta^m x + \sum_{k=0}^m \eta^k$$

and, by putting  $S_\eta = \sum_{k=0}^{\infty} \eta^k$ , it provides the bound

$$0 \leq \sup_{m,x} ((f^m \circ \phi)(x) - \eta^m x) \leq S_\eta.$$

For fixed  $x$  and for every integer  $i \geq 0$  put  $b_i = (f^i \circ \phi)(x)$  and

$$a_i = \eta b_i = \psi(b_i) = (\psi \circ f^i \circ \phi)(x);$$

notice that  $b_i$  is an integer and  $b_0 = \phi(x) = \lceil x \rceil$ ; notice also that, for every  $i \geq 1$ ,

$$\begin{aligned} b_i &= (f^i \circ \phi)(x) = f((f^{i-1} \circ \phi)(x)) = (\phi \circ \psi)((f^{i-1} \circ \phi)(x)) = \phi((\psi \circ f^{i-1} \circ \phi)(x)) \\ &= \phi(a_{i-1}) = \lceil a_{i-1} \rceil. \end{aligned} \tag{16}$$

Last, denote

$$r(t) = \frac{1 - \eta^{t+1}}{\eta^{t-1}(1 - \eta)^2}, \quad t \in [1, +\infty);$$

the function  $r$  is increasing, as one can check easily by writing it in the form

$$r(t) = \frac{1}{(1 - \eta)^2} \left( \eta e^{t \log \frac{1}{\eta}} - \eta^2 \right).$$

**Lemma 15.** *Let  $x \geq r(q)$  for some integer  $q \geq 1$ . Then, for every  $i = 1, \dots, q$ , the following inequalities hold*

$$\eta^{i+1} x \leq a_i \leq \eta^i x \leq a_{i-1} \leq b_i \leq \eta^{i-1} x \leq b_{i-1}.$$

*Proof.*

(a) That  $a_{i-1} \leq b_i$  follows from (ii) of Lemma 14, since, by the above definitions and putting  $t = (f^{i-1} \circ \phi)(x)$ ,

$$a_{i-1} = \psi(t) \leq f(t) = b_i.$$

(b) Now we prove that  $\eta^{i-1}x \leq b_{i-1}$ , i.e.  $\psi^{i-1}(x) \leq f^{i-1} \circ \phi(x)$ , which follows from

$$\psi^{i-1}(x) \underbrace{\leq}_{\text{Lemma 14,(ii)}} f^{i-1}(x) \underbrace{\leq}_{\text{Lemma 14,(i)}} f^{i-1} \circ \phi(x),$$

(since  $f^m$  is non-decreasing for every  $m$ ).

(c) We prove that  $b_i \leq \eta^{i-1}x$ , which means  $(f^i \circ \phi)(x) \leq \eta^{i-1}x$ .

From Lemma 14 (iii) we know that

$$f^i(\phi(x)) \leq \eta^i x + \sum_{k=0}^i \eta^k$$

and the inequality  $\eta^i x + \sum_{k=0}^i \eta^k \leq \eta^{i-1}x$  is equivalent to  $x \geq r(i)$ ; thus the claim follows from  $x \geq r(q) \geq r(i)$ , recalling that  $r$  is increasing.

(d) From the preceding points (b) and (c) we have

$$b_i \leq \eta^{i-1}x \leq b_{i-1}$$

and multiplying by  $\eta$  we get

$$a_i = \eta b_i \leq \eta^i x \leq \eta b_{i-1} = a_{i-1}.$$

(e) In order to show that  $\eta^{i+1}x \leq a_i$  it suffices to multiply by  $\eta$  each side of the inequality (already proved)  $\eta^i x \leq b_i$ .

The proof is complete. □

**Lemma 16.** *Let  $\xi \in (0, 1)$  and  $\eta \in (\xi, 1)$  be fixed; assume that  $x \geq r(q)$  for some integer  $q \geq \frac{\log \xi}{\log \eta}$ . Then the set  $\{i \geq 0 : a_i > \xi x\}$  is a finite interval of integers  $\{0, 1, \dots, d-1\}$  for some integer  $d$  with  $1 \leq d \leq q$ .*

*Proof.* From Lemma 15 we know that the sequence  $\{a_i, i = 0, \dots, q\}$  is non-increasing. Moreover  $a_0 = \eta \lceil x \rceil > \xi x$ , while  $a_q \leq \eta^q x < \xi x$ , since  $q \geq \frac{\log \xi}{\log \eta}$  by the assumption. This proves the statement. □

Let  $\xi \in (0, 1)$  and  $\eta \in (\xi, 1)$  be fixed and assume that  $x \geq r(q)$  for some integer  $q \geq \frac{\log \xi}{\log \eta}$ . From now on, by the symbol  $d$  we denote the first integer such that  $a_d \leq \xi x$ , i.e.  $d$  verifies

$$a_d \leq \xi x < a_{d-1}; \tag{17}$$

further we recall that  $b_0 = \lceil x \rceil \geq x$ . Thus

$$(\xi x, x] \subseteq \bigcup_{i=0}^d (a_i, b_i].$$

Motivated by these remarks, we can give the following

**Definition 17.** By the  $\eta$ -covering of  $(\xi x, x]$  we mean the family of intervals

$$\{(a_i, b_i], i = 0, \dots, d\},$$

constructed as shown above.

## 6.2 Some properties of the $\eta$ -covering of $(\xi x, x]$

**Lemma 18.** *Let  $\xi \in (0, 1)$  and  $\eta \in (\xi, 1)$  be fixed; assume that  $x \geq \frac{1-\eta^2\xi}{\xi(1-\eta)^2}$ . Let*

$$\{(a_i, b_i], i = 0, 1, \dots, d\}$$

*be the  $\eta$ -covering of  $(\xi x, x]$ . Then  $d \leq \lceil \frac{\log \xi}{\log \eta} \rceil$ .*

*Proof.* In the proof we denote  $\lceil \frac{\log \xi}{\log \eta} \rceil =: q$  for simplicity. Notice that

$$x \geq \frac{1-\eta^2\xi}{\xi(1-\eta)^2} = r\left(\frac{\log \xi}{\log \eta} + 1\right) > r(q).$$

Hence Lemma 15 is in force for the integers  $1, \dots, q$ . Applying this lemma with  $i = q$  and by the definition of the ceiling function, we find

$$a_q \leq \eta^q x \leq \xi x.$$

By the definition of  $d$ , this relation says that  $d \leq q$ . □

*Remark 3.* Actually, we can prove even more, precisely that  $d \in \{\lfloor \frac{\log \xi}{\log \eta} \rfloor - 1, \lfloor \frac{\log \xi}{\log \eta} \rfloor\}$ . The proof of this fact is rather complicated, and is postponed in the last Section 7. Here Lemma 18 will be sufficient for our scopes.

**Lemma 19.** *Let  $\xi \in (0, 1)$  and  $\eta \in (\xi, 1)$  be fixed. Denote*

$$M = M(\xi, \eta) := \frac{1-\eta^2\xi}{\xi(1-\eta)^2} \vee \frac{\frac{\log \xi}{\log \eta} + 2}{\eta^2\xi(1-\eta)}.$$

*Let  $x \geq M$  and let  $\{(a_i, b_i], i = 0, 1, \dots, d\}$  be the  $\eta$ -covering of  $(\xi x, x]$ .*

$$\sum_{i=0}^d (b_i - a_i) \leq (1 - \xi\eta^3)x.$$

*Proof.* Once more by Lemma 15, for every  $i$  the two intervals  $(a_{i-1}, b_{i-1}]$  and  $(a_i, b_i]$  overlap on the interval  $(a_{i-1}, b_i]$ , the length of which is  $b_i - a_{i-1} = \lceil a_{i-1} \rceil - a_{i-1} \leq 1$ , due to (16). Hence (recall that  $b_0 = \lceil x \rceil$ )

$$\sum_{i=0}^d (b_i - a_i) \leq b_0 - a_d + d = \lceil x \rceil - a_d + d \leq x + 1 - a_d + d \leq x + 1 - \eta^{d+1}x + d,$$

where we have used the left-hand inequality in Lemma 15. Continuing and using Lemma 18, we find

$$x + 1 - \eta^{d+1}x + d = (1 - \eta^{d+1})x + d + 1 \leq (1 - \eta^2\xi)x + d + 1 \leq (1 - \eta^2\xi)x + \frac{\log \xi}{\log \eta} + 2 \leq (1 - \eta^3\xi)x,$$

where the first and second inequalities come from  $d \leq \frac{\log \xi}{\log \eta} + 1$  (by Lemma 18), and the last one holds since  $x \geq M$ . □



### 6.3 Use of the $\eta$ -covering of $(\xi x, x]$

Now we are in a position to prove Theorem 13.

First notice the obvious relations

$$\begin{aligned} \liminf_{\eta \rightarrow 1^-} \left( \limsup_n \frac{F(n) - F(\eta n)}{(1 - \eta)n} \right) &\leq \limsup_{\eta \rightarrow 1^-} \left( \limsup_n \frac{F(n) - F(\eta n)}{(1 - \eta)n} \right) \\ &\leq \sup_{\eta \in (0,1)} \left( \limsup_n \frac{F(n) - F(\eta n)}{(1 - \eta)n} \right) \leq \sup_{\xi \in (0,1)} \left( \limsup_x \frac{F(x) - F(\xi x)}{(1 - \xi)x} \right); \end{aligned}$$

hence, in order to prove Theorem 13, it suffices to show that

**Proposition 20.** *We have*

$$\liminf_{\eta \rightarrow 1^-} \left( \limsup_{n \rightarrow \infty} \frac{F(n) - F(\eta n)}{(1 - \eta)n} \right) \geq \sup_{\xi \in (0,1)} \left( \limsup_{x \rightarrow \infty} \frac{F(x) - F(\xi x)}{(1 - \xi)x} \right).$$

*Proof.* We shall use the following Lemma, the proof of which is postponed at the end.

**Lemma 21.** *Let  $\xi \in (0, 1)$  and  $\epsilon \in (0, 1)$  be fixed. Then there exists  $\delta = \delta(\xi, \epsilon)$  with the following property: for every  $\eta \in (\delta, 1)$  there exists  $M = M(\xi, \epsilon, \eta)$  such that for every  $x > M$  there exists an integer  $n = n(\xi, \epsilon, \eta, x)$*

$$\frac{F(n) - F(\eta n)}{(1 - \eta)n} \geq \frac{F(x) - F(\xi x)}{(1 - \xi)x} (1 - \epsilon).$$

Now, in order to prove Proposition 20, observe that in Lemma 21 the integer  $n$  depends on  $x$ , so here we denote it by  $n_x$ . Thus, passing to the limsup as  $x \rightarrow \infty$  in the relation of Lemma 21 we obtain that, for any fixed  $\xi$  and  $\epsilon$ , there exists  $\delta(\xi, \epsilon)$  such that, for every  $\eta \in (\delta(\xi, \epsilon), 1)$ ,

$$\limsup_{x \rightarrow \infty} \frac{F(n_x) - F(\eta n_x)}{(1 - \eta)n_x} \geq \limsup_{x \rightarrow \infty} \frac{F(x) - F(\xi x)}{(1 - \xi)x} (1 - \epsilon).$$

Consequently we have also

$$\limsup_{n \rightarrow \infty} \frac{F(n) - F(\eta n)}{(1 - \eta)n} \geq \limsup_{x \rightarrow \infty} \frac{F(x) - F(\xi x)}{(1 - \xi)x} (1 - \epsilon).$$

Denote provisorily

$$A(\xi) = \limsup_{x \rightarrow \infty} \frac{F(x) - F(\xi x)}{(1 - \xi)x}; \quad B(\eta) = \limsup_{n \rightarrow \infty} \frac{F(n) - F(\eta n)}{(1 - \eta)n}.$$

Then we have that, for any fixed  $\xi$  and  $\epsilon$ , there exists  $\delta(\xi, \epsilon)$  such that, for every  $\eta \in (\delta(\xi, \epsilon), 1)$ ,

$$B(\eta) \geq A(\xi)(1 - \epsilon);$$

hence, for every  $\xi$  and  $\epsilon$ ,

$$\liminf_{\eta \rightarrow 1^-} B(\eta) \geq A(\xi)(1 - \epsilon),$$

and now the statement follows by the arbitrariness of  $\xi$  and  $\epsilon$ .

This concludes the proof of Proposition 20. □

It remains to give the proof of Lemma 21. To this extent, we need the following

**Lemma 22.** *Let  $\xi \in (0, 1)$  and  $\eta \in (\xi, 1)$  be fixed. Then there exists  $M = M(\xi, \eta) > 0$  such that, for every  $x > M$ , there exists an integer  $n = n(\xi, x, \eta)$  that verifies the inequality*

$$\frac{F(n) - F(\eta n)}{(1 - \eta)n} \geq \frac{F(x) - F(\xi x)}{(1 - \xi\eta^3)x}.$$

*Proof of Lemma 22.* Let  $M$  be as in Lemma 19. If  $x > M$ , the  $\eta$ -covering of the interval  $(\xi x, x]$ ,  $\{(a_i, b_i], i = 0, \dots, d\}$  is such that

$$\sum_i (b_i - a_i) \leq (1 - \xi\eta^3)x.$$

By Lemma 5(ii), there exists  $j \in \{0, \dots, d\}$  such that

$$\frac{F(b_j) - F(a_j)}{b_j - a_j} \geq \frac{F(x) - F(\xi x)}{\sum_i (b_i - a_i)}.$$

By construction of an  $\eta$ -covering,  $b_j$  is an integer and  $a_j = \eta b_j$ ; thus, if we call  $b_j = n$ , we have obtained

$$\frac{F(n) - F(\eta n)}{(1 - \eta)n} \geq \frac{F(x) - F(\xi x)}{\sum_i (b_i - a_i)} \geq \frac{F(x) - F(\xi x)}{(1 - \xi\eta^3)x}.$$

□

*Proof of Lemma 21.* Since

$$\lim_{\eta \rightarrow 1^-} \frac{1 - \xi}{1 - \xi\eta^3} = 1,$$

there exists  $\bar{\delta}(\xi, \epsilon)$  such that, for every  $\eta \in (\bar{\delta}, 1)$

$$1 - \xi\eta^3 < \frac{1 - \xi}{1 - \epsilon}.$$

Now take  $\delta(\xi, \epsilon) = \bar{\delta}(\xi, \epsilon) \vee \xi$  and apply Lemma 22 to conclude. □

## 7 On the number $d$

This Section is devoted to the proof of the announced result concerning  $d$  (see Remark 3). Precisely

**Proposition 23.** (i) *Assume that  $\frac{\log \xi}{\log \eta}$  is not an integer and*

$$x \geq \frac{1 - \eta^2 \xi}{\xi(1 - \eta)^2} \vee \frac{\lfloor \frac{\log \xi}{\log \eta} \rfloor + 2}{\xi - \eta^{\lfloor \frac{\log \xi}{\log \eta} \rfloor + 1}} \vee \frac{1}{\eta^{\lfloor \frac{\log \xi}{\log \eta} \rfloor} - \xi}. \quad (18)$$

Then  $d = \lfloor \frac{\log \xi}{\log \eta} \rfloor$ .

(ii) If  $\frac{\log \xi}{\log \eta}$  is an integer and

$$x \geq \frac{1 - \eta^2 \xi}{\xi(1 - \eta)^2} \vee \frac{\frac{\log \xi}{\log \eta} + 2}{\xi(1 - \eta)},$$

then  $d \in \{ \frac{\log \xi}{\log \eta} - 1, \frac{\log \xi}{\log \eta} \}$ .

*Proof.* In the proof we denote  $\lfloor \frac{\log \xi}{\log \eta} \rfloor =: p$  for simplicity. Notice also that Lemma 15 is applicable for  $i = 1, \dots, p + 1$ , since, in both (i) and (ii),

$$x \geq \frac{1 - \eta^2 \xi}{\xi(1 - \eta)^2} = r \left( \frac{\log \xi}{\log \eta} + 1 \right) \geq r(p + 1).$$

Last, in case (i),  $\eta^{p+1} < \xi < \eta^p$ , hence the denominators  $\xi - \eta^{p+1}$  and  $\eta^p - \xi$  appearing in (18) are strictly positive.

The trivial observation that  $\eta^m x \leq b_m$  and Lemma 14 (iii) yield for every  $m \geq 0$

$$\eta^m x \leq b_m \leq \eta^m x + m + 1,$$

and by (16), for every  $m \geq 0$

$$b_m - 1 < a_{m-1} \leq b_m.$$

Putting  $m = p$  and  $m = p + 1$  we obtain in turn

$$\eta^p x \leq b_p \leq \eta^p x + p + 1, \quad b_p - 1 < a_{p-1} \leq b_p \quad (19)$$

and

$$\eta^{p+1} x \leq b_{p+1} \leq \eta^{p+1} x + p + 2, \quad b_{p+1} - 1 < a_p \leq b_{p+1}. \quad (20)$$

Since  $x \geq \frac{p+2}{\xi - \eta^{p+1}}$ , we have, by (20) and Lemma 15,

$$a_p \leq b_{p+1} \leq \eta^{p+1} x + p + 2 \leq \xi x, \quad (21)$$

which says that  $d \leq p$ .

Now we distinguish the two cases (i) and (ii).

(i) If  $\frac{\log \xi}{\log \eta}$  is not an integer, by (19),

$$a_{p-1} > b_p - 1 \geq \eta^p x - 1 > \xi x,$$

Then, by (21),

$$a_p \leq \xi x < a_{p-1}, \quad (22)$$

implying that  $d = p$ .

(ii) If  $\frac{\log \xi}{\log \eta}$  is an integer, then  $\xi x = \eta^p x$  and, by Lemma 15,  $\eta^p x \leq a_{p-1}$ .

If  $\xi x (= \eta^p x) < a_{p-1}$ , we have the same inequalities as before in (22) and  $d = p$  again. Otherwise  $\eta^p x (= \xi x) = a_{p-1}$ . We prove that in this case  $d = p - 1$ , which means that

$$a_{p-1} \leq \xi x < a_{p-2}. \quad (23)$$

The proof of (23) follows from

**Lemma 24.** Assume that for some  $i \geq 0$  we have  $\eta^{i+1}x = a_i$ . Then

$$\eta = \frac{b_1}{b_0}, \quad x = b_0, \quad a_j = \frac{b_1^{j+1}}{b_0^j}, \quad j = 0, \dots, i.$$

*Proof* (of Lemma 24). First, recall the definition of every  $a_j$ , i.e.

$$a_j = \eta b_j. \tag{24}$$

Now, from  $a_i = \eta^{i+1}x$  we deduce by (24) that  $b_i = \eta^i x$ . Since  $\eta^i x \leq a_{i-1} \leq b_i$  (by Lemma 15), we obtain that  $a_{i-1} = b_i$ , whence  $\eta b_i = \eta a_{i-1}$  and (by (24))  $a_i = \eta a_{i-1}$ . Repeating this argument we get that  $a_j = \eta a_{j-1}$  for every  $j = 1, \dots, i$ ; from  $a_1 = \eta a_0$  and (24) (applied to  $a_0$  and  $a_1$ ) we deduce that  $b_1 = \eta b_0$ , hence  $\eta = \frac{b_1}{b_0}$ , which implies that

$$a_0 = \eta b_0 = \frac{b_1}{b_0} \cdot b_0 = b_1, \quad a_1 = \eta a_0 = \frac{b_1}{b_0} \cdot b_1 = \frac{b_1^2}{b_0}, \dots, a_j = \frac{b_1^{j+1}}{b_0^j}, \quad j = 2, \dots, i;$$

further

$$x = \frac{a_i}{\eta^{i+1}} = \frac{b_1^{i+1}}{b_0^i} \cdot \frac{b_0^{i+1}}{b_1^{i+1}} = b_0.$$

□

We go back to the proof of Proposition 23; the left inequality in (23) holds obviously (it is nothing but the equality  $a_{p-1} = \eta^p x = \xi x$ ). Concerning the right one, by the same equality and Lemma 24, it is equivalent to  $\frac{b_1^p}{b_0^{p-1}} < \frac{b_1^{p-1}}{b_0^{p-2}}$  and, after simplification, to  $\frac{b_1}{b_0} < 1$ , which is true since it is nothing but  $\eta < 1$  (by Lemma 24 again).

□

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