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Morse Sequences: A simple approach to discrete Morse theory

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Abstract

In this paper, we develop the notion of a Morse sequence, which provides an alternative approach to discrete Morse theory, and which is both simple and effective. A Morse sequence on a finite simplicial complex is a sequence composed solely of two elementary operations, that is, expansions (the inverse of a collapse), and fillings (the inverse of a perforation). In a dual manner, a Morse sequence may be obtained by considering only collapses and perforations. Such a sequence is another way to represent the gradient vector field of an arbitrary discrete Morse function. To each Morse sequence, we assign a reference map and an extension map. A reference map associates a set of critical simplexes to each simplex of a given complex, and an extension map associates a set of simplexes to each critical simplex. By considering the boundary of each critical simplex, we obtain a chain complex from these maps, which corresponds precisely to the Morse complex. We show that, when restricted to homology, an extension map is the inverse of a reference map. Also we show that these two maps allow us to recover directly the isomorphism theorem between the homology of an object and the homology of its Morse complex. At last, we introduce the notion of a flow complex, which is based solely on extension maps. We prove that this notion is equivalent to the classical one based on gradient flows.

Keywords: Discrete Morse theory, Simplicial complex, Expansions and collapses, Fillings and perforations, Homology and cohomology

1 Introduction

Discrete Morse theory is a combinatorial adaptation of classical Morse theory, applicable to a wide range of topological problems. Initially developed by Robin Forman

[17, 19], this theory was first introduced through the use of discrete Morse functions that identify particular cells, known as critical cells, which encapsulate the essential topological characteristics of a given object. In more recent treatments [28, 29], discrete Morse theory is presented through the notion of a combinatorial matching, which is equivalent to the gradient vector field of a discrete Morse function.

In this paper ¹, we propose an approach where, instead of a Morse function or a gradient vector field, a sequence of elementary operators is used for a simple representation of an object. This sequence, that we called a *Morse sequence*, is composed solely of two elementary operations, that is, expansions (the inverse of a collapse), and fillings (the inverse of a perforation). These operations correspond exactly to the ones introduced by Henry Whitehead [37].

After some basic definitions (Section 2), we introduce Morse sequences and give two meaningful examples (Sections 3). We provide two computational schemes for extracting a Morse sequence from a given simplicial complex. Also, we introduce a notion of equivalence between two Morse sequences, this notion is based on the *gradient vector field* of a Morse sequence.

In section 4, we present *reference maps*, which are maps that associate a set of critical simplexes to each simplex. These maps allow adding some crucial information to Morse sequences. In Section 5, we show that reference maps may be fully characterized by counting the number of gradient paths between each simplex and each critical face.

By simply considering the reference map of the boundary of each critical simplex, we obtain the *critical complex* of a Morse sequence (Section 6), which corresponds precisely to the so-called Morse complex.

In Section 7, we introduce *arranged sequences* which are obtained through a rearrangement of Morse sequences. We also introduce the related notions of *lower and upper skeletons* of a Morse sequence; these skeletons explicitly appear in arranged sequences.

We then define *extension maps* (Section 8) which associate a set of simplexes to each critical simplex. We show that, when restricted to homology, an extension map is the inverse of a reference map. Also, we show that reference and extension maps allow us to recover directly an isomorphism between the homology of an object and the homology of its critical complex.

In Section 9, we propose a definition of an *extension complex*, which is based solely on extension maps. Again, we show that there is an isomorphism between the homology of an extension complex and the homology of a critical complex. Furthermore, we prove that this notion is equivalent to the classical one based on gradient flows.

After the conclusion, we give two appendices. In the first one, we emphasize that a Morse sequence may represent the gradient vector field of any arbitrary discrete Morse function (Appendix A). In the second one, we make clear the relation between Morse sequences and different kinds of Morse functions (Appendix B).

Note that the paper is self-contained, the only external result which is used is Theorem 31 from [18], located at the very end of the paper.

¹A preliminary version of some parts of this paper appears in [5] and [8]

2 Simplicial complexes, homology, and cohomology

2.1 Simplicial complexes

Let K be a finite family composed of non-empty finite sets. The family K is a (*simplicial*) *complex* if $\sigma \in K$ whenever $\sigma \neq \emptyset$ and $\sigma \subseteq \tau$ for some $\tau \in K$.

An element of a simplicial complex K is a *simplex of K* , or a *face of K* . A *facet of K* is a simplex of K that is maximal for inclusion. The *dimension* of $\sigma \in K$, written $\dim(\sigma)$, is the number of its elements minus one. If $\dim(\sigma) = p$, we say that σ is a *p -simplex*. The *dimension of K* , written $\dim(K)$, is the largest dimension of its simplices, the *dimension of \emptyset* , the void complex, being defined to be -1 .

We recall the definitions of the collapses/expansions operators [37].

Let K, L be simplicial complexes. Let $\sigma, \tau \in K$. The couple (σ, τ) is a *free pair for K* , if τ is the only face of K that contains σ . Thus, τ is necessarily a facet of K . The *dimension of a free pair (σ, τ)* is defined by $\dim(\sigma, \tau) = \dim(\tau)$. Thus, $\dim(\sigma, \tau) = \dim(\sigma) + 1$. If (σ, τ) is a free pair for K , then $L = K \setminus (\sigma, \tau)$ is an *elementary collapse of K* , and K is an *elementary expansion of L* .

We say that K *collapses onto L* , or that L *expands onto K* , if there exists a sequence $\langle K = K_0, \dots, K_k = L \rangle$, such that K_i is an elementary collapse of K_{i-1} , $i \in [1, k]$; we say that such a sequence is a *collapse sequence (from K to L)*. The complex K is *collapsible* if K collapses onto a *vertex*, that is, onto a complex of the form $\{\{a\}\}$.

2.2 Chain and cochain complexes

A *chain complex* (C_p, d_p) is a sequence of vector spaces C_p , with $p \in \mathbf{Z}$, connected by linear transformations $d_p : C_p \rightarrow C_{p-1}$, called *boundary operators* with the property that $d_{p-1} \circ d_p = 0$. Each element of C_p is a *p -chain*.

In a dual way, a *cochain complex* (C^p, d^p) is a sequence of vector spaces C^p , with $p \in \mathbf{Z}$, connected by linear transformations $d^p : C^p \rightarrow C^{p+1}$ called *coboundary operators* with the property that $d^{p+1} \circ d^p = 0$. Each element of C^p is a *p -cochain*.

Let (C_p, d_p) and (C^p, d^p) be a chain and a cochain complex. We define the four vector spaces:

- the set $Z_p(C)$ of *p -cycles of (C_p, d_p)* , $Z_p(C)$ is the kernel of d_p ;
- the set $B_p(C)$ of *p -boundaries of (C_p, d_p)* , $B_p(C)$ is the image of d_{p+1} ;
- the set $Z^p(C)$ of *p -cocycles of (C^p, d^p)* , $Z^p(C)$ is the kernel of d^p ;
- the set $B^p(C)$ of *p -coboundaries of (C^p, d^p)* , $B^p(C)$ is the image of d^{p-1} .

By the definition of (co)boundary operators, all (co)boundaries are (co)cycles. We obtain homology and cohomology with the following quotient vector spaces:

- $H_p(C) = Z_p(C) \setminus B_p(C)$, which is the p^{th} *homology vector space of (C_p, d_p)* ;
- $H^p(C) = Z^p(C) \setminus B^p(C)$, which is the p^{th} *cohomology vector space of (C^p, d^p)* .

An element h in $H_p(C)$ is such that $h = z + B_p(C)$ for some $z \in Z_p(C)$. We write $h = [z]$, which is the *homology class of the cycle z* .

Similarly, an element h in $H^p(C)$ is such that $h = z + B^p(C)$ for some $z \in Z^p(C)$. We also write $h = [z]$, which is the *cohomology class of the cocycle z* .

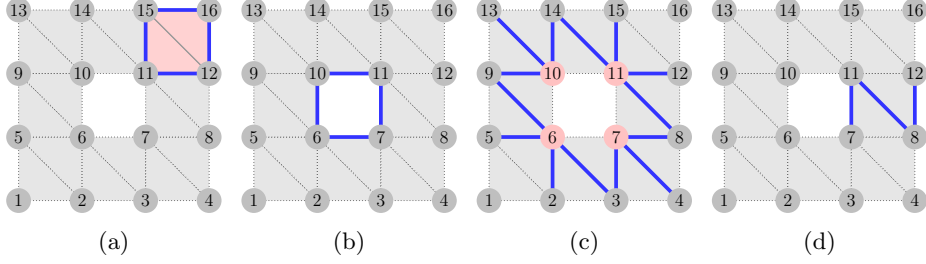


Fig. 1: An annulus, with various cycles and cocycles. See text for details.

2.3 Homology and cohomology modulo 2

Let K be a simplicial complex. We write $K^{(p)}$ for the set of all p -simplexes of K , and $K[p]$ for the set composed of all subsets of $K^{(p)}$. Thus $K^{(p)} = \emptyset$ and $K[p] = \{\emptyset\}$ whenever $p < 0$ or $p > \dim(K)$. Each element of $K[p]$ is a p -chain of K . The symmetric difference of two elements of $K[p]$ endows $K[p]$ with the structure of a vector space over the field $\mathbb{Z}_2 = \{0, 1\}$. The set $K^{(p)}$ is a basis for this vector space. Within this structure, a chain $c \in K[p]$ may be written as a sum $\sum_{\sigma \in c} \sigma$, the chain $c = \emptyset$ being written 0. The sum of two chains is obtained using the modulo 2 arithmetic. See Chapter 8 of [21] and see [26] for a general presentation of modulo 2 homology.

If $S \subseteq K$, we write $S^{(p)} = \{\nu \in S \mid \nu \in K^{(p)}\}$ and $S[p] = \{c \subseteq S^{(p)}\}$.

As we are dealing with a finite simplicial complex, boundary and coboundary operators can be both defined as operators on $K[p]$. If $\sigma \in K^{(p)}$, we set:

$$\partial(\sigma) = \{\tau \in K^{(p-1)} \mid \tau \subset \sigma\} \text{ and } \delta(\sigma) = \{\tau \in K^{(p+1)} \mid \sigma \subset \tau\}.$$

The *boundary operator* $\partial_p : K[p] \rightarrow K[p-1]$ is such that, for each $c \in K[p]$, $\partial_p(c) = \sum_{\sigma \in c} \partial(\sigma)$, with $\partial_p(\emptyset) = 0$.

The *coboundary operator* $\delta^p : K[p] \rightarrow K[p+1]$ is such that, for each $c \in K[p]$, $\delta^p(c) = \sum_{\sigma \in c} \delta(\sigma)$, with $\delta^p(\emptyset) = 0$.

For each $p \in \mathbf{Z}$, we have $\partial_p \circ \partial_{p+1} = 0$ and $\delta^{p+1} \circ \delta^p = 0$.

Thus, $(C_p, d_p) = (K[p], \partial_p)$ is a chain complex and $(C^p, d^p) = (K[p], \delta^p)$ is a cochain complex. We write $Z_p(K)$, $B_p(K)$, $H_p(K)$ for $Z_p(C)$, $B_p(C)$, $H_p(C)$, and $Z^p(K)$, $B^p(K)$, $H^p(K)$ for $Z^p(C)$, $B^p(C)$, $H^p(C)$.

Let $\beta_p(K) = \dim(H_p(K))$ and $\beta^p(K) = \dim(H^p(K))$. We have $\beta_p(K) = \beta^p(K)$ (See [15, Sec. V.1]). The number $\beta_p(K) = \beta^p(K)$ is the p^{th} Betti number (mod 2) of K . Informally, the p^{th} Betti number of K corresponds to the number of “ p -dimensional holes” of the complex K .

Fig. 1 depicts an annulus, with various cycles and cocycles, coloured in blue. In Fig. 1.a, we see a 1-cycle that is the 1-boundary of the two pink triangles. In Fig. 1.b, we have a 1-cycle that is not a 1-boundary. Such a cycle detects a “hole” by “contouring” it. In Fig. 1.c, we see a 1-cocycle which is the 1-coboundary of the four pink points. In Fig. 1.d, we have a 1-cocycle that is not a 1-coboundary. Such a cocycle detects a “hole” by “cutting” the annulus.

3 Morse sequences

Let us start first with the definition of perforations and fillings.

Let K, L be simplicial complexes. If $\nu \in K$ is a facet of K and if $L = K \setminus \{\nu\}$, we say that L is an *elementary perforation* of K , and that K is an *elementary filling* of L .

These transformations were introduced by Whitehead in a seminal paper [37]. Combined with collapses and expansions, it has been shown that we obtain four operators that correspond to the homotopy equivalence between two simplicial complexes (Th. 17 of [37]). See also [4] which provides another kind of equivalence based on a variant of these operators.

We introduce the notion of a ‘‘Morse sequence’’ by simply considering expansions and fillings of a simplicial complex.

Definition 1. *Let K be a simplicial complex. A Morse sequence (on K) is a sequence $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ of simplicial complexes such that, for each $i \in [1, k]$, K_i is either an elementary expansion or an elementary filling of K_{i-1} . We also write $\vec{W}(K)$ for a Morse sequence \vec{W} on K .*

Let $\vec{W} = \langle K_0, \dots, K_k \rangle$ be a Morse sequence. For each $i \in [1, k]$, let k_i be such that:

- If K_i is an elementary filling of K_{i-1} and $K_i = K_{i-1} \cup \{\nu\}$, then $\kappa_i = \nu$. We say that the face κ_i is *critical* for \vec{W} .
- If K_i is an elementary expansion of K_{i-1} and $K_i = K_{i-1} \cup \{\sigma, \tau\}$, with $\sigma \subseteq \tau$, then $\kappa_i = (\sigma, \tau)$. The pair κ_i is *regular* for \vec{W} , the face σ is *lower regular* for \vec{W} , and the face τ is *upper regular* for \vec{W} .

We write $\diamond \vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$, and we say that $\diamond \vec{W}$ is a *simplex-wise (Morse) sequence*.

We will use also the four following notations:

- $\widehat{W} = \{\nu \in K \mid \nu \text{ is critical for } \vec{W}\}$,
- $\check{W} = \{(\sigma, \tau) \mid (\sigma, \tau) \text{ is a regular pair for } \vec{W}\}$,
- $\overline{W} = \{\nu \in K \mid \nu \text{ is upper regular for } \vec{W}\}$,
- $\underline{W} = \{\nu \in K \mid \nu \text{ is lower regular for } \vec{W}\}$.

Thus, $\diamond \vec{W}$ is a sequence of faces of \widehat{W} and pairs of \check{W} . Clearly, \vec{W} and $\diamond \vec{W}$ are two equivalent forms. That is, one form is determined by the other. Also, we observe that $K = \widehat{W} \cup \underline{W} \cup \overline{W}$. This partition of a complex corresponds to a classification which is often associated to a matching in the Hasse diagram of a complex or a poset, see for example Section 10.2 of [29].

Observe that, if $\vec{W} = \langle K_0, \dots, K_k \rangle$ is a Morse sequence, with $k \geq 1$, then K_1 is necessarily a filling of \emptyset . Thus, K_1 is necessarily a vertex. That is, K_1 is made of a single 0-simplex that is critical for \vec{W} .

There are several ways to obtain a Morse sequence \vec{W} from a given complex K . The two following schemes are basic ones to achieve this goal, see [3, 33, 24, 20, 14] for similar schemes given in the context of discrete Morse theory:

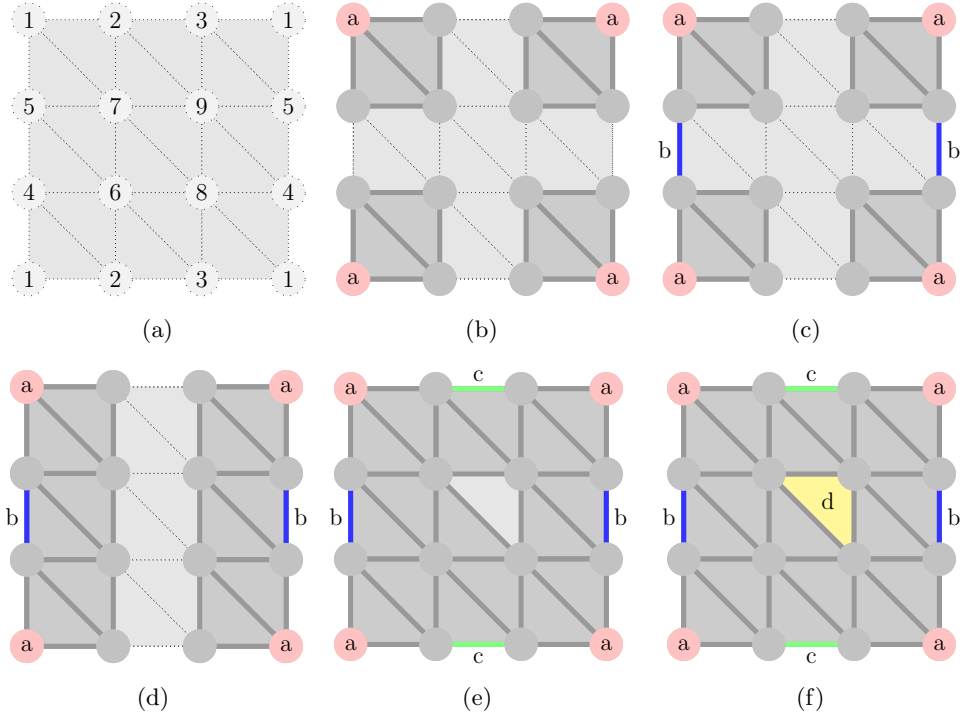


Fig. 2: A Morse sequence on the torus. (a) A triangulation, points with the same label are identified. (b) The sequence begins with the critical 0-simplex a . Elementary expansions are added to the sequence until we obtain a maximal expansion from a . (c) The critical 1-simplex b is added to the sequence. (d) A maximal expansion from b is done. (e) The second critical 1-simplex c is added, and a maximal expansion from c is done. (f) The critical 2-simplex d is added.

1. *The increasing scheme.* We build \vec{W} from the left to the right. Starting from \emptyset , we obtain K by iterative expansions and fillings. We say that this scheme is *maximal* if we make a filling only if no expansion can be made.
2. *The decreasing scheme.* We build \vec{W} from the right to the left. Starting from K , we obtain \emptyset by iterative collapses and perforations. We say that this scheme is *maximal* if we make a perforation only if no collapse can be made.

Fig. 2 presents an example of a Morse sequence on a torus. The sequence has been obtained by following the maximal increasing scheme. Using the same scheme, an example of a Morse sequence on the dunce hat is given Fig. 3. The dunce hat is a well-known example of a contractible but not collapsible object, see [38]. Contractibility can be seen by noting that the dunce hat embeds in a 3-ball which both collapses onto a vertex and onto the dunce hat. We observe that:

- The Morse sequence on the torus leads to two critical 1-faces and one critical 2-face. These numbers correspond exactly to the numbers of 1D and 2D “holes” of the torus.

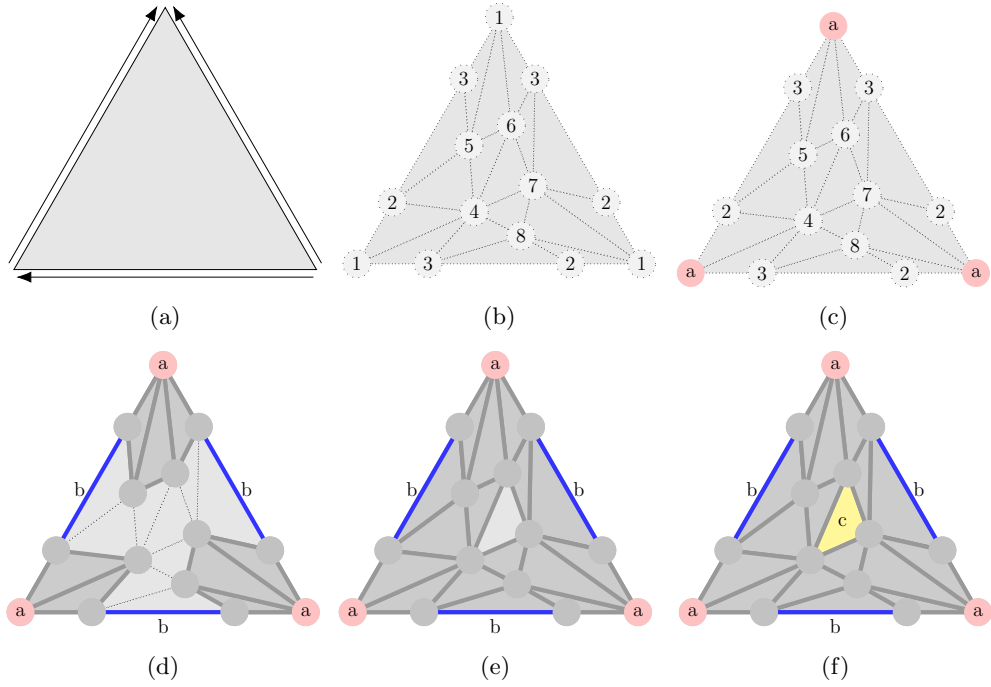


Fig. 3: A Morse sequence on the dunce hat. (a) the dunce hat, the three edges of the triangle have to be identified with the arrows. (b) A triangulation of the dunce hat. (c) The sequence begins with the critical 0-simplex a . (d) A maximal expansion from a is done, then the 1-critical simplex b is added. (e) A maximal expansion from b . (f) The critical 2-simplex c is added.

- The dunce hat has no 1D or 2D “hole”, nevertheless the Morse sequence leads to one critical 1-face and one critical 2-face. These numbers are the smallest we can obtain. In this paper, we use the torus and the dunce hat as our canonical examples, as they exemplify two contrasting cases which arise in the context of discrete Morse theory.

Remark 1. The above maximal increasing and decreasing schemes are two methods which try to minimize the number of critical simplexes of a Morse sequence. By the results given in [27, 30], this problem is NP-hard. Therefore, these methods do not, in general, give optimal results.

Remark 2. Let $\vec{W} = \langle K_0, \dots, K_k \rangle$ be a Morse sequence, let $\diamond \vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$, and let $\kappa_i, \kappa_j, j > i$, be two consecutive critical faces of $\diamond \vec{W}$, that is, $\kappa_{i+1}, \dots, \kappa_{j-1}$ are regular pairs. Then, as a direct consequence of the definition of a Morse sequence, the complex K_{j-1} collapses onto K_i . This property is the core of a fundamental theorem, called *the collapse theorem*, which makes the link between the basic definitions of discrete Morse theory and discrete homotopy (See Theorem 3.3 of [16] and Theorem

4.27 of [35]). In a certain sense, we can say that Morse sequences provide an introduction to discrete Morse theory by starting from this property.

Remark 3. Any Morse sequence \vec{W} on K is a *filtration on K* , that is a sequence of nested complexes $\langle \emptyset = K_0, \dots, K_k = K \rangle$ such that, for $i \in [0, k - 1]$, we have $K_i \subseteq K_{i+1}$; see [13]. Also any *simplex-wise filtration on K* is a special case of a Morse sequence where, for $i \in [0, k - 1]$, $K_{i+1} \setminus K_i$ is made of a single simplex. That is, a simplex-wise filtration is a Morse sequence which is made solely of fillings; all faces of K are critical for such a sequence.

Definition 2. Let $\vec{W}(K)$ be a Morse sequence. The gradient vector field of \vec{W} is the set composed of all regular pairs for \vec{W} , that is, the set \vec{W} . We say that two Morse sequences $\vec{V}(K)$ and $\vec{W}(K)$ are equivalent if they have the same gradient vector field.

Let $\vec{V}(K)$ and $\vec{W}(K)$ be two equivalent sequences. Clearly, since $\vec{V} = \vec{W}$, we have $\widehat{V} = \widehat{W}$. It should be noted that the converse of this proposition is, in general, not true. For example, let us consider a collapsible complex K . There exists a Morse sequence \vec{W} on K which has a single critical simplex κ . The simplex κ is a vertex, and \vec{W} may be obtained by a collapse sequence onto κ . Now, suppose that $\dim(K) \geq 2$. By considering another collapse sequence onto κ , we see that we may obtain a Morse sequence \vec{V} , which has the same critical point as \vec{W} , and which is not equivalent to \vec{W} .

To conclude this section, it is worth mentioning that there is no loss of generality when using Morse sequences as a presentation of acyclic vector fields and Morse functions. We can prove that there is an equivalence between the gradient vector field of an arbitrary Morse function and the gradient vector field of a Morse sequence, and the same holds true for an arbitrary acyclic vector field: see Appendix A and B which give more details about these correspondences. In fact, a Morse sequence may be simply seen as a total ordering that respects the partial ordering induced by a Morse function or the one induced by an acyclic vector field. See Definition 5.38 of [35] which introduces some linear extensions of a poset that respect these partial orderings; see also Theorem 11.9 of [29] which provides an equivalence between a linear extension and an acyclic vector field.

4 The reference and co-reference maps

A reference or a co-reference map is a linear map which assigns, to each simplex of a simplicial complex, a set of critical simplexes. These maps provide Morse sequences with a structure, they will be the main ingredient of this paper.

If $\vec{W}(K)$ is a Morse sequence, we write $\widehat{W}[p] = \{c \in K[p] \mid c \subseteq \widehat{W}\}$. Thus, $\widehat{W}[p]$ is a vector subspace of $K[p]$. Also, in the sequel of the paper, we often omit the subscript or the superscript p if the variable p is clear from the context.

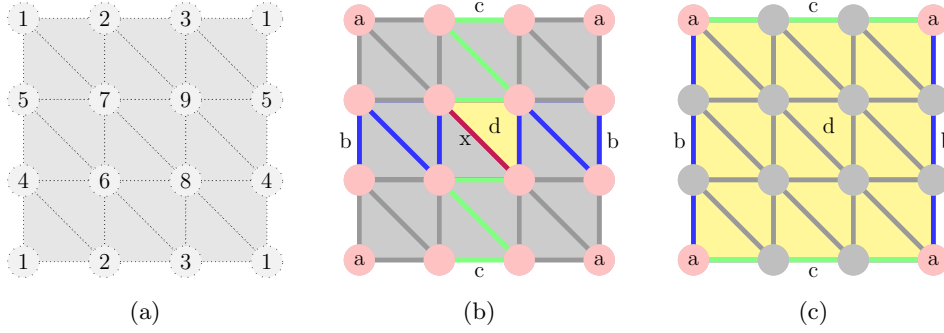


Fig. 4: The reference map (b) and the co-reference map (c) of the Morse sequence of Fig. 2. See text for details.

Definition 3. Let $\vec{W}(K)$ be a Morse sequence. Let Υ_p, λ_p be two linear maps:

$$\Upsilon_p : K[p] \rightarrow \widehat{W}[p] \text{ and } \lambda_p : K[p] \rightarrow \widehat{W}[p],$$

such that, for each critical simplex ν of \vec{W} , we have $\Upsilon(\nu) = \lambda(\nu) = \nu$.

We say that Υ is a reference map for \vec{W} if, for each regular pair (σ, τ) of \vec{W} , we have $\Upsilon(\tau) = 0$ and $\Upsilon(\sigma) = \Upsilon(\partial(\tau) + \sigma)$.

We say that λ is a co-reference map for \vec{W} if, for each regular pair (σ, τ) of \vec{W} , we have $\lambda(\sigma) = 0$ and $\lambda(\tau) = \lambda(\delta(\sigma) + \tau)$.

Let $\vec{W}(K) = \langle K_0, \dots, K_k \rangle$ and let $\diamond \vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$.

1) Since κ_1 is critical, we have $\Upsilon(\kappa_1) = \kappa_1$. Suppose $i \geq 2$. If κ_i is critical, then $\Upsilon(\kappa_i) = \kappa_i$. If κ_i is a regular pair (σ, τ) , then $\Upsilon(\tau) = 0$ and we observe that $\partial(\tau) + \sigma \subseteq K_{i-1}$. Thus a reference map on K may be computed by scanning the sequence $\diamond \vec{W}$ from the left to the right.

2) Also, if κ_k is critical, then $\lambda(\kappa_k) = \kappa_k$. If κ_k is a regular pair (σ, τ) , then $\lambda(\sigma) = 0$ and $\lambda(\tau) = \lambda(\tau + \tau) = 0$. Suppose $i \leq k - 1$. If κ_i is critical, then $\lambda(\kappa_i) = \kappa_i$. If $\kappa_i = (\sigma, \tau)$, then $\lambda(\sigma) = 0$ and we observe that $\delta(\sigma) + \tau \subseteq K_{i+1}$. Thus a co-reference map on K may be computed by scanning the sequence $\diamond \vec{W}$ from the right to the left. In this way, we obtain two maps Υ and λ on $K[p]$, we see that these maps are the only possible reference and co-reference maps for \vec{W} . Thus, we have the following result.

Theorem 1. Any Morse sequence admits a unique reference map and a unique co-reference map.

If Υ and λ are respectively the reference and the co-reference maps of $\vec{W}(K)$, we say that (Υ, λ) is the reference pair of $\vec{W}(K)$. If $c \in K[p]$, we say that $\Upsilon(c)$ is the reference of c and $\lambda(c)$ is the co-reference of c (for \vec{W}).

We illustrate, Fig. 4 (b), the reference map Υ of the Morse sequence of the torus depicted Fig. 2 (f). In this figure, any simplex ν in grey is such that $\Upsilon(\nu) = 0$. We have

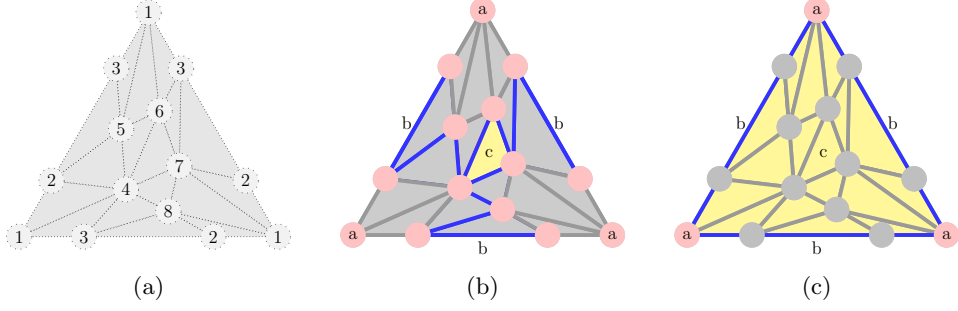


Fig. 5: The reference map (b) and the co-reference map (c) of the Morse sequence of Fig. 3. See text for details.

$\Upsilon(\nu) = a$ (in pink) for all simplexes ν of dimension 0. All the simplexes ν coloured in blue (resp. in green) are such that $\Upsilon(\nu) = b$ (resp. $\Upsilon(\nu) = c$). We have $\Upsilon(x) = b+c$ (in red) and $\Upsilon(d) = d$ (in yellow). Using the same colouring conventions, the co-reference map λ of the same Morse sequence is given Fig. 4 (c). In Fig. 5, again with these conventions, we give the reference and the co-reference maps of the Morse sequence of the dunce hat depicted Fig. 3 (f).

The following proposition is easy to prove. It provides another example of a reference and a co-reference map with the simple case where the complex K has a unique critical face, that is, when K is collapsible.

Proposition 2. Let $\vec{W}(K)$ be a Morse sequence such that $\widehat{W} = \{\kappa\}$. Thus, the simplex κ is a vertex, and K collapses onto κ . Then, for each $\nu \in K$,

- we have $\Upsilon(\nu) = \kappa$ if $\dim(\nu) = 0$, and $\Upsilon(\nu) = 0$ if $\dim(\nu) \geq 1$,
- we have $\lambda(\nu) = \kappa$ if $\nu = \kappa$, and $\lambda(\nu) = 0$ if $\nu \neq \kappa$.

Proposition 3. Let $\vec{W}(K)$ and $\vec{W}'(K)$ be two Morse sequences. Let (Υ, λ) be the reference pair of \vec{W} , and (Υ', λ') be the reference pair of \vec{W}' . If \vec{W} and \vec{W}' are equivalent, then we have $\Upsilon = \Upsilon'$ and $\lambda = \lambda'$.

Proof. Let \vec{W} and \vec{W}' be two equivalent sequences. We have $\ddot{W} = \ddot{W}'$ and $\widehat{W} = \widehat{W}'$. We set $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ and $\diamond\vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$. The simplex κ_1 is critical for both \vec{W} and \vec{W}' , thus $\Upsilon(\kappa_1) = \Upsilon'(\kappa_1)$. Let i such that $2 \leq i \leq k$. Suppose we have $\Upsilon(\nu) = \Upsilon'(\nu)$ for each $\nu \in K_{i-1}$. If $\kappa_i \in \widehat{W}$, then $\kappa_i \in \widehat{W}'$, thus we have $\Upsilon(\kappa_i) = \Upsilon'(\kappa_i) = \kappa_i$. If $\kappa_i = (\sigma, \tau)$ is in \ddot{W} , then $\Upsilon(\tau) = 0$ and $\Upsilon(\sigma) = \Upsilon(\partial(\tau) + \sigma)$. Since κ_i is in \ddot{W}' , we have $\Upsilon'(\tau) = 0$. We have $\partial(\tau) + \sigma \subseteq K_{i-1}$, thus by the induction hypothesis $\Upsilon(\sigma) = \Upsilon'(\sigma)$. Therefore, we have $\Upsilon(\nu) = \Upsilon'(\nu)$ for each $\nu \in K_i$ and we obtain $\Upsilon = \Upsilon'$. The equality $\lambda = \lambda'$ may be derived in the same way. \square

It should be noted that the converse of the previous proposition is, in general, not true. Let us consider again a collapsible complex K , with $\dim(K) \geq 2$. We can build two Morse sequences \vec{W} and \vec{W}' that are not equivalent by following two distinct

collapse sequences. By Proposition 2, these two sequences have the same reference and co-reference maps.

Let (Υ, λ) be the reference pair of $\vec{W}(K)$. Let (σ, τ) be a regular pair of \vec{W} . By linearity of the operators Υ and λ , we may write:

$$\Upsilon(\partial(\tau)) = \Upsilon(\partial(\tau) + \sigma) + \Upsilon(\sigma) \text{ and } \lambda(\delta(\sigma)) = \lambda(\delta(\sigma) + \tau) + \lambda(\tau).$$

By the definitions of Υ and λ , we obtain $\Upsilon(\partial(\tau)) = 0$ and $\lambda(\delta(\sigma)) = 0$. It means that, if $C \subseteq \overline{W}(p)$ and $D \subseteq \underline{W}(p)$ are made of a single simplex, then $\Upsilon(\partial(C)) = 0$ and $\lambda(\delta(D)) = 0$. By linearity, we obtain the following elementary result.

Proposition 4. *Let (Υ, λ) be the reference pair of a Morse sequence $\vec{W}(K)$. If $C \subseteq \overline{W}(p)$, then $\Upsilon(\partial(C)) = 0$. If $D \subseteq \underline{W}(p)$, then $\lambda(\delta(D)) = 0$.*

5 Gradient paths and cogradient paths

Reference maps are closely related to the notion of a gradient path. We first recall the classical definition of such a path (see also [18]).

Let $\vec{W}(K)$ be a Morse sequence.

1. Let $\pi = \langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$, $k \geq 0$, be a sequence with $\sigma_i \in K^{(p)}$, $\tau_i \in K^{(p+1)}$. We say that π is a *gradient path in \vec{W}* (from σ_0 to σ_k) if, for any $i \in [0, k-1]$, the pair (σ_i, τ_i) is regular for \vec{W} and $\sigma_{i+1} \in \partial(\tau_i)$, with $\sigma_{i+1} \neq \sigma_i$. The path π is *trivial* if $k = 0$, that is, if $\pi = \langle \sigma_0 \rangle$ with $\sigma_0 \in K^{(p)}$.
2. Let $\pi = \langle \tau_0, \sigma_1, \tau_1, \dots, \sigma_k, \tau_k \rangle$, $k \geq 0$, be a sequence with $\tau_i \in K^{(p)}$, $\sigma_i \in K^{(p-1)}$. We say that π is a *cogradient path in \vec{W}* (from τ_0 to τ_k) if, for any $i \in [1, k]$, the pair (σ_i, τ_i) is regular for \vec{W} and $\tau_{i-1} \in \delta(\sigma_i)$, with $\tau_i \neq \tau_{i-1}$. The path π is *trivial* if $k = 0$, that is, if $\pi = \langle \tau_0 \rangle$ with $\tau_0 \in K^{(p)}$.

Observe that a gradient path may end at a critical face, and a cogradient path may begin at a critical face. Also note that the sequence obtained by reversing a gradient path is not a cogradient path unless the path is trivial.

A gradient vector field which may be obtained from a Morse sequence is given Figure 6 (a). The 1D (resp. 2D) regular pairs correspond to black (resp. green) arrows, the critical faces are in red. There are two gradient paths from the face a to the critical face b . The gradient path beginning at a splits into two gradient paths at a triangle, and these two paths merge at a segment. Note that this gradient vector field is not obtained by a maximal increasing or decreasing scheme, otherwise we would have only one critical face.

The parity of the number of gradient paths between two simplices is a classical tool for extracting information about the homology of a complex. With our definitions, this may be accomplished through the following result.

Theorem 5. *Let (Υ, λ) be the reference pair of a Morse sequence $\vec{W}(K)$. If $\kappa \in \widehat{W}$ and $\nu \in K$, then:*

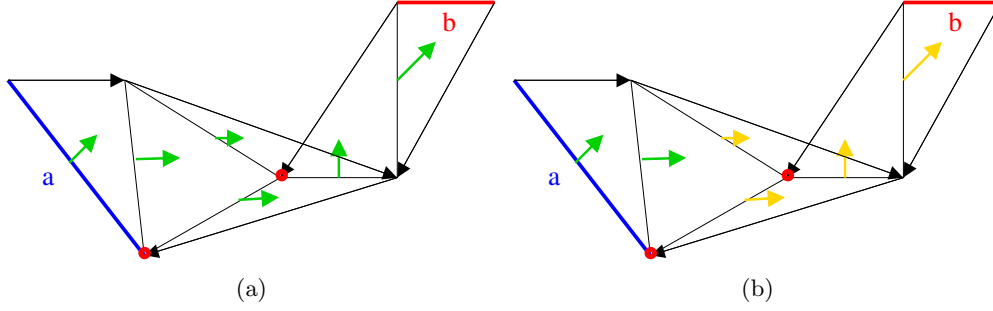


Fig. 6: (a) A gradient vector field: there are two gradient paths from a to b .
(b) The only regular pairs involved in a Υ -path ending at b are in yellow: thus there is no Υ -path from a to b . See text for details.

- 1) $\kappa \in \Upsilon(\nu)$ if and only if the number of gradient paths from ν to κ is odd.
- 2) $\kappa \in \lambda(\nu)$ if and only if the number of cogradient paths from κ to ν is odd.

Proof. Let $\vec{W}(K) = \langle \emptyset = K_0, \dots, K_i, \dots, K_k = K \rangle$.

1) For the first statement we proceed by induction on i , the base case $\nu \in K_0$ is trivial. Suppose the property is true for each $\nu \in K_{i-1}$, $i \geq 1$. We write $N(\nu, \kappa)$ for the number of gradient paths from ν to κ . Let $\nu \in K_i \setminus K_{i-1}$.

i) Suppose $\nu \in \widehat{W}$. We have $\Upsilon(\nu) = \nu$. If $\nu \neq \kappa$ there is no gradient path from ν to κ , thus $N(\nu, \kappa) = 0$. If $\nu = \kappa$ there is one and only one gradient path from ν to κ , that is, the trivial path $\langle \kappa \rangle$, thus $N(\nu, \kappa) = 1$.

ii) If $\nu \notin \widehat{W}$, we have $K_i \setminus K_{i-1} = \{\sigma, \tau\}$ where (σ, τ) is a regular pair for \vec{W} .

- Suppose $\nu = \tau$. We have $\Upsilon(\tau) = 0$ and $\tau \neq \kappa$. Furthermore, there is no non-trivial gradient path starting at τ . Thus we have $N(\nu, \kappa) = 0$.

- Suppose $\nu = \sigma$. We have $\sigma \neq \kappa$. We consider the sets:

$$A = \{\nu' \in \sigma + \partial\tau\}, B = \{\nu' \in A \mid N(\nu', \kappa) \text{ is odd}\}, C = \{\nu' \in A \mid \kappa \in \Upsilon(\nu')\}.$$

Any gradient path from ν to κ is obtained by concatenating (σ, τ) to a gradient path from a simplex $\nu' \in A$ to κ . Thus we have $N(\nu, \kappa) = \sum_{\nu' \in A} N(\nu', \kappa)$. By parity considerations, we obtain:

$$N(\nu, \kappa) \text{ is odd if and only if } \text{Card}(B) \text{ is odd.} \quad (P)$$

By the definition of Υ we have:

$$\kappa \in \Upsilon(\nu) \text{ if and only if } \text{Card}(C) \text{ is odd.} \quad (Q)$$

We have $A \subseteq K_{i-1}$, thus $B \subseteq K_{i-1}$ and $C \subseteq K_{i-1}$. By our induction hypothesis we obtain $B = C$. The result follows from (P) and (Q).

2) For the second statement we consider the sequence

$$\overleftarrow{W}(K) = \langle \overline{K}_0 = K \setminus K_k, \dots, \overline{K}_j = K \setminus K_{k-j}, \dots, \overline{K}_k = K \setminus K_0 \rangle.$$

We proceed by induction on the number j , the base case $\nu \in \overline{K}_0$ is trivial. Then we consider $\nu \in \overline{K}_j \setminus \overline{K}_{j-1}$. We use the same kind of arguments as above. The set A is replaced by the set $A' = \{\nu' \in \tau + \delta\sigma\}$, thus $A' \subseteq \overline{K}_{j-1}$. \square

Now, we introduce a refinement of gradient and cogradient paths.

Definition 4. Let (Υ, λ) be the reference pair of a Morse sequence \overrightarrow{W} .

We say that a gradient path $\langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$ in \overrightarrow{W} is a Υ -path if, for each $i \in [0, k]$, we have $\sigma_i \in \Upsilon(\sigma_i)$.

We say that a cogradient path $\langle \tau_0, \sigma_1, \tau_1, \dots, \sigma_k, \tau_k \rangle$ in \overrightarrow{W} is a λ -path if, for each $i \in [0, k]$, we have $\tau_i \in \lambda(\tau_i)$.

We observe that a Υ -path (resp. a λ -path) necessarily ends (resp. begins) at a critical face. By using the same arguments as in the proof of Theorem 5, we derive:

Proposition 6. Let (Υ, λ) be the reference pair of a Morse sequence $\overrightarrow{W}(K)$.

If $\kappa \in \widehat{W}$ and $\nu \in K$, then:

- 1) $\kappa \in \Upsilon(\nu)$ if and only if the number of Υ -paths from ν to κ is odd.
- 2) $\kappa \in \lambda(\nu)$ if and only if the number of λ -paths from κ to ν is odd.

Therefore, we can deduce that:

Corollary 7. Let (Υ, λ) be the reference pair of a Morse sequence $\overrightarrow{W}(K)$.

If $\kappa \in \widehat{W}$ and $\nu \in K$, then:

- 1) We have $\kappa \in \Upsilon(\nu)$ if and only if there exists a Υ -path from ν to κ .
- 2) We have $\kappa \in \lambda(\nu)$ if and only if there exists a λ -path from κ to ν .

In Figure 6 (b), the only regular pairs involved in a Υ -path ending at the critical face b are in yellow: we can check that there are precisely four segments ν such that $b \in \Upsilon(\nu)$, with $\nu \neq b$. We can also check that there is no Υ -path from a to b . By the above corollary, it means that $b \notin \Upsilon(a)$. Since there are precisely two gradient paths from a to b , this fact is also a consequence of Theorem 5.

Thus, the property given in Corollary 7 does not hold if we consider arbitrary gradient (or co-gradient) paths instead of Υ -paths (or λ -paths): Υ - and λ -paths allow us to discard some paths that do not convey information about homology.

6 The critical and cocritical complexes

Now, thanks to the reference and the coreference of a Morse sequence \overrightarrow{W} , we introduce two maps $\widehat{\partial}_p$ and $\widehat{\delta}_p$ that are restricted to \widehat{W} . With these maps we obtain the critical and the cocritical complexes of a Morse sequence.

In the sequel of the paper, if we write \overrightarrow{W} for a Morse sequence, the symbols Υ and λ stand respectively for the reference and co-reference maps of \overrightarrow{W} .

Definition 5. Let \overrightarrow{W} be a Morse sequence. Let $\widehat{\partial}_p$ and $\widehat{\delta}_p$ be the two linear maps:

$$\widehat{\partial}_p : \widehat{W}[p] \rightarrow \widehat{W}[p-1] \text{ and } \widehat{\delta}_p : \widehat{W}[p] \rightarrow \widehat{W}[p+1],$$

such that, for each $\nu \in \widehat{W}(p)$, $\widehat{\partial}(\nu) = \Upsilon_{p-1}(\partial(\nu))$ and $\widehat{\delta}(\nu) = \lambda_{p+1}(\delta(\nu))$.

If $c \in \widehat{W}[p]$, then $\widehat{\partial}_p(c)$ is the Υ -boundary of c , $\widehat{\delta}_p(c)$ is the λ -coboundary of c .

- Thus, for each $c \in \widehat{W}[p]$, we have:
- $\widehat{\partial}_p(c) = \sum_{\nu \in c} \Upsilon_{p-1}(\partial(\nu)) = \Upsilon_{p-1}(\sum_{\nu \in c} \partial(\nu)) = \Upsilon_{p-1}(\partial_p(c))$, and
 - $\widehat{\delta}_p(c) = \sum_{\nu \in c} \lambda_{p+1}(\delta(\nu)) = \lambda_{p+1}(\sum_{\nu \in c} \delta(\nu)) = \lambda_{p+1}(\delta_p(c))$.

As an illustration, let us consider our two canonical examples:

- In Fig. 4 (b), we have $\widehat{\partial}(d) = \Upsilon_1(\partial(d)) = b + c + \Upsilon(x) = 0$,
- In Fig. 4 (c), we have $\widehat{\delta}(b) = \lambda_2(\delta(b)) = d + d = 0$,
- In Fig. 5 (b), we have $\widehat{\partial}(c) = \Upsilon_1(\partial(c)) = b + b + b = b$.
- In Fig. 5 (c), we have $\widehat{\delta}(b) = \lambda_2(\delta(b)) = c + c + c = c$.

This last example suggests us the following theorem, which reflects an important duality relation between the reference and the co-reference maps of a Morse sequence.

Theorem 8. *Let \vec{W} be a Morse sequence on K and let $\sigma, \tau \in \widehat{W}$. We have:
 $\sigma \in \widehat{\partial}(\tau)$ if and only if $\tau \in \widehat{\delta}(\sigma)$.*

Proof. Let $\sigma, \tau \in \widehat{W}$. We write:

- Ω_σ for the set of all gradient paths from a simplex $\nu \in \partial(\tau)$ to σ ,
 - Ω^τ for the set of all cogradient paths from τ to a simplex $\nu \in \delta(\sigma)$.
- We have $\sigma \in \widehat{\partial}(\tau)$ if and only if $\text{Card}(\{\nu \in \partial(\tau) \text{ and } \sigma \in \Upsilon(\nu)\})$ is odd.

Thus, by Theorem 5, we have $\sigma \in \widehat{\partial}(\tau)$ if and only if $\text{Card}(\Omega_\sigma)$ is odd.

Similarly we have $\tau \in \widehat{\delta}(\sigma)$ if and only if $\text{Card}(\Omega^\tau)$ is odd.

We will establish the result by showing that $\text{Card}(\Omega_\sigma) = \text{Card}(\Omega^\tau)$.

1) Let π be a gradient path in Ω_σ . Then π is of the form

$$\pi = \langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle,$$

with $\sigma_0 \in \partial(\tau)$ and $\sigma_k = \sigma$. Let $f(\pi)$ be the sequence

$$f(\pi) = \langle \tau, \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1} \rangle.$$

We can check that $f(\pi)$ is a cogradient path from τ to τ_{k-1} . Furthermore we have $\tau_{k-1} \in \delta(\sigma)$, thus $f(\pi)$ is in Ω^τ .

Let f be the map $\pi \in \Omega_\sigma \mapsto f(\pi) \in \Omega^\tau$ and let

$$\pi = \langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle \text{ and } \pi' = \langle \sigma'_0, \tau'_0, \dots, \sigma'_{l-1}, \tau'_{l-1}, \sigma'_l \rangle$$

be two distinct paths in Ω_σ . Since $\sigma = \sigma_k = \sigma'_l$, the two sequences

$$\langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1} \rangle \text{ and } \langle \sigma'_0, \tau'_0, \dots, \sigma'_{l-1}, \tau'_{l-1} \rangle$$

must be distinct. It follows that $f(\pi) \neq f(\pi')$. Therefore the map f is injective, which means that $\text{Card}(\Omega_\sigma) \leq \text{Card}(\Omega^\tau)$.

2) Now, let $\pi = \langle \tau_0, \sigma_1, \tau_1, \dots, \sigma_k, \tau_k \rangle$ be a path in Ω^τ . Thus, we have $\tau_0 = \tau$ and $\tau_k \in \delta(\sigma)$. We proceed as above:

- The sequence $g(\pi) = \langle \sigma_1, \tau_1, \dots, \sigma_k, \tau_k, \sigma \rangle$ is a gradient path in Ω_σ ,
- The map $g : \pi \in \Omega^\tau \mapsto g(\pi) \in \Omega_\sigma$ is injective,
- Therefore we have $\text{Card}(\Omega^\tau) \leq \text{Card}(\Omega_\sigma)$. □

Theorem 9. *Let \vec{W} be a Morse sequence. We have:*

$$\widehat{\partial}_p \circ \Upsilon_p = \Upsilon_{p-1} \circ \partial_p \text{ and } \widehat{\delta}_p \circ \lambda_p = \lambda_{p+1} \circ \delta_p.$$

We will only give the first part of the proof of the theorem. The second part may be derived by simply exchanging the role of the operators ∂ and δ .

In the sequel of the paper, we will proceed in the same manner for all propositions and theorems where we have such a duality.

Proof. Let $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ be a Morse sequence on K , and let $\diamond \vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$. We consider the statement (S_i) : For each $c \in K_i[p]$, we have $\widehat{\partial}_p(\Upsilon_p(c)) = \Upsilon_{p-1}(\partial_p(c))$. We have $K_0[p] = \{\emptyset\}$. Thus (S_0) holds. Suppose (S_{i-1}) holds, with $0 \leq i-1 \leq k-1$, and let $c \in K_i[p]$.

1) Suppose $\kappa_i = \nu$, with $\nu \in \widehat{W}$. If $\nu \notin c$, then we are done. Otherwise, we have $c = c' \cup \{\nu\}$, with $c' \in K_{i-1}[p]$. Thus $\partial_p(c) = \partial_p(c') + \partial(\nu)$ and $\Upsilon_{p-1}(\partial_p(c)) = \Upsilon_{p-1}(\partial_p(c')) + \Upsilon_{p-1}(\partial(\nu))$. By the induction hypothesis and by the definition of $\widehat{\partial}(\nu)$, we obtain $\Upsilon_{p-1}(\partial_p(c)) = \widehat{\partial}_p(\Upsilon_p(c')) + \widehat{\partial}(\nu)$. Since the reference map Υ is the identity on critical faces, we may write:

$$\Upsilon_{p-1}(\partial_p(c)) = \widehat{\partial}_p(\Upsilon_p(c')) + \widehat{\partial}_p(\Upsilon_p(\nu)).$$

By linearity we obtain $\Upsilon_{p-1}(\partial_p(c)) = \widehat{\partial}_p(\Upsilon_p(c))$.

2) Suppose $\kappa_i = (\sigma, \tau)$ is a free pair. If $\sigma \notin c$ and $\tau \notin c$, then we are done.

2.1) Suppose $\sigma \in c$. Let $c' = c + \partial_{p+1}(\tau)$. By Proposition 4, we have $\Upsilon_p(c') = \Upsilon_p(c) + \Upsilon_p(\partial_{p+1}(\tau)) = \Upsilon_p(c)$. We also have $\partial_p(c') = \partial_p(c) + \partial_p(\partial_{p+1}(\tau)) = \partial_p(c)$.

But we see that $\sigma \notin c'$. By the induction hypothesis, it follows that $\widehat{\partial}_p(\Upsilon_p(c')) = \Upsilon_{p-1}(\partial_p(c'))$. By the previous equalities, we obtain $\widehat{\partial}_p(\Upsilon_p(c)) = \Upsilon_{p-1}(\partial_p(c))$.

2.2) Suppose $\tau \in c$. We have $c = c' \cup \{\tau\}$, with $c' \in K_{i-1}[p]$. Since $\Upsilon(\tau) = 0$, we obtain $\Upsilon_p(c) = \Upsilon_p(c')$. Furthermore $\Upsilon_{p-1}(\partial_p(c)) = \Upsilon_{p-1}(\partial_p(c')) + \Upsilon_{p-1}(\partial_p(\tau)) = \Upsilon_{p-1}(\partial_p(c'))$. By the induction hypothesis, we have $\widehat{\partial}_p(\Upsilon_p(c')) = \Upsilon_{p-1}(\partial_p(c'))$. Therefore $\widehat{\partial}_p(\Upsilon_p(c)) = \Upsilon_{p-1}(\partial_p(c))$.

We conclude that, in all cases, the statement (S_i) holds. \square

The two following results are direct consequences of Theorem 9.

Proposition 10. *Let $\vec{W}(K)$ be a Morse sequence and let $c, c' \in K[p]$.*

- 1) *We have $\Upsilon_{p-1}(\partial_p(c)) = \Upsilon_{p-1}(\partial_p(c'))$ whenever $\Upsilon_p(c) = \Upsilon_p(c')$.*
- 2) *We have $\lambda_{p+1}(\delta_p(c)) = \lambda_{p+1}(\delta_p(c'))$ whenever $\lambda_p(c) = \lambda_p(c')$.*

Proof. Let $c, c' \in K[p]$ with $\Upsilon_p(c) = \Upsilon_p(c')$. Thus $\widehat{\partial}_p(\Upsilon_p(c)) = \widehat{\partial}_p(\Upsilon_p(c'))$. By Theorem 9, we have $\Upsilon_{p-1}(\partial_p(c)) = \Upsilon_{p-1}(\partial_p(c'))$. \square

Proposition 11. *If \vec{W} is a Morse sequence, then:*

- 1) *The maps $\widehat{\partial}_p$ are boundary operators. That is, we have $\widehat{\partial}_p \circ \widehat{\partial}_{p+1} = 0$.*
- 2) *The maps $\widehat{\delta}_p$ are coboundary operators. That is, we have $\widehat{\delta}_{p+1} \circ \widehat{\delta}_p = 0$.*

Proof. Let $\nu \in \widehat{W}$, with $\nu \in K^{(p+1)}$. We have $\widehat{\partial}_{p+1}(\nu) = \Upsilon_p(\partial_{p+1}(\nu))$. By Theorem 9, we have $\widehat{\partial}_p(\Upsilon_p(\partial_{p+1}(\nu))) = \Upsilon_{p-1}(\partial_p(\partial_{p+1}(\nu))) = \Upsilon_{p-1}(0) = 0$.

Thus $\widehat{\partial}_p(\widehat{\partial}_{p+1}(\nu)) = 0$, which gives the result by linearity. \square

Since $\widehat{\partial}_p \circ \widehat{\partial}_{p+1} = 0$, the pair $(\widehat{W}[p], \widehat{\partial}_p)$ satisfies the definition of a chain complex, see Section 2. Similarly, $(\widehat{W}[p], \widehat{\delta}_p)$ satisfies the definition of a cochain complex.

Definition 6. Let \vec{W} be a Morse sequence. We say that the chain complex $(\widehat{W}[p], \widehat{\partial}_p)$ is the critical complex of \vec{W} and the cochain complex $(\widehat{W}[p], \widehat{\delta}_p)$ is the cocritical complex of \vec{W} .

This notion of a critical complex is equivalent to the classical notion, also sometimes called *Morse complex*, given in the context of discrete Morse theory. This fact may be verified using Theorem 8.31 of [35], Theorem 5, and the very definition of the differential $\widehat{\partial}_p$.

By Theorem 9, the map γ is a *chain map* [25] from the chain complex $(K[p], \partial_p)$ to the chain complex $(\widehat{W}[p], \widehat{\partial}_p)$. The map λ is a *cochain map* [25] from the chain complex $(K[p], \delta_p)$ to the chain complex $(\widehat{W}[p], \widehat{\delta}_p)$.

Also, we easily derive from Theorem 9 the following facts, which are basic properties of chain maps, we follow the notations given in subsection 2.2:

- If $c \in Z_p(K)$, then $\gamma_p(c) \in Z_p(\widehat{W})$, if $c \in Z^p(K)$, then $\lambda_p(c) \in Z^p(\widehat{W})$.
- If $c \in B_p(K)$, then $\gamma_p(c) \in B_p(\widehat{W})$, if $c \in B^p(K)$, then $\lambda_p(c) \in B^p(\widehat{W})$.

It follows that we have linear maps (homomorphisms) between $H_p(K)$ and $H_p(\widehat{W})$, and between $H^p(K)$ and $H^p(\widehat{W})$:

$$\begin{aligned} \gamma_p^H : [z] \in H_p(K) &\mapsto \gamma_p^H([z]) = [\gamma_p(z)] \in H_p(\widehat{W}), \\ \lambda_p^H : [z] \in H^p(K) &\mapsto \lambda_p^H([z]) = [\lambda_p(z)] \in H^p(\widehat{W}). \end{aligned}$$

Thus, the maps γ and λ allow to carry out the homology of the complex K to the smaller space \widehat{W} . We will see in section 8 how to go in the other direction in order to obtain an isomorphism between the vector spaces $H^p(K)$ and $H^p(\widehat{W})$.

7 Arranged Morse sequences and skeletons

In this section, we introduce *arranged sequences* which may be obtained thanks to a reordering of Morse sequences. Also we introduce the related notions of *lower* and *upper skeletons* of an arbitrary Morse sequence. In particular, the properties of skeletons and arranged sequences are crucial for the theorems of the next sections.

Recall that the dimension of a free pair (σ, τ) is equal to $\dim(\tau)$. An arranged sequence is simply a Morse sequence where the critical faces and the free pairs are partially ordered according to their dimensions ².

Definition 7. Let \vec{W} be a Morse sequence and let $\diamond\vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$. We say that \vec{W} is an *arranged* (Morse) sequence if for each $i \in [1, k-1]$, we have:

- 1) $\dim(\kappa_i) \leq \dim(\kappa_{i+1})$, and
- 2) $\dim(\kappa_i) < \dim(\kappa_{i+1})$ if κ_i is critical and κ_{i+1} is a regular pair.

²See also the rephrasing of the proof of Proposition 13.1 given in [29] where a similar ordering on the simplexes is used.

Let us write $\widehat{W}^{(p)} = \{\nu \in \widehat{W} \mid \dim(\nu) = p\}$ and $\check{W}^{(p)} = \{(\sigma, \tau) \in \check{W} \mid \dim(\tau) = p\}$. If \vec{W} is an arranged sequence, each of the two sets $\widehat{W}^{(p)}$ and $\check{W}^{(p)}$ constitutes a substring of $\diamond \vec{W}$, that is a contiguous sequence. Furthermore, $\widehat{W}^{(p)} \cup \check{W}^{(p)}$ constitutes also a substring of $\diamond \vec{W}$ where $\check{W}^{(p)}$ is a prefix.

Let $\diamond \vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$ be a simplex-wise Morse sequence. If κ_i and κ_{i+1} do not satisfy one of the two above conditions, it can be checked that the sequence obtained by swapping κ_i and κ_{i+1} is still a simplex-wise Morse sequence. By induction, we have the following result.

Theorem 12. *If \vec{W} is a Morse sequence, then there exists an arranged sequence \vec{V} such that \vec{W} and \vec{V} are equivalent.*

Remark 4. It should be noted that the maximal increasing scheme introduced in Section 3 does not, in general, provide directly an arranged Morse sequence. A sorting has to be made in order to obtain an equivalent arranged sequence. The same holds true for the maximal decreasing scheme.

Recall that the p -skeleton of a simplicial complex K is the set composed of all the q -simplices of K such that $q \leq p$. The following may be seen as a refinement of this notion for Morse sequences.

Definition 8. *Let \vec{W} be a Morse sequence. We write:*

$$- W_p^- = \{\nu \in \overline{W} \mid \dim(\nu) \leq p\} \cup \{\nu \in \widehat{W} \cup \underline{W} \mid \dim(\nu) \leq p-1\}.$$

$$- W_p^+ = \{\nu \in \widehat{W} \cup \overline{W} \mid \dim(\nu) \leq p\} \cup \{\nu \in \underline{W} \mid \dim(\nu) \leq p-1\}.$$

The sets W_p^- and W_p^+ are, respectively, the lower and the upper p -skeleton of \vec{W} . Also, if $d = \dim(K)$, we say that the sequence:

$$\vec{S}(W) = \langle W_0^-, W_0^+, \dots, W_p^-, W_p^+, \dots, W_d^-, W_d^+ \rangle$$

is the skeleton sequence of \vec{W} .

Thus, W_p^- is the simplicial complex that contains the $(p-1)$ -skeleton of K and the p -simplexes that are upper regular for \vec{W} ; W_p^+ is the simplicial complex that contains the p -skeleton of K except the p -simplices of K that are lower regular for \vec{W} . We observe that:

- We have $W_0^- = \emptyset$ and $W_d^+ = K$. The set W_0^+ is made of all vertices of K that are critical for \vec{W} .

- The skeleton sequence $\vec{S}(W)$ of \vec{W} is a filtration (see Remark 3).

- If two Morse sequences \vec{V} and \vec{W} are equivalent, then $\vec{S}(V) = \vec{S}(W)$.

- We have $W_p^+ \setminus W_p^- = \widehat{W}^{(p)}$.

- We have $W_{p+1}^- \setminus W_p^+ = \{\nu \in \overline{W} \mid \dim(\nu) = p+1\} \cup \{\nu \in \underline{W} \mid \dim(\nu) = p\}$. That is, $W_{p+1}^- \setminus W_p^+$ is made of all simplexes that are in $\check{W}^{(p)}$.

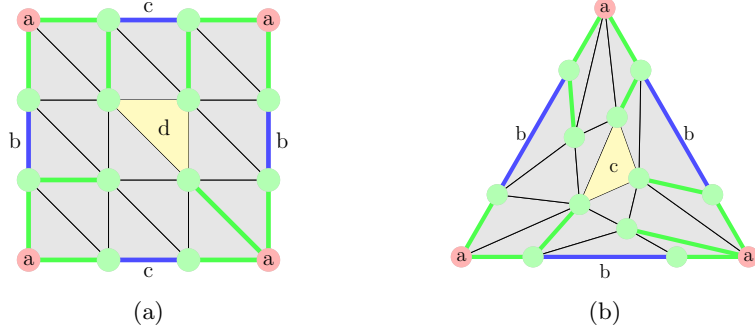


Fig. 7: Skeleton sequences of two Morse sequences $\vec{W}(K)$. In (a) the sequence $\vec{W}(K)$ is the one corresponding to Fig 2, and in (b) the sequence $\vec{W}(K)$ corresponds to Fig 3. See text for details.

An illustration of two skeleton sequences $\vec{S}(W) = \langle W_0^-, W_0^+, \dots, W_2^-, W_2^+ \rangle$ is given Figure 7. In each of the figures (a) and (b), we have:

- $W_0^- = \emptyset$ and $W_0^+ = \{a\}$, where a is the critical vertex.
- W_1^- is the tree composed of W_0^+ , all the green vertices, and all the green edges,
- W_1^+ is composed of W_1^- and all the critical edges, that is, all the blue edges,
- W_2^- is composed of W_1^+ , all the dark edges, and all the grey triangles.
- $W_2^+ = K$, that is, $W_2^+ = W_2^- \cup \{\kappa\}$, where κ is the critical yellow triangle.

The two next propositions are direct consequences of the above definitions.

Proposition 13. *Let \vec{W} be an arranged Morse sequence and let $\vec{S}(W)$ be the skeleton sequence of \vec{W} . Then each complex in $\vec{S}(W)$ is a complex in \vec{W} .*

Note that we can also affirm that $\vec{S}(W)$ is a subsequence of an arranged sequence \vec{W} . That is, $\vec{S}(W)$ can be obtained from \vec{W} by removing some elements without changing the order of the remaining elements.

Proposition 14. *Let $\vec{W}(K) = \langle K_0, \dots, K_k \rangle$ be an arranged Morse sequence. Let $K_i = W_p^+$ and $K_j = W_{p+1}^-$. Then the sequence $\langle K_j, \dots, K_i \rangle$ is a collapse sequence.*

The following result is a direct consequence of Theorem 12 and Proposition 14. It may be seen as another aspect of the fundamental collapse theorem of discrete Morse theory; see Remark 2 given in Section 3. A specificity of this formulation is that all regular pairs (and not only a part of them) of a given dimension are removed by the collapse sequence.

Theorem 15. *If \vec{W} is a Morse sequence, then W_{p+1}^- collapses onto W_p^+ .*

We now give some properties of reference and co-reference maps. Theorems 17 and 18 will be essential for some crucial proofs in the sequel of this paper. These theorems are obtained thanks to arranged sequences that may be derived from arbitrary Morse sequences (Theorem 12), and thanks to the skeletons which appear in such sequences (Proposition 13).

Let $\nu \in K^{(p)}$ such that $\nu \in W_p^+$. Thus $\nu \in \widehat{W} \cup \overline{W}$. We have $\Upsilon(\nu) = \nu$ if $\nu \in \widehat{W}$, and $\Upsilon(\nu) = 0$ if $\nu \in \overline{W}$. This leads us to the following:

Proposition 16. *Let $\overrightarrow{W}(K)$ be a Morse sequence and let $c \in K[p]$.*

- 1) *If $c \subseteq W_p^+$, then $\Upsilon(c) = c \cap \widehat{W}$.*
- 2) *If $c \subseteq K \setminus W_p^-$, then $\lambda(c) = c \cap \widehat{W}$.*

Theorem 17. *Let $\overrightarrow{W}(K)$ be a Morse sequence.*

- 1) *Let $z, z' \in Z_p(K)$. If $\Upsilon(z) = \Upsilon(z')$ and $z, z' \subseteq W_p^+$, then $z = z'$.*
- 2) *Let $z, z' \in Z^p(K)$. If $\lambda(z) = \lambda(z')$ and $z, z' \subseteq K \setminus W_p^-$, then $z = z'$.*

Proof. By Theorem 12 and by Proposition 3, we may suppose that $\overrightarrow{W}(K)$ is an arranged sequence, we set $\overrightarrow{W}(K) = \langle K_0, \dots, K_k \rangle$.

1) We will first prove that $z = 0$ whenever $z \in Z_p(K)$ and $z \subseteq W_p^-$. Since $W_0^- = \emptyset$, the property is trivially true whenever $p = 0$. Suppose $p \geq 1$. By Proposition 13, there exist K_i and K_j such that $K_i = W_{p-1}^+$ and $K_j = W_p^-$. Since $\dim(K_i) \leq p-1$, the property is trivially true whenever $z \subseteq K_i$. Suppose the property is true whenever $z \subseteq K_{l-1}$, with $i \leq l-1 \leq j-1$, and let $z \in Z_p(K)$ with $z \subseteq K_l$. By Proposition 14 we have $K_l = K_{l-1} \cup \{\sigma, \tau\}$, where (σ, τ) is a free pair and $\dim(\tau) = p$. But it is not possible that z contains the simplex τ otherwise we would have $\sigma \in \partial(z)$ and z would not be a cycle. Thus, $z \subseteq K_{l-1}$. By the induction hypothesis, it means that $z = 0$. Therefore the property is true whenever $z \subseteq K_j$. Since $K_j = W_p^-$, we obtain the desired conclusion.

2) Now, suppose $z \in Z_p(K)$, $z \subseteq W_p^+$ and $\Upsilon(z) = 0$. By Proposition 16, it means that $z \subseteq W_p^-$. By the result obtained in 1), we deduce that $z = 0$. Let $z, z' \in Z_p(K)$ such that $z, z' \subseteq W_p^+$, and $\Upsilon(z) = \Upsilon(z')$. We have $\Upsilon(z + z') = 0$. By the previous result, this gives $z = z'$. \square

By the previous statement, two distinct p -cycles in W_p^+ must have distinct references. Of course this property is not true if we consider arbitrary p -cycles. For example, let us consider the 1-cycle z that is the boundary of the dunce hat depicted Figure 5 (b). We have $\Upsilon(z) = b$. But the 1-cycle $z' = \partial(c)$, the boundary of the critical simplex c , is made of three simplices ν such that $\Upsilon(\nu) = b$. Thus we have $z \neq z'$ but $\Upsilon(z) = \Upsilon(z') = b$.

In the next result, we consider the general case where the p -cycles are not necessarily in W_p^+ . We write $z \sim z'$ if z and z' are two cycles (or two cocycles) that are in the same homology (or cohomology) class.

Theorem 18. *Let $\vec{W}(K)$ be a Morse sequence.*

- 1) *Let $z, z' \in Z_p(K)$. If $\Upsilon(z) = \Upsilon(z')$, then $z \sim z'$.*
- 2) *Let $z, z' \in Z^p(K)$. If $\lambda(z) = \lambda(z')$, then $z \sim z'$.*

Proof. Again, by Theorem 12 and by Proposition 3, we may suppose that $\vec{W}(K)$ is an arranged sequence, we set $\vec{W}(K) = \langle K_0, \dots, K_k \rangle$. We will prove that we have $z \sim 0$ whenever $z \in Z_p(K)$ and $\Upsilon(z) = 0$. The result will follow by linearity.

Let $z \in Z_p(K)$. We first observe that $z \subseteq W_{p+1}^-$. By Proposition 13, there exist K_i and K_j such that $K_i = W_p^+$ and $K_j = W_{p+1}^-$. By Theorem 17, the property is true whenever $z \subseteq K_i$. Suppose it is true for any $z \subseteq K_{l-1}$, with $i \leq l-1 \leq j-1$, and let $z \in Z_p(K)$ such that $z \subseteq K_l$ and $\Upsilon(z) = 0$. By Proposition 14 we have $K_l = K_{l-1} \cup \{\sigma, \tau\}$, where (σ, τ) is a free pair. We have $\dim(\tau) = p+1$, thus the simplex τ is not in z . Also by the induction hypothesis, we have $z \sim 0$ whenever $\sigma \notin z$.

Suppose now that $\sigma \in z$. Let $z' = z + \partial(\tau)$. We have $\Upsilon(z') = \Upsilon(z) + \Upsilon(\partial(\tau)) = \Upsilon(z) = 0$. We have $z' \subseteq K_{l-1}$. Therefore, by the induction hypothesis, we have $z' \sim 0$. But $\partial(\tau)$ is a boundary, thus $z \sim z'$. Therefore $z \sim 0$. \square

It should be noted that the converse of the previous statement is, in general, not true. For example, let us consider again the cycle z that is the boundary of the dunce hat depicted Figure 5 (b). We have $\Upsilon(z) = b$, but $z \sim 0$ and $\Upsilon(0) = 0$.

8 The extension and co-extension maps

In this section, we introduce two maps $\tilde{\Upsilon}$ and $\tilde{\lambda}$ which, at the level of homology, may be seen as the inverses of the maps Υ and λ . Each couple $(\Upsilon, \tilde{\Upsilon})$ and $(\lambda, \tilde{\lambda})$ allows us to establish a relationship between a complex and its critical complex. In particular, this relation includes the equality of their homology.

Definition 9. *Let $\vec{W}(K)$ be a Morse sequence. Let $\tilde{\Upsilon}_p$ and $\tilde{\lambda}_p$ be the two linear maps $\tilde{\Upsilon}_p : \widehat{W}[p] \rightarrow K[p]$ and $\tilde{\lambda}_p : \widehat{W}[p] \rightarrow K[p]$ defined by:*

$\tilde{\Upsilon}(\kappa) = \{\nu \in K \mid \kappa \in \lambda(\nu)\}$ and $\tilde{\lambda}(\kappa) = \{\nu \in K \mid \kappa \in \Upsilon(\nu)\}$, where $\kappa \in \widehat{W}$, $\tilde{\Upsilon}$ and $\tilde{\lambda}$ are, respectively, the extension and the co-extension map of \vec{W} . If $c \in \widehat{W}[p]$, $\tilde{\Upsilon}_p(c)$ and $\tilde{\lambda}_p(c)$ are, respectively, the extension and the co-extension of c (for \vec{W}).

In other words, if $\kappa \in \widehat{W}^{(p)}$, we have:

$$\nu \in \tilde{\Upsilon}(\kappa) \text{ if and only if } \kappa \in \lambda(\nu) \text{ and } \nu \in \tilde{\lambda}(\kappa) \text{ if and only if } \kappa \in \Upsilon(\nu).$$

Also, the two maps $\tilde{\Upsilon}$ and $\tilde{\lambda}$ may be described by λ - and Υ -paths:

- We have $\nu \in \tilde{\Upsilon}(\kappa)$ if and only if there exists a λ -path from κ to ν .
- We have $\nu \in \tilde{\lambda}(\kappa)$ if and only if there exists a Υ -path from ν to κ .

An illustration of two extension maps is directly given by the co-reference maps of the torus and the dunce hat of Figure 4 (c) and 5 (c). Another example is given Figure 8 where the effect of the modulo 2 arithmetic is emphasized.

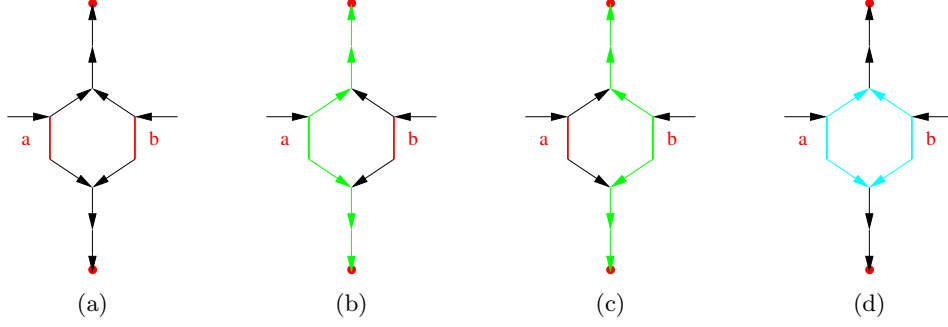


Fig. 8: (a) The gradient vector field of a Morse sequence with 4 critical faces. (b) The extension of the face a . (c) The extension of the face b . (d) The extension of $a + b$.

With the notion of an extension map, we retrieve the closure operator ϕ introduced in [29], Chapter 12. This fact can be checked by using Proposition 12.7 of [29] and Theorem 5 of this paper. It has been proved that the image of the operator ϕ induces a chain complex which leads to an isomorphism with the critical complex, see Chapter 13 of [29]. Perhaps the main difference between $\tilde{\gamma}$ and ϕ is that the map $\tilde{\gamma}$ is defined with the co-reference map λ , while ϕ is built with a decomposition of a graph, see Definition 12.3 of [29]. In the following, we will see that the link between $\tilde{\gamma}$ and λ allows us to highlight the strong relationships between the maps $\tilde{\gamma}$, γ , $\hat{\partial}$ and, in a dual way, between the maps $\tilde{\lambda}$, λ , $\hat{\delta}$. We will recover, in a different manner, and without the image of $\tilde{\gamma}$, an isomorphism with the critical complex. In the next section, we will use the chain complex induced by the image of $\tilde{\gamma}$ for an equivalence with the flow complex.

Let $\kappa \in \widehat{W}$. We have $\lambda(\kappa) = \gamma(\kappa) = \kappa$, therefore $\kappa \in \tilde{\gamma}(\kappa)$ and $\kappa \in \tilde{\lambda}(\kappa)$. Also, if $\kappa' \in \widehat{W}$ and $\kappa \in \lambda(\kappa')$ or $\kappa \in \gamma(\kappa')$, we have $\kappa = \kappa'$ (since $\lambda(\kappa') = \gamma(\kappa') = \kappa'$). Furthermore:

- if $\nu \in \tilde{\gamma}(\kappa)$, then ν cannot be lower regular (since $\lambda(\nu) \neq 0$).
- if $\nu \in \tilde{\lambda}(\kappa)$, then ν cannot be upper regular (since $\gamma(\nu) \neq 0$).

Thus, we have the two following results.

Proposition 19. *Let \vec{W} be a Morse sequence on K and let $\kappa \in \widehat{W}$. Then:*

- 1) $\tilde{\gamma}(\kappa)$ and $\tilde{\lambda}(\kappa)$ each contains a unique critical simplex which is precisely κ .
- 2) If $\nu \in \tilde{\gamma}(\kappa)$ and $\nu \neq \kappa$, then ν is upper regular for \vec{W} . That is $\tilde{\gamma}(\kappa) + \kappa \subseteq \overline{W}$.
- 3) If $\nu \in \tilde{\lambda}(\kappa)$ and $\nu \neq \kappa$, then ν is lower regular for \vec{W} . That is $\tilde{\lambda}(\kappa) + \kappa \subseteq \underline{W}$.

Proposition 20. *Let \vec{W} be a Morse sequence and let $c \in \widehat{W}[p]$. We have:*

$$\tilde{\gamma}_p(c) \subseteq W_p^+ \text{ and } \tilde{\lambda}_p(c) \subseteq K \setminus W_p^-.$$

Let $\kappa \in \widehat{W}$. We have $\gamma(\nu) = 0$ whenever ν is upper regular. Thus, by Proposition 19, we have $\gamma(\tilde{\gamma}(\kappa)) = \kappa$. If we denote by Id_S the identity map on a set S , we obtain

by linearity:

Proposition 21. *Let \vec{W} be a Morse sequence. We have:*

$$\gamma_p \circ \tilde{\gamma}_p = Id_{\widehat{W}_{[p]}} \text{ and } \lambda_p \circ \tilde{\lambda}_p = Id_{\widehat{W}_{[p]}}.$$

Let $\kappa \in \widehat{W}$. With Propositions 4 and 19, we derive $\gamma(\partial(\tilde{\gamma}(\kappa) + \kappa)) = 0$. It means that $\gamma(\partial(\tilde{\gamma}(\kappa))) = \gamma(\partial(\kappa))$ where the last expression corresponds to the definition of the map $\hat{\partial}$. By linearity we obtain:

Proposition 22. *Let \vec{W} be a Morse sequence. We have:*

$$\gamma_{p-1} \circ \partial_p \circ \tilde{\gamma}_p = \hat{\partial}_p \text{ and } \lambda_{p+1} \circ \delta_p \circ \tilde{\lambda}_p = \hat{\delta}_p$$

Proposition 23. *Let \vec{W} be a Morse sequence and let $\kappa \in \widehat{W}$.*

- 1) *If $\nu \in \partial(\tilde{\gamma}(\kappa))$, then ν is either critical or upper regular for \vec{W} .*
- 2) *If $\nu \in \delta(\tilde{\lambda}(\kappa))$, then ν is either critical or lower regular for \vec{W} .*

Proof. Let $\kappa \in \widehat{W}$ and let σ be a simplex that is lower regular for \vec{W} . Let $T = \{\nu \in \tilde{\gamma}(\kappa) \mid \sigma \in \partial(\nu)\}$. By the definition of the boundary operator, we have $\sigma \in \partial(\tilde{\gamma}(\kappa))$ if and only if $\text{card}(T)$ is odd. But we may also write $T = \{\nu \in \delta(\sigma) \mid \kappa \in \lambda(\nu)\}$. By the definition of λ , since σ is lower regular, we have $\lambda(\delta(\sigma)) = 0$ (Proposition 4). Therefore, the number $\text{card}(T)$ must be even, otherwise we would have $\kappa \in \lambda(\delta(\sigma))$. It means that $\partial(\tilde{\gamma}(\kappa))$ cannot contain a simplex that is lower regular for \vec{W} . \square

Proposition 24. *Let \vec{W} be a Morse sequence and let $\kappa \in \widehat{W}$. We have:*

$$\hat{\partial}(\kappa) = \partial(\tilde{\gamma}(\kappa)) \cap \widehat{W} \text{ and } \hat{\delta}(\kappa) = \delta(\tilde{\lambda}(\kappa)) \cap \widehat{W}.$$

Proof. By Proposition 22, we have $\hat{\partial}(\kappa) = \gamma(\partial(\tilde{\gamma}(\kappa)))$. Let $\nu \in K$. We can write:

$$\nu \in \gamma(\partial(\tilde{\gamma}(\kappa))) \Leftrightarrow \text{card}\{\nu' \mid \nu' \in \partial(\tilde{\gamma}(\kappa)) \text{ and } \nu \in \gamma(\nu')\} \text{ is odd.}$$

If ν' is an upper regular simplex of $\partial(\tilde{\gamma}(\kappa))$, we have $\gamma(\nu') = 0$. Furthermore, by Proposition 23, $\partial(\tilde{\gamma}(\kappa))$ does not contain any lower regular simplex. Therefore, if $\nu' \in \partial(\tilde{\gamma}(\kappa))$ and $\nu \in \gamma(\nu')$, we must have $\nu' \in \widehat{W}$. In this case, we also have $\gamma(\nu') = \nu'$. We obtain $\nu = \nu'$. Thus, we can write:

$$\nu \in \gamma(\partial(\tilde{\gamma}(\kappa))) \Leftrightarrow \text{card}\{\nu \mid \nu \in \partial(\tilde{\gamma}(\kappa)) \cap \widehat{W}\} \text{ is odd,}$$

This last number must be equal to 1. That is, we have $\nu \in \gamma(\partial(\tilde{\gamma}(\kappa)))$ if and only if $\nu \in \partial(\tilde{\gamma}(\kappa)) \cap \widehat{W}$, which gives the desired result. \square

Proposition 25. *Let \vec{W} be a Morse sequence on K .*

- 1) *If $z \in Z_p(\widehat{W})$, then $\tilde{\gamma}_p(z) \in Z_p(K)$.*
- 2) *If $z \in Z^p(\widehat{W})$, then $\tilde{\lambda}_p(z) \in Z^p(K)$.*

Proof. If $z \in Z_0(\widehat{W})$, then $z \in K[0]$, $\tilde{\gamma}_0(z) = z$, and $\partial(z) = 0$. Thus the property is true for $p = 0$. Suppose the property is true for any $z \in Z_{p-1}(\widehat{W})$, with $p \geq 1$.

1) Let $\kappa \in \widehat{W}$ with $\kappa \in K^{(p)}$. By Proposition 24, we have $\hat{\partial}(\kappa) = \partial_p(\tilde{\gamma}(\kappa)) \cap \widehat{W}$. By Proposition 23, if $\nu \in \partial_p(\tilde{\gamma}(\kappa))$, then ν is either critical or upper regular for \vec{W} . Therefore, $\gamma_{p-1}(\partial_p(\tilde{\gamma}(\kappa))) = \partial_p(\tilde{\gamma}(\kappa)) \cap \widehat{W}$. Thus we obtain:

$$\Upsilon_{p-1}(\partial_p(\tilde{\Upsilon}(\kappa))) = \widehat{\partial}(\kappa) \text{ and } \partial_p(\tilde{\Upsilon}(\kappa)) \subseteq W_{p-1}^+.$$

Also, by Proposition 21 and by Proposition 19, we have:

$$\Upsilon_{p-1}(\tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa))) = \widehat{\partial}(\kappa) \text{ and } \tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa)) \subseteq W_{p-1}^+.$$

We have $\partial_p(\tilde{\Upsilon}(\kappa)) \in Z_{p-1}(K)$. Since $\widehat{\partial}(\kappa) \in Z_{p-1}(\widehat{W})$, by the induction hypothesis, we also have $\tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa)) \in Z_{p-1}(K)$. Thus, by considering the two above expressions of $\widehat{\partial}(\kappa)$, Theorem 17 leads to $\tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa)) = \partial_p(\tilde{\Upsilon}(\kappa))$.

2) Let $z \in Z_p(\widehat{W})$, that is $\widehat{\partial}_p(z) = 0$. We have:

$$\partial_p(\tilde{\Upsilon}_p(z)) = \sum \{\partial_p(\tilde{\Upsilon}(\kappa)) \mid \kappa \in z\}.$$

With the above result we obtain:

$$\partial_p(\tilde{\Upsilon}_p(z)) = \sum_{\kappa \in z} \tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa)) = \tilde{\Upsilon}_{p-1}(\sum_{\kappa \in z} \widehat{\partial}(\kappa)) = \tilde{\Upsilon}_{p-1}(\widehat{\partial}_p(z)) = 0. \quad \square$$

Proposition 26. Let \vec{W} be a Morse sequence.

1) If $z \in Z_p(K)$, then $\tilde{\Upsilon}_p(\Upsilon_p(z)) \sim z$.

2) If $z \in Z^p(K)$, then $\tilde{\lambda}_p(\lambda_p(z)) \sim z$.

Proof. Let $z \in Z_p(K)$. By Theorem 9, $\Upsilon_p(z) \in Z_p(\widehat{W})$, and by Proposition 25 $\tilde{\Upsilon}_p(\Upsilon_p(z)) \in Z_p(K)$. By Proposition 21, we have $\Upsilon_p(\tilde{\Upsilon}_p(\Upsilon_p(z))) = \Upsilon_p(z)$. Now, by applying Theorem 18, we obtain $\tilde{\Upsilon}_p(\Upsilon_p(z)) \sim z$. \square

Theorem 27. Let \vec{W} be a Morse sequence. We have:

$$\partial_p \circ \tilde{\Upsilon}_p = \tilde{\Upsilon}_{p-1} \circ \widehat{\partial}_p \text{ and } \delta_p \circ \tilde{\lambda}_p = \tilde{\lambda}_{p+1} \circ \widehat{\delta}_p$$

Proof. Let $\kappa \in \widehat{W}$ with $\kappa \in K^{(p)}$.

- By Proposition 23, we have $\partial_p(\tilde{\Upsilon}(\kappa)) \subseteq W_{p-1}^+$. By Proposition 16, we obtain $\Upsilon_{p-1}(\partial_p(\tilde{\Upsilon}(\kappa))) = \partial_p(\tilde{\Upsilon}(\kappa)) \cap \widehat{W}$. Thus, by Proposition 24,

$$\Upsilon_{p-1}(\partial_p(\tilde{\Upsilon}(\kappa))) = \widehat{\partial}(\kappa).$$

- By Proposition 21, we have $\Upsilon_{p-1}(\tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa))) = \widehat{\partial}(\kappa)$. Thus:

$$\Upsilon_{p-1}(\partial_p(\tilde{\Upsilon}(\kappa))) = \Upsilon_{p-1}(\tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa))).$$

Furthermore, by Proposition 25, we have $\tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa)) \in Z_{p-1}(K)$ and, by Proposition 20, $\tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa)) \subseteq W_{p-1}^+$. Thus, by Theorem 17, we obtain:

$$\partial_p(\tilde{\Upsilon}(\kappa)) = \tilde{\Upsilon}_{p-1}(\widehat{\partial}(\kappa)),$$

which gives the result by linearity. \square

Thus, the previous theorem is a counter-part of Theorem 9 given for Υ and λ : the map $\tilde{\Upsilon}$ is a chain map from the chain complex $(\widehat{W}[p], \widehat{\partial}_p)$ to the chain complex $(K[p], \partial_p)$, and the map $\tilde{\lambda}$ is a cochain map from $(\widehat{W}[p], \widehat{\delta}_p)$ to $(K[p], \delta_p)$.

The counter-part of Υ_p^H and λ_p^H are two linear maps $\tilde{\Upsilon}_p^H$ and $\tilde{\lambda}_p^H$ between $H_p(\widehat{W})$ and $H_p(K)$, and between $H^p(\widehat{W})$ and $H^p(K)$.

From Propositions 21 and 26, we deduce that:

- 1) $\tilde{\Upsilon}_p^H \circ \Upsilon_p^H = Id_{H_p(K)}$ and $\Upsilon_p^H \circ \tilde{\Upsilon}_p^H = Id_{H_p(\widehat{W})}$,
- 2) $\tilde{\lambda}_p^H \circ \lambda_p^H = Id_{H^p(K)}$ and $\lambda_p^H \circ \tilde{\lambda}_p^H = Id_{H^p(\widehat{W})}$.

Therefore, we have the following result which relates the homology of a complex with the homology of its critical complex.

Theorem 28. *Let \vec{W} be a Morse sequence. For all p :*

- 1) *The vector spaces $H_p(K)$ and $H_p(\widehat{W})$ are isomorphic.*
- 2) *The vector spaces $H^p(K)$ and $H^p(\widehat{W})$ are isomorphic.*

We have seen that the critical complex $(\widehat{W}[p], \widehat{\partial}_p)$ is equivalent to the classical notion of a Morse complex. Thus, the previous theorem allows us to retrieve a basic fact of discrete Morse theory. There are several presentations of this fact, see [16], [29]. A specificity of the above results is that the isomorphism between $H_p(K)$ and $H_p(\widehat{W})$ is given directly with the maps Υ and $\widetilde{\Upsilon}$.

9 The flow complex and gradient flows

In discrete Morse theory, gradient flows and flow complexes are basic ingredients for setting the fundamental property of a Morse complex, that is, the equality of homology between a complex and its Morse complex. In this section, we first give the definition of chain complexes based on the images of $\widetilde{\Upsilon}$ and $\widetilde{\lambda}$, which was mentioned in the last section. Then, we outline a deep link between reference and extension maps of a Morse sequence, and gradient flows (Theorem 33). We conclude by showing the equivalence between the classical definition of a flow complex and the definition of a chain complex based on extension maps.

Let \vec{W} be a Morse sequence. We write:

$$\overline{O}[p] = \{\widetilde{\Upsilon}_p(c) \mid c \in \widehat{W}[p]\} \text{ and } \underline{O}[p] = \{\widetilde{\lambda}_p(c) \mid c \in \widehat{W}[p]\}.$$

That is, $\overline{O}[p]$ is the image of the map $\widetilde{\Upsilon}_p$, and $\underline{O}[p]$ is the image of the map $\widetilde{\lambda}_p$.

The sets $\overline{O}[p]$ and $\underline{O}[p]$ are two vector spaces. In the example of Figure 8, $\overline{O}[1]$ is composed of four vectors: we have $\overline{O}[1] = \{0, \widetilde{\Upsilon}_p(a), \widetilde{\Upsilon}_p(b), \widetilde{\Upsilon}_p(a+b)\}$.

Let $d \in \overline{O}[p]$ and let $c \in \widehat{W}[p]$ such that $d = \widetilde{\Upsilon}(c)$. We have $\partial_p(d) = \partial_p(\widetilde{\Upsilon}_p(c))$ and, by Theorem 27, $\partial_p(d) = \widetilde{\Upsilon}_{p-1}(\widehat{\partial}_p(c))$. Since $\widehat{\partial}_p(c) \in \widehat{W}[p-1]$, we obtain $\partial_p(d) \in \overline{O}[p-1]$. Thus, the boundary operator $\partial_p : K[p] \rightarrow K[p-1]$ can be restricted to $\partial_p : \overline{O}[p] \rightarrow \overline{O}[p-1]$. This leads us to the following definition:

Definition 10. *Let $\vec{W}(K)$ be a Morse sequence. We say that the chain complex $(\overline{O}[p], \partial_p)$ is the extension complex (of \vec{W}), and the cochain complex $(\underline{O}[p], \delta_p)$ is the coextension complex (of \vec{W}). We write $H_p(\overline{O})$ and $H^p(\underline{O})$ for the corresponding homology vector spaces.*

The map $\widetilde{\Upsilon}_p$ can be considered as a map $\widetilde{\Upsilon}_p : \widehat{W}[p] \rightarrow \overline{O}[p]$; with an abuse of notation, we use the same name for this map. Also, we may consider the restriction of the map Υ_p to $\overline{O}[p]$, we write $\Upsilon_p : \overline{O}[p] \rightarrow \widehat{W}[p]$; again we use the same name for this map. We proceed in the same manner with the maps $\widetilde{\lambda}_p$ and λ_p .

By Proposition 21, we have $\Upsilon_p \circ \widetilde{\Upsilon}_p = Id_{\widehat{W}[p]}$. Now, let $d \in \overline{O}[p]$, we have $d = \widetilde{\Upsilon}_p(c)$, with $c \in \widehat{W}[p]$. Again by Proposition 21, we have $\Upsilon_p(d) = c$. Thus,

$\tilde{\Upsilon}_p(\Upsilon_p(d)) = d$. Hence, $\tilde{\Upsilon}_p \circ \Upsilon_p = Id_{\widehat{\mathcal{O}}[p]}$. Since Υ_p and $\tilde{\Upsilon}_p$ are linear, we obtain:

Proposition 29. *Let \vec{W} be a Morse sequence. For all p :*

- 1) *The map $\tilde{\Upsilon}_p : \widehat{W}[p] \rightarrow \widehat{\mathcal{O}}[p]$ is a vector space isomorphism. Its inverse is given by the map $\Upsilon_p : \widehat{\mathcal{O}}[p] \rightarrow \widehat{W}[p]$.*
- 2) *The map $\tilde{\lambda}_p : \widehat{W}[p] \rightarrow \widehat{Q}[p]$ is a vector space isomorphism. Its inverse is given by the map $\lambda_p : \widehat{Q}[p] \rightarrow \widehat{W}[p]$.*

By Theorem 9, the map Υ is a chain map from the chain complex $(\widehat{\mathcal{O}}[p], \partial_p)$ to $(\widehat{W}[p], \partial_p)$, and by Theorem 27 the map $\tilde{\Upsilon}$ is a chain map from the chain complex $(\widehat{W}[p], \partial_p)$ to $(\widehat{\mathcal{O}}[p], \partial_p)$. As in the previous section, we obtain from Propositions 21 and 26 two isomorphisms Υ_p^H and $\tilde{\Upsilon}_p^H$ between $H_p(\widehat{W})$ and $H_p(\widehat{\mathcal{O}})$:

Theorem 30. *Let \vec{W} be a Morse sequence. For all p :*

- 1) *The vector spaces $H_p(\widehat{\mathcal{O}})$ and $H_p(\widehat{W})$ are isomorphic.*
- 2) *The vector spaces $H^p(\widehat{Q})$ and $H^p(\widehat{W})$ are isomorphic.*

Now, we recall the definition of a gradient flow, see [18]; we just adapt this definition in the context of Morse sequences. See also [31, 32] for an interpretation in terms of chain homotopy equivalences.

Let $\vec{W}(K)$ be a Morse sequence.

We write V_p, V_p^* for the linear maps $V_p, V_p^* : K[p] \rightarrow K[p]$ such that:

- $V(\kappa) = V^*(\kappa) = 0$ if κ is critical for \vec{W} .
- $V(\sigma) = \tau, V(\tau) = 0, V^*(\sigma) = 0, V^*(\tau) = \sigma$ if (σ, τ) is a regular pair for \vec{W} .

We write Φ_p, Φ_p^* for the linear maps $\Phi_p, \Phi_p^* : K[p] \rightarrow K[p]$ such that:

- for each $\nu \in K^{(p)}$, $\Phi_p(\nu) = \nu + \partial_{p+1}(V_p(\nu)) + V_{p-1}(\partial_p(\nu))$.
- for each $\nu \in K^{(p)}$, $\Phi_p^*(\nu) = \nu + \delta_{p-1}(V_p^*(\nu)) + V_{p+1}^*(\delta_p(\nu))$.

The maps Φ and Φ^* are, respectively, the (gradient) flow and coflow of \vec{W} .

If $c \in K[p]$, then there are integers i, j such that $\Phi^{i+1}(c) = \Phi^i(c)$ and $(\Phi^*)^{j+1}(c) = (\Phi^*)^j(c)$, we write $\overline{\Phi}_p(c) = \Phi^i(c)$ and $\underline{\Phi}_p(c) = (\Phi^*)^j(c)$.

In this way we obtain two linear maps $\overline{\Phi}_p, \underline{\Phi}_p : K[p] \rightarrow K[p]$.

Let $\pi = \langle \pi_0, \dots, \pi_k \rangle$ be a sequence where each π_i is either a gradient or a cogradient path in $\vec{W}(K)$. We say that π is a composite (gradient) path (from ν to μ) if:

- 1) The path π_0 begins at ν and the path π_k ends at μ ,
- 2) For each $i \in [1, k]$, the path π_i begins where π_{i-1} ends.

We say that π is *nontrivial* if π contains more than one simplex.

For example, if $\pi_0 = \langle \sigma_0, \tau_0, \nu \rangle$ is a gradient path and if $\pi_1 = \langle \nu, \sigma_1, \tau_1 \rangle$ is a cogradient path, then the sequence $\pi = \langle \pi_0, \pi_1 \rangle = \langle \sigma_0, \tau_0, \nu, \sigma_1, \tau_1 \rangle$ is a composite path from σ_0 to τ_1 . Note that we have $\dim(\sigma_0) = \dim(\tau_1)$.

Let $\nu, \mu \in K^{(p)}$. We let Ω_ν^μ denote the set of all composite paths from ν to μ which include at least one critical simplex. Then, we have the following characterization of the maps $\overline{\Phi}$ and $\underline{\Phi}$.

Theorem 31 (from [18]). *Let \overrightarrow{W} be a Morse sequence on K and let $\nu \in K$.*

- We have $\mu \in \overline{\Phi}(\nu)$ if and only if $\text{Card}(\Omega_\nu^\mu)$ is odd.
- We have $\mu \in \underline{\Phi}(\nu)$ if and only if $\text{Card}(\Omega_\nu^\mu)$ is odd.

From Lemma 1.3 of [18], we also have three basic properties of composite paths from which we easily derive Proposition 32:

- If $\kappa, \kappa' \in \widehat{W}$ and $\kappa \neq \kappa'$, then there are no composite paths from κ to κ' .
- If $\kappa \in \widehat{W}$, then any composite path ending at κ is a gradient path.
- If $\kappa \in \widehat{W}$, then any composite path beginning at κ is a cogradient path.

Proposition 32. *Let \overrightarrow{W} be a Morse sequence on K . Let $\nu, \mu \in K^{(p)}$ and let $\pi \in \Omega_\nu^\mu$. Then there exists $\kappa \in \widehat{W}$, a gradient path π' from ν to κ , and a cogradient path π'' from κ to μ , such that $\pi = \langle \pi', \pi'' \rangle$.*

Let $\nu, \mu \in K^{(p)}$. Suppose $\nu \in \widehat{W}$. Then, Ω_ν^μ is necessarily a cogradient path. By Theorem 31, Theorem 5, and Definition 9, we obtain $\overline{\Phi}(\nu) = \widetilde{\gamma}(\nu)$. In fact, we have the following more general result which shows that $\overline{\Phi}$ may be decomposed into the maps $\widetilde{\gamma}$ and γ .

Theorem 33. *If \overrightarrow{W} is a Morse sequence, then we have:*

$$\overline{\Phi} = \widetilde{\gamma} \circ \gamma \text{ and } \underline{\Phi} = \widetilde{\lambda} \circ \lambda.$$

Proof. Let $\nu, \mu \in K^{(p)}$. If $\kappa \in \widehat{W}$, we write $\Omega_\nu^\mu(\kappa)$ for the set composed of all paths in Ω_ν^μ which contain κ . Since each path in Ω_ν^μ contains a single critical simplex, by Theorem 31, we have:

$$\mu \in \overline{\Phi}(\nu) \text{ if and only if } \sum \{ \text{Card}(\Omega_\nu^\mu(\kappa)) \mid \kappa \in \widehat{W} \} \text{ is odd.}$$

But $\text{Card}(\Omega_\nu^\mu(\kappa)) = \text{Card}(\Omega_\nu^\kappa) \times \text{Card}(\Omega_\kappa^\mu)$. Therefore, by parity considerations, we have $\mu \in \overline{\Phi}(\nu)$ if and only if:

$$\sum \{ \text{Card}(\Omega_\nu^\kappa) \mid \kappa \in \widehat{W} \text{ and } \text{Card}(\Omega_\kappa^\mu) \text{ is odd} \} \text{ is odd.}$$

By Proposition 32, Ω_ν^κ is the set of all gradient paths from ν to κ . Thus, by Theorem 5, we have $\gamma(\nu) = \{ \kappa \in \widehat{W} \mid \text{Card}(\Omega_\nu^\kappa) \text{ is odd} \}$. Therefore:

$$\mu \in \overline{\Phi}(\nu) \text{ if and only if } \sum \{ \text{Card}(\Omega_\kappa^\mu) \mid \kappa \in \gamma(\nu) \} \text{ is odd.}$$

By Proposition 32, Ω_κ^μ is the set of all cogradient paths from κ to μ . By Definition 9, and by Theorem 5, we obtain:

$$\mu \in \overline{\Phi}(\nu) \text{ if and only if } \mu \in \widetilde{\gamma}(\gamma(\nu)). \quad \square$$

Classically, the flow complex of a complex K is defined with the vector space $F_p = \{c \in K[p] \mid \Phi_p(c) = c\}$, see Section 7 of [16] and Section 8.2 of [35]. We have $F_p = \{c \in K[p] \mid \overline{\Phi}_p(c) = c\}$. Also, we observe that:

- If $c \in F_p$, then by Theorem 33, $\widetilde{\gamma}_p(\gamma_p(c)) = c$. Therefore we have $c = \widetilde{\gamma}_p(d)$, with $d \in \widehat{W}[p]$. It follows that $c \in \overline{O}[p]$.
- If $c \in \overline{O}[p]$, then $c = \widetilde{\gamma}_p(d)$ with $d \in \widehat{W}[p]$. By Theorem 33, we

have $\overline{\Phi}_p(c) = \tilde{\Upsilon}_p(\Upsilon_p(c)) = \tilde{\Upsilon}_p(\Upsilon_p(\tilde{\Upsilon}_p(d)))$. By Proposition 21, we derive $\overline{\Phi}_p(c) = \tilde{\Upsilon}_p(d) = c$. Thus $c \in F_p$.

Therefore, as a direct consequence of the previous theorem, we obtain the equivalence between a flow complex and an extension complex.

Corollary 34. *If \overrightarrow{W} is a Morse sequence, then we have:*

$$\overline{O}[p] = \{c \in K[p] \mid \overline{\Phi}(c) = c\} \text{ and } \underline{O}[p] = \{c \in K[p] \mid \underline{\Phi}(c) = c\}.$$

10 Conclusion

In the final chapter of his gentle introduction to discrete Morse theory [35], Nicholas Scoville wrote, “As is now apparent, the idea behind discrete Morse theory is extremely simple: every simplicial complex can be broken down (or, equivalently, built up) using only two moves: 1) perform an elementary collapse, and 2) remove a facet.” In this paper, we just start from this basic fact for an alternative approach to this theory. A Morse sequence is simply made of two operators which provide a direct introduction to discrete Morse theory with two fundamental aspects: homotopy with the expansion operator, and homology with the filling operator.

A key aspect of a Morse sequence is that it gives rise to the maps Υ and $\tilde{\Upsilon}$. These two maps, along with their dual λ and $\tilde{\lambda}$, contain the crucial homological information of the sequence.

The reference map Υ assigns, to each simplex, a set of critical simplexes. This leads us to a chain complex, the critical complex, which corresponds precisely to the classical definition of such a complex. The map Υ is a chain map: it allows to carry out the homology of the original complex to the critical complex.

The extension map $\tilde{\Upsilon}$ assigns, to each critical simplex, a set of simplexes. It is noteworthy that Υ and $\tilde{\Upsilon}$ are “homology inverses”. This fact allows us to recover a fundamental property, that is, the equality of homology between a complex and its critical complex. Also, extension maps leads directly to a chain complex which fully characterizes a flow complex.

Arranged Morse sequences constitute a special class of sequences which contains different kinds of skeletons of the complex. Any Morse sequence may be transformed to an arranged sequence by a simple rearrangement which preserves the gradient vector field. The skeleton sequence which is included in an arranged sequence allows us to derive some crucial properties of the map Υ . Furthermore, it leads to a result which provides another aspect of the fundamental collapse theorem of discrete Morse theory.

Reference and extension maps, as well as their duals, can be computed with solely two local operators. For future work [7], we will take advantage of this fact to derive new algorithms for topological data analysis and for digital image segmentation [34, 12, 11]. Also, we will make a link between Morse sequences and some notions of Mathematical Morphology such as the watersheds [10, 9, 36, 6]. On the theoretical side, we aim at extending our framework to other fields than the mod 2 arithmetic and to other kind of complexes.

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A Discrete vector fields

We first recall the definitions of a discrete vector field and a p -gradient path, see Definitions 2.43 and 2.46 of [35]. See also [22, 23] where a generalization of gradient vector fields is introduced, which allows cycles under a certain algebraic condition.

Let K be a complex and V be a set of pairs (σ, τ) , with $\sigma, \tau \in K$ and $\sigma \in \partial\tau$. We say that V is a *(discrete) vector field on K* if each simplex of K is in at most one pair of V . We say that $\sigma \in K$ is *critical for V* if σ is not in a pair of V .

Let V be a vector field on a complex K . A *(p -)gradient path in V (from σ_0 to σ_k)* is a sequence $\pi = \langle \sigma_0, \tau_0, \sigma_1, \tau_1, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$, with $k \geq 0$, composed of faces $\sigma_i \in K^{(p)}$, $\tau_i \in K^{(p+1)}$ such that, for all $i \in [0, k-1]$, (σ_i, τ_i) is in V , $\sigma_{i+1} \subset \tau_i$, and $\sigma_{i+1} \neq \sigma_i$. This sequence π is said to be *trivial* if $k = 0$, that is, if $\pi = \langle \sigma_0 \rangle$; otherwise, if $k \geq 1$, we say that π is *non-trivial*. Also, the sequence π is *closed* if $\sigma_0 = \sigma_k$. We say that a vector field V is *acyclic* if V contains no non-trivial closed p -gradient path.

Now, let \vec{W} be a Morse sequence on K . Then, the gradient vector field of \vec{W} is clearly a vector field. Also, a path is a gradient path in this vector field if and only if it is a gradient path in \vec{W} .

In the sequel of this section, we want to emphasize that a Morse sequence may be seen as an alternative way to represent the gradient vector field of an arbitrary discrete Morse function. A classical result of discrete Morse theory states that a discrete vector field V is the gradient vector field of a discrete Morse function if and only if V is acyclic (Theorem 2.51 of [35]). Thus, in order to achieve this goal, it is sufficient to establish the equivalence between gradient vector fields of Morse sequences and acyclic vector fields. In fact, this equivalence is a direct consequence of Theorem 11.9 of [29] which is formulated in terms of linear extensions associated to a poset. We give hereafter a proof which is based on the notion of a maximal gradient path. Such a path allows us to extract, in the top dimension of a complex K , either a critical simplex or a free pair for K (Lemma 35). This formalizes an incremental deconstruction of a complex, which is usually given with certain Morse functions, see Remark 13 of [1].

Let V be a vector field on K and let $\pi = \langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$ be a p -gradient path in V . We say that a pair of simplexes (η, ν) is an *extension of π (in V)* if $\langle \eta, \nu, \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$ or if $\langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k, \nu, \eta \rangle$ is a p -gradient path in V . We say that π is *maximal (in V)* if π has no extension in V . If V is acyclic, it can be checked that, for any $p \geq 0$, there exists a maximal p -gradient path in V . To see this point, we can pick an arbitrary (possibly trivial) p -gradient path and extend it iteratively with extensions. If V is acyclic, we obtain a maximal path after a finite number of extensions.

Lemma 35. *Let V be an acyclic vector field on K , with $\dim(K) = d$. Then, at least one of the following holds:*

- 1) There exists a facet τ of K , with $\dim(\tau) = d$, that is critical for V .
- 2) There exists a pair (σ, τ) in V , with $\dim(\tau) = d$, that is a free pair for K .

Proof. If K has a d -face that is critical for V , then we are done. Suppose there is no such faces in K . If $d = 0$, then all the 0-faces of K are critical, thus we must have $d \geq 1$. Let τ be an arbitrary d -face of K . Since τ is not critical, there exists a pair (σ, τ) that is in V . Since $d \geq 1$, there is a face $\sigma' \in K$ such that $\pi' = \langle \sigma, \tau, \sigma' \rangle$ is a $(d-1)$ gradient path in V . By iteratively extending π' with extensions we obtain a maximal $(d-1)$ -gradient path in V that is non-trivial. Let $\pi = \langle \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$ be such a path, we have $k \geq 1$. If (σ_0, τ_0) is a free pair for K , then we are done. Otherwise, σ_0 must be a subset of a d -simplex ν , with $\nu \neq \tau_0$. By our hypothesis ν is not critical for V . Since ν is a facet for K , there must exist a $(d-1)$ -simplex η , $\eta \neq \sigma_0$, such that (η, ν) is in V . In this case, the path $\pi' = \langle \eta, \nu, \sigma_0, \tau_0, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$ would be a $(d-1)$ -gradient path in V . Thus, the path π would not be maximal, a contradiction: the pair (σ_0, τ_0) must be a free pair for K . \square

Theorem 36. *Let K be a simplicial complex. A vector field V on K is acyclic if and only if V is the gradient vector field of a Morse sequence on K .*

Proof. i) Let $\vec{W} = \langle \emptyset = K_0, \dots, K_i, \dots, K_l = K \rangle$ be a Morse sequence on K , and let V be the gradient vector field of \vec{W} . For each $\nu \in K$, let $\rho(\nu)$ be the index i such that $\nu \in K_i$ and $\nu \notin K_{i-1}$. Now, let $\pi = \langle \sigma_0, \tau_0, \sigma_1, \tau_1, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$, $k \geq 1$, be a non-trivial p -gradient path in V . For all $i \in [0, k-1]$, (σ_i, τ_i) is in V , thus $\rho(\sigma_i) = \rho(\tau_i)$. Since $\sigma_{i+1} \subset \tau_i$ and since \vec{W} is a filtration, we have $\rho(\sigma_{i+1}) \leq \rho(\tau_i)$. Since $\sigma_{i+1} \neq \sigma_i$ the pair (σ_{i+1}, τ_i) is not a regular pair for \vec{W} , thus we have $\rho(\sigma_{i+1}) < \rho(\tau_i)$. It follows that, for all $i \in [0, k-1]$, we have $\rho(\sigma_{i+1}) < \rho(\sigma_i)$. This gives $\rho(\sigma_k) < \rho(\sigma_0)$. It means that $\sigma_k \neq \sigma_0$, and the path π cannot be closed. Thus the vector field V is acyclic.

ii) Let V be an acyclic vector field on K , with $\dim(K) = d$.

1) Suppose there exists a facet τ of K , with $\dim(\tau) = d$, that is critical for V .

Let $K' = K \setminus \{\tau\}$ and $V' = V$. Then, the set V' is also an acyclic vector field on K' .

2) Suppose there exists a pair (σ, τ) in V , with $\dim(\tau) = d$, that is a free pair for K . Clearly, the set $V' = V \setminus \{(\sigma, \tau)\}$ is also an acyclic vector field on $K' = K \setminus \{\sigma, \tau\}$.

By 1), 2), and by Lemma 35, we can build inductively two sequences $\overleftarrow{W} = \langle K = K_0, \dots, K_k = \emptyset \rangle$ and $\langle V = V_0, \dots, V_k = \emptyset \rangle$ such that, for each $i \in [1, k]$:

- either K_i is an elementary perforation of K_{i-1} and $V_i = V_{i-1}$,

- or $K_i = K_{i-1} \setminus \{\sigma, \tau\}$ is an elementary collapse of K_{i-1} and $V_i = V_{i-1} \setminus \{(\sigma, \tau)\}$.

By considering the inverse of \overleftarrow{W} we obtain the sequence $\vec{W}' = \langle K_k = K'_k, \dots, K'_0 = K_0 \rangle$, which is such that, for each $i \in [1, k]$, either K'_i is an elementary expansion of K'_{i-1} , or K'_i is an elementary filling of K'_{i-1} . In other words, \vec{W}' is a Morse sequence on $K_0 = K$; the gradient field of \vec{W}' is precisely V , as required. \square

B Morse functions and Morse sequences

Discrete Morse theory is classically introduced through the concept of a discrete Morse function. In this section we show that it is possible, in a straightforward manner, to make a link between Morse sequences and these functions.

We first introduce the notion of a Morse function on a Morse sequence \vec{W} .

Definition 11. Let $\vec{W}(K)$ be a Morse sequence, let $\diamond\vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$. We say that a map $f: K \rightarrow \mathbb{Z}$ is a Morse function on \vec{W} if f satisfies the two conditions:

- 1) If κ_i is critical for \vec{W} and $\nu \in \partial(\kappa_i)$, then $f(\kappa_i) > f(\nu)$.
- 2) If $\kappa_i = (\sigma, \tau)$ is regular for \vec{W} , then $f(\sigma) \geq f(\tau)$.

Now, we can check that the following definition of a Morse function on a simplicial complex K is equivalent to the classical one [17, 19].

Let K be a simplicial complex and let $f: K \rightarrow \mathbb{Z}$ be a map on K . Let V be the set of all pairs (σ, τ) , with $\sigma, \tau \in K$, such that $\sigma \in \partial(\tau)$ and $f(\sigma) \geq f(\tau)$. If each $\nu \in K$ is in at most one pair in V , we say that f is a Morse function on K , and V is the gradient vector field of f . We say that two Morse functions on K are equivalent if they have the same gradient vector field.

Let f be a Morse function on K , and V be the gradient vector field of f . From the above definition, the set V is a discrete vector field on K . If $\pi = \langle \sigma_0, \tau_0, \sigma_1, \tau_1, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$ is a p -gradient path in V , we have $f(\sigma_i) \geq f(\tau_i)$, and also $f(\tau_i) > f(\sigma_{i+1})$. Thus, $f(\sigma_0) > f(\sigma_k)$ whenever $k \geq 1$. It means that V contains no non-trivial closed p -gradient path. In other words, we have the classical result:

Proposition 37. If f is a Morse function on K , then the gradient vector field of f is an acyclic vector field.

Let \vec{W} be a Morse sequence on K . We see that a Morse function on \vec{W} is indeed a Morse function on K , the gradient vector field of this Morse function is precisely the gradient vector field of \vec{W} . Conversely, by Proposition 37 and by Theorem 36, if f is a Morse function on K , then there exists a Morse sequence \vec{W} on K which has the same gradient vector field as f . It is easy to check that f is also a Morse function on \vec{W} . This leads us to the following result.

Theorem 38. If f is a Morse function on K , then there exists a Morse sequence \vec{W} on K such that f is a Morse function on \vec{W} . Furthermore, any Morse function on \vec{W} is equivalent to f .

We introduce hereafter a particular kind of Morse function. Since a Morse sequence is a filtration, the following function f is indeed a Morse function on \vec{W} .

Definition 12. Let \vec{W} be a Morse sequence on K and let $\diamond\vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$. The canonical Morse function of \vec{W} is the map $f: K \rightarrow \mathbb{Z}$ such that:

- 1) If κ_i is critical for \vec{W} , then $f(\kappa_i) = i$.
- 2) If $\kappa_i = (\sigma, \tau)$ is regular for \vec{W} , then $f(\sigma) = f(\tau) = i$.

As a consequence of Theorem 38, any Morse function on K is equivalent to a canonical Morse function.

We note that a canonical Morse function f is *flat*, that is, we have $f(\sigma) = f(\tau)$ whenever (σ, τ) is in the gradient vector field of f (Definition 4.14 of [35]). Also f is *excellent*, that is, all values of the critical simplexes are distinct (Definition 2.31 of [35]). In fact, a canonical Morse function has the three properties which define a basic Morse function (see [2] and also Definition 2.3 of [35]).

Let $f: K \rightarrow \mathbb{Z}$ be a map on K . We say that f is a *basic Morse function* if f satisfies the properties:

- 1) *monotonicity*: we have $f(\sigma) \leq f(\tau)$ whenever $\sigma \subseteq \tau$;
- 2) *semi-injectivity*: for each $i \in \mathbb{Z}$, the cardinality of $f^{-1}(i)$ is at most 2;
- 3) *genericity*: if $f(\sigma) = f(\tau)$, then either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

We observe that, if f is a basic Morse function on K , then we can build a Morse sequence \vec{W} if we pick the simplexes of K by increasing values of f . For each i , $f^{-1}(i)$ gives a critical simplex if the cardinality of $f^{-1}(i)$ is one, and $f^{-1}(i)$ gives a regular pair if the cardinality of $f^{-1}(i)$ is two.

Let f and g be two basic Morse functions on K . We say that f and g are *strongly equivalent* if f and g induce the same order on K . That is, we have $f(\sigma) \leq f(\tau)$ if and only if $g(\sigma) \leq g(\tau)$.

With the above scheme for building a Morse sequence from a basic Morse function, we obtain the following result.

Proposition 39. *Let f be a basic Morse function on K . There exists one and only one Morse sequence \vec{W} such that the canonical Morse function of \vec{W} is strongly equivalent to f .*