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Solving viability problems in dynamic games using individual strategies derived from guaranteed viability kernels: application to an agricultural cooperative model [★]

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Abstract

In this paper we address decentralized control problems in the context of sustainable management of discrete-time deterministic dynamical systems. We consider N players, each with their own controlled dynamical system, which are linked through the dynamics of a global variable. We consider sustainable constraints as constraints in the state space. From this general problem defined in both viability theory and game theory, we define N viability sub-problems. For each of these, we consider the guaranteed viability kernel, subset of the constraint set where it is possible to maintain the subsystem whatever the strategy of other players. When these kernels are not empty, we propose a general algorithm to compute individual strategies that ensure the sustainability of all. Numerical experiments on a model of agricultural cooperative show the feasibility of our approach and suggest that it can be the basis for the definition of new operating rules when the sustainability constraints cannot be ensured.

Key words: N -player discrete-time dynamic games; viability problem; decentralization; guaranteed viability kernel; agricultural cooperative model; control of constrained systems.

1 Introduction

Problems dealing with the compatibility between dynamics and constraints over time are present in renewable resource management and especially in the regulation of fisheries [30,11,18,20], urban pigeon population [23], forest preservation [2,14] or lake eutrophication management [28], see [33] for a review. They also arise in broader(eco)-system dynamics [19,22,37], in pure economic or social ones [4,13], or in a variety of engineering problems [25,36,35,6].

In all these works, systems are described by global variables and the control regulation also operates at the global scale. But, for example, the total amount of harvest is the sum of harvests of individual fisheries that have their own controls and viability constraints. They make their decisions with an imperfect knowledge of the whole system and without having the ability of predicting others' behaviors. Hence, an important issue is how to find individual strategies that ensure the viability of all when the system under consideration involves

many individual entities whose state evolutions are intertwined.

Addressing this issue with a direct approach consists in testing by simulations the viability of evolutions governed by prespecified individual strategies. For instance, [29] have proposed to combine viability theory and simulations to study viability properties of an individual-based model facing both local and global constraints: they used viability theory tools to derive individual feedback strategies according to assumptions made on others' possible behaviors (myopic or not). And they studied by simulations the impact of these strategies on collective viability. Such strategies reinforce viability only when the anticipations on others' behaviors are correct.

The inverse problem of the individual strategy design in order to reach particular individual objectives is addressed by both differential game theory as stated in [9] and viability theory [7,3].

Actually, in the framework of game theory, Nash [32] provided the concept of solution for general noncooperative games for N players. When the players' state depends on time, the games are called differential games, first developed by Isaacs [24]. Viability problems can be

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considered as particular differential games (with a particular payoff function). But, assumptions are needed to find feedback Nash equilibria such as nonzero-sum differential games with a very large number of players in the case of mean-field games [26,12] or nonzero-sum linear quadratic games with N players [16]. Here we are interested in games with N players with more general dynamics.

In the viability theory framework, the viability of each player is represented by a constraint set, a subset of the state space inside which the system state must remain. The classical answer to viability problems [7] is the viability kernel which is a subset of the state space which gathers all states from which there exists at least one set of individual control functions that governs a viable evolution. Hence, a coordination between all individual entities is required to follow such a viable path. In the context of two-player noncooperative differential games, the concept of guaranteed viability kernel introduced by [8] was defined as the subset of the state space that gathers states from which one player can ensure its viability over time whatever the other player's actions. Guaranteed viability is the appropriate framework when other players are considered as perturbations (such as move by nature in [10] or others' models in [1]).

In the context of N -player discrete-time deterministic infinite non-linear dynamic games, the objective of this paper is to show how individual strategies derived from individual guaranteed viability kernels can ensure all players' viability. Hence, designing mechanisms that ensure that all individual guaranteed viability kernels are non-empty emerges as a way of building collective viability.

The rest of the paper is organized as follows. In Section 2, we specify the generic problem addressed in this paper in both frameworks of game theory and viability theory. We specify the particular assumptions we make and highlight the classical assumptions we do not need. In Section 3, we propose our resolution approach based on individual guaranteed viability kernels computation. We also design an algorithm to approximate them. And we exhibit sufficient conditions to allow to derive from these guaranteed viability kernels, N sets of feedback strategies that govern system evolutions in which all individual constraints are satisfied over time. Section 4 shows experiments on an agricultural cooperative model from literature [29]. We detail the implementation of our approach illustrated with numerical results on computed individual strategies that ensure the viability of all. When our approach results in empty guaranteed viability kernels, and then failed at providing the desired individual strategies, the definition of the game must change. Here we propose to change the rules. Ensuring the non-emptiness of the individual guaranteed viability kernels then becomes a guide to design new operating rules. The final section discusses the contributions of the paper.

2 Problem formulation

Let us consider a global entity described by the macroscopic variables $x \in \mathbb{R}^n$ which gathers N individuals (or players). Each member $i \in \mathcal{N} := \{1, \dots, N\}$ is described by the state variables $x_i \in \mathbb{R}^{n_i}$.

Let us assume that the evolution of state variables x_i of member i is governed by the discrete-time controlled dynamical system (f_i, U_i) :

$$\begin{aligned} x_i(t+1) &= f_i(x_i(t), u_i(t), x(t)) \\ u_i(t) &\in U_i(x_i(t), x(t)) \subset \mathbb{R}^{p_i}. \end{aligned} \quad (1)$$

The map $U_i : \mathbb{R}^{n_i} \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^{p_i}$ associates the set of admissible controls with the state of member i , x_i , and the global state, x . The function $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ associates the new values of local state variables with the current values of the local state variables, the global state variables, and the local control variables.

The variations of the macroscopic variables $x \in \mathbb{R}^n$ are also governed by a discrete-time controlled dynamical system $(f, (U_i)_{i \in \mathcal{N}})$:

$$\begin{aligned} x(t+1) &= f(x(t), (x_i(t))_{i \in \mathcal{N}}, (u_i(t))_{i \in \mathcal{N}}) \\ u_i(t) &\in U_i(x_i(t), x(t)) \quad \forall i \in \mathcal{N}. \end{aligned} \quad (2)$$

The function $f : \mathbb{R}^n \times (\prod_{i=1}^N \mathbb{R}^{n_i}) \times (\prod_{i=1}^N \mathbb{R}^{p_i})^1 \rightarrow \mathbb{R}^n$ associates the new values of the global state variables, with the current values of the global state variables, all local state variables and all local control variables.

Besides, let us assume that each individual $i \in \mathcal{N}$ faces viability constraints described by a subset of the local state space $\mathcal{L}_i \subset \mathbb{R}^{n_i}$ and the global entity also faces viability constraints described by a subset of the state space $L \subset \mathbb{R}^n$.

Definition 1 *An evolution of the whole system will be considered as viable if and only if*

$$\forall t \in \mathbb{N}, \quad x(t) \in L \text{ and } \forall i \in \mathcal{N}, \quad x_i(t) \in \mathcal{L}_i. \quad (3)$$

The particular viability problem we address in this paper is to find a subset $V \subset L \times (\prod_{i=1}^N \mathcal{L}_i) \subset \mathbb{R}^n \times (\prod_{i=1}^N \mathbb{R}^{n_i})$ of initial states and N feedback functions $\tilde{U}_i : \mathbb{R}^{n_i} \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^{p_i}$ such that all solutions of the dynamical system (\mathcal{S}) below starting at $(x(0), x_1(0), \dots, x_N(0)) \in V$

¹ We denote by $\prod_{i=1}^n X_i$ the n -ary Cartesian product over the n sets X_1, \dots, X_n

$\prod_{i=1}^n X_i := X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i, \forall i \in \{1, \dots, n\}\}$.

are viable.

$$S \begin{cases} x(t+1) = f(x(t), (x_i(t))_{i \in \mathcal{N}}, (u_i(t))_{i \in \mathcal{N}}) \\ x_i(t+1) = f_i(x_i(t), u_i(t), x(t)), \forall i \in \mathcal{N} \\ u_i(t) \in \tilde{U}_i(x_i(t), x(t)), \forall i \in \mathcal{N}. \end{cases} \quad (4)$$

When there exist such feedback functions $(\tilde{U}_i)_{i \in \mathcal{N}}$, it means that if all the players play according to their own feedback function \tilde{U}_i , although they don't know the actions of others, they are all viable.

2.1 Problem description in the game theory framework

Following the definitions of [9], the problem we address belongs to infinite dynamic games: the action sets of the players comprise an infinite number (in fact a continuum) of elements (alternatives) (Eq. (1)), and the players gain some dynamic information throughout the decision process. More precisely, we consider a discrete-time deterministic infinite dynamic game with infinite horizon since the time is discretized, the dynamics are deterministic and the time horizon is not prespecified. Following again the definitions of [9], in the dynamic game we consider,

- (i) the players' set is \mathcal{N} , the stage is described by $t \in \mathbb{N}$, the state space of the game is $\mathbb{R}^n \times (\prod_{i=1}^N \mathbb{R}^{n_i})$, the product set of the global and individual variables $(x, (x_i)_{i \in \mathcal{N}})$ (Eqs. (1)-(2)),
- (ii) the action (control) set (the permissible actions) of player i at stage t is \mathbb{R}^{p_i} , to which the action u_i of player i belongs, and the admissible strategies of player i at all stages are mappings u_i from $\mathbb{R}^{n_i} \times \mathbb{R}^n$ to \mathbb{R}^{p_i} that satisfy $u_i(x_i, x) \in U_i(x_i, x) \subset \mathbb{R}^{p_i}$ (Eq. (1)),
- (iii) the state equation of the dynamic game, describing the evolution of the underlying decision process is the function which associates any state and any action

$$(x, (x_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}}) \in \mathbb{R}^n \times \left(\prod_{i=1}^N \mathbb{R}^{n_i} \right) \times \left(\prod_{i=1}^N \mathbb{R}^{p_i} \right)$$

with the next system state

$$(f(x, (x_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}}), (f_i(x_i, u_i, x))_{i \in \mathcal{N}})$$

(this function does not depend on the stage of the game) (Eqs. (1)-(2)),

- (iv) the observation set of player i at all stages is $\mathbb{R}^{n_i} \times \mathbb{R}^n$, the product space of the state space of player i and the global state space, to which the observation (x_i, x) of player i belongs at all stages (Eq. (1)). The state-measurement (-observation) equation of player i is then at all stages the projection of the

system state on the product of the state space of player i and the global state space,

- (v) the information structure of the game is a feedback imperfect state information (FIS) pattern: the information gained and recalled by any player i at any stage t is $(x_i(t), x(t))$, it has no additional information about the strategy of the other players. In particular, he can not predict the future actions of the other players. The information space of any player i at any stage t , induced by his information $(x_i(t), x(t))$ is then $\mathbb{R}^{n_i} \times \mathbb{R}^n$,
- (vi) the cost functional of player i in this game of infinite duration derives from the objective (Eq.(3)):

$$G_i((x(t))_{t \in \mathbb{N}}, (x_i(t))_{i \in \mathcal{N}, t \in \mathbb{N}}, (u_i(t))_{i \in \mathcal{N}, t \in \mathbb{N}}) = \sum_{t \in \mathbb{N}} g_i(x(t), x_i(t))$$

where $g_i(x, x_i) = 0$ if $x_i \in \mathcal{L}_i$ and $x \in L$, and 1 otherwise.

We recall that a N -uplet of deterministic control functions is a pure Nash equilibrium for the game within the class of feedback strategies if for all $i \in \mathcal{N}$, the i^{th} control function provides an optimal feedback in connection with the cost functional of player i .

Hence, in the game we consider, the classical game theory approach by looking for pure feedback Nash equilibria allows to find N -uplets of control functions $(u_i^*(x_i, x))_{i \in \mathcal{N}}$ such that for all $i \in \mathcal{N}$, the control $(x_i, x) \rightarrow u_i^*(x_i, x)$ provides an optimal feedback in connection with the optimal control problem for player i . Let us consider the set

$$D := \{(x, (x_i)_{i \in \mathcal{N}}) \in \mathbb{R}^n \times (\prod_{i=1}^N \mathbb{R}^{n_i}) | \forall i \in \mathcal{N}, G_i((x^*(t))_{t \in \mathbb{N}}, (x_i^*(t))_{i \in \mathcal{N}, t \in \mathbb{N}}, (u_i^*(x(t), x_i(t)))_{i \in \mathcal{N}, t \in \mathbb{N}}) = 0\}$$

with $(x^*(\cdot), (x_i^*(\cdot))_{i \in \mathcal{N}})$ the evolution of the game governed by the individual feedback strategies $(u_i^*)_{i \in \mathcal{N}}$ starting at $(x, (x_i)_{i \in \mathcal{N}})$. Then, from states belonging to D , the N feedback functions u_i^* govern viable evolutions of the game since G_i is null for all $i \in \mathcal{N}$.

Hence, pure feedback Nash equilibria associated with the subset of the state space where all costs are null provide solutions to the problem addressed in this paper.

However, apart from the linear quadratic case and the mean-field cases, the Nash equilibria cannot be described by analytical expressions. In the general case where approximate solutions must be sought, the concepts and algorithms of viability theory make it possible to propose solutions.

2.2 Problem description in the viability theory framework

The problem addressed in this paper can also be stated in the framework of viability theory. Following [7,3] notations,

- (i) the system state belongs to $\mathbb{R}^n \times (\prod_{i=1}^N \mathbb{R}^{n_i})$, the product set of the global and individual variables $(x, (x_i)_{i \in \mathcal{N}})$ (Eqs. (1)-(2)),
- (ii) each individual i can choose its control values u_i in the set of admissible controls which depends on the global variables and on its own individual variables $(x, (x_i)_{i \in \mathcal{N}})$ (Eq. (1))

$$(u_i)_{i \in \mathcal{N}} \in \prod_{i=1}^N U_i(x_i, x), \quad (5)$$

- (iii) given the system state at time $t \in \mathbb{N}$ and the controls chosen by each individual, the dynamics governing the system state values at time $t + 1$ are deterministic (Eqs. (1)-(2))

$$\begin{cases} x(t+1) = f(x(t), (x_i(t))_{i \in \mathcal{N}}, (u_i(t))_{i \in \mathcal{N}}) \\ (x_i(t+1))_{i \in \mathcal{N}} = (f_i(x_i(t), u_i(t), x(t)))_{i \in \mathcal{N}}, \end{cases} \quad (6)$$

- (iv) from Eq. (3), the viability constraints are described by the subset of the state space

$$L \times \left(\prod_{i=1}^N \mathcal{L}_i \right). \quad (7)$$

The classical viability analysis determines the viability kernel of dynamics (Eqs. (5)-(6)) subject to constraints (Eq. (7)). It allows to find the subset of $L \times (\prod_{i=1}^N \mathcal{L}_i)$ gathering all states $(x, (x_i)_{i \in \mathcal{N}})$ from which controls can be chosen to make the system evolution remain in the constraint set. Moreover, feedback strategies governing these viable evolutions can be derived from the viability kernel through the regulation map [7]. But, these feedback strategies, U^* , depend on the whole system state, $(u_i)_{i \in \mathcal{N}} \in U^*(x, (x_i)_{i \in \mathcal{N}})$. Whereas in the problem addressed in this paper we aim at finding feedback strategies that only depend on (x, x_i) for each u_i since individual i does not know the other individuals' state (the information space of player i is $\mathbb{R}^{n_i} \times \mathbb{R}^n$ in the game theory terminology). So the classical viability approach is not relevant in this case.

We recall that a viability domain is a closed subset of the state space which has the property that from each element of it starts an evolution viable in this domain [7]. Hence, in the framework of viability theory, solutions to the particular viability problem we address in this paper are provided by subsets $D \subset L \times (\prod_{i=1}^N \mathcal{L}_i)$ which are

viability domains with feedback strategies that can be written split into \mathcal{N} parts:

$$U^*(x, (x_i)_{i \in \mathcal{N}}) := \prod_{i=1}^N U_i^*(x_i, x). \quad (8)$$

Actually, from initial states belonging to D solutions governed by N feedback functions u_i satisfying $u_i(x, (x_j)_{j \in \mathcal{N}}) \in U_i^*(x_i, x)$ remain in D which is a subset of the constraint set, and are then viable.

3 The proposed decentralized algorithm

To find individual strategies that ensure the viability of all members, we propose to split the whole problem into individual problems in which the others' actions are considered as perturbations.

3.1 Individual problem formulation

From the N player game with $(n + \sum_{i=1}^N n_i)$ -dimensional state space (Eqs. (1)-(2) section 2), let us define N two-player games with $(n + n_i)$ -dimensional state spaces and dynamics described by \mathcal{S}_i :

$$\mathcal{S}_i \begin{cases} x(t+1) = f(x(t), (x_i(t))_{i \in \mathcal{N}}, (u_i(t))_{i \in \mathcal{N}}) \\ x_i(t+1) = f_i(x_i(t), u_i(t), x(t)) \\ u_i(t) \in U_i(x_i(t), x(t)) \\ x_{j, j \neq i}(t) \in \mathcal{L}_j \\ u_{j, j \neq i}(t) \in U_j(x_j(t), x(t)), \end{cases} \quad (9)$$

where $u_i \in U_i(x_i, x)$ is the action or control of player i considered as the first player of this new game, and

$$v_i := ((x_j, u_j))_{j \in \mathcal{N}, j \neq i} \in V_i(x) := \prod_{\substack{j=1 \\ j \neq i}}^N \left(\bigcup_{x_j \in \mathcal{L}_j} (x_j \times U_j(x_j, x)) \right) \quad (10)$$

which gathers all other members' actions (considered as perturbations in the viability theory framework since member i can not control them) is the action of the second player.

The objective of player i is to ensure its own viability which means from Eq. (3)

$$(x(t), x_i(t)) \in L \times \mathcal{L}_i, \quad \forall t \in \mathbb{N}. \quad (11)$$

3.2 Guaranteed viability kernels

3.2.1 A concept of the mathematical viability theory

When dynamical systems are described by ordinary differential equations, a *positively invariant set* has the

property that, if it contains the system state at some time, then it will contain the system state also in the future. When controlled dynamical systems are concerned, a set is *controlled invariant* if for all initial conditions chosen among its elements, the trajectory remains inside the set by means of a proper control action [15]. In the differential inclusion framework (which encompasses ordinary differential equations and controlled dynamical systems), a viable set gathers states from which at least one solution to the differential inclusion remains inside it. The largest viable set inside a prescribed domain is called the viability kernel [7]. When uncertain systems are considered, the concept of *invariance domain* [7] requires that all solutions remain inside the set whatever the perturbation; when a control input is present, the concept of *discriminating domain* [17] or *guaranteed viability domain* [3] deals with the possibility of finding a control law that governs viable evolutions despite the perturbations.

3.2.2 Definition and main property of a guaranteed viability kernel in the discrete-time case

We recall the definition of the guaranteed viability kernel in the discrete-time dynamics case [3].

Definition 2 A discrete-time dynamic game (U, V, Φ) is defined by a set-valued control map $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$, a perturbation set-valued map $V : \mathbb{R}^n \rightsquigarrow \mathbb{R}^q$ and a function $\Phi : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^n$ governing evolutions satisfying:

$$\begin{cases} x(t+1) = \Phi(x(t), u(t), v(t)) \\ u(t) \in U(x(t)) \\ v(t) \in V(x(t)) \end{cases} \quad (12)$$

Definition 3 We associate with any subset $L \subset \text{Dom}(U)$ ² the guaranteed regulation map

$$\Gamma_L(x) := \{u \in U(x) \mid \Phi(x, u, V(x)) \in L\}.$$

We shall say that L is a guaranteed viability domain if

$$\forall x \in L, \Gamma_L(x) \neq \emptyset.$$

Moreover, we can define the largest guaranteed viability domain contained in a closed set with additional assumptions on the dynamics.

Proposition 4 Let L be a closed subset of $\text{Dom}(U)$.

² Let U be a set-valued map $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$, $\text{Dom}(U) := \{x \in \mathbb{R}^n \mid U(x) \neq \emptyset\}$.

When U is upper semicompact³, V is lower semicontinuous⁴ and Φ is continuous, the union of all guaranteed viability domains contained in L is a closed guaranteed viability domain called the guaranteed viability kernel. We denote it $\text{Guar}_{(U,V,\Phi)}(L)$.

PROOF. We first prove that if L is closed and if $D \subset L$ is a guaranteed viability domain for the game (U, V, Φ) , then the closed subset $\bar{D} := \{x \in L \mid \exists (x_n)_{n \in \mathbb{N}} \in D^{\mathbb{N}} \mid x = \lim_{n \rightarrow +\infty} x_n\} \subset L$ is a guaranteed viability domain. Actually, let us consider $x \in \bar{D}$, there exists a sequence $x_n \in D$ such that x_n converges to x . Since $x_n \in D$, there exists $u_n \in U(x_n)$ such that $\Phi(x_n, u_n, V(x_n)) \in D$. Since U is upper semicompact, there exists a subsequence u_{n_p} converging to some $u \in U(x)$. Since V is lower semicontinuous, for any $v \in V(x)$, there exists a sequence $v_{n_j} \in V(x_{n_j})$ converging to v . But, $\Phi(x_{n_j}, u_{n_j}, v_{n_j}) \in D$ and Φ is continuous, so $\Phi(x, u, v) \in \bar{D}$ for all $v \in V(x)$.

We finally remark that the union of all guaranteed viability domains contained in L is a guaranteed viability domain.

3.2.3 Guaranteed viability kernel approximation algorithm

Analytical definitions of viability kernels are rarely achievable. But, algorithms have been proposed to compute approximations of viable sets (using Euler methods [34], level set approaches [31], Lagrangian methods [27], and classification functions [21]).

In the discrete-time dynamics case, approximations of guaranteed viability kernels can be computed with the following algorithm inspired by [5].

Let $L \subset \mathbb{R}^n$ be a closed subset and let us consider

$$\Delta_{U,V,\Phi}^{\ominus 1}(L) := \{x \in L \mid \exists u \in U(x) \mid \Phi(x, u, V(x)) \in L\}.$$

Starting with $L_0 := L$, we define recursively the subsets L_n :

$$\forall n \geq 0, L_{n+1} := \Delta_{U,V,\Phi}^{\ominus 1}(L_n).$$

Proposition 5 Let us assume that L is closed and that U is upper semicompact, V is lower semicontinuous and

³ A set-valued map $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ is said to be upper semicompact at x if for every sequence $x_n \in \text{Dom}(U)$ converging to x and for every sequence $y_n \in U(x_n)$, there exists a subsequence y_{n_p} converging to some $y \in U(x)$.

⁴ A set-valued map $V : \mathbb{R}^n \rightsquigarrow \mathbb{R}^q$ is said to be lower semicontinuous at x if for any sequence $x_n \in \text{Dom}(V)$ converging to x , for any $y \in V(x)$, there exists $y_n \in V(x_n)$ converging to y .

Φ is continuous, then the subsets L_n are closed and

$$\text{Guar}_{(U,V,\Phi)}(L) = \bigcap_{n \geq 0} L_n.$$

PROOF. We know from Proposition 4 that $\text{Guar}_{(U,V,\Phi)}(L)$ is a viability domain so $\text{Guar}_{(U,V,\Phi)}(L) \subset \bigcap_{n \geq 0} L_n$.

Let $x \in L_\infty := \bigcap_{n \geq 0} L_n$. Then for every $n \geq 0$, $x \in \Delta_{U,V,\Phi}^{\ominus 1}(L_n)$, and thus, there exists $u_n \in U(x)$ such that $\Phi(x, u_n, V(x)) \subset L_n$. Since $U(x)$ is compact, a subsequence (again denoted u_n) converges to some $u^* \in U(x)$. But, when $m \geq n$, $\Phi(x, u_m, V(x)) \subset L_m \subset L_n$, so $\Phi(x, u^*, V(x)) \subset \overline{L_n}$ and $\Phi(x, u^*, V(x)) \subset \bigcap_{n \geq 0} \overline{L_n}$. Hence, whenever the subsets L_n are closed, $x \in \overline{L_\infty} \cap \Delta_{U,V,\Phi}^{\ominus 1}(L_\infty)$ and thus, L_∞ is a guaranteed viability domain and consequently, it is contained in $\text{Guar}_{(U,V,\Phi)}(L)$.

To prove recursively that the subsets L_n are closed, it is enough to prove that L_{n+1} is closed whenever L_n is closed. Let us consider a sequence $x_j \in L_{n+1}$ which converges to x , that belongs to L_n . There exists $u_j \in U(x_j)$ such that $\Phi(x_j, u_j, V(x_j)) \subset L_n$. Since U is assumed to be upper semicontact, a subsequence (again denoted by u_j) converges to $u \in U(x)$. Since V is lower semicontinuous, for any $v \in V(x)$, there exists a sequence $v_j \in V(x_j)$ converging to v . Since $\Phi(x_j, u_j, V(x_j)) \subset L_n$, we know that $\Phi(x_j, u_j, v_j)$ belongs to L_n , and consequently, since Φ is continuous, that $\Phi(x, u, v)$ belongs to L_n . Hence, $\Phi(x, u, V(x)) \subset L_n$, so that x belongs to L_{n+1} .

3.3 From individual guaranteed viability kernels to a global viability domain

Considering the N viability problems of subsection 3.1, if \mathcal{L}_i and L are closed and if the dynamics \mathcal{S}_i (Eq. (9)) satisfy the assumptions of proposition 4, the guaranteed viability kernel of $\mathcal{L}_i \times L$ exists and we denote it by $\text{Guar}_{\mathcal{S}_i}(\mathcal{L}_i \times L) \subset \mathcal{L}_i \times L \subset \mathbb{R}^{n_i} \times \mathbb{R}^n$.

From these N guaranteed viability kernels, we can build a subset of the $(n + \sum_{i=1}^N n_i)$ -dimensional state space $L \times (\prod_{i=1}^N \mathcal{L}_i)$.

Definition 6 The subset D of $L \times (\prod_{i=1}^N \mathcal{L}_i)$ defined by

$$D := \bigcup_{x \in L} (\{x\} \times (\prod_{i=1}^N (\text{Guar}_{\mathcal{S}_i}(\mathcal{L}_i \times L) \cap (\mathbb{R}^{n_i} \times \{x\}))_{|\mathbb{R}^{n_i}}^5)) \quad (13)$$

is called the global viability domain associated with the N guaranteed viability kernels.

⁵ The function $|\mathbb{R}^{n_i}$ is the canonical projection onto \mathbb{R}^{n_i}

D gathers all states $(x, (x_i)_{i \in \mathcal{N}}) \in L \times (\prod_{i=1}^N \mathcal{L}_i)$ for which $(x, x_i) \in \text{Guar}_{\mathcal{S}_i}(\mathcal{L}_i \times L)$ for all $i \in \mathcal{N}$.

When the global viability domain is non empty, it allows to solve the problem addressed in this paper, to design individual strategies that ensure the viability of all with information space of payer i reduced to $\mathbb{R}^{n_i} \times \mathbb{R}^n$ (Eq.(8)).

Theorem 7 The subset D of $L \times (\prod_{i=1}^N \mathcal{L}_i)$ defined by Eq. (5) is a viability domain for dynamics (Eqs. (1)-(2)) facing constraints (Eq. (3)).

Moreover, an evolution starting in D and governed by Eqs. (1)-(2) with $\forall t \in \mathbb{N}$, $\forall i \in \mathcal{N}$, $u_i(t) \in \Gamma_{\text{Guar}_{\mathcal{S}_i}(\mathcal{L}_i \times L)}(x_i(t), x(t))$ is viable for constraints described by Eq. (3).

PROOF. Let us assume that $(x(t), (x_i(t))_{i \in \mathcal{N}}) \in D$, then for all $i \in \mathcal{N}$, $(x_i(t), x(t)) \in \text{Guar}_{\mathcal{S}_i}(\mathcal{L}_i \times L)$. Let us choose

$$(u_i(t))_{i \in \mathcal{N}} \in \prod_{i=1}^N \Gamma_{\text{Guar}_{\mathcal{S}_i}(\mathcal{L}_i \times L)}(x_i(t), x(t)).$$

Then $(x_i(t+1), x(t+1)) \in \text{Guar}_{\mathcal{S}_i}(\mathcal{L}_i \times L)$ for all $i \in \mathcal{N}$ and $(x(t+1), (x_i(t+1))_{i \in \mathcal{N}}) \in D$.

4 Experiments with an agricultural cooperative model

An agricultural cooperative and its members may constitute a relatively simple example of two-scale systems. Actually, these farmer organizations are characterized by open membership societies, collective ownership, equal voting power, principles of equality and solidarity. Cooperative members, however, are rather heterogeneous. In general, members can be differentiated according to geographic dispersion, variances in age and education, farm size and type, as well as business objectives and strategies.

4.1 The model description

[29] consider a stylized agricultural cooperative with N members. Each member i of the cooperative commits to bring a fixed amount of its production, w_i , to the cooperative each year. The cooperative transforms the whole material brought by its members and sells its production. The cooperative pays each member according to the provided amount, and the price per quantity unit is the same for all members. Nevertheless, this price, $p(t)$, may evolve each year. Hence, the cooperative pays $p(t)w_i$ to member i each year.

To complete the description of the evolution of the capital, K_i , of member i , we add that members have charges the amount of which is proportional to the quantity produced with proportional coefficient, $\frac{C_i}{u_i(t)}$ where C_i is fixed and $u_i(t) \in [1; \bar{\alpha}]$ is related to the production process choice of member i at time t : a high quality production corresponds to a low process tolerance level which is associated with $u_i(t) = 1$ and consequently with high charges per unit quantity. The lowest quality corresponds to $u_i(t) = \bar{\alpha} > 1$. Finally,

$$K_i(t+1) = K_i(t) + p(t+1)w_i - C_i \frac{w_i}{u_i(t)}. \quad (14)$$

The price $p(t)$ is the price at which the cooperative is able to sell its production. We consider that there is a matter of reputation. This inertia is represented by a weight allocated to the previous price, $p(t)$, when evaluating $p(t+1)$:

$$p(t+1) = \delta(p(t), p_{ex}(t))p(t) + (1 - \delta(p(t), p_{ex}(t)))p_{ex}(t) \quad (15)$$

where $p_{ex}(t)$ is directly linked with the process qualities used by all members during period t to $t+1$ and $\delta(p, p_{ex}) := \delta \in [0, 1[$ if $p \leq p_{ex}$ and $\delta(p, p_{ex}) := 0$ otherwise: when the quality decreases, the effect on the price is instantaneous; however, when the quality increases, the price will increase less quickly since building up a reputation as quality producer takes time.

$p_{ex}(t)$ depends on the process qualities of all members between t and $t+1$, hence depends on $(u_i(t))_{i \in \mathcal{N}}$. It equals the maximal price denoted by p_0 when $u_i(t) = 1$ for all members i . However, if all members choose least quality processes ($u_i(t) = \bar{\alpha}, \forall i \in \mathcal{N}$), it drops to $p_0/\bar{\alpha}^2$:

$$\begin{aligned} p_{ex} &:= p_0 - A(u_{mean} - 1) = f(u_{mean}) \\ \text{with } u_{mean} &:= \frac{\sum_{i=1}^N w_i}{\sum_{i=1}^N (w_i/u_i)} \\ \text{and } A &:= \frac{1+\bar{\alpha}}{\bar{\alpha}^2} p_0. \end{aligned} \quad (16)$$

Consequently, the evolutions of the cooperative and its N members are described by the following $(N+1)$ -dimensional discrete-time controlled dynamical systems:

$$\mathcal{S} \begin{cases} p(t+1) = \delta(p(t), f(\frac{\sum_{i=1}^N w_i}{\sum_{i=1}^N (w_i/u_i(t))}))p(t) \\ \quad + (1 - \delta(p(t), f(\frac{\sum_{i=1}^N w_i}{\sum_{i=1}^N (w_i/u_i(t))})))f(\frac{\sum_{i=1}^N w_i}{\sum_{i=1}^N (w_i/u_i(t))}), \\ K_i(t+1) = K_i(t) + p(t+1)w_i - C_i \frac{w_i}{u_i(t)}, \forall i \in \mathcal{N} \\ u_i(t) \in [1; \bar{\alpha}], \forall i \in \mathcal{N}. \end{cases} \quad (17)$$

4.2 The constraint set definition

The global variable, the price p , must fluctuate between the price of the lowest quality production $\frac{p_0}{\bar{\alpha}^2}$ and the price of the highest quality production p_0 .

Moreover, we assume that the minimal condition for member i to remain in the cooperative is to preserve a minimal capital amount $K_{i,min}$ the value of which may depend on the member.

Then, along an evolution, the N members satisfy their viability constraint if, for all $t \in \mathbb{N}$, the whole system state $(p(t), K_1(t), \dots, K_N(t))$ belongs to

$$[\frac{p_0}{\bar{\alpha}^2}; p_0] \times \left(\prod_{i=1}^N [K_{i,min}; +\infty[\right). \quad (18)$$

4.3 Definition of the individual guaranteed viability kernels

Following the approach of subsection 3.1, from the $(N+1)$ -dimensional discrete-time dynamics (Eq. (17)) describing the agricultural cooperative dynamics, we define N two-player games with the following 2-dimensional dynamics $S_{i,W}$:

$$S_{i,W} \begin{cases} p(t+1) = \delta(p(t), f(\frac{W}{\frac{w_i}{u_i(t)} + \frac{W-w_i}{v_i(t)}}))p(t) \\ \quad + (1 - \delta(p(t), f(\frac{W}{\frac{w_i}{u_i(t)} + \frac{W-w_i}{v_i(t)}})))f(\frac{W}{\frac{w_i}{u_i(t)} + \frac{W-w_i}{v_i(t)}}) \\ \quad =: \mathcal{P}(p(t), u_i(t), v_i(t)) \\ K_i(t+1) = K_i(t) + p(t+1)w_i - C_i \frac{w_i}{u_i(t)} \\ \quad =: K_i(t) + \mathcal{K}_i(\mathcal{P}(p(t), u_i(t), v_i(t)), u_i(t)) \\ u_i(t) \in [1; \bar{\alpha}] \\ v_i(t) \in [1; \bar{\alpha}] \end{cases} \quad (19)$$

with $W := \sum_i w_i$.

At each time t , $u_i(t)$ is the control member i (first player) can choose and $v_i(t)$ the result of the other members' choices (regarded as second player action). Actually, in this particular case, the influence of all the other members on the dynamics can be captured by a one-dimensional variable, $v_i \in [1; \bar{\alpha}]$, since

$$\cup_{u_j \neq i \in [1; \bar{\alpha}]} \left\{ \sum_{j=1, j \neq i}^N \frac{w_j}{u_j} \right\} = \cup_{v_i \in [1; \bar{\alpha}]} \frac{W - w_i}{v_i}.$$

Applying the approach of subsection 3.1 to the constraints described by Eq. (18), the objective of the first player is to keep the system state inside the constraint set L_i :

$$L_i := [K_{i,min}; +\infty[\times [\frac{p_0}{\bar{\alpha}^2}; p_0]. \quad (20)$$

Since $S_{i,W}$ (Eq. (19)) and L_i (Eq. (20)) satisfy the assumptions of Proposition 4 (L_i is closed, U and V are constant, and \mathcal{K}_i and \mathcal{P} are continuous), the guaranteed viability kernel associated with dynamics $S_{i,W}$ and the constraint set L_i exists, it is a closed subset of \mathbb{R}^2 and we denote it by $Guar_{S_{i,W}}(L_i)$.

Considering $K_i(t) - K_{i,min}$ instead of $K_i(t)$, we can set $K_{i,min} = 0$ without loss of generality, $S_{i,W}$ is unchanged, L_i becomes $[0; +\infty[\times[\frac{p_0}{\alpha^2}; p_0]$, and $Guar_{S_{i,W}}(L_i)$ depends then on six parameters

$$\bar{\alpha}, p_0, \delta, w_i, C_i, W.$$

$\bar{\alpha}$ depends on the production process, p_0 and δ depend on the market dynamics, W depend on the cooperative size and w_i and C_i are individual characteristics of member i . From our study viewpoint, $\bar{\alpha}$, p_0 and δ are environment parameters, and we put the focus on the relationship between individual (w_i and C_i) and global (W) parameters.

4.4 The definition of the global viability domain

Following the approach of subsection 3.3, from the N individual guaranteed viability kernels, we define the subset $D \subset [p_0/\alpha^2; p_0] \times (\prod_{i=1}^N [0; +\infty[) \subset \mathbb{R}^{N+1}$ by :

$$D := \bigcup_{p \in [p_0/\alpha^2; p_0]} (\{p\} \times (\prod_{i=1}^N (Guar_{S_{i,W}}(L_i) \cap ([0; +\infty[\times\{p\})_{|K})^6)) \quad (21)$$

Theorem 8 *The subset $D \subset [p_0/\alpha^2; p_0] \times (\prod_{i=1}^N [0; +\infty[)$ is a guaranteed viability domain for dynamics of Eq. (17) facing constraints described by Eq. (18).*

Moreover, an evolution $(p(t), K_1(t), \dots, K_N(t))$ starting in D and governed by Eq. (17) with $\forall t \in \mathbb{N}, \forall i \in \mathcal{N}, u_i(t) \in \Gamma_{Guar_{S_{i,W}}(L_i)}(K_i(t), p(t))$ is viable in the constraint set described by Eq. (18).

PROOF. It is a particular case of Theorem 7.

4.5 Necessary and sufficient condition for the global viability domain to be non-empty

Corollary 9 *D is empty if and only if there exists $i \in \mathcal{N}$ such that $Guar_{S_{i,W}}(L_i)$ is empty.*

⁶ The function $_{|K}$ is defined by:

$$\begin{aligned} proj_{|K} : \mathbb{R} \times [p_0/\alpha^2; p_0] &\rightarrow \mathbb{R} \\ (K, p) &\rightarrow K. \end{aligned}$$

PROOF. This is a consequence of the shape of the individual guaranteed viability kernels $Guar_{S_{i,W}}(L_i)$ (Lemma 10).

Lemma 10 *If $(K, p) \in Guar_{S_{i,W}}(L_i)$, then for all $(K^*, p^*) \in L_i$ with $K^* \geq K$ and $p^* \geq p$, $(K^*, p^*) \in Guar_{S_{i,W}}(L_i)$.*

PROOF. Let us consider $(K, p) \in Guar_{S_{i,W}}(L_i)$ and $\tilde{v}_i(\cdot) : \mathbb{N} \rightarrow [1; \bar{\alpha}]$. There exists $\tilde{u}_i(\cdot) : \mathbb{N} \rightarrow [1; \bar{\alpha}]$ such that the evolution $(K_i(\cdot), p(\cdot))$ governed by Eq. (19) with $u_i(\cdot) = \tilde{u}_i(\cdot)$, $v_i(\cdot) = \tilde{v}_i(\cdot)$, and $(K_i(0), p(0)) = (K, p)$, remains in L_i for all $t \in \mathbb{N}$. But $K_i(t+1) - K_i(t)$ and $p(t+1) - p(t)$ do not depend on $K_i(t)$ (Eq. (19)), so the evolution governed by Eq. (19) with $u_i(\cdot) = \tilde{u}_i(\cdot)$, $v_i(\cdot) = \tilde{v}_i(\cdot)$ and starting at (K^*, p) with $K^* \geq K$ will be $(K_i(\cdot) + K^* - K, p(\cdot))$ which also remains inside L_i for all $t \in \mathbb{N}$. So, $(K^*, p) \in Guar_{S_{i,W}}(L_i)$.

Moreover, since $K_i(t+1) - K_i(t)$ and $p(t+1)$ increases with $p(t)$ (Eq. 19), the evolution governed by Eq. (19) with $u_i(\cdot) = \tilde{u}_i(\cdot)$, $v_i(\cdot) = \tilde{v}_i(\cdot)$ and starting at (K^*, p^*) with $p_0 \geq p^* \geq p$ will be $(K_i^*(t), p^*(t))$ with $p_0 \geq p^*(t) \geq p(t)$ and $K_i^*(t) \geq K_i(t)$ which also remains inside L_i for all $t \in \mathbb{N}$. So, $(K, p^*) \in Guar_{S_{i,W}}(L_i)$.

Consequently, if all the individual guaranteed viability kernels are not empty, there exist initial situations for which all the individual constraints are satisfied over time if all the members follow the guaranteed strategies derived from their individual guaranteed viability problems.

4.6 Numerical results in a case when the global viability domain is non-empty

The environment parameter values are $\bar{\alpha} = 2$, $p_0 = 12$ and $\delta = 0.1$, and the cooperative is made of three members whose parameter values are $w_1 = 2$, $C_1 = 6.35$, $w_2 = 3$, $C_2 = 7.5$, $w_3 = 4$ and $C_3 = 8.55$ (and consequently, $W = w_1 + w_2 + w_3 = 9$).

4.6.1 Numerical approximations of the three individual guaranteed viability kernels

Fig. 1 displays approximations of these guaranteed viability kernels. These approximations are obtained using algorithm of section 3.2.3 and discretization in space from [34].

4.6.2 Evolution with time of the whole cooperative with members following feedback strategies derived from the regulation map of their guaranteed viability kernels

Fig. 2 displays the evolutions with time of the capital of members 1, 2 and 3 with initial conditions $p(0) = 5$,

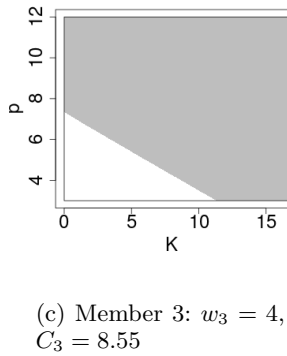
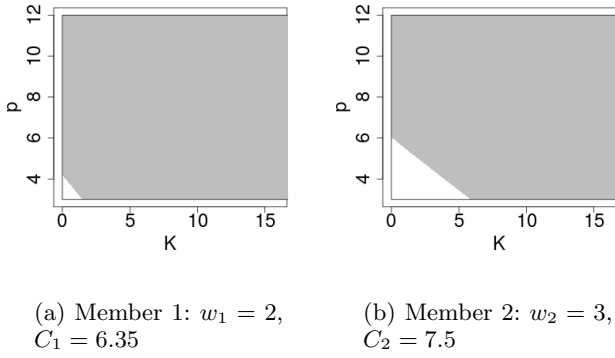


Fig. 1. Approximations of the guaranteed viability kernels of the three members. $\bar{\alpha} = 2$, $p_0 = 12$, $\delta = 0.1$ and $W = w_1 + w_2 + w_3 = 9$.

$K_1(0) = 1$, $K_2(0) = 7$ and $K_3(0) = 10$ which belong to the viability domain

$$\bigcup_{p \in [p_0/\alpha^2; p_0]} (\{p\} \times \left(\prod_{i=1}^3 (\text{Guar}_{S_i, W}(L_i) \cap ([0; +\infty[\times \{p\})|_K) \right)). \quad (22)$$

These evolutions are governed by Eq. (19) with $u_i(t) \in \Gamma_{\text{Guar}_{S_i, W}(L_i)}(K_i(t), p(t))$ for $i = 1, 2, 3$. We ran three simulations modifying the way the members choose among the process tolerance levels that guarantee their viability:

- in the experiment (a), the three members choose the level that maximizes their worst next income

$$u_i^*(t) := \operatorname{argmax}_{v_i(t) \in [0; \bar{\alpha}]} \min_{v_i(t) \in [0; \bar{\alpha}]} \mathcal{K}_i(\mathcal{P}(p(t), u_i(t), v_i(t)), u_i(t)),$$

when $u_i(t)$ ranges over $\Gamma_{\text{Guar}_{S_i, W}(L_i)}(K_i(t), p(t))$

- in the experiment (b), the lowest level

$$u_i^*(t) = \min_{u_i(t) \in \Gamma_{\text{Guar}_{S_i, W}(L_i)}(K_i(t), p(t))} u_i(t),$$

- in the experiment (c), an arbitrary level among subset $\Gamma_{\text{Guar}_{S_i, W}(L_i)}(K_i(t), p(t))$.

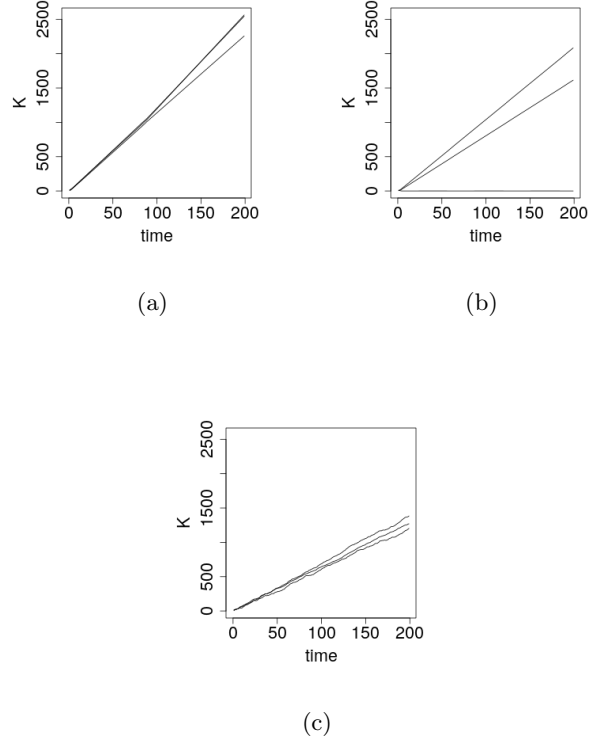


Fig. 2. Evolutions with time of the capital of the three members according to their strategy (a), (b) or (c). The initial conditions are $p(0) = 5$, $K_1(0) = 1$, $K_2(0) = 7$ and $K_3(0) = 10$ and the dynamics governed by Eq. (19). $\bar{\alpha} = 2$, $p_0 = 12$, $\delta = 0.1$.

Obviously, each member capital evolution depends on its strategy and on those of the other members but since all the members choose their controls among the guaranteed viable ones, all member capitals remain in the constraint set (i.e. bigger than 0) (Theorem 8).

4.7 Mechanism design when the global viability domain is empty

When at least one individual guaranteed viability kernel is empty, the global viability domain defined by Eq. (6) is empty (Corollary 9). A modification of the cooperative rules is then needed to allow the design of individual strategies that ensure the viability of all members. The study of the individual guaranteed viability kernel properties can provide a straightforward way of designing new cooperative operating rules (from which new individual dynamic games S_i^* will derive) so that the derived global viability domain D^* will be non-empty provided that the individual charges are not too high given the environment parameters $\bar{\alpha}$ and p_0 .

Lemma 11 Given $p_0, \bar{\alpha}$ with $\bar{\alpha}^2 - \bar{\alpha} - 1 \geq 0$ and a cooperative member i characterized by its pair (C_i, w_i) , if $C_i \leq \frac{p_0}{\bar{\alpha}} \frac{(1+\bar{\alpha}+\bar{\alpha}^2)^2}{4\bar{\alpha}(1+\bar{\alpha})}$, then there exists a total production amount of the cooperative $W(i) \geq w_i$ such that $\text{Guar}_{S_i, W}(L_i)$ is non-empty for $W \in [w_i; W(i)]$.

PROOF. This is a consequence of both following Proposition 12 and Lemma 13.

Proposition 12 The guaranteed viability kernel $\text{Guar}_{S_i, W}(L_i)$ is not empty if there exists $u_i \in [1; \bar{\alpha}]$ such that

$$\mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i) \geq 0.$$

PROOF. See Appendix A.1.

Lemma 13 When $\bar{\alpha}^2 - \bar{\alpha} - 1 \geq 0$, if $C_i \leq \frac{p_0}{\bar{\alpha}} \frac{(1+\bar{\alpha}+\bar{\alpha}^2)^2}{4\bar{\alpha}(1+\bar{\alpha})}$, then there exists a total production amount of the cooperative $W(i) \geq w_i$ such that for all $W \in [w_i; W(i)]$, there exists $u_i \in [1; \bar{\alpha}]$ such that

$$\mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i) \geq 0$$

PROOF. See Appendix A.2.

Moreover, when the guaranteed viability kernel is non-empty, there exists a lower bound on K_i , let us denote it \underline{K}_i , such that all pairs (K, p) belong to the guaranteed viability kernel when $K \geq \underline{K}_i$:

Corollary 14 If there exists $u_i \in [1; \bar{\alpha}]$ such that $\mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i) \geq 0$, then $(K, p) \in \text{Guar}_{S_i, W}(L_i)$ for all $p \in [p_0/\bar{\alpha}^2; p_0]$ when $K \geq w_i \frac{1}{1-\delta} \frac{p_0}{\bar{\alpha}} (\bar{\alpha} - 1)$.

PROOF. See Appendix A.3.

Then, let us consider a cooperative in which all pairs (C_i, w_i) satisfy the assumption of Lemma 11 with the additional hypothesis:

$$W^* := \min_{i \in \mathcal{N}} W(i) \geq \max_{i \in \mathcal{N}} w_i. \quad (23)$$

As a consequence of Lemma 11, in such a cooperative, for all $i \in \mathcal{N}$, $\text{Guar}_{S_i, W^*}(L_i)$ is non-empty. Moreover, as a consequence of Corollary 14, for all $i \in \mathcal{N}$, $(K, p) \in \text{Guar}_{S_i, W^*}(L_i)$ when $K \geq \max_{i \in \mathcal{N}} \underline{K}_i$.

4.7.1 New rules for the cooperative

The central idea comes from this property that each member would be able to ensure its viability in a cooperative with smaller total production quantity (W^*). Several mechanism designs can be derived from this property. We propose here the following one: when one cooperative member economic health becomes worrying ($K_i(t) < \underline{K}$ with $\underline{K} \geq \max_{i \in \mathcal{N}} \underline{K}_i$ to be chosen), the production of all members are no longer mixed all together but split into N^* groups of production size smaller or equal to W^* , in which each member can ensure its viability. N^* may take several values. Splitting into N groups is always a theoretical solution (although not compatible with the definition of a cooperative). Lower values can be reached depending on W, W^* but also on all the individual production values, but $\frac{W}{W^*}$ is a lower bound of N^* .

Let $(I_j)_{j \in \mathcal{N}^* := \{1, \dots, N^*\}}$ be the member indexes of the N^* groups and let d be the function from \mathcal{N} to \mathcal{N}^* which associates with i the index of the group of member i . The total production of group j equals $W_j \leq W^*$. To describe the dynamics, we need to add the variable b which can take three values: when $b(t) = 1$, the cooperative production is split (and consequently, there are as many prices as groups), when $b(t) = 0$, all individual productions are mixed (and there is only one price), and $b(t) = 2$ at the grouping time. We make the assumption that at the grouping time the one price for the new group is defined with the average price over former subgroups. We also assume that the splitting and grouping processes do not generate additional costs for the cooperative members.

The discrete-time system governing the new cooperative dynamics is then defined on $\mathbb{R}^{N^*+1} \times \{0, 1, 2\} \times \mathbb{R}^N$ by:

- if $b(t) = 1$, cooperative is split, there are as many prices, p_j , as groups,

$$\begin{aligned} \forall j \in \mathcal{N}^*, \forall i \in I_j, \\ K_i(t+1) &= K_i(t) + p_j(t+1)w_i - C_i \frac{w_i}{u_i(t)} \\ p_j(t+1) &= \delta(p_j(t), f(\frac{W_j}{\sum_{i \in I_j} \frac{W_j}{u_i(t)}})p_j(t) \\ &\quad + (1 - \delta(p_j(t), f(\frac{W_j}{\sum_{i \in I_j} \frac{W_j}{u_i(t)}})))f(\frac{W_j}{\sum_{i \in I_j} \frac{W_j}{u_i(t)}}) \\ b(t+1) &= 1 \text{ if } \min_{i \in \mathcal{N}} K_i(t+1) < \underline{K} \\ &= 2 \text{ otherwise.} \end{aligned} \quad (24)$$

- if $b(t) = 2$, this is grouping time,

$$\begin{aligned}
& \forall i \in \mathcal{N}, \\
& K_i(t+1) = K_i(t) + p(t+1)w_i - C_i \frac{w_i}{u_i(t)} \\
& p(t+1) = \delta(p(t), f(\frac{W}{\sum_{i=1}^N \frac{w_i}{u_i(t)}})p(t) \\
& \quad + (1 - \delta(p(t), f(\frac{W}{\sum_{i=1}^N \frac{w_i}{u_i(t)}})))f(\frac{W}{\sum_{i=1}^N \frac{w_i}{u_i(t)}}) \\
& \text{with } p(t) = \sum_{j \in \mathcal{N}^*} p_j(t)/N^* \\
& b(t+1) = 1 \text{ if } \min_{i \in \mathcal{N}} K_i(t+1) < \underline{K} \\
& \quad = 0 \text{ otherwise.}
\end{aligned} \tag{25}$$

- if $b(t) = 0$, all individual productions are mixed, there is only one price, p ,

$$\begin{aligned}
& \forall i \in \mathcal{N}, \\
& K_i(t+1) = K_i(t) + p(t+1)w_i - C_i \frac{w_i}{u_i(t)} \\
& p(t+1) = \delta(p(t), \frac{W}{\sum_{i=1}^N \frac{w_i}{u_i(t)}})p(t) \\
& \quad + (1 - \delta(p(t), \frac{W}{\sum_{i=1}^N \frac{w_i}{u_i(t)}}))f(\frac{W}{\sum_{i=1}^N \frac{w_i}{u_i(t)}}) \\
& b(t+1) = 1 \text{ if } \min_{i \in \mathcal{N}} K_i(t+1) < \underline{K} \\
& \quad = 0 \text{ otherwise.}
\end{aligned} \tag{26}$$

The objective remains to ensure all the capitals to remain positive, but, to include the prices of the different groups and the three-valued variable which describes the cooperative organisation phase, the constraint set is now a subset of $\mathbb{R}^{N^*+1} \times \{0, 1, 2\} \times \mathbb{R}^N$:

$$[\frac{p_0}{\alpha^2}; p_0]^{N^*+1} \times \{0, 1, 2\} \times \left(\prod_{i=1}^N [0; +\infty[\right). \tag{27}$$

4.7.2 Definition of the individual viability problems

Following the approach of subsection 3.1, from the discrete-time dynamics of Eqs. (24)-(25)-(26) describing the new cooperative dynamics, we define N two-player games which are 3-dimensional: one individual variable K_i and two global variables p and b . The new global variable b evolves as follows: $b(t+1) = 1$ if $K_i(t+1) < \underline{K}$, otherwise, $b(t+1) \in \{1, 2\}$ if $b(t) = 1$, and $b(t+1) \in \{0, 1\}$ otherwise (since member i can not predict if the next capital values of all other member capitals will be above the threshold \underline{K}). W is replaced by the total production of the group agent i belongs to $W_{d(i)} := \sum_{j \in I_{d(i)}} w_j$ when $b(t) = 1$ (the cooperative is split). All possible previous prices must be considered when $b(t) = 2$ since the other group prices are unknown to member i . Then, the dynamics of $(K_i, p, b) \in \mathbb{R} \times [\frac{p_0}{\alpha^2}; p_0] \times \{0, 1, 2\}$ are described by

$S_{i,W,W_{d(i)},\underline{K}}^*$:

$$\left\{ \begin{aligned}
& K_i(t+1) = K_i(t) + p(t+1)w_i - C_i \frac{w_i}{u_i(t)} \\
& p(t+1) = \delta(v_p(t), f(\frac{W(t)}{\frac{w_i}{u_i(t)} + \frac{W(t)-w_i}{v_i(t)}})v_p(t) \\
& \quad + (1 - \delta(v_p(t), f(\frac{W(t)}{\frac{w_i}{u_i(t)} + \frac{W(t)-w_i}{v_i(t)}})))f(\frac{W(t)}{\frac{w_i}{u_i(t)} + \frac{W(t)-w_i}{v_i(t)}}) \\
& b(t+1) = 1 \text{ if } K_i(t+1) < \underline{K} \\
& \quad = v_b(t) \text{ otherwise.} \\
& \text{with} \\
& W(t) = W_{d(i)} \text{ if } b(t) = 1 \text{ and } W \text{ otherwise.} \\
& u_i(t) \in [1; \bar{\alpha}] \\
& v_i(t) \in [1; \bar{\alpha}] \\
& v_p(t) \in V_p(b(t), p(t)) := [\frac{p_0}{\alpha^2}; p_0] \text{ if } b(t) = 2 \text{ and } p(t) \text{ otherwise.} \\
& v_b(t) \in V_b(b(t)) := \{1, 2\} \text{ if } b(t) = 1 \text{ and } \{0, 1\} \text{ otherwise.}
\end{aligned} \right. \tag{28}$$

The control remains $u_i(t) \in [1; \bar{\alpha}]$. The perturbations now embed the result of other member choices ($v_i(t) \in [1; \bar{\alpha}]$), but also the uncertainty on the cooperation configuration (split or not) ($v_b(t) \in V_b(b(t))$) and the uncertainty on the other group prices at the grouping time ($v_p(t) \in V_p(b(t), p(t))$).

The state constraints become $L_{i,\underline{K}}^*$:

$$L_{i,\underline{K}}^* := ([0; \underline{K}] \times [\frac{p_0}{\alpha^2}; p_0] \times \{1\}) \cup ([\underline{K}; +\infty[\times [\frac{p_0}{\alpha^2}; p_0] \times \{0; 1; 2\}), \tag{29}$$

since all the capitals, especially the one of member i , are greater than \underline{K} when the cooperative is not split, i.e. $b(t) \in \{0, 2\}$.

4.7.3 Non-emptiness of the individual guaranteed viability kernels

Since $S_{i,W,W_{d(i)},\underline{K}}^*$ and $L_{i,\underline{K}}^*$ satisfy the assumptions of Proposition 4, $\text{Guar}_{S_{i,W,W_{d(i)},\underline{K}}^*}(L_{i,\underline{K}}^*)$, the guaranteed viability kernel associated with dynamics $S_{i,W,W_{d(i)},\underline{K}}^*$ and constraint set $L_{i,\underline{K}}^*$, is closed.

Moreover, for a good choice of \underline{K} , $\text{Guar}_{S_{i,W,W_{d(i)},\underline{K}}^*}(L_{i,\underline{K}}^*)$ is non-empty.

Proposition 15 *When $W_{d(i)} \leq W^*$ defined by Eq. (23), if $\underline{K} \geq w_i \frac{1}{1-\delta} \frac{p_0}{\alpha} (\bar{\alpha} - 1) + w_i (\frac{p_0}{\alpha} - C_i)$, then $\text{Guar}_{S_{i,W,W_{d(i)},\underline{K}}^*}(L_{i,\underline{K}}^*)$ is non-empty and*

$$\begin{aligned}
& \text{Guar}_{S_{i,W,W_{d(i)},\underline{K}}^*}(L_{i,\underline{K}}^*) \supset (\text{Guar}_{S_{i,W_{d(i)}}}(L_i) \times \{1\}) \\
& \quad \cup ([\underline{K}; +\infty[\times [\frac{p_0}{\alpha^2}; p_0] \times \{0; 2\}).
\end{aligned}$$

PROOF. Let us consider

$$\begin{aligned}
& (K_i(t), p(t), b(t)) \in (\text{Guar}_{S_{i,W_{d(i)}}}(L_i) \times \{1\}) \\
& \quad \cup ([\underline{K}; +\infty[\times [\frac{p_0}{\alpha^2}; p_0] \times \{0; 2\}).
\end{aligned}$$

If $b(t) = 1$, $(K_i(t), p(t)) \in \text{Guar}_{S_i, W}(L_i)$ and there exists $u_i(t) \in [1; \bar{\alpha}]$, such that for all $v_i(t) \in [1; \bar{\alpha}]$, $(K_i(t+1), p(t+1)) \in \text{Guar}_{S_i, W_{d(i)}}(L_i)$. If $b(t) = 0$ or 2 , $K_i(t) \geq \underline{K}$ and $K_i(t+1) \geq \underline{K} - w_i(\frac{p_0}{\alpha} - C_i) \geq w_i \frac{1}{1-\delta} \frac{p_0}{\alpha} (\bar{\alpha} - 1)$. And, from Corollary 14, $(K_i(t+1), p(t+1)) \in \text{Guar}_{S_i, W_{d(i)}}(L_i)$. Then, if $b(t+1) = 1$, $(K_i(t+1), p(t+1), b(t+1)) \in \text{Guar}_{S_i, W_{d(i)}}(L_i) \times \{1\}$ and if $b(t+1) = 0$ or 2 , $K_i(t+1) \geq \underline{K}$ and $(K_i(t+1), p(t+1), b(t+1)) \in [\underline{K}; +\infty[\times[\frac{p_0}{\alpha^2}; p_0] \times \{0; 2\}$.

Theorem 16 *Given p_0 , $\bar{\alpha}$, δ and a cooperative of N members characterized by their pairs (C_i, w_i) satisfying Lemma 11 condition, if for all members i , $W_{d(i)} \leq W^*$ defined by Eq. (23), and*

$$\underline{K} \geq \max_{i \in \mathcal{N}} w_i \frac{1}{1-\delta} \frac{p_0}{\alpha} (\bar{\alpha} - 1) + w_i (\frac{p_0}{\alpha} - C_i)$$

then the subset

$$D^* := \bigcup_{(p,b) \in [p_0/\alpha^2; p_0] \times \{0;1;2\}} (\{(p,b)\} \times (\prod_{i=1}^N (\text{Guar}_{S_i, W_{d(i)}, \underline{K}}^*(L_i, \underline{K}) \cap ([0; +\infty[\times\{(p,b)\}]))_{|K})) \quad (30)$$

is non-empty.

Moreover, D^* is a guaranteed viability domain for the cooperative governed by the new operating rules (Eqs. (24)-(25)-(26)) and facing constraints described by Eq. (27).

PROOF. The non-emptiness of D^* is a consequence of Proposition 15 and Corollary 9. D^* being a guaranteed viability domain for the cooperative governed by the new operating rules (Eqs. (24)-(25)-(26)) and facing constraints described by Eq. (27) is a particular case of Theorem 7.

Hence, adding the mechanism of production split into parts smaller than W^* when someone faces economic difficulties ($K_i < \underline{K}$), allows individuals to derive collective viable evolutions from individual guaranteed strategies when the initial conditions belong to D^* .

5 Discussion

We have shown how and under which conditions sets of individual guaranteed strategies can produce viable evolutions for all players in discrete-time N -player deterministic infinite dynamic games with infinite horizon. The particularity of this approach is that the strategies are computed locally without making assumptions on the others' behavior or on coordination among individuals. We have illustrated the implementation of such strategies on a model of an agricultural cooperative. In

this case, we have shown the conditions under which the relevant intersection of guaranteed viability kernels is non-empty. For some ranges of parameter values, a modification of the operating rules of the cooperative was necessary.

Two extensions of this work are worth considering. First, the results were obtained under the assumption of discrete time. It would be of interest to consider continuous time dynamics. Second, designing mechanisms in order to build non-empty individual guaranteed viability kernel intersections as a way of ensuring the viability of all individuals is a challenging and relevant research question in a range of problems broader than the particular agricultural cooperative case.

References

- [1] I. Alvarez, L. Zaleski, J.-P. Briot, and M. de A. Irving. Collective management of environmental commons with multiple usages: A guaranteed viability approach. *Ecological Modelling*, 475:110186, 2023.
- [2] P. Andrés Domenech, P. Saint-Pierre, and G. Zaccour. Forest conservation and co2 emissions: A viable approach. *Environ. Model. Assess*, 16:519–539, 2011.
- [3] J.-P. Aubin, A. Bayen, and P. Saint-Pierre. *Viability Theory. New Directions*. Springer, Berlin, 2011.
- [4] J.-P. Aubin, N. Bonneuil, F. Maurin, and P. Saint-Pierre. Viability of pay-as-you-go systems. *J. Evol. Econ.*, 11(5):555–571, 2001.
- [5] J.-P. Aubin and F. Catté. Bilateral fixed-points and algebraic properties of viability kernels and capture basins of sets. *Set-Valued Anal.*, 10(4):379–416, 2002.
- [6] Jean-Pierre Aubin. Regulation of viable and optimal cohorts. *Appl. Math. Optim.*, 72(2):203–232, Oct 2015.
- [7] J.P. Aubin. *Viability Theory*. Birkhäuser, Basel, 1991.
- [8] J.P. Aubin. *Dynamic Economic Theory : A Viability Approach*. Springer-Verlag, Berlin, 1997.
- [9] T. Basar and G.J. Olsder. *Dynamic Noncooperative Game Theory*. SIAM, Philadelphia, 1999.
- [10] S. Bates and P. Saint-Pierre. Adaptive policy framework through the lens of the viability theory: A theoretical contribution to sustainability in the anthropocene era. *Ecological Economics*, 145:244–262, 2018.
- [11] C. Béné, L. Doyen, and D. Gabay. A viability analysis for a bio-economic model. *Ecol. Econ.*, 36(3):385 – 396, 2001.
- [12] A. Bensoussan and J. Frehse. Control and nash games with mean field effect. *Chinese Ann. Math. B*, 34(2):161–192, 2013.
- [13] C. Bernard and S. Martin. Building strategies to ensure language coexistence in presence of bilingualism. *Appl. Math. Comput.*, 218(17):8825 – 8841, 2012.
- [14] C. Bernard and S. Martin. Comparing the sustainability of different action policy possibilities: Application to the issue of both household survival and forest preservation in the corridor of fianarantsoa. *Math. Biosci.*, 245(2):322 – 330, 2013.
- [15] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747 – 1767, 1999.

- [16] R. Buckdahn, P. Cardaliaguet, and M. Quincampoix. Some recent aspects of differential game theory. *Dyn. Games Appl.*, 1(1):74–114, 2011.
- [17] P. Cardaliaguet. A differential game with two players and one target. *SIAM J. Control Optim.*, 34(4):1441–1460, 1996.
- [18] L. Chapel, G. Deffuant, S. Martin, and C. Mullon. Defining yield policies in a viability approach. *Ecol. Model.*, 212(1):10 – 15, 2008.
- [19] P. M. Cury, C. Mullon, S. M. Garcia, and L. J. Shannon. Viability theory for an ecosystem approach to fisheries. *ICES J. Mar. Sci.*, 62(3):577–584, 2005.
- [20] M. De Lara, L. Doyen, T. Guilbaud, and M.-J. Rochet. Is a management framework based on spawning-stock biomass indicators sustainable? A viability approach. *ICES J. Mar. Sci.*, 64(4):761–767, 2007.
- [21] G. Deffuant, L. Chapel, and S. Martin. Approximating viability kernels with support vector machines. *IEEE T. Automat Contr.*, 52(5):933–937, 2007.
- [22] L. Doyen, M. De Lara, J. Ferraris, and D. Pelletier. Sustainability of exploited marine ecosystems through protected areas: A viability model and a coral reef case study. *Ecol. Model.*, 208(2):353 – 366, 2007.
- [23] I. Haidar, I. Alvarez, and A.C. Prévot. Mathematical modeling of an urban pigeon population subject to local management strategies. *Math. Biosci.*, 288:71 – 83, 2017.
- [24] R. Isaacs. *Differential Games*. Wiley, New York, 1965.
- [25] M. Kalisiak and M. van de Panne. Approximate safety enforcement using computed viability envelopes. In *IEEE Int. Conf. Robot.*, volume 5, pages 4289–4294, 2004.
- [26] J.-M. Lasry and P.-L. Lions. Mean field games. *Jap. J. Math.*, 2(1):229–260, 2007.
- [27] J. N. Maidens, S. Kaynama, I. M. Mitchell, M. M.K. Oishi, and G. A. Dumont. Lagrangian methods for approximating the viability kernel in high-dimensional systems. *Automatica*, 49(7):2017 – 2029, 2013.
- [28] S. Martin. The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication. *Ecol. Soc.*, 9(2), 2004.
- [29] S. Martin, I. Alvarez, and J.-D. Kant. Micro/macro viability analysis of individual-based models: Investigation into the viability of a stylized agricultural cooperative. *Complexity*, 21(2):276–296, 2015.
- [30] V. Martinet, O. Thebaud, and L. Doyen. Defining viable recovery paths toward sustainable fisheries. *Ecol. Econ.*, 64(2):411–422, 2007.
- [31] I. M. Mitchell, A. M. Bayen, and C. J. Tomlin. A time-dependant Hamilton-Jacobi formulation of reachable sets for continuous dynamics games. *IEEE Trans. Automat. Control*, 50(7):947 – 957, 2005.
- [32] J. Nash. Non-cooperative games. *Ann. Math.*, 2:286 – 295, 1951.
- [33] A. Oubraham and G. Zaccour. A survey of applications of viability theory to the sustainable exploitation of renewable resources. *Ecological Economics*, 145:346–367, 2018.
- [34] P. Saint-Pierre. Approximation of viability kernel. *Appl. Math. Optim.*, 29:187–209, 1994.
- [35] M. Sicard, N. Perrot, R. Reuillon, S. Mesmoudi, I. Alvarez, and S. Martin. A viability approach to control food processes: Application to a camembert cheese ripening process. *Food Control*, 23(2):312 – 319, 2012.
- [36] C. J. Tomlin, I. Mitchell, A. M. Bayen, and M. Oishi. Computational techniques for the verification of hybrid systems. *Proc. IEEE*, 91(7):986–1001, 2003.
- [37] W. Wei, I. Alvarez, and S. Martin. Sustainability analysis: Viability concepts to consider transient and asymptotical dynamics in socio-ecological tourism-based systems. *Ecol. Model.*, 251:103 – 113, 2013.

A Proofs

A.1 Proof of Proposition 12

Let us consider $u_i \in [1; \bar{\alpha}]$ such that

$$\mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i) \geq 0.$$

Let us consider $p := f(g(u_i))$ and $K \geq 0$. The evolution starting at (K, p) governed by dynamics of Eq. (19) with $\forall t \geq 0, u_i(t) = u_i$ and $v_i(t) = \bar{\alpha}$ is such that $p(t) = p$ and $K_i(t+1) - K_i(t) = \mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i) \geq 0, \forall t \geq 0$. Then this evolution is viable.

Now, the evolutions starting at (K, p) governed by dynamics of Eq. (19) with $\forall t \geq 0, u_i(t) = u_i$ and $v_i(t) : \mathbb{N} \rightarrow [0; \bar{\alpha}]$ are such that $p(t) \geq p$ and $K_i(t+1) - K_i(t) = \mathcal{K}_i(\mathcal{P}(p(t), u_i, v_i(t)), u_i) \geq \mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i) \geq 0, \forall t \geq 0$. Then these evolutions are also viable. Consequently, (K, p) belongs to the guaranteed viability kernel $\text{Guar}_{S_i, w}(L_i)$ which is then non-empty.

A.2 Proof of Lemma 13

First,

$$\begin{aligned} \mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i) &= w_i(f(g(u_i)) - \frac{C_i}{u_i}) \\ &= \frac{w_i}{\bar{\alpha}^2 u_i (\bar{\alpha} + (k-1)u_i)} G(u_i) \end{aligned}$$

with $G : [1; \bar{\alpha}] \rightarrow \mathbb{R}$ defined by:

$$G(u_i) := p_0(k - (1 + \bar{\alpha} + \bar{\alpha}^2))u_i^2 + (p_0\bar{\alpha}(1 + \bar{\alpha} + \bar{\alpha}^2) - C_i\bar{\alpha}^2(k-1))u_i - C_i\bar{\alpha}^3. \quad (\text{A.1})$$

$$\text{and } k := \frac{W}{w_i} \geq 1.$$

Both signs of $\mathcal{K}_i(\mathcal{P}(f(g(u_i)), u_i, \bar{\alpha}), u_i)$ and $G(u_i)$ are the same since $k \geq 1$.

The function $k \rightarrow \max_{u_i \in [1; \bar{\alpha}]} G(u_i)$ is continuous. Moreover, when $k = 1$,

$$G(u_i) = -p_0(\bar{\alpha} + \bar{\alpha}^2)u_i^2 + p_0\bar{\alpha}(1 + \bar{\alpha} + \bar{\alpha}^2)u_i - C_i\bar{\alpha}^3.$$

G is a polynomial of degree 2 which has a positive value in $[1; \bar{\alpha}]$ if and only if

- $G(\bar{\alpha}) \geq 0$, that is

$$C_i \leq \frac{p_0}{\bar{\alpha}}$$

- or $G(1) \geq 0$, that is

$$C_i \leq p_0$$

- or the discriminant of G is positive, that is

$$p_0^2 \bar{\alpha}^2 (1 + \bar{\alpha} + \bar{\alpha}^2)^2 - 4p_0 \bar{\alpha}^4 (1 + \bar{\alpha}) C_i \geq 0,$$

$$C_i \leq \frac{p_0 (1 + \bar{\alpha} + \bar{\alpha}^2)^2}{\bar{\alpha} 4\bar{\alpha} (1 + \bar{\alpha})},$$

and $\operatorname{argmax}_{u_i \in \mathbb{R}} G(u_i) = \frac{1 + \bar{\alpha} + \bar{\alpha}^2}{2(1 + \bar{\alpha})} \in [1; \bar{\alpha}]$, when $\bar{\alpha}^2 - \bar{\alpha} - 1 \geq 0$.

A.3 Proof of Corrolary 14

Let us consider u_i^* such that

$$\mathcal{K}_i(\mathcal{P}(f(g(u_i^*)), u_i^*, \bar{\alpha}), u_i^*) \geq 0,$$

and let us define $p^* := f(g(u_i^*)) \in [p_0/\bar{\alpha}^2; p_0]$. Let us consider an evolution $(K_i(\cdot), p(\cdot))$ governed by Eq. (19) along which $u_i(t) = u_i^*$ for all t . Then, for all $T \in \mathbb{N}$ and all $v_i(\cdot) : \mathbb{N} \rightarrow [1; \bar{\alpha}]$,

$$\begin{aligned} K_i(T) - K_i(0) &= w_i \sum_{t=0}^T (p(t) - \frac{C_i}{u_i^*}) \\ &= w_i T (p^* - \frac{C_i}{u_i^*}) - w_i \sum_{t=0}^T (p^* - p(t)) \\ &= T \mathcal{K}_i(\mathcal{P}(f(g(u_i^*)), u_i^*, \bar{\alpha}), u_i^*) - w_i \sum_{t=0}^T (p^* - p(t)) \\ &\geq -w_i \sum_{t=0}^T (p^* - p(t)), \end{aligned} \tag{A.2}$$

since $\mathcal{K}_i(\mathcal{P}(f(g(u_i^*)), u_i^*, \bar{\alpha}), u_i^*) \geq 0$. But,

$$\begin{aligned} p^* - p(t+1) &= p^* - \delta(p(t), f(\frac{W}{\frac{w_i}{u_i^*} + \frac{W-w_i}{v_i(t)}})p(t)) \\ &\quad - (1 - \delta(p(t), f(\frac{w_i}{\frac{w_i}{u_i^*} + \frac{W-w_i}{v_i(t)}}))) f(\frac{W}{\frac{w_i}{u_i^*} + \frac{W-w_i}{v_i(t)}}) \\ &\leq p^* - \delta(p(t), f(\frac{W}{\frac{w_i}{u_i^*} + \frac{W-w_i}{v_i(t)}})p(t)) \\ &\quad - (1 - \delta(p(t), f(\frac{w_i}{\frac{w_i}{u_i^*} + \frac{W-w_i}{v_i(t)}}))) p^* \\ &\leq \delta(p(t), f(\frac{W}{\frac{w_i}{u_i^*} + \frac{W-w_i}{v_i(t)}})p(t)) (p^* - p(t)) \\ &\leq \max(0, \delta(p^* - p(t))). \end{aligned} \tag{A.3}$$

So, if $p(0) \in [p^*; p_0]$, $K_i(T) - K_i(0) \geq 0$ and if $p(0) \in [\frac{p_0}{\bar{\alpha}^2}; p^*]$,

$$\begin{aligned} K_i(T) - K_i(0) &\geq -w_i (p^* - p(0)) \sum_{t=0}^T \delta^t \\ &\geq -w_i \frac{p_0}{\bar{\alpha}} (\bar{\alpha} - 1) \frac{1}{1 - \delta}. \end{aligned} \tag{A.4}$$

So, $K_i(T) \geq 0$ for all $T \in \mathbb{N}$, when $K_i(0) \geq w_i \frac{p_0}{\bar{\alpha}} (\bar{\alpha} - 1) \frac{1}{1 - \delta}$ whatever $p(0) \in [\frac{p_0}{\bar{\alpha}^2}; p_0]$.