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A CONNECTION BETWEEN THE RANDOM PINNING MODEL AND RANDOM WALKS IN SPARSE RANDOM ENVIRONMENTS

JULIEN POISAT

ABSTRACT. The purpose of this short note is to establish a connection between a one-dimensional random walk in a random sparse environment and the random pinning model. We show that the grand canonical partition function of the pinning model coincides with the mean number of returns to the origin for a random walk in a random sparse environment averaged on the randomness location. We obtain thereof some information on the integrability of the number of return times in the annealed and partially annealed setups.

CONTENTS

1. Introduction	1
2. Reminder on random walks in a potential	2
3. Random walk in a random environment (RWRE)	3
4. Random walk in a sparse random environment (RWsRE)	3
5. Connection with the random pinning model	4
References	7

1. INTRODUCTION

Random walk in a *sparse* random environment (RWsRE) is a variant of the more common random walk in random environment (RWRE) that behaves as a simple random walk everywhere but at *random locations* where random transition probabilities are drawn. This model with a double layer of randomness has recently received some interest [6, 7, 8, 10, 19, 22]. In this short paper we establish a connection between the RWsRE and the so-called random pinning model [14, 15]. We first remind the reader of standard facts about the random walk in a general potential and in a random environment, in Sections 2 and 3 respectively. We then recall some (non-exhaustive) results on the RWsRE in Section 4 and finally make the connection with the random pinning model in Section 5. The key relation, displayed in (5.10), is an equality between the grand canonical partition function of the random pinning model and the mean number of return times to the origin for the law of the RWsRE averaged on the *location* of the randomness only (partially annealed setup). Our observations eventually lead to Theorem 3: the annealed and (more interestingly) quenched critical curves of the pinning model are shown to correspond to thresholds for the (respectively annealed and partially annealed) integrability of the mean

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number of return times of the RWsRE.

Notation. Throughout the paper, \mathbb{N}_0 and \mathbb{N} respectively denote the set of non-negative and positive integers.

2. REMINDER ON RANDOM WALKS IN A POTENTIAL

Let us first recall a few standard facts about random walks in a potential. Let $(V_i)_{i \in \mathbb{Z}}$ be a real-valued sequence with $V_0 = 0$ and $\Delta V_i := V_i - V_{i-1}$ for $i \in \mathbb{Z}$. The *random walk in potential* V is the Markov chain $X = (X_n)_{n \geq 0}$ on \mathbb{Z} , started at $X_0 = 0$ unless stated otherwise, with transition matrix:

$$(2.1) \quad p(i, i+1) = 1 - p(i, i-1) = \frac{1}{1 + e^{\Delta V_i}} \in (0, 1), \quad i \in \mathbb{Z}.$$

Its law shall be denoted by \mathcal{P}_V in the sequel, with \mathcal{E}_V the corresponding expectation. Alternatively to (2.1),

$$(2.2) \quad \Delta V_i = \log \left[\frac{p(i, i-1)}{p(i, i+1)} \right], \quad i \in \mathbb{Z}.$$

Set $W(0) = 0$ and

$$(2.3) \quad W(n) := \sum_{0 \leq k < n} e^{V_k}, \quad W(-n) := - \sum_{1 \leq k \leq n} e^{V_{-k}} \quad n \geq 1.$$

The following fact is standard: using that $(W(X_n))_{n \geq 0}$ is a martingale adapted to X , an application of Doob's optional stopping theorem yields

$$(2.4) \quad \mathcal{P}_V(H_a > \tilde{H}_c | X_0 = b) = \frac{W(b) - W(a)}{W(c) - W(a)}, \quad a < b \leq c,$$

where

$$(2.5) \quad \begin{aligned} \tilde{H}_x &:= \inf\{n \geq 0: X_n = x\} \quad (\text{hitting time}), \\ H_x &:= \inf\{n \geq 1: X_n = x\} \quad (\text{entrance time}), \end{aligned}$$

see e.g. [27]. In particular,

$$(2.6) \quad \mathcal{P}_V(H_y > \tilde{H}_x | X_0 = y+1)^{-1} = \sum_{y \leq i < x} e^{V_i - V_y}, \quad 0 \leq y < x.$$

Choosing $y = 0$ and letting $x \rightarrow \infty$ in the line above, we obtain

$$(2.7) \quad \mathcal{P}_V(H_0 = +\infty | X_0 = 1)^{-1} = \sum_{i \geq 0} e^{V_i},$$

and in a similar way,

$$(2.8) \quad \mathcal{P}_V(H_0 = +\infty | X_0 = -1)^{-1} = \sum_{i \geq 0} e^{V_{-i-1} - V_{-1}}.$$

If we assume moreover that V is symmetric w.r.t to $-1/2$, so that $V_i = V_{-i-1} - V_{-1}$ for every $i \geq 0$, then the quantities in (2.7) and (2.8) coincide. Consequently, we obtain the following expression for the mean number of return times to the origin:

$$(2.9) \quad \mathcal{E}_V(\text{card}\{n \geq 0: X_n = 0\}) = \sum_{i \geq 0} e^{V_i}.$$

In particular, X is *transient* (resp. *recurrent*) if and only if the above sum converges (resp. diverges). Throughout the paper, we assume the aforementioned symmetry on the potential V , which amounts to replacing the \mathbb{Z} -valued Markov chain by an \mathbb{N}_0 -valued chain (a.k.a. birth-and-death process).

3. RANDOM WALK IN A RANDOM ENVIRONMENT (RWRE)

A random walk in a random environment is obtained when the potential is drawn *randomly*. This is a very active field of research, see [12, 13, 16, 17, 18, 21, 23, 24, 25, 27] for a selection of seminal papers and reviews on the topic. A common assumption made in the literature is that the increments $(\Delta V_i)_{i \geq 1}$ form an i.i.d. sequence. Therefore, we let $(\omega_i)_{i \geq 1}$ be a sequence of (integrable) i.i.d. random variables with law \mathbb{P} and define

$$(3.1) \quad \Delta V_i := \beta \omega_i + h, \quad i \geq 1,$$

where $h \in \mathbb{R}$ and (w.l.o.g.) $\beta \geq 0$ and $\mathbb{E}(\omega_1) = 0$. In that setup, it is known since Solomon [24] that X is transient under \mathcal{P}_V if and only if $h < 0$ and that X is transient with *positive* speed if and only if $h < -\log \mathbb{E}(e^{\beta \omega_1})$, provided that the exponential moment exists.

4. RANDOM WALK IN A SPARSE RANDOM ENVIRONMENT (RWsRE)

More recently, several authors [6, 7, 8, 10, 19, 22] have introduced a variant of RWRE, by assuming that the randomness of the potential is only seen at *random locations*, outside which the potential is flat. This means defining the potential by $V_0 = 0$ and

$$(4.1) \quad \Delta V_i = (h + \beta \omega_i) \mathbf{1}_{\{i \in \tau\}}, \quad i \geq 1,$$

where $\tau = \{\tau_i\}_{i \geq 0}$ is a certain random subset of \mathbb{N}_0 . A common assumption, that we shall adopt in the sequel, is that $(\tau_i)_{i \geq 0}$ is a *renewal* process starting at $\tau_0 = 0$ with independent \mathbb{N} -valued increments $(\tau_i - \tau_{i-1})_{i \geq 1}$. Its law shall be denoted by \mathbb{P} . In that case, we write

$$(4.2) \quad \mathcal{P}_{\beta, h}^{\omega, \tau} := \text{law of the quenched RWsRE when (4.1) holds,}$$

instead of \mathcal{P}_V . Also, we assume here that τ and ω are independent, which is not necessarily the case in the RWsRE literature. Such a random environment is called *moderately* (resp. *strongly*) sparse if $\mathbb{E}(\tau_1)$ is finite (resp. infinite). By a result of Matzavinos, Roitershtein and Seol [19, Theorem 3.1], we have the following:

Proposition 1. *The RWsRE is transient under the annealed law, i.e.*

$$(4.3) \quad \mathbb{E} \mathbb{E} \left[\mathcal{P}_{\beta, h}^{\omega, \tau}(\text{card}\{n \geq 0 : X_n = 0\} < \infty) \right] = 1,$$

whenever $h < 0$ and $\mathbb{E}[\log \tau_1] < \infty$.

In complement to Proposition 1, the same authors [19, Theorem 3.3] proved under the same assumptions, and still considering the annealed law, that the RWsRE is transient with *positive* speed if $h < -\log \mathbb{E}(e^{\beta \omega_1})$ and $\mathbb{E}(\tau_1^2) < \infty$, or transient with *zero* speed otherwise. See [7, Proposition 2.1] in the case where the vectors $(\omega_k, \tau_k - \tau_{k-1})_{k \geq 1}$ are independent and identically distributed (i.i.d.) but ω_k and $\tau_k - \tau_{k-1}$ are allowed to be dependent.

Note that the equality in (4.3) readily implies that

$$(4.4) \quad \mathcal{P}_{\beta, h}^{\omega, \tau}(\text{card}\{n \geq 0 : X_n = 0\} < \infty) = 1, \quad \mathbb{P} \otimes \mathbb{P} - \text{a.s.},$$

that is transience for almost every quenched sparse environment. This also implies that

$$(4.5) \quad \mathcal{E}_{\beta, h}^{\omega, \tau}(\text{card}\{n \geq 0 : X_n = 0\}) < \infty, \quad \mathbb{P} \otimes \mathbb{P} - \text{a.s.}$$

However, (4.3) does *not* imply that

$$(4.6) \quad \mathbb{E} \mathbb{E} \mathcal{E}_{\beta, h}^{\omega, \tau}(\text{card}\{n \geq 0: X_n = 0\}) < \infty.$$

We will see in Section 5 that the mean number of return times of RWsRE to the origin, once averaged over the renewal times, coincides with the grand canonical partition function of the so-called random pinning model. This will allow us to investigate the integrability of the former quantity w.r.t. the law $\mathbb{P} \otimes \mathbb{P}$ and (more interestingly) the law \mathbb{P} , for \mathbb{P} -a.e. sequence ω .

5. CONNECTION WITH THE RANDOM PINNING MODEL

Let us now introduce the random pinning model, to which we want to connect the RWsRE. This is a statistical mechanics model of a polymer pinned along a (one-dimensional) defect line that is motivated for instance by interface models or the phenomenon of DNA denaturation (a.k.a. Poland-Scheraga models in this context). There has been an intense and fruitful activity on this model lately, in particular around the issue of disorder relevance, which we shortly come back to below. We keep our exposition as concise as possible but invite the reader to refer to [14, 15] or [11, Section 11], and references therein for a more thorough introduction to this model. From now on we will assume that

$$(5.1) \quad \mathbb{P}(\tau_1 = n) = L(n)n^{-(1+\alpha)}, \quad n \in \mathbb{N},$$

where L is a slowly varying function and $\alpha \geq 0$, as well as

$$(5.2) \quad \mathbb{E}(\omega_1) = 0, \quad \mathbb{E}(e^{\beta\omega_1}) < \infty, \quad \forall \beta \geq 0.$$

The quenched *partition function* of the random pinning model is defined as

$$(5.3) \quad Z_{n, \omega}^{\beta, h} := \mathbb{E} \left[\exp \left(\sum_{i=1}^n (\beta\omega_i + h) \mathbf{1}_{\{i \in \tau\}} \right) \right], \quad n \geq 1,$$

with the convention $Z_{0, \omega}^{\beta, h} := 0$. Here, the renewal set τ can be thought of as the random set of contact points between a linear polymer chain and a one-dimensional defect line, with $\beta\omega_i + h$ the random energy reward/penalty gathered from a contact at site $i \in \mathbb{N}$. It is known that the following limit, called quenched *free energy*, exists \mathbb{P} -a.s.

$$(5.4) \quad f(\beta, h) := \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_{n, \omega}^{\beta, h} \geq 0,$$

and that the following dichotomy holds: there exists a critical curve $-\infty < h_c(\beta) \leq 0$ such that $f(\beta, h) = 0$ for every $h \leq h_c(\beta)$ and $f(\beta, h) > 0$ for every $h > h_c(\beta)$. This is a signature of a phase transition between a delocalized phase and a localized phase, and it is known that that (i) $h_c(0) = 0$ and (ii) $h_c(\beta) < 0$ if $\beta > 0$, see [1] or [14, Chap. 5, Sect. 2]. One of the major challenge in this model has been to determine whether disorder is *relevant* or not, that is whether the critical curve coincides with the critical curve for the *annealed* model for at least sufficiently small values of β (disorder strength). Note that the annealed critical curve, denoted by $h_c^a(\beta)$, is easy to compute, since the annealed partition function equals

$$(5.5) \quad \mathbb{E} Z_{n, \omega}^{\beta, h} = \mathbb{E} \left[\exp \left(\sum_{i=1}^n (\lambda(\beta) + h) \mathbf{1}_{\{i \in \tau\}} \right) \right], \quad \text{where } \lambda(\beta) := \log \mathbb{E}(e^{\beta\omega_1}).$$

Therefore $h_c^a(\beta) = -\lambda(\beta)$ is simply the critical point of an adequately shifted homogeneous (i.e. $\beta = 0$) model. The following equivalence for disorder relevance in the sense of critical point shifts is now known to hold:

$$(5.6) \quad \{\forall \beta > 0, h_c(\beta) > h_c^a(\beta)\} \Leftrightarrow \sum_{n \geq 1} \frac{1}{n^{2(1-\alpha)} L(n)^2} = +\infty,$$

that is, if and only if the renewal process $\tau^{(1)} \cap \tau^{(2)}$, obtained by intersecting two independent copies of τ , is transient, see [5, Theorem 2.2] and references therein. The result displayed in (5.6) finally put on firm mathematical grounds a prediction made by physicists (the Harris criterion) and resolved a debate about the delicate marginal case $\alpha = 1/2$. Let us stress however that the strict inequality between the quenched and annealed critical points always holds for large enough β [26, Corollary 3.2] unless $\alpha = 0$, in which case $h_c(\beta) = h_c^a(\beta)$ for every $\beta \geq 0$ [2].

Before getting to our main observation, we define the *grand canonical* partition function of the model by

$$(5.7) \quad \mathcal{Z}_{\beta,h}^\omega(f) := \sum_{n \geq 0} Z_{n,\omega}^{\beta,h} e^{-fn}, \quad f \in \mathbb{R}.$$

Note that

$$(5.8) \quad f(\beta, h) = \inf\{f \geq 0: \mathcal{Z}_{\beta,h}^\omega(f) < \infty \text{ } \mathbb{P}\text{-a.s.}\}.$$

It turns out that this is related to the RWsRE with an additional drift. To this end, we let

$$(5.9) \quad \mathcal{P}_{\beta,h,f}^{\omega,\tau} := \mathcal{P}_V \quad \text{when} \quad \Delta V_i := (h + \beta\omega_i) \mathbf{1}_{\{i \in \tau\}} - f, \quad i \geq 1.$$

Averaging (2.9) over the renewal process τ leads to the relation

$$(5.10) \quad \mathbb{E} \mathcal{E}_{\beta,h,f}^{\omega,\tau}(\text{card}\{n \geq 0: X_n = 0\}) = \mathcal{Z}_{\beta,h}^\omega(f),$$

that is the key for the remaining discussion. From (5.8) we obtain the following:

- If $h > h_c(\beta)$ then there exists $\varepsilon_0 > 0$ such that $\mathcal{Z}_{\beta,h}^\omega(\varepsilon) = \mathbb{E} \mathcal{E}_{\beta,h,\varepsilon}^{\omega,\tau}(\text{card}\{n \geq 0: X_n = 0\}) = \infty$ for every $0 \leq \varepsilon < \varepsilon_0$.
- If $h \leq h_c(\beta)$ then $\mathcal{Z}_{\beta,h}^\omega(\varepsilon) = \mathbb{E} \mathcal{E}_{\beta,h,\varepsilon}^{\omega,\tau}(\text{card}\{n \geq 0: X_n = 0\}) < \infty$ for every $\varepsilon > 0$.

It is however not straightforward to deduce a result for $\mathcal{P}_{\beta,h,f}^{\omega,\tau}$ in the case $f = 0$ and $h \leq h_c(\beta)$ from the mere definition of the free energy. Fortunately, we may use a result from Mourrat [20, Theorem 1] about the *pinned* version of the grand canonical partition function, that is

$$(5.11) \quad \mathcal{Z}_{\beta,h}^{\omega,c}(f) := \sum_{n \geq 0} Z_{n,\omega}^{\beta,h}(n \in \tau) e^{-fn}, \quad f \in \mathbb{R},$$

where

$$(5.12) \quad Z_{n,\omega}^{\beta,h}(n \in \tau) := \mathbb{E} \left[\exp \left(\sum_{i=1}^n (\beta\omega_i + h) \mathbf{1}_{\{i \in \tau\}} \right) \mathbf{1}_{\{n \in \tau\}} \right], \quad n \geq 1, \quad (0 \text{ if } n = 0),$$

is the *pinned* partition function (compare to (5.3)). With this notation in hand,

$$(5.13) \quad h < h_c(\beta) \quad \Rightarrow \quad \mathcal{Z}_{\beta,h}^{\omega,c}(0) < \infty,$$

according to [20, Theorem 1]. We can now relate the grand canonical partition function to its pinned version. By decomposing the partition function according to the last renewal point before n , we obtain

$$(5.14) \quad Z_{n,\omega}^{\beta,h} = \sum_{k \geq 0} \mathbb{E} \left(e^{\sum_{1 \leq i \leq k} (\beta \omega_{\tau_i} + h)} \mathbf{1}_{\{\tau_k \leq n < \tau_{k+1}\}} \right),$$

(the sum in the above exponential is understood as zero if $k = 0$). Therefore,

$$(5.15) \quad \begin{aligned} \mathcal{Z}_{\beta,h}^{\omega}(0) &= \sum_{n \geq 0} Z_{n,\omega}^{\beta,h} = \mathbb{E} \left[\sum_{k \geq 0} e^{\sum_{1 \leq i \leq k} (\beta \omega_{\tau_i} + h)} (\tau_{k+1} - \tau_k) \right] \\ &= \mathbb{E}(\tau_1) \sum_{k \geq 0} \mathbb{E} \left(e^{\sum_{1 \leq i \leq k} (\beta \omega_{\tau_i} + h)} \right) = \mathbb{E}(\tau_1) \mathcal{Z}_{\beta,h}^{\omega,c}(0). \end{aligned}$$

We deduce thereby:

- If $\mathbb{E}(\tau_1) = \infty$ then $\mathcal{Z}_{\beta,h}^{\omega}(0) = \infty$ for every $\beta \geq 0$ and $h \in \mathbb{R}$.
- If $\mathbb{E}(\tau_1) < \infty$ then $\mathcal{Z}_{\beta,h}^{\omega}(0) < \infty$ for every $\beta > 0$ and $h < h_c(\beta)$.

Remark 2. Note that the first item in the above dichotomy may be obtained in a different manner. Indeed, by a standard ruin probability estimate,

$$(5.16) \quad \mathcal{P}_{\beta,h}^{\omega,\tau}(H_0 = \infty) \leq \mathcal{P}_{\beta,h}^{\omega,\tau}(H_0 > H_{\tau_1}) = 1/\tau_1,$$

which leads, using (5.10), to

$$(5.17) \quad \mathcal{Z}_{\beta,h}^{\omega}(0) = \mathbb{E} \mathcal{E}_{\beta,h}^{\omega,\tau}(\text{card}\{n \geq 0 : X_n = 0\}) \geq \mathbb{E}(\tau_1).$$

Summarizing all what precedes, we obtain the three following cases:

Theorem 3. Let $\beta > 0$.

- (1) If $h_c(\beta) < h < 0$ then the RWsRE is transient, so $\mathcal{E}_{\beta,h}^{\omega,\tau}(\text{card}\{n \geq 0 : X_n = 0\})$ is finite $\mathbb{P} \otimes \mathbb{P}$ -a.s, but $\mathbb{E} \mathcal{E}_{\beta,h,\varepsilon}^{\omega,\tau}(\text{card}\{n \geq 0 : X_n = 0\})$ is infinite \mathbb{P} -a.s. for every $0 \leq \varepsilon < f(\beta, h)$.
- (2) If $h_c^a(\beta) < h \leq h_c(\beta)$ then $\mathbb{E} \mathcal{E}_{\beta,h,\varepsilon}^{\omega,\tau}(\text{card}\{n \geq 0 : X_n = 0\})$ is finite \mathbb{P} -a.s. for every $\varepsilon > 0$. The result also holds for $\varepsilon = 0$ if $\mathbb{E}(\tau_1)$ is finite and $h < h_c(\beta)$. However, $\mathbb{E} \mathcal{E}_{\beta,h,\varepsilon}^{\omega,\tau}(\text{card}\{n \geq 0 : X_n = 0\})$ is infinite for every $0 \leq \varepsilon < f(0, h - h_c^a(\beta))$.
- (3) If $h \leq h_c^a(\beta)$ then $\mathbb{E} \mathcal{E}_{\beta,h,\varepsilon}^{\omega,\tau}(\text{card}\{n \geq 0 : X_n = 0\})$ is finite for every $\varepsilon > 0$. The result also holds for $\varepsilon = 0$ if $\mathbb{E}(\tau_1)$ is finite and $h < h_c^a(\beta)$.

Case (2) in Theorem 3 is non-empty (i) for large enough β when $\alpha > 0$ or (ii) for every $\beta > 0$ if $\alpha > 1/2$ or $\alpha = 1/2$ and $\sum n^{-1} L(n)^{-2} = +\infty$, see the discussion around (5.6).

Remark 4. The relation in (5.10) could be written for other one-dimensional statistical mechanics models based on renewal processes, like the copolymer model [11, Section 10] or to the random pinning model with other types of disorder sequences, like correlated random sequences [4, 9]. In particular, [3, Theorem 2.5] asserts that when ω is a Gaussian sequence with non-summable non-negative correlations, then the free energy of the random pinning model is positive for every $\beta > 0$ and $h \in \mathbb{R}$, i.e. $h_c(\beta) = -\infty$. Consequently, in that case $\mathbb{E} \mathcal{E}_{\beta,h}^{\omega,\tau}(\text{card}\{n \geq 0 : X_n = 0\})$ is infinite \mathbb{P} -a.s. for every $\beta > 0$, no matter how small h is.

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