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Well-posedness and geometric optics expansions for semi-linear hyperbolic boundary value problems in a strip

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Abstract

The present article deals with semi-linear hyperbolic boundary value problems defined in the strip geometry. After having established a strong well-posedness result by using a rather classical linear scheme of approximation, we address the question of the construction of approximate solutions by geometric optics expansions methods. On the one hand because the problem is semi-linear, the non linearity creates one self-interaction of the phases: the generation of harmonics. Such a self-interaction is classical. But, on the other hand, because of the geometry of the strip, a phase can regenerate itself by repeated reflections against the two sides of the boundary. Consequently we have an other self-interaction phenomenon. A natural question is to ask if these two phenomena can interact the one with the other. We here show and rigorously justify that the answer of the above question. The reason being essentially that one phenomenon occurs in the interior while the second one is located at the boundary.

AMS subject classification: 35L50, 35L60, 78A05

Key words: Hyperbolic system of PDE/Boundary value problem/Geometric optics expansions/Semi-linear problems/Geometric self-interaction/Generation of harmonics

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1 Introduction

This article deals with the construction of geometric optics expansions construction and the analysis of semi-linear hyperbolic boundary value problems defined in a strip. We will thus consider throughout the text the system of partial differential equations: for $T > 0$, $d \geq 1$,

$$\begin{cases} L(\partial)u := \partial_t u + \sum_{j=1}^d A_j \partial_j u = f + \mathbf{F}(u) & \text{in } \Omega_T :=]0, T[\times \mathbb{R}^{d-1} \times]0, 1[, \\ B_0 u|_{x_d=0} = g_0 & \text{on } \partial\Omega_{0,T} :=]0, T[\times \mathbb{R}^{d-1} \times \{0\}, \\ B_1 u|_{x_d=1} = g_1 & \text{on } \partial\Omega_{1,T} :=]0, T[\times \mathbb{R}^{d-1} \times \{1\}, \\ u|_{t=0} = 0 & \text{on } \Gamma := \mathbb{R}^{d-1} \times]0, 1[, \end{cases} \quad (1)$$

where the coefficients $A_j \in \mathbf{M}_{N \times N}(\mathbb{R})$ are given fixed matrices, for some fixed $N \geq 1$. The unknown u has its values in \mathbb{R}^N . The non linearity \mathbf{F} being a function of class \mathcal{C}^∞ of its arguments or entire and satisfying $\mathbf{F}(0) = 0$.

In system (1), the boundary conditions B_0 and B_1 are given matrices in $\mathbf{M}_{p \times N}(\mathbb{R})$ and $\mathbf{M}_{(N-p) \times N}(\mathbb{R})$ respectively, where p stands for the number of (strictly) positive eigenvalues of the normal coefficient A_d . Consequently the number of boundary conditions in (1) is suitably chosen in such a way that the system (1) is not over or under-determined.

The aims of the article are twofold, firstly we briefly describe a strong well-posedness theory for the semi-linear system (1). Such well-posedness question for the underlying linear problem associated to (1) has been addressed in [2]. In this contribution, the author shows that in order to obtain a sharp strong well-posedness theory, that is to say a theory characterizing the minimal growth of the solution with respect to time, some extra condition issued from the analysis of [20] in the quarter-space is necessary and ("almost") sufficient.

This extra condition encodes the self-interaction of the solution by repeated reflections against the sides $\partial\Omega_{0,T}$ and $\partial\Omega_{1,T}$. What we mean here is that in the strip geometry the solution never spreads to infinity. It suffers repeated reflections against the two sides of the boundary periodically with respect to time. Consequently, the solution essentially behaves like C^t where $C > 0$ is a numerical constant corresponding to the coefficient of amplification during a return travel of the solution to the side $\partial\Omega_{0,T}$. When $C > 1$, the solution then has a non trivial exponential growth with respect to time, while when $C \leq 1$ such a growth does not occurs and the problem is said to be sharply well-posed¹. Let us however insist on the fact that such a characterization is really pertinent for unbounded times of resolution. Indeed for a fixed finite time of resolution $T > 0$, then the solution just behaves like $C^T < \infty$.

As a consequence, in the present article, because we are interested in non linear problems, we should in particular consider *a priori* finite times of resolution. So we will not use the sharp theory of [2] to establish the L^2 strong well-posedness of the linear problem, but we will just consider a simple and straightforward argument of localization.

Because the above argument is not sharp, one can not prevent the solution to have a non trivial exponential growth with respect to the time variable. But such a growth will not influence the resolution of system (1) by using iterative schemes of approximation. Consequently the proof exposed here for the strong well-posedness is not very new but it has the advantage of simplicity and to be (in the author's knowledge) the first well-posedness result for hyperbolic problems combining the two difficulties of having a non linearity and having several components in the boundary of the domain of resolution. The relations that these two difficulties share are then studied in more details thanks to geometric optics expansions.

We have reasons to believe that taking into account the sharp estimates of [2] can be of some help to obtain sharp times of existence for the semi-linear problem (1). In particular, we believe that is shall be of particular interest to determine and characterize the couple of boundary conditions which act as damping terms and which can lead to a longer time of existence than the one in the half-space geometry. However

¹In [2] we used the heavy terminology low exponentially well-posed.

such a precise analysis is left for future studies.

Once that we have established the existence (and uniqueness) of a solution to the semi-linear problem (1) a natural question is then the existence of geometric optics expansions approximating the exact solution. In a second time, we will thus restrict our attention to semi-linear problems with highly oscillating sources terms and we aim to describe an algorithm permitting to construct approximate solutions for such systems. Let $0 < \varepsilon \ll 1$ be the typical wavelength, we consider

$$\begin{cases} L(\partial)u^\varepsilon = \mathbf{F}(u^\varepsilon) & \text{in } \Omega_T, \\ B_0 u^\varepsilon|_{x_d=0} = g_0^\varepsilon & \text{on } \partial\Omega_{0,T}, \\ B_1 u^\varepsilon|_{x_d=1} = 0 & \text{on } \partial\Omega_{1,T}, \\ u^\varepsilon|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (2)$$

where the source g_0^ε admits fast oscillations with respect to the small parameter ε .

Geometric optics expansions for hyperbolic systems of equations have a long story and are described in an abundant literature. We refer to the seminal series of papers of Joly, Métivier and Rauch of the 90' (parts of such results being summarized in the book [22]) for the Cauchy problem and to the series of works of Coulombel, Guès and Williams of the 10' for problems in the half-space geometry (see also the earlier results of Williams [23] and [24]).

In the not so considered geometry of the strip, the construction of such geometric optics expansions for the underlying linear problem associated to (2) has been considered by the author in [3]. The main point of this article is to show that in order to start the resolution of the cascade of equations of geometric optics, a new invertibility condition has to be imposed (we refer to [3] for more details). Such a condition is a microlocalized version of the global condition ensuring the sharp strong well-posedness in [2]. Consequently it is linked to the self-interaction of the phases caused by the repeated reflections against the two sides of the strip.

Because we are now dealing with non linear problem other self-interaction phenomena of the phases composing the expansion occur. They are caused by the generation of the harmonics and the possible resonances² of the phases. So we have *a priori* three sources of self-interaction appearing in our framework of study.

The main question that we would like to answer here is: "Do these self-interaction phenomena can interact the one with the other?".

The answer of this question will be that the above phenomena do not interact the one with the other to create some new qualitative phenomena. The reason can simply be summarized has follows. Resonances and generation of harmonics form self-interaction phenomena that come into play at the level of the interior equation while the self-interaction phenomenon caused by repeated reflections come into play at the level of the boundary conditions. Consequently these two phenomena decouple the one from the other.

With more details, because we are not interested here in a precise description of the profiles equations induced by resonances (see for example [7]) we can use the so-called adapted basis of [25] and [24] (see also [11]) which consists in a reduced choice of the basis where the resonances do not occur³. So we just have to deal with generation of harmonics and repeated reflection self-interaction phenomena.

The main point here is that the amplitudes solve non linear equations essentially reading under the form (we refer to equation (36) for more details):

$$\begin{cases} \text{transport } u^b = \tilde{\mathbf{F}}^b(u) & \text{in } \Omega_T, \\ u^b|_{x_d=0} = G_0 u^{\sharp}|_{x_d=0} & \text{on } \partial\Omega_{0,T}, \text{ and } \\ u^b|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} \text{transport } u^\sharp = \tilde{\mathbf{F}}^\sharp(u) & \text{in } \Omega_T, \\ u^\sharp|_{x_d=1} = G_1 u^b|_{x_d=1} & \text{on } \partial\Omega_{1,T}, \\ u^\sharp|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (3)$$

²We shortly recall that the generation of harmonics means that we have to consider the phases $n\varphi$, $n \in \mathbb{Z}$ where φ is a characteristic phase appearing in the linear expansion, while resonances occur when two such phases, namely φ_a and φ_b , interact to produce a third one, namely φ_c . More precisely if one can find three integers $n_a, n_b, n_c \in \mathbb{Z}$ such that $n_a\varphi_a + n_b\varphi_b = n_c\varphi_c$.

³Crudely speaking if a resonance $n_a\varphi_a + n_b\varphi_b = n_c\varphi_c$ occurs then the adapted basis consists in considering only the family (φ_a, φ_b) instead of $(\varphi_a, \varphi_b, \varphi_c)$ because at the end of the day the contribution of φ_c is already taken into account by φ_a and φ_b .

where G_0 and G_1 are linear and where we used a decomposition⁴ $u^\varepsilon = u^\sharp + u^b$.

Consequently in (3) we have an interior coupling coming from the non linearity and a boundary coupling coming from the repeated reflections⁵. To deal with these two coupling especially the one in the interior induced by the non linearity, iterative scheme shall be considered. The main point being here that we do not require to use an approximation scheme to deal with the boundary coupling. Such a coupling being handled by assuming the invertibility of an operator reading under the form $I - \mathbf{T}$ as in the linear framework (we refer to [3] or to Paragraph 3.3) to determine the value of the trace $u^b|_{x_d=0}$ from which we can then deduce the one of $u^\sharp|_{x_d=1}$. So that, we can define the iterative scheme of approximation: let $u^{b,0}, u^{\sharp,0}$ be given and for $\nu \geq 1$, let $u^{b,\nu}, u^{\sharp,\nu}$ be the solutions to:

$$\left\{ \begin{array}{ll} \text{transport } u^{b,\nu} = \widetilde{\mathbf{F}}^b(u^{\nu-1}) & \text{in } \Omega_T, \\ u^b|_{x_d=0} = \text{known} & \text{on } \partial\Omega_{0,T}, \text{ and } \\ u^b|_{t=0} = 0 & \text{on } \Gamma, \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \text{transport } u^{\sharp,\nu} = \widetilde{\mathbf{F}}^\sharp(u^{\nu-1}) & \text{in } \Omega_T, \\ u^{\sharp,\nu}|_{x_d=1} = \text{known} & \text{on } \partial\Omega_{1,T}, \\ u^{\sharp,\nu}|_{t=0} = 0 & \text{on } \Gamma, \end{array} \right.$$

and the classical analysis of convergence (see for instance [24]) follows.

The fact that the non linear self-interactions do not influence the repeated reflection one is on the one hand a positive result because geometric optics expansions will be easy to construct. But on the other hand we can be a little disappointed, because nothing new happen. In order to force things in such a way that new phenomena appear we have two ideas. The first one is of course to consider instead of a linear boundary condition a non linear one. In such a way, both equations in the interior and at the boundaries need to be approximated by the iterative scheme of resolution and it is possible that now the repeated self-interaction phenomenon interacts with such an approximate boundary condition. A second alternative might possibly be to consider multiphase (at the boundary) geometric optics expansion as in [12]. In both cases, constructing geometric expansions for such hyperbolic boundary value problems in a strip are left for future studies.

We also think that it shall be very interesting in the future to consider quasi-linear problems which are completely motivated by their physical relevance. By first considering here the toy models of semi-linear problems, the results of the paper can be seen as a first step in such a program.

The article is organized as follows. Section 2 gives the first main result of the article, namely Theorem 2.1 establishing the local in time well-posedness of the semi-linear problem (1). Then Section 3 deals with the associated geometric optics expansions. The main results being Theorem 3.1 (whose proof occupies Paragraphs 3.2, 3.3 and 3.4) which gives the construction of the geometric optics expansion and Theorem 3.2 (shown in Paragraph 3.5) which justifies the fact that the constructed geometric optics expansions is effectively an approximate solution to (2).

2 Well-posedness theory

This section concerns the existence and uniqueness of the solution to the semi-linear problem (1). In order to state our main result precisely, let us define for some $s \in \mathbb{N}$ and some $T > 0$,

$$\mathbf{W}_T^s := \{u \in \mathcal{C}^0([0, T]; H^s(\Gamma)) \mid \forall 0 \leq j \leq s, \text{ we have } \partial_t^j u \in \mathcal{C}^0([0, T]; H^{s-j}(\Gamma))\},$$

⁴To be precise this decomposition corresponds to consider separately incoming and outgoing modes.

⁵In the half-space geometry, equation (3) simply reads

$$\left\{ \begin{array}{ll} \text{transport } u^b = \widetilde{\mathbf{F}}^b(u) & \text{in } \Omega_T, \\ u^b|_{x_d=0} = G_0 u^\sharp|_{x_d=0} & \text{on } \partial\Omega_{0,T}, \text{ and } \\ u^b|_{t=0} = 0 & \text{on } \Gamma, \end{array} \right. \text{ and } \left\{ \begin{array}{ll} \text{transport } u^\sharp = \widetilde{\mathbf{F}}^\sharp(u) & \text{in } \Omega_T, \\ u^\sharp|_{t=0} = 0 & \text{on } \Gamma, \end{array} \right.$$

so that we do not have such a boundary coupling. We can determine u^\sharp first and then proceed to the determination of u^b , by using the value of $u^\sharp|_{x_d=0}$ as a source term.

where for $k \geq 0$, H^k stands for the usual Sobolev space. The space \mathbf{W}_T^s is equipped with the following norm: for $u \in \mathbf{W}_T^s$ we denote

$$\|u(t)\|_s := \sum_{j=0}^s \|\partial_t^j u(t)\|_{H^{s-j}(\Gamma)} < \infty.$$

We remark that we have the inclusions $H^{s+1}(\Omega_T) \subset \mathbf{W}_T^s \subset H^s(\Omega_T)$. We will also consider $\mathbf{W}_T^\infty := \bigcap_{s \geq 0} \mathbf{W}_T^s$. For compatibility reasons and because of the homogeneous initial condition⁶ in (1), we require some flatness of the sources. More precisely for $X \subset \Omega_T$, we introduce $H_b^s(X)$ the subset of functions of $H^s(X)$ with flatness at the time origin:

$$H_b^s(X) := \{u \in H^s(X) \mid \forall 0 \leq j \leq s, (\partial_t^j u)|_{t=0} = 0\}.$$

The well-posedness result for (1) is the following

Theorem 2.1 *Let $s \in \mathbb{N}$ be such that $s > \frac{d}{2}$, and let $T > 0$ be such that the sources $f \in H_b^s(\Omega_T)$, $(g_0, g_1) \in H_b^s(\partial\Omega_{0,T}) \times H_b^s(\partial\Omega_{1,T})$. Then there exists some $T_0 \in]0, T[$ such that the semi-linear problem (1) admits a unique solution $u \in \mathbf{W}_{T_0}^s$.*

The proof of this theorem occupies the end of the current section. It is a classical proof based upon the convergence of a linear scheme of approximation which decouples to two sides of the interior equation. The linear hyperbolic boundary value problem associated to (1) is studied in Paragraph 2.1. Then with such a well-posedness result in hand, we investigate the non linear problem in Paragraph 2.2.

2.1 Well-posedness of the linear problem

Until the end of the paragraph, let us consider the linear problem associated to (1), namely:

$$\begin{cases} L(\partial)u = f & \text{in } \Omega_T, \\ B_0 u|_{x_d=0} = g_0 & \text{on } \partial\Omega_{0,T}, \\ B_1 u|_{x_d=1} = g_1 & \text{on } \partial\Omega_{1,T}, \\ u|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (4)$$

where f , g_0 and g_1 are given sources in the suitable Sobolev spaces H_b^s .

We aim to show the following strong well-posedness result:

Theorem 2.2 *Assume that the linear hyperbolic boundary value problem (4) is symmetric (Assumption 2.1), with non characteristic boundary (Assumption 2.2) and finally that the two sides of the boundary have strictly dissipative boundary conditions (Assumption 2.3). Then there exists $0 < \underline{T} \leq T$ such that if the sources $f \in H_b^s(\Omega_T)$, $(g_0, g_1) \in H_b^s(\partial\Omega_{0,T}) \times H_b^s(\partial\Omega_{1,T})$, then (4) admits a unique solution $u \in \mathbf{W}_{\underline{T}}^s$. Moreover u satisfies the energy estimate, for all $0 < t \leq \underline{T}$ one can find a constant $C_t > 0$ such that*

$$\|u(t)\|_s + \|u|_{x_d=0}\|_{H^s(\partial\Omega_{0,t})} + \|u|_{x_d=1}\|_{H^s(\partial\Omega_{1,t})} \leq C_t \left(\int_0^t \|f(r)\|_s dr + \|g_0\|_{H^s(\partial\Omega_{0,t})} + \|g_1\|_{H^s(\partial\Omega_{1,t})} \right). \quad (5)$$

Before to give the proof of Theorem 2.2 in Paragraphs 2.1.1 and 2.1.2, let us conclude this introduction by a clarification of the above assumptions. These three assumptions are performed in order to ensure *a priori* energy estimates for the solutions the three problems constituting the linear strip problem (4). More precisely, the two problems involving the boundaries $\partial\Omega_{0,T}$ and $\partial\Omega_{1,T}$ respectively and the problem in the interior of the strip.

The strong well-posedness of the problem in the interior is ensured under some hyperbolicity assumption. Here we choose to restrict our study to symmetric operators

Assumption 2.1 (Symmetric operators) *The operator $L(\partial)$ is symmetric. That is to say that for all $j \in \llbracket 1, d \rrbracket$, we have $A_j^T = A_j$, where \cdot^T stands for the transpose operator.*

⁶The following analysis can be extended in a rather straightforward manner to non homogeneous initial conditions if one assumes the suitable compatibility conditions.

While the strong well-posedness of the half-space problems is facilitated under Assumption 2.1 and under the assumption:

Assumption 2.2 (Non characteristic boundary) *We assume that the normal coefficient matrix A_d is invertible.*

The strong well-posedness result for the half-spaces problems also requires some properties on the boundary conditions. Here to simplify as much as possible the exposition and because we do not have in mind a sharp well-posedness result, we will assume that the boundary conditions are strictly dissipative in the following sense:

Assumption 2.3 (Strictly dissipative boundary conditions) *We assume that the boundary condition B_0 (resp. B_1) on $\partial\Omega_{0,T}$ (resp. $\partial\Omega_{1,T}$) is strictly dissipative meaning that*

$$\forall v \in \ker B_0 \text{ (resp. } \ker B_1) \text{ we have } \langle A_d v, v \rangle < 0 \text{ (resp. } \langle A_d v, v \rangle > 0).$$

Following [13], [[4]-Chapter 4], Assumptions 2.1, 2.2 and 2.3 do not constitute the most general framework where strong well-posedness holds. Indeed the symmetry assumption can be transformed into the one of constant hyperbolicity, while Assumption 2.3 can be turned into the (sharp) uniform Kreiss-Lopatinskii condition [13] or [[4]-Chapter 4]. Assumption 2.2 can also be removed of the analysis (we refer for example to [[4]-Chapter 6]). Some comments about the sharp estimate in such a general framework are given at the end of Paragraph 2.1.1.

2.1.1 A simple L^2 well posedness theory

In this paragraph we give a proof of Theorem 2.2 in the particular framework where $s = 0$ that is to say that we are interested in the L^2 strong well-posedness. In the following, we only describe the establishment of the *a priori* energy estimate. The existence of a solution is then classically deduced from the dual *a priori* energy estimate as it is done first in [15], then in [2] (see also [[4]-Sections 4.4 and 4.5]). Similarly the "weak=strong" lemma establishing that the solution satisfies the *a priori* energy estimate is a straightforward regularization argument by mollification with respect to the tangential variables (t, x') .

Because we are interested in non linear problems, we should *a priori* assume that we are in finite time, so that we will not follow the analysis of [2] and we will not use the weighted L^2 spaces $L^2_\gamma := \{u \in \mathcal{D}' \setminus e^{-\gamma t} u \in L^2\}$. As we will see the following localization argument however gives rise to a non optimal maximal time of resolution and we have good reasons to believe that the analysis of [2] could be of some help to recover the sharp maximal time of existence. This is however left for future studies.

The main result of the paragraph is the following:

Proposition 2.1 *Under Assumptions 2.1, 2.2 and 2.3, if u is regular enough then it satisfies the *a priori* energy estimate: there exists $0 < \underline{T} \leq T$ such that, for all $t \leq \underline{T}$ we have:*

$$\begin{aligned} & \|u(t, \cdot)\|_{L^2(\Gamma)} + \|u|_{x_d=0}\|_{L^2(\partial\Omega_{0,t})} + \|u|_{x_d=1}\|_{L^2(\partial\Omega_{1,t})} \\ & \leq C \left(\int_0^t \|\{L(\partial)u\}(s, \cdot)\|_{L^2(\Gamma)} ds + \|B_0 u|_{x_d=0}\|_{L^2(\partial\Omega_{0,t})} + \|B_1 u|_{x_d=1}\|_{L^2(\partial\Omega_{1,t})} \right). \end{aligned} \quad (6)$$

Proof : The proof exposed here is a straightforward localization argument. Let us introduce two cut-off functions χ_0 and $\chi_1 \in \mathcal{C}^\infty([-1, 2])$ satisfying

$$\chi_0(x_d) := \begin{cases} 1 & \text{for } 0 \leq x_d \leq \frac{1}{3}, \\ 0 & \text{for } \frac{2}{3} \leq x_d \leq 1, \end{cases} \text{ and } \chi_1(x_d) := \begin{cases} 0 & \text{for } 0 \leq x_d \leq \frac{1}{3}, \\ 1 & \text{for } \frac{2}{3} \leq x_d \leq 1. \end{cases}$$

We define $\chi_{1/2} := 1 - \chi_0 - \chi_1$ and for $k \in \{0, \frac{1}{2}, 1\}$, $u_k := \chi_k u$. Clearly $u_{1/2}$ satisfies the Cauchy problem:

$$\begin{cases} L(\partial)u_{1/2} = \chi'_{1/2} A_d u + \chi_{1/2} L(\partial)u & \text{for } (t, x) \in]0, T[\times \mathbb{R}^d, \\ u_{1/2}|_{t=0} = 0 & \text{on } \mathbb{R}^d, \end{cases} \quad (7)$$

while u_0 and u_1 satisfy the half-space boundary value problems

$$\begin{cases} L(\partial)u_0 = \chi'_0 A_d u + \chi_0 L(\partial)u & \text{in } \bar{\Omega}_T, \\ B_0 u_0|_{x_d=0} = g_0 & \text{on } \partial\Omega_{0,T}, \\ u_0|_{t=0} = 0 & \text{on } \bar{\Gamma}, \end{cases} \quad (8)$$

and

$$\begin{cases} L(\partial)u_1 = \chi'_1 A_d u + \chi_1 L(\partial)u & \text{in } \underline{\Omega}_T, \\ B_1 u_1|_{x_d=1} = g_1 & \text{on } \partial\Omega_{1,T}, \\ u_1|_{t=0} = 0 & \text{on } \underline{\Gamma}, \end{cases} \quad (9)$$

where we defined

$$\underline{\Omega}_T :=]0, T[\times \underline{\Gamma}, \bar{\Omega}_T :=]0, T[\times \bar{\Gamma}, \text{ with } \underline{\Gamma} := \mathbb{R}^{d-1} \times \{x_d < 1\} \text{ and } \bar{\Gamma} := \mathbb{R}^{d-1} \times \{x_d > 0\}.$$

Because of Assumption 2.1 we have the following *a priori* energy estimate for the Cauchy problem (7): for all $0 < t \leq T$

$$\begin{aligned} \|u_{1/2}(t, \cdot)\|_{L^2(\mathbb{R}^d)} &\leq C \int_0^t \|\{\chi'_{1/2} A_d u + \chi_{1/2} L(\partial)u\}(s, \cdot)\|_{L^2(\mathbb{R}^d)} ds \\ &\leq C_{1/2} \left(T \sup_{t \in]0, T[} \|u(t, \cdot)\|_{L^2(\Gamma)} + \int_0^t \|\{L(\partial)u\}(s, \cdot)\|_{L^2(\Gamma)} ds \right). \end{aligned} \quad (10)$$

Using Assumptions 2.1, 2.2 and 2.3, Kreiss's theory for half-space hyperbolic boundary problems applies (see [13] and [21]) and we deduce the two *a priori* energy estimates: for all $0 < t \leq T$,

$$\begin{aligned} \|u_0(t, \cdot)\|_{L^2(\bar{\Gamma})} + \|u|_{x_d=0}\|_{L^2(\partial\Omega_{0,t})} \\ \leq C_0 \left(T \sup_{t \in]0, T[} \|u(t, \cdot)\|_{L^2(\Gamma)} + \int_0^t \|\{L(\partial)u\}(s, \cdot)\|_{L^2(\Gamma)} ds + \|B_0 u|_{x_d=0}\|_{L^2(\partial\Omega_{0,T})} \right), \end{aligned} \quad (11)$$

and

$$\begin{aligned} \|u_1(t, \cdot)\|_{L^2(\underline{\Gamma})} + \|u|_{x_d=1}\|_{L^2(\partial\Omega_{1,t})} \\ \leq C_1 \left(T \sup_{t \in]0, T[} \|u(t, \cdot)\|_{L^2(\Gamma)} + \int_0^t \|\{L(\partial)u\}(s, \cdot)\|_{L^2(\Gamma)} ds + \|B_1 u|_{x_d=1}\|_{L^2(\partial\Omega_{1,T})} \right). \end{aligned} \quad (12)$$

Estimate (6) follows from the triangle inequality where we restrict the maximal time $t \leq \underline{T}$ with for instance $\underline{T} := \frac{1}{6 \max(C_0, C_{1/2}, C_1)}$, in order to absorb the three terms $\sup_{t \in]0, T[} \|u(t, \cdot)\|_{L^2(\Gamma)}$ appearing in the right-hand side of (10), (12) and (11) by the one in the left-hand side of (6). □

This ends up the proof of Theorem 2.2 when $s = 0$. The next paragraph then uses such a well-posedness result to deduce the result for $s > 0$.

Before that, let us just mention how we can also use the mentioned L^2 weighted spaces introduced at the beginning of this paragraph to derive *a priori* energy estimates. Because the above localization result only requires the well-posedness of the three problems (7), (8) and (9), then it can be extended to the most generic class of systems which are constantly hyperbolic and satisfy the so-called uniform Kreiss-Lopatinskii condition of [13]. In such a framework, using the results of [13], or the complete expositions of [[5]-Chapter 7] and [[4]-Chapters 4 and 5], or [21] to incorporate non homogeneous sources, estimate (6) becomes: there exists $\gamma_0 > 0$ such that if the sources $f \in L^2_\gamma(\Omega_T)$, $(g_0, g_1) \in L^2_\gamma(\partial\Omega_{0,T}) \times L^2_\gamma(\partial\Omega_{1,T})$ for $\gamma \geq \gamma_0$, then

$$\begin{aligned} e^{-2\gamma T} \|u(T, \cdot)\|_{L^2(\Gamma)}^2 + \gamma \|u\|_{L^2_\gamma(\Omega_T)}^2 + \|u|_{x_d=0}\|_{L^2_\gamma(\partial\Omega_{0,T})}^2 + \|u|_{x_d=1}\|_{L^2_\gamma(\partial\Omega_{1,T})}^2 \\ \leq C \left(\frac{1}{\gamma} \|L(\partial)u\|_{L^2_\gamma(\Omega_T)}^2 + \|B_0 u|_{x_d=0}\|_{L^2_\gamma(\partial\Omega_{0,T})}^2 + \|B_1 u|_{x_d=1}\|_{L^2_\gamma(\partial\Omega_{1,T})}^2 \right), \end{aligned} \quad (13)$$

where we stress that the above threshold γ_0 is **a priori** strictly positive (we refer to the introduction of [2] for more details).

Such an estimate can then be extended to higher order Sobolev spaces and can be used in place of the energy estimate of Theorem 2.2 to deal with non linear problems. If one does so then to make the iterations converge he/she will have to ensure that $\frac{e^{2\gamma T}}{\gamma}$ is small (see Paragraph 2.2). It can be ensured by assuming first that $T \sim \frac{1}{\gamma}$ to kill the exponential factor and then by considering $\gamma \geq \gamma_0$ large enough. Consequently we also obtain a finite time of resolution.

In [2], by using an extra assumption asking the invertibility of some operator reading $I - \mathbf{T}$ we characterized the boundary conditions for which the estimate (13) holds for $\gamma_0 = 0$. Consequently in the above discussion at the end of the day one only has to impose that $\gamma > 0$ is large enough and *a priori* the final time of resolution is longer.

2.1.2 Persistence of regularity

Once again we will only show the energy estimate. Because tangential derivatives commute with the initial and boundary conditions of (4) we can apply Proposition 2.1 which gives the estimate for the $H_{x'}^s$ -norms of u in terms of the analogous norms of the sources. Consequently in order to close the estimate for the first term in $\|u(t)\|_s$ it is sufficient to estimate the norms $H_{x_d}^j(H_{x'}^{s-j})$ of u . In order to do so, we consider first the second term appearing in $\|u(t)\|_s$, that is to say the term involving the first time derivative. Classically we use the interior equation to isolate the time derivative $\partial_t u$ and write $\partial_t u = f - \sum_{j=1}^d A_j \partial_j u$ from which we evaluate the value of the trace $(\partial_t u)|_{t=0}$. Because of the flatness of the source terms it vanishes.

Let $\beta \in \mathbb{N}^{d-1}$ be such that $|\beta| \leq s - 1$, we have that $\partial_t \partial_{x'}^\beta u$ solves the system

$$\begin{cases} L(\partial) \partial_t \partial_{x'}^\beta u = \partial_t \partial_{x'}^\beta L(\partial) u & \text{in } \Omega_T, \\ B_0(\partial_t \partial_{x'}^\beta u)|_{x_d=0} = \partial_t \partial_{x'}^\beta B_0 u|_{x_d=0} & \text{on } \partial\Omega_{0,T}, \\ B_1(\partial_t \partial_{x'}^\beta u)|_{x_d=1} = \partial_t \partial_{x'}^\beta B_1 u|_{x_d=1} & \text{on } \partial\Omega_{1,T}, \\ (\partial_t \partial_{x'}^\beta u)|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (14)$$

so that Proposition 2.1 applies. It gives the control of the first part of the second term in $\|u(t)\|$ namely the control of $\partial_t u$ in $H_{x'}^{s-1}$. We can use such a control to close the estimate for the first term of $\|u(t)\|_s$. Using Assumption 2.2, we can now isolate the normal derivative. Indeed we can write

$$\partial_{x'}^\beta \partial_d u = A_d^{-1} \left[\underbrace{\partial_{x'}^\beta L(\partial) u - \partial_{x'}^\beta \partial_t u - \sum_{j=1}^{d-1} A_j \partial_j \partial_{x'}^\beta u}_{\in L^2(\Gamma)} \right].$$

We can conclude iteratively. Indeed let $\beta \in \mathbb{N}^{d-1}$ be such that $|\beta| \leq s - 2$, then we have

$$\begin{aligned} \partial_{x'}^\beta \partial_d^2 u &= A_d^{-1} \left[\underbrace{\partial_{x'}^\beta \partial_d L(\partial) u - \sum_{j=1}^{d-1} A_j \partial_j \partial_{x'}^\beta \partial_d u}_{\in L^2(\Gamma)} \right] - A_d^{-1} \partial_t \partial_d u \\ &= A_d^{-1} \left[\partial_d L(\partial) u - \sum_{j=1}^{d-1} A_j \partial_j \partial^\beta \partial_d u \right] - (A_d)^{-2} \left[\underbrace{\partial_t \partial_{x'}^\beta L(\partial) u - \sum_{j=1}^d A_j \partial_j \partial_{x'}^\beta \partial_t u}_{\in L^2(\Gamma)} \right] + A_d^{-2} \partial_{tt}^2 \partial_{x'}^\beta u, \end{aligned}$$

where to estimate the last term in the right-hand side we use (14). A simple induction then ends the proof of estimate (6). This completes the discussion of the main lines of the proof of Theorem 2.2.

2.2 Well-posedness of the non linear problem

The proof of the well-posedness of the non linear problem (1) follows from the one of the associated linear problem (4) by a straightforward contraction argument. Such a method being classical, we will here only insist on the main ideas in the proof.

Let us first recall the following non linear estimates (see for instance [[19]-Paragraph 2.5.1])

Proposition 2.2 *Consider a \mathcal{C}^∞ function \mathbf{F} satisfying $\mathbf{F}(0) = 0$. Then for all $s > \frac{d}{2}$, there is a non decreasing function $C(\cdot)$ on \mathbb{R}_+ such that for all $T > 0$ and for all $u \in \mathbf{W}_T^s$, $\mathbf{F}(u) \in \mathbf{W}_T^s$. In addition for all $0 \leq t \leq T$:*

$$\|u(t)\|_s \leq R \implies \|\{\mathbf{F}(u)\}(t)\|_s \leq C(R).$$

Moreover we also have the lipschitzian estimate, for all $u, v \in \mathbf{W}_T^s$ satisfying $\|u(t)\|_s, \|v(t)\|_s \leq R$ then

$$\|\{F(u) - F(v)\}(t)\|_s \leq C(R)\|u - v\|_s.$$

We define the sequence $(u^\nu)_{\nu \in \mathbb{N}}$ as follows: u^0 as the solution to

$$\begin{cases} L(\partial)u^0 = f & \text{in } \Omega_T, \\ B_0 u^0|_{x_d=0} = g_0 & \text{on } \partial\Omega_{0,T}, \\ B_1 u^0|_{x_d=1} = g_1 & \text{on } \partial\Omega_{1,T}, \\ u^0|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (15)$$

and for all $\nu \geq 1$, u^ν as the solution to

$$\begin{cases} L(\partial)u^\nu = f + \mathbf{F}(u^{\nu-1}) & \text{in } \Omega_T, \\ B_0 u^\nu|_{x_d=0} = g_0 & \text{on } \partial\Omega_{0,T}, \\ B_1 u^\nu|_{x_d=1} = g_1 & \text{on } \partial\Omega_{1,T}, \\ u^\nu|_{t=0} = 0 & \text{on } \Gamma. \end{cases} \quad (16)$$

Let us stress here that such a sequence is well-defined at least for short times. Indeed, Theorem 2.2 applies to justify the existence of u^0 on the time interval $[0, \underline{T}]$. Then because $\partial_t^j \mathbf{F}(u)$ reads under the form

$$\partial_t^j \mathbf{F}(u)|_{t=0} = \sum_{k=1}^j \sum_{j=j_1+\dots+j_k} C_{j_1, \dots, j_k}^k d_u^k \mathbf{F}(u_0)(\partial_t^{j_1} u, \dots, \partial_t^{j_k} u),$$

where the C_{\dots} stand for some numerical constants, then the flatness of $u^{\nu-1}$ (see (14)) spreads to the interior source term of (16). The regularity of the source being given by Proposition 2.2. As a consequence, Theorem 2.2 applies and shows that u^ν is well-defined for all $\nu \in \mathbb{N}$.

The next point aims to obtain uniform bounds on the sequence $(u^\nu)_{\nu \in \mathbb{N}}$. More precisely

Lemma 2.1 *We can choose $R > 0$ large enough and $0 < T_0 \leq \underline{T}$ small enough such that for all $\nu \in \mathbb{N}$ and for all $0 \leq t \leq T_0$, we have $\|u^\nu(t)\|_s \leq R$*

Proof : We proceed by induction. Firstly for $\nu = 0$, the energy estimate of Theorem 2.2 gives that for all $0 < t \leq \underline{T}$

$$\|u^0(t)\|_s \leq C \underbrace{\left(\int_0^T \|f(s)\|_{L^2(\Gamma)} ds + \|g_0\|_{H^s(\partial\Omega_{0,T})}^2 + \|g_1\|_{H^s(\partial\Omega_{1,T})}^2 \right)}_{:= \mathbf{C}_0}.$$

So that, we have the desired property for the initial step as soon as we choose $R \geq \mathbf{C}_0$.

Then we assume that at some step $\nu \in \mathbb{N}$, we have $\|u^\nu(t)\|_s \leq R$, for all $t \leq \underline{T}$. The energy estimate of Theorem 2.2 gives

$$\|u^{\nu+1}(t)\|_s \leq \mathbf{C}_0 + C \int_0^T \|\{\mathbf{F}(u^\nu)\}(r)\|_s dr \leq \mathbf{C}_0 + TC(R),$$

where we used the first property of Proposition 2.2. We fix R such that $R \geq \mathbf{C}_0 + 1$ and we then choose T small enough in order to ensure that $TC(R) \leq 1$. So that for $0 < t \leq T_0 = \min(\underline{T}, \frac{1}{C(R)})$ we have the desired property.

□

Once we have shown uniform bounds on the sequence $(u^\nu)_{\nu \in \mathbb{N}}$ then the convergence of this sequence follows easily. Indeed consider the sequence $(v^\nu)_{\nu \in \mathbb{N}}$ defined by $v^\nu := u^{\nu+1} - u^\nu$. Such a sequence satisfies

$$\begin{cases} L(\partial)v^\nu = \mathbf{F}(u^\nu) - \mathbf{F}(u^{\nu-1}) & \text{in } \Omega_T, \\ B_0 v^\nu|_{x_d=0} = 0 & \text{on } \partial\Omega_{0,T}, \\ B_1 v^\nu|_{x_d=1} = 0 & \text{on } \partial\Omega_{1,T}, \\ v^\nu|_{t=0} = 0 & \text{on } \Gamma, \end{cases}$$

so that the energy estimate of Theorem 2.2 gives the bound (we use the second point of Proposition 2.2): for all $t \leq T_0$

$$\|v^\nu(t)\|_s \leq C(R) \int_0^t \|v^{\nu-1}(r)\|_s dr.$$

We have $\|v^0(t)\|_s \leq 2R$ so that iterative integrations lead to the estimate

$$\|v^\nu(t)\|_s \leq K^{\nu+1} \frac{t^{\nu-1}}{(\nu-1)!} \rightarrow_{\nu \rightarrow \infty} 0,$$

for some $K > 0$. As a consequence $(u^\nu)_{\nu \in \mathbb{N}}$ converges in $\mathbf{W}_{T_0}^s$, its limit being a solution to (1). The proof of the uniqueness of the solution follows the same lines and is omitted here. It ends up the proof of Theorem 2.1.

□

3 Geometric optics expansions

This section is devoted to the asymptotic study of the following highly oscillating problem associated to (1). For $0 < \varepsilon \ll 1$ and $T > 0$, we consider

$$\begin{cases} L(\partial)u^\varepsilon = \mathbf{F}(u^\varepsilon) & \text{in } \Omega_T, \\ B_0 u^\varepsilon|_{x_d=0} = g^\varepsilon(t, x') & \text{on } \partial\Omega_{0,T}, \\ B_1 u^\varepsilon|_{x_d=1} = 0 & \text{on } \partial\Omega_{1,T}, \\ u^\varepsilon|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (17)$$

where the given source g^ε reads

$$g^\varepsilon(t, x') := G\left(t, x'; \underbrace{\frac{\zeta \cdot (t, x')}{\varepsilon}}_{:=\psi(t, x')=\theta_0}\right),$$

for a given smooth function G which is periodic with respect to the last variable $\theta_0 \in \mathbb{R}$, and where $\zeta := (\mathcal{I}, \eta) \in \mathbb{R}^{1+d-1}$ is a given boundary frequency.

Of course Theorem 2.1 applies to system (17) and shows that this problem is locally well-posed in time. However *a priori* the final time of resolution T given by Theorem 2.1 depends on ε . An interesting question is consequently the one of the existence of a solution to (17) on a time interval independent of ε to make sure that such an interval does not shrink to $\{0\}$ when $\varepsilon \downarrow 0$. Such a question will be answered in Theorem 3.2 by constructing an exact solution near an approximate one.

In order to do so, in the following we will discuss and describe the solution u^ε to (17) by using geometric optics expansions. In order to do this, the current section is divided as follow: Paragraph 3.1 recalls some materials which are classically used in the construction of geometric optics expansions and states the main results, namely Theorems 3.1 and 3.2. Then the proof of Theorem 3.1 (existence of the expansion) occupies Paragraphs 3.2, 3.3 and 3.4, while the one of Theorem 3.2 (justification of the expansion) occupies Paragraph

3.5.

Here our aim was to recover the same kind of results as the ones obtained in [24]. We will however simplify the framework of study by considering only hyperbolic frequencies (see Definition 3.1) whereas Williams achieved the construction in the difficult and technical case of *glancing* frequencies (see Lemma 3.1). Such a simplification will be explained and motivated in the core of the text. It is fair to say that our analysis and exposition closely follows the one of [24].

3.1 Preliminaries and main results

3.1.1 Mandatory materials

This paragraph regroups the several main objects and assumptions needed in order to perform geometric optics expansion for the strip problem (17). We tried to be as brief as possible and we refer to [3] for a longer exposition of the phase generation process in the strip geometry.

Let us first describe the expected phases appearing in the geometric optics expansion. In order to do so, it is convenient to have a quick spectral analysis of the hyperbolic operator $L(\partial)$. We perform a Laplace transform with respect to time $t \rightsquigarrow \sigma := \gamma + i\tau$, with $\gamma > 0$ and $\tau \in \mathbb{R}$ and a Fourier transform with respect to $x' \rightsquigarrow \eta \in \mathbb{R}^{d-1}$, in the evolution equation $L(\partial)u = 0$. At least at the formal level, it leads us to consider the so-called resolvent strip problem

$$\begin{cases} \frac{d}{dx_d} \widehat{u}(\zeta, x_d) = \mathcal{A}(\zeta) \widehat{u}(\zeta, x_d) & \text{for } x_d \in]0, 1[, \\ B_0 \widehat{u}|_{x_d=0}(\zeta) = \widehat{g}_0, \\ B_1 \widehat{u}|_{x_d=1}(\zeta) = 0, \end{cases} \quad (18)$$

where ζ is now a shorthand notation for (σ, η) and where the resolvent matrix \mathcal{A} is defined by:

$$\mathcal{A}(\zeta) := -A_d^{-1} \left(\sigma I + \sum_{j=1}^{d-1} \eta_j A_j \right).$$

It will be convenient to define the frequency set

$$\Xi := \{ \zeta := (\sigma := \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus \{ \gamma \geq 0 \} \},$$

and its boundary $\Xi_0 := \Xi \cap \{ \gamma = 0 \}$.

The eigenvalues of \mathcal{A} then encode the behaviour of the solution to (18). A well-known result of Hersh [10] establishes that as long as the Laplace frequency σ has non vanishing real part, then the eigenvalues of $\mathcal{A}(\zeta)$ have non vanishing real part. With more details p (resp. $N - p$) of them have negative (resp. positive) real part. In the following we denote by $\mathbf{E}^s(\zeta)$ (resp. $\mathbf{E}^u(\zeta)$) the associated eigenspace. We thus have the decomposition

$$\mathbb{C}^N = \mathbf{E}^s(\zeta) \oplus \mathbf{E}^u(\zeta). \quad (19)$$

When studying geometric optics expansions a precise analysis of the boundary frequencies, namely the ones satisfying $\gamma = 0$ is required (recall the definition of ζ in (17)). Of course, Hersh's result [10] does not hold any more for these boundary frequencies, because in the limit $\gamma \downarrow 0$, a signed quantity will not be signed any more. Consequently the real part of some of the eigenvalues will degenerate to zero. The following lemma, the so-called block structure lemma of [13] (see also [18]), describes how such degeneracies occur.

Lemma 3.1 *If the boundary value problem (17) is strictly hyperbolic and admits a non characteristic boundary (see Assumption 2.2 below), then for all $\underline{\zeta} \in \Xi$, one can find a neighbourhood \mathcal{V} of $\underline{\zeta}$ in Ξ , a positive integer $L \geq 0$, integers $\varrho_1, \dots, \varrho_L \geq 1$ such that $N = \sum_{k=1}^L \varrho_k$ and finally a matrix $T \in \mathbf{GL}_N(\mathbb{C})$ defined and regular on \mathcal{V} such that*

$$\forall \zeta \in \mathcal{V}, T^{-1}(\zeta) \mathcal{A}(\zeta) T(\zeta) = \text{diag}(\mathcal{A}_1(\zeta), \dots, \mathcal{A}_L(\zeta)),$$

where the blocks $\mathcal{A}_j \in \mathbf{M}_{\varrho_j \times \varrho_j}(\mathbb{C})$ satisfy one of the following alternatives:

1. All the elements in the spectrum of \mathcal{A}_j have strictly negative real part.
2. All the elements in the spectrum of \mathcal{A}_j have strictly positive real part.
3. We have $\varrho_j = 1$, $\mathcal{A}_j(\underline{\zeta}) \in i\mathbb{R}$ and $\partial_\gamma \mathcal{A}_j(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$.
4. We have $\varrho_j > 1$ and there exists $\mu_j \in \mathbb{R}$ such that the matrix $\mathcal{A}_j(\underline{\zeta})$ reads

$$\mathcal{A}_j(\underline{\zeta}) = i \begin{bmatrix} \mu_j & 1 & 0 \\ & \ddots & 1 \\ 0 & & \mu_j \end{bmatrix}$$

where the South-West corner coefficient of $\partial_\gamma \mathcal{A}_j(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$.

With such lemma in hand, we can define the four possible kinds of frequencies.

Definition 3.1 Let $\underline{\zeta} \in \Xi_0$ we say that such a boundary frequency is

- elliptic if Lemma 3.1 (for $\underline{\zeta}$) holds with blocks of type 1. and 2. only.
- mixed if Lemma 3.1 holds with blocks of type 1., 2. and at least one block of type 3..
- hyperbolic if Lemma 3.1 holds with blocks of type 3. only.
- glancing if Lemma 3.1 holds with one block of type 4..

Each of the above frequencies have a distinct behaviour in geometric optics expansions. The hyperbolic ones, associated to type 3. blocks in Lemma 3.1 are the most frequently studied. They are linked with the well-known transport phenomena in geometric optics expansions. The elliptic and glancing ones, associated to blocks of type 1., 2. and 4., are considered for instance in [1] (see also [17] and [16]) for elliptic modes or in the seminal works of Williams [23] and [24] for glancing modes. In the geometric optic expansion, such modes are not associated to transport phenomena, but with boundary layers. That is to say that they create terms reading under the form $U(t, x; \frac{x_d}{\varepsilon^\alpha})$ (with $\alpha = 1$ for elliptic modes and $\alpha = \frac{1}{2}$ for glancing ones) appearing to correct the expansion near the boundary in order to force it to satisfy the boundary conditions.

As pointed in [3], the self-interaction phenomenon induced by the geometry of the strip is linked to hyperbolic modes only. It is due to the fact that boundary layers have fast decay with respect to the normal variable x_d . So that, when an information on $\partial\Omega_{0,T}$ is propagated *via* such layers it hits the other side of the boundary $\partial\Omega_{1,T}$ with amplitude $O(\varepsilon^\infty)$, so that it is negligible from a geometric optics point of view. As a consequence, such boundary layers are not reflected back and can not lead to self-interaction.

Because in this contribution we are particularly interested in an analysis where non linear self-interaction and geometric self-interaction coexist, we will restrict our study to the case of a hyperbolic boundary frequency. In such a setting, the decomposition (19) is still valid in the limit $\gamma \downarrow 0$ and it is well-known that the spaces $\mathbf{E}^s(\underline{\zeta})$ and $\mathbf{E}^u(\underline{\zeta})$ can be made more precise. Following for instance [6], we have

$$\mathbf{E}^s(\underline{\zeta}) := \oplus_{k \in \mathcal{S}} \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \xi_k) \text{ and } \mathbf{E}^u(\underline{\zeta}) := \oplus_{\ell \in \mathcal{O}} \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \xi_\ell), \quad (20)$$

where \mathcal{L} denotes the symbol of $L(\partial)$, that is to say for $(\xi_0, \xi^\dagger) \in \mathbb{R}^{1+d}$, $\mathcal{L}(\xi_0, \xi^\dagger) := \xi_0 I + \sum_{j=1}^d \xi_j^\dagger A_j$, and where the set \mathcal{S} (resp. \mathcal{O}) is the set of incoming (resp. outgoing) indices. These sets are defined in terms of the group velocities as follows.

Let us first remark that $i\xi^m$ is a purely imaginary eigenvalue of $\mathcal{A}(\underline{\zeta})$ if and only if we have the dispersion relation

$$\det(\underline{\tau}I + A(\underline{\eta}, \xi^m)) = 0. \quad (21)$$

where $A(\xi^\dagger) := \sum_{j=1}^d \xi_j^\dagger A_j$ stands for the spatial symbol of $L(\partial)$.

So that if we use the following hyperbolicity assumption

Assumption 3.1 *The operator $L(\partial)$ is strictly hyperbolic in the sense that there exists N analytic functions $\lambda_1 < \dots < \lambda_N$ on $\mathbb{R}^d \setminus \{0\}$ such that*

$$\forall \xi^\dagger \in \mathbb{S}^{d-1}, \det \left(\tau I + A(\xi^\dagger) \right) = \prod_{k=1}^N (\tau + \lambda_k(\xi^\dagger)).$$

Then one can find some index $k_m \in \llbracket 1, N \rrbracket$ such that $\tau + \lambda_{k_m}(\underline{\eta}, \xi^m) = 0$. This motivates the definition:

Definition 3.2 (Group velocities) *We define*

- \mathcal{I} the set of incoming⁷ indices as the set formed by the indices $m \in \llbracket 1, N \rrbracket$ such that $\partial_d \lambda_{k_m}(\underline{\eta}, \xi^m) > 0$.
- \mathcal{O} the set of outgoing indices as the set formed by the indices $m \in \llbracket 1, N \rrbracket$ such that $\partial_d \lambda_{k_m}(\underline{\eta}, \xi^m) < 0$.
- For $j \in \mathcal{I} \cup \mathcal{O}$, we denote $\mathbf{v}_j := \nabla \lambda_{k_j}(\underline{\eta}, \xi^j)$ the so-called group velocity associated to the index j .

This motivates the following definition for the phase functions: for all $j \in \llbracket 1, M \rrbracket$, $\varphi_j := \varphi_j(t, x) := \psi(t, x') + x_d \xi_j$, where the $(\xi_j)_{j \in \llbracket 1, N \rrbracket}$ denote the roots in the ξ -variable of the dispersion relation

$$\det \left(\tau I + \sum_{j=1}^{d-1} \eta_j A_j + \xi A_d \right) = 0.$$

We regroup the several phases in the following notation: for all $j \in \llbracket 1, N \rrbracket$, let θ_j be a placeholder for $\frac{1}{\varepsilon} \varphi_j$ and

$$\theta := (\theta_1, \dots, \theta_N), \quad \varphi = (\varphi_1, \dots, \varphi_N). \quad (22)$$

We assume that the boundary frequency $\underline{\zeta}$ satisfies:

Assumption 3.2 *Let $\underline{\zeta} := (i\tau, \underline{\eta}) \in \Xi_0$ be a given boundary frequency. We assume that $\underline{\zeta}$ satisfies the requirements:*

- $\underline{\zeta}$ is hyperbolic in the sense of Definition 3.1.
- We have the small divisor condition that there exists $C > 0$ and $d \in \mathbb{R}$ such that

$$\forall \alpha \in \mathbb{Z}^N \setminus \{0\}, |\nabla_{(t,x)} \alpha \cdot \varphi| \geq C |\alpha|^d. \quad (23)$$

Let us stress that condition (23) implies in particular that the phases contained in φ are \mathbb{Q} -independent. It will be particularly useful in the derivation of the cascade of equations (see Paragraph 3.2). As a consequence, (23) excludes real resonances of the form $n_a \varphi_a + n_b \varphi_b = n_c \varphi_c$, with $(n_a, n_b, n_c) \in \mathbb{Z}^3$, so that it *a priori* excludes one type of non-linear selfinteraction that we wanted to study.

Let us however point that this is not really an obstruction nor a restriction. Indeed, if a resonance of the form $n_a \varphi_a + n_b \varphi_b = n_c \varphi_c$, with $(n_a, n_b, n_c) \in \mathbb{Z}^3$, occurs then one can use *adapted basis* as in [24] in order to rearrange the things in such a way that (23) holds.

We give some more details about such *adapted basis* as introduced in [[25]-Section 3] and then considered in [24]-Paragraph 8.2].

A real resonance occurs when we have a relation $\sum_{j \in \mathcal{I} \cup \mathcal{O}} n_j \varphi_j = 0$ for some (non trivial) integers $n_j \in \mathbb{Z}$. In such a situation, we will consider a so-called adapted basis for the \mathbb{Q} -span of $\{\varphi_j\}_{j \in \mathcal{I} \cup \mathcal{O}}$. Consider a basis $B(\underline{\zeta})$ of the \mathbb{Q} -span of $\{\varphi_j\}_{j \in \mathcal{I} \cup \mathcal{O}}$, because of the resonances such a basis contains $b \leq M$ elements and we parametrize $B(\underline{\zeta}) := \text{span}\{\psi_k\}_{1 \leq k \leq b}$. We then introduce the shorthand notation $\psi := (\psi_1, \dots, \psi_b)$ and consider

$$\mathcal{D}_j := \{\alpha \in \mathbb{Z}^b \setminus \alpha \cdot \psi \text{ is in the } \mathbb{R} - \text{span of } \varphi_j\}.$$

Let α_j^* be an element with minimal length in \mathcal{D}_j , such that $\alpha_j^* \cdot \psi = \lambda_j \varphi_j$, $\lambda_j > 0$. With such tools in hand we can precise the notion of adapted basis⁸ as a straightforward modification of the adapted basis of [[24]-Definition 8.2].

⁷We choose as a convention incoming and outgoing for the side $\partial\Omega_{0,T}$. Of course, if one denotes by \mathcal{I}_1 (resp. \mathcal{O}_1) the set of incoming (resp. outgoing) indices for the side $\partial\Omega_{0,T}$, then he/she obtains $\mathcal{I}_1 = \mathcal{O}$ and $\mathcal{O}_1 = \mathcal{I}$.

⁸Maybe the study of some example can here be of some help. Assume that we have the resonance relations $\varphi_1 + 2\varphi_2 = \varphi_3$ and $\varphi_2 + 2\varphi_3 = \varphi_4$. We claim that (φ_1, φ_2) is an adapted basis. Indeed, the boundary values are satisfied and moreover in such a setting we have $\alpha_1^* = (1, 0)$, $\alpha_2^* = (0, 1)$, $\alpha_3^* = (1, 2)$ and $\alpha_4^* = (2, 5)$.

Definition 3.3 [Adapted basis] A basis $B(\zeta) = \{\psi_k\}_{1 \leq k \leq b}$ of the \mathbb{Q} -span of $\{\varphi_j\}_{j \in \mathcal{S} \cup \mathcal{O}}$ is said to be adapted if

1. For all $k \in \llbracket 1, b \rrbracket$, $\psi_{k|_{x_d=0}} = m_k \psi$ for some $m_k \in \mathbb{Z}$.
2. For all $k \in \llbracket 1, b \rrbracket$, one can find some $j \in \mathcal{S} \cup \mathcal{O}$, such that $\psi_{k|_{x_d=1}} = m_k(\psi + \xi^j)$ for some $m_k \in \mathbb{Z}$.
3. For all $j \in \mathcal{S} \cup \mathcal{O}$, $\varphi_j = \alpha_j^* \cdot \psi$.

Points 1. and 3 of Definition 3.3 appear in Definition 8.2 of [24]. Point 2. is just a straightforward adaptation made to ensure that the adapted basis has a good behaviour on the second side of the strip $\partial\Omega_{1,T}$.

So that if for a resonant family of $(\varphi_j)_{j \in \mathcal{S} \cup \mathcal{O}}$ we consider the family $(\psi_k)_{1 \leq k \leq b}$ instead of $(\varphi_j)_{j \in \mathcal{S} \cup \mathcal{O}}$ then because the new family is not resonant any more it satisfies the small divisor Assumption 3.2.

Of course the question of the existence of such adapted basis should be addressed. The interesting fact is here that such basis can always be constructed at least for hyperbolic frequencies. We refer the interested reader to [[25]-Proposition 3.7]. Because we modified a little the definition of adapted basis, one has to verify that the constructed basis in [[25]-Proposition 3.7] satisfies the extra condition 2. of Definition 3.3, but it is straightforward.

So that, such basis permit to deal with resonances, but let us point that compared to some more recent works of the literature (see for instance [7]), then such a way to take into account resonances it not very descriptive. Indeed we can find in [7] (see equations (2.63) and (2.64)) some profiles equations which clearly take into account and describe the influence of such resonances. Such equations are non-linear so that they are solved inductively by using iterative schemes (see equations (2.71) and (2.72) of [7]).

As we will see in Paragraph 3.3 the main point in the present construction is to justify that in the analysis of such schemes, one can decouples the geometric self-interaction (which acts in the boundary conditions) from the non-linear one (which acts in the interior equation). So that, even if we do not have a precise description of resonances here because of the use of adapted basis, then we however succeeded in our objective which was to justify that the two self-interaction phenomena do not interact the one with the other.

We have reasons to believe that in the framework of [7] the non linear coupling in the interior and the one at the boundary do not interact the one with the other exactly as it will be described below. So that the combination of the profiles equations of [7] with the present article shall probably give a descriptive description of resonances for hyperbolic strip problems. Our aim being here to determine if some new phenomena come into play, we postpone this precise description for later studies.

We then consider our second small divisor assumption.

Assumption 3.3 Consider a boundary frequency ζ , its associated phases functions φ and let $\mathcal{P}(\alpha)$ be the symbol of the operator of derivation \mathcal{P} appearing in the cascade of equations (27) (see equations (28) and (31)). Then we assume that there exists two numerical constants $C > 0$ and $d \in \mathbb{R}$ such that for all $\alpha \in \mathbb{Z}^N$ satisfying that $\det \mathcal{P}(\alpha) \neq 0$ we have

$$|\det \mathcal{P}(\alpha)| \geq C|\alpha|^d.$$

Such an assumption is classically used in non linear geometric optics expansions in order to be able to construct correctors. With more details the above assumption ensures the existence of some pseudo-inverse operators (see the operator \mathcal{Q} in the following) which are required to express the non polarized part of the correctors (and consequently to construct such higher order terms). By the way let us introduce the following usual projections in geometric optics expansions:

Definition 3.4 For all $j \in \llbracket 1, N \rrbracket$ we consider

- π_j the projection on $\ker \mathcal{L}(d\varphi_j)$ along the range of $\mathcal{L}(d\varphi_j)$.
- $r_j \in \mathbb{C}^N$ a generator of $\ker \mathcal{L}(d\varphi_j)$.

Of course, one can ask when the small divisors Assumptions 3.3 holds. The answer is that for hyperbolic frequencies one can *generically* assume that Assumption 3.3 holds. Our aim is not here to be very precise about what we mean by "generically" and we refer to [[25]-Proposition 4.5] for a precise statement and proof.

To construct geometric optics expansions for boundary value problems, a microlocalized version of the condition ensuring the strong well-posedness, namely the uniform Kreiss-Lopatinskii condition of [13], is mandatory. We will consequently assume in the following that we have

Assumption 3.4 [*Localized Kreiss-Lopatinskii condition*] *Let $\underline{\zeta}$ be a boundary frequency. We assume that the boundary condition B_0 (resp. B_1) satisfies the localized Kreiss-Lopatinskii condition for the side $\partial\Omega_{0,T}$ (resp. $\partial\Omega_{1,T}$) at $\underline{\zeta}$ meaning that*

$$\ker B_0 \cap \mathbf{E}^s(\underline{\zeta}) = \{0\} \text{ (resp. } \ker B_1 \cap \mathbf{E}^u(\underline{\zeta}) = \{0\}\text{)}.$$

In particular, the matrix B_0 (resp. B_1) is invertible when restricted to $\mathbf{E}^s(\underline{\zeta})$ (resp. $\mathbf{E}^u(\underline{\zeta})$). We denote such an inverse by ϕ_0 (resp. ϕ_1).

At last, the following definition gives a rigorous framework for the set of profiles and the set of remainders.

Definition 3.5 (Profiles and remainders) *To define precisely the set of profiles we first introduce*

$$\tilde{\mathbf{P}}_T := \left\{ \tilde{u} := \tilde{u}(t, x; \tilde{\theta}) \in \mathcal{C}^\infty(\Omega_T \times \mathbb{T}^N), \tilde{u} := \sum_{\alpha \in \mathbb{Z}^N} u_\alpha(t, x) e^{i\alpha \cdot \tilde{\theta}} \text{ s.t. } \forall \alpha \in \mathbb{N}^N, \partial_{t, x, \tilde{\theta}} \tilde{u} \text{ is bounded} \right\}.$$

- *The set of profiles \mathbf{P}_T is the set of functions in $\tilde{\mathbf{P}}_T$ which have been evaluated in the suitable phases described above. More precisely*

$$\mathbf{P}_T := \left\{ u = u(t, x; \theta) \text{ s.t. } u = \tilde{u}_{|\tilde{\theta}=\theta} \text{ with } \tilde{u} \in \tilde{\mathbf{P}}_T \text{ and } \theta \text{ as in (22)} \right\}.$$

- *For $s \in \mathbb{N}$ and $T, \varrho > 0$. We define:*

$$\begin{aligned} \mathbf{B}_{\varrho, T}^s &:= \{u^\varepsilon := u^\varepsilon(t, x) \in H^s(\Omega_T) \setminus \forall \alpha \in \mathbb{N}^{1+d} \text{ with } |\alpha| \leq s, \|\partial^\alpha u^\varepsilon\|_{L^2(\Omega_T)} \leq \varrho \varepsilon^{-|\alpha|}\}, \\ \mathbf{D}_{\varrho, T}^s &:= \{u^\varepsilon := u^\varepsilon(t, x) \in W^{s, \infty}(\Omega_T) \setminus \forall \alpha \in \mathbb{N}^{1+d} \text{ with } |\alpha| \leq s, \|\partial^\alpha u^\varepsilon\|_{L^\infty(\Omega_T)} \leq \varrho \varepsilon^{-|\alpha|}\}, \end{aligned}$$

and we define similarly the sets $\mathbf{B}_{\varrho, T, 0}^s$ and $\mathbf{D}_{\varrho, T, 0}^s$ (resp $\mathbf{B}_{\varrho, T, 1}^s$ and $\mathbf{D}_{\varrho, T, 1}^s$) as the sets obtained when one uses $\partial\Omega_{0,T}$ (resp. $\partial\Omega_{1,T}$) instead of Ω_T in the above definitions. The set of remainders in the interior (resp. at the boundary) $\mathbf{R}_{\varrho, T}^s$ (resp. $\mathbf{r}_{\varrho, T, 0}^s$ and $\mathbf{r}_{\varrho, T, 1}^s$) is thus defined by $\mathbf{R}_{\varrho, T}^s := \mathbf{B}_{\varrho, T}^s \cap \mathbf{D}_{\varrho, T}^s$ (resp. $\mathbf{r}_{\varrho, T, 0}^s := \mathbf{B}_{\varrho, T, 0}^s \cap \mathbf{D}_{\varrho, T, 0}^s$ and $\mathbf{r}_{\varrho, T, 1}^s := \mathbf{B}_{\varrho, T, 1}^s \cap \mathbf{D}_{\varrho, T, 1}^s$).

- *For all the above functional spaces, here symbolized by X , we will use the notation X_\flat for the subspace of X formed of functions which are flat at the time origin.*

3.1.2 Main results.

The main results of the article about geometric optics expansions are Theorems 3.1 and 3.2. The first one establishes the existence of the expansion. The second one, namely Theorem 3.2 is a Guès's type theorem (see [9]) showing the existence of a solution to (17) lying near an approximated one. As a consequence, combining the two above theorems, we end up with Corollary 3.1 which in particular justifies that the constructed geometric optics expansion is effectively an approximate solution to (17).

Theorem 3.1 *Under Assumptions 2.1, 2.2 and 2.3 to ensure the well-posedness and Assumptions 3.1, 3.2, 3.3 and 3.4 to construct the expansion. Consider an entire non linearity \mathbf{F} satisfying $\mathbf{F}(0) = 0$ and a boundary source g^ε in (17) satisfying that G is regular, flat at the time origin, with bounded derivatives. Choose a couple of parameters (M, m_1) in such a way that $M - 1 \geq m_1 \geq \frac{d+1}{2}$. Then there exists a time*

$0 < T_0 \leq T$, profiles $u_n \in \mathbf{P}_{T_0, b}$, remainders $R^\varepsilon \in \mathbf{R}_{\varrho, T_0, b}^{m_1}$ and $(r_0^\varepsilon, r_1^\varepsilon) \in \mathbf{r}_{\varrho, T_0, 0, b}^{m_1} \times \mathbf{r}_{\varrho, T_0, 1, b}^{m_1}$ such that the ansatz

$$\tilde{u}^\varepsilon(t, x) := \sum_{n=0}^N \varepsilon^n u_n\left(t, x; \frac{\varphi(t, x)}{\varepsilon}\right) \quad (24)$$

satisfies

$$\begin{cases} L(\partial)\tilde{u}^\varepsilon = \mathbf{F}(\tilde{u}^\varepsilon) + \varepsilon^{M-1}R^\varepsilon & \text{in } \Omega_T, \\ B_0\tilde{u}^\varepsilon = g^\varepsilon + \varepsilon^{M-1}r_0^\varepsilon & \text{on } \partial\Omega_{0, T}, \\ B_1\tilde{u}^\varepsilon = \varepsilon^{M-1}r_1^\varepsilon & \text{on } \partial\Omega_{1, T}, \\ \tilde{u}^\varepsilon|_{t=0} = 0 & \text{on } \Gamma. \end{cases} \quad (25)$$

Theorem 3.2 (Exact solutions near approximate ones) Under Assumptions 2.1, 2.2 and 2.3, choose two parameters (M_1, m_1) in such a way that $M_1 \geq m_1 > \frac{d+1}{2}$ and let be given an approximate solution \tilde{u}^ε to (1) in the sense that $\tilde{u}^\varepsilon \in \mathbf{D}_{\varrho, T, b}^{m_1}$, for some $\varrho, T > 0$ and that we have

$$\begin{cases} L(\partial)\tilde{u}^\varepsilon = \mathbf{F}(\tilde{u}^\varepsilon) + \varepsilon^{M_1}R^\varepsilon & \text{in } \Omega_T, \\ B_0\tilde{u}^\varepsilon = g^\varepsilon + \varepsilon^{M-1}r_0^\varepsilon & \text{on } \partial\Omega_{0, T}, \\ B_1\tilde{u}^\varepsilon = \varepsilon^{M-1}r_1^\varepsilon & \text{on } \partial\Omega_{1, T}, \\ \tilde{u}^\varepsilon|_{t=0} = 0 & \text{on } \Gamma, \end{cases}$$

for some terms $R^\varepsilon \in \mathbf{B}_{\varrho, T, b}^{m_1}$, $(r_0^\varepsilon, r_1^\varepsilon) \in \mathbf{B}_{\varrho, T, 0, b}^{m_1} \times \mathbf{B}_{\varrho, T, 1, b}^{m_1}$. Then there exists $\varepsilon_0 > 0$ and $\varrho_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the problem (17) has a unique solution $u^\varepsilon = \tilde{u}^\varepsilon + \varepsilon^{M_1}\mathbf{B}_{\varrho_0, T, b}^{m_1}$.

Combining Theorems 3.1 and 3.2 we then end up with

Corollary 3.1 Under Assumptions of Theorems 3.1, let the several parameters $T_0, (M, m_1)$ be as above. We consider the profiles u_n given by Theorem 3.1, then there exists $\varepsilon_0, \varrho_0 > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, the problem (17) admits a unique solution $u^\varepsilon \in H^{m_1}(\Omega_{T_0})$. This solution is given by $u^\varepsilon = \tilde{u}^\varepsilon + \varepsilon^{M-1}\mathbf{B}_{\varrho_0, T_0, b}^{m_1}$ where \tilde{u}^ε is given by (24).

3.2 The ansatz and the cascade of equations

We recall that θ is a placeholder for $\frac{\varphi}{\varepsilon}$, θ_0 is one for $\frac{1}{\varepsilon}\psi(t, x')$. For $M \in \mathbb{N}$, we define as an ansatz

$$\tilde{u}^\varepsilon(t, x) := \sum_{n=0}^M \varepsilon^n u_n\left(t, x; \frac{\varphi(t, x)}{\varepsilon}\right). \quad (26)$$

Injecting the ansatz (26) in the interior equation of (17), Taylor expanding the non-linearity around u_0 and regrouping the different terms according to their powers of ε gives a cascade of equations in the interior. With more details, we obtain the cascade:

$$\begin{cases} \mathcal{P}u_0 = 0 & \text{for } \varepsilon^{-1}, \\ \mathcal{P}u_1 + L(\partial)u_0 = f(u_0) & \text{for } \varepsilon^0, \\ \mathcal{P}u_{n+1} + L(\partial)u_n = f'(u_0)u_n + \mathcal{F}_n(u_0, \dots, u_{n-1}) & \text{for } \varepsilon^n, \forall n \geq 1 \end{cases} \quad (27)$$

where the operator \mathcal{P} is defined by

$$\mathcal{P} := \mathcal{P}(\partial_\theta) := \sum_{j \in \mathcal{I} \cup \mathcal{O}} \mathcal{L}(d\varphi_j)\partial_{\theta_j}. \quad (28)$$

and where the term \mathcal{F}_n depends on the derivatives of \mathbf{F} . Using for instance [[19]-equation (2.5.5)], such terms can be made precise. However such explicit values will be of little interest in the following.

Injecting the *ansatz* in the boundary conditions of (17) gives the boundary cascade of equations

$$\forall n \geq 0, \begin{cases} B_0 u_{n|_{x_d=0}}(t, x'; \theta_0, \dots, \theta_0) = \delta_{n,0} G(t, x'; \theta_0) & \text{on } \partial\Omega_{0,T}, \\ B_1 u_{n|_{x_d=0}}(t, x'; \theta_0 + \xi_1, \dots, \theta_0 + \xi_N) = 0 & \text{on } \partial\Omega_{1,T} \end{cases} \quad (29)$$

where $\delta_{\cdot, \cdot}$ stands for Kronecker's symbol.

At last, injecting the *ansatz* in the initial condition gives:

$$\forall n \geq 0, u_{n|_{t=0}} = 0 \text{ on } \Gamma. \quad (30)$$

The two next paragraphs describe the construction of solutions to the cascades (27), (29) and (30). We first consider the leading order term in Paragraph 3.3, while Paragraph 3.4 is devoted to higher order terms.

3.3 The leading order term

Let us use the classical notation: for a periodic function f with respect to some variable θ we write $f(\theta) := \underline{f} + f^*(\theta)$ where \underline{f} stands for the mean of f and f^* denotes the oscillating part.

3.3.1 Analysis of the profiles equations

The first equation of (27) implies that the leading order amplitude u_0 lies in $\ker \mathcal{P}$. So that we have to consider \mathbf{E} the projection onto $\ker \mathcal{P}$. To precise this operator, let us remark that for a profile $u \in \mathbf{P}_T$, we have

$$\mathcal{P}u = \sum_{j=1}^N \mathcal{L}(d\varphi_j) \partial_{\theta_j} \left(\sum_{\alpha \in \mathbb{Z}^N} u_{\alpha}(t, x) e^{i\alpha \cdot \theta} \right) = i \sum_{\alpha \in \mathbb{Z}^N} \underbrace{\sum_{j=1}^N \alpha_j \mathcal{L}(d\varphi_j)}_{:= \mathcal{P}(\alpha)} u_{\alpha}(t, x) e^{i\alpha \cdot \theta}. \quad (31)$$

So that a non trivial amplitude will contribute in the *ansatz* if and only it lies in a trivial component of $\ker \mathcal{P}(\alpha)$. This motivates the following definition;

Definition 3.6 [*Characteristic frequencies*] Let $\alpha \in \mathbb{Z}^M$, we say that α is a characteristic frequency and we denote $\alpha \in \mathcal{C}$ if we have $\det \mathcal{P}(\alpha) = 0$. With the material of Paragraph 3.1.1 in hand, we can describe $\mathcal{C} := \cup_{j=1}^N \mathcal{C}_j$, where

$$\mathcal{C}_j := \{\alpha \in \mathbb{Z}^N \setminus \alpha \cdot d\varphi \in \text{span}_{\mathbb{R}}\{d\varphi_j\}\}.$$

Because of the small divisor Assumption 3.2, the several sheets of \mathcal{C} do not intersect (because we do not have resonance), except for $\alpha = 0$, so that

$$\mathcal{C}_j := \{(0, \dots, 0, \alpha_j, 0, \dots, 0) \in \mathbb{Z}^N \text{ with } \alpha_j \in \mathbb{Z}\}.$$

Then we define properly the projection $\mathbf{E} : \mathbf{P}_T \rightarrow \mathbf{P}_T$ on $\ker \mathcal{P}$. In order to do so, we first extend the definitions of the projections π_j on $\ker \mathcal{L}(d\varphi_j)$ to the whole set of frequencies, by setting: for $\alpha \in \mathbb{Z}^N$

$$\begin{cases} \pi_{\alpha} = 0 & \text{if } \alpha \notin \mathcal{C}, \\ \pi_0 = I, \\ \pi_{\alpha} = \pi_j & \text{if } \alpha \in \mathcal{C}_j \setminus \{0\}. \end{cases} \quad (32)$$

Because of the remark of Definition 3.6, we can write $\mathbf{E} := \mathbf{E}_0 + \sum_{j=1}^N \mathbf{E}_j$, where the projections \mathbf{E}_j do not interact the one with the others and where the operators \mathbf{E}_j have the following action on monomials:

$$\begin{cases} \mathbf{E}_0(u_{\alpha} e^{i\alpha \cdot \theta}) = u_{\alpha} \text{ if } \alpha = 0 \text{ and } \mathbf{E}_0(u_{\alpha} e^{i\alpha \cdot \theta}) = 0 \text{ otherwise,} \\ \mathbf{E}_j(u_{\alpha} e^{i\alpha \cdot \theta}) = (\pi_{\alpha} u_{\alpha}) e^{i\alpha \cdot \theta} \text{ if } \alpha \in \mathcal{C}_j \setminus \{0\} \text{ and } \mathbf{E}_j(u_{\alpha} e^{i\alpha \cdot \theta}) = 0 \text{ otherwise.} \end{cases}$$

As already mentioned with Assumption 3.3 in hand, we can construct (we refer for instance to [[22]-Paragraph 7.4] an operator $\mathcal{Q} : \mathbf{P}_T \rightarrow \mathbf{P}_T$ which is a pseudo-inverse of \mathcal{P} in the sense that

$$\begin{cases} \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = I - \mathbf{E}, \\ \text{Im } \mathbf{E} = \ker \mathcal{P} = \ker \mathcal{Q}, \\ \ker \mathbf{E} = \text{Im } \mathcal{P} = \text{Im } \mathcal{Q}. \end{cases} \quad (33)$$

Applying \mathcal{Q} to the first equation of the cascade (27) gives $(I - \mathbf{E})u_0 = 0$, so that $\mathbf{E}u_0 = u_0$ and we recover the usual polarization property of the leading order amplitude.

Because of the definition of \mathbf{E} , we can write

$$\mathbf{E}u_0(t, x; \theta) := \underline{u}_0(t, x) + \sum_{j=1}^M \sigma_{0,j}(t, x; \theta_j) r_j, \quad (34)$$

where the $\sigma_{0,j}$ are some (unknown) real valued functions and where we recall, see Definition 3.4, that $\text{span}\{r_j\} := \ker \mathcal{L}(d\varphi_j)$. So that, in order to construct the leading order term it is sufficient to determine its mean \underline{u}_0 and the amplitudes $\sigma_{0,j}$.

To obtain evolution equations for such quantities, we consider the second equation in the interior cascade (27). Applying \mathbf{E} to cancel the first term in the left-hand side, we obtain $\mathbf{E}L(\partial)\mathbf{E}u_0 = \mathbf{E}\mathbf{F}(u_0)$. Using the decomposition of $\mathbf{E} = \mathbf{E}_0 + \sum_{j=1}^N \mathbf{E}_j$, the expression (34), and Lax lemma⁹ [14], we end up with the (coupled via the non linearity) system of equations in the interior for the leading order term

$$\begin{cases} L(\partial)\underline{u}_0 = \mathbf{F}(\underline{u}_0) & \text{in } \Omega_T, \\ \forall j \in \llbracket 1, N \rrbracket, (\partial_t + \mathbf{v}_j \cdot \nabla_x)\sigma_{0,j} r_j = \mathbf{E}_j \mathbf{F}(\underline{u}_0) & \text{in } \Omega_T \times \mathbb{T}, \end{cases} \quad (35)$$

and we have to precise the required boundary conditions. Let us first recall here that depending on the kind of \mathbf{v}_j , the transport equation in (35) can be incoming ($j \in \mathcal{I}$), and it only requires a boundary condition on $\partial\Omega_{0,T}$, or outgoing ($j \in \mathcal{O}$), and it only requires a boundary condition on $\partial\Omega_{1,T}$.

So that with the expression (34), the first couple of boundary conditions in the boundary cascade becomes:

$$\begin{cases} B_0 [\underline{u}_0 + \sum_{k \in \mathcal{I}} \sigma_{0,k} r_k]_{|x_d=0, \theta=(\theta_0, \dots, \theta_0)} = G - B_0 [\sum_{\ell \in \mathcal{O}} \sigma_{0,\ell} r_\ell]_{|x_d=0, \theta=(\theta_0, \dots, \theta_0)}, \\ B_1 [\underline{u}_0 + \sum_{\ell \in \mathcal{O}} \sigma_{0,\ell} r_\ell]_{|x_d=1, \theta=(\theta_0 + \xi_1, \dots, \theta_0 + \xi_N)} = -B_1 [\sum_{k \in \mathcal{I}} \sigma_{0,k} r_k]_{|x_d=1, \theta=(\theta_0 + \xi_1, \dots, \theta_0 + \xi_N)}, \end{cases}$$

so that separating the mean value from the oscillations and from the uniform Kreiss-Lopatinskii condition (see Assumption 3.4), we obtain the quadruplet of boundary conditions

$$\begin{cases} B_0 \underline{u}_0|_{x_d=0} = \underline{G}, \\ B_1 \underline{u}_0|_{x_d=1} = 0, \\ \forall k \in \mathcal{I}, \sigma_{0,k}|_{x_d=0}(\theta_0) r_k = \pi_k \phi_0 \left[G^*(\theta_0) - B_0 \sum_{\ell \in \mathcal{O}} \sigma_{0,\ell}|_{x_d=0}(\theta_0) r_\ell \right], \\ \forall \ell \in \mathcal{O}, \sigma_{0,\ell}|_{x_d=1}(\theta_0 + \xi^\ell) r_\ell = -\pi_\ell \phi_1 B_1 \sum_{k \in \mathcal{I}} \sigma_{0,k}|_{x_d=1}(\theta_0 + \xi^k), \end{cases}$$

where we recall that ϕ_0 and ϕ_1 stands for the inverse given by the localized Kreiss-Lopatinskii condition (see Assumption 3.4). As a consequence, adding the initial conditions we end up with the following system to

⁹Such a classical lemma ensures that

$\forall j \in \llbracket 1, d \rrbracket, \forall k \in \llbracket 1, N \rrbracket$ we have $\pi_k A_j \pi_k = \mathbf{v}_{k,j} \pi_k$.

solve in order to determine the several components of the leading order amplitude

$$\begin{cases}
L(\partial)\underline{u}_0 = \mathbf{F}(u_0) & \text{in } \Omega_T, \\
B_0\underline{u}_0|_{x_d=0} = \underline{G} & \text{on } \partial\Omega_{0,T}, \\
B_1\underline{u}_0|_{x_d=1} = 0 & \text{on } \partial\Omega_{1,T}, \\
\underline{u}_0|_{t=0} = 0 & \text{on } \Gamma, \\
\forall j \in \mathcal{I} \cup \mathcal{O}, (\partial_t + \mathbf{v}_j \cdot \nabla_x)\sigma_{0,j}r_j = \mathbf{E}_j\mathbf{F}(u_0) & \text{in } \Omega_T \times \mathbb{T}, \\
\forall k \in \mathcal{I}, \sum_{k \in \mathcal{I}} \sigma_{0,k}|_{x_d=0}(\theta_0)r_k = \pi_k\phi_0 \left[G^*(\theta_0) - B_0 \sum_{\ell \in \mathcal{O}} \sigma_{0,\ell}|_{x_d=0}(\theta_0)r_\ell \right] & \text{on } \partial\Omega_{0,T} \times \mathbb{T}, \\
\forall \ell \in \mathcal{O}, \sigma_{0,\ell}|_{x_d=1}(\theta_0 + \xi^\ell)r_\ell = -\pi_\ell\phi_1 B_1 \sum_{k \in \mathcal{I}} \sigma_{0,k}|_{x_d=1}(\theta_0 + \xi^k) & \text{on } \partial\Omega_{1,T} \times \mathbb{T}, \\
\forall j \in \mathcal{I} \cup \mathcal{O}, \sigma_{0,j}|_{t=0} = 0 & \text{on } \Gamma \times \mathbb{T}.
\end{cases} \quad (36)$$

Such a system is studied in the following paragraph. Before its resolution, let us end up the present paragraph by a short discussion describing the main ideas in the forthcoming proof. The system (36) is classically (see for instance [24], [7]) composed of a boundary value problem determining the mean \underline{u}_0 with transport phenomena determining the unknowns $\sigma_{j,0}$. Such two distinct behaviours being coupled the one with the other via the non linearity $\mathbf{F}(u_0)$ appearing in the interior equations.

However, compared with the classical geometry of the half-space¹⁰, we have an extra coupling of the amplitudes at the level of the boundary conditions.

Such a coupling was first encountered in the linear case in [3]. What will be convenient for our following analysis is that in order to deal with the boundary coupling in [3], no iterative scheme was used. Indeed, such a coupling was ruled out by imposing the invertibility of some operator reading under the form $I - \mathbf{T}$ (we refer to [3] or to the following paragraph for more details). As a consequence, there is no interaction between the two coupling. Indeed, to deal with the non linear interior we use a usual iterative scheme, it fixes the source term in the interior. The invertibility of the above operator, then decouples the boundary conditions and determines the values of the required trace.

3.3.2 Solving the two coupled non linear systems

The following paragraph justifies that system (36) admits a unique solution at least for a short time of existence $T_0 \leq T$. In order to state the result precisely it will be convenient to introduce the following modification of the spaces \mathbf{W}_T^s : for $T > 0$ and $s \in \mathbb{N}$, let

$$\underline{\mathbf{W}}_T^s := \{u \in \mathcal{C}^0([0, T]; H^s(\Gamma \times \mathbb{T})) \setminus \forall j \in \llbracket 0, s \rrbracket \text{ we have } \partial_t^j u \in \mathcal{C}^0([0, T]; H^{s-j}(\Gamma \times \mathbb{T}))\}.$$

It is equipped with the obvious norm and we define $\overline{\mathbf{W}}_T^s := (\underline{\mathbf{W}}_T^s)^N$. At last, we consider $\overline{\mathbf{W}}_T^\infty := \bigcap_{s \geq 0} \overline{\mathbf{W}}_T^s$.

Proposition 3.1 *Under Assumptions 2.1, 2.2 and 2.3 and 3.4 if the source G is regular and flat at the origin there exists a time $0 < T_0 \leq T$ such that system (36) has a unique solution $(\underline{u}_0, (\sigma_{0,j})_{j \in \mathcal{I} \cup \mathcal{O}}) \in \mathbf{W}_{T_0}^\infty \times \overline{\mathbf{W}}_{T_0}^\infty$. Moreover, the solution $(\underline{u}_0, (\sigma_{0,j})_{j \in \mathcal{I} \cup \mathcal{O}})$ is flat at the time origin.*

The proof of Proposition 3.1 uses the study of the convergence of the following iterative scheme of approximation. Let $\underline{u}_0^0 \equiv 0$ and for all $j \in \mathcal{I} \cup \mathcal{O}$, $\sigma_{0,j}^0 \equiv 0$. We define the sequence $(\underline{u}_0^\nu, \sigma_{0,j}^\nu)_{\nu \in \mathbb{N}^*, j \in \mathcal{I} \cup \mathcal{O}}$ as the solution (given by Theorem 2.1 and Proposition 3.2) of the following (decoupled) system of equations:

¹⁰In such a geometry, the outgoing modes do not require any boundary conditions, the associated transport equations can be solved at the beginning and the associated traces can then be used in the boundary condition determining the incoming modes.

for all $\mu \geq 0$,

$$\begin{cases} L(\partial)\underline{u}_0^{\nu+1} = \underline{\mathbf{F}}(u_0^\nu) & \text{in } \Omega_T, \\ B_0\underline{u}_0^{\nu+1}|_{x_d=0} = \underline{\mathbf{G}} & \text{on } \partial\Omega_{0,T}, \\ B_1\underline{u}_0^{\nu+1}|_{x_d=1} = 0 & \text{on } \partial\Omega_{1,T}, \\ \underline{u}_0^{\nu+1}|_{t=0} = 0 & \text{on } \Gamma, \\ \forall j \in \mathcal{I} \cup \mathcal{O}, (\partial_t + \mathbf{v}_j \cdot \nabla_x)\sigma_{0,j}^{\nu+1}r_j = \mathbf{E}_j\mathbf{F}(u_0^\nu) & \text{in } \Omega_T \times \mathbb{T}, \\ \forall k \in \mathcal{I}, \sigma_{0,k|_{x_d=0}}^{\nu+1}(\theta_0)r_k = \pi_k\phi_0 \left[G^*(\theta_0) - B_0 \sum_{\ell \in \mathcal{O}} \sigma_{0,\ell|_{x_d=0}}^{\nu+1}(\theta_0)r_\ell \right] & \text{on } \partial\Omega_{0,T} \times \mathbb{T}, \\ \forall \ell \in \mathcal{O}, \sigma_{0,\ell|_{x_d=1}}^{\nu+1}(\theta_0 + \xi^\ell)r_\ell = -\pi_\ell\phi_1 B_1 \sum_{k \in \mathcal{I}} \sigma_{0,k|_{x_d=1}}^{\nu+1}(\theta_0 + \xi^k) & \text{on } \partial\Omega_{1,T} \times \mathbb{T}, \\ \forall j \in \mathcal{I} \cup \mathcal{O}, \sigma_{0,j|_{t=0}}^{\nu+1} = 0 & \text{on } \Gamma \times \mathbb{T}. \end{cases} \quad (37)$$

The precise study of the convergence can be made exactly as the one described in Paragraph 2.2, so that we will feel free to omit the details here and we will only give the well-posedness of the two linear problems constituting scheme (37). We consider the two linear problems

$$\begin{cases} L(\partial)u = f & \text{in } \Omega_T, \\ B_0u|_{x_d=0} = g & \text{on } \partial\Omega_{0,T}, \\ B_1u|_{x_d=1} = 0 & \text{on } \partial\Omega_{1,T}, \\ u|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (38)$$

whose well-posedness has been studied in Section 2, and

$$\begin{cases} \forall j \in \mathcal{I} \cup \mathcal{O}, (\partial_t + \mathbf{v}_j \cdot \nabla_x)\sigma_j r_j = \pi_j f & \text{in } \Omega_T \times \mathbb{T}, \\ \forall k \in \mathcal{I}, \sigma_{k|_{x_d=0}}(\theta_0)r_k = \pi_k\phi_0 \left[g_0(\theta_0) - B_0 \sum_{\ell \in \mathcal{O}} \sigma_{\ell|_{x_d=0}}(\theta_0)r_\ell \right] & \text{on } \partial\Omega_{0,T} \times \mathbb{T}, \\ \forall \ell \in \mathcal{O}, \sigma_{\ell|_{x_d=1}}(\theta_0 + \xi^\ell)r_\ell = -\pi_\ell\phi_1 B_1 \sum_{k \in \mathcal{I}} \sigma_{k|_{x_d=1}}(\theta_0 + \xi^k) & \text{on } \partial\Omega_{1,T} \times \mathbb{T}, \\ \forall j \in \mathcal{I} \cup \mathcal{O}, \sigma_{j|_{t=0}} = 0 & \text{on } \Gamma \times \mathbb{T}, \end{cases} \quad (39)$$

whose well-posedness is considered in the following proposition.

Proposition 3.2 *Under Assumptions 2.1, 2.2 and 3.4, let $T > 0$ be given. Then for all $f \in L_b^2(\Omega_T \times \mathbb{T})$, $g_0 \in L_b^2(\partial\Omega_{0,T} \times \mathbb{T})$, system (39) admits a unique solution $(\sigma_j)_{j \in \mathcal{I} \cup \mathcal{O}}$ in $[L_b^2(\Omega_T \times \mathbb{T})]^N$. Such a solution satisfies the energy estimate: for all $0 < t \leq T$, for all $j \in \llbracket 1, N \rrbracket$:*

$$\begin{aligned} \|\sigma_j(t, \cdot)\|_{L^2(\Gamma \times \mathbb{T})} + \|\sigma_{j|_{x_d=0}}\|_{L^2(\partial\Omega_{t,0} \times \mathbb{T})} + \|\sigma_{j|_{x_d=1}}\|_{L^2(\partial\Omega_{t,1} \times \mathbb{T})} \\ \leq C_T \left[\int_0^T \|f(s, \cdot)\|_{L^2(\Gamma \times \mathbb{T})} ds + \|g_0\|_{L^2(\partial\Omega_{0,T} \times \mathbb{T})} \right]. \end{aligned} \quad (40)$$

Moreover, for all $s \in \mathbb{N}$, if the sources $f \in H_b^s(\Omega_T \times \mathbb{T})$, $g_0 \in H_b^s(\partial\Omega_{0,T} \times \mathbb{T})$, the solution $(\sigma_j)_{j \in \mathcal{I} \cup \mathcal{O}} \in \overline{\mathbf{W}}_{T,b}^s$, and the energy estimate (40) holds in high order Sobolev norm (that is the \mathbf{W}_T^s -norm).

In particular if $f \in H_b^\infty(\Omega_T)$ and $g_0 \in H_b^\infty(\partial\Omega_{0,T})$, then the solution $(\sigma_j)_{j \in \mathcal{I} \cup \mathcal{O}} \in \overline{\mathbf{W}}_{T,b}^\infty$.

Proof of Proposition 3.2 We construct by hand an explicit solution to (39). We then verify that this solution verify (40) and that it check the persistence of regularity and flatness properties.

In order to construct the solution, we follow the method of [3]. We recall such a method because it is the core of the proof, but let us stress that in the present contribution we just give a rather straightforward extension of [3]. More precisely we extend the method of [3] to a periodic setting with non trivial interior sources.

Let us assume for a while that $\mathbf{I} := \mathbf{I}(t, x'; \theta_0) = \sum_{\ell \in \mathcal{O}} \sigma_{\ell|_{x_d=0}}(\theta_0)r_\ell$ is known, so that the right-hand side of the boundary condition on $\partial\Omega_{0,T}$ is a known function. For incoming modes, we thus have to solve

$$\forall k \in \mathcal{I}, \begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x)\sigma_k r_k = \pi_k f & \text{in } \Omega_T \times \mathbb{T}, \\ \sigma_{k|_{x_d=0}}(\theta_0)r_k = \pi_k\phi_0 [g_0(\theta_0) - B_0\mathbf{I}(\theta_0)] & \text{on } \partial\Omega_{0,T} \times \mathbb{T}, \\ \sigma_{k|_{t=0}} = 0 & \text{on } \Gamma \times \mathbb{T}. \end{cases}$$

Integrate along the characteristics, thus gives the explicit formula

$$\begin{aligned} \sigma_k(t, x, \theta_0)r_k &:= \pi_k \phi_0 [g_0 - B_0 \mathbf{I}] \left(t - \frac{x_d}{\mathbf{v}_{k,d}}, x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} x_d; \theta_0 \right) \\ &+ \mathbf{1}_{\{t - \frac{x_d}{\mathbf{v}_{k,d}} \geq 0\}} \frac{1}{\mathbf{v}_{k,d}} \int_0^{x_d} \pi_k f_k \left(t - \frac{1}{\mathbf{v}_{k,d}} (x_d - y), x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} (x_d - y), y; \theta_0 \right) dy \\ &+ \mathbf{1}_{\{t - \frac{x_d}{\mathbf{v}_{k,d}} < 0\}} \int_0^t \pi_k f(s, x - \mathbf{v}_k(t - s); \theta_0) ds \end{aligned} \quad (41)$$

so that we deduce the values of the terms $\sigma_{k|x_d=1}(\theta_0 + \xi^k)$ appearing in the right-hand side of the boundary condition on $\partial\Omega_{1,T}$ of (39). We have

$$\begin{aligned} \sum_{k \in \mathcal{J}} \sigma_{k|x_d=1}(\theta_0 + \xi_k)r_k &= \sum_{k \in \mathcal{J}} \pi_k \phi_0 [g_0 - B_0 \mathbf{I}] \left(t - \frac{1}{\mathbf{v}_{k,d}}, x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}}; \theta_0 + \xi_k \right) \\ &+ \sum_{k \in \mathcal{J}} \mathbf{1}_{\{t - \frac{1}{\mathbf{v}_{k,d}} \geq 0\}} \frac{1}{\mathbf{v}_{k,d}} \int_0^1 \pi_k f_k \left(t - \frac{1}{\mathbf{v}_{k,d}} (1 - y), x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} (1 - y), y; \theta_0 + \xi_k \right) dy \\ &+ \sum_{k \in \mathcal{J}} \mathbf{1}_{\{t - \frac{1}{\mathbf{v}_{k,d}} < 0\}} \int_0^t \pi_k f_k(s, x' - \mathbf{v}'_k(t - s), 1 - \mathbf{v}_{k,d}(t - s); \theta_0 + \xi_k) ds. \end{aligned} \quad (42)$$

Clearly when $t < \frac{1}{\mathbf{v}_{k,d}}$ the information initially lying at $\partial\Omega_{0,T}$, because it travels with finite speed \mathbf{v}_k , has not hit the side $\partial\Omega_{1,T}$ yet. So, there is no loss of generality to consider for such short times the problem in the half-space $\{x_d > 0\}$ instead of the strip problem. Consequently to consider a "true" strip problem there is no loss of generality by assuming that $t \geq \frac{1}{\mathbf{v}_{k,d}}$ and (42) simplifies into:

$$\begin{aligned} \sum_{k \in \mathcal{J}} \sigma_{k|x_d=1}(\theta_0 + \xi_k)r_k &= \sum_{k \in \mathcal{J}} \pi_k \phi_0 [g_0 - B_0 \mathbf{I}] \left(t - \frac{1}{\mathbf{v}_{k,d}}, x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}}; \theta_0 + \xi_k \right) \\ &+ \sum_{k \in \mathcal{J}} \frac{1}{\mathbf{v}_{k,d}} \int_0^1 \pi_k f_k \left(t - \frac{1}{\mathbf{v}_{k,d}} (1 - y), x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} (1 - y), y; \theta_0 + \xi_k \right) dy \\ &:= \sum_{k \in \mathcal{J}} \pi_k \phi_0 [g_0 - B_0 \mathbf{I}] \left(t - \frac{1}{\mathbf{v}_{k,d}}, x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}}; \theta_0 + \xi_k \right) + (\mathcal{F}_1 f)(t, x'; \theta_0). \end{aligned} \quad (43)$$

So that at this step, we know (in terms of the unknown \mathbf{I}) the boundary source term in the outgoing transport equations:

$$\forall \ell \in \mathcal{O}, \begin{cases} (\partial_t + \mathbf{v}_\ell \cdot \nabla_x) \sigma_\ell r_\ell = \pi_\ell f & \text{in } \Omega_T \times \mathbb{T}, \\ \sigma_{\ell|x_d=1}(\theta_0 + \xi_\ell)r_\ell = -\pi_\ell \phi_1 B_1 \sum_{k \in \mathcal{J}} \sigma_{k|x_d=1}(\theta_0 + \xi_k)r_k & \text{on } \partial\Omega_{0,T} \times \mathbb{T}, \\ \sigma_{\ell|t=0} = 0 & \text{on } \Gamma \times \mathbb{T}. \end{cases}$$

We integrate along the characteristics to obtain with the notation of (43) in hand,

$$\begin{aligned} \sigma_\ell(t, x, \theta_0)r_\ell &:= -\pi_\ell \phi_1 B_1 \sum_{k \in \mathcal{J}} \pi_k \phi_0 \\ &[g_0 - B_0 \mathbf{I}] \left(t - \frac{1}{\mathbf{v}_{k,d}} - \frac{1}{\mathbf{v}_{\ell,d}} (x_d - 1), x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} - \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}} (x_d - 1); \theta_0 + \xi_k - \xi_\ell \right) \\ &- \pi_\ell \phi_1 B_1 (\mathcal{F}_1 f) \left(t - \frac{1}{\mathbf{v}_{\ell,d}} (x_d - 1), x' - \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}} (x_d - 1); \theta_0 - \xi_\ell \right) \\ &- \mathbf{1}_{\{t - \frac{1}{\mathbf{v}_{\ell,d}} (x_d - 1) \geq 0\}} \frac{1}{\mathbf{v}_{\ell,d}} \int_{x_d}^1 \pi_\ell f \left(t - \frac{1}{\mathbf{v}_{\ell,d}} (x_d - y), x' - \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}} (x_d - y), y; \theta_0 \right) dy \\ &+ \mathbf{1}_{\{t - \frac{1}{\mathbf{v}_{\ell,d}} (x_d - 1) < 0\}} \int_0^t \pi_\ell f(s, x - \mathbf{v}_\ell(t - s); \theta_0) ds. \end{aligned} \quad (44)$$

Once again there is no loss of generality by assuming that the time is taken sufficiently large to ensure that the last term does not contribute in $\sigma_{\ell}|_{x_d=0}$.

As a consequence, we have that \mathbf{I} is solution to the equation

$$(I - \mathbf{T})\mathbf{I} = \mathcal{G}g_0 + \mathcal{F}f, \quad (45)$$

where we defined

$$\begin{aligned} (\mathbf{T}u)(t, x'; \theta_0) &:= \sum_{\ell \in \mathcal{O}} \sum_{k \in \mathcal{J}} \pi_\ell \phi_1 B_1 \pi_\ell \phi_0 B_0 u \left(t - \frac{1}{\mathbf{v}_{k,d}} + \frac{1}{\mathbf{v}_{\ell,d}}, x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} + \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}}; \theta_0 + \xi_k - \xi_\ell \right), \\ (\mathcal{G}g)(t, x'; \theta_0) &:= \sum_{\ell \in \mathcal{O}} \sum_{k \in \mathcal{J}} \pi_\ell \phi_1 B_1 \pi_\ell \phi_0 g \left(t - \frac{1}{\mathbf{v}_{k,d}} + \frac{1}{\mathbf{v}_{\ell,d}}, x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} + \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}}; \theta_0 + \xi_k - \xi_\ell \right), \\ (\mathcal{F}f)(t, x'; \theta_0) &:= - \sum_{\ell \in \mathcal{O}} \pi_\ell \phi_1 B_1 (\mathcal{F}_1 f) \left(t + \frac{1}{\mathbf{v}_{\ell,d}}, x' + \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}}; \theta_0 - \xi_\ell \right) \\ &\quad - \sum_{\ell \in \mathcal{O}} \frac{1}{\mathbf{v}_{\ell,d}} \int_{x_d}^1 \pi_\ell f \left(t + \frac{y}{\mathbf{v}_{\ell,d}}, x' + y \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}}, y; \theta_0 \right) dy \end{aligned} \quad (46)$$

We have the following lemma

Lemma 3.2 • *If the sources $f \in L^2_b(\Omega_T \times \mathbb{T})$ and $g_0 \in L^2_b(\partial\Omega_{0,T} \times \mathbb{T})$, then the terms $\mathcal{F}f$ and $\mathcal{G}g_0$ are in $L^2_b(\partial\Omega_{0,T} \times \mathbb{T})$. Moreover for all $T > 0$, and for all $\mathbf{g} \in L^2_b(\partial\Omega_{0,T} \times \mathbb{T})$, the equation*

$$(I - \mathbf{T})\mathbf{u} = \mathbf{g},$$

has a unique solution $\mathbf{u} \in L^2_b(\partial\Omega_{0,T} \times \mathbb{T})$. Such a solution satisfies

$$\|\mathbf{u}\|_{L^2(\partial\Omega_{0,T} \times \mathbb{T})} \leq C \|\mathbf{g}\|_{L^2(\partial\Omega_{0,T} \times \mathbb{T})}. \quad (47)$$

- *The above result also holds in high order Sobolev spaces. More precisely, for all $s \in \mathbb{N}$, for all $T > 0$, if the sources $f \in H^s_b(\Omega_T \times \mathbb{T})$ and $g_0 \in H^s_b(\partial\Omega_{0,T} \times \mathbb{T})$, then the terms $\mathcal{F}f$ and $\mathcal{G}g_0$ are in $H^s_b(\partial\Omega_{0,T} \times \mathbb{T})$. Moreover for all $T > 0$, and for all $\mathbf{g} \in H^s_b(\partial\Omega_{0,T} \times \mathbb{T})$ the equation*

$$(I - \mathbf{T})u = \mathbf{g},$$

has a unique solution $u \in H^s_b(\partial\Omega_{0,T} \times \mathbb{T})$. It satisfies

$$\|\mathbf{u}\|_{H^s(\partial\Omega_{0,T} \times \mathbb{T})} \leq C \|\mathbf{g}\|_{H^s(\partial\Omega_{0,T} \times \mathbb{T})}.$$

Let us postpone the proof of this lemma and conclude the proof of Proposition 3.2. We inject the expression of $\mathbf{I} \in H^s(\partial\Omega_{0,T} \times \mathbb{T})$ given in the explicit formulas defining the amplitudes $\sigma_{0,j}$, namely (41) for incoming modes and (44). It is then clear that the contribution of \mathbf{I} in such formulas gives rise to a term in $H^s(\partial\Omega_{0,T} \times \mathbb{T})$. For the other contributions, namely the ones depending on the sources f and g_0 , we can show that they also give rise to $H^s(\partial\Omega_{0,T} \times \mathbb{T})$ terms (we refer to the proof of Lemma 3.2 for more details). As a consequence, we have justified the existence and the regularity of the amplitudes $\sigma_{0,j} \in H^s(\Omega_T \times \mathbb{T})$. From the explicit formulas (41) and (44), such amplitudes are flat at the origin if the sources are. Finally, the energy estimate (40) can be shown by hand by using the explicit formulas (41) and (44) or by repeating the kind of energy estimates of Paragraph 2.1 in the simpler setting of transport equations.

Consequently, with Proposition 3.2 in hand to estimate the amplitudes $\sigma_{0,j}^\mu$ and Theorem 2.1 to estimate u_0^μ , we can reproduce the convergence analysis of Paragraph 2.2 to the scheme (37) and it ends up the proof of Proposition 3.1. It concludes the construction of the leading order term in the expansion.

We end up with the proof of Lemma 3.2¹¹

¹¹Let us point here that the unique resolvability of $(I - \mathbf{T})\mathbf{u} = \mathbf{g}$ in some regular functional space was assumed in [3] in order to compute the expansion. Indeed for infinite time problems, such an invertibility condition is not automatic. However, here we take advantage of the fact that we have *a priori* a finite time of resolution because we are dealing with non linear problems and the result becomes automatic.

Proof of Lemma 3.2 We start by justifying the regularity of the modifications of the sources. From the explicit formulas (46), we have first for the boundary contribution:

$$\|\mathcal{G}g_0\|_{L^2(\partial\Omega_0, T \times \mathbb{T})} \leq C\|g_0\|_{L^2(\partial\Omega_0, T \times \mathbb{T})}$$

by the simple change of variables $t \rightsquigarrow t - \frac{1}{\mathbf{v}_{k,d}} + \frac{1}{\mathbf{v}_{\ell,d}}$, $x' \rightsquigarrow x' - \frac{\mathbf{v}'_k}{\mathbf{v}_{k,d}} + \frac{\mathbf{v}'_\ell}{\mathbf{v}_{\ell,d}}$. Then the study of the interior contribution follows the same lines thanks to Jensen's inequality.

We then turn to the resolution of the equation $(I - \mathbf{T})\mathbf{u} = \mathbf{g}$. The source term \mathbf{g} being flat at the time origin, its extension to the whole time line by zero for negative times (still denoted by \mathbf{g}) is regular. At the formal level, let us consider the Neumann series expansion

$$\mathbf{u} := \sum_{p=0}^{\infty} \mathbf{T}^p \mathbf{g}. \quad (48)$$

The aim of the following is to justify that the above series simplifies into a finite sum when we consider finite final time of resolution $T < \infty$. Let us define $\underline{\alpha} := \min_{k \in \mathcal{I}, \ell \in \mathcal{O}} \frac{1}{\mathbf{v}_{k,d}} - \frac{1}{\mathbf{v}_{\ell,d}} > 0$ because of the kinds of the group velocities. The real $\underline{\alpha}$ corresponds to the minimal time of regeneration of the paths of phases by reflections against $\partial\Omega_{0,T}$ and $\partial\Omega_{1,T}$.

Then the time evaluation in $\mathbf{T}\mathbf{g}$ can be bounded by $T - \underline{\alpha}$. In particular, it vanishes for $T < \underline{\alpha}$, meaning that in such a configuration the considered final time is too short to ensure that the phases have been regenerated.

A simple induction then shows that the time variable in $\mathbf{T}^p \mathbf{g}$ can be bounded by $T - p\underline{\alpha}$. Consequently such a term is negative if $p \geq \left\lceil \frac{T}{\underline{\alpha}} \right\rceil + 1$ and consequently for such p , $\mathbf{T}^p \mathbf{g}$ vanishes because $g \in L^2_b(\partial\Omega_{0,T} \times \mathbb{T})$. As a consequence, the series in (48) simplifies into a finite sum, the associated solution $\mathbf{u} \in L^2_b(\partial\Omega_{0,T} \times \mathbb{T})$ is well-defined and the energy estimate (47) trivially holds.

The previous results can easily be extended to higher order Sobolev norms, because from the explicit formulas (46) we have $\|\mathbf{T}\mathbf{u}\|_{H^s(\partial\Omega_{0,T})} \leq C_s \|\mathbf{u}\|_{H^s(\partial\Omega_{0,T})}$ and analogous formulas for the other operators. The details are left to the interested reader.

□

3.4 Higher order terms

Once the leading order term in the expansion has been constructed, because of our small divisor Assumption 3.3 such a method of construction can easily be adapted to determine as many correctors as required. On the one hand the proof simplifies because the correctors now solve linear problems, on the other hand it complicates because the correctors are not polarized any more. We here describe the modifications in the determination of the first corrector, namely u_1 . The analysis can then be used without modifications for the higher order correctors.

For convenience, let us recall that u_1 appears in the two interior equations:

$$\begin{cases} \mathcal{P}u_1 + L(\partial)u_0 = \mathbf{F}(u_0) & \text{in } \Omega_T, \\ \mathcal{P}u_2 + L(\partial)u_1 = \mathbf{F}'(u_0)u_1 & \text{in } \Omega_T. \end{cases} \quad (49)$$

We first apply the pseudo-inverse \mathcal{Q} to the first equation in (49). It gives the value of the unpolarized part of u_1 . More precisely:

$$(I - \mathbf{E})u_1 = \mathcal{Q}[\mathbf{F}(u_0) - L(\partial)u_0], \quad (50)$$

where the right-hand side is known, regular and flat at the time origin because of the construction of Paragraph 3.3. As a consequence, to conclude the construction of u_1 it is sufficient to construct $\mathbf{E}u_1$. In order to do, we reproduce the construction of Paragraph 3.3. Let us write

$$\mathbf{E}u_1(t, x; \theta) := \underline{u}_1(t, x) + \sum_{j=1}^M \sigma_{1,j}(t, x; \theta_j) r_j,$$

where the terms $\sigma_{1,j}$ are real value (unknown) functions. Reiterating the same kind of reasoning as the one performed for the leading order term, we obtain that the quantities determining the polarized part $\mathbf{E}u_1$ satisfy the three linear and uncoupled (in the interior) systems

$$\begin{cases} L(\partial)\underline{u}_1 + \mathbf{F}'(u_0)\mathbf{E}u_1 = \underline{f} & \text{in } \Omega_T, \\ B_0\underline{u}_1|_{x_d=0} = -\mathbf{E}_0B_0(I - \mathbf{E})u_1|_{x_d=0} & \text{on } \partial\Omega_{0,T}, \\ B_1\underline{u}_1|_{x_d=1} = -\mathbf{E}_0B_1(I - \mathbf{E})u_1|_{x_d=1} & \text{on } \partial\Omega_{1,T}, \\ \underline{u}_1|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (51)$$

$$\forall k \in \mathcal{I}, \begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x)\sigma_{1,k}r_k + \mathbf{E}_k\mathbf{F}'(u_0)\mathbf{E}u_1 = \mathbf{E}_kf & \text{in } \Omega_T, \\ \sigma_{1,k}|_{x_d=0}r_k = g_{0,k} - \pi_k\phi_0B_0 \sum_{\ell \in \mathcal{O}} \sigma_{1,\ell}|_{x_d=0} & \text{on } \partial\Omega_{0,T}, \\ \sigma_{1,k}|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (52)$$

and

$$\forall \ell \in \mathcal{O} \begin{cases} (\partial_t + \mathbf{v}_\ell \cdot \nabla_x)\sigma_{1,\ell}r_\ell + \mathbf{E}_\ell\mathbf{F}'(u_0)\mathbf{E}u_1 = \mathbf{E}_\ell f & \text{in } \Omega_T, \\ \sigma_{1,\ell}|_{x_d=1}r_\ell = g_{1,\ell} - \pi_\ell\phi_1B_1 \sum_{k \in \mathcal{I}} \sigma_{1,k}|_{x_d=1} & \text{on } \partial\Omega_{1,T}, \\ \sigma_{1,\ell}|_{t=0} = 0 & \text{on } \Gamma, \end{cases} \quad (53)$$

where we defined $f := -L(\partial)\mathcal{Q}(I - \mathbf{E})u_1$, and similarly $g_0 := -\mathbf{E}_kB_0(I - \mathbf{E})u_1|_{x_d=0}$, $g_{1,\ell} := -\mathbf{E}_\ell B_1(I - \mathbf{E})u_1|_{x_d=1}$. These three terms are regular and depends explicitly on u_0 via the formula (50). So that we now have to give some more details about the adding of the zero order terms in (51), (52) and (53). The classical manner to deal with such terms is to consider the three above equations as a single one in the unknowns $\mathbf{U} := (\underline{u}_1, (\sigma_{1,j}r_j)_{j \in \mathcal{I} \cup \mathcal{O}})$. We here propose an other approach more in the spirit of the above in order to deal with the coupling introduced by the zero order term. We treat it as a source term and we solve by using a scheme of approximation which update this coupling at each step. With more details we solve (51), (52) and (53) as the limit of the solutions $(\underline{u}^\nu, (\sigma_j^\nu)_{j \in \mathcal{I} \cup \mathcal{O}})$ to the equations: for $\nu \geq 1$

$$\begin{cases} L(\partial)\underline{u}^\nu + = \underline{f} - \mathbf{F}'(u_0)\mathbf{E}u^{\nu-1} & \text{in } \Omega_T, \\ B_0\underline{u}^\nu|_{x_d=0} = -\mathbf{E}_0B_0(I - \mathbf{E})u|_{x_d=0} & \text{on } \partial\Omega_{0,T}, \\ B_1\underline{u}^\nu|_{x_d=1} = -\mathbf{E}_0B_1(I - \mathbf{E})u|_{x_d=1} & \text{on } \partial\Omega_{1,T}, \\ \underline{u}^\nu|_{t=0} = 0 & \text{on } \Gamma, \end{cases}$$

$$\forall k \in \mathcal{I}, \begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x)\sigma_k^\nu r_k = \mathbf{E}_kf - \mathbf{E}_k\mathbf{F}'(u_0)\mathbf{E}u^{\nu-1} & \text{in } \Omega_T, \\ \sigma_k^\nu|_{x_d=0}r_k = g_{0,k} - \pi_k\phi_0B_0 \sum_{\ell \in \mathcal{O}} \sigma_\ell^\nu|_{x_d=0} & \text{on } \partial\Omega_{0,T}, \\ \sigma_k^\nu|_{t=0} = 0 & \text{on } \Gamma, \end{cases}$$

and

$$\forall \ell \in \mathcal{O} \begin{cases} (\partial_t + \mathbf{v}_\ell \cdot \nabla_x)\sigma_\ell^\nu r_\ell = \mathbf{E}_\ell f - \mathbf{E}_\ell\mathbf{F}'(u_0)\mathbf{E}u^{\nu-1} & \text{in } \Omega_T, \\ \sigma_\ell^\nu|_{x_d=1}r_\ell = g_{1,\ell} - \pi_\ell\phi_1B_1 \sum_{k \in \mathcal{I}} \sigma_k^\nu|_{x_d=1} & \text{on } \partial\Omega_{1,T}, \\ \sigma_\ell^\nu|_{t=0} = 0 & \text{on } \Gamma, \end{cases}$$

The first system is then solved by using the results of Section 3 (see Theorem 2.1). The two other systems determining the $(\sigma_j)_{j \in \mathcal{I} \cup \mathcal{O}}$ are then solved exactly as in Paragraph 3.3 (see Proposition 3.2). In particular to deal with the coupling of the boundary conditions we can effectively reproduce the analysis of Paragraph 3.3 which gives the operator $I - \mathbf{T}$ and which essentially relies on explicit formula¹².

It concludes the construction of the first corrector u_1 and the one of all the higher order ones.

¹²The use of a scheme of approximation is not necessary here because the above explicit formula holds for the transport equation with zero order term λ . Indeed, we can obtain an explicit formula for the $\sigma_{1,k}$ by multiplying the two first lines of (41) by $e^{-\frac{1}{\mathbf{v}_{k,d}} \int_0^{x_d} \lambda(t, x', y) dy}$ and the last line by $e^{-\int_0^t \lambda(s, x) ds}$. Such factors being harmless and do not require some sign properties on the zero order term λ , because x_d and t are both bounded.

3.5 Error analysis, proof of Theorem 3.2

The proof exposed here closely follows the one given in [[24]-Section 8.3] which simplifies the one of [8]. We however reproduce the main lines here for a sake of completeness.

Let \tilde{u}^ε be an approximate solution as in Theorem 3.2, we want to construct an exact solution to (17) reading $u^\varepsilon = \tilde{u}^\varepsilon + w^\varepsilon$. The second term in the right-hand side satisfies the system of equations

$$\begin{cases} L(\partial)w^\varepsilon = \mathbf{F}(w^\varepsilon + \tilde{u}^\varepsilon) - \mathbf{F}(\tilde{u}^\varepsilon) - \varepsilon^{M_1} R^\varepsilon & \text{in } \Omega_T, \\ B_0 w^\varepsilon|_{x_d=0} = -\varepsilon^{M_1} r_0^\varepsilon & \text{on } \partial\Omega_{0,T}, \\ B_1 w^\varepsilon|_{x_d=1} = -\varepsilon^{M_1} r_1^\varepsilon & \text{on } \partial\Omega_{1,T}, \\ w^\varepsilon|_{t=0} = 0 & \text{on } \Gamma. \end{cases} \quad (54)$$

The existence of such a w^ε is obtained as the limit when $\nu \uparrow \infty$ of the sequence $(w^{\varepsilon,\nu})_{0 < \varepsilon < 1, \nu \geq 0}$ (ε fixed) define iteratively as the solution to

$$\begin{cases} L(\partial)w^{\varepsilon,\nu+1} = \mathbf{F}(w^{\varepsilon,\nu} + \tilde{u}^\varepsilon) - \mathbf{F}(\tilde{u}^\varepsilon) - \varepsilon^{M_1} R^\varepsilon & \text{in } \Omega_T, \\ B_0 w^{\varepsilon,\nu+1}|_{x_d=0} = -\varepsilon^{M_1} r_0^\varepsilon & \text{on } \partial\Omega_{0,T}, \\ B_1 w^{\varepsilon,\nu+1}|_{x_d=1} = -\varepsilon^{M_1} r_1^\varepsilon & \text{on } \partial\Omega_{1,T}, \\ w^{\varepsilon,\nu+1}|_{t=0} = 0 & \text{on } \Gamma. \end{cases} \quad (55)$$

In order to establish the convergence of such a sequence, one needs to analyse the linear problem associated to (55), namely

$$\begin{cases} L(\partial)u = f & \text{in } \Omega_T, \\ B_0 u|_{x_d=0} = g_0 & \text{on } \partial\Omega_{0,T}, \\ B_1 u|_{x_d=1} = g_1 & \text{on } \partial\Omega_{1,T}, \\ u|_{t=0} = 0 & \text{on } \Gamma. \end{cases} \quad (56)$$

Such a linearised problem has already been considered in Section 2. However here in order to compensate the losses of powers with respect to ε appearing when one computes H^s -norm, we have to make more precise the basic energy estimates, by using the weighted norms of Guès [8]. This motivates the definition: let $m \in \mathbb{N}$, $\gamma > 0$ and $\mu > 0$ then we define for $X \subset \Omega_T$:

$$\|u\|_{m,\mu,\gamma,X} := \sum_{|\alpha| \leq m} \mu^{m-|\alpha|} \|e^{-\gamma t} \partial^\alpha u\|_{L^2(X)}. \quad (57)$$

Because in terms of ε we have that $\partial^\alpha u \sim \frac{1}{\varepsilon^{|\alpha|}}$ then for $\mu \sim \frac{1}{\varepsilon}$, $(\mu)^{m-|\alpha|} \partial^\alpha u \sim \varepsilon^m$, so that the above loss is compensated.

As already mentioned in the end of Paragraph 2.1 the basic L^2 estimate for the linearised problem (56) for γ -weighted norms reads: there exists $C > 0$ and $\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$ and for all $\mu > 0$ we have

$$\begin{aligned} \|u\|_{0,\mu,\gamma,\Omega_T} + \frac{1}{\sqrt{\gamma}} (\|u|_{x_d=0}\|_{0,\mu,\gamma,\partial\Omega_{0,T}} + \|u|_{x_d=1}\|_{0,\mu,\gamma,\partial\Omega_{1,T}}) \\ \leq C \left(\frac{1}{\gamma} \|f\|_{0,\mu,\gamma,\Omega_T} + \frac{1}{\sqrt{\gamma}} (\|g_0\|_{0,\mu,\gamma,\partial\Omega_{0,T}} + \|g_1\|_{0,\mu,\gamma,\partial\Omega_{1,T}}) \right) \end{aligned} \quad (58)$$

Such an estimate is satisfied if one considers hyperbolic value problems with boundary conditions satisfying the uniform Kreiss-Lopatinskii condition of [13]. So that it is in particular satisfied under our triplet of Assumptions (2.1, 2.2 and 2.3) ensuring strong well-posedness.

Let us note that from (55) the adding of the second source term on $\partial\Omega_{1,T}$ will not create any troubles because it has exactly the same size than the one on $\partial\Omega_{0,T}$.

With such a L^2 estimate in hand, one can derive the following estimate (see Proposition 8.2 of [24]):

Proposition 3.3 Let $m_1 > \frac{d+1}{2}$ and ϱ be as specified in Theorem 3.2. Let us assume that for some $\nu \geq 0$, the solution $w^{\varepsilon,\nu} \in H_b^{m_1}(\Omega_T)$ satisfies the bound

$$\|w^{\varepsilon,\nu}\|_{L^\infty(\Omega_T)} \leq K$$

then we have the existence of two constants $C = C(K, \varrho) > 0$ and $\gamma_0 := \gamma_0(K, \varrho)$ and of a positive function θ such that:

$$\|w^{\varepsilon,\nu+1}\|_{m_1, \frac{\gamma}{\varepsilon}, \gamma, \Omega_T} \leq \frac{C}{\gamma} \|w^{\varepsilon,\nu}\|_{m_1, \frac{\gamma}{\varepsilon}, \gamma, \Omega_T} + \varepsilon^{M_1 - m_1} \theta(\gamma).$$

As a consequence, using the Sobolev inequality of [8]: for $u \in H_b^{m_1}(\Omega_T)$ with $m_1 > \frac{d+1}{2} + \delta$, $\delta > 0$ we have

$$\|u\|_{L^\infty(\Omega_T)} \leq \mu^{-\delta} C(\gamma) \|u\|_{m_1, \mu, \gamma, \Omega_T},$$

we deduce the following boundedness properties for the iterates $(w^{\varepsilon,\nu})_{\nu \geq 0}$: there exists $\gamma_1 := \gamma_1(K, \varrho)$ and $\varepsilon_1 := \varepsilon_1(\gamma)$ such that if $\gamma \geq \gamma_1$ and if $0 < \varepsilon < \varepsilon_1$ we have that for all $\nu \geq 0$:

$$\|w^{\varepsilon,\nu}\|_{m_1, \frac{\gamma}{\varepsilon}, \gamma, \Omega_T} \leq 2\varepsilon^{M_1 - m_1} \theta(\gamma) \tag{59}$$

$$\|w^{\varepsilon,\nu}\|_{L^\infty(\Omega_T)} \leq K. \tag{60}$$

Clearly, the proof of Proposition 3.3 does not see the geometry. It is obtained by applying the tangential operators of differentiation, namely the ones involving ∂_t and ∂_j for $1 \leq j < d$, to (56), estimating the commutators and then by using the basic L^2 estimate (58). The normal derivative is then estimated in a second time and takes advantage of the fact that the problem has non characteristic boundary (see Assumption 2.2). We refer the interested reader to the proof of Proposition 8.2 in [24] or to [8].

As a corollary of the uniform estimates (59) and (60) we obtain the concluding convergence result

Proposition 3.4 One can find $\gamma_2 := \gamma_2(K, \varrho) \geq \gamma_1$ and $0 < \varepsilon_2 := \varepsilon_2(\gamma) \leq \varepsilon_1$ such that if $\gamma \geq \gamma_2$ and $0 < \varepsilon \leq \varepsilon_2$, the sequence $(w^{\varepsilon,\nu})_{\nu \in \mathbb{N}}$ converges uniformly with respect to ε in the Guès norm $\|\cdot\|_{m_1, \frac{\gamma}{\varepsilon}, \gamma, \Omega_T}$. The limit w satisfies

$$\|w\|_{m_1, \frac{\gamma}{\varepsilon}, \gamma, \Omega_T} \leq 2\varepsilon^{M_1 - m_1} \theta(\gamma),$$

so that, for the definition of $\|\cdot\|_{m_1, \frac{\gamma}{\varepsilon}, \gamma, \Omega_T}$ (see (57)) it is clear that $(w, w|_{x_d=0}, w|_{x_d=1}) \in \varepsilon^{M_1} \mathbf{B}_{\varrho', T, b}^{m_1} \times \varepsilon^{M_1} \mathbf{B}_{\varrho', T, 0, b}^{m_1} \times \varepsilon^{M_1} \mathbf{B}_{\varrho', T, 1, b}^{m_1}$, for some $\varrho' \geq \varrho$.

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