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# Analysis of a toy model for optimal crop protection

Luis Almeida\*, Aymeric Jacob de Cordemoy†, Ayman Moussa‡, Nicolas Vauchelet§

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## Abstract

In this paper we investigate an optimal control problem involving a toy model for the protection on a crop field. Precisely, we consider a protection on a crop field and we want to place intervention zones represented by a control, in order to maximise the protection on the field during a given period. Using a relaxation method, we prove that there exists a control which maximises the protection and, moreover, it must be a bang-bang control. Furthermore, with additional assumptions on the crop field geometry, some results on the shape of the optimal intervention are proved using comparison results for elliptic equations via Schwarz and Steiner symmetrizations. Finally, some numerical simulations are performed in order to illustrate those results.

**Keywords:** Optimal control, bang-bang controls, Schwarz symmetrization, Steiner symmetrization, elliptic equations, population dynamics.

**AMS Classification:** 35Q92; 49J99; 49J30; 49K20; 49Q10.

## 1 Introduction

**Motivation** *Population dynamics* is a branch of mathematical biology that studies the fluctuation over time of the number of individuals in a population of living beings. The encounter of optimal control techniques with population dynamics led to remarkable examples of applications in applied mathematics. For instance: optimal intervention strategies have been studied to control epidemics (see, e.g., [11, 22] for more general settings and [36] for an application to COVID19); in [37] the use of insecticide is optimized in the fight against arboviruses; in [2, 13, 19] the authors design release protocols in a population replacement strategy. We refer also to [31] where the author studies how migration and spatial heterogeneity of the environment affect the total population of a single and multiple species, to [32] that deals with the problem of optimising the total population size for a standard logistic-diffusive model, to [33] that investigates the optimisation of the carrying capacity in logistic diffusive models, or to [35] which focuses on how to maximize the total population of a single species with logistic growth living in a patchy environment.

In this paper, we are interested in controlling crop diseases such as *Cercospora* leaf spot, beet yellows virus, powdery mildew, etc., which are one of the main causes of yield loss in vegetable

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crops. Phytosanitary products are classically used to fight their spread, but due to their negative impact on the environment it is important to reduce their use and to find alternative strategies. For instance, in [21] the authors develop an agro-ecological approach to control aphid populations in sugar beet fields using natural predators in order to prevent the spread of viruses transmitted by these pests. Inspired by this problematic and the references cited above, we consider in this paper an optimal control problem involving a toy model for the protection on a crop field.

**Description of the optimal control problem** Let  $n \in \mathbb{N}^*$  be a positive integer and  $\Omega$  be a nonempty bounded connected open subset of  $\mathbb{R}^n$ , with a lipschitz boundary  $\partial\Omega$ . Let  $T > 0$  be the final time and consider the following toy model for the protection on a crop field given by

$$\begin{cases} p_u - D\Delta p_u = \phi(u) & \text{in } \Omega \times ]0, T[, \\ p_u = 1 & \text{on } \partial\Omega \times ]0, T[, \end{cases} \quad (1.1)$$

where  $\phi(u) : \Omega \times ]0, T[ \rightarrow \phi(u)(x, t) := e^{-\alpha(1-u(x))t} \in \mathbb{R}^+$ , for some  $u \in L^\infty(\Omega)$  such that  $0 \leq u \leq 1$  *a.e.* on  $\Omega$  and  $D > 0$  is the diffusion coefficient. This model has the following interpretation. The domain  $\Omega$  represents a crop field, the function  $\phi(u)$  represents the active ammount of a phytosanitary product and  $p_u(x, t)$  the protection due to  $\phi(u)$  at time  $t > 0$  and at a position  $x \in \Omega$  in the field. This protection is assumed to be perfect at initial time, i.e.  $p_u(x, 0) = 1$  *a.e.* on  $\Omega$ , and degrades over time with a rate denoted  $\alpha > 0$ . We want to place some intervention zones (for instance a predator reservoir) in the domain such that the protection remains as high as possible throughout the season. Then, we introduce a control variable  $u \in L^\infty(\Omega)$  such that  $0 \leq u \leq 1$  *a.e.* on  $\Omega$ , describing those interventions: in particular if  $u = 1$  the intervention is maximal and if  $u = 0$  there is no intervention. Obviously at the position where  $u = 1$  the protection does not degrade, whereas at the position  $u = 0$  the degradation of the protection is maximal. Given an amount of possible total intervention, i.e.  $\int_\Omega u \leq L$ , where  $L \in [0, |\Omega|]$ , we focus on the question of knowing where and how to intervene in order to maximise the protection during a given period of time  $[0, T]$ . Thus this leads to the following optimal control problem

$$\max_{u \in \mathcal{U}_{ad}} \mathcal{J}(u), \quad (1.2)$$

where

$$\mathcal{U}_{ad} := \left\{ u \in L^\infty(\Omega) \mid 0 \leq u \leq 1 \text{ a.e. on } \Omega, \text{ and } \int_\Omega u(x) dx \leq L \right\},$$

and where  $\mathcal{J}$  is the cost functional defined by

$$\begin{aligned} \mathcal{J} : L^\infty(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \mathcal{J}(u) := \int_0^T \int_\Omega p_u(x, t) dx dt, \end{aligned}$$

and  $p_u$  is the solution of the protection problem (1.1) for the control  $u \in \mathcal{U}_{ad}$ .

**Notations** We denote by  $L^2(\Omega), H^1(\Omega), H_0^1(\Omega)$ , the usual Lebesgue and Sobolev spaces endowed with their standard norms, by  $B_n(0, r)$  the open ball of  $\mathbb{R}^n$  centered at 0 with radius  $r > 0$ , and by  $\chi_C$  the characteristic function of  $C \subset \mathbb{R}^n$  defined by  $\chi_C(x) = 1$  if  $x \in C$ , and 0 if  $x \notin C$ . At some points of this paper, we will use the notation  $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ , if  $n \geq 2$ , where  $n_1 \in \mathbb{N}^*, n_2 \in \mathbb{N}^*$ , such that  $n_1 + n_2 = n$ . In that case  $x = (x_1, \dots, x_{n_1})$  will refer to the variables in  $\mathbb{R}^{n_1}$  and  $y = (y_1, \dots, y_{n_2})$

to the variables in  $\mathbb{R}^{n_2}$ . Finally, we denote  $\mathcal{V}_{ad} \subset \mathcal{U}_{ad}$  the set of controls that saturate the integral constraint, i.e.,

$$\mathcal{V}_{ad} := \left\{ u \in L^\infty(\Omega) \mid 0 \leq u \leq 1 \text{ a.e. on } \Omega, \text{ and } \int_{\Omega} u(x) \, dx = L \right\}, \quad (1.3)$$

and we call bang-bang control an element  $u \in \mathcal{V}_{ad}$  such that  $u \in \{0, 1\}$  a.e. on  $\Omega$  and  $\int_{\Omega} u(x) \, dx = L$ .

**Main results** Our main results are summarized in the following theorem.

**Theorem 1.1.** *There exists at least one solution to the optimal control problem (1.2), and solutions are bang-bang controls. Moreover:*

- (i) *If  $\Omega := B_n(0, r)$ , where  $r > 0$ . Then  $u^* := \chi_{B_n(0, (\frac{L}{C_n})^{\frac{1}{n}})}$  is the unique solution of the optimal control problem (1.2), where  $C_n := |B_n(0, 1)|$ ;*
- (ii) *If  $n \geq 2$ , and  $\Omega = B_{n_1}(0, r_1) \times \Omega_{n_2} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ , where  $r_1 > 0$ , and  $\Omega_{n_2}$  is a nonempty bounded connected open subset of  $\mathbb{R}^{n_2}$ , then there exists a subset  $E \subset \Omega$  satisfying  $|E| = L$ , which is symmetric with respect to the hyperplane  $x_i = 0$  and that is convex in the  $x_i$ -direction, for all  $i \in \{1, \dots, n_1\}$ , such that  $u^* = \chi_E$  is solution of the optimal control problem (1.2);*
- (iii) *If  $n \geq 2$ , and  $\Omega = B_{n_1}(0, r_1) \times B_{n_2}(0, r_2) \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ , where  $(r_1, r_2) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$ , then there exists a subset  $E \subset \Omega$  satisfying  $|E| = L$ , which is symmetric with respect to the hyperplane  $x_i = 0$  and with respect to the hyperplane  $y_j = 0$ , and that is convex in the  $x_i$ -direction, convex in the  $y_j$ -direction, for all  $(i, j) \in \{1, \dots, n_1\} \times \{1, \dots, n_2\}$ , and star-shaped at  $0 \in E$ , such that  $u^* = \chi_E$  is solution of the optimal control problem (1.2).*

**Symmetrization methods** In this paper, symmetrization methods are used in order to determine the shape of the optimal intervention for particular crop field geometry. They consist in transforming one set into another with symmetry properties. Two methods are mainly used: the *Schwarz symmetrization* and the *Steiner symmetrization*. In particular, they are used in the literature to compare solutions of partial differential equations, establishing relations between the norm of the original solutions and that of their symmetrized counterparts. The Schwarz symmetrization (see, e.g., [12, 24, 26, 27]) consists in rearranging the level sets of a function in balls of the same measure in order to create a new function that is radially symmetric and non-increasing. In particular, this method is used to prove the well-known Faber-Krahn inequality (see, e.g. [24, Chapter 3, Section 3.2]). Comparison results using Schwarz symmetrization started with the pioneering work [38] where the symmetrized problem is given by the Dirichlet Laplacian operator. This result is extended in [6, 10, 17, 30, 40] and [39, Section 9]) for second-order elliptic operators including lower-order terms. We also refer to [4, 7, 28] for Neumann and Robin boundary conditions and to [6, 9, 34, 41] for parabolic equations. The Steiner symmetrization (see, e.g., [8, 24, 26]) is a partial symmetrization which consists in symmetrizing a set with respect to a hyperplane. In particular, when applied to a level set of a function leads to a new function that is symmetric with respect to a hyperplane. In [5] a comparison result is proved for Dirichlet boundary conditions and second-order elliptic operators without lower-order terms. Generalizations can be found, for instance, in [15, 16], where the authors deal with the first and zeroth-order term. We also refer to [20, 29] for Neumann boundary conditions and to [18] for parabolic equations. We conclude this paragraph by mentioning that in [42] the authors maximize a linear cost functional which depends

on a solution of an elliptic equation on a ball, with controls acting on the divergence operator. Using Schwarz symmetrization they prove that the optimal control is the characteristic function of a ball.

**Summary of the methodology** Our methodology is based on the theory of optimal control, the relaxation methods and on comparison results for elliptic equations via symmetrization methods. Precisely, we prove that if there exists a solution to the optimal control problem (1.2), then it must saturate the integral constraint and, on the other hand, we prove that the cost functional  $\mathcal{J}$  is strictly convex by computing its twice Fréchet differential. Thus, the supremum of  $\mathcal{J}$  on  $\mathcal{U}_{ad}$  coincides with the supremum of  $\mathcal{J}$  on the bang-bang controls set. Then, using a relaxation method (see, e.g., [25, Chapter 7, Section 7.2]), we prove the existence of a solution among bang-bang controls. For particular domains, we are able to prove some results about the shape of the optimal intervention using symmetrization methods.

**Paper structure** The paper is organized as follows. In Section 2, we prove all preliminary results required to tackle Theorem 1.1. In Section 3, we focus on the proof of Theorem 1.1 and, in Section 4, additional results and remarks are presented. In Section 5, numerical simulations are performed in order to illustrate our results. Finally, some notions and results on Schwarz and Steiner symmetrizations are recalled in Appendix A.

## 2 Preliminary results

This section is devoted to the statement of some preliminary results needed to prove the main Theorem 1.1. In a nutshell, we first prove here that if solutions to the optimal control problem (1.2) exist, then they must saturate the integral constraint. Thus, our analysis is reduced to the set  $\mathcal{V}_{ad} \subset \mathcal{U}_{ad}$  which is the set of controls that saturate the integral constraint (see (1.3)). Then, we prove that the cost functional  $\mathcal{J}$  is strictly convex on  $\mathcal{V}_{ad}$  and thus its supremum on  $\mathcal{V}_{ad}$  coincides with its supremum on the set of extreme points of  $\mathcal{V}_{ad}$ , i.e. on bang-bang controls set.

Let us recall that, for all  $u \in \mathcal{U}_{ad}$ , the existence and uniqueness of the weak solution  $p_u \in L^2(]0, T[, H^1(\Omega))$  of the protection problem (1.1) are guaranteed by the Riesz representation theorem.

**Proposition 2.1.** *Let us assume that there exists  $u^* \in \mathcal{U}_{ad}$  such that  $\mathcal{J}(u^*) \geq \mathcal{J}(u)$  for all  $u \in \mathcal{U}_{ad}$ . Then,*

$$\int_{\Omega} u^*(x) dx = L.$$

*Proof.* Assume that  $\int_{\Omega} u^*(x) dx < L$ . Then there exists  $\lambda > 0$  such that  $\int_{\Omega} (u^* + \lambda) = L$ . Let us denote  $u_{\lambda} := u^* + \lambda \chi_{\{x \in \Omega \mid u^*(x) + \lambda \leq 1\}} \in \mathcal{U}_{ad}$ , then  $u_{\lambda} \geq u^*$  a.e. on  $\Omega$  and  $u_{\lambda} > u^*$  on a set of positive measure. Thus, since  $\phi(u^*) \leq \phi(u_{\lambda})$ , one deduces from the maximum principle that  $p_{u_{\lambda}} \geq p_{u^*}$  a.e. on  $\Omega \times ]0, T[$ . Hence  $\mathcal{J}(u_{\lambda}) \geq \mathcal{J}(u^*)$ . It follows that  $\mathcal{J}(u_{\lambda}) = \mathcal{J}(u^*)$ , therefore  $p_{u_{\lambda}} = p_{u^*}$  a.e. on  $\Omega \times ]0, T[$  and one deduces that  $u_{\lambda} = u^*$  a.e. on  $\Omega \times ]0, T[$  which is a contradiction.  $\square$

From Proposition 2.1, it follows that

$$\sup_{u \in \mathcal{U}_{ad}} \mathcal{J}(u) = \sup_{u \in \mathcal{V}_{ad}} \mathcal{J}(u).$$

Now let us prove that  $\mathcal{J}$  is Fréchet differentiable and twice Fréchet differentiable in  $L^{\infty}(\Omega)$ .

**Proposition 2.2.** For all  $u \in L^\infty(\Omega)$ , the cost functional  $\mathcal{J}$  is Fréchet differentiable at  $u$  and its differential  $d\mathcal{J}(u)$  is given by

$$d\mathcal{J}(u)(h) = \int_{\Omega} h(x)q(x)w \circ u(x) dx, \quad \forall h \in L^\infty(\Omega),$$

where  $q \in H_0^1(\Omega)$  denotes the adjoint state solving the Dirichlet problem

$$\begin{cases} q - D\Delta q = 1 & \text{in } \Omega, \\ q = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

and  $w \in L^\infty(\mathbb{R})$  is the switching function given by

$$w : \mathbb{R} \longrightarrow \mathbb{R}^+ \\ s \longmapsto \begin{cases} \alpha \frac{T^2}{2} & \text{if } s = 1, \\ \frac{(1+e^{-\alpha(1-s)T}(-\alpha(1-s)T-1))}{\alpha(1-s)^2} & \text{if } s \neq 1. \end{cases}$$

*Proof.* By standard computations, the map  $u \in L^\infty(\Omega) \mapsto p_u \in L^2(]0, T[, H^1(\Omega))$  is Fréchet differentiable at every  $u \in L^\infty(\Omega)$  with its differential at  $u$  given, for all  $h \in L^\infty(\Omega)$ , by  $\dot{p}_u(h) \in L^2(]0, T[, H^1(\Omega))$  solution of

$$\begin{cases} \dot{p}_u(h) - D\Delta \dot{p}_u(h) = \alpha h t e^{-\alpha(1-u)t} & \text{on } \Omega \times ]0, T[, \\ \dot{p}_u(h) = 0 & \text{on } \partial\Omega \times ]0, T[. \end{cases}$$

Then, it follows that  $\mathcal{J}$  is Fréchet differentiable at any  $u \in L^\infty(\Omega)$  with its differential at  $u$  given by

$$d\mathcal{J}(u)(h) = \int_0^T \int_{\Omega} \dot{p}_u(h)(x, t) dx dt, \quad \forall h \in L^\infty(\Omega).$$

For all  $(u, h) \in L^\infty(\Omega) \times L^\infty(\Omega)$ , let us take the adjoint state  $q \in H_0^1(\Omega)$ , solution to (2.1), as test function in the weak variational formulation of  $\dot{p}_u(h)$  and, for almost all  $t \in ]0, T[, \dot{p}_u(h)(t, \cdot) \in H_0^1(\Omega)$  as test function in the weak variational formulation of  $q$ . Then, one can deduce that

$$\int_0^T \int_{\Omega} \dot{p}_u(h)(x, t) dx dt = \int_0^T \int_{\Omega} \alpha h(x) t e^{-\alpha(1-u(x))t} q(x) dx dt.$$

We define the switching function by  $w(s) := \alpha \int_0^T t e^{-\alpha(1-s)t} dt \in \mathbb{R}^+$ , for almost all  $s \in \mathbb{R}$ , thus

$$d\mathcal{J}(u)(h) = \int_{\Omega} h(x)q(x)w \circ u(x) dx,$$

which concludes the proof.  $\square$

**Proposition 2.3.** For all  $u \in L^\infty(\Omega)$ , the cost functional  $\mathcal{J}$  is twice Fréchet differentiable at  $u$  and its second differential  $d^2\mathcal{J}(u)$  is given by

$$d^2\mathcal{J}(u)(h_1, h_2) = \int_{\Omega} \alpha^2 q(x) h_1(x) h_2(x) \left( \int_0^T t^2 e^{-\alpha(1-u(x))t} dt \right) dx, \quad \forall (h_1, h_2) \in L^\infty(\Omega) \times L^\infty(\Omega).$$

Moreover, for all  $u \in L^\infty(\Omega)$ ,  $d^2\mathcal{J}(u)$  is positive definite thus  $\mathcal{J}$  is a strictly convex function on  $L^\infty(\Omega)$ .

*Proof.* One can easily prove that the map  $u \in L^\infty(\Omega) \mapsto p_u \in L^2(]0, T[, H^1(\Omega))$  is twice Fréchet differentiable at every  $u \in L^\infty(\Omega)$  with its second differential at  $u$  given, for all  $(h_1, h_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$ , by  $\ddot{p}_u(h_1, h_2) \in L^2(]0, T[, H^1(\Omega))$  solution of

$$\begin{cases} \ddot{p}_u(h_1, h_2) - D\Delta\ddot{p}_u(h_1, h_2) = \alpha^2 h_1 h_2 t^2 e^{-\alpha(1-u)t} & \text{in } \Omega \times ]0, T[, \\ \ddot{p}_u(h_1, h_2) = 0 & \text{on } \partial\Omega \times ]0, T[. \end{cases}$$

Thus, it can be deduced that  $\mathcal{J}$  is twice Fréchet differentiable at any  $u \in L^\infty(\Omega)$  with its second differential at  $u$  given by

$$d^2\mathcal{J}(u)(h_1, h_2) = \int_0^T \int_\Omega \ddot{p}_u(h_1, h_2)(x, t) dx dt, \quad \forall (h_1, h_2) \in L^\infty(\Omega) \times L^\infty(\Omega).$$

Then, using the adjoint state  $q \in H_0^1(\Omega)$  one can prove in the same way as in Proposition 2.2, that

$$\int_0^T \int_\Omega \ddot{p}_u(h_1, h_2)(x, t) dx dt = \int_\Omega \alpha^2 q(x) h_1(x) h_2(x) \left( \int_0^T t^2 e^{-\alpha(1-u(x))t} dt \right) dx,$$

for all  $u, h_1, h_2 \in L^\infty(\Omega)$ . Finally, by considering  $h_2 = h_1$ , one gets

$$d^2\mathcal{J}(u)(h_1, h_1) = \int_\Omega \alpha^2 q(x) h_1(x)^2 \left( \int_0^T t^2 e^{-\alpha(1-u(x))t} dt \right) dx.$$

From the strong maximum principle one has  $q > 0$  *a.e.* on  $\Omega$ , thus  $d^2\mathcal{J}(u)(h_1, h_1) \geq 0$  with equality if and only if  $h_1 = 0$  *a.e.* on  $\Omega$ . Thus  $d^2\mathcal{J}(u)$  is positive definite and, since  $u \in L^\infty(\Omega)$  is arbitrary, one concludes that  $\mathcal{J}$  is a strictly convex function.  $\square$

Since  $\mathcal{J}$  is strictly convex on  $\mathcal{V}_{ad}$ , it is well known that

$$\sup_{u \in \mathcal{V}_{ad}} \mathcal{J}(u) = \sup_{u \in \text{Extr}(\mathcal{V}_{ad})} \mathcal{J}(u),$$

where  $\text{Extr}(\mathcal{V}_{ad})$  is the set of extreme points of  $\mathcal{V}_{ad}$  defined by

$$\text{Extr}(\mathcal{V}_{ad}) := \left\{ u \in \mathcal{V}_{ad} \mid v_1, v_2 \in \mathcal{V}_{ad}, u = \frac{v_1 + v_2}{2} \implies u = v_1 = v_2 \right\},$$

which corresponds to the set of bang-bang controls (see, e.g., [14]),

$$\text{Extr}(\mathcal{V}_{ad}) = \left\{ u \in L^\infty(\Omega) \mid u \in \{0, 1\} \text{ a.e. on } \Omega, \text{ and } \int_\Omega u(x) dx = L \right\}.$$

### 3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. From the previous section, we know that if there exists a control which maximizes  $J$  on the bang-bang controls set, then it is also solution to the optimal control problem (1.2). Thus, inspired by [33], let us prove that there exists a solution among bang-bang controls.

*Proof of the existence of a bang-bang solution.* Let us consider  $P_u := \int_0^T p_u(\cdot, t) dt / T \in H^1(\Omega)$ , for all  $u \in \text{Extr}(\mathcal{V}_{ad})$ . Then  $P_u$  is solution of the Dirichlet problem

$$\begin{cases} P_u - D\Delta P_u = \frac{1}{T} \int_0^T e^{-\alpha(1-u)t} dt & \text{in } \Omega, \\ P_u = 1 & \text{on } \partial\Omega. \end{cases}$$

Denote  $\varphi(u) := \int_0^T e^{-\alpha(1-u)t} dt / T \in L^\infty(\Omega)$ , for all  $u \in \text{Extr}(\mathcal{V}_{ad})$ . Therefore,

$$\varphi(u) := \begin{cases} 1 & \text{if } u = 1, \\ \frac{1-e^{-\alpha T}}{\alpha T} & \text{if } u = 0, \end{cases}$$

and

$$\int_{\Omega} \varphi(u)(x) dx = \int_{\{u=1\}} dx + \int_{\{u=0\}} \frac{1-e^{-\alpha T}}{\alpha T} dx = L + \frac{1-e^{-\alpha T}}{\alpha T} (|\Omega| - L) =: M.$$

Now consider

$$\mathcal{F} := \left\{ f \in L^\infty(\Omega) \mid f \in \left\{ \frac{1-e^{-\alpha T}}{\alpha T}, 1 \right\} \text{ a.e. on } \Omega, \text{ and } \int_{\Omega} f(x) dx = M \right\}.$$

Then  $\varphi$  is a bijective function from  $\text{Extr}(\mathcal{V}_{ad})$  to  $\mathcal{F}$  with its inverse given by

$$\varphi^{-1}(f) := \begin{cases} 1 & \text{if } f = 1, \\ 0 & \text{if } f = \frac{1-e^{-\alpha T}}{\alpha T}, \end{cases}$$

for all  $f \in \mathcal{F}$ . By [25, Proposition 7.2.17] the closed convex hull of  $\mathcal{F}$  for the weak  $L^\infty(\Omega) - *$  topology is

$$\overline{\text{conv}}^*(\mathcal{F}) := \left\{ f \in L^\infty(\Omega) \mid \frac{1-e^{-\alpha T}}{\alpha T} \leq f \leq 1 \text{ a.e. on } \Omega, \text{ and } \int_{\Omega} f(x) dx = M \right\},$$

and  $\mathcal{F} = \text{Extr}(\overline{\text{conv}}^*(\mathcal{F}))$ . Let us introduce the relaxed problem given by

$$\max_{f \in \overline{\text{conv}}^*(\mathcal{F})} J(f), \quad (3.1)$$

where  $J : f \in \overline{\text{conv}}^*(\mathcal{F}) \mapsto \int_{\Omega} Z_f(x) dx \in \mathbb{R}$ , and where  $Z_f \in H_0^1(\Omega)$  is the unique solution of

$$\begin{cases} Z_f - D\Delta Z_f = f & \text{in } \Omega, \\ Z_f = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

Since  $J$  is linear and continuous on  $\overline{\text{conv}}^*(\mathcal{F})$  for the weak  $L^\infty(\Omega) - *$  topology, there exists  $f^* \in \mathcal{F}$  such that  $J(f^*) \geq J(f)$ , for all  $f \in \overline{\text{conv}}^*(\mathcal{F})$  (see, e.g., [25, Chapter 7, Section 7.2 Remark p.293]). Moreover, one has  $P_u = Z_{\varphi(u)} + W$ , for all  $u \in \text{Extr}(\mathcal{V}_{ad})$ , where  $W \in H^1(\Omega)$  is the unique solution of

$$\begin{cases} W - D\Delta W = 0 & \text{in } \Omega, \\ W = 1 & \text{on } \partial\Omega. \end{cases} \quad (3.3)$$

Therefore, by considering  $u^* := \varphi^{-1}(f^*) \in \text{Extr}(\mathcal{V}_{ad})$ , it follows that

$$\begin{aligned} \int_{\Omega} P_{u^*}(x) dx &= \int_{\Omega} Z_{\varphi(u^*)}(x) dx + \int_{\Omega} W(x) dx \\ &\geq \int_{\Omega} Z_{\varphi(u)}(x) dx + \int_{\Omega} W(x) dx = \int_{\Omega} P_u(x) dx, \quad \forall u \in \text{Extr}(\mathcal{V}_{ad}). \end{aligned}$$

Thus, by definition of  $P_u$ , one gets

$$\int_0^T \int_{\Omega} p_{u^*}(x, t) dx dt \geq \int_0^T \int_{\Omega} p_u(x, t) dx dt, \quad \forall u \in \text{Extr}(\mathcal{V}_{ad}),$$

i.e.,

$$\mathcal{J}(u^*) = \max_{u \in \text{Extr}(\mathcal{V}_{ad})} \mathcal{J}(u),$$

which concludes the proof.  $\square$



**Remark 3.1.** Note that a nonhomogeneous Dirichlet condition in problem (3.2) could be considered in order to avoid introducing  $W \in H^1(\Omega)$  solution to problem (3.3). Nevertheless, this homogeneous Dirichlet condition will be required to prove items (i),(ii) and (iii) of Theorem 1.1.

Now let us prove item (i) using the Schwarz symmetrization which is recalled in Appendix A.1.

*Proof of item (i).* In the proof of the existence of a bang-bang solution, we deduced that there exists  $u^* \in \text{Extr}(\mathcal{V}_{ad})$  satisfying  $\mathcal{J}(u^*) \geq \mathcal{J}(u)$  for all  $u \in \text{Extr}(\mathcal{V}_{ad})$ . Let  $C \subset \Omega$ ,  $|C| = L$  such that  $u^* = \chi_C$ . By denoting  $f^* = \varphi(\chi_C) \in \mathcal{F}$ , it follows that

$$f^* = \chi_C + \frac{(1 - e^{-\alpha T})}{\alpha T} \chi_{\Omega \setminus C},$$

and

$$\int_{\Omega} Z_f(x) dx \leq \int_{\Omega} Z_{f^*}(x) dx, \quad \forall f \in \mathcal{F},$$

where  $Z_f \in H_0^1(\Omega)$  is the unique solution of problem (3.2) for the control  $f \in \mathcal{F}$ . Then, using the Schwarz symmetrization comparison theorem (see Theorem A.4), one deduces that,

$$\int_{\Omega} Z_{f^*}(x) dx \leq \int_{\Omega^{\#}} v(x) dx, \quad (3.4)$$

where  $v \in H_0^1(\Omega)$  is the unique solution of

$$\begin{cases} v - D\Delta v = f^{*\#} & \text{in } \Omega^{\#}, \\ v = 0 & \text{on } \partial\Omega^{\#}, \end{cases}$$

and where  $\Omega^{\#}$  is the symmetric rearrangement of  $\Omega$  and  $f^{*\#} = \chi_{C^{\#}} + ((1 - e^{-\alpha T})/\alpha T)\chi_{\Omega^{\#} \setminus C^{\#}}$  is the Schwarz symmetrization of  $f$  (see Example A.3). Moreover, since  $\Omega = B_n(0, r)$ , then  $\Omega^{\#} = \Omega$ ,  $f^{*\#} \in \mathcal{F}$  and  $v = Z_{f^{*\#}}$ . Thus, by Inequality (3.4), one has,

$$\int_{\Omega} Z_{f^*}(x) dx = \int_{\Omega} Z_{f^{*\#}}(x) dx \geq \int_{\Omega} Z_f(x) dx, \quad \forall f \in \mathcal{F}.$$

By the equality case in Theorem A.4, it follows that  $Z_{f^*} = Z_{f^{*\#}}$  and  $f^* = f^{*\#}$ . Thus  $C = C^{\#} = B_n\left(0, \left(\frac{L}{C_n}\right)^{\frac{1}{n}}\right)$  and then  $u^* = \chi_{B_n\left(0, \left(\frac{L}{C_n}\right)^{\frac{1}{n}}\right)}$ . Since it is true for any  $u^*$  solution to  $\max_{u \in \text{Extr}(\mathcal{V}_{ad})} \mathcal{J}(u)$ , ones concludes that  $\chi_{B_n\left(0, \left(\frac{L}{C_n}\right)^{\frac{1}{n}}\right)}$  is the unique solution of the optimal control problem (1.2).  $\square$

To prove items (ii) and (iii) some properties and results on the Steiner symmetrization are used, and we refer to Appendix A.2, for more details.

*Proof of item (ii).* The proof is close to the one of item (i). One knows that there exists  $C \subset \Omega$ ,  $|C| = L$  such that

$$f^* = \chi_C + \frac{(1 - e^{-\alpha T})}{\alpha T} \chi_{\Omega \setminus C},$$

and  $f^*$  is a solution of the relaxed problem (3.1). From Theorem A.11, it follows that,

$$\int_{B_{n_1}(0, r_1)} Z_{f^*}(x, y) dx \leq \int_{B_{n_1}(0, r_1)} v(x, y) dx,$$

for almost all  $y \in \Omega_{n_2}$ , where  $Z_{f^*} \in H_0^1(\Omega)$  is the unique solution of Problem (3.2) for the control  $f^*$ , and  $v \in H_0^1(S_{(n_1, \cdot)}(\Omega))$  is the unique solution of

$$\begin{cases} v - D\Delta v = S_{(n_1, \cdot)}(f^*) & \text{in } S_{(n_1, \cdot)}(\Omega), \\ v = 0 & \text{on } \partial S_{(n_1, \cdot)}(\Omega), \end{cases}$$

where  $S_{(n_1, \cdot)}(\Omega) = B_{n_1}(0, r_1) \times \Omega_{n_2}$  is the  $(n_1, \cdot)$ -Steiner symmetrization of  $\Omega$  (see Example A.6) and where

$$S_{(n_1, \cdot)}(f^*) = \chi_{S_{(n_1, \cdot)}(C)} + ((1 - e^{-\alpha T})/\alpha T)\chi_{S_{(n_1, \cdot)}(\Omega) \setminus S_{(n_1, \cdot)}(C)},$$

is the  $(n_1, \cdot)$ -Steiner symmetrization of  $f^*$  (see Example A.10). Furthermore,  $|S_{(n_1, \cdot)}(C)| = |C|$  and  $S_{(n_1, \cdot)}(\Omega) = \Omega$ , thus  $S_{(n_1, \cdot)}(f^*) \in \mathcal{F}$  and  $v = Z_{S_{(n_1, \cdot)}(f^*)}$ . Hence,

$$\int_{\Omega_{n_2}} \int_{B_{n_1}(0, r_1)} Z_{f^*}(x, y) dx dy = \int_{\Omega_{n_2}} \int_{B_{n_1}(0, r_1)} Z_{S_{(n_1, \cdot)}(f^*)}(x, y) dx dy,$$

and one concludes that  $u^* = \chi_{S_{(n_1, \cdot)}(C)} \in \text{Extr}(\mathcal{V}_{ad})$  is a solution of the optimal control problem (1.2). The symmetry of  $S_{(n_1, \cdot)}(C)$  with respect to the hyperplane  $x_i = 0$  and the convexity in the  $x_i$ -direction for all  $i \in \{1, \dots, n_1\}$ , follow from Proposition A.7.  $\square$

*Proof of item (iii).* In the proof of item (ii), one constructs

$$S_{(n_1, \cdot)}(f^*) = \chi_{S_{(n_1, \cdot)}(C)} + \frac{1 - e^{-\alpha T}}{\alpha T} \chi_{\Omega \setminus S_{(n_1, \cdot)}(C)},$$

solution of the relaxed problem (3.1). Thus, by considering  $S_{(n_2, \cdot)}(\Phi(\Omega))$  and  $Z_{S_{(n_1, \cdot)}(f^*)} \circ \Phi^{-1}$ , where  $\Phi : (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mapsto (y, x) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_1}$ , it follows from Theorem A.11 that

$$\int_{B_{n_2}(0, r_2)} Z_{S_{(n_1, \cdot)}(f^*)}(x, y) dy \leq \int_{B_{n_2}(0, r_2)} v(x, y) dy,$$

for almost all  $x \in B_{n_1}(0, r_1)$ , where  $v \in H^1(S_{(\cdot, n_2)}(\Omega))$  is the unique solution of

$$\begin{cases} v - D\Delta v = S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(f^*)) & \text{in } S_{(\cdot, n_2)}(\Omega), \\ v = 0 & \text{on } \partial S_{(\cdot, n_2)}(\Omega), \end{cases}$$

where  $S_{(\cdot, n_2)}(\Omega) := \Phi^{-1}(S_{(n_2, \cdot)}(\Phi(\Omega))) = B_{n_1}(0, r_1) \times B_{n_2}(0, r_2)$  is the  $(\cdot, n_2)$ -Steiner symmetrization of  $\Omega$ , and  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(f^*)) := S_{(n_2, \cdot)}(S_{(n_1, \cdot)}(f^*) \circ \Phi^{-1}) \circ \Phi$  is the  $(\cdot, n_2)$ -Steiner symmetrization of  $S_{(n_1, \cdot)}(f^*)$ . One deduces that

$$S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(f^*)) = \chi_{S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(C))} + \frac{1 - e^{-\alpha T}}{\alpha T} \chi_{S_{(\cdot, n_2)}(\Omega) \setminus S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(C))}.$$

From Proposition A.7, one has  $|S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(C))| = |S_{(n_1, \cdot)}(C)| = |C|$  and, since  $S_{(\cdot, n_2)}(\Omega) = \Omega$ , then  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(f^*)) \in \mathcal{F}$  and  $v = Z_{S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(f^*))}$ . Thus, one can conclude that  $u^* = \chi_{S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(C))} \in \text{Extr}(\mathcal{V}_{ad})$  is a solution of the optimal control problem (1.2). From Proposition A.8 it follows that  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(C))$  is symmetric with respect to the hyperplane  $x_i = 0$  and with respect to the hyperplane  $y_j = 0$ , and it is a convex set in the  $x_i$ -direction and in the  $y_j$ -direction, for all  $(i, j) \in \{1, \dots, n_1\} \times \{1, \dots, n_2\}$ , and also star-shaped at  $0 \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(C))$ .  $\square$

**Remark 3.2.** To the best of our knowledge, we do not know if the optimal control obtained in item (ii) and (iii) of Theorem 1.1 is unique. Indeed, unlike the comparison result using Schwarz symmetrization (Theorem A.4) we have not found any results on the equality case occurring in Theorem A.11. Note that in the particular case where the second-order elliptic operator does not contain a zero order term, the equality case is mentioned in [5], therefore we can legitimately wonder whether this result can be extended to our case which would be an interesting topic for further work.

## 4 Additional results and remarks

In this section some additional results and remarks are given. Among domains with the same measure, it is possible to determine on which domain and for which bang-bang control the cost  $\mathcal{J}$  is the lowest.

**Proposition 4.1.** *Let  $A$  be the annulus  $A := \Omega^\# \setminus B_n(0, (\frac{|\Omega| - L}{C_n})^{\frac{1}{n}})$ . Then,*

$$\inf_{u \in \text{Extr}(\mathcal{V}_{ad})} \mathcal{J}(u) \geq \int_0^T \int_{\Omega^\#} p_{\chi_A}^{\Omega^\#}(x, t) dx dt,$$

where  $p_{\chi_A}^{\Omega^\#} \in L^2(]0, T[, H^1(\Omega^\#))$  is the solution of the protection problem (1.1) defined on  $\Omega^\# \times ]0, T[$  for the control  $u = \chi_A$ .

*Proof.* Let  $u \in \text{Extr}(\mathcal{V}_{ad})$  and  $W_u := 1 - \int_0^T p_u/T \in H^1(\Omega)$  which is the unique solution of

$$\begin{cases} W_u - D\Delta W_u = 1 - \frac{1}{T} \int_0^T e^{-\alpha(1-u)t} dt & \text{in } \Omega, \\ W_u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $u \in \text{Extr}(\mathcal{V}_{ad})$ , there exists  $C \subset \Omega$ ,  $|C| = L$  such that

$$1 - \frac{1}{T} \int_0^T e^{-\alpha(1-u)t} dt = \left(1 - \frac{1 - e^{-\alpha T}}{\alpha T}\right) \chi_{\Omega \setminus C},$$

*a.e.* on  $\Omega$ . Therefore, by Theorem A.4, one deduces that  $\int_{\Omega} W_u(x) dx \leq \int_{\Omega^\#} v(x) dx$ , where  $v \in H_0^1(\Omega^\#)$  is the unique solution of

$$\begin{cases} v - D\Delta v = \left(1 - \frac{1 - e^{-\alpha T}}{\alpha T}\right) \chi_{(\Omega \setminus C)^\#} & \text{in } \Omega^\#, \\ v = 0 & \text{on } \partial\Omega^\#, \end{cases}$$

*i.e.*,

$$\int_{\Omega} dx - \frac{1}{T} \int_{\Omega} \int_0^T p_u(x, t) dx dt \leq \int_{\Omega^\#} dx - \int_{\Omega^\#} v_1(x) dx,$$

where  $v_1 \in H^1(\Omega^\#)$  is the unique solution of

$$\begin{cases} v_1 - D\Delta v_1 = \frac{1 - e^{-\alpha T}}{\alpha T} \chi_{(\Omega \setminus C)^\#} + \chi_{\Omega^\# \setminus (\Omega \setminus C)^\#} & \text{in } \Omega^\#, \\ v_1 = 1 & \text{on } \partial\Omega^\#, \end{cases}$$

and where  $\Omega^\# \setminus (\Omega \setminus C)^\# = \Omega^\# \setminus B_n(0, (\frac{|\Omega| - L}{C_n})^{\frac{1}{n}}) =: A$  since  $|\Omega \setminus C| = |\Omega| - L$ . Moreover,

$$\frac{1 - e^{-\alpha T}}{\alpha T} \chi_{(\Omega \setminus C)^\#} + \chi_{\Omega^\# \setminus (\Omega \setminus C)^\#} = \frac{1}{T} \int_0^T e^{-\alpha(1-\chi_A)t} dt,$$

*a.e.* on  $\Omega^\#$ . Thus,  $v_1 = \int_0^T p_{\chi_A}^{\Omega^\#}/T \in H^1(\Omega^\#)$  which concludes the proof.  $\square$

**Remark 4.2.** Note that, if we consider a homogeneous Dirichlet boundary condition in the protection problem (1.1), one can determine similarly to Proposition 4.1, on which domain and for which control the cost  $\mathcal{J}$  is the highest, i.e.,

$$\max_{u \in \mathcal{U}_{ad}} \mathcal{J}(u) \leq \int_0^T \int_{\Omega^\#} p_{\chi_{B_n}^{\Omega^\#} \left(0, \left(\frac{t}{c_n}\right)^{\frac{1}{n}}\right)} dx dt,$$

where  $p_{\chi_{B_n}^{\Omega^\#} \left(0, \left(\frac{t}{c_n}\right)^{\frac{1}{n}}\right)} \in L^2(]0, T[, H_0^1(\Omega^\#))$  is the solution of the protection problem (1.1) defined on  $\Omega^\# \times ]0, T[$  with a homogeneous Dirichlet boundary condition and for the control  $u = \chi_{B_n}^{\Omega^\#} \left(0, \left(\frac{t}{c_n}\right)^{\frac{1}{n}}\right)$  (see [42] for a similar result on an optimal control problem where the control acts on the divergence operator).

**Remark 4.3.** If one does not restrict Proposition 4.1 to bang-bang controls, then one can prove in the same way that,

$$\min_{u \in \mathcal{U}_{ad}} \mathcal{J}(u) = \mathcal{J}(0) \geq \int_0^T \int_{\Omega^\#} p_0^{\Omega^\#}(x, t) dx dt,$$

where  $p_0^{\Omega^\#}$  is the solution of the protection problem (1.1) defined on  $\Omega^\# \times ]0, T[$  for the null control  $u = 0$  a.e. on  $\Omega$ .

We conclude this section with the following remark.

**Remark 4.4.** By integrating on  $\Omega \times ]0, T[$  the protection problem (1.1) and using Green's formula, we obtain

$$\int_0^T \int_{\Omega} p_u(x, t) dx dt = \int_0^T \int_{\Omega} e^{-\alpha(1-u(x))t} dx dt + D \int_0^T \int_{\partial\Omega} \partial_n p_u(x, t) dx dt,$$

for all  $u \in \text{Extr}(\mathcal{V}_{ad})$ . Since

$$\int_0^T \int_{\Omega} e^{-\alpha(1-u(x))t} dx dt = T \int_{\{u=1\}} dx + \int_0^T \int_{\{u=0\}} e^{-\alpha t} dx dt = TL + (|\Omega| - L) \frac{1 - e^{-\alpha T}}{\alpha},$$

for all  $u \in \text{Extr}(\mathcal{V}_{ad})$ , then the optimal control problem (1.2) is equivalent to maximizing the functional  $\int_0^T \int_{\partial\Omega} \partial_n p_u(x, t) dx dt$  on  $\text{Extr}(\mathcal{V}_{ad})$ . In particular, if the Dirichlet boundary condition is replaced by a Neumann boundary condition in the protection problem (1.1), then all bang-bang controls are solutions of the optimal control problem (1.2). Indeed, if  $\partial_n p_u = g$  on  $\partial\Omega \times ]0, T[$ , where  $g \in L^2(]0, T[, H^{-1/2}(\Omega))$ , then we can still prove that  $\mathcal{J}$  attains its maximum on the set of bang-bang controls and, moreover,

$$\int_0^T \int_{\Omega} p_u(x, t) dx dt = C, \quad \forall u \in \text{Extr}(\mathcal{V}_{ad}),$$

where  $C > 0$  is a constant independant of  $u \in \text{Extr}(\mathcal{V}_{ad})$ .

## 5 Numerical simulations

In this section we numerically solve an example of the optimal control problem (1.2) in the two-dimensional case  $n = 2$ , in order to illustrate Theorem 1.1. The numerical simulations have been performed using Freefem++ software [23] with P1-finite elements and standard affine mesh.

## 5.1 Numerical methodology

Starting from an initial data  $u_0 \in \mathcal{U}_{ad}$ , note that Proposition 2.2 allows to obtain an ascent direction  $h_d(u_0)$  of the cost functional  $\mathcal{J}$  at  $u_0$ , given by  $h_d(u_0) := qw \circ u_0 \in L^\infty(\Omega)$ , since it satisfies  $d\mathcal{J}(u_0)(h_d(u_0)) = \|qw \circ u_0\|_{L^2(\Omega)}^2 \geq 0$ . To deal with the inequality constraint  $\int_\Omega u(x)dx \leq L$ , the Uzawa algorithm (see, e.g., [1, Chapter 3 p.64]) is used, while the bound constraints are dealt with the projected gradient. In a nutshell, we consider the augmented functional  $\mathcal{J} - \lambda_0 F$ , where  $F : u \in L^\infty(\Omega) \mapsto \int_\Omega u - L \in \mathbb{R}$  and  $\lambda_0 \in \mathbb{R}^+$  is an initial Lagrange multiplier. Since the Fréchet differential of  $F$  at  $u_0$  is the map  $h \in L^\infty(\Omega) \mapsto \lambda_0 \int_\Omega h \in \mathbb{R}$ , one can obtain an ascent direction of the augmented functional at  $u_0$  given by  $h_d(u_0) - \lambda_0$ . Then the new control is given by

$$u_1 = \text{proj}_{\mathcal{V}}(u_0 + \eta(h_d(u_0) - \lambda_0)),$$

where  $\eta > 0$  is a fixed parameter and  $\text{proj}_{\mathcal{V}}$  is the projection operator onto  $\mathcal{V} := \{u \in L^\infty(\Omega) \mid 0 \leq u \leq 1 \text{ a.e. on } \Omega\}$  considered in  $L^2(\Omega)$ . Finally, the Lagrange multiplier is updated as follows

$$\lambda_1 := \lambda_0 + \mu F(u_1),$$

where  $\mu > 0$  is a fixed parameter, and the algorithm restarts with  $u_1$  and  $\lambda_1$ , and so on. To conclude, note that the algorithm stops when, for all  $i \in \mathbb{N}^*$ , the difference between the cost functional  $\mathcal{J}$  at the iteration  $20 \times i$  and at the iteration  $20 \times (i - 1)$  is small enough.

## 5.2 Numerical results

For numerical simulations, we consider a diffusion coefficient  $D = 0.01$  and a death rate  $\alpha = 1$ . In the following, the domains considered have the same measure  $|\Omega| = 4$  and the maximal surface of the intervention zones is  $L = |\Omega|/4 = 0.4$ .

In Figure 1 (resp. Figure 2), the numerical simulation is performed on the domain  $\Omega_1 = ]-1, 1[ \times ]0, -1[ \cup ]0, 1[ \times ]0, 2[$  (resp.  $\Omega_2 = ]-3c, 3c[ \times ]-c, c[ \cup ]-c, c[ \times ]c, 3c[ \cup ]-c, c[ \times ]-3c, -c[$  for  $c = 1/\sqrt{5}$ ). Figure (1a) (resp. (2a)) shows the optimal control, (1b) (resp. (2b)) the optimal protection at the final time  $T$ . One observes that the optimal control takes exclusively the two values 0 (in orange) and 1 (in red) in the domain and thus it is a bang-bang control which is in accordance with Theorem 1.1. Furthermore, one sees that the zone of intervention is connected, and in the case of the symmetric domain  $\Omega_2$ , it is also symmetric. (1c) (resp. (2c)) shows the evolution of the value of  $\mathcal{J}$  and (1d) (resp. (2d)) the surface of the intervention zone with respect to the iterations. We observe that  $\mathcal{J}$  seems to converge with some oscillations due to the Lagrange multiplier in order to satisfy the surface constraint.

In Figure 3, a circular domain is considered  $\Omega_3 = B_2(0, 2/\sqrt{\pi})$ . As in Figure 1 and Figure 2, one observes that the optimal control is bang-bang. The zone of intervention is a disk concentrated in the center of the domain with a surface close to 0.4 as stated in item (i) of Theorem 1.1.

In Figure 4, we consider a rectangular domain  $\Omega_4 = ]-2, 2[ \times ]-\frac{1}{2}, \frac{1}{2}[$ . As in previous figures, the optimal control is bang-bang. Moreover the zone of intervention is symmetric with respect to the axis  $(Ox)$  and  $(Oy)$ , convex in the  $x$ -direction and in the  $y$ -direction and also star-shaped at 0 as expected from item (iii) of Theorem 1.1.

In Figure 5, in (5a) are summarized the values of the cost functional at the optimal control on their respective domains. The circular domain and the annular intervention zone described in Proposition 4.1 are computed in (5b), with the value of  $\mathcal{J}$  who is lower than in the other numerical simulations, as claimed.

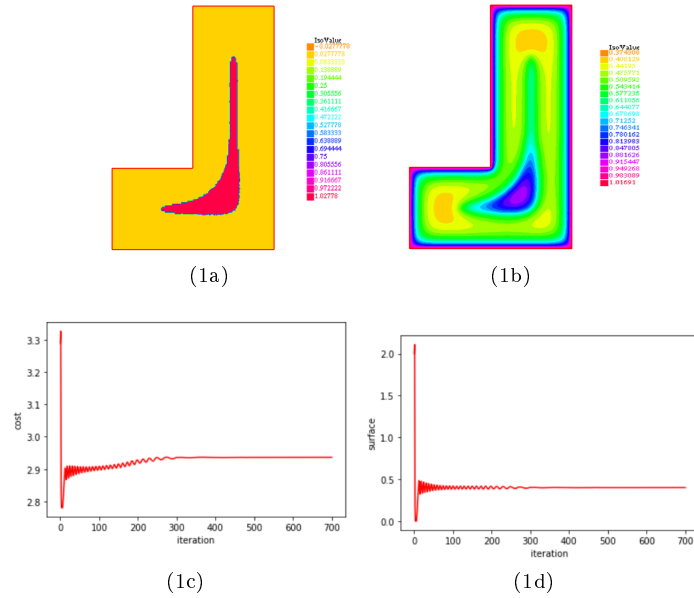


Figure 1: (1a) The optimal control on  $\Omega_1$ . (1b) The protection for the optimal control at the final time  $T$ . The values of the cost functional (1c) and the surface of the intervention zone (1d) with respect to the iterations.

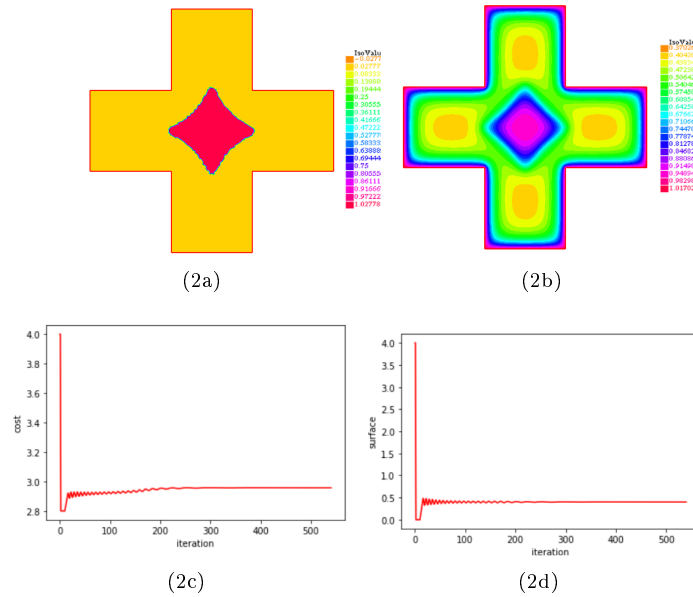
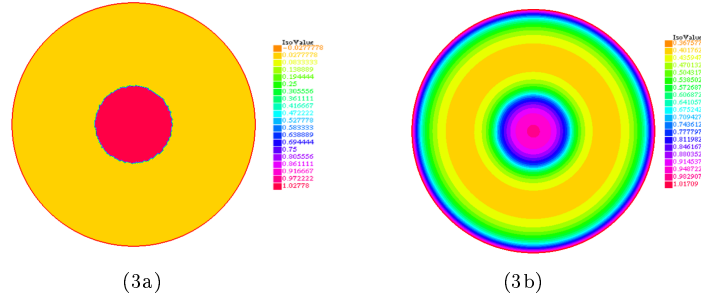
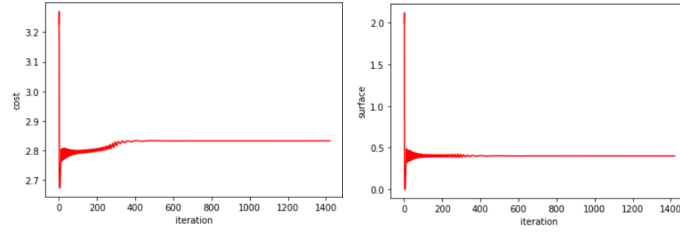


Figure 2: (2a) The optimal control on  $\Omega_2$ . (2b) The protection for the optimal control at the final time  $T$ . The values of the cost functional (2c) and the surface of the intervention zone (2d) with respect to the iterations.



(3a)

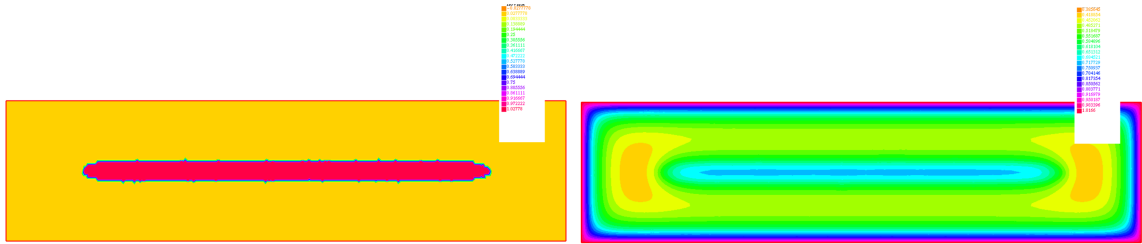
(3b)



(3c)

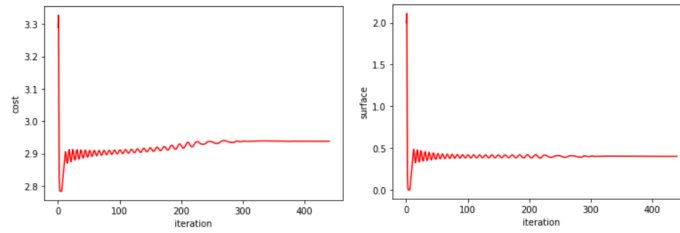
(3d)

Figure 3: (3a) The optimal control on the circular domain  $\Omega_3$ . (3b) The protection for the optimal control at the final time  $T$ . The values of the cost functional (3c) and the surface of the intervention zone (3d) with respect to the iterations.



(4a)

(4b)



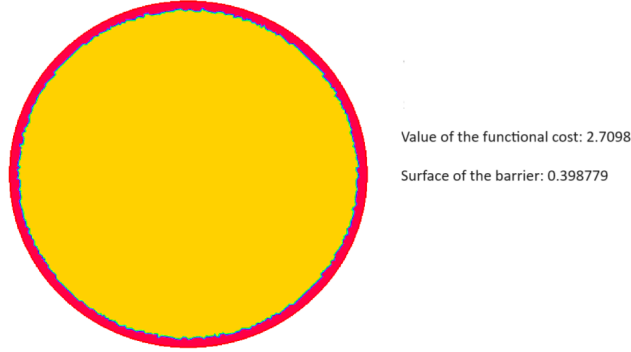
(4c)

(4d)

Figure 4: (4a) The optimal control on the rectangular domain  $\Omega_4$ . (4b) The protection for the optimal control at the final time  $T$ . The values of the cost functional (4c) and the surface of the intervention zone (4d) with respect to the iterations.

The domain	$\Omega_1$	$\Omega_2$	$\Omega_3$	$\Omega_4$
value of $\mathcal{J}$ at the optimal control	2.93643	2.95918	2.83285	2.93773
Surface of the intervention zone	0.400416	0.399502	0.399777	0.399541

(5a)



(5b)

Figure 5: (5a) The value of  $\mathcal{J}$  at the optimal control and the surface of the intervention zone on the different domains considered. (5b) A circular domain with an annular intervention zone and the value of  $\mathcal{J}$  for this control.

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## A Notions and results on Schwarz and Steiner symmetrizations

The aim of this section is to recall and to prove some results on the Schwarz and Steiner symmetrizations. For more details we refer to standard references such as [8, 12, 24, 26, 27].

### A.1 Schwarz symmetrization

**Definition A.1** (Symmetric rearrangement). *Let  $A$  be a measurable subset of  $\mathbb{R}^n$  such that  $|A| < \infty$ . The symmetric rearrangement of  $A$  is the subset  $A^\# \subset \mathbb{R}^n$  defined by*

$$A^\# := B_n \left( 0, \left( \frac{|A|}{C_n} \right)^{\frac{1}{n}} \right),$$

where  $C_n := |B_n(0, 1)|$ .

**Definition A.2** (Schwarz symmetrization). *Let  $A$  be a measurable subset of  $\mathbb{R}^n$  such that  $|A| < \infty$ , and let  $f : A \rightarrow \mathbb{R}$  be a nonnegative measurable function. The Schwarz symmetrization of  $f$  is the function  $f^\#$  defined by*

$$f^\#(x) := \int_0^{+\infty} \chi_{\{f>t\}^\#}(x) dt,$$



for all  $x \in A^\#$ .

**Example A.3.** Let  $\Omega$  be a nonempty bounded open subset  $\mathbb{R}^n$  and let  $A$  be a measurable subset of  $\Omega$ . Consider the map  $f : x \in \Omega \mapsto \alpha\chi_A(x) + \beta\chi_{\Omega \setminus A}(x) \in \mathbb{R}^+$ , where  $\alpha, \beta$  are constants such that  $\alpha \geq \beta \geq 0$ , and  $\chi_A$  (resp.  $\chi_{\Omega \setminus A}$ ) is the characteristic function of  $A$  (resp. of  $\Omega \setminus A$ ). Then the Schwarz symmetrization of  $f$  is  $f^\# : x \in \Omega^\# \mapsto \alpha\chi_{A^\#}(x) + \beta\chi_{\Omega^\# \setminus A^\#}(x)$ .

The next comparison theorem for elliptic equations is proved in [6, 17, 40] or [39, Section 9]), and it is one of the key points to derive item (i) of Theorem 1.1. The proof for the necessary and sufficient conditions of the equality case can be found in [3, Theorem 4].

**Theorem A.4.** Let us consider  $\Omega$  a nonempty bounded connected open subset of  $\mathbb{R}^n$ , with a Lipschitz boundary  $\partial\Omega$ ,  $f \in L^2(\Omega)$  a nonnegative function,  $D > 0$  a constant,  $Z_f \in H_0^1(\Omega)$  the unique solution of the Dirichlet problem,

$$\begin{cases} Z_f - D\Delta Z_f = f & \text{in } \Omega, \\ Z_f = 0 & \text{on } \partial\Omega, \end{cases}$$

and  $v \in H_0^1(\Omega^\#)$  the unique solution of the Dirichlet problem

$$\begin{cases} v - D\Delta v = f^\# & \text{in } \Omega^\#, \\ v = 0 & \text{on } \partial\Omega^\#, \end{cases}$$

where  $\Omega^\#$  is the symmetric rearrangement of  $\Omega$  and  $f^\#$  is the Schwarz symmetrization of  $f$ . Then,

$$\int_{\Omega} Z_f(x) dx \leq \int_{\Omega^\#} v(x) dx,$$

with equality if and only if  $\Omega = x_0 + \Omega^\#$ ,  $Z_f(\cdot + x_0) = v$ ,  $f(\cdot + x_0) = f^\#$  for some  $x_0 \in \mathbb{R}^n$ .

## A.2 Steiner symmetrization

In this subsection, let us assume that  $n \geq 2$  and let us denote  $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ , where  $n_1 \in \mathbb{N}^*$ ,  $n_2 \in \mathbb{N}^*$ , such that  $n_1 + n_2 = n$ . We refer to the variables in  $\mathbb{R}^{n_1}$  as  $x = (x_1, \dots, x_{n_1})$ , and to the variables in  $\mathbb{R}^{n_2}$  as  $y = (y_1, \dots, y_{n_2})$ .

**Definition A.5** (Steiner symmetrization of sets). Let  $A \subset \mathbb{R}^n$  be a measurable set such that  $|A| < \infty$ . The  $(n_1, \cdot)$ -Steiner symmetrization of  $A$  is defined by

$$S_{(n_1, \cdot)}(A) := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid x \in A(y)^\#, y \in \mathbb{R}^{n_2}\},$$

where, for all  $y \in \mathbb{R}^{n_2}$ ,  $A(y) = \{x \in \mathbb{R}^{n_1} \mid (x, y) \in A\}$  is the  $y$ -slice of  $A$ , and  $A(y)^\#$  is the symmetric rearrangement of  $A(y)$ . One defines the  $(\cdot, n_2)$ -Steiner symmetrization of  $A$  as  $S_{(\cdot, n_2)}(A) := \Phi^{-1}(S_{(n_2, \cdot)}(\Phi(A)))$  where  $\Phi : (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mapsto (y, x) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_1}$ .

The next example is involved in item (ii) and item (iii) of Theorem 1.1.

**Example A.6.** Let  $A = A_{n_1} \times A_{n_2} \subset \mathbb{R}^n$  such that  $A_{n_1} \subset \mathbb{R}^{n_1}$  (resp.  $A_{n_2} \subset \mathbb{R}^{n_2}$ ) is a measurable set of  $\mathbb{R}^{n_1}$  (resp. of  $\mathbb{R}^{n_2}$ ) with a finite measure. Then the  $(n_1, \cdot)$ -Steiner symmetrization of  $A$  is given by  $S_{(n_1, \cdot)}(A) = A_{n_1}^\# \times A_{n_2}$ , where  $A_{n_1}^\#$  is the symmetric rearrangement of  $A_{n_1}$ .

From the Steiner symmetrization definition, one deduces the following proposition.

**Proposition A.7.** Let  $A \subset \mathbb{R}^n$  be a measurable set such that  $|A| < \infty$ . Then:

- 1)  $|S_{(n_1, \cdot)}(A)| = |S_{(\cdot, n_2)}(A)| = |A|$ ;
- 2)  $S_{(n_1, \cdot)}(A)$  (resp.  $S_{(\cdot, n_2)}(A)$ ) is symmetric with respect to the hyperplane  $x_i = 0$  for all  $i \in \{1, \dots, n_1\}$  (resp. with respect to the hyperplane  $y_j = 0$  for all  $j \in \{1, \dots, n_2\}$ );
- 3)  $S_{(n_1, \cdot)}(A)$  (resp.  $S_{(\cdot, n_2)}(A)$ ) is convex in the  $x_i$ -direction for all  $i \in \{1, \dots, n_1\}$  (resp. in the  $y_j$ -direction for all  $j \in \{1, \dots, n_2\}$ ).

**Proposition A.8.** Let  $A \subset \mathbb{R}^n$  be a measurable set such that  $|A| < \infty$ . Then the  $(\cdot, n_2)$ -Steiner symmetrization of  $S_{(n_1, \cdot)}(A)$ , i.e.  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$ , is:

- A) symmetric in the hyperplane  $x_i = 0$  for all  $i \in \{1, \dots, n_1\}$  and in the hyperplane  $y_j = 0$  for all  $j \in \{1, \dots, n_2\}$ ;
- B) convex in the  $x_i$ -direction for all  $i \in \{1, \dots, n_1\}$  and convex in the  $y_j$ -direction for all  $j \in \{1, \dots, n_2\}$ ;
- C) star-shaped at  $0 \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$ .

*Proof.* A) can be easily proved using item 2) of Proposition A.7.

B) By item 3) of Proposition A.7, the set  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$  is convex in the  $y_j$ -direction for all  $j \in \{1, \dots, n_2\}$ . Let  $i \in \{1, \dots, n_1\}$  be fixed and consider  $(x', y), (x'', y) \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$  where  $x' = (x'_k)_{k \in \{1, \dots, n_1\}}$  and  $x'' = (x''_k)_{k \in \{1, \dots, n_1\}}$ , such that  $x'_k = x''_k$ , for all  $k \in \{1, \dots, n_1\} \setminus \{i\}$ . Then  $y \in S_{(n_1, \cdot)}(A)(x')^\#$  and  $y \in S_{(n_1, \cdot)}(A)(x'')^\#$ . Without loss of generality, we can assume that  $|x''_i| \leq |x'_i|$ , then since  $S_{(n_1, \cdot)}(A)$  is symmetric with respect to the hyperplane  $x_i = 0$  and convex in the  $x_i$ -direction, then

$$S_{(n_1, \cdot)}(A)(x') \subset S_{(n_1, \cdot)}(A)(x'_1, \dots, x'_{i-1}, z_i, x'_{i+1}, \dots, x'_n),$$

for all  $z_i \in [-x'_i, x'_i]$ . In particular, since  $|x''_i| \leq |x'_i|$ , then  $x''_i \in [-x'_i, x'_i]$  and  $[x''_i, x'_i] \subset [-x'_i, x'_i]$ . It follows that

$$S_{(n_1, \cdot)}(A)(x') \subset S_{(n_1, \cdot)}(A)(x'_1, \dots, x'_{i-1}, z_i, x'_{i+1}, \dots, x'_n),$$

for all  $z_i \in [x''_i, x'_i]$ , thus

$$S_{(n_1, \cdot)}(A)(x')^\# \subset S_{(n_1, \cdot)}(A)(x'_1, \dots, x'_{i-1}, tx'_i + (1-t)x''_i, x'_{i+1}, \dots, x'_n)^\#,$$

and then  $y \in S_{(n_1, \cdot)}(A)(x'_1, \dots, x'_{i-1}, tx'_i + (1-t)x''_i, x'_{i+1}, \dots, x'_n)^\#$  for all  $t \in [0, 1]$ . One gets that

$$(x'_1, \dots, x'_{i-1}, tx'_i + (1-t)x''_i, x'_{i+1}, \dots, x'_n, y) = t(x', y) + (1-t)(x'', y) \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A)),$$

therefore  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$  is convex in the  $x_i$ -direction.

C) Let  $(x', y') \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$ . Since  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$  is symmetric with respect to the hyperplanes  $x_i = 0$  and convex in the  $x_i$ -direction, for all  $i \in \{1, \dots, n_1\}$ , then one gets  $(0_{n_1}, y') \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$ , where  $0_{n_1}$  is the null vector of  $\mathbb{R}^{n_1}$ . In the same way, using the symmetry with respect to the hyperplane  $y_j = 0$  and the convexity of  $S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$  in the  $y_j$ -direction for all  $j \in \{1, \dots, n_2\}$ , it follows that  $0 = (0_{n_1}, 0_{n_2}) \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$ . Now, let us consider  $z \in [0, (x', y')]$ . Then, there exists  $t \in [0, 1]$  such that  $z = t(x', y')$ . Using the symmetry with respect to the hyperplanes  $x_i = 0$  and the convexity in the  $x_i$ -direction, for all  $i \in \{1, \dots, n_1\}$ , one gets that  $(tx', y') \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$ . Then, in the same way one gets  $(tx', ty') \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$  and it follows that  $z \in S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$ , thus  $[0, (x', y')] \subset S_{(\cdot, n_2)}(S_{(n_1, \cdot)}(A))$  which concludes the proof.  $\square$

**Definition A.9** (Steiner symmetrization of functions). *Let  $A$  be a measurable subset of  $\mathbb{R}^n$  such that  $|A| < \infty$ , and let  $f : A \rightarrow \mathbb{R}$  be a nonnegative measurable function. The  $(n_1, \cdot)$ -Steiner symmetrization of  $f$  is the function  $S_{(n_1, \cdot)}(f)$  defined by*

$$S_{(n_1, \cdot)}(f)(x, y) := \int_0^{+\infty} \chi_{S_{(n_1, \cdot)}(\{f>t\})}(x, y) dt,$$

for all  $(x, y) \in S_{(n_1, \cdot)}(A)$ .

**Example A.10.** *Let  $\Omega$  be a nonempty bounded open subset  $\mathbb{R}^n$  and let  $A$  be a measurable subset of  $\Omega$ . Consider the map  $f : (x, y) \in \Omega \mapsto \alpha \chi_A(x, y) + \beta \chi_{\Omega \setminus A}(x, y) \in \mathbb{R}^+$ , where  $\alpha, \beta$  are constants such that  $\alpha \geq \beta \geq 0$ , and  $\chi_A$  (resp.  $\chi_{\Omega \setminus A}$ ) is the characteristic function of  $A$  (resp. of  $\Omega \setminus A$ ). Then the  $(n_1, \cdot)$ -Steiner symmetrization of  $f$  is  $S_{(n_1, \cdot)}(f) : (x, y) \in S_{(n_1, \cdot)}(\Omega) \mapsto \alpha \chi_{S_{(n_1, \cdot)}(A)}(x, y) + \beta \chi_{S_{(n_1, \cdot)}(\Omega) \setminus S_{(n_1, \cdot)}(A)}(x, y) \in \mathbb{R}$ .*

The next Steiner symmetrization comparison theorem for elliptic equations (see [16]) is the main theorem needed to prove item (ii) and item (iii) of Theorem 1.1.

**Theorem A.11.** *Let  $\Omega \subset \mathbb{R}^n$  and assume that  $\Omega = \Omega_{n_1} \times \Omega_{n_2}$ , where  $\Omega_{n_1} \subset \mathbb{R}^{n_1}$  (resp.  $\Omega_{n_2} \subset \mathbb{R}^{n_2}$ ) is a nonempty bounded open subset of  $\mathbb{R}^{n_1}$  (resp. of  $\mathbb{R}^{n_2}$ ). Let  $f \in L^2(\Omega)$  be a nonnegative function,  $D > 0$ , and consider  $Z_f \in H_0^1(\Omega)$  the unique solution of the Dirichlet problem*

$$\begin{cases} Z_f - D\Delta Z_f = f & \text{in } \Omega, \\ Z_f = 0 & \text{on } \partial\Omega, \end{cases}$$

and  $v \in H_0^1(S_{(n_1, \cdot)}(\Omega))$  the unique solution of the Dirichlet problem

$$\begin{cases} v - D\Delta v = S_{(n_1, \cdot)}(f) & \text{in } S_{(n_1, \cdot)}(\Omega), \\ v = 0 & \text{on } \partial S_{(n_1, \cdot)}(\Omega), \end{cases}$$

where  $S_{(n_1, \cdot)}(\Omega) = \Omega_{n_1}^\# \times \Omega_{n_2}$ . Then, one has

$$\int_{\Omega_{n_1}} Z_f(x, y) dx \leq \int_{\Omega_{n_1}^\#} v(x, y) dx,$$

for almost all  $y \in \Omega_{n_2}$ .

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