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Matrix elements of $SO(3)$ in sl_3 representations as bispectral multivariate functions

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Abstract

We compute the matrix elements of $SO(3)$ in any finite-dimensional irreducible representation of sl_3 . They are expressed in terms of a double sum of products of Krawtchouk and Racah polynomials which generalize the Griffiths–Krawtchouk polynomials. Their recurrence and difference relations are obtained as byproducts of our construction. The proof is based on the decomposition of a general three-dimensional rotation in terms of elementary planar rotations and a transition between two embeddings of sl_2 in sl_3 . The former is related to monovariate Krawtchouk polynomials and the latter, to monovariate Racah polynomials. The appearance of Racah polynomials in this context is algebraically explained by showing that the two sl_2 Casimir elements related to the two embeddings of sl_2 in sl_3 obey the Racah algebra relations. We also show that these two elements generate the centralizer in $U(sl_3)$ of the Cartan subalgebra and its complete algebraic description is given.

1 Introduction

Consider the particular irreducible representation of the Lie algebra sl_3 given by the n^{th} symmetric power of its three-dimensional defining representation. The group $SO(3)$ acts naturally on this representation and its matrix elements are given (up to some normalization) by the bivariate Griffiths–Krawtchouk polynomials. This was proven in [1] by using oscillator algebras to realize the n^{th} symmetric power representation.

The goal of this paper is to consider an arbitrary finite-dimensional irreducible representation of sl_3 . The group $SO(3)$ still acts naturally on this representation and our main result is an expression of its matrix elements. They are expressed as a novel family of 3-variable functions, enjoying many nice properties: they are bispectral, orthogonal and are given as sums of products of univariate Krawtchouk and Racah polynomials.

On Krawtchouk and Racah polynomials

The Krawtchouk and Racah polynomials have appeared in various aspects of representation theory. The univariate Krawtchouk polynomials appear as matrix elements of irreducible representations of $SU(2)$ [2]. They also appear as $3j$ and $6j$ coefficients of the oscillator algebra [3, 4] and as spherical functions on wreath products of symmetric groups [5]. Meanwhile, the Racah polynomials have been identified as the $6j$ coefficients of the algebra sl_2 [6, 7] and an algebraic interpretation has been given in [8].

These objects and their connections have been extended to higher dimensions in multiple ways. Griffiths introduced bivariate (and multivariate) Krawtchouk polynomials in 1971 [9]. Twenty years later, Tratnik defined multivariate Racah polynomials [10] from which d -variable Krawtchouk polynomials could be obtained through a limit. The d -variable Griffiths–Krawtchouk polynomials depend on $\frac{1}{2}d(d+1)$ parameters while the d -variable Tratnik–Krawtchouk polynomials depend on d parameters. Although one could expect that the Krawtchouk polynomials of Griffiths type are more general than those of Tratnik type due to their larger number of parameters, the precise relationship between the two remained unclear for quite some time. This is in part due to the very different parametrizations of both families. Things became clear in [1] where an algebraic interpretation cemented the fact that the $\frac{1}{2}d(d+1)$ parameters play the role of the Euler angles that parametrize general $SO(d+1)$ rotations. The Tratnik–Krawtchouk case could then be seen as a special case where some Euler angles vanish. The Griffiths–Krawtchouk polynomials have also appeared in the study of character algebras and been given an expression in terms of Aomoto–Gel’fand series [11], the bivariate case was rediscovered from a probabilistic perspective [12] and various algebraic interpretations of the bivariate and multivariate Krawtchouk polynomials have been presented [13–17]. The multivariate Krawtchouk polynomials have also been studied in the context of birth and death processes [18], Markov chains (see [19] for a review) and their bispectrality has been established [1, 20]. As for the multivariate Racah polynomials, they have appeared as $3nj$ coefficients of the algebra sl_2 [21] and the bivariate Racah polynomials have been studied in details in [22]. We note that, so far, the Racah polynomials had never been given a direct interpretation in terms of matrix elements of representations of the rotation group, even if such a result could be deduced from [23–25] where the transition coefficients between different Gelfand–Tsetlin bases are related to $6j$ -symbols of sl_2 . We use a more modern approach involving explicitly the Racah polynomials and hope that this will pave the way for a subsequent multivariate extension.

In this paper, the matrix elements of $SO(3)$ rotations in any representation of sl_3 are expressed as “hybrid” multivariate Krawtchouk and Racah polynomials. Here is what we mean by “hybrid”. In the d -variable extensions of Tratnik (respectively Griffiths), the multivariate Racah (Krawtchouk) polynomials can be expressed as sums of products of univariate Racah (Krawtchouk) polynomials. Meanwhile, the family that we obtain here is written as sums of products of both univariate Racah and Krawtchouk polynomials.

Such hybrid multivariate polynomials appeared before (see for example [5, 26, 27]), but they did not attract that much attention. Nevertheless, they have recently resurfaced as a byproduct of the definition of bivariate P -polynomial association schemes [28, 29]. Hybrid multivariate polynomial families seem to be more and more an important aspect of future research in the field of special functions and their appearance in a very natural setting in the present paper is an indication of that.

On the Racah algebra and bispectrality

The Racah polynomials (like all families of the Askey scheme) possess a bispectrality property [30, 31] as they are simultaneously eigenfunctions of two eigenvalue problems in two different variables. The first eigenvalue problem is written as the three term recurrence relation in n , with eigenvalue in x , and the second one as the difference equation in x , with eigenvalue in n . We refer to the recurrence and difference operators as the bispectral operators.

When both taken in the variable or degree representation, these operators do not commute with each other and the algebraic relations that they obey are the so-called Racah algebra relations. These

read

$$\begin{aligned} [K_2, [K_1, K_2]] &= K_2^2 + \{K_1, K_2\} + dK_2 + e_1, \\ [[K_1, K_2], K_1] &= K_1^2 + \{K_1, K_2\} + dK_1 + e_2, \end{aligned} \quad (1)$$

where K_1, K_2 are generators of this algebra and d, e_1 and e_2 are some parameters (or central elements).

The Racah relations first appeared in the study of angular momentum recoupling [32] and were connected to the Racah polynomials in [8]. Hence, whenever the Racah algebra appears, the Racah polynomials will be lurking, and vice-versa. Remarkably, even though, the Racah algebra is quadratic, it can be studied in detail. Its representation theory is well-known [33, 34], it has been embedded in various algebraic structures [35–42] and it encapsulated the properties of the eponymous polynomials. It has been connected to physics and plays in particular an important role as the symmetry algebra of the generic superintegrable system on the 2-sphere [43] (see also [44] for a review). Generalizations of this algebra to describe multivariate Racah polynomials lead to the higher rank Racah algebras, which have also been extensively studied [17, 21, 45–49].

The Racah polynomials will appear here in a new fashion as matrix elements of some particular rotation in sl_3 representations. From the above, one would expect the existence of a realization of the Racah algebra in $U(sl_3)$, and indeed we exhibit explicitly such a realization.

Outline

We first define the algebra $U(sl_3)$ and introduce its Gelfand–Tsetlin basis in Section 2. In Section 3, we pose the main problem precisely. Inner automorphisms of $U(sl_3)$ corresponding to $SO(3)$ rotations are introduced and the strategy to compute the matrix elements by decomposing the general rotation as a product of two types of rotations is explained. In Section 4, we look at a first type of rotation, from which we extract Krawtchouk polynomials. A second type of rotation analyzed in Section 5 leads to Racah polynomials. Explicit expressions for the matrix elements associated to a generic rotation are presented in Section 6. Special cases of interest are displayed in Section 7. In Section 8, the realization of the Racah algebra in $U(sl_3)$ is presented, its connection with a centralizer is explained and its Hilbert–Poincaré series is computed. Closing remarks and perspectives conclude the paper.

2 The algebra $U(sl_3)$ and its Gelfand–Tsetlin basis

Let us first introduce the algebra $U(sl_3)$ and its finite-dimensional irreducible representations in the Gelfand–Tsetlin bases.

Definition. The enveloping algebra $U(sl_3)$ of the Lie algebra sl_3 is generated by the elements e_{ij} with $1 \leq i, j \leq 3$ satisfying the defining relations

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{jk}, \quad \sum_{i=1}^3 e_{ii} = 0. \quad (2)$$

The following Casimir elements generate the center of $U(sl_3)$:

$$C_2 = \sum_{i,j=1}^3 e_{ij}e_{ji} \quad \text{and} \quad C_3 = \sum_{i,j,k=1}^3 e_{ij}e_{jk}e_{ki}. \quad (3)$$

We shall take J to be the Casimir element of the sl_2 subalgebra generated by $e_{11} - e_{22}$, e_{12} and e_{21} :

$$J = \frac{(e_{11} - e_{22})^2 + 2(e_{11} - e_{22})}{4} + e_{21}e_{12}. \quad (4)$$

Finite-dimensional irreducible representations. Finite-dimensional irreducible representations of sl_3 are in one-to-one correspondence with 3-tuples of complex numbers $\lambda = (\lambda_{31}, \lambda_{32}, \lambda_{33})$, called the highest weight, such that $\lambda_{31} - \lambda_{32} \in \mathbb{Z}_{\geq 0}$, $\lambda_{32} - \lambda_{33} \in \mathbb{Z}_{\geq 0}$ and $\lambda_{31} + \lambda_{32} + \lambda_{33} = 0$. The representation associated to the highest weight λ contains a unique (up to normalization) non-zero vector ξ , called the highest weight vector, such that

$$e_{ii} \xi = \lambda_{3,i} \xi \quad \text{for } 1 \leq i \leq 3, \quad \text{and} \quad e_{ij} \xi = 0 \quad \text{for } 1 \leq i < j \leq 3. \quad (5)$$

This representation is also characterized by a two-row Young tableau with $\lambda_{31} - \lambda_{33}$ boxes in the first row and $\lambda_{32} - \lambda_{33}$ boxes in the second row.

This representation can also be described by a Gelfand–Tsetlin (GT) pattern which is given by $\Lambda = (\lambda_{11}, \lambda_{21}, \lambda_{22}; \lambda_{31}, \lambda_{32}, \lambda_{33})$ with the conditions

$$\lambda_{31} - \lambda_{21}, \quad \lambda_{21} - \lambda_{32}, \quad \lambda_{32} - \lambda_{22}, \quad \lambda_{22} - \lambda_{33}, \quad \lambda_{21} - \lambda_{11}, \quad \lambda_{11} - \lambda_{22} \in \mathbb{Z}_{\geq 0}. \quad (6)$$

In the following the three last numbers in Λ are fixed and we write only $\Lambda = (\lambda_{11}, \lambda_{21}, \lambda_{22})$. The set of GT patterns for this highest weight λ is denoted \mathcal{P}_λ . For a GT pattern Λ , one associates the representation basis vectors, called GT vectors (see [50] and references therein)¹

$$\xi_\Lambda = \left| \begin{array}{ccc} \lambda_{31} & \lambda_{32} & \lambda_{33} \\ & \lambda_{21} & \lambda_{22} \\ & & \lambda_{11} \end{array} \right\rangle. \quad (7)$$

The sl_3 generators e_{ij} act as follows on these vectors:

$$e_{11} \xi_\Lambda = \lambda_{11} \xi_\Lambda, \quad e_{22} \xi_\Lambda = (\lambda_{21} + \lambda_{22} - \lambda_{11}) \xi_\Lambda, \quad e_{33} \xi_\Lambda = -(\lambda_{21} + \lambda_{22}) \xi_\Lambda, \quad (8a)$$

$$e_{12} \xi_\Lambda = (\lambda_{21} - \lambda_{11})(\lambda_{11} - \lambda_{22} + 1) \xi_{\Lambda + \delta^{11}}, \quad e_{21} \xi_\Lambda = \xi_{\Lambda - \delta^{11}}, \quad (8b)$$

$$e_{23} \xi_\Lambda = \frac{\lambda_{31} - \lambda_{21}}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda + \delta^{21}} + \frac{\lambda_{31} - \lambda_{22} + 1}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda + \delta^{22}}, \quad (8c)$$

$$e_{32} \xi_\Lambda = \frac{(\lambda_{21} - \lambda_{32})(\lambda_{21} - \lambda_{33} + 1)(\lambda_{21} - \lambda_{11})}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda - \delta^{21}} \\ + \frac{(\lambda_{11} - \lambda_{22} + 1)(\lambda_{32} - \lambda_{22} + 1)(\lambda_{22} - \lambda_{33})}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda - \delta^{22}}, \quad (8d)$$

where $\xi_{\Lambda \pm \delta^{ij}}$ is either the basis element associated to the GT pattern $\Lambda \pm \delta^{ij}$ where the value of λ_{ij} has become $\lambda_{ij} \pm 1$, or 0 if the resulting pattern is not a valid GT pattern. One can deduce the actions of the remaining generators:

$$e_{13} \xi_\Lambda = \frac{(\lambda_{11} - \lambda_{22} + 1)(\lambda_{31} - \lambda_{21})}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda + \delta^{21} + \delta^{11}} - \frac{(\lambda_{21} - \lambda_{11})(\lambda_{31} - \lambda_{22} + 1)}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda + \delta^{22} + \delta^{11}}, \quad (8e)$$

$$e_{31} \xi_\Lambda = \frac{(\lambda_{21} - \lambda_{32})(\lambda_{21} - \lambda_{33} + 1)}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda - \delta^{21} - \delta^{11}} - \frac{(\lambda_{32} - \lambda_{22} + 1)(\lambda_{22} - \lambda_{33})}{\lambda_{21} - \lambda_{22} + 1} \xi_{\Lambda - \delta^{22} - \delta^{11}}. \quad (8f)$$

The Casimir elements of $U(sl_3)$ are proportional to the identity in this representation and, using (8), one gets

$$C_2 \xi_\Lambda = 2(\lambda_{31}^2 + \lambda_{31} \lambda_{32} + \lambda_{32}^2 + 2\lambda_{31} + \lambda_{32}) \xi_\Lambda, \quad (9a)$$

$$C_3 \xi_\Lambda = 3\lambda_{31}(1 - \lambda_{32})(2 + \lambda_{31} + \lambda_{32}) \xi_\Lambda. \quad (9b)$$

The element J defined by (4) acts diagonally on this basis

$$J \xi_\Lambda = \frac{1}{4}(\lambda_{21} - \lambda_{22})(\lambda_{21} - \lambda_{22} + 2) \xi_\Lambda. \quad (10)$$

¹For later convenience, we change the normalization of the vectors in comparison with [50]:
 $\xi_{\lambda \text{ old}} \rightarrow \frac{(\lambda_{21} - \lambda_{32})!(\lambda_{21} - \lambda_{33} + 1)!(\lambda_{22} - \lambda_{33})!}{(\lambda_{32} - \lambda_{22})!} \xi_\lambda.$

In fact, this eigenvalue of the sl_2 Casimir element J , with the ones of the hypercharge $Y = \frac{1}{3}(e_{11} + e_{22} - 2e_{33})$ and the Cartan element $H = e_{11} - e_{22}$:

$$Y \xi_\Lambda = (\lambda_{21} + \lambda_{22}) \xi_\Lambda, \quad (11)$$

$$H \xi_\Lambda = (2\lambda_{11} - \lambda_{21} - \lambda_{22}) \xi_\Lambda, \quad (12)$$

completely characterize the vectors ξ_Λ (if the highest weight λ is given).

Let us introduce the normalized vectors

$$\zeta_\Lambda = \frac{1}{N_\Lambda} \xi_\Lambda, \quad (13)$$

with

$$(N_\Lambda)^2 = \frac{1}{\lambda_{21} - \lambda_{22} + 1} (\lambda_{21} - \lambda_{11})! (\lambda_{31} - \lambda_{21})! (\lambda_{31} - \lambda_{22} + 1)! (\lambda_{31} - \lambda_{32})! (\lambda_{32} - \lambda_{33})! \\ \times \frac{(\lambda_{22} - \lambda_{33})! (\lambda_{31} - \lambda_{33} + 1)! (\lambda_{21} - \lambda_{32})! (\lambda_{21} - \lambda_{33} + 1)!}{(\lambda_{11} - \lambda_{22})! (\lambda_{32} - \lambda_{22})!}. \quad (14)$$

In this basis, the anti-automorphism \star of sl_3 defined by $\star : e_{ij} \mapsto e_{ji}$ corresponds to the transposition.

3 The general problem: rotations and change of basis coefficients

In this Section, we introduce the inner automorphisms of sl_3 associated to elements of the group $SO(3)$. The matrix elements of $SO(3)$ are interpreted as the overlap coefficients between different bases related by these inner automorphisms. We provide the decomposition of a general rotation of $SO(3)$ into a product of two types of rotations R^z and T , which will be examined separately in the following two Sections.

Inner automorphism. Let $R \in SO(3)$ be a 3×3 -matrix whose real entries satisfy

$$\sum_{j=1}^3 R_{ij} R_{kj} = \sum_{j=1}^3 R_{ji} R_{jk} = \delta_{ik}. \quad (15)$$

The map Ψ_R , labeled by $R \in SO(3)$ and acting on $g \in sl_3$ by $\Psi_R(g) = RgR^{-1}$, provides an automorphism of $U(sl_3)$ given explicitly by:

$$\Psi_R : e_{ij} \mapsto \sum_{k,\ell=1}^3 R_{ki} R_{\ell j} e_{k\ell}. \quad (16)$$

One gets that $\Psi_R \circ \Psi_{R'} = \Psi_{RR'}$ for any $R, R' \in SO(3)$.

From now on, let us fix a highest weight $\lambda = (\lambda_{31}, \lambda_{32}, \lambda_{33})$ and let ξ_Λ be vectors of the representation space. For any rotation R , we denote ρ the operator representing R^{-1} . It satisfies:

$$\Psi_R(g) \cdot \xi_\Lambda = \rho^{-1} \cdot g \cdot \rho \cdot \xi_\Lambda \quad \text{for any } g \in sl_3, \quad (17)$$

where

$$\rho \cdot \xi_\Lambda = \sum_{\Lambda' \in \mathcal{P}_\lambda} \rho_{\Lambda', \Lambda} \xi_{\Lambda'}. \quad (18)$$

This means that $\Psi_R(g)$ and g are two equivalent representations and the change of basis is provided by ρ . The coefficients $\rho_{\Lambda', \Lambda}$ are the matrix elements of R^{-1} in the Gelfand–Tsetlin basis.

In the basis of normalized vectors ζ_Λ (13), the matrices corresponding to ρ are orthogonal. Therefore, due to the change of normalization, one gets the following orthogonality relation

$$\sum_{\Lambda' \in \mathcal{P}_\lambda} (N_{\Lambda'})^2 \rho_{\Lambda', \Lambda} \rho_{\Lambda', \Lambda''} = (N_\Lambda)^2 \delta_{\Lambda, \Lambda''}. \quad (19)$$

Change of basis for any rotation. The goal of this paper is to provide an explicit formula for the entries of ρ for any R and any representation. When the representation is a symmetric representation of $U(sl_3)$ (i.e. $\lambda_{32} = \lambda_{33}$), the entries of ρ are given in terms of the Griffiths–Krawtchouk polynomials [1, 15]. As explained in the introduction, we generalize this result to any finite-dimensional irreducible representation of sl_3 .

Following [1], to compute the change of basis σ for any $S \in SO(3)$, we decompose S into three elementary rotations: two rotations around the z axis and one around the y axis.

$$S = R_\chi^z R_\theta^y R_\phi^z, \quad (20)$$

with

$$R_\phi^z = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_\theta^y = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}. \quad (21)$$

The rotations around the z axis are easy to deal with since they leave the Casimir element J of sl_2 invariant and the calculation is done in Section 4. For the rotation around the y axis, the computation is more involved but the idea consists in writing this rotation as follows:

$$R_\theta^y = T R_\theta^z T^{-1}, \quad (22)$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (23)$$

The change of basis τ associated to the transformation T is computed in Section 5 in terms of Racah polynomials.

After the changes of basis ρ_ϕ and τ associated to R_ϕ^z and T have been computed, one can write the whole change of basis $\sigma = \rho_\phi \tau^{-1} \rho_\theta \tau \rho_\chi$ associated to the rotation $S = R_\chi^z T R_\theta^z T^{-1} R_\phi^z$, since

$$\begin{aligned} \Psi_S(g) \cdot \xi_\Lambda &= \Psi_{R_\chi^z T R_\theta^z T^{-1} R_\phi^z}(g) \cdot \xi_\Lambda = \Psi_{R_\chi^z} \left(\Psi_T \left(\Psi_{R_\theta^z} \left(\Psi_{T^{-1}} \left(\Psi_{R_\phi^z}(g) \right) \right) \right) \right) \cdot \xi_\Lambda \\ &= (\rho_\phi \tau^{-1} \rho_\theta \tau \rho_\chi)^{-1} \cdot g \cdot (\rho_\phi \tau^{-1} \rho_\theta \tau \rho_\chi) \cdot \xi_\Lambda. \end{aligned} \quad (24)$$

We shall show that this full transformation is given as a double sum of a product of three Krawtchouk and two Racah polynomials (see Proposition 6.1).

4 The change of basis for R_ϕ^z and the Krawtchouk polynomials

In this section, we focus on the case where $R = R_\phi^z$ and we denote the change of basis by ρ . Imposing relation (17) for $g = Y, H$ and J constrains the operator ρ . Let us remark that $\Psi_R(Y) = Y$ and $\Psi_R(J) = J$, therefore relation (17) reduces to

$$Y \cdot \rho \cdot \xi_\Lambda = \rho \cdot Y \cdot \xi_\Lambda \quad \text{and} \quad J \cdot \rho \cdot \xi_\Lambda = \rho \cdot J \cdot \xi_\Lambda. \quad (25)$$

Using the explicit expressions of the action of sl_3 on the GT basis, one gets

$$\begin{aligned} 0 &= \rho_{\Lambda' \Lambda} (\lambda'_{21} + \lambda'_{22} - \lambda_{21} - \lambda_{22}), \\ 0 &= \rho_{\Lambda' \Lambda} ((\lambda'_{21} - \lambda'_{22})(\lambda'_{21} - \lambda'_{22} + 2) - (\lambda_{21} - \lambda_{22})(\lambda_{21} - \lambda_{22} + 2)), \end{aligned} \quad (26)$$

which imply that

$$\rho_{\Lambda' \Lambda} = \delta_{\lambda'_{21}, \lambda_{21}} \delta_{\lambda'_{22}, \lambda_{22}} r(\lambda_{11}, \lambda'_{21}, \lambda'_{11}, \lambda'_{22}). \quad (27)$$

One also gets

$$\Psi_R(H) = (\cos^2(\phi) - \sin^2(\phi)) H - 2 \cos(\phi) \sin(\phi) (e_{12} + e_{21}). \quad (28)$$

Relation (17) for $g = H$ provides the following constraint (and using relation (27)):

$$\begin{aligned} (2\lambda'_{11} - \lambda'_{21} - \lambda'_{22})\rho_{\Lambda',\Lambda} &= -2\cos(\phi)\sin(\phi)(\lambda_{21} - \lambda_{11})(\lambda_{11} - \lambda_{22} + 1)\rho_{\Lambda',\Lambda+\delta^{11}} \\ &\quad + (\cos^2(\phi) - \sin^2(\phi))(2\lambda_{11} - \lambda_{21} - \lambda_{22})\rho_{\Lambda',\Lambda} \\ &\quad - 2\cos(\phi)\sin(\phi)\rho_{\Lambda',\Lambda-\delta^{11}}. \end{aligned} \quad (29)$$

This relation is similar to the recurrence relation of the Krawtchouk polynomials $K_n(x; p, N)$:

$$K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \middle| \frac{1}{p}\right), \quad (30)$$

where ${}_2F_1$ is the usual hypergeometric function [51]. We use the convention that $K_n(x; p, N) = 0$ if $n, x < 0$ or $n, x > N$. By direct check, we see that

$$r(\lambda_{11}, \lambda'_{21}, \lambda'_{11}, \lambda'_{22}) = \frac{\tan^{\lambda_{11}-\lambda'_{22}}(\phi)}{(\lambda_{11}-\lambda'_{22})!} K_{\lambda_{11}-\lambda'_{22}}\left(\lambda'_{11}-\lambda'_{22}; \sin^2(\phi), \lambda'_{21}-\lambda'_{22}\right) r(\lambda'_{21}, \lambda'_{11}, \lambda'_{22}) \quad (31)$$

is the solution of (29). Here the indices $\lambda_{11} - \lambda'_{22}$ and variables $\lambda'_{11} - \lambda'_{22}$ both go from 0 to $\lambda'_{21} - \lambda'_{22}$.

It remains to determine the function r . By using the orthogonality relation (19) satisfied by ρ , one can show

$$r(x, y, z)^2 = \left(\frac{(x-z)!}{(x-y)!} \tan^{y-z}(\phi) \cos^{x-z}(\phi)\right)^2 \quad (32)$$

by making use of the orthogonality relation of the Krawtchouk polynomials [51]

$$\binom{N}{n} \sum_{X=0}^N \binom{N}{X} p^{X+n} (1-p)^{N-X-n} K_m(X; p, N) K_n(X; p, N) = \delta_{n,m}. \quad (33)$$

The previous results lead to the following proposition.

Proposition 4.1. *The change of basis ρ associated to R_ϕ^z is given by*

$$\begin{aligned} \rho_{\Lambda',\Lambda} &= \delta_{\lambda'_{21},\lambda_{21}} \delta_{\lambda'_{22},\lambda_{22}} (-1)^{\lambda'_{11}-\lambda_{22}} \frac{(\lambda_{21} - \lambda_{22})! \tan^{\lambda'_{11}+\lambda_{11}-2\lambda_{22}}(\phi) \cos^{\lambda_{21}-\lambda_{22}}(\phi)}{(\lambda_{11} - \lambda_{22})!(\lambda_{21} - \lambda'_{11})!} \\ &\quad \times K_{\lambda_{11}-\lambda_{22}}\left(\lambda'_{11} - \lambda_{22}; \sin^2(\phi), \lambda_{21} - \lambda_{22}\right). \end{aligned} \quad (34)$$

Proof. By combining (31) and taking the square root of (32), we can obtain the change of basis ρ up to signs. The signs may be fixed by using equation (17) for the other elements of sl_3 but it is simpler to remark that the operator ρ tends continuously to the identity operator when R tends to the identity (*i.e.* $\phi \rightarrow 0$):

$$\rho_{\Lambda',\Lambda} \Big|_{\phi=0} = \delta_{\lambda'_{21},\lambda_{21}} \delta_{\lambda'_{11},\lambda_{11}} \delta_{\lambda'_{22},\lambda_{22}}. \quad (35)$$

Using

$$p^{(x+n)/2} \binom{N}{n} K_n(x; p, N) \Big|_{p=0} = (-1)^x \delta_{x,n}, \quad (36)$$

the result (34) follows. \square

5 The change of basis corresponding to T and the Racah polynomials

Under the sl_3 automorphism associated to the rotation T , the sl_2 subalgebra on indices 1, 2 is sent to the subalgebra on indices 1, 3. Indeed, after a short calculation, one can show that the sl_2 Casimir element J is mapped to

$$\Psi_T(J) = \frac{(e_{11} - e_{33})^2 + 2(e_{11} - e_{33})}{4} + e_{31}e_{13}, \quad (37)$$

which is precisely the Casimir element of the subalgebra sl_2 generated by $e_{11} - e_{33}$, e_{13} and e_{31} .

Relations (17) for $R = T$ and $g = H, Y$ provide the following two constraints on the entries of the change of basis τ :

$$0 = (\lambda'_{11} - \lambda_{11})\tau_{\Lambda'\Lambda}, \quad (38a)$$

$$0 = (\lambda_{21} + \lambda_{22} - \lambda'_{11} + \lambda'_{21} + \lambda'_{22})\tau_{\Lambda'\Lambda}. \quad (38b)$$

These constraints lead to the following form for τ :

$$\tau_{\Lambda'\Lambda} = \delta_{\lambda'_{11}, \lambda_{11}} \delta_{\lambda_{21} + \lambda_{22}, \lambda'_{11} - \lambda'_{21} - \lambda'_{22}} t(\lambda_{21}, \lambda'_{21}, \lambda'_{11}, \lambda'_{22}). \quad (39)$$

Similarly, equation (17) for $g = J$ provides the following constraint (where we use relations (38) to replace λ_{11} by λ'_{11} and $\lambda_{21} + \lambda_{22}$ by $\lambda'_{11} - \lambda'_{21} - \lambda'_{22}$):

$$A_\Lambda \tau_{\Lambda', \Lambda - \delta^{21} + \delta^{22}} + (A_\Lambda + C_\Lambda) \tau_{\Lambda'\Lambda} + C_\Lambda \tau_{\Lambda', \Lambda + \delta^{21} - \delta^{22}} = (\lambda'_{21} - \lambda_{31})(\lambda'_{22} - \lambda_{31} - 1)\tau_{\Lambda'\Lambda}, \quad (40)$$

with

$$A_\Lambda = \frac{(\lambda_{21} - \lambda_{32})(\lambda_{31} - \lambda_{22} + 1)(\lambda_{21} - \lambda_{11})(\lambda_{21} - \lambda_{33} + 1)}{(\lambda_{21} - \lambda_{22} + 1)(\lambda_{21} - \lambda_{22})}, \quad (41a)$$

$$C_\Lambda = \frac{(\lambda_{11} - \lambda_{22} + 1)(\lambda_{22} - \lambda_{33})(\lambda_{32} - \lambda_{22} + 1)(\lambda_{31} - \lambda_{21})}{(\lambda_{21} - \lambda_{22} + 1)(\lambda_{21} - \lambda_{22} + 2)}. \quad (41b)$$

Relation (40) is the recurrence relation of the Racah polynomial [51].

$$R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) = {}_4F_3 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix} \middle| 1 \right). \quad (42)$$

To simplify the notation, one defines $\tilde{R}_n(x; \alpha, \beta, \gamma, \delta) = R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta)$. We use also the convention that $\tilde{R}_n(x; \alpha, \beta, \gamma, \delta) = 0$ if $n, x < 0$ or $n, x > N$ where $N = \min(-\alpha - 1, -\beta - \delta - 1, -\gamma - 1)$. One gets explicitly:

$$t(\lambda_{21}, \lambda'_{21}, \lambda'_{11}, \lambda'_{22}) = t(\lambda'_{21}, \lambda'_{11}, \lambda'_{22}) (-1)^{\lambda_{31} - \lambda_{21}} \tilde{R}_{\lambda_{31} - \lambda_{21}}(\lambda_{31} - \lambda'_{21}; \alpha, \beta, \gamma, \delta), \quad (43a)$$

with

$$\alpha = \lambda_{32} - \lambda_{31} - 1, \quad \beta = \lambda'_{11} - \lambda'_{21} - \lambda'_{22} + \lambda_{33} - 1, \quad (43b)$$

$$\gamma = \lambda'_{11} - \lambda_{31} - 1, \quad \delta = \lambda'_{21} + \lambda'_{22} - \lambda'_{11} - \lambda_{31} - 1. \quad (43c)$$

The degree $\lambda_{31} - \lambda_{21}$ of the Racah polynomials is in the set $\{0, 1, \dots, \min(-\alpha_\Lambda - 1, -\gamma_\Lambda - 1)\}$ and its variable $\lambda_{31} - \lambda'_{21}$ is in the set $\{0, 1, \dots, \min(-\alpha_{\Lambda'} - 1, -\gamma_{\Lambda'} - 1)\}$. After the identification (38), both sets are identical.

As in the previous section, the factor $t(\lambda'_{21}, \lambda'_{11}, \lambda'_{22})$ is determined by using the orthogonality relation satisfied by τ and the one of the Racah polynomials [51]. After a few manipulations on the parameters, one obtains

$$t(x, y, z)^2 = \left((x - z + 1) \frac{(\lambda_{31} - \lambda_{32})!(\lambda_{31} - \lambda_{33} + 1)!(\lambda_{31} - y)!(\lambda_{32} - z)!(y - z)!}{(x - \lambda_{32})!(x - \lambda_{33} + 1)!(x - y)!(\lambda_{31} - z + 1)!(\lambda_{31} - x)!(z - \lambda_{33})!} \right)^2. \quad (44)$$

The previous results lead to the following proposition

Proposition 5.1. *The matrix elements of the change of basis τ corresponding to the rotation T are given (up to a global undetermined sign) by*

$$\begin{aligned} \tau_{\Lambda'\Lambda} &= \delta_{\lambda'_{11}, \lambda_{11}} \delta_{\lambda_{21} + \lambda_{22}, \lambda'_{11} - \lambda'_{21} - \lambda'_{22}} t(\lambda'_{21}, \lambda'_{11}, \lambda'_{22}) (-1)^{\lambda'_{22} - \lambda_{21}} \\ &\quad \times \tilde{R}_{\lambda_{31} - \lambda_{21}}(\lambda_{31} - \lambda'_{21}; \lambda_{32} - \lambda_{31} - 1, \lambda_{21} + \lambda_{22} + \lambda_{33} - 1, \lambda_{11} - \lambda_{31} - 1, -\lambda_{21} - \lambda_{22} - \lambda_{31} - 1), \end{aligned} \quad (45)$$

with

$$t(x, y, z) = (x - z + 1) \frac{(\lambda_{31} - \lambda_{32})!(\lambda_{31} - \lambda_{33} + 1)!(\lambda_{31} - y)!(\lambda_{32} - z)!(y - z)!}{(x - \lambda_{32})!(x - \lambda_{33} + 1)!(x - y)!(\lambda_{31} - z + 1)!(\lambda_{31} - x)!(z - \lambda_{33})!}. \quad (46)$$

Proof. From equations (39), (43), (44), $\tau_{\Lambda', \Lambda}$ is determined up to a sign $\sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}}$ which could depend on the values of λ'_{21} , λ'_{11} , λ'_{22} :

$$\tau_{\Lambda'\Lambda} = \delta_{\lambda'_{11}, \lambda_{11}} \delta_{\lambda_{21} + \lambda_{22}, \lambda'_{11} - \lambda'_{21} - \lambda'_{22}} t(\lambda'_{21}, \lambda'_{11}, \lambda'_{22}) (-1)^{\lambda_{31} - \lambda_{21}} \tilde{R}_{\lambda_{31} - \lambda_{21}}(\lambda_{31} - \lambda'_{21}; \alpha, \beta, \gamma, \delta), \quad (47a)$$

with

$$t(\lambda'_{21}, \lambda'_{11}, \lambda'_{22}) = T(\lambda'_{21}, \lambda'_{11}, \lambda'_{22}) \sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}}, \quad (47b)$$

and

$$T(x, y, z) = (x - z + 1) \frac{(\lambda_{31} - \lambda_{32})!(\lambda_{31} - \lambda_{33} + 1)!(\lambda_{31} - y)!(\lambda_{32} - z)!(y - z)!}{(x - \lambda_{32})!(x - \lambda_{33} + 1)!(x - y)!(\lambda_{31} - z + 1)!(\lambda_{31} - x)!(z - \lambda_{33})!}. \quad (47c)$$

Once the sign function has been determined, one immediately recovers the full expression of $\tau_{\Lambda', \Lambda}$ from equations (39), (43), (47b).

The expression for the sign function $\sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}}$ is obtained from constraints following from (17) for other elements of the algebra $U(\mathfrak{sl}_3)$.

First, use $g = e_{12}$ in relation (17) to get

$$\begin{aligned} 0 &= (\lambda_{21} - \lambda_{11})(\lambda_{11} - \lambda_{22} + 1) \tau_{\Lambda', \Lambda + \delta^{11}} - \frac{(\lambda'_{11} - \lambda'_{22})(\lambda'_{31} - \lambda'_{21} + 1)}{(\lambda'_{21} - \lambda'_{22})} \tau_{\Lambda' - \delta^{21} - \delta^{11}, \Lambda} \\ &\quad + \frac{(\lambda'_{21} - \lambda'_{11} + 1)(\lambda'_{31} - \lambda'_{22} + 2)}{(\lambda'_{21} - \lambda'_{22} + 2)} \tau_{\Lambda' - \delta^{22} - \delta^{11}, \Lambda}. \end{aligned}$$

Substituting (47), and using the following relation between Racah polynomials [52, equation (4.13)]

$$\begin{aligned} (n + \gamma)(n - \gamma + \alpha + \beta + 1) \tilde{R}_n(x; \alpha, \beta, \gamma, \delta) \\ = \frac{\gamma}{2x + \gamma + \delta + 1} \left[(x + \alpha + 1)(x + \beta + \delta + 1) \tilde{R}_n(x + 1; \alpha, \beta, \gamma - 1, \delta) \right. \\ \left. - (x - \alpha + \gamma + \delta)(x - \beta + \gamma) \tilde{R}_n(x; \alpha, \beta, \gamma - 1, \delta) \right] \end{aligned} \quad (48)$$

allows one to fix

$$\sigma_{\lambda'_{21} - 1, \lambda'_{11} - 1, \lambda'_{22}} = (+1) \sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}}, \quad \sigma_{\lambda'_{21}, \lambda'_{11} - 1, \lambda'_{22} - 1} = (-1) \sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}}. \quad (49)$$

Next, use $g = e_{23}$ in relation (17). Substituting (47), and using the following relation between Racah polynomials²

$$\begin{aligned} 0 &= \frac{n}{2n + \alpha + \beta} \tilde{R}_{n-1}(x; \alpha, \beta, \gamma, \delta) + \frac{n + \alpha + \beta}{2n + \alpha + \beta} \tilde{R}_n(x; \alpha, \beta, \gamma, \delta) \\ &\quad - \frac{x}{2x + \gamma + \delta + 1} \tilde{R}_n(x - 1; \alpha, \beta - 1, \gamma, \delta + 1) - \frac{x + \gamma + \delta + 1}{2x + \gamma + \delta + 1} \tilde{R}_n(x; \alpha, \beta - 1, \gamma, \delta + 1) \end{aligned} \quad (51)$$

²This relation follows from applying the following hypergeometric contiguity relation to simplify the first two terms and the last two terms of (51)

$$\frac{\alpha_1}{\alpha_2 - \alpha_1} {}_4F_3 \left(\begin{matrix} \alpha_1 + 1, \alpha_2, \alpha_3, \alpha_4 \\ \beta_1, \beta_2, \beta_3 \end{matrix} \middle| z \right) - \frac{\alpha_2}{\alpha_2 - \alpha_1} {}_4F_3 \left(\begin{matrix} \alpha_1, \alpha_2 + 1, \alpha_3, \alpha_4 \\ \beta_1, \beta_2, \beta_3 \end{matrix} \middle| z \right) = {}_4F_3 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3, \alpha_4 \\ \beta_1, \beta_2, \beta_3 \end{matrix} \middle| z \right) \quad (50)$$

allows one to fix

$$\sigma_{\lambda'_{21}+1, \lambda'_{11}, \lambda'_{22}} = (+1)\sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}}, \quad \sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}+1} = (-1)\sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}}. \quad (52)$$

Finally, combining (49), (52), one can write (up to a global undetermined sign)

$$\sigma_{\lambda'_{21}, \lambda'_{11}, \lambda'_{22}} = (-1)^{\lambda'_{22} - \lambda_{31}}. \quad (53)$$

□

Let us emphasize that the undetermined global sign does not play a role in the upcoming computations since only the product of T and T^{-1} is involved.

For particular representations, the previous result simplifies.

Corollary 5.2. *For the symmetric representation (i.e. $\lambda_{32} = \lambda_{33} = \lambda_{22}$, $\lambda_{31} = -2\lambda_{32}$), τ is given (up to a global sign) by*

$$\tau_{\Lambda' \Lambda} = \delta_{\lambda'_{11}, \lambda_{11}} \delta_{\lambda_{21} + 2\lambda_{32}, \lambda'_{11} - \lambda'_{21}} \delta_{\lambda'_{22}, \lambda_{32}} \delta_{\lambda_{22}, \lambda_{32}} (-1)^{\lambda_{32} - \lambda_{21}} \frac{(\lambda_{21} - \lambda_{32})!}{(\lambda_{11} - \lambda_{21} - 3\lambda_{32})!}. \quad (54)$$

For the representation characterized by $\lambda_{31} = \lambda_{32} = \lambda_{21}$, $\lambda_{33} = -2\lambda_{31}$, one gets

$$\tau_{\Lambda' \Lambda} = \delta_{\lambda'_{11}, \lambda_{11}} \delta_{\lambda_{11} - \lambda_{22} - 2\lambda_{31}, \lambda'_{22}} \delta_{\lambda'_{21}, \lambda_{31}} \delta_{\lambda_{21}, \lambda_{31}} (-1)^{\lambda'_{22} - \lambda_{31}} \frac{(\lambda_{22} + 2\lambda_{31})!}{(\lambda_{11} - \lambda_{22})!}. \quad (55)$$

Proof. For the symmetric representation, the Racah polynomial in (45) reduces to

$$R_{-2\lambda_{32} - \lambda_{21}}(X) = {}_3F_2 \left(\begin{matrix} 2\lambda_{32} + \lambda_{21}, 3\lambda_{32} - 1, \lambda_{11} - \lambda_{21} \\ \lambda_{11} + 2\lambda_{32}, 3\lambda_{32} \end{matrix} \middle| 1 \right). \quad (56)$$

Using Thomae's transformation formula (see [53, relation (3.1.1)] for example)

$${}_3F_2 \left(\begin{matrix} -n, a, b \\ c, d \end{matrix} \middle| 1 \right) = \frac{(d-b)_n}{(d)_n} {}_3F_2 \left(\begin{matrix} -n, c-a, b \\ c, 1+b-d-n \end{matrix} \middle| 1 \right), \quad (57)$$

relation (56) becomes

$$R_{-2\lambda_{32} - \lambda_{21}}(X) = \frac{(3\lambda_{32} - \lambda_{11} + \lambda_{21})_{-\lambda_{21} - 2\lambda_{32}}}{(3\lambda_{32})_{-\lambda_{21} - 2\lambda_{32}}} {}_2F_1 \left(\begin{matrix} 2\lambda_{32} + \lambda_{21}, \lambda_{11} - \lambda_{21} \\ \lambda_{11} + 2\lambda_{32} \end{matrix} \middle| 1 \right), \quad (58)$$

$$= \frac{(3\lambda_{32} - \lambda_{11} + \lambda_{21})_{-\lambda_{21} - 2\lambda_{32}} (2\lambda_{32} + \lambda_{21})_{-\lambda_{21} - 2\lambda_{32}}}{(3\lambda_{32})_{-\lambda_{21} - 2\lambda_{32}} (\lambda_{11} + 2\lambda_{32})_{-\lambda_{21} - 2\lambda_{32}}}. \quad (59)$$

This proves relation (54). Relation (55) is proven by remarking that $R_0(X) = 1$. □

6 General matrix elements for $SO(3)$ rotations in a general sl_3 irrep and a new family of bispectral trivariate hybrid functions

Theorem 6.1. *The matrix elements of the change of basis σ induced by the rotation S given by (20) are, for Λ, Λ' GT patterns,*

$$\begin{aligned} \sigma_{\Lambda', \Lambda} = & \sum_{n=\max\{-\lambda_{21}, -\lambda'_{21}\}}^{\min\{-\lambda_{22}, -\lambda'_{22}\}} \sum_{\ell=\ell_{\min}}^{\min\{\lambda_{31}, n-\lambda_{33}\}} \mu_{\Lambda', \Lambda}(\ell, n) K_{n+\lambda'_{21}}(\lambda'_{11} - \lambda'_{22}; \sin^2(\phi), \lambda'_{21} - \lambda'_{22}) \\ & \times \tilde{R}_{\lambda_{31} - \lambda'_{21}}(\lambda_{31} - \ell; \lambda_{32} - \lambda_{31} - 1, \lambda'_{21} + \lambda'_{22} + \lambda_{33} - 1, n + \lambda'_{21} + \lambda'_{22} - \lambda_{31} - 1, -\lambda'_{21} - \lambda'_{22} - \lambda_{31} - 1) \\ & \times K_{\ell + \lambda_{21} + \lambda_{22}}(\ell + \lambda'_{21} + \lambda'_{22}; \sin^2(\theta), 2\ell - n) \\ & \times \tilde{R}_{\lambda_{31} - \lambda_{21}}(\lambda_{31} - \ell; \lambda_{32} - \lambda_{31} - 1, \lambda_{21} + \lambda_{22} + \lambda_{33} - 1, n + \lambda_{21} + \lambda_{22} - \lambda_{31} - 1, -\lambda_{21} - \lambda_{22} - \lambda_{31} - 1) \\ & \times K_{\lambda_{11} - \lambda_{22}}(n + \lambda_{21}; \sin^2(\chi), \lambda_{21} - \lambda_{22}), \end{aligned} \quad (60)$$

where $\ell_{min} = \max\{\lambda_{32}, n - \lambda_{32}, -\lambda_{21} - \lambda_{22}, n + \lambda_{21} + \lambda_{22}, -\lambda'_{21} - \lambda'_{22}, n + \lambda'_{21} + \lambda'_{22}\}$,

$$\begin{aligned} \mu_{\Lambda', \Lambda}(\ell, n) &= (-1)^{\lambda'_{11} + \ell - n} t(\lambda'_{21}, n + \lambda'_{21} + \lambda'_{22}, \lambda'_{22}) t(\ell, n + \lambda_{21} + \lambda_{22}, n - \ell) \\ &\times \frac{(\lambda'_{21} - \lambda'_{22})! (2\ell - n)! (\lambda_{21} - \lambda_{22})! \cos^{\lambda'_{21} - \lambda'_{22}}(\phi) \cos^{2\ell - n}(\theta) \cos^{\lambda_{21} - \lambda_{22}}(\chi)}{(n + \lambda'_{21})! (\lambda'_{21} - \lambda'_{11})! (\ell + \lambda_{21} + \lambda_{22})! (\ell - n - \lambda'_{21} - \lambda'_{22})! (\lambda_{11} - \lambda_{22})! (-n - \lambda_{22})!} \\ &\times \tan^{n + \lambda'_{11} + \lambda'_{21} - \lambda'_{22}}(\phi) \tan^{2\ell + \lambda'_{21} + \lambda'_{22} + \lambda_{21} + \lambda_{22}}(\theta) \tan^{n + \lambda_{11} + \lambda_{21} - \lambda_{22}}(\chi), \end{aligned} \quad (61)$$

and $t(x, y, z)$ is defined by (46).

Proof. From the discussion at the end of Section 2, one gets

$$\sigma_{\Lambda', \Lambda} = \sum_{\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)} \in \mathcal{P}_\lambda} (\rho_\phi)_{\Lambda', \Lambda^{(1)}} (\tau^{-1})_{\Lambda^{(1)}, \Lambda^{(2)}} (\rho_\theta)_{\Lambda^{(2)}, \Lambda^{(3)}} (\tau)_{\Lambda^{(3)}, \Lambda^{(4)}} (\rho_\chi)_{\Lambda^{(4)}, \Lambda} \quad (62)$$

The components of ρ_ϕ , ρ_θ , ρ_χ are given by (34) and the ones of τ by (45). The ones of τ^{-1} are computed as follows. Using the orthogonality relation (19), one gets

$$(\tau^{-1})_{\Lambda, \Lambda'} = \left(\frac{N_{\Lambda'}}{N_\Lambda} \right)^2 \tau_{\Lambda', \Lambda} \quad (63)$$

which leads to

$$\begin{aligned} (\tau^{-1})_{\Lambda, \Lambda'} &= \delta_{\lambda'_{11}, \lambda_{11}} \delta_{\lambda_{21} + \lambda_{22}, \lambda'_{11} - \lambda'_{21} - \lambda'_{22}} t(\lambda_{21}, \lambda_{11}, \lambda_{22}) (-1)^{\lambda'_{22} - \lambda_{21}} \\ &\times \tilde{R}_{\lambda_{31} - \lambda_{21}}(\lambda_{31} - \lambda'_{21}; \lambda_{32} - \lambda_{31} - 1, \lambda_{21} + \lambda_{22} + \lambda_{33} - 1, \lambda_{11} - \lambda_{31} - 1, -\lambda_{21} - \lambda_{22} - \lambda_{31} - 1). \end{aligned} \quad (64)$$

The expression (62) can be expressed now as follows

$$\begin{aligned} \sigma_{\Lambda', \Lambda} &= \sum_{\lambda_{21}^{(1)}, \lambda_{11}^{(1)}, \lambda_{22}^{(1)}, \lambda_{21}^{(2)}, \dots, \lambda_{22}^{(4)}} \delta_{\lambda'_{21}, \lambda_{21}^{(1)}} \delta_{\lambda'_{22}, \lambda_{22}^{(1)}} \bar{\rho}_{\Lambda', \Lambda^{(1)}} \delta_{\lambda_{11}^{(1)}, \lambda_{11}^{(2)}} \delta_{\lambda_{21}^{(1)} + \lambda_{22}^{(1)}, \lambda_{11}^{(2)} - \lambda_{21}^{(2)} - \lambda_{22}^{(2)}} \tilde{\tau}_{\Lambda^{(1)}, \Lambda^{(2)}} \\ &\times \delta_{\lambda_{21}^{(2)}, \lambda_{21}^{(3)}} \delta_{\lambda_{22}^{(2)}, \lambda_{22}^{(3)}} \bar{\rho}_{\Lambda^{(2)}, \Lambda^{(3)}} \delta_{\lambda_{11}^{(3)}, \lambda_{11}^{(4)}} \delta_{\lambda_{21}^{(3)} + \lambda_{22}^{(3)}, \lambda_{11}^{(4)} - \lambda_{21}^{(4)} - \lambda_{22}^{(4)}} \bar{\tau}_{\Lambda^{(3)}, \Lambda^{(4)}} \\ &\times \delta_{\lambda_{21}^{(4)}, \lambda_{21}^{(4)}} \delta_{\lambda_{22}^{(4)}, \lambda_{22}^{(4)}} \bar{\rho}_{\Lambda^{(4)}, \Lambda}, \end{aligned} \quad (65)$$

where $\bar{\rho}_{\Lambda, \Lambda'}$ (resp. $\bar{\tau}_{\Lambda, \Lambda'}$, $\tilde{\tau}_{\Lambda, \Lambda'}$) are the expressions following the δ 's in (34) (resp. (45), (64)). Combining everything together and setting $n = \lambda_{11}^{(1)} - \lambda'_{21} - \lambda'_{22}$, $\ell = \lambda_{21}^{(2)}$, the theorem is proven. \square

The element $\sigma_{\Lambda', \Lambda}$ given in (60) can be seen as a function of the three variables $\Lambda' = (\lambda'_{21}, \lambda'_{11}, \lambda'_{22})$. In the following, we shall give a difference relation satisfied by this function. We shall also provide a relation between the functions for different values of $\Lambda = (\lambda_{21}, \lambda_{11}, \lambda_{22})$, called recurrence relation, in reference to the similar relations satisfied by the Krawtchouk or Racah polynomials.

Recurrence relations. To simplify the notations, we denote by S_{ij} the entries of the rotation S given by (20). Relation (17) for this rotation reads as follows, for any $g \in sl_3$,

$$\sigma \cdot \Psi_S(g) \cdot \xi_\Lambda = g \cdot \sigma \cdot \xi_\Lambda, \quad (66)$$

and provides three different recurrence relations for $\sigma_{\Lambda', \Lambda}$ when one chooses either $g = H, Y, J$.

Explicitly, for $g = H$, it reads:

$$\begin{aligned}
& \left((S_{11}S_{11} - S_{12}S_{12})\lambda_{11} + (S_{21}S_{21} - S_{22}S_{22})(\lambda_{21} + \lambda_{22} - \lambda_{11}) - (S_{31}S_{31} - S_{32}S_{32})(\lambda_{21} + \lambda_{22}) \right) \sigma_{\Lambda', \Lambda} \\
& + (S_{11}S_{21} - S_{12}S_{22})(\lambda_{21} - \lambda_{11})(\lambda_{11} - \lambda_{22} + 1) \sigma_{\Lambda', \Lambda + \delta^{11}} + (S_{21}S_{11} - S_{22}S_{12}) \sigma_{\Lambda', \Lambda - \delta^{11}} \\
& + (S_{21}S_{31} - S_{22}S_{32}) \left(\frac{\lambda_{31} - \lambda_{21}}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda + \delta^{21}} + \frac{\lambda_{31} - \lambda_{22} + 1}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda + \delta^{22}} \right) \\
& + (S_{31}S_{21} - S_{32}S_{22}) \left(\frac{(\lambda_{21} - \lambda_{32})(\lambda_{21} - \lambda_{33} + 1)(\lambda_{21} - \lambda_{11})}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda - \delta^{21}} \right. \\
& \quad \left. + \frac{(\lambda_{11} - \lambda_{22} + 1)(\lambda_{32} - \lambda_{22} + 1)(\lambda_{22} - \lambda_{33})}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda - \delta^{22}} \right) \\
& + (S_{11}S_{31} - S_{12}S_{32}) \left(\frac{(\lambda_{11} - \lambda_{22} + 1)(\lambda_{31} - \lambda_{21})}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda + \delta^{11} + \delta^{21}} - \frac{(\lambda_{21} - \lambda_{11})(\lambda_{31} - \lambda_{22} + 1)}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda + \delta^{11} + \delta^{22}} \right) \\
& + (S_{31}S_{11} - S_{32}S_{12}) \left(\frac{(\lambda_{21} - \lambda_{32})(\lambda_{21} - \lambda_{33} + 1)}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda - \delta^{11} - \delta^{21}} - \frac{(\lambda_{32} - \lambda_{22} + 1)(\lambda_{22} - \lambda_{33})}{\lambda_{21} - \lambda_{22} + 1} \sigma_{\Lambda', \Lambda - \delta^{11} - \delta^{22}} \right) \\
& = (2\lambda'_{11} - \lambda'_{21} - \lambda'_{22}) \sigma_{\Lambda', \Lambda}. \tag{67}
\end{aligned}$$

For $g = Y, J$, the recurrence relations can be computed similarly. The ‘hybrid’ property of the functions $\sigma_{\Lambda', \Lambda}$ shows up in particular in the fact that the eigenvalue in the R.H.S. is also linear in Λ' for $g = Y$ while it is quadratic for $g = J$.

Difference relations. To get the difference relations, replace g by $\Psi_{S^t}(g)$ in relation (66) to find

$$\sigma \cdot g \cdot \xi_{\Lambda} = \Psi_{S^t}(g) \cdot \sigma \cdot \xi_{\Lambda}, \tag{68}$$

where S^t stands for the transposition of S and we have used that $\Psi_S(\Psi_{S^t}(g)) = g$. Using this relation for $g = H, Y, J$, one gets three difference relations. Namely, for $g = H$, it becomes

$$\begin{aligned}
& (2\lambda_{11} - \lambda_{21} - \lambda_{22}) \sigma_{\Lambda', \Lambda} = \tag{69} \\
& \left((S_{11}S_{11} - S_{21}S_{21})\lambda'_{11} + (S_{12}S_{12} - S_{22}S_{22})(\lambda'_{21} + \lambda'_{22} - \lambda'_{11}) - (S_{13}S_{13} - S_{23}S_{23})(\lambda'_{21} + \lambda'_{22}) \right) \sigma_{\Lambda', \Lambda} \\
& + (S_{11}S_{12} - S_{21}S_{22})(\lambda'_{21} - \lambda'_{11} + 1)(\lambda'_{11} - \lambda'_{22}) \sigma_{\Lambda' - \delta^{11}, \Lambda} + (S_{12}S_{11} - S_{22}S_{21}) \sigma_{\Lambda' + \delta^{11}, \Lambda} \\
& + (S_{12}S_{13} - S_{22}S_{23}) \left(\frac{\lambda'_{31} - \lambda'_{21} + 1}{\lambda'_{21} - \lambda'_{22}} \sigma_{\Lambda' - \delta^{21}, \Lambda} + \frac{\lambda'_{31} - \lambda'_{22} + 2}{\lambda'_{21} - \lambda'_{22} + 2} \sigma_{\Lambda' - \delta^{22}, \Lambda} \right) \\
& + (S_{13}S_{12} - S_{23}S_{22}) \left(\frac{(\lambda'_{21} - \lambda'_{32} + 1)(\lambda'_{21} - \lambda'_{33} + 2)(\lambda'_{21} - \lambda'_{11} + 1)}{\lambda'_{21} - \lambda'_{22} + 2} \sigma_{\Lambda' + \delta^{21}, \Lambda} \right. \\
& \quad \left. + \frac{(\lambda'_{11} - \lambda'_{22})(\lambda'_{32} - \lambda'_{22})(\lambda'_{22} - \lambda'_{33} + 1)}{\lambda'_{21} - \lambda'_{22}} \sigma_{\Lambda' + \delta^{22}, \Lambda} \right) \\
& + (S_{11}S_{13} - S_{21}S_{23}) \left(\frac{(\lambda'_{11} - \lambda'_{22})(\lambda'_{31} - \lambda'_{21} + 1)}{\lambda'_{21} - \lambda'_{22}} \sigma_{\Lambda' - \delta^{11} - \delta^{21}, \Lambda} - \frac{(\lambda'_{21} - \lambda'_{11} + 1)(\lambda'_{31} - \lambda'_{22} + 2)}{\lambda'_{21} - \lambda'_{22} + 2} \sigma_{\Lambda' - \delta^{11} - \delta^{22}, \Lambda} \right) \\
& + (S_{13}S_{11} - S_{23}S_{21}) \left(\frac{(\lambda'_{21} - \lambda'_{32} + 1)(\lambda'_{21} - \lambda'_{33} + 2)}{\lambda'_{21} - \lambda'_{22} + 2} \sigma_{\Lambda' + \delta^{11} + \delta^{21}, \Lambda} - \frac{(\lambda'_{32} - \lambda'_{22})(\lambda'_{22} - \lambda'_{33} + 1)}{\lambda'_{21} - \lambda'_{22}} \sigma_{\Lambda' + \delta^{11} + \delta^{22}, \Lambda} \right).
\end{aligned}$$

For $g = Y, J$, similar relations are obtained. Again, the ‘hybrid’ property of the functions $\sigma_{\Lambda', \Lambda}$ is illustrated by the fact that the eigenvalue in the L.H.S. is also linear in Λ for $g = Y$ while it is quadratic for $g = J$.

These sets of recurrence relations and difference relations establish the bispectrality of the functions introduced in (60).

7 Particular examples

7.1 Bivariate Krawtchouk polynomials of Griffiths type

Let us consider the case of symmetric representations of sl_3 which has also been looked at in Corollary 5.2. We recall that the GT pattern of the symmetric representations are such that $\lambda_{32} = \lambda_{33}$, $\lambda_{31} = -2\lambda_{33}$ which implies that $\lambda_{22} = \lambda_{32} = \lambda_{33}$.

Theorem 7.1. *The matrix elements of the change of basis σ induced by the rotation S given by (20) are, for Λ, Λ' GT patterns of symmetric representations,*

$$\begin{aligned} \sigma_{\Lambda, \Lambda'} = & \sum_{\ell=\max\{0, \lambda_{21}-\lambda'_{21}\}}^{\lambda_{21}-\lambda_{33}} f_{\ell}(\lambda_{21}, \lambda_{11}, \lambda'_{21}, \lambda'_{11}) K_{\ell}(\lambda_{11} - \lambda_{33}; \sin^2 \phi, \lambda_{21} - \lambda_{33}) \\ & \times K_{\ell+\lambda'_{21}-\lambda_{21}}(\ell; \sin^2 \theta, \ell - \lambda_{21} - 2\lambda_{33}) K_{\lambda'_{11}-\lambda_{33}}(\ell + \lambda'_{21} - \lambda_{21}, \sin^2 \chi, \lambda'_{21} - \lambda_{33}), \end{aligned} \quad (70)$$

with

$$\begin{aligned} f_{\ell}(\lambda_{21}, \lambda_{11}, \lambda'_{21}, \lambda'_{11}) = & (-1)^{\lambda_{11}-2\lambda_{21}+2\lambda_{33}-\lambda'_{22}} (\tan \phi)^{\ell+\lambda_{11}-\lambda_{33}} (\tan \theta)^{2\ell+\lambda'_{21}-\lambda_{21}} (\tan \chi)^{\ell+\lambda'_{21}-\lambda_{21}+\lambda'_{11}-\lambda_{33}} \\ & \times \frac{(\cos \phi)^{\lambda_{21}-\lambda_{33}} (\cos \theta)^{\ell-\lambda_{21}-2\lambda_{33}} (\cos \chi)^{\lambda'_{21}-\lambda_{33}}}{\ell!(\ell + \lambda'_{21} - \lambda_{21})!(\ell - \lambda_{21} - 2\lambda_{33})!(-\ell + \lambda_{21} - \lambda_{33})!} \\ & \times \frac{[(\lambda_{21} - \lambda_{33})!(\lambda'_{21} - \lambda_{33})!]^2}{(\lambda_{21} - \lambda_{11})!(\lambda'_{11} - \lambda_{33})!(-2\lambda_{33} - \lambda_{21})!}. \end{aligned} \quad (71)$$

Proof. The calculation proceeds similarly to what was done in Theorem 6.1, this time using the expressions for the symmetric representation in Corollary 5.2. Combining everything together and setting $\ell = \lambda'_{11} - \lambda_{33}$, the result follows. \square

Making use of the symmetry property of Krawtchouk polynomials

$$K_n(x; p, N) = \frac{n!}{(N-n)!} (-1)^{x+n} p^{n-x} (1-p)^{x+n-N} K_{N-n}(N-x; p, N), \quad (72)$$

one can compare with the expressions found in [1, Equations (10.4)–(10.5)] for the bivariate Griffiths–Krawtchouk polynomials.

7.2 Bivariate hybrid functions in terms of Krawtchouk and Racah polynomials

We consider the transformation T followed by a rotation around the z -axis:

$$S = R_{\eta}^z T = \begin{pmatrix} \cos(\eta) & 0 & \sin(\eta) \\ -\sin(\eta) & 0 & \cos(\eta) \\ 0 & -1 & 0 \end{pmatrix}. \quad (73)$$

It can be obtained with the following choice of angles in the parametrization of a general rotation used in the preceding sections:

$$\theta = \frac{\pi}{2}, \quad \phi = -\frac{\pi}{2}, \quad \chi = \eta + \frac{\pi}{2}. \quad (74)$$

It is simpler to calculate the matrix elements of S directly rather than putting these parameters in the general formula. Indeed, we have:

$$\sigma_{\Lambda', \Lambda} = \sum_{\Lambda^{(1)}} \tau_{\Lambda', \Lambda^{(1)}}(\rho_{\eta})_{\Lambda^{(1)}, \Lambda}. \quad (75)$$

Using the explicit expressions found for the matrix elements τ and ρ_{η} , we get the following result.

Proposition 7.2. For Λ, Λ' GT patterns, the matrix elements of the change of basis σ corresponding to the rotation S are given (up to a global undetermined sign) by ,

$$\begin{aligned} \sigma_{\Lambda' \Lambda} = & \delta_{\lambda_{21} + \lambda_{22}, \lambda'_{11} - \lambda'_{21} - \lambda'_{22}} t(\lambda'_{21}, \lambda'_{11}, \lambda'_{22}) (-1)^{2\lambda_{21} + \lambda'_{21}} \frac{(\lambda_{21} - \lambda_{22})! \tan^{\lambda'_{11} + \lambda_{11} - 2\lambda_{22}}(\eta) \cos^{\lambda_{21} - \lambda_{22}}(\eta)}{(\lambda_{11} - \lambda_{22})! (\lambda_{21} - \lambda'_{11})!} \\ & \times K_{\lambda_{11} - \lambda_{22}}(\lambda'_{11} - \lambda_{22}; \sin^2(\eta), \lambda_{21} - \lambda_{22}) \widetilde{R}_{\lambda_{31} - \lambda_{21}}(\lambda_{31} - \lambda'_{21}; \alpha, \beta, \gamma, \delta), \end{aligned} \quad (76)$$

with

$$\begin{aligned} \alpha &= \lambda_{32} - \lambda_{31} - 1, & \beta &= \lambda'_{11} - \lambda'_{21} - \lambda'_{22} + \lambda_{33} - 1, \\ \gamma &= \lambda'_{11} - \lambda_{31} - 1, & \delta &= \lambda'_{21} + \lambda'_{22} - \lambda'_{11} - \lambda_{31} - 1. \end{aligned} \quad (77)$$

In the previous proposition, a product of Krawtchouk and Racah polynomials appears in each sector where $\lambda_{21} + \lambda_{22} = \lambda'_{11} - \lambda'_{21} - \lambda'_{22}$. It naturally suggests to consider the following bivariate function:

$$P_{n_1, n_2}(x_1, x_2) = K_{n_1}(x_1 - n_2; \sin^2(\eta), N - 2n_2) \widetilde{R}_{n_2}(x_2; \alpha, \beta, x_1 - N - 1, \delta), \quad (78)$$

where N is a positive integer and $\alpha, \beta, \delta, \eta$ are parameters. In the formula of the proposition, N corresponds to $2\lambda_{31} - \lambda_{21} - \lambda_{22}$ and x_1, x_2, n_1 and n_2 to

$$x_1 = \lambda'_{22} + \lambda'_{21} + \lambda_{31}, \quad x_2 = \lambda_{31} - \lambda'_{21}, \quad (79a)$$

$$n_1 = \lambda_{11} - \lambda_{22}, \quad n_2 = \lambda_{31} - \lambda_{21}. \quad (79b)$$

For these choices, the L.H.S. of the recurrence relations of $P_{n_1, n_2}(x_1, x_2)$ obtained by (66) involves only terms of the form $P_{n'_1, n'_2}(x_1, x_2)$. More precisely, (66) for $g = e_{11}$ provides a recurrence relation involving only $P_{n_1 + \varepsilon, n_2}(x_1, x_2)$ and (66) for $g = e_{21}e_{12}$ provides a recurrence relation involving only $P_{n_1 + \varepsilon, n_2}(x_1, x_2)$, $P_{n_1, n_2 + \varepsilon}(x_1, x_2)$, $P_{n_1 + \varepsilon, n_2 - \varepsilon}(x_1, x_2)$ and $P_{n_1 + 2\varepsilon, n_2 - \varepsilon}(x_1, x_2)$, with $\varepsilon \in \{-1, 0, 1\}$. Similarly, the R.H.S. of the difference relations of $P_{n_1, n_2}(x_1, x_2)$ obtained by (68) involves only $P_{n_1, n_2}(x'_1, x'_2)$: for $g = e_{11}$, one obtains $P_{n_1, n_2}(x_1 + \varepsilon, x_2)$ and $P_{n_1, n_2}(x_1 + \varepsilon, x_2 - \varepsilon)$ and for $g = J$, one obtains $P_{n_1, n_2}(x_1, x_2 + \varepsilon)$, with $\varepsilon \in \{-1, 0, 1\}$.

Formula (78) resembles very much the Tratnik construction of bivariate orthogonal polynomials, except that here it is an hybrid of two different univariate polynomials. Such hybrid constructions already appeared, see for example [5, 29] for a formula like (78) using dual Hahn polynomials instead of Racah polynomials.

8 The Racah algebra in $U(sl_3)$ and the centralizer of the Cartan sub-algebra

This section contains some algebraic results concerning the elements J and $\Psi_T(J)$ introduced previously in Section 5.

Racah algebra. The observation that the change of basis associated to T is related to the Racah polynomials can be understood algebraically by remarking that J and $\bar{J} = \Psi_T(J)$ satisfy the relations of the Racah algebra (1). Indeed, let us define

$$K := [J, \bar{J}] = e_{31}e_{12}e_{23} - e_{32}e_{21}e_{13}. \quad (80a)$$

Then, by a lengthy but straightforward calculation using the defining relations of $U(sl_3)$, one can show that

$$[J, K] = 2J^2 + 2\{J, \bar{J}\} - aJ + b^+, \quad (80b)$$

$$[K, \bar{J}] = 2\bar{J}^2 + 2\{J, \bar{J}\} - a\bar{J} + b^-, \quad (80c)$$

where

$$a = h^2 + 3y^2 + C_2, \quad (81a)$$

$$b^\pm = (3y \mp h) \left(\frac{(2 - y \mp h)C_2}{4} - \frac{C_3}{3} + \frac{(y - 2 \pm h)(y + 2 \pm h)(y \mp h)}{8} \right). \quad (81b)$$

The Casimir elements C_2, C_3 of $U(sl_3)$ are defined by (3) and $h = \frac{1}{2}(e_{22} - e_{33})$, $y = \frac{1}{6}(2e_{11} - e_{22} - e_{33})$ are Cartan elements stable under T .

Relations (80) are those of the Racah algebra³. This realization interestingly adds to others previously found, namely between the recurrence and the difference operators of the Racah polynomials [44, 54, 55], in terms of the intermediate Casimir elements of $U(sl_2)^{\otimes 3}$ [8] or in terms of the elements of $U(sl_2)$ [56, 57]. The result here provides a realization of the Racah algebra in terms of the elements of $U(sl_3)$ (also see [37, 49]).

It is well-known that there exists a central element in the Racah algebra given by

$$\Gamma = 2\{J^2, \bar{J}\} + 2\{J, \bar{J}^2\} - K^2 - 4(J + \bar{J})^2 - a\{J, \bar{J}\} + 2(b^- + a)J + 2(b^+ + a)\bar{J}. \quad (82)$$

In the realization (80)-(81) introduced here, this central element takes a definite value in terms of the other central elements of the Racah algebra:

$$\begin{aligned} \Gamma = & \frac{1}{2}(y^2 + h^2)^2(1 - C_2) - \frac{1}{8}(h^2 - y^2)^3 + 2h^2y^2 - \left(\frac{C_3}{3} + 2y + yC_2\right) \left(\frac{C_3}{3} + 2y + yC_2 - C_2\right) \\ & - \frac{y}{6}(3h^2 - 11y^2)(3C_2 - 2C_3) + \frac{C_2}{8}(h^2 + 3y^2)(5h^2 - y^2 + 4). \end{aligned} \quad (83)$$

The quotient of the Racah algebra by the previous relation (83) is called the special Racah algebra [48]. The terminology ‘‘special’’ comes from [58], where a similar quotient, called ‘‘special Askey–Wilson algebra’’, was introduced for the Askey–Wilson algebra. This type of quotient appears in various other situations and, in particular, as the diagonal centralizer of $U(sl_2)$ in $U(sl_2)^{\otimes 3}$ [48].

Isomorphism with a centralizer. In this paragraph, the centralizer \mathcal{Z} of the Cartan subalgebra in $U(sl_3)$ is described using the elements previously introduced. The centralizer is defined as follows:

$$\mathcal{Z} = \{x \in U(sl_3) \mid [x, H_i] = 0, i = 1, 2\}, \quad (84)$$

where $H_1 = \frac{1}{2}(e_{11} - e_{22})$ and $H_2 = \frac{1}{2}(e_{22} - e_{33})$. Assigning degree 1 to each generator e_{ij} , the algebra $U(sl_3)$ is filtered and so is its subalgebra \mathcal{Z} , which allows to describe \mathcal{Z} as in the following proposition.

Proposition 8.1. *The algebra \mathcal{Z} is generated by*

$$J, \bar{J}, K, H_1, H_2, C_2, C_3, \quad (85)$$

and its Hilbert–Poincaré series is:

$$F_{\mathcal{Z}}(t) = \frac{1 + t^3}{(1 - t)^2(1 - t^2)^3(1 - t^3)}. \quad (86)$$

Proof. A monomial $H_1^a H_2^b e_{21}^c e_{12}^d e_{32}^e e_{23}^f e_{31}^g e_{13}^h$ commutes with H_1 and H_2 if and only if:

$$\begin{cases} 2(d - c) + (e - f) + (h - g) = 0, \\ (c - d) + 2(f - e) + (h - g) = 0, \end{cases} \quad d - c = f - e = h - g.$$

Considering either $d - c \geq 0$ or $d - c \leq 0$, it follows that \mathcal{Z} is spanned by the following elements:

$$\{H_1^a H_2^b (e_{21}e_{12})^i (e_{32}e_{23})^j (e_{31}e_{13})^k x\}_{a,b,i,j,k \geq 0}, \quad \text{with } x \in \{(e_{12}e_{23}e_{31})^\ell, (e_{32}e_{21}e_{13})^\ell\}_{\ell \geq 0}. \quad (87)$$

³Up to a trivial rescaling of the generators such as $J = -2K_2$, $\bar{J} = -2K_1$, $a = 4d$, $b^+ = 8e_1$, $b^- = 8e_2$.

This shows that \mathcal{Z} is generated as an algebra by:

$$H_1, H_2, e_{21}e_{12}, e_{32}e_{23}, e_{31}e_{13}, e_{31}e_{12}e_{23}, e_{32}e_{21}e_{13}. \quad (88)$$

By direct computations, one extracts

$$e_{21}e_{12} = J - H_1(H_1 + 1), \quad (89a)$$

$$e_{31}e_{13} = \bar{J} - (H_1 + H_2)(H_1 + H_2 + 1), \quad (89b)$$

$$e_{32}e_{23} = \frac{1}{2}C_2 - J - \bar{J} - H_2 + \frac{1}{3}(2H_1H_2 + 2H_1^2 - H_2^2), \quad (89c)$$

$$e_{31}e_{12}e_{23} - e_{32}e_{21}e_{13} = K, \quad (89d)$$

$$e_{31}e_{12}e_{23} + e_{32}e_{21}e_{13} = \frac{1}{3}C_3 - 2J(H_1 + H_2 + 1) - 2\bar{J}H_1 + \frac{1}{3}C_2(2H_1 + H_2) \\ + \frac{2}{27}(H_2 + 2H_1)(11H_1^2 + 11H_1H_2 - 4H_2^2) - \frac{2}{3}(2H_1H_2 - H_1^2 + 2H_2^2 + 2H_2 - H_1). \quad (89e)$$

It follows easily that \mathcal{Z} is generated by the elements in (85).

To calculate the Hilbert–Poincaré series, we sum the degrees of the elements in (87), which are linearly independent in $U(\mathfrak{sl}_3)$. Namely, one gets

$$F_{\mathcal{Z}}(t) = \sum_{a,b,i,j,k \geq 0} t^{a+b+2(i+j+k)} \left(1 + 2 \sum_{\ell \geq 1} t^{3\ell} \right) = \frac{1}{(1-t)^2(1-t^2)^3} \left(1 + \frac{2t^3}{(1-t^3)} \right), \quad (90)$$

which leads to the desired result. \square

Recall that the elements H_1, H_2, C_2, C_3 are central in the subalgebra \mathcal{Z} . Moreover, we have found some generators of \mathcal{Z} together with the relations (80) and (83). These relations allow to write any element of \mathcal{Z} as a linear combination of

$$\{H_1^a H_2^b C_2^c C_3^d J^i \bar{J}^j K^k\}_{a,b,c,d,i,j \geq 0, k \in \{0,1\}}. \quad (91)$$

This is true since we can reorder any product using (80), the elements H_1, H_2, C_2, C_3 are central, and K^2 is rewritten using the special relation (83).

Thus, the above set is a spanning set of \mathcal{Z} , and comparing with the Hilbert–Poincaré series found previously, we arrive at the following corollary.

Corollary 8.2. *The centralizer \mathcal{Z} is isomorphic to the algebra generated by J, \bar{J}, K and the central elements H_1, H_2, C_2, C_3 with the defining relations (80) and (83). A basis is given by the set (91).*

9 Conclusion

The realization of the rank one Racah algebra as the centralizer in $U(\mathfrak{sl}_3)$ of the Cartan subalgebra naturally brings to mind its generalization to $U(\mathfrak{sl}_n)$ for any n . This suggests a way of realizing higher rank Racah algebras which is different from the generalization related to tensor products of $U(\mathfrak{sl}_2)$. Centralizers of the Cartan subalgebras were studied in [49] but a description of the resulting algebra is still not known for arbitrary n . Similar results for any simple Lie (super)-algebra should be also very interesting. We have so far focused on finite-dimensional representations of the algebra and obtained correspondingly finite families of functions. Infinite-dimensional representations should provide similar results, involving infinite families of functions.

Studying the cases associated to the quantum groups can also be envisaged. The results of this paper should generalize: the centralizer of the Cartan subalgebra of $U_q(\mathfrak{sl}_3)$ should be associated to the Askey–Wilson algebra, the Krawtchouk polynomials should be replaced by certain q -Krawtchouk polynomials (see [59, 60]) and the Racah polynomials, by the q -Racah polynomials. We plan on pursuing these questions.

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Data Availability

Data sharing not applicable – no new data generated.

References

- [1] V. X. Genest, L. Vinet, and A. Zhedanov, *The multivariate Krawtchouk polynomials as matrix elements of the rotation group representations on oscillator states*, Journal of Physics A: Mathematical and Theoretical **46**(50), 505203 (2013), arXiv:1306.4256.
- [2] T. H. Koornwinder, *Krawtchouk Polynomials, a Unification of Two Different Group Theoretic Interpretations*, SIAM Journal on Mathematical Analysis **13**(6), 1011–1023 (1982).
- [3] N. J. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions*, Mathematics and Its Applications, volume 72 (Springer, Dordrecht, 1991), 612 pp.
- [4] J. Van der Jeugt, *Coupling coefficients for Lie algebra representations and addition formulas for special functions*, Journal of Mathematical Physics **38**, 2728 (1997).
- [5] C. F. Dunkl, *A Krawtchouk Polynomial Addition Theorem and Wreath Products of Symmetric Groups*, Indiana University Mathematics Journal **25**(4), 335–358 (1976).
- [6] J. A. Wilson, *Some Hypergeometric Orthogonal Polynomials*, SIAM Journal on Mathematical Analysis **11**(4), 690–701 (1980).
- [7] A. F. Nikiforov, V. B. Uvarov, and S. V. Suslov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer Series in Computational Physics, [Klassicheskie ortogonal’nye polinomy diskretnoi peremennoi, Moscow, Nauka, 1985] (Springer-Verlag, Berlin Heidelberg, 1991), 374 pp.
- [8] Y. A. Granovskii and A. S. Zhedanov, *Nature of the symmetry group of the $6j$ -symbol*, Journal of Experimental and Theoretical Physics **94**(10), 49–54 (1988).
- [9] R. C. Griffiths, *Orthogonal Polynomials on the Multinomial Distribution*, Australian & New Zealand Journal of Statistics **13**(1), 27–35 (1971).
- [10] M. V. Tratnik, *Some multivariable orthogonal polynomials of the Askey tableau-discrete families*, Journal of Mathematical Physics **32**, 2337–2342 (1991).
- [11] H. Mizukawa and H. Tanaka, *$(n + 1, m + 1)$ -Hypergeometric Functions Associated to Character Algebras*, Proceedings of the American Mathematical Society **132**(9), 2613–2618 (2004).
- [12] M. R. Hoare and M. Rahman, *A Probabilistic Origin for a New Class of Bivariate Polynomials*, Symmetry, Integrability and Geometry: Methods and Applications **4**, 089 (2008), arXiv:0812.3879.
- [13] A. V. Rosenblyum, *Spectral analysis of generators of representations of the group $U(3)$* , Theoretical and Mathematical Physics **73**, 1352–1356 (1987).
- [14] A. Zhedanov, *$9j$ -symbols of the oscillator algebra and Krawtchouk polynomials in two variables*, Journal of Physics A: Mathematical and General **30**(23), 8337 (1997).

- [15] P. Iliev and P. Terwilliger, *The Rahman polynomials and the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$* , Transactions of the American Mathematical Society **364**(8), 4225–4238 (2012), arXiv:1006.5062.
- [16] P. Iliev, *A Lie-theoretic interpretation of multivariate hypergeometric polynomials*, Compositio Mathematica **148**(3), 991–1002 (2012), arXiv:1101.1683.
- [17] N. Crampé, W. van de Vijver, and L. Vinet, *Racah problems for the oscillator algebra, the Lie algebra \mathfrak{sl}_n , and multivariate Krawtchouk polynomials*, Annales Henri Poincaré **21**, 3939–3971 (2020), arXiv:1909.12643.
- [18] P. R. Milch, *A Multi-dimensional Linear Growth Birth and Death Process*, The Annals of Mathematical Statistics **39**(3), 727–754 (1968).
- [19] P. Diaconis and R. Griffiths, *An introduction to multivariate Krawtchouk polynomials and their applications*, Journal of Statistical Planning and Inference **154**, 39–53 (2014), arXiv:1309.0112.
- [20] J. S. Geronimo and P. Iliev, *Bispectrality of Multivariable Racah–Wilson Polynomials*, Constructive Approximation **31**(1), 417–457 (2010), arXiv:0705.1469.
- [21] H. De Bie and W. van de Vijver, *A discrete realization of the higher rank Racah algebra*, Constructive Approximation **52**, 1–29 (2020), arXiv:1808.10520.
- [22] N. Crampé, L. Frappat, and E. Ragoucy, *Representations of the rank two Racah algebra and orthogonal multivariate polynomials*, Linear Algebra and its Applications **664**, 165–215 (2023), arXiv:2206.01031.
- [23] E. Chacón and M. Moshinsky, *Representations of finite U_3 transformations*, Physics Letters **23**(10), 567–569 (1966).
- [24] Z. Pluhař, Y. F. Smirnov, and V. N. Tolstoy, *Simple construction of $SU(3)$ representations using the $SU(2)$ projection technique*, Czechoslovak Journal of Physics B **35**, 593–601 (1985).
- [25] I. Gelfand and A. Zelevinsky, *Canonical basis in irreducible representations of \mathfrak{gl}_3 and its applications*, in *Group theoretical methods in physics, Vol. II* (VNU Science Press, Utrecht, 1986), pp. 127–146.
- [26] H. Tarnanen, M. J. Aaltonen, and J.-M. Goethals, *On the Nonbinary Johnson Scheme*, European Journal of Combinatorics **6**(3), 279–285 (1985).
- [27] V. X. Genest, H. Miki, L. Vinet, and A. Zhedanov, *The multivariate Charlier polynomials as matrix elements of the Euclidean group representation on oscillator states*, Journal of Physics A: Mathematical and Theoretical **47**, 215204 (2014), arXiv:1401.7901.
- [28] P.-A. Bernard, N. Crampé, L. Poulain d’Andecy, L. Vinet, and M. Zaimi, *Bivariate P -polynomial association schemes*, (2023), arXiv:2212.10824.
- [29] N. Crampé, L. Vinet, M. Zaimi, and X. Zhang, *A bivariate Q -polynomial structure for the non-binary Johnson scheme*, (2023), arXiv:2306.01882.
- [30] D. A. Leonard, *Orthogonal Polynomials, Duality and Association Schemes*, SIAM Journal on Mathematical Analysis **13**(4), 656–663 (1982).
- [31] F. A. Grünbaum, *The Bispectral Problem: An Overview*, in *Special Functions 2000: Current Perspective and Future Directions* (Springer, Dordrecht, 2001), pp. 129–140.
- [32] J.-M. Lévy-Leblond and M. Lévy-Nahas, *Symmetrical Coupling of Three Angular Momenta*, Journal of Mathematical Physics **6**(9), 1372–1380 (1965).
- [33] H.-W. Huang and S. Bockting-Conrad, *Finite-Dimensional Irreducible Modules of the Racah Algebra at Characteristic Zero*, Symmetry, Integrability and Geometry: Methods and Applications **16**, 018–034 (2020), arXiv:1910.11446.
- [34] H.-W. Huang and S. Bockting-Conrad, *The Casimir elements of the Racah algebra*, Journal of Algebra and Its Applications, 2150135 (2020), arXiv:1711.09574.

- [35] V. X. Genest, L. Vinet, and A. Zhedanov, *The equitable Racah algebra from three $\mathfrak{su}(1, 1)$ algebras*, Journal of Physics A: Mathematical and Theoretical **45**(2), 025203 (2013), arXiv:1309.3540.
- [36] V. X. Genest, L. Vinet, and A. Zhedanov, *Embeddings of the Racah Algebra into the Bannai–Ito Algebra*, Symmetry, Integrability and Geometry: Methods and Applications **11**, 050 (2015), arXiv:1504.00558.
- [37] F. Correa, M. A. del Olmo, I. Marquette, and J. Negro, *Polynomial algebras from $\mathfrak{su}(3)$ and a quadratically superintegrable model on the two sphere*, Journal of Physics A: Mathematical and Theoretical **54**, 015205 (2021), arXiv:2007.11163.
- [38] P. Terwilliger and R. Vidunas, *Leonard pairs and the Askey–Wilson relations*, Journal of Algebra and Its Applications **03**(04), 411–426 (2004), arXiv:math/0305356.
- [39] T. H. Koornwinder, *The Relationship between Zhedanov’s Algebra $AW(3)$ and the Double Affine Hecke Algebra in the Rank One Case*, Symmetry, Integrability and Geometry: Methods and Applications **3**, 063 (2007), arXiv:math/0612730.
- [40] N. Crampé, L. Poulain d’Andecy, and L. Vinet, *Temperley–Lieb, Brauer and Racah algebras and other centralizers of $\mathfrak{su}(2)$* , Transactions of the American Mathematical Society **373**(7), 4907–4932 (2020), arXiv:1905.06346.
- [41] S. Bockting-Conrad and H.-W. Huang, *The universal enveloping algebra of \mathfrak{sl}_2 and the Racah algebra*, Communications in Algebra **48**, 1022–1040 (2020), arXiv:1907.02135.
- [42] J. Gaboriaud, L. Vinet, S. Vinet, and A. Zhedanov, *The Racah algebra as a commutant and Howe duality*, Journal of Physics A: Mathematical and Theoretical **51**(50), 50LT01 (2018), arXiv:1808.05261.
- [43] E. G. Kalnins, W. Miller Jr., and S. Post, *Wilson polynomials and the generic superintegrable system on the 2-sphere*, Journal of Physics A: Mathematical and Theoretical **40**(38), 11525–11538 (2007).
- [44] V. X. Genest, L. Vinet, and A. Zhedanov, *The Racah algebra and superintegrable models*, Journal of Physics: Conference Series **512**, 012011 (2014), arXiv:1312.3874.
- [45] E. G. Kalnins, W. Miller Jr., and S. Post, *Two-Variable Wilson Polynomials and the Generic Superintegrable System on the 3-Sphere*, Symmetry, Integrability and Geometry: Methods and Applications **7**, 051 (2011), arXiv:1010.3032.
- [46] S. Post, *Racah Polynomials and Recoupling Schemes of $\mathfrak{su}(1, 1)$* , Symmetry, Integrability and Geometry: Methods and Applications **11**, 057 (2015), arXiv:1504.03705.
- [47] H. De Bie, V. X. Genest, W. van de Vijver, and L. Vinet, *A higher rank Racah algebra and the \mathbb{Z}_2^n Laplace–Dunkl operator*, Journal of Physics A: Mathematical and Theoretical **51**(2), 025203 (2017), arXiv:1610.02638.
- [48] N. Crampé, J. Gaboriaud, L. Poulain d’Andecy, and L. Vinet, *Racah algebras, the centralizer $Z_n(\mathfrak{sl}_2)$ and its Hilbert–Poincaré series*, Annales Henri Poincaré, 10.1007/s00023-021-01152-y (2022), arXiv:2105.01086.
- [49] R. Campoamor-Stursberg, D. Latini, I. Marquette, and Y.-Z. Zhang, *Algebraic (super-)integrability from commutants of subalgebras in universal enveloping algebras*, Journal of Physics A: Mathematical and Theoretical **56**, 045202 (2023), arXiv:2211.04664.
- [50] A. I. Molev, *Gelfand–Tsetlin bases for classical Lie algebras*, Handbook of Algebra **4**, 109–170 (2006), arXiv:math/0211289.
- [51] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q -Analogues*, Springer Monographs in Mathematics (Springer-Verlag, Berlin Heidelberg, 2010), 578 pp.
- [52] J. Wilson, *Hypergeometric series recurrence relations and some new orthogonal functions*, PhD thesis (University of Wisconsin, 1978).

- [53] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia of Mathematics and its Applications (Cambridge University Press, Cambridge, 2004), 428 pp.
- [54] A. S. Zhedanov, “*Hidden symmetry*” of Askey–Wilson polynomials, Theoretical and Mathematical Physics **89**, [Teoreticheskaya i Matematicheskaya Fizika, **89**(2), 190–204, 1991], 1146–1157 (1992).
- [55] V. X. Genest, L. Vinet, and A. Zhedanov, *Superintegrability in Two Dimensions and the Racah–Wilson Algebra*, Letters in Mathematical Physics **104**, 931–952 (2014), arXiv:1307.5539.
- [56] Y. I. Granovskii and A. S. Zhedanov, *Linear covariance algebra for $SL_q(2)$* , Journal of Physics A: Mathematical and General **26**(7), L357–L359 (1993).
- [57] N. Crampé, D. Shaaban Kabakibo, and L. Vinet, *New realizations of algebras of the Askey–Wilson type in terms of Lie and quantum algebras*, Reviews in Mathematical Physics **33**, 2150002 (2021), arXiv:2005.06957.
- [58] N. Crampé, L. Frappat, J. Gaboriaud, L. Poulain d’Andecy, E. Ragoucy, and L. Vinet, *The Askey–Wilson algebra and its avatars*, Journal of Physics A: Mathematical and Theoretical **54**(6), 063001 (2021), arXiv:2009.14815.
- [59] V. X. Genest, S. Post, L. Vinet, G.-F. Yu, and A. Zhedanov, *q -Rotations and Krawtchouk polynomials*, The Ramanujan Journal **40**, 335–357 (2016), arXiv:1408.5292.
- [60] V. X. Genest, S. Post, and L. Vinet, *An algebraic interpretation of the multivariate q -Krawtchouk polynomials*, The Ramanujan Journal **43**, 415–445 (2017), arXiv:1508.07770.