A quasilinear elliptic equation with absorption term and Hardy potential

Marie-Françoise Bidaut-Véron, Huyuan Chen

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Abstract

Here we study the positive solutions of the equation

$$-\Delta_p u + \mu \frac{u^{p-1}}{|x|^p} + |x|^\theta u^q = 0, \qquad x \in \mathbb{R}^N \setminus \{0\}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ and $1 < p < N, q > p-1, \mu, \theta \in \mathbb{R}$. We give a complete description of the existence and the asymptotic behaviour of the solutions near the singularity 0, or in an exterior domain. We show that the global solutions $\mathbb{R}^N \setminus \{0\}$ are radial and give their expression according to the position of the Hardy coefficient μ with respect to the critical exponent $\mu_0 = -(\frac{N-p}{p})^p$. Our method consists into proving that any nonradial solution can be compared to a radial one, then making exhaustive radial study by phase-plane techniques. Our results are optimal, extending the known results when $\mu = 0$ or $p = 2$, with new simpler proofs.They make in evidence interesting phenomena of nonuniqueness when $\theta + p = 0$, and of existence of locally constant solutions when moreover $p > 2$.

Contents

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1 Introduction

Here we study the positive solutions u in a domain Ω of $\mathbb{R}^N$ of the quasilinear equation with a Hardy term

$$-\Delta_p u + \mu \frac{u^{p-1}}{|x|^p} + |x|^\theta u^q = 0, \tag{1.1}$$

where Δ_p is the p-Laplace operator $u \longmapsto \Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ and

$$1 < p < N, \qquad q > p-1, \qquad \mu, \theta \in \mathbb{R}.$$

We consider the problem of the isolated singularities at 0 with $\Omega = B_{r_0} \setminus \{0\}$ (a priori estimates, description of the asymptotic behaviour of the solutions, existence of local solutions of any possible type), the problem in an exterior domain with $\Omega = \mathbb{R}^N \backslash \overline{B_{r_0}}$ (same questions), and the global problem in $\mathbb{R}^N \setminus \{0\}$ (existence of solutions, radiality or nonradiality of the possible solutions).

The case $p = 2$ and $q > 1, \theta = 0$ was studied by Guerch and Veron [18], who gave the precise behaviour near 0 of the solutions of any sign of equation

$$-\Delta u + \mu \frac{u}{|x|^2} + g(u) = 0 \tag{1.2}$$

and isotropy results when $g(u)$ has a power-like growth. They extended many properties the solutions of the classical problem

$$-\Delta u + u^q = 0$$

object of an impressive number of articles, starting to the pionneer papers [7],[29], and also [6].It is clear that problem (1.2) is deeply linked to the Hardy operator

$$v \longmapsto \mathcal{L}_{2,\mu} v = -\Delta v + \mu \frac{v}{|x|^2}$$

for which the value $\mu = \mu_0 = -(\frac{N-2}{2})^2$ plays an essential role, due to the Hardy inequality in bounded Ω

$$\int_\Omega |\nabla u|^2 \, dx \geq (\frac{N-2}{2})^2 \int_\Omega \frac{u^2}{|x|^2} dx, \qquad \forall u \in W_0^{1,2}(\Omega).$$

In [9], Cirstea who considered the positive solutions of

$$-\Delta u + \mu \frac{u}{|x|^2} + b(x)g(u) = 0$$

where $g(u)/u^q$ and $b(x)/|x|^\theta$ may have a logarithmic behaviour, giving a precise behaviour near 0, in the case $\mu \geq -(\frac{N-2}{2})^2$, and near ∞ by Kelvin transform. In the case of equation

$$-\Delta u + \mu \frac{u}{|x|^2} + |x|^\theta u^q = 0 \tag{1.3}$$

twith $\mu < -(\frac{N-2}{2})^2$ and $\theta + 2 > 0$, Wei and Du [30] gave a precise behaviour near 0, and a uniqueness result of global solutions in $\mathbb{R}^N \setminus \{0\}$. The problem of existence and radiality of global solutions was then solved in [11] for any value of θ, by using thin techniques of supersolutions and subsolutions.

The case $p = 2$ and $q < 1$ was treated in [5], showing in particular the existence of many nonradial solutions, with possible dead cores.

In case $p > 1$, the quasilinear equation where $\mu = 0, \theta = 0, q > p - 1$,

$$-\Delta_p u + u^q = 0 \tag{1.4}$$

was studied by [15] and [28]. Concerning the behaviour of any solution u in $B_{r_0} \setminus \{0\}$, it was shown that three eventualities occur when $q < q_c = \frac{N(p-1)}{N-p}$: either $\lim_{|x| \longrightarrow 0} |x|^{\frac{p}{q+1-p}} \; u = C_{N,p,q} > 0$, or $\lim_{|x| \longrightarrow 0} |x|^{N-p} \; u = k > 0$, or u can be extended as a solution in B_{r_0}; and the singularity is removable if and only if $q \geq q_c$.

In case of quasilinear equation (1.1), the Hardy operator

$$v \longmapsto \mathcal{L}_{p,\mu} v = -\Delta_p v + \mu \frac{v^{p-1}}{|x|^p} \tag{1.5}$$

plays an essential role, and the critical value is

$$\mu_0 = -(\frac{N-p}{p})^p, \tag{1.6}$$

coming from the extended Hardy inequality

$$\int_\Omega |\nabla u|^p \, dx \geq (\frac{N-p}{p})^p \int_\Omega \frac{|u|^p}{|x|^p} dx, \qquad \forall u \in W_0^{1,p}(\Omega).$$

For a precise study of the Hardy operator, we refer for exemple to [14].

Quasilinear equations involving this operator with a source term, of type

$$-\Delta_p u + \mu \frac{u^{p-1}}{|x|^p} = u^q$$

have been the object if an intensive study, starting from the pioneer articles of Serrin [22],[23], and [16], [8], [24], when $\mu = 0$; in particular in the case of critical growth $q = q_s = \frac{N(p-1)+p}{N-p.}$, for solutions with finite energy: the problem of existence, uniqueness and behaviour of the radial solutions were established in [1], sharp asymptotic estimates of the solutions were given in [31], [32], and their radiality was obtained by moving planes technique in [21].

To our knowledge, equation (1.1) has been much less studied. We can mention an article of [13] which extends the results of [30] in restrictive conditions on the parameters. The Dirichlet problem in bounded Ω for the nonhomogeneous equation $\mathcal{L}_{p,\mu} v + v^q = h$ where $h \in L^1(\Omega)$, $h \neq 0$ was considered in [20].

Here our purpose is to extend to equation (1.1) for any $p > 1$ and $q > p - 1$, the study of [9], [11] relative to the case $p = 2$, and simplify the proofs, by a **quite different approach**. 0ur line of attack is new, even in the case of equation (1.4). It is built on a complete study of the radial case by phase-plane techniques. Indeed, equation (1.1) is invariant by the scaling T_k defined for any $k > 0$ and $x \in \mathbb{R}^N \setminus \{0\}$ by

$$T_k u(x) = k^\gamma u(kx), \tag{1.7}$$

where

$$\gamma = \frac{p + \theta}{q + 1 - p}. \tag{1.8}$$

Then its radial formulation can be reduced to an autonomous system of order 2. A precise analysis of the phase-plane first implies many properties of existence and possible uniqueness of local and global radial solutions. Then our **key point** is to show that any nonradial solution can be compared from above and below with a radial one, as shown at Theorem 5.10. In that way we avoid a construction of supersolutions and subsolutions, in general hard with a quasilinear equation such as (1.1). In particular we show that **all the global solutions in $\mathbb{R}^N \setminus \{0\}$ are radial.** Our results are optimal.

2 Main results

2.1 Parameters of the study

• We first consider the equation

$$\mathcal{L}_{p,\mu} u = 0 \tag{2.1}$$

and search of the form $u(x) = C\,|x|^{-S}$, with $C > 0$; then

$$\mathcal{L}_{p,\mu} u = 0 \Longleftrightarrow \varphi(S) = 0; \tag{2.2}$$

where

$$\varphi(S) = (p-1)\,|S|^p - (N-p)\,|S|^{p-2}\,S - \mu; \tag{2.3}$$

the function φ admits a minimum value $\varphi(\frac{N-p}{p}) = -(\frac{N-p}{p})^p - \mu = \mu_0 - \mu$, thus such solutions exist if and only if $\mu \geq \mu_0$. In any case

$$S_1 > 0, \qquad S_2 \leq \frac{N-p}{p} \leq S_1. \tag{2.4}$$

If $\mu > 0$ we obtain two roots $S_2 < 0 < S_1$. If $\mu_0 < \mu < 0$ we obtain two roots $0 < S_2 < S_1$. If $\mu = 0$, $S_2 = 0 < S_1 = \frac{N-p}{p-1}$. If $\mu = \mu_0$, we find only one root $S_1 = S_2 = \frac{N-p}{p}$, corresponding to solutions $u(x) = C\,|x|^{-\frac{N-p}{p}}$. Note that for $\mu = \mu_0$, equation (2.1) admits also radial local positive solutions with a logarithmic behaviour :

$$\lim_{x \longrightarrow 0} |x|^{\frac{N-p}{p}} \, |\ln |x||^{-\frac{2}{p}} \, u(x) = C_1 > 0, \qquad \text{or} \quad \lim_{|x| \longrightarrow \infty} |x|^{\frac{N-p}{p}} \, |\ln |x||^{-\frac{2}{p}} \, u(x) = C_2 > 0,$$

(explicit and well known when $p = 2$) see **[19]** and a short proof at Lemma 4.5 below.

• Next we consider equation (1.1), which takes the form

$$\mathcal{L}_{p,\mu} u + |x|^{\theta} \, u^q = 0 \tag{2.5}$$

and search a particular power solution of the form $u^*(x) = a^* \, |x|^{-\lambda}$. If it exists, then necessarily $\lambda = \gamma$, defined at (1.8) and

$$u^*(x) = a^* \, |x|^{-\gamma} \text{ and } (a^*)^{q+1-p} = \varphi(\gamma) > 0, \tag{2.6}$$

and the existence of u only depends on the position of γ with respect to S_1, S_2.

• In the sequel we divide our analysis into 5 assumptions:

$$\begin{cases} (\mathcal{H}_1): & \mu \geq \mu_0 \quad \text{and } \gamma > S_1, \\ (\mathcal{H}_2): & \mu \geq \mu_0 \quad \text{and } \gamma < S_2, \\ (\mathcal{H}_3): & \mu > \mu_0 \quad \text{and } S_2 \leq \gamma \leq S_1, \\ (\mathcal{H}_4): & \mu = \mu_0 \quad \text{and } \gamma = \frac{N-p}{p}, \\ (\mathcal{H}_5): & \mu < \mu_0. \end{cases} \tag{2.7}$$

Then the solution u^* exists exclusively in cases $(\mathcal{H}_1)$,$(\mathcal{H}_2)$ $(\mathcal{H}_5)$.

We note that γ has the sign of $p + \theta$. These different cases do not involve the sign of γ. but a main difference is the behaviour of the solution u^*, when it exists: when $p + \theta > 0$, then u^* is singular, decreasing from ∞ as $r \to 0$ to 0 as $r \to \infty$; when $p + \theta < 0$, then $u^* \in C([0, \infty)$, increasing from 0 to ∞.

When $p + \theta = 0$, that means $\gamma = 0$, equation (1.1) takes the form

$$-\Delta_p u + \frac{u^{p-1}(u^{q+1-p} + \mu)}{|x|^p} = 0, \tag{2.8}$$

so that for $\mu < 0, \mu \neq \mu_0$, the function $u^* \equiv |\mu|^{\frac{1}{q+1-p}}$ is a **constant solution**. This case has a specific interest. Indeed there hold phenomena of **nonuniqueness** of solutions in a neighborhood of 0, such that $\lim_{r \longrightarrow 0} u = |\mu|^{\frac{1}{q+1-p}}$, possibly global, see Remark 4.11. Moreover when $p > 2$, we show the existence of **locally constant solutions**, but not identically constant, near 0 or ∞, and possibly global. It can happen under $(\mathcal{H}_2)$ when $\mu_0 \leq \mu < 0$, or $(\mathcal{H}_5)$.

2.2 Local and global behaviour of the solutions

Next we give our main results on the behaviour near 0 or ∞ of the solutions, and on the existence and possible uniqueness of the global solutions. All the following theorems extend to the case $p > 1$ the results of [9] and [11] relative to equation (1.3). Note that the behaviour of the solutions in $\mathbb{R}^N \backslash \overline{B_{r_0}}$ cannot be obtained from the ones in $B_{r_0} \backslash \{0\}$ by Kelvin transform, see also Remark 7.9 below.

We say that $u \geq 0$ is a solution of equation (1.1) in $\Omega = B_{r_0} \backslash \{0\}$ (resp. $\Omega = \mathbb{R}^N \backslash \overline{B_{r_0}}$) if $u \in L^\infty_{loc}(\Omega)$, $|\nabla u| \in L^p_{loc}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi dx + \int_\Omega (\mu \frac{u^{p-1}}{|x|^p} + |x|^\theta u^q) \varphi dx = 0, \qquad \forall \varphi \in C^1_c(\Omega).$$

It follows from [25, Theorem 1] that $u \in C^1(\Omega)$.

Our results are given following the different assumptions (2.7)

Theorem 2.1 ***Case*** *($\mathcal{H}_1$) Let $\mu \geq \mu_0$ and $\gamma > S_1$.*
(i) Let u be any positive solution in $B_{r_0} \backslash \{0\}$. Then

- *either*
$$\lim_{|x| \to 0} |x|^\gamma u = a^*, \tag{2.9}$$
- *or*
$$\begin{cases} \lim_{r \to 0} |x|^{S_1} u = k_1 > 0 \text{ if } \mu > \mu_0 \\ \lim_{x \to 0} |x|^{\frac{N-p}{p}} (|\ln |x||^{\frac{2}{p}}) u(x) = l > 0, \text{ if } \mu = \mu_0 \end{cases} \tag{2.10}$$
- *or*
$$\lim_{|x| \to 0} |x|^{S_2} u = k_2 > 0. \tag{2.11}$$

(ii) Let u be any positive solution in $\mathbb{R}^N \backslash \overline{B_{r_0}}$. Then

$$\lim_{|x| \to \infty} |x|^\gamma u = a^*. \tag{2.12}$$

There exist solutions of each type.

(iii) There exists global solutions in $\mathbb{R}^N \backslash \{0\}$, all of them are radial:

- $u^* = a^* |x|^{-\gamma}$,
- *if $\mu > \mu_0$, for any $k_1 > 0$,there exist a unique global solution such that*
$$\lim_{r \to 0} |x|^{S_1} u = k_1 > 0, \qquad \lim_{r \to \infty} |x|^\gamma u = a^*, \tag{2.13}$$
- *if $\mu = \mu_0$ there exists a unique global solution such that*
$$\lim_{x \to 0} |x|^{\frac{N-p}{p}} |\ln |x||^{\frac{2}{p}} u(x) = l, \qquad \lim_{|x| \to \infty} |x|^\gamma u = a^*. \tag{2.14}$$

Theorem 2.2 ***Case*** *($\mathcal{H}_2$): Let $\mu \geq \mu_0$ and $\gamma < S_2$*
(i) Let u be any positive solution in $B_{r_0} \backslash \{0\}$. Then

$$\lim_{|x| \to 0} |x|^\gamma u = a^*. \tag{2.15}$$

(ii) Let u be any positive solution in $\mathbb{R}^N\backslash\overline{B_{r_0}}$. Then

- *either*

$$\lim_{|x|\to\infty} |x|^{\gamma}\, u = a^*, \tag{2.16}$$

- *or*

$$\begin{cases} \lim_{|x|\to\infty} |x|^{S_2}\, u = k_2 > 0 & \text{if } \mu > \mu_0, \\ \lim_{x\longrightarrow\infty} |x|^{\frac{N-p}{p}} \left(|\ln|x||^{\frac{2}{p}}\right) u(x) = \ell > 0 & \text{if } \mu = \mu_0, \end{cases} \tag{2.17}$$

- *or*

$$\lim_{|x|\to\infty} |x|^{S_1}\, u = k_1 > 0.$$

there exist solutions of each type.
(iii) There exist global solutions in $\mathbb{R}^N\backslash\{0\}$, given by:

- $u^* = a^*\,|x|^{-\gamma}$.
- *if $\mu > \mu_0$, for any $k_2 > 0$, there exist a unique global solution (then radial), such that*

$$\lim_{x\longrightarrow 0} |x|^{\gamma}\, u = a^*. \qquad \lim_{|x|\to\infty} |x|^{S_2}\, u = k_2 > 0, \tag{2.18}$$

- *if $\mu = \mu_0$, for any $\ell > 0$, there exists a unique global solution such that*

$$\lim_{x\longrightarrow 0} |x|^{\gamma}\, u = a^*, \qquad \lim_{|x|\to\infty} |x|^{\frac{N-p}{p}} \left(|\ln|x||^{\frac{2}{p}}\right) u(x) = \ell > 0. \tag{2.19}$$

Theorem 2.3 ***Case*** *($\mathcal{H}_3$) Let $\mu > \mu_0$ and $S_2 \le \gamma \le S_1$. Then*
(i) Let u be any positive solution in $B_{r_0}\backslash\{0\}$. Then $\gamma < S_1$ and

$$\begin{aligned} \lim_{x\longrightarrow 0} |x|^{S_2}\, u &= k_2 > 0 \qquad \text{if } S_2 < \gamma < S_1, \\ \lim_{x\longrightarrow 0} |x|^{S_2} \left(|\ln|x||\right)^{-\frac{1}{q+1-p}} u &= \alpha_{N,p,q} > 0 \qquad \text{if } \gamma = S_2 \neq 0, \\ \lim_{x\longrightarrow 0} |\ln|x||^{-\frac{p-1}{q+1-p}}\, u &= \delta_{N,p,q} > 0, \qquad \text{if } \gamma = S_2 = 0. \end{aligned}$$

(ii) Let u be any positive solution in $\mathbb{R}^N\backslash\overline{B_{r_0}}$. Then $S_2 < \gamma$ and

$$\begin{aligned} \lim_{|x|\longrightarrow\infty} |x|^{S_1}\, u &= k_1 > 0 \qquad \text{if } S_2 < \gamma < S_1, \\ \lim_{|x|\longrightarrow\infty} |x|^{S_1} \left(|\ln|x||\right)^{-\frac{1}{q+1-p}} u &= \beta_{N,p,q}, \qquad \text{if } \gamma = S_1. \end{aligned}$$

There exist solutions of each type.
(iii) There is no global positive solution.

Theorem 2.4 ***Case*** *($\mathcal{H}_4$) Let $\mu = \mu_0$ and $\gamma = \frac{N-p}{p}$. Then*
(i) there is no global positive solution.
(ii) All the solutions in $B_{r_0}\backslash\{0\}$. (resp. $\mathbb{R}^N\backslash\overline{B_{r_0}}$) satisfy

$$\lim_{x\longrightarrow 0} |x|^{\frac{N-p}{p}} \left|\ln|x|\right|^{\frac{2}{q+1-p}} u = c_{N,p,q} > 0, \ \text{(resp. } \lim_{|x|\longrightarrow\infty} r\,|x|^{\frac{N-p}{p}} \left|\ln|x|\right|^{\frac{2}{q+1-p}} u = c_{N,p,q}).$$

There exist solutions of each type for any $p > 1$, moreover ***explicit*** *if $p = 2$.*

The last result concerns the case $(\mathcal{H}_5)$ where $\mu < \mu_0$. It extends the main result of [30, Theorem 1.1] and the one of [11, Theorem 1.1] to the case of quasilinear equation (1.1), and the proof is quite shorter.

Theorem 2.5 ***Case*** *$(\mathcal{H}_5)$ Let $\mu < \mu_0$. Then*
(i) there is the unique global solution in $\mathbb{R}^N \setminus \{0\}$, radial:

$$u^*(x) = a^* \, |x|^{-\gamma} ; \tag{2.20}$$

(ii) any solution in $B_{r_0} \setminus \{0\}$. (resp. $\mathbb{R}^N \setminus \overline{B_{r_0}}$) satisfies

$$\lim_{|x| \longrightarrow 0} |x|^{\gamma} \, u = a^* \ \textit{(resp.} \lim_{|x| \longrightarrow \infty} |x|^{\gamma} \, u = a^* \textit{).} \tag{2.21}$$

Moreover if $\gamma = 0$ and $p > 2$, then u is constant for small $|x|$ small enough (resp. for large $|x|$). There exist local solutions of each type.

2.3 Other formulation of the classification (2.7)

We can formulate the different assumptions of (2.7) in another way, distinguish according to the sign of γ: when $\mu \geq \mu_0$, we define

$$\mathbf{q}_1 = p - 1 + \frac{p+\theta}{S_1}; \qquad \mathbf{q}_2 = p - 1 + \frac{p+\theta}{S_2}, \quad \text{if } \mu \neq 0.$$

These two critical values of q can be involved, according to the value of θ and μ. For the case $\mu = \mu_0$,

$$\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{q}_s := \frac{N(p-1) + p + p\theta}{N - p}$$

is the Sobolev radial exponent.

We get the following equivalences:
• when $p + \theta > 0$, then $\mathbf{q}_1 \leq \mathbf{q}_2$, and there holds $\mathbf{q}_1 > p - 1$, and $\mathbf{q}_2 > p - 1 \Longleftrightarrow \mu < 0$, and

$$\begin{aligned}
&(\mathcal{H}_1) \Longleftrightarrow \ \mu \geq \mu_0 \qquad \text{and } 1 < q < \mathbf{q}_1, \\
&(\mathcal{H}_2) \Longleftrightarrow 0 > \mu \geq \mu_0 \qquad \text{and } \mathbf{q}_2 < q, \\
&(\mathcal{H}_3) \Longleftrightarrow \ (\mu \geq 0 \text{ and } \mathbf{q}_1 \leq q) \qquad \text{or } (0 > \mu > \mu_0 \text{ and } \mathbf{q}_1 \leq q \leq \mathbf{q}_2), \\
&(\mathcal{H}_4) \Longleftrightarrow \mu = \mu_0 \qquad \text{and } q = \mathbf{q}_s;
\end{aligned}$$

• when $p + \theta < 0$, there holds $\mathbf{q}_1 < p - 1$, and $\mathbf{q}_2 > p - 1 \Longleftrightarrow \mu > 0$, then $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$ are empty,

$$\begin{aligned}
(\mathcal{H}_2) \ &\Longleftrightarrow \ \mu \leq 0 \qquad \text{or } (\mu > 0 \text{ and } 1 < q \leq \mathbf{q}_2), \\
(\mathcal{H}_3) \ &\Longleftrightarrow \ \mu > 0 \qquad \text{and } \mathbf{q}_2 \leq q;
\end{aligned}$$

• when $p + \theta = 0$, then $\gamma = 0$, so $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$ are still empty, and

$$(\mathcal{H}_2) \Longleftrightarrow 0 > \mu \geq \mu_0, \qquad (\mathcal{H}_3) \ \Longleftrightarrow \mu \geq 0,$$

and u^* exists for any $\mu < 0$ and is constant: $u^* \equiv a^* = (-\mu)^{\frac{1}{q+1-p}}$.

Let us add a simple corollary relative to the case $\mu = 0$, still studied in [15] for $\theta = 0$, and [10] for $\theta + p > 0$. We find again and complete their results, in particular when $\theta + p \leq 0$, which to our knowledge, is new: the existence of solutions in an exterior domain such that $\lim_{|x|\to\infty} u = k_2 > 0$, first obtained in [3] is a **surprising** significant result. Here $S_2 = 0$, $S_1 = \frac{N-p}{p-1}$, and there is only one critical value

$$\mathbf{q}_1 = \mathbf{q}_c = \frac{(p-1)(N+\theta)}{N-p}. \tag{2.22}$$

When $\theta + p > 0$, $(\mathcal{H}_1) \Longleftrightarrow 1 < q < \mathbf{q}_1$, $(\mathcal{H}_3) \Longleftrightarrow \mathbf{q}_1 \leq q$, and the other cases do not hold. When $\theta + p < 0$, then $(\mathcal{H}_2)$ always hold.

Corollary 2.6 *Consider the Henon type equation with $\theta \in \mathbb{R}$:*

$$-\Delta_p u + |x|^\theta u^q = 0. \tag{2.23}$$

(i) Suppose $\theta + p > 0$ and $q < \mathbf{q}_c$, where $\mathbf{q}_c$ is defined at (2.22).Then any solution $u \not\equiv 0$ in $B_{r_0} \setminus \{0\}$ satisfies either $\lim_{|x|\to 0} |x|^\gamma u = a^*$, or $\lim_{|x|\to 0} |x|^{N-p} u = k_1 > 0$, or u extends as a solution in B_{r_0}. Any positive solution in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ satisfies $\lim_{|x|\to\infty} |x|^\gamma u = a^*$. The global solutions in $\mathbb{R}^N \setminus \{0\}$ are radial, given by

- $u^* = a^* |x|^{-\gamma}$.
- *for any $k_1 > 0$,there exist a unique global solution such that*

$$\lim_{r\to 0} |x|^{N-p} u = k_1 > 0, \qquad \lim_{r\to\infty} |x|^\gamma u = a^*.$$

(ii) Suppose $\theta + p > 0$ and $q \geq \mathbf{q}_c$. Any solution $u \not\equiv 0$ in $B_{r_0} \setminus \{0\}$ extends a solution in B_{r_0}. Any positive solution in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ satisfies $\lim_{|x|\to\infty} |x|^{N-p} u = k_1 > 0$. There is no positive solution in $\mathbb{R}^N \setminus \{0\}$.

(iii) Suppose $\theta + p < 0$, hence $\gamma < 0$. Then any solution $u \not\equiv 0$ in $B_{r_0} \setminus \{0\}$ satisfies $\lim_{|x|\to 0} |x|^\gamma u = a^*$. Any solution in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ satisfies either $\lim_{|x|\to\infty} |x|^\gamma u = a^*$, or $\lim_{|x|\to\infty} u = k_2 > 0$, or $\lim_{|x|\to\infty} |x|^{N-p} u = k_1 > 0$. The global solutions are radial:

- $u^* = a^* |x|^{-\gamma}$.
- *For any $k_1 > 0$,there exist a unique global (increasing) solution such that*

$$\lim_{x\longrightarrow 0} |x|^\gamma u = a^*. \qquad \lim_{|x|\to\infty} u = k_1 > 0.$$

(iv) Suppose $\theta + p = 0$. Then there is no global solution $u \not\equiv 0$. Any solution in $B_{r_0} \setminus \{0\}$ satisfies

$$\lim_{x\longrightarrow 0} |\ln |x||^{-\frac{p-1}{q+1-p}} u = \delta_{N,p,q} > 0.$$

Any solution $u \not\equiv 0$ in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ satisfies

$$\lim_{|x|\to\infty} |x|^{N-p} u = k_1 > 0,$$

and there exist such solutions for any $k_1 > 0$.

Remark 2.7 *Moreover, when* $\theta = -\frac{p(N-1)}{p-1}$, *thus* $\theta + p < 0$, *there exist* ***explicit*** *global and local solutions, in* $B_{r_0} \setminus \{0\}$ *and in* $\mathbb{R}^N \setminus \overline{B_{r_0}}$ *of equation (2.23), given by*

$$u(r) = (c \pm d_{p,q,N} r^{\frac{p-N}{p-1}})^{-\frac{p}{q+1-p}},$$

where $c > 0$ *and* $d_{p,q,N} = \frac{(p-1)(q-p+1)}{p(N-p)} (\frac{p}{(p-1)(q+1)})^{\frac{1}{p}}$, *see [3, Theorem 4.17].*

3 First specific radial cases

Here we only indicate the method of the proofs, since the results will be covered by the description of the general case.

3.1 The radial case $p > 1, \mu = 0, \theta \in \mathbb{R}$

In the case $\mu = 0$, equation (1.1) reduces to the equation (2.23). The radial solutions of Henon type equations with the two signs

$$-\Delta_p u - \varepsilon |x|^\theta u^q = 0, \qquad \varepsilon = \pm 1 \tag{3.1}$$

have been exhaustively studied for $q > p - 1$ in [3], for any value of N, p, θ, by a phase-plane technique. Here we are concerned by the case $p < N$, see [3, Theorems 4.9,4.14,4.15] for noncritical cases, and [3, Theorems 4.10,4.18,4.19] for critical ones.

3.2 The radial case $p = 2, \mu, \theta \in \mathbb{R}$

When $p = 2 < N$, equation (1.1) reduces to equation (1.3), studied in [9],[30],[11]. In the radial case, the study can also be reduced completely to the case $\mu = 0$, by an elementary proof:

Lemma 3.1 *Let* $p = 2 < N$ *and* $\mu \geq \mu_0 = -\left(\frac{N-2}{2}\right)^2$. *Let*

$$u = r^{-S_i} w, \quad \text{with } i = 1 \text{ or } 2,$$

where S_1, S_2 *are the roots of (2.3), given by*

$$\begin{aligned} S_1 &= \frac{N-2+\sqrt{(N-2)^2+4\mu}}{2} = \frac{N-2}{2} + \sqrt{\mu - \mu_0}, \\ S_2 &= \frac{N-2-\sqrt{(N-2)^2+4\mu}}{2} = \frac{N-2}{2} - \sqrt{\mu - \mu_0}. \end{aligned}$$

Then u *is a radial solution of equation (1.3) if and only if* w *satisfies a Henon type equation in another dimension (possibly not an integer) with a Henon-factor* $|x|^\sigma$, *where* σ *depends on* θ,μ *and also* q*:*

$$-\Delta^{(\mathbf{N})} w + r^\sigma w^q = 0, \qquad \mathbf{N} = N - 2S_i, \qquad \sigma = \theta - S_i(q-1). \tag{3.2}$$

In particular, for $i = 2$, *then* $\mathbf{N} > 2$.

Proof. By definition, $S^2-(N-2)S-\mu=0$. Setting $u=r^{-S}w$, we find

$$u'=r^{-S}w'-Sr^{-S-1}w,\qquad u''=r^{-S}w''-2Sr^{-S-1}w'+S(S+1)r^{-S-2}w,$$

then u satisfies equation (1.3) if and only if

$$\begin{aligned}0&=-r^{-S}w''+2Sr^{-S-1}w'+S(S+1)r^{-S-2}w-(N-1)(r^{-S-1}w'-Sr^{-S-2}w)\\&\quad+\mu r^{-S-2}w-r^{-Sq+\theta}w^q\\&=r^{-S}(-w''-(N-2S-1)\frac{w'}{r}+\frac{w}{r^2}(\mu-S^2+(N-2)S)-r^{\theta-S(q-1)}w^q)\\&=r^{-S}(-w''-(N-2S-1)\frac{w'}{r}-r^{\theta-S(q-1)}w^q)=r^{-S}(-\Delta^{(\mathbf{N})}w-r^{\sigma}w^q),\end{aligned}$$

hence the conclusion. If $S=S_2$, then $\mathbf{N}-2=N-2S_2-2=\frac{1}{2}\sqrt{(N-2)^2+4\mu}>0$. ∎

As a consequence, all the radial study of equation (1.3) can be obtained by making for example the change $u=r^{-S_2}w$, and then applying the results of [3, Theorems 4.9,4.14;4.15] and [3, Theorems 4.10,4.18,4.19] with $p=2$. We find again the study of [9, Chapter 7] concerning the radial solutions of equation (1.3).

Remark 3.2 *The introduction of equation 3.2 for the radial solutions is new, even if the change of unknown, first introduced in [5] for $q<1$, was used in [9] and [11]. It gives an interesting explanation of the link between the case $\mu=0$ and the general case. For example: by the change $u=r^{-S_2}w$, the radial solutions u in $B_{r_0}\setminus\{0\}$ such that the solutions $\lim_{r\longrightarrow 0}r^{S_1}u=k_1>0$ correspond to solutions w such that $\lim_{r\longrightarrow 0}r^{\mathbf{N}-2}w=k_1$, and the solutions u such that $\lim_{r\longrightarrow 0}r^{S_2}u=k_2>0$ correspond to the solutions solutions w such that $\lim_{r\longrightarrow 0}w=k_2>0$ and then w extend to a solution in B_{r_0}.*

Remark 3.3 *From this change of unknown, and Remark 2.7 we also deduce explicit solutions of equation (1.3):*

$$u(r)=r^{-S_i}(c\pm dr^{2-\mathbf{N}})^{-\frac{2}{q-1}}=r^{-S_i}(c\pm dr^{2-N+2S_i})^{-\frac{2}{q-1}}\ \textit{when}\ \sigma=-2(\mathbf{N}-1),\ \textit{that means}$$

$$u_1(r)=r^{-S_1}(c\pm dr^{2\sqrt{\mu-\mu_0}})^{-\frac{2}{q-1}}\,\textit{when}\ \theta=S_1(q-1)-2(N-1-2S_1),$$

$$u_2(r)=r^{-S_2}(c\pm dr^{-2\sqrt{\mu-\mu_0}})^{-\frac{2}{q-1}}\ \textit{when}\ \theta=S_2(q-1)-2(N-1-2S_2),$$

where $c>0$ and $d=d_{N,q}>0$. In particular we find again the global solutions given in [11, p.470].

4 General radial case $p>1,\mu,\theta\in\mathbb{R}$

Here we make a full description of the radial solutions of equation (1.1), where we find again in particular the specific cases above. **In case $p\neq 2$, we cannot reduce the study to the case $\mu=0$**, as we did at Lemma 3.1 in case $p=2$.

The radial study by phase-plane techniques reducing the problem to an autonomous system of two equations has the advantage that it gives a generally **exhaustive description of the solutions**, and in a very precise way. And it is the keypoint for obtaining the behaviours of all the possibly nonradial solutions, see Theorem 5.10. We consider the radial form of equation (1.1):

$$\mathcal{L}^{rad}_{p,\mu}u+r^{\theta}u^q=0,\ \text{where}\ \mathcal{L}^{rad}_{p,\mu}u=-\frac{d}{dr}(\left|u'\right|^{p-2}u')-\frac{N-1}{r}\left|u'\right|^{p-2}u'+\mu\frac{u^{p-1}}{r^p}.\qquad(4.1)$$

4.1 A common formulation of the problem

For studying the l solutions of (4.1), and more generally of

$$\mathcal{L}^{rad}_{p,\mu} u = \varepsilon r^{\theta} u^q, \qquad \varepsilon = \pm 1, \tag{4.2}$$

the first attempt is to make a classical transformation, introduced in [2] for the equation with a source term $-\Delta_p u = u^q$,

$$X = r^{\gamma} u = e^{\gamma t} U, \qquad Y = -r^{(\gamma+1)(p-1)} \left|u'\right|^{p-2} u' = -e^{\gamma(p-1)t} \left|U_t\right|^{p-2} U_t, \qquad t = \ln r, \tag{4.3}$$

where γ has been defined at (1.8). For equation (4.2), it leads to the autonomous system

$$\left\{ \begin{array}{rcl} X_t &=& \gamma X - |Y|^{\frac{2-p}{p-1}} Y, \\ Y_t &=& -(N-p-\gamma(p-1))Y - \varepsilon X^q - \mu X^{p-1}. \end{array} \right. \tag{4.4}$$

A system of this type was still used in [1] for equation (4.2) with $\varepsilon = 1, \theta = 0$, when $q = \frac{N(p-1)+p}{N-p}$ is the critical exponent (which was the object of many works). In particular system (4.4) admits a Pohozaev type energy function, above all used for $\varepsilon = 1$:

Lemma 4.1 *Consider the system (4.4). Let*

$$D = N - p - p\gamma = \frac{(N-p)(q-\mathbf{q}_s)}{q+1-p}, \ \textit{with } \mathbf{q}_s = \frac{N(p-1)+p+p\theta}{N-p},$$

$$\mathcal{E} = \frac{p-1}{p} |Y|^{\frac{p}{p-1}} - \gamma XY + (D|\gamma|^{p-2}\gamma - \mu)\frac{X^p}{p} + \varepsilon \frac{X^{q+1}}{q+1}. \tag{4.5}$$

Then

$$\mathcal{E}_t = -D(\gamma X - |Y|^{\frac{2-p}{p-1}} Y)(|\gamma X|^{p-2}(\gamma X) - Y) \tag{4.6}$$

has the sign of $-D$, that means $\mathcal{E}$ is increasing $q < q_s$, decreasing for $q > \mathbf{q}_s$ and constant for $q = \mathbf{q}_s$.

Proof. By simple computation: we set $k = N-p-\gamma(p-1)$ and $D = k - \gamma = N - p - p\gamma$; then

$$\begin{aligned} \mathcal{E}_t &= |Y|^{\frac{1}{p-1}} Y_t - \gamma X Y_t - \gamma X_t Y + (D|\gamma|^{p-2}\gamma - \mu) X^{p-1} X_t + \varepsilon X^q X_t \\ &= (|Y|^{\frac{2-p}{p-1}} Y - \gamma X)(-kY + \varepsilon X^q - \mu X^{p-1}) \\ &+ (\gamma X - |Y|^{\frac{2-p}{p-1}} Y)(\varepsilon X^q - \gamma Y + (D|\gamma|^{p-2}\gamma - \mu) X^{p-1}) \\ &= (\gamma X - |Y|^{\frac{2-p}{p-1}} Y)(kY - \varepsilon X^q + \mu X^{p-1} + \varepsilon X^q - \gamma Y + (D|\gamma|^{p-2}\gamma - \mu) X^{p-1}) \\ &= (\gamma X - |Y|^{\frac{2-p}{p-1}} Y)(A|\gamma|^{p-2}\gamma X^{p-1} + DY). \end{aligned}$$

■

4.2 New formulation as an autonomous system

The system (4.4) has the drawback to be singular at $X = 0$ or $Y = 0$, according to the value of p. But the main lack is that it has at most two fixed points; and the study at $(0, 0)$ is difficult, and does not show easily the multiple possibilities of behaviour of the solutions. In [4] we have shown that an introduction of the function slope $\frac{ru'}{u}$in such kind of problems allows to give much more informations on the solutions, that is what we do in the sequel. Nevertheless, the change of unknown that we had introduced in the case $\mu = 0$ of equation (3.1) with $\varepsilon = -1$, defined by

$$S = -\frac{ru_r}{u}, \qquad Z = r^{\theta+1}\frac{u^q}{|u_r|^{p-2}\,u_r}, \qquad t = \ln r,$$

leading to the system

$$\left\{\begin{array}{rcl} S_t &=& S(S - \frac{N-p}{p-1} + \frac{Z}{p-1}),\\ Z_t &=& Z(N + \theta - qS - Z),\end{array}\right.$$

cannot be adapted to the case $\mu \neq 0$. That is why we introduce another form allowing to treat equation (1.1) **for any** $\mu \in \mathbb{R}$.

Lemma 4.2 *Let u be a radial positive solution of (1.1). For any $r > 0$, let*

$$G(t) = |S|^{p-2}\,S(r), \text{ where } S = -\frac{ru'}{u}, \qquad V = r^{\theta+p}u^{q+1-p}, \qquad t = \ln r. \tag{4.7}$$

Then (G, V) satisfies the system

$$\left\{\begin{array}{rcl} G_t &=& (p-1)\,|G|^{\frac{p}{p-1}} - (N-p)G - \mu - V,\\ V_t &=& (q+1-p)V(\gamma - |G|^{\frac{2-p}{p-1}}\,G).\end{array}\right. \tag{4.8}$$

Proof. We set $t = \ln r$ and $u(r) = U(t)$, then $S = -\frac{U_t}{U}$,

$$u'(r) = \frac{U_t}{r}, \quad \left|u'\right|^{p-2}u' = \frac{|U_t|^{p-2}\,U_t}{r^{p-1}}, \quad \frac{d}{dr}(\left|u'\right|^{p-2}u') = \frac{(|U_t|^{p-2}\,U_t)_t - (p-1)\,|U_t|^{p-2}\,U_t}{r^p}$$

$$-(|U_t|^{p-2}\,U_t)_t + (p-N)\,|U_t|^{p-2}\,U_t + \mu U^{p-1} + r^{\theta+p}U^q = 0,$$

Since U is positive, dividing by U^{p-1} we get

$$\frac{-(|U_t|^{p-2}\,U_t)_t}{U^{p-1}} + (N-p)\frac{|U_t|^{p-2}\,U_t}{U^{p-1}} + \mu + e^{(\theta+p)t}U^{q-p+1} = 0.$$

Defining G and V by (4.7) we obtain the system (4.8). ∎

Remark 4.3 *Note the relations between systems (4.4) and (4.8): there holds*

$$V = X^{q+1-p}, \qquad Y = GX^{p-1}, \tag{4.9}$$

and the energy function is expressed in terms of G, V by

$$\mathcal{E} = V^{\frac{p}{q+1-p}}(\frac{p-1}{p}\,|G|^{\frac{p}{p-1}} - \gamma G + \frac{D\,|\gamma|^{p-2}\,\gamma - \mu}{p} - \frac{V}{q+1}) = V^{\frac{p}{q+1-p}}(\frac{F(G) + D\,|\gamma|^{p-2}\,\gamma}{p} - \frac{V}{q+1}). \tag{4.10}$$

Remark 4.4 *Note that the relation (2.2) giving the possible roots S_1,S_2 can be written in terms of G by*

$$\varphi(\mathcal{S}) = 0 \Longleftrightarrow (p-1)\,|G|^{\frac{p}{p-1}} - (N-p)G - \mu = 0,$$

and the difference $(p-1)\,|G|^{\frac{p}{p-1}} - (N-p)G - \mu$ clearly appears in system (4.8). The choice of the variable G compared to S is motivated by the fact that it leads to a simpler system: indeed the system relative to S, V is

$$\left\{ \begin{array}{rcl} S_t & = & S^2 - \frac{(N-p)}{p-1}S - \frac{1}{p-1}\,|S|^{2-p}\,(\mu + V), \\ V_t & = & (q+1-p)V(\gamma - S), \end{array} \right.$$

where the term $|S|^{2-p}$ does not make the study easy.

Next we apply the logarithmic transformation to the radial solutions u of equation (2.1). We consider in particular the case $\mu = \mu_0$, and find again a result of [19, Theorem 2.1] with a simple proof :

Lemma 4.5 *Let $p > 1$ and $\mu = \mu_0$. Then for any $\ell > 0$ there exists a radial solution u of equation (2.1) defined near $r = 0$ (resp. near ∞) such that*

$$\lim_{r \longrightarrow 0} r^{\frac{N-p}{p}}\,|\ln r|^{-\frac{2}{p}}\,u(r) = \ell \qquad (resp.\ \lim_{r \longrightarrow \infty} r^{\frac{N-p}{p}}\,|\ln r|^{-\frac{2}{p}}\,u(r) = \ell) \tag{4.11}$$

Proof. Let again $G(t) = |S|^{p-2}\,S(r)$, where $S = -\frac{ru'}{u}$ and $t = \ln r$. We get

$$\mathcal{L}_{p,\mu}u = 0 \Longleftrightarrow G_t = F(G) = (p-1)\,|G|^{\frac{p}{p-1}} - (N-p)G - \mu \tag{4.12}$$

• In the case $p = 2$ then $G = S$ and $S_t = (S - \frac{N-2}{2})^2$, and by integration we obtain explicitely all the solutions of any sign of the equation by $u(r) = r^{-\frac{N-2}{2}}(C_1 + C_2 \ln r)$, depending on two parameters $C_1, C_2 \in \mathbb{R}$.
• In the general case $p > 1$, the equation still admits the solution $G \equiv G_0$, that means $-\frac{U_t}{U} = \frac{N-p}{p}$, which corresponds to the solutions $u(r) = Cr^{-\frac{N-p}{p}\cdot}$. The other solutions are obtained by quadrature. Setting $G_0 = (\frac{N-p}{p})^{p-1}$ and $G = G_0 + \overline{G}$, there holds

$$F(G_0 + \overline{G}) = \frac{1}{2}F''(G_0 + \tau\overline{G})\overline{G}^2 = \frac{p}{2(p-1)}(G_1 + \tau\overline{G})^{\frac{2-p}{p-1}}\overline{G}^2$$

for some $\tau \in (0,1)$, hence $F(G) \sim_{G \longrightarrow G_0} c\overline{G}^2$ with $c = \frac{p}{2}(\frac{N-p}{p})^{2-p}$; so we obtain a family with one parameter of solutions such that

$$t = C + \int_{G_0}^{G} \frac{dg}{F(g)}.$$

defined on (∞, C), or in (C, ∞) and then $\overline{G} \sim_{t \longrightarrow \pm\infty} \frac{1}{ct}$; then $G_t = \frac{p}{2(p-1)}G_0^{\frac{2-p}{p-1}}(\overline{G}^2 + O(\overline{G}^3) = \frac{p}{2(p-1)}G_0^{\frac{2-p}{p-1}}\overline{G}^2(1 + O(\frac{1}{|t|})$; by integration $\overline{G}(t) = -\frac{1}{ct} + O(\frac{|\ln|t||}{|t|^2})$; then

$$S = -\frac{U_t}{U} = \frac{N-p}{p} - \frac{(\frac{N-p}{p})^{2-p}}{(p-1)c}\frac{1}{t} + O(\frac{\ln|t|}{|t|^2}) = \frac{N-p}{p} - \frac{2}{p}\frac{1}{t} + O(\frac{|\ln|t||}{|t|^2}),$$

thus $\ln(Ue^{\frac{N-p}{p}t}|t|^{-\frac{2}{p}}) = O(\frac{\ln|t|}{|t|^2})$, and $\frac{\ln|t|}{|t|^2}$ is integrable, then we obtain, $\lim_{t\longrightarrow\pm\infty} Ue^{\frac{N-p}{p}t}|t|^{-\frac{2}{p}} = \ell_\pm \neq 0$. By a scaling $u(r) = v(ar)$ letting the equation invariant, we obtain the existence for any $\ell > 0$. ∎

4.3 Main radial results

The proofs of the following theorems are postponed at Section 7.

Theorem 4.6 ***Case*** *($\mathcal{H}_1$): Let $\mu \geq \mu_0$ and $\gamma > S_1$. Then*

- *There exists a global particular solution $u^* = a^* r^{-\gamma}$.*
- *If $\mu > \mu_0$, for any $k_1 > 0$, there exist a unique global solution such that*

$$\lim_{r\to 0} r^{S_1} u = k_1, \qquad \lim_{r\to\infty} r^\gamma u = a^*. \tag{4.13}$$

- *If $\mu = \mu_0$, for any $\ell > 0$ there exists a unique global solution such that*

$$\lim_{r\to 0} r^{\frac{N-p}{p}}(|\ln(r)|^{\frac{2}{p}})u(r) = \ell, \qquad \lim_{r\to\infty} r^\gamma u = a^*, \qquad \text{if } \mu = \mu_0. \tag{4.14}$$

All the other solutions are local.

- *There exist solutions in a finite interval $(0, R)$, such that respectively*

$$\lim_{r\to 0} r^\gamma u = a^*, \qquad \lim_{r\to R} u = \infty, \tag{4.15}$$

$$\lim_{r\to 0} r^\gamma u = a^*, \qquad u(R) = 0, \tag{4.16}$$

$$\begin{cases} \lim_{r\to 0} r^{S_1} u = k_1 > 0 \text{ if } \mu > \mu_0 \\ \lim_{r\to 0} r^{\frac{N-p}{p}}(|\ln(r)|^{\frac{2}{p}})u(r) = \ell > 0, \text{ if } \mu = \mu_0 \end{cases}, \qquad \lim_{r\to R} u = \infty, \tag{4.17}$$

$$\begin{cases} \lim_{r\to 0} r^{S_1} u = k_1 > 0 \text{ if } \mu > \mu_0 \\ \lim_{r\to 0} r^{\frac{N-p}{p}}(|\ln(r)|^{\frac{2}{p}})u(r) = \ell > 0, \text{ if } \mu = \mu_0 \end{cases} \qquad \lim_{r\to R} u = 0, \tag{4.18}$$

$$\lim_{r\to 0} r^{S_2} u = k_2 > 0, \qquad \lim_{r\to R} u = \infty. \tag{4.19}$$

- *There exist solutions in an interval (R, ∞) such that*

$$\lim_{r\to R} u = \infty, \qquad \lim_{r\to\infty} r^\gamma u = a^*. \tag{4.20}$$

Any radial solution, local near 0 or ∞ or global, has one of these types.

Theorem 4.7 ***Case*** *($\mathcal{H}_2$): Let $\mu \geq \mu_0$ and $\gamma < S_2$. Then*

- *There exists a global particular solution $u^* = a^* r^{-\gamma}$.*
- *If $\mu > \mu_0$, for any $k_2 > 0$, there exist a unique global solution such that*

$$\lim_{r\to 0} r^\gamma u = a^*, \qquad \lim_{r\to\infty} r^{S_2} u = k_2. \tag{4.21}$$

- *If $\mu = \mu_0$, for any $\ell > 0$ there exists a unique global solution such that*

$$\lim_{r\to 0} r^\gamma u = a^*, \qquad \lim_{r\to\infty} r^{\frac{N-p}{p}} |\ln(r)|^{\frac{2}{p}} u(r) = \ell. \tag{4.22}$$

All the other solutions are local, each of them has one of the following types.

- *There exists solutions in a finite interval (R,∞), such that respectively*

$$\lim_{r\to R} u=\infty, \qquad \lim_{r\to\infty} r^{\gamma}u=a^*, \tag{4.23}$$

$$\lim_{r\to R} u=0, \qquad \lim_{r\to\infty} r^{\gamma}u=a^*, \tag{4.24}$$

$$\lim_{r\to R} u=\infty, \qquad \begin{cases} \lim_{r\to\infty} r^{S_2}u=k_2>0 \text{ if } \mu>\mu_0 \\ \lim_{r\to\infty} r^{\frac{N-p}{p}}(|\ln(r)|^{\frac{2}{p}})u(r)=\ell>0, \text{ if } \mu=\mu_0 \end{cases}, \tag{4.25}$$

$$\lim_{r\to R} u=0, \qquad \begin{cases} \lim_{r\to\infty} r^{S_2}u=k_2>0 \text{ if } \mu>\mu_0 \\ \lim_{r\to\infty} r^{\frac{N-p}{p}}(|\ln(r)|^{\frac{2}{p}})u(r)=\ell>0, \text{ if } \mu=\mu_0 \end{cases}, \tag{4.26}$$

$$\lim_{r\to R} u=\infty, \qquad \lim_{r\to 0} r^{S_1}u=k_1>0, \tag{4.27}$$

- *There exist solutions in an interval $(0,R)$ such that*

$$\lim_{r\to 0} r^{\gamma}u=a^*, \qquad \lim_{r\to R} u=\infty. \tag{4.28}$$

Moreover, if $p>2$ and $\gamma=0$, the solutions satisfying (4.21),(4.22),(4.28) are constant near 0, the solutions satisfying (4.23),(4.24) are constant near ∞.

Theorem 4.8 ***Case*** *($\mathcal{H}_3$) Let $\mu>\mu_0$ and $S_2\le\gamma\le S_1$. Then there is no global solution. There exists solutions respectively on $(0,R)$ such that*

$$\lim_{r\longrightarrow 0} r^{S_2}u=k_2>0, \qquad \lim_{r\to R} u=\infty, \qquad \text{if } S_2<\gamma<S_1, \tag{4.29}$$

$$\lim_{r\longrightarrow 0} r^{S_2}(|\ln r|)^{\frac{1}{q+1-p}}u=\alpha_{N,p,q}, \qquad \lim_{r\to R} u=\infty, \qquad \text{if } \gamma=S_2\neq 0, \tag{4.30}$$

$$\lim_{r\longrightarrow 0} |\ln r|^{-\frac{p-1}{q+1-p}}u=\delta_{N,p,q}, \qquad \lim_{r\to R} u=\infty, \qquad \text{if } \gamma=S_2=0, \tag{4.31}$$

with $\alpha_{N,p,q}=(\frac{(p-1)(N-p-pS_2)S_2^{p-2}}{q+1-p})^{\frac{1}{q+1-p}}$, $\delta_{N,p,q}=(\frac{(q+1-p)(N-p)^{-\frac{1}{p-1}}}{p-1})^{\frac{p-1}{q+1-p}}$; and on (R,∞) such that

$$\lim_{r\to R} u=\infty, \qquad \lim_{r\longrightarrow\infty} r^{S_1}u=k_2>0, \qquad \text{if } S_1<\gamma<S_2, \tag{4.32}$$

$$\lim_{r\to R} u=\infty, \qquad \lim_{r\longrightarrow\infty} r^{S_1}(|\ln r|)^{\frac{1}{q+1-p}}u=\beta_{N,p,q}, \qquad \text{if } \gamma=S_1, \tag{4.33}$$

with $\beta_{N,p,q}=(\frac{(p-1)(pS_1-N+p)S_2^{p-2}}{q+1-p})^{\frac{1}{q+1-p}}$. Any local solution near 0 or ∞ has one of these types; and it is unique when $S_2<\gamma<S_1$.

Theorem 4.9 ***Case*** *($\mathcal{H}_4$) Suppose $\mu=\mu_0$ and $\gamma=\frac{N-p}{p}$. Then there is no global solution. For any $R>0$,there exists solutions such that*

$$\lim_{r\longrightarrow R} u=\infty, \qquad \lim_{r\longrightarrow\infty} r^{\frac{N-p}{p}}(\ln r)^{\frac{2}{q+1-p}}u=c_{N,p,q}, \qquad R>0, \tag{4.34}$$

$$\lim_{r\longrightarrow 0} r^{\frac{N-p}{p}} \left|\ln r\right|^{\frac{2}{q+1-p}} u = c_{N,p,q}, \qquad \lim_{r\longrightarrow R} u = \infty, \qquad R>0, \tag{4.35}$$

with $c_{N,p,q} = (\frac{2(q+1)}{(q+1-p)^2}(\frac{N-p}{p})^{p-2})^{\frac{1}{q+1-p}}$. *All of the solutions have one of these types, and can be obtained by quadratures. If* $p=2$, *the solutions are* ***explicit,*** *given by*

$$u(r) = (\frac{2(q+1)}{(q-1)^2})^{\frac{1}{q-1}} r^{\frac{2-N}{2}} (\ln r \pm \ln R)^{-\frac{2}{q-1}}, \qquad R>0. \tag{4.36}$$

Theorem 4.10 ***Case*** *($\mathcal{H}_5$). Suppose* $\mu<\mu_0$. *Then there exists a unique global solution:* $u=u^*$.
Moreover, there exist solutions satisfying any of the behaviours

$$\lim_{r\to R} u = \infty, \qquad \lim_{r\to\infty} r^\gamma u = a^*, \tag{4.37}$$

$$\lim_{r\to R} u = 0, \qquad \lim_{r\to\infty} r^\gamma u = a^*, \tag{4.38}$$

$$\lim_{r\to 0} r^\gamma u = a^*, \qquad \lim_{r\to R} u = \infty, \tag{4.39}$$

$$\lim_{r\to 0} r^\gamma u = a^*, \qquad u(R)=0. \tag{4.40}$$

Any local solution near 0 *or* ∞ *has one of these types. Moreover, if* $p>2$ *and* $\gamma=0$, *the solutions are constant near* ∞, *or near* 0.

Remark 4.11 *When* $\gamma=0$, *Theorem 4.7 in case* $\mu_0<\mu<0$ *(resp. Theorem 4.10 in case* $\mu<\mu_0$*) shows in evidence a phenomena of* ***nonuniquess*** *of solutions* u *in* $C^1(B_{r_0}\setminus\{0\})\cap C(B_{r_0})$ *of equation (2.8),* ***valid for any*** $p>1$*: besides the constant solution* $u^*(r)\equiv|\mu|^{\frac{1}{q+1-p}}$, *it admits solutions satisfying (4.21),(4.22),(4.28), (resp. (4.39),(4.40)). Considering again* $u=|\mu|^{\frac{1}{q-1}}(1+\overline{u})$, *this result can be compared to the nonuniqueness result of [17, Remark 5.1] relative to the equation* $-\Delta_p w + w = 0$ *for* $p>2$.

5 Basic arguments for the nonradial case

5.1 The strong maximum principle

Theorem 5.1 *(Strong Maximum Principle) Let* u *be any nonnegative* C^1*solution of (1.1) in a domain* ω *such that* $\overline{\omega}\subset\mathbb{R}^N\setminus\{0\}$. *Then either* u *is positive in* ω, *or* $u\equiv 0$. *As a consequence, any nonnegative solution in* $B_{r_0}\setminus\{0\}$ *(resp. in* $\mathbb{R}^N\setminus\overline{B_{r_0}}$*) is positive, or* $u\equiv 0$.

Proof. If $\mu\le 0$, then setting $C_1=\max_{x\in\overline{\omega}}|x|^\theta$, and $\beta_1(u)=C_1u^q$,

$$-\Delta_p u+\beta_1(u)\ge -\Delta_p u+|x|^\theta u^q = |\mu|\,\frac{u^{p-1}}{|x|^p}\ge 0.$$

If $\mu>0$ then setting $C_2=\max_{x\in\overline{\omega}}|x|^{-p}$ and $\beta_2(u)=\mu C_2u^{p-1}+u^q$,

$$-\Delta_p u+\beta_2(u)\ge 0.$$

Then the result comes from the Strong Maximum Principle of [27, Theorem 5], since in any case $\int_0^1(s\beta_i(s))^{-\frac{1}{p}}ds=\infty$. ∎

5.2 The Weak Comparison Principle

We recall a main argument, due to [12, Proposition 2.2], leading to a Weak Comparison Principle:

Proposition 5.2 *[12, Proposition 2.2] Let ω be a bounded domain in $\mathbb{R}^N$, $A \in C_b(\omega)$ and $B \in C^+(\omega)$, $B \not\equiv 0$. Let $u,v \in C^1(\omega)$ be two positive functions such that*

$$-\Delta_p v + Av^{p-1} + Bg(v) \geq 0 \geq -\Delta_p u + Au^{p-1} + Bg(u)$$

in $\mathcal{D}'(\omega)$, where $g \in C([0,\infty)$, and $s \longmapsto g(s)/s^{p-1}$ is increasing on $(\inf_\omega(u,v), \sup_\omega(u,v))$. If $\limsup_{d(x,\partial\omega)\longrightarrow 0}(u-v)(x) \leq 0$, then $u \leq v$ in ω.

We apply this theorem to problem (1.1), with $A(x) = \mu\,|x|^{-p}$ and $g(s) = s^q$, when $q > p-1$, with $B(x) = |x|^\theta$ and ω is a domain such that $\overline{\omega} \subset \Omega \setminus \{0\}$:

Corollary 5.3 *Let $\omega = B_{r_2} \setminus \overline{B_{r_1}}$, with $0 < r_1 < r_2$. Let $q > p-1$,and $\mathcal{L}_{p,\mu}$ be defined by (1.5) for any $\mu,\theta \in \mathbb{R}$. Let $u,v \in C^1(\omega) \cap C(\overline{\omega})$ such that*

$$\mathcal{L}_{p,\mu} v + |x|^\theta v^q \geq 0 \geq \mathcal{L}_{p,\mu} u + |x|^\theta u^q$$

a.e.in ω, and $u \leq v$ on $\partial\omega$. Then $u \leq v$ in ω.

5.3 A priori Osserman's estimate near 0 or ∞

Proposition 5.4 *Let $q > p-1$, and $\mu,\theta \in \mathbb{R}$. Then there exists a constant $C = C_{N,p,q,\theta,\mu} > 0$ such that, for any solution u of (1.1) in $B_{2r_0} \setminus \{0\}$, (resp. in $\mathbb{R}^N \setminus \overline{B_{\frac{r_0}{2}}}$, resp. in $\mathbb{R}^N \setminus \{0\}$) and for any $x \in B_{r_0} \setminus \{0\}$, (resp. $x \in \mathbb{R}^N \setminus \overline{B_{r_0}}$, resp. $x \in \mathbb{R}^N \setminus \{0\}$),*

$$u(x) \leq C_{N,p,q,\theta,\mu}\,|x|^{-\frac{p+\theta}{q+1-p}}, \qquad \forall x \in B_{r_0} \setminus \{0\} \ \ (\textit{resp. } x \in \mathbb{R}^N \setminus \overline{B_{r_0}}, \ \textit{resp. } x \in \mathbb{R}^N \setminus \{0\}). \tag{5.1}$$

Proof. The estimate is classical in case $\mu > 0$. In the general case, we write, since $q > p-1$,

$$\begin{aligned} -\Delta_p u + |x|^\theta u^q &\leq |\mu|\,\frac{u^{p-1}}{|x|^p} = |\mu|\,(|x|^\theta u^q)^{\frac{p-1}{q}}\,|x|^{-\frac{pq+\theta(p-1)}{q}} \\ &\leq |\mu|\,(\varepsilon^{\frac{q}{p-1}}\,|x|^\theta u^q + \varepsilon^{-\frac{q}{q-p+1}}(|x|^{-\frac{pq+\theta(p-1)}{q}})^{\frac{q}{q-p+1}} \leq \frac{|x|^\theta u^q}{2} + c_{p,q,\mu}\,|x|^{-\frac{pq+\theta(p-1)}{q-p+1}}, \end{aligned}$$

where $c_{p,q,\mu}$ only depends on p,q,μ from Hölder inequality with $\varepsilon = (\frac{1}{2|\mu|})^{\frac{p-1}{q}}$, then

$$-\Delta_p u + \frac{1}{2}\,|x|^\theta u^q \leq C_{p,q,\mu}\,|x|^{-\frac{pq+\theta(p-1)}{q-p+1}}.$$

Next we apply the following estimate, see for example [28]: if ψ is a function such that in a ball B_R

$$-\Delta_p \psi + a\psi^q \leq b$$

for some $a,b > 0$, and $q > p-1$, then there exists $C_{N,p,q} > 0$ depending on N,p,q such that

$$\psi(0) \leq C_{N,p,q}\left(\frac{1}{aR^p}\right)^{\frac{1}{q+1-p}} + (\frac{b}{a})^{\frac{1}{q}} \tag{5.2}$$

Applying (5.2) to the function $\psi(x) = u(x - x_0)$ for any $x_0 \in B_{\frac{r_0}{2}} \setminus \{0\}$, and $|x - x_0| < R = \frac{|x_0|}{2}$, thus $\frac{|x_0|}{2} \le |x| \le \frac{3|x_0|}{2}$, and

$$a = |x_0|^\theta \min\left\{2^{-(\theta+1)}, 3.2^{\theta-1}\right\}, \qquad b = |x_0|^{\frac{pq+\theta(p-1)}{q-p+1}} \max\left\{2^{-\frac{pq+\theta(p-1)}{q-p+1}}, (\frac{3}{2})^{\frac{pq+\theta(p-1)}{q-p+1}}\right\},$$

we get

$$\begin{aligned} u(x_0) &\le C_{N,p,q,\mu}\left(\frac{1}{a(\frac{|x_0|}{2})^p}\right)^{\frac{1}{q+1-p}} + (\frac{b}{a})^{\frac{1}{q}} \\ &\le C_{N,p,q,\theta,\mu}(|x_0|^{-\frac{p+\theta}{q+1-p}} + C'_{N,p,q,\theta,\mu}\,|x_0|^{-(\frac{pq+\theta(p-1)}{q-p+1}+\theta)\frac{1}{q}} = C_{N,p,q,\theta,\mu}\,|x_0|^{-\frac{p+\theta}{q+1-p}}, \end{aligned}$$

where $C_{N,p,q,\theta,\mu}$ only depends on the parameters. Then we get (5.1).The estimate is also valid for the exterior problem in $\mathbb{R}^N \setminus \overline{B_{\frac{r_0}{2}}}$, and the global problem in $\mathbb{R}^N \setminus \{0\}$. ∎

Remark 5.5 *Note that Osserman's estimate holds without any restriction assumption: it is valid for any $\mu \in \mathbb{R}$, and any $\theta \in \mathbb{R}$; in particular u is bounded when $p + \theta = 0$, and u tends to 0 as $|x| \to 0$ when $p + \theta < 0$.*

5.4 Regularity result

Followinq the method of [15], we check that estimates of the function u imply estimates of the gradient:

Proposition 5.6 *Let $q > p - 1$. Let u be a solution of (1.1) in $B_{r_0} \setminus \{0\}$, such that*

$$u(x) \le C_1\,|x|^{-\delta} \qquad \text{in } B_{r_0} \setminus \{0\}$$

for some $\delta \le \gamma$. Then

$$|\nabla u(x)| \le C_2\,|x|^{-(\delta+1)} \qquad \text{in } B_{\frac{r_0}{2}} \setminus \{0\}, \tag{5.3}$$

and there exists $\alpha \in (0,1)$ only depending of N, p, q, δ such that, for any $x, x' \in B_{\frac{r_0}{2}} \setminus \{0\}$,

$$\left|\nabla u(x) - \nabla u(x')\right| \le C_3(\left|x'\right|^{-\delta} + |x|^{-\delta})\,|x|^{-(1+\alpha)}\left|x - x'\right|^\alpha. \tag{5.4}$$

Analogous results hold in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ when $\delta \ge \gamma$.

Proof. For $x \in B_{\frac{r_0}{2}} \setminus \{0\}$ there exists $R \in (0, \frac{r_0}{6})$ such that $2R \le |x| \le 3R$. Let $\Gamma = B_7 \setminus \overline{B_1}$ and $\Gamma' = B_6 \setminus \overline{B_2}$. We set $u_R(\xi) = R^\delta u(R\xi)$ for any $\xi \in \Gamma$. Then

$$-\Delta_p u_R(\xi) + \frac{\mu u_R^{p-1}(\xi)}{|\xi|^p} + R^{(q+1-p)(\gamma-\delta)} u_R^q = 0$$

for $\xi \in \Gamma$. Now $R^{(q+1-p)(\gamma-\delta)}$ stays bounded as $R \to 0$, thanks to $\delta \le \gamma$ and $q > p - 1$; and $\|u_R\|_{L^\infty(\Gamma)} \le C_1\,|\xi|^\delta \le C'_1$, thus $\Delta_p u_R$ is bounded in Γ. Then from Tolksdorf results of [25,

Theorem 1], $|\nabla u_R|$ is bounded in $C^{0,\alpha}(\Gamma')$ independently of R for some $\alpha \in (0,1)$. Then (5.3) holds. Moreover, let $x, x' \in B_{\frac{r_0}{2}} \setminus \{0\}$. First assume $|x| \leq |x'| \leq 2|x|$. Then $\xi = \frac{x}{R}$ and $\xi' = \frac{x'}{R} \in \Gamma'$, and

$$\left|\nabla u(x) - \nabla u(x')\right| = R^{-1-\delta} \left|\nabla u_R(\xi) - \nabla u(\xi')\right| \leq CR^{-1-\delta} \left|\xi - \xi'\right|^{\alpha} \leq C_2 |x|^{-(\delta+1+\alpha)} \left|x - x'\right|^{\alpha}.$$

Next assume $|x'| \geq 2|x|$. Then

$$\left|\nabla u(x) - \nabla u(x')\right| \leq C(\frac{|x|^{-\delta}}{|x|} + \frac{|x'|^{-\delta}}{|x|}) \leq 2C \frac{|x'|^{-\delta} + |x|^{-\delta}}{|x|^{\alpha+1}} \left|x' - x\right|^{\alpha},$$

implying (5.4). ∎

Remark 5.7 *In the same way, if u is a solution in $B_{r_0} \setminus \{0\}$, with $r_0 < 1$, such that*

$$u(x) \leq C_1 |x|^{-\delta} |\ln |x|| \qquad \text{in } B_{r_0} \setminus \{0\}$$

for some $\delta < \gamma$. Then for any $x, x' \in B_{\frac{r_0}{2}} \setminus \{0\}$,

$$|\nabla u(x)| \leq C_2 |x|^{-(\delta+1)} |\ln |x||$$

$$\left|\nabla u(x) - \nabla u(x')\right| \leq C_3(\left|x'\right|^{-\delta} + |x|^{-\delta}) |x|^{-(1+\alpha)} |\ln |x|| \left|x - x'\right|^{\alpha}.$$

5.5 Harnack inequality

This argument is fundamental, valid for $q > p - 1$:

Proposition 5.8 *.There exists a constant $c = c_{N,p,q,\mu,\theta} > 0$ such that for any $r_0 > 0$ and any solution u of (1.1) in $B_{r_0} \setminus \{0\}$ (resp. $\mathbb{R}^N \setminus \overline{B_{r_0}}$, resp. $\mathbb{R}^N \setminus \{0\}$)*

$$\sup_{|x|=r} u(x) \leq c \inf_{|x|=r} u(x), \qquad \forall r \in (0, \frac{r_0}{2}) \text{ (resp. } (2r_0, \infty), \text{resp. } (0, \infty)). \tag{5.5}$$

Proof. It is analogous to the proof of [15, Lemma 2.2] given in case $\mu = \theta = 0$ and is a consequence of the Osserman's estimate: for any $x_0 \in B_{\frac{r_0}{2}} \setminus \{0\}$, we write the equation (1.1) in a ball $B(x_0, \frac{|x_0|}{2})$ under the form

$$-\Delta_p u + \psi^p u^{p-1} = 0,$$

where

$$\psi^p = \frac{\mu}{|x|^p} + |x|^{\theta} u^{q+1-p}.$$

From Trudinger [26, p.724], there exists a constant C_1 depending on $N, p, |x_0| \sup |\varphi|_{B(x_0, \frac{|x_0|}{2})}$ such that

$$\sup_{x \in B(x_0, \frac{|x_0|}{6})} u(x) \leq C \inf_{x \in B(x_0, \frac{|x_0|}{6})} u(x).$$

From (5.1), $|x_0| \sup |\psi|_{B(x_0, \frac{|x_0|}{2})}$ is bounded by a constant only depending on N, p, q, θ; then C only depends on N, p, q, θ, μ. Then (5.5) holds by connecting two points x_1, x_2 such that $|x_1| = |x_2| = r < \frac{r_0}{2}$ by 10 connected balls of radius $\frac{r}{6}$. ∎

5.6 Existence of radial solutions in $\omega = B_{r_2} \backslash \overline{B_{r_1}}$

Proposition 5.9 *Let $\mu \in \mathbb{R}$. Then for any $0 < R_1 < R_2$ and any $\ell_1 \geq 0, \ell_2 \geq 0$, with $\ell_1 + \ell_2 \neq 0$, there exists a unique positive function $v = v_{\ell_1,\ell_2}$ in $\omega = B_{R_2} \backslash \overline{B_{R_1}}$ such that $v \in C(\overline{\omega})$ and*

$$\begin{cases} -\Delta_p v + \mu \frac{v^{p-1}}{|x|^p} + |x|^\theta v^q = 0 \text{ in } \omega = B_{R_2} \backslash \overline{B_{R_1}}, \\ v(x) = \ell_1 \text{ for } |x| = R_1, \\ v(x) = \ell_2 \text{ for } |x| = R_2, \end{cases} \tag{5.6}$$

and it is radial.

Proof. From Corollary 5.3, if such a positive solution exists, it is unique, then it is radial. We proceed by minimisation in a space of radial functions. Let $\varphi(x) = \ell_1 + (\ell_2 - \ell_1)\frac{|x|-r_1}{r_2-r_1}$, be a smooth radial function in $\overline{\omega}$ satisfying the boundary conditions. We define a function J on $W_\varphi = \left\{ v \in W^{1,p}_{rad}(\omega) \mid v = \varphi \text{ on } \partial\omega \right\}$ by

$$J(v) = \int_\omega (|\nabla v|^p + \mu \frac{(v^+)^p}{|x|^p} + \frac{p}{q+1} |x|^\theta |v|^{q+1}) dx.$$

Then

$$J(v) \geq \int_\omega (|\nabla(v)|^p - |\mu| c_1^p |v|^p + c_2 |v|^q) dx,$$

with two constants c_1, c_2 only depending on r_1, r_2 and θ, p, q. For $q > p-1$, from the Hölder inequality, with c_3 depending on c_1, c_2 and $|\mu|$

$$J(v) \geq \int_\omega (|\nabla v|^p + \frac{c_2}{2} |v|^q - c_3) dx.$$

Then $\lim_{v \in W_\varphi, \|v\|_{W^{1,p(\omega)}} \longrightarrow \infty} J(v) = \infty$, and by compacity, $\inf_{w \in W_\varphi} J$ is attained at least at some $\widetilde{v} \in W_\varphi$. When $\mu \geq 0$, it is clear that the problem of minimisation admits a unique solution, but not when $\mu < 0$. Concerning the question of positivity, we observe that the function $\widetilde{v}^{\,+} \in W_\varphi$, thus $\widetilde{v}^{\,+} \in C(\overline{\omega})$, and satisfies $J(\widetilde{v}) \geq J(\widetilde{v}_+)$; then $J(\widetilde{v}_+) = \inf_{v \in W_\varphi} J(v)$, and $\widetilde{v}^{\,+}$ satisfies the equation (1.1) in $\mathcal{D}'(\omega)$, and the conditions on $\partial\omega$. From [25, Theorem 1], $\widetilde{v}^{\,+} \in C^1(\omega)$. Hence $\widetilde{v}_+$ is a nonnegative solution of problem (5.6). Moreover, from Theorem 5.1, either $\widetilde{v}^+ \equiv 0$, which contradicts the assumption $\ell_1 + \ell_2 \neq 0$; then $\widetilde{v}^{\,+} > 0$. Then $\widetilde{v}^{\,+}$ is the unique solution of (5.6). ∎

Next we give **the key-point of our study**, consequence of Osserman's estimate and Harnack inequality. It is is a comparison from above and below with radial solutions with the same behaviour

Theorem 5.10 *Let u be any positive solution of (1.1) in $B_{r_0} \setminus \{0\}$ (resp. $\mathbb{R}^N \backslash \overline{B_{r_0}}$) (resp. $\mathbb{R}^N \setminus \{0\}$). Then there exist radial solutions v and w such that*

$$v \leq u \leq w \leq cv \qquad \text{in } B_{\frac{r_0}{2}} \setminus \{0\} \quad \text{(resp.}\mathbb{R}^N \backslash \overline{B_{2r_0}}\text{),(resp.}\mathbb{R}^N \setminus \{0\}\text{)},$$

where $c = c_{N,p,q,\mu,\theta}$ is the Harnack constant defined at Proposition 5.8.

Proof. Let u be any positive solution of (1.1) in $B_{r_0}\setminus\{0\}$. From Proposition 5.9, for any *integer* $n \geq 1$, there exist radial positive solutions of (1.1) in $B_{r_0}\setminus\overline{B_{r_n}}$ such that

$$v_n(x) = \min_{|y|=\frac{1}{n}} u(y) \text{ for } |x| = \frac{1}{n}, \qquad v_n(x) = \min_{|y|=\frac{r_0}{2}} u(y) \text{ for } |x| = \frac{r_0}{2}, \tag{5.7}$$

$$w_n(x) = \max_{|y|=\frac{1}{n}} u(y) \text{ for } |x| = \frac{1}{n}, \qquad w_n(x) = \max_{|y|=\frac{r_0}{2}} u(y) \text{ for } |x| = \frac{r_0}{2}, \tag{5.8}$$

Then from Corollary 5.3 we get $v_n \leq u \leq w_n$. Moreover there holds from Proposition 5.8

$$w_n(x) \leq c_{N,p,q,\mu,\theta} v_n, \qquad \text{for } |x| = \frac{1}{n} \text{and for } |x| = \frac{r_0}{2},$$

We consider the function $y_n = c_{N,p,q,\mu,\theta} v_n$, which satisfies

$$-\Delta_p y_n + \mu \frac{y_n^{p-1}}{|x|^p} + c_{N,p,q,\mu,\theta}^{p-1-q} |x|^\theta y_n^q = 0.$$

Since $c_{N,p,q,\mu,\theta} \geq 1$ and $q > p-1$, the function y_n is a **supersolution** of the equation, and greater than w_n for $|x| = \frac{1}{n}$and for $|x| = r_0$. From Corollary 5.3, we get $w_n \leq y_n$, then

$$v_n \leq u \leq w_n \leq c_{N,p,q,\mu,\theta} v_n \leq c_{N,p,q,\mu,\theta} u$$

in $\overline{B}_{\frac{r_0}{2}}\setminus B_{\frac{1}{n}}$. Then from Theorem 5.4,

$$v_n \leq C_{N,p,q,\mu,\theta} |x|^{-\gamma}, \qquad w_n \leq c_{N,p,q,\mu,\theta} C_{N,p,q,\mu,\theta} |x|^{-\gamma}.$$

For any fixed $\varepsilon \in (0, \frac{r_0}{8})$, and $n > \frac{1}{\varepsilon}$, the sequences v_n and w_n are uniformly bounded in $C(\overline{B}_{\frac{r_0}{2}}\setminus B_\varepsilon)$ and in $W^{1,p}(\overline{B}_{\frac{r_0}{2}}\setminus B_\varepsilon)$. As a consequence, $\Delta_p v_n$ and $\Delta_p w_n$ are uniformly bounded in $\overline{B}_{\frac{r_0}{2}-\varepsilon}\setminus B_\varepsilon$, thus from [25, Theorem 1], v_n and w_n are uniformly bounded in $C^{1,\alpha}(\overline{B}_{\frac{r_0}{2}-\varepsilon}\setminus B_\varepsilon)$ for some $\alpha > 0$. By a diagonal process, there exists subsequences v_ν and w_ν converging strongly in $C^1_{loc}(B_{\frac{r_0}{2}}\setminus\{0\})$ and weakly in $W^{1,p}_{loc}(B_{\frac{r_0}{2}}\setminus\{0\}))$ to radial functions v and w, solutions of (1.1), such that $v \leq u \leq w \leq cv$ in $B_{\frac{r_0}{2}}\setminus\{0\}$.

We get the same conclusions in $\mathbb{R}^N\setminus\overline{B_{r_0}}$ (resp. $\mathbb{R}^N\setminus\{0\}$) by appying Proposition 5.9 in $B_n\setminus\overline{B_{2r_0}}$ (resp. in $B_n\setminus\overline{B_{\frac{1}{n}}}$). ∎

5.7 Precise convergence results

Finally give a more precise behaviour of the possibly nonradial solutions of Hardy type, extending some results of [15].

Notation: $u \asymp_{x\longrightarrow 0} \widetilde{u}$ means that there exists constants $C_1, C_2 > 0$ such that $C_1\widetilde{u} \leq u \leq C_2\widetilde{u}$ near 0.

Proposition 5.11 *Suppose that u is a solution in $B_{r_0}\setminus\{0\}$ of (1.1) such that $u \asymp_{x\longrightarrow 0} |x|^{-S_i}$ ($i = 1$ or $2, S_i \neq 0$), where $S_i < \gamma$ (resp. $u \asymp_{x\longrightarrow\infty} |x|^{-S_i}$ where $S_i > \gamma$).Then there exists $k_i > 0$ such that*

$$\lim_{x\longrightarrow 0} |x|^{S_i} u = k_i \text{ (resp. } \lim_{x\longrightarrow\infty} |x|^{S_i} u = k_i)$$

Proof. Here we proceed as in [15] for the case $\mu = 0$, $i = 1$. We consider for example the case where $u \asymp_{x \longrightarrow 0} |x|^{-S_i}$, with $S_i < \gamma$. Let $v_i = |x|^{-S_i}$, and

$$k = \limsup_{x \longrightarrow 0} \frac{u}{v_i}(x) \qquad \text{and} \qquad \widetilde{k}(r) = \sup_{|x|=r} \frac{u}{v_i}(x), \quad \forall r \in (0, r_0)\,.$$

(i) We first check that

$$\lim_{r \longrightarrow 0} \widetilde{k}(r) = k. \tag{5.9}$$

Indeed there holds $\limsup_{r \longrightarrow 0} \widetilde{k}(r) = k$. Suppose that (5.9) is false, then $\liminf_{r \longrightarrow 0} \widetilde{k}(r) = k_0 < k$. There exists $r^\star < r_0$ such that $\widetilde{k}(r^*) > \frac{k_0+k}{2}$. There exists a decreasing sequence $(r_n)_{n\geq 1} \to 0$ such that $\widetilde{k}(r_n) \to k_0$, then $\widetilde{k}(r_n) < \frac{k_0+k}{2}$ and $r_n < r^\star < r_0$ for $n \geq n_0$ if n_0 is large enough. Then $M = \sup_{\overline{B_{r_{n_0}}} \setminus B_{r_n}} \frac{u}{v_i}$ is not attained on the boundary, but in an interior point $x_0 \in B_{r_{n_0}} \setminus \overline{B_{r_n}}$. But there holds

$$\mathcal{L}_{p,\mu}(Mv_i) = -\Delta_p(Mv_i) + \mu \frac{(Mv_i)^{p-1}}{|x|^p} = 0 \geq \mathcal{L}_{p,\mu} u,$$

and $|\nabla v_i|$ does not vanish in $B_{r_{n_0}} \setminus \overline{B_{r_n}}$, and $u \leq Mv_i$ in $B_{r_{n_0}} \setminus \overline{B_{r_n}}$ with $u(x_0) = Mv_i(x_0)$. Then

$$-div(|\nabla Mv_i|^{p-2} \nabla Mv_i - |\nabla u|^{p-2} \nabla u + \Phi(Mv_i - u) \geq 0,$$

where $\Phi = \mu \frac{(Mv_i)-u^{p-1}}{|x|^p (Mv_i - u)}$ is bounded in $B_{r_{n_0}} \setminus \overline{B_{r_n}}$, since u and v are positive. As in [15, Lemma 1.3], we get $u \equiv Mv_i$ in $B_{r_{n_0}} \setminus \overline{B_{r_n}}$, which is contradictory.

(ii) Next we consider for some $r_0 \in (0, 1)$ the scaled function for given $r > 0$

$$u_r(\xi) = \frac{u(r\xi)}{v_i(r)} \qquad \text{for } 0 < |\xi| < \frac{r_0}{r}.$$

and choose $\xi_r \in S^{N-1}$ such that $\widetilde{k}(r) = r^{S_i} u(r\xi_r)$. By computation, u_r satisfy the equation

$$-\Delta_p u_r(\xi) + \mu \frac{u_r^{p-1}}{|\xi|^p} + r^{d_i} u_r^q = 0$$

with $d_i = (q - p + 1)(\gamma - S_i) > 0$, then $\lim_{r \longrightarrow 0} r^{d_i} = 0$ as $r \longrightarrow 0$. Moreover, by assumption there holds $u(x) \leq C\,|x|^{-S_i}$ in $B_{\frac{r_0}{2}} \setminus \{0\}$, implying precise estimates of the gradient from Proposition 5.6 with $\delta = S_i$: for any $|\xi| < \frac{r_0}{2r}, |\xi'| < \frac{r_0}{2r}$,

$$\begin{aligned} u_r(\xi) &\leq C\,|\xi|^{-S_i}\,, \qquad |\nabla u_r(\xi)| \leq C\,|x|^{-(S_i+1)}\,, \\ \left|\nabla u_r(\xi) - \nabla u_r(\xi')\right| &\leq C \frac{|\xi|^{-S_i} + |\xi'|^{-S_i}}{|\xi|^{\alpha+1}} \left|\xi - \xi'\right|^\alpha. \end{aligned}$$

Otherwise, by definition,

$$\frac{u_r(\xi)}{v_i(\xi)} = (r\xi)^{S_i} u(r\xi) \leq \widetilde{k}(r\,|\xi|), \qquad \frac{u_r(\xi_r)}{v_i(\xi_r)} = r^{S_i} u(r\xi_r) = \widetilde{k}(r).$$

Thus for any sequence $\widetilde{r_n} \to 0$ we can extract a subsequence r_n such that u_{r_n} converges in $C^{1,loc}(\mathbb{R}^N \setminus \{0\}$ to a function $w \geq 0$, and ξ_{r_n} converges to $\widetilde{\xi} \in S^{N-1}$. Then

$$-\Delta_p w + \mu \frac{w^{p-1}}{|\xi|^p} = 0 \text{ in } \mathbb{R}^N \setminus \{0\};$$

and $w(\xi) \leq k v_i(\xi)$ for any $\xi \in \mathbb{R}^N \setminus \{0\}$, with $w(\widetilde{\xi}) = k$ from (5.9). Note that kv_i is also a solution of this equation, and $|\nabla v_i|$ does not vanishes in $\mathbb{R}^N \setminus \{0\}$, then as above $w \equiv kv_i$ from [15, Lemma 1.3]. Since $\widetilde{r_n}$ is arbitrary, $\lim_{r \longrightarrow 0} u_r(\xi) = k\,|\xi|^{-S_i}$, then taking $|\xi| = 1$, we obtain

$$\lim_{x \longrightarrow 0} |x|^{S_i}\, u(x) = k.$$

We get analogous results near ∞, by using Proposition 5.6 in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ with $\delta = S_i > \gamma$. ∎

In the particular case $\mu = \mu_0$ where $S_1 = S_2 = \frac{N-p}{p} < \gamma$ (resp. $\frac{N-p}{p} > \gamma$) there exists two types of radial singular solutions near 0 (resp. near ∞) there still exists the functions $r \longmapsto \ell r^{-\frac{N-p}{p}}$, $\ell > 0$ for which the proposition is still valid, but also some functions v such that $\lim_{r \longrightarrow 0} r^{\frac{N-p}{p}} |\ln r|^{-1} v(r) = \ell$, see Lemma 4.5. We get the following:

Proposition 5.12 *Let $\mu = \mu_0$. Suppose that u is a solution in $B_{r_0} \setminus \{0\}$ (resp $\mathbb{R}^N \setminus \overline{B_{r_0}}$) of (1.1) such that $u \asymp_{x \longrightarrow 0} |x|^{-\frac{N-p}{p}} |\ln |x||^{\frac{2}{p}}$ and $\frac{N-p}{p} < \gamma$ (resp. $u \asymp_{x \longrightarrow \infty} |x|^{-\frac{N-p}{p}} (\ln |x|)^{\frac{2}{p}}$ and $\frac{N-p}{p} > \gamma$).Then there exists $k > 0$ such that*

$$\lim_{x \longrightarrow 0} |x|^{-\frac{N-p}{p}} |\ln |x||^{\frac{2}{p}} = k \text{ (resp. } \lim_{|x| \longrightarrow \infty} |x|^{-\frac{N-p}{p}} (\ln |x|)^{\frac{2}{p}} = k)$$

Proof. Consider for example a solution u in $B_{r_0} \setminus \{0\}$ with such a behaviour, let v_0 be the unique radial solution of the equation

$$\mathcal{L}_{p,\mu}(v_0) = -\Delta_p v_0 + \mu_0 \frac{v_0^{p-1}}{|x|^p} = 0$$

such that $\lim_{r \longrightarrow 0} r^{\frac{N-p}{p}} |\ln r|^{-1} v_0(r) = 1$, given at Lemma 4.5. Let

$$k = \limsup_{x \longrightarrow 0} \frac{u}{v_0}(x) > 0 \qquad \text{and} \qquad \widetilde{k}(r) = \sup_{|x| = r} \frac{u}{v_0}(x), \quad \forall r \in (0, r_0).$$

(i) We first check that $\lim_{r \longrightarrow 0} \widetilde{k}(r) = k$, exactly as before, where v_i is replaced by v_0.

(ii) Next consider for some $r_0 \in (0,1)$ and given $r > 0$

$$u_r(\xi) = \frac{u(r\xi)}{v_0(r)} \qquad \text{for } 0 < |\xi| < \frac{r_0}{r}.$$

and we choose $\xi_r \in S^{N-1}$ such that $\widetilde{k}(r) = v_0(r) u(r\xi_r)$. Then u_r satisfies the equation

$$-\Delta_p u_r(\xi) + \mu \frac{u_r^{p-1}}{|\xi|^p} + (r^\gamma v_0)^{q+1-p} u_r^q = 0$$

Since $\frac{N-p}{p} < \gamma$ there holds $\lim_{r\longrightarrow 0}(r^{\gamma}v_0)^{q+1-p} = 0$. By hypothesis, for $u(x) \leq Cv_0(|x|) \leq C_1 |x|^{-\frac{N-p}{p}} |\ln|x||$ in $B_{\frac{r_0}{2}} \setminus \{0\}$. From Remark 5.7, for any $x, x' \in B_{\frac{r_0}{2}} \setminus \{0\}$,

$$|\nabla u(x)| \leq C_2 |x|^{-\frac{N}{p}} |\ln|x||$$

$$\left|\nabla u(x) - \nabla u(x')\right| \leq C_3(\left|x'\right|^{-\frac{N-p}{p}} + |x|^{-\frac{N-p}{p}}) |x|^{-(1+\alpha)} |\ln|x|| \left|x - x'\right|^{\alpha}.$$

then for $|\xi|, |\xi'| < \frac{r_0}{2r} < \frac{1}{2r}$

$$u_r(\xi) \leq C\frac{v_0(r\xi)}{v_0(r)} \leq C_1 |\xi|^{-\frac{N-p}{p}} \frac{|\ln r\xi|}{|\ln r|} \leq C_1 |\xi|^{-\frac{N-p}{p}} (1 + \frac{|\ln \xi|}{|\ln r|})$$

$$|\nabla u_r(\xi)| = \frac{r}{v_0(r)} |\nabla u(r\xi)| \leq C_4 r |r\xi|^{-\frac{N}{p}} \frac{|\ln r\xi|}{r^{-\frac{N-p}{p}} |\ln r|} = C_4 |\xi|^{-\frac{N}{p}} \frac{|\ln r\xi|}{|\ln r|}$$

$$\left|\nabla u_r(\xi) - \nabla u_r(\xi')\right| \leq C_3 r^{\alpha}(\left|\xi'\right|^{-\frac{N-p}{p}} + |\xi|^{-\frac{N-p}{p}}) |\xi|^{-(1+\alpha)} |\ln r\xi| \left|\xi - \xi'\right|^{\alpha}.$$

Then we still have that for any sequence $\widetilde{r_n} \longrightarrow 0$, there exists a subsequence r_n such that u_{r_ν} converges in $C^1_{loc}(\mathbb{R}^N \setminus \{0\})$ to a function w, such that

$$\mathcal{L}_{p,\mu}(w) = -\Delta_p w(\xi) + \mu_0 \frac{w^{p-1}}{|\xi|^p} = 0$$

in $\mathbb{R}^N \setminus \{0\}$, and $\xi_{r_n} \longrightarrow \xi_0 \in S^{n-1}$, $u_{r_n}(\xi_{r_n}) \longrightarrow w(\xi_0)$ and $\widetilde{k}(r_n) = \frac{u_{r_n}(\xi_{r_n})}{v_0(r_n)}$. But from [14, Example 1.1], the unique solutions of this equation are the functions $\xi \longmapsto \ell |\xi|^{-\frac{N-p}{p}}$, $\ell > 0$. Then $w(\xi_0) = \ell |\xi_0|^{-\frac{N-p}{p}} = \ell$ and $\widetilde{k}(r_n) = u_{r_n}(\xi_{r_n}) \longrightarrow k$. Then $\ell = k$, thus w is independant of the choice of r_n, then $\lim_{r\longrightarrow 0} u(r\xi) = k$ any $\xi \in \mathbb{R}^N \setminus [0]$, in particular for $\xi \in S^{n-1}$. Then $\lim_{x\longrightarrow 0} \frac{u}{v_0}(x) = k$. We get similar results in $\mathbb{R}^N \setminus \overline{B_{r_0}}$ when $\frac{N-p}{p} > \gamma$. ∎

6 Proof of the nonradial results

We begin by Theorem 2.5, which is the simplest case, and the proof is quite short.

Proof of Theorem 2.5. Let u be any solution in $B_{r_0} \setminus \{0\}$ in $B_{r_0} \setminus \{0\}$. (resp. $\mathbb{R}^N \setminus \overline{B_{r_0}}$). From Theorem 5.10, there exist two radial solutions such that $v \leq u \leq w$ in $B_{\frac{r_0}{2}} \setminus \{0\}$ (resp. in $\mathbb{R}^N \setminus \overline{B_{r_0}}$), (resp. $\mathbb{R}^N \setminus \{0\}$). From Theorem 4.10, where all the possible behaviours are described, there holds $\lim_{x\longrightarrow 0} |x|^{\gamma} v(x) = a^*, \lim_{x\longrightarrow 0} |x|^{\gamma} w(x) = a^*$ (resp $\lim_{x\longrightarrow \infty} |x|^{\gamma} v(x) = a^*, \lim_{x\longrightarrow \infty} |x|^{\gamma} w(x) = a^*$, resp. $|x|^{\gamma} v(x) = a^* = |x|^{\gamma} w(x)$). Hence we find $\lim_{x\longrightarrow 0} |x|^{\gamma} u(x) = a^*$ (resp $\lim_{x\longrightarrow \infty} |x|^{\gamma} u(x) = a^*$, resp $|x|^{\gamma} u(x) = a^*$). ∎

Next we prove Theorem 2.1 (resp. 2.2).

Proof of Theorem 2.1. (i) Let u be a solution in $B_{r_0} \setminus \{0\}$. From Theorem 5.10, there exists radial functions such that $v \leq u \leq w \leq cv$ in $B_{r_0} \setminus \{0\}$. First suppose $\mu > \mu_0$. From Theorem 4.6, either $\lim_{r\longrightarrow 0} r^{\gamma} v = a^*$, then also $\lim_{r\longrightarrow 0} r^{\gamma} w = a^*$, then by squeezing $\lim_{x\longrightarrow 0} |x|^{\gamma} u = a^*$; or $\lim_{r\longrightarrow 0} r^{S_i} v = k_i' > 0$, with $i = 1$ or 2, then also $\lim_{r\longrightarrow 0} r^{S_i} w = k_i'' > 0$, implying that $u \asymp |x|^{-S_i}$, then $\lim_{x\longrightarrow 0} |x|^{S_i} u = k_i > 0$ from Proposition 5.11. For $\mu = \mu_0$ we still have a possibility that

$\lim_{x\to 0}|x|^{\frac{N-p}{p}}(|\ln|^{-\frac{2}{p}}|x|)v(|x|)=\ell_1>0$, and $\lim_{x\to 0}|x|^{\frac{N-p}{p}}(|\ln|^{-\frac{2}{p}}|x|)w(|x|)=\ell_2>0$, then we get (2.17) from Proposition 5.12.

(ii) Let u be a solution in $\mathbb{R}^N\backslash\overline{B_{r_0}}$. Then from 4.6, $\lim_{r\longrightarrow\infty}r^\gamma v=a^*=\lim_{r\longrightarrow\infty}r^\gamma w$, thus $\lim_{|x|\longrightarrow\infty}|x|^\gamma u=a^*$.

(iii) Let u be a solution in $\mathbb{R}^N\backslash\{0\}$. Then $v\le u\le w\le cv$ in $\mathbb{R}^N\backslash\{0\}$. From Theorem 4.6, all the behaviours of radial solutions in $\mathbb{R}^N\backslash\{0\}$ are still described. Either $\mu>\mu_0$ and $r^\gamma v\equiv r^\gamma v=a^*$, thus $|x|^\gamma u\equiv a^*$. Or $\mu>\mu_0$, and $\lim_{r\longrightarrow 0}r^{S_1}v=k_i'>0$, and $\lim_{r\longrightarrow\infty}r^\gamma v=a^*$; then $\lim_{x\longrightarrow 0}|x|^{S_1}u=k_i>0$ from Proposition 5.11, and $\lim_{|x|\longrightarrow\infty}|x|^\gamma u=a^*$. For given $k_1>0$, there exists a unique function u satisfying these two conditions. Indeed if $\widetilde{u}$ is another such solution, then $(1+\varepsilon)\widetilde{u}$ is a supersolution, and near 0 and ∞, it is greater than u, then $(1+\varepsilon)\widetilde{u}\ge u$; then $\widetilde{u}\ge u$, and $\widetilde{u}=u$. By uniqueness u is radial. Finally if $\mu=\mu_0$, then $\lim_{x\to 0}|x|^{\frac{N-p}{p}}(|\ln|^{-\frac{2}{p}}|x|)u(x)=\ell>0$ from Proposition 5.12, and $\lim_{|x|\longrightarrow\infty}|x|^\gamma u(x)=a^*$. By comparison as above, it is unique and radial. ■

The proofs of Theorem 2.2, 2.3, 2.4 are analogous .

Remark 6.1 *Compared to the proofs of [15] in the case $\mu=0$ and of [11] in the case $p=2$, our proofs are much shorter: we do not need any discussion on the cases where $\limsup_{|x|\longrightarrow 0}|x|^{S_1}u=\infty$ or $\liminf_{|x|\longrightarrow 0}u=0$; and we do not require a comparison with radial subsolutions or supersolutions which existence is difficult to obtain, see [15, Lemma 1.4] and [9, Propositions 3.1,3.4].*

7 Proofs of the radial results

7.1 Fixed points of system (4.8)

We consider the system (4.8), which takes the form

$$\left\{\begin{array}{rcl} G_t &=& F(G)-V,\\ V_t &=& (q+1-p)V(\gamma-|G|^{\frac{2-p}{p-1}}G),\end{array}\right. \tag{7.1}$$

where we recall that $V>0$, $G=|S|^{p-2}S$, and

$$F(G)=\varphi(S)=(p-1)|G|^{\frac{p}{p-1}}-(N-p)G-\mu, \tag{7.2}$$

so we study the system for

$$(G,V)\in\mathbb{R}\times[0,\infty).$$

- The system has **three possible fixed points**: when $\mu\ge\mu_0$ we find two first points

$$\mathbf{A}_1=(G_1,0)=(|S_1|^{p-2}S_1,0),\qquad \mathbf{A}_2=(G_2,0)=(|S_2|^{p-2}S_2,0),$$

eventually confounded when $\mu=\mu_0$, and eventually a third point in $\mathbb{R}\times(0,\infty)$,

$$\mathbf{M}_0=(G_0,V_0)=(|\gamma|^{p-2}\gamma,(p-1)|\gamma|^p-(N-p)|\gamma|^{p-2}\gamma-\mu)$$

under the condition $F(G_0) > 0$, corresponding to the solution u^* given by (2.6); if $\gamma = 0$, $\mu < 0$, then $\mathbf{M}_0 = (0, |\mu|)$.

• In the sequel we use the **vanishing curves of the vector field** in the phase-plane $\mathbb{R}\times(0,\infty)$

$$\mathcal{C} = \{G_t = 0\} = \{V = F(G)\},$$

$$\{V_t = 0\} = \{V = 0\} \cup \mathcal{L}, \qquad \mathcal{L} = \left\{G = G_0 = |\gamma|^{p-2}\gamma\right\}.$$

Obviously $\mathcal{C}$ and $\mathcal{L}$ meet at $\mathbf{M}_0$ when it exists. And $\{V = 0\}$ and $\mathcal{L}$ meet at point $L_0 = (G_0, 0) = (|\gamma|^{p-2}\gamma, 0)$. Moreover when $\mu \geq \mu_0$, then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1$ is the graph of a nonincreasing function F_1 such that $F_1(G_1) = 0$ and $\mathcal{C}_2$ is the graph of an nondecreasing function F_2 such that $F_2(G_2) = 0$. When $\mu > \mu_0$, the slope of F_1 at $\mathbf{A}_1$ (resp. of F_2 at G_2) is

$$F_1'(G_1) = p\,|G_1|^{\frac{p-2}{p-1}}\,G_1 - (N-p) > 0 \quad (\text{resp. } F_2'(G_2) = pG_2^{\frac{1}{p-1}} - (N-p) < 0).$$

When $\mu < \mu_0$ (resp. $\mu = \mu_0$) the function F has a positive (resp. zero) minimum at $(\frac{N-p}{p})^{p-1}$.

Remark 7.1 *Consider the eventual trajectories $\mathcal{T}$ located on the axis $\{V = 0\}$, which are **nonadmissible** for our purpose. We claim that the union of their adherence covers the axis, so that **any trajectory with a point in** $\mathbb{R}\times(0,\infty)$ **stays in it**. Indeed in case $V = 0$, the system reduces to equation*

$$G_t = (p-1)\,|G|^{\frac{p}{p-1}} - (N-p)G - \mu = F(G),$$

corresponding to the solutions of the Hardy equation $\mathcal{L}_{p,\mu}(u) = 0$, see (4.12). Thus in any interval $\mathcal{I}$ where $F(G) \neq 0$, we can express t as a function of G: for fixed $\widetilde{G} \in \mathcal{I}$, we write $t(G) = t(\widetilde{G}) + \int_{\widetilde{G}}^{G} \frac{dg}{F(g)}$. If $\mu < \mu_0$, then $\mathcal{I} =\mathbb{R}$, and the integral converges at $\pm\infty$, then G describes $\mathbb{R}$ as t describes $(t(-\infty), t(\infty))$. When $\mu \geq \mu_0$, the integral still converges as $G \to \pm\infty$, and diverges at G_1, G_2; taking $\mathcal{I} = (-\infty, G_2)$ (resp. (G_1, ∞)) G describes $\mathcal{I}$ as t describes $(t(-\infty), \infty)$ (resp. $(-\infty, t(\infty))$); if $G_1 \neq G_2$, taking $\mathcal{I} =(G_1, G_2)$, G describes $\mathcal{I}$ as t describes $\mathbb{R}$.

Next we give a general property of the trajectories of system (4.8).

Lemma 7.2 *For any solution u of (4.1) defined near $r = 0$ (resp. near $r = \infty$), then (G, V) converge to one of the fixed points $\mathbf{M}_0, \mathbf{A}_1, \mathbf{A}_2$ as $t \to -\infty$ (resp. $t \to \infty$).*

Proof. From the Osserman's estimate (5.1), for such a solution, V is bounded near $t = -\infty$ (resp. near $t = \infty$). Suppose that G is unbounded near $t = -\infty$ (resp. near $t = \infty$). Either G is monotone near $t = -\infty$ (resp. near $t = \infty$) and G tends to $\pm\infty$, then $G_t \sim (p-1)\,|G|^{\frac{p}{p-1}}$; by integration, we deduce that G tends to 0, which is contradictory. Or there exists a monotone sequence $t_n \to -\infty$ (resp. $t_n \to \infty$) such that $G_t(t_n) = 0$, and $|G(t_n)| \to \infty$ then $V(t_n) = F(G(t_n)) \to \infty$, which is still contradictory. Then (V, G) is bounded. From the Poincaré-Bendixon Theorem, either it converges to a fixed point, or it has a limit cycle around a fixed point. If it is $\mathbf{A}_1$ or $\mathbf{A}_2$, it converges to the point. If it is $\mathbf{M}_0 = (G_0, V_0)$, then $F(G_0) > 0$. Then either $\mu > \mu_0$

and then $\gamma \neq \frac{N-p}{p}$, or $\mu < \mu_0$. Suppose that there exists a periodic trajectory $\mathcal{T}$ around $\mathbf{M}_0$. It is impossible when $\gamma \neq \frac{N-p}{p}$: indeed the energy function $\mathcal{E}$ defined at (4.10) satisfies

$$\mathcal{E}_t = -DV^{\frac{p}{p-1}}(\gamma - |G|^{\frac{2-p}{p-1}}\, G)(|\gamma|^{p-2}\,\gamma - G)$$

with $D = N - p - p\gamma \neq 0$, then $D\mathcal{E}_t < 0$, up to a finite number of points where $G = |\gamma|^{p-2}\,\gamma$. Then (V, G) converges to $\mathbf{M}_0$. Next assume $\mu < \mu_0$ and $\gamma = \frac{N-p}{p}$. Consider a solution $t \in \mathbb{R} \longmapsto (G(t), V(t))$ with period $\mathcal{P}$ on the trajectory $\mathcal{T}$. There exist at least a value $t_1 \in [0, \mathcal{P}]$ where $G(t_1) = G_0$ and $V(t_1) > V_0$; then at $P(t_1) = (G_0, V(t_1))$, $\mathcal{T}$ enters the region $\{V > F(g), G < G_0\}$; this region is positively invariant since the field on the curve $\mathcal{C}_2$ is directed by $(0, 1)$ and the field on $\mathcal{L}$ by $(-1, 0)$ (see Figure 8); so $\mathcal{T}$ stays in it for $t > t_1$, and cannot turn around $\mathbf{M}_0$, and we still get a contradiction. ∎

7.1.1 Behaviour near the fixed point $\mathbf{M}_0$

Next we precise the nature of the point $\mathbf{M}_0$. In that case some new phenomena appear in the particular case $\theta + p = 0$, equivalently $\gamma = 0$.

Lemma 7.3 *Suppose that $\mathbf{M}_0$ exists, and $\gamma \neq 0$. Then $\mathbf{M}_0$ is a saddle point: the eigenvalues $\lambda_1 < 0 < \lambda_2$ are the roots of the trinom*

$$T(\lambda) = \lambda^2 - (p\gamma - N + p)\lambda - \frac{q+1-p}{p-1}\,|\gamma|^{2-p}\,V_0.$$

Some corresponding eigenvectors are

$$\overrightarrow{w_1} = (1, p\gamma - N + p - \lambda_1) = (1, \lambda_2), \qquad \overrightarrow{w_2} = (1, p\gamma - N + p - \lambda_2) = (1, \lambda_1).$$

There exist precisely two trajectories $\mathcal{T}_1, \mathcal{T}_2$,converging to $\mathbf{M}_0$ as $t \to \infty$, directed by $\overrightarrow{w_1}$, with slope λ_2 : a trajectory $\mathcal{T}_1$ with $G < G_0$ (resp. $\mathcal{T}_2$ with $G > G_0$) as $t \to \infty$; and two trajectories $\mathcal{T}_3, \mathcal{T}_4$,converging to $\mathbf{M}_0$ as $t \to -\infty$, directed by $\overrightarrow{w_2}$, with slope λ_1 :a trajectory $\mathcal{T}_3$ such that $G < G_0$ (resp. $\mathcal{T}_4$ with $G > G_0$) as $t \to -\infty$.

Proof. We set $G = (G_0 + \overline{G}, V_0 + \overline{V})$, then the linearized system at $\mathbf{M}_0$ is given by

$$\left\{ \begin{array}{rcl} \overline{G}_t &=& (p\gamma - N + p)\overline{G} - \overline{V}, \\ \overline{V}_t &=& -\frac{q+1-p}{p-1}\,|\gamma|^{2-p}\,V_0\overline{G}, \end{array} \right. \tag{7.3}$$

so the eigenvalues, given by

$$\det\begin{pmatrix} p\gamma - N + p - \lambda & -1 \\ -\frac{q+1-p}{p-1}\,|\gamma|^{2-p}\,V_0 & -\lambda \end{pmatrix} = 0,$$

are the roots of the trinom $T(\lambda)$; and the product of the roots is negative, thus $\mathbf{M}_0$ is saddle point. The eigenvectors can be computed easily. ∎

Next we study the particular case $\gamma = 0$, equivalently $\theta + p = 0$. Equation (1.1) takes the form (2.8), and the solution u^* exists when $\mu < 0$, and it is constant: $u^* \equiv |\mu|^{\frac{1}{q+1-p}}$. Here the study depends on p, in particular we find the existence of **locally constant solutions near** 0 **or near** ∞ **when** $p > 2$:

Lemma 7.4 *Assume $\gamma = 0$. When $\mu < 0$, $\mathbf{M}_0 = (0, |\mu|)$ is well defined.*

(i) For $p = 2$, Lemma 7.3 still applies without change. Denoting by $\lambda_1 < 0 < \lambda_2$ the roots of $\lambda^2 + (N-2)\lambda - (q-1)|\mu| = 0$, the solutions corresponding to trajectories $\mathcal{T}_3$ and $\mathcal{T}_4$ satisfy

$$u - |\mu|^{\frac{1}{q-1}} \sim_{r \longrightarrow 0} C_i r^{\lambda_2}, \quad i = 3, 4, \quad C_3 > 0 > C_4,$$

those corresponding to trajectories $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy

$$u - |\mu|^{\frac{1}{q-1}} \sim_{r \longrightarrow \infty} C_i r^{\lambda_1}, \quad i = 1, 2, \quad C_2 > 0 > C_1.$$

(ii) For $p \neq 2$ there exist at least a trajectory $\mathcal{T}_1$ with $G < 0$ (resp. trajectory $\mathcal{T}_2$ with $G > 0$) converging to $\mathbf{M}_0$ as $t \to \infty$; and a trajectory$\mathcal{T}_3$ such that $G < 0$ (resp. a trajectory $\mathcal{T}_4$ with $G > 0$) converging to $\mathbf{M}_0$ as $t \to -\infty$.

- *For $p < 2$, the solutions corresponding to trajectories of types $\mathcal{T}_3$ and $\mathcal{T}_4$ satisfy*

$$u - |\mu|^{\frac{1}{q+1-p}} \sim_{r \longrightarrow 0} C_i \left|\ln r\right|^{-\frac{p-1}{2-p}}, \quad i = 3, 4, \quad C_3 > 0 > C_4;$$

the solutions corresponding to trajectories of types $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy

$$u - |\mu|^{\frac{1}{q+1-p}} \sim_{r \longrightarrow \infty} C_i r^{-\frac{N-p}{p-1,}} \quad i = 1, 2, \quad C_2 > 0 > C_1.$$

- *For $p > 2$, the solutions u corresponding to trajectories of types $\mathcal{T}_3$ and $\mathcal{T}_4$ are constant near $r = 0$, there holds $\rho > 0$ such that*

$$u(r) - |\mu|^{\frac{1}{q+1-p}} \sim_{r \longrightarrow \rho} C_i((r-\rho)^+)^{\frac{p}{p-2}}, \quad i = 3, 4, \quad C_3 > 0 > C_4.$$

The solutions u corresponding to trajectories $\mathcal{T}_1$ and $\mathcal{T}_2$ are constant near $r = \infty$, and there holds $R > 0$ such that

$$u(r) - |\mu|^{\frac{1}{q+1-p}} = C_i((R-r)^+)^{\frac{p}{p-2}}, \quad i = 1, 2, \quad C_2 > 0 > C_1.$$

Proof. We are in the case $(\mathcal{H}_2)$ with $\gamma = 0 < S_2 < S_1$, $\mu_0 < \mu < 0$, or in the case $(\mathcal{H}_5)$ $\mu < \mu_0$. We consider the regions in $\mathbb{R} \times (0, \infty)$

$$\begin{array}{rcl} \mathcal{J}_1 & = & \{0 < V < F(G), G < 0\}, \qquad \mathcal{J}_2 = \{V > F(G), G > 0\}, \\ \mathcal{J}_3 & = & \{V > F(G), G < 0\}, \qquad \mathcal{J}_4 = \{0 < V < F(G), G > 0\}. \end{array}$$

(i) Case $p = 2$. The former linearization analysis still applies: the eigenvalues are the roots $\lambda_1 < 0 < \lambda_2$ of

$$T(\lambda) = \lambda^2 + (N-2)\lambda - (q-1)|\mu|$$

so $\mathbf{M}_0$ is still a saddle point as above, defining four trajectories $\mathcal{T}_1$,$\mathcal{T}_2$,$\mathcal{T}_3$,$\mathcal{T}_4$, respectively in regions $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$. System (7.3) becomes

$$\left\{ \begin{array}{rcl} \overline{G}_t & = & (p\gamma - N + p)\overline{G} - \overline{V}, \\ \overline{V}_t & = & -(q-1)|\mu|\,\overline{G}. \end{array} \right.$$

By reduction to the diagonal form we deduce that $\lim_{t \longrightarrow -\infty} \overline{V}\ e^{\lambda_2 t} = l \neq 0$ on $\mathcal{T}_3, \mathcal{T}_4$ (resp. $\lim_{t \longrightarrow \infty} \overline{V}\ e^{\lambda_1 t} = \lambda \neq 0$ on $\mathcal{T}_1, \mathcal{T}_2$). Since $U(t) = (|\mu| + \overline{V})^{\frac{1}{q-1}} = (|\mu| + \overline{V})^{\frac{1}{q-1}}$, there holds

$U(t) - |\mu|^{\frac{1}{q-1}} \sim_{t\longrightarrow-\infty} \frac{1}{q-1}\,|\mu|^{\frac{2-q}{q-1}}\,\overline{V} \sim_{t\longrightarrow-\infty} C_i e^{\lambda_2 t}$ on $\mathcal{T}_3, \mathcal{T}_4$ (resp. $U(t) - |\mu|^{\frac{1}{q-1}} \sim_{t\longrightarrow\infty} C_i e^{\lambda_1 t}$ on $\mathcal{T}_1$,$\mathcal{T}_2$).

(ii) Cases $p \neq 2$.

• Case $p < 2$. Then the linearization is still valid, with $T(\lambda) = \lambda^2 + (N-p)\lambda$, that means $\lambda_1 = -(N-p) < 0 = \lambda_2$. Relatively to the eigenvalue λ_1, there is an eigenvector $(1, 0)$, there exists precisely two trajectories with slope 0, $\mathcal{T}_2$ in the region $\mathcal{J}_2$ and $\mathcal{T}_1$ in the region $\mathcal{J}_1$, ending at the point as $t \to \infty$. Relatively to the eigenvalue 0, the central manifold, of dimension 1, is directed by $(1, -(N-p))$, we find at least two trajectories on it, converging to $\mathbf{M}_0$ as $t \to \infty$ or $t \to \infty$. Since $\lim_{t\longrightarrow\infty} \frac{V_t}{G_t} = -(N-p)$, then (V, G) is necessarily in one of the regions $\mathcal{J}_3, \mathcal{J}_4$; as a consequence, the convergence holds as $t \to -\infty$, so we find two trajectories $\mathcal{T}_3, \mathcal{T}_4$. Moreover, consider for example $\mathcal{T}_4$; we know that $\lim_{t\longrightarrow-\infty} \frac{\overline{V}}{G} = -(N-p)$, then

$$\overline{V}_t = -(q+1-p)(|\mu| + \overline{V})\,|G|^{\frac{2-p}{p-1}}\,G) \sim_{t\longrightarrow-\infty} -k\,|G|^{\frac{2-p}{p-1}}\,G \sim k(N-p)^{-\frac{1}{p-1}}\overline{V}^{\frac{1}{p-1}},$$

thus by integration, $\overline{V} \sim_{t\longrightarrow-\infty} c\,|t|^{-\frac{p-1}{2-p}}$, hence $u^{q+1-p} - |\mu| \sim_{t\longrightarrow-\infty} c_1\,|\ln r|^{-\frac{p-1}{2-p}}$, so that $u - |\mu|^{\frac{1}{q+1-p}} \sim_{r\longrightarrow 0} C_4\,|\ln r|^{-\frac{p-1}{2-p}}$. We obtain a similar result for the trajectory $\mathcal{T}_3$. Next consider the trajectory $\mathcal{T}_2$; there holds $G_t < 0$,$V_t < 0$ and $\lim_{t\longrightarrow\infty} \frac{\overline{V}}{G} = 0$, then $G_t \sim_{t\longrightarrow\infty} -(N-p)G$, thus $G = O(e^{(-(N-p)+\varepsilon)t})t$. Moreover, we still have $\overline{V}_t \sim_{t\longrightarrow-\infty} -k\,|G|^{\frac{2-p}{p-1}}\,G$, then $\overline{V}_t = O(e^{\frac{-(N-p)+\varepsilon}{p-1}t})$, thus $\overline{V} = O(e^{\frac{-(N-p)+\varepsilon}{p-1}t})$, and $|G|^{\frac{p}{p-1}} = O(e^{\frac{-(N-p)+\varepsilon}{p-1}pt})$; then $G_t = -(N-p)G + O(e^{\frac{-(N-p)+\varepsilon}{p-1}t})$, which implies that $\lim_{t\longrightarrow\infty} e^{(N-p)t}G = C > 0$; since $\lim_{t\longrightarrow\infty} U = |\mu|^{\frac{1}{q+1-p}}$, we get $U_t \sim_{t\longrightarrow\infty} C^{\frac{1}{p-1}}e^{-\frac{(N-p)t}{p-1}}$, and $u - |\mu|^{\frac{1}{q+1-p}} \sim_{r\longrightarrow\infty} C_2 r^{-\frac{N-p}{p-1}}$, where $C_2 > 0$. Similarly for $\mathcal{T}_1$.

• Case $p > 2$. Here we cannot linearize the system. We first consider the region $\mathcal{J}_4$, which is positively invariant: all the trajectories with one point in $\mathcal{J}_4$ converge to $\mathbf{A}_2$ as $t \to \infty$. Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of points P of $\mathcal{J}_4$ such that the trajectory passing by P cuts the curve $\mathcal{C}_2$ (resp. the line $\mathcal{L}$); then $\mathcal{U}$ (resp. $\mathcal{V}$) is an open set in $\mathcal{J}_4$, since the intersections are transverse. Then $\mathcal{U} \cup \mathcal{V} \neq \mathcal{J}_4$. Thus there exists at least a trajectory $\mathcal{T}_4$ in $\mathcal{J}_4$ converging to $\mathbf{M}_0$ as $t \to \infty$. In the same way the region $\mathcal{J}_3$ is positivement invariant, and there exists at least a trajectory $\mathcal{T}_3$ in $\mathcal{J}_3$ converging to $\mathbf{M}_0$ as $t \to -\infty$. The region $\mathcal{J}_1$ (resp. $\mathcal{J}_2$) is negatively invariant and as before it contains at least a trajectory $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) converging to $\mathbf{M}_0$ as $t \to \infty$.

Setting again $V = |\mu| + \overline{V}$, there holds

$$\left\{\begin{array}{rcl} G_t &=& (p-1)\,|G|^{\frac{p}{p-1}} - (N-p)G - \overline{V}, \\ \overline{V}_t &=& -(q+1-p)(|\mu| + \overline{V})\,|G|^{\frac{2-p}{p-1}}\,G). \end{array}\right.$$

By derivation, setting $k = (q+1-p)\,|\mu|$

$$G_{tt} = (p\,|G|^{\frac{2-p}{p-1}}\,G - (N-p))G_t - \overline{V}_t,$$

$$G_{tt} + (N-p))G_t = |G|^{\frac{2-p}{p-1}}\,G(pG_t + (q+1-p)(|\mu| + \overline{V}));$$

near the fixed point, as $t \to \pm\infty$, G and $\overline{V}$, then also G_t tends to 0, then

$$G_{tt} + (N-p)G_t = k\,|G|^{\frac{2-p}{p-1}}\,G(1+o(1)), \qquad k = (q+1-p)\,|\mu|\,.$$

First consider the trajectory $\mathcal{T}_4$ where $G>0$, $G_t>0$:

$$\frac{k}{2}G^{\frac{1}{p-1}} \leq G_{tt}+(N-p)G_t \leq 2kG^{\frac{1}{p-1}};$$

by multiplication by G_t we deduce that

$$(\frac{G_t^2}{2}-2k\frac{p-1}{p}G^{\frac{p}{p-1}})_t \leq 0.$$

Then $\frac{G_t^2}{2}-2k\frac{p-1}{p}G^{\frac{p}{p-1}}$ is nonincreasing, and tends to 0 at $-\infty$, then $\frac{G_t^2}{2} \leq 2k\frac{p-1}{p}G^{\frac{p}{p-1}}$, thus $G^{-\frac{p}{2(p-1)}}G_t \leq c$, hence $\frac{2(p-1)}{p-2}G^{\frac{p-2}{2(p-1)}}-ct$ is nonincreasing; this is impossible if $t\to-\infty$. Denoting by (T,τ) the maximal interval where the solution is positive, T is finite, and as $t\to T$, then for $p>2$, $G^{\frac{p-2}{2(p-1)}}-c(t-T)\leq 0$. We obtain a solution u such that u_r is zero for $r\leq e^T$, that means u is constant near the origin. Moreover $G_t \leq cG^{\frac{p}{2(p-1)}}=o(G^{\frac{1}{p-1}})$, then $G_{tt}\sim_{t\longrightarrow T} kG^{\frac{1}{p-1}}$, then for any $\varepsilon>0$, $(k-\varepsilon)G^{\frac{1}{p-1}} \leq G_{tt} \leq (k+\varepsilon)G^{\frac{1}{p-1}}$ for $t-T\leq t_\varepsilon$, and by new integrations we get consecutively $G\sim_{t\longrightarrow T} c_k(t-T)^{\frac{2(p-1)}{p-2}}$,

$$\frac{U_t}{U}\sim_{t\longrightarrow T}|\mu|^{-\frac{1}{q+1-p}}U_t\sim_{t\longrightarrow T}-c_k(t-T)^{\frac{2}{p-2}},$$

$$U-|\mu|^{\frac{1}{q+1-p}}\sim_{t\longrightarrow T}\frac{p-2}{p}c_k(t-T)^{\frac{p}{p-2}}\sim_{t\longrightarrow T}C_4(r-e^T)^{\frac{p}{p-2}},$$

where $C_4<0$.. We get a similar result for the trajectory $\mathcal{T}_3$: in this case $G<0$, $G_t<0$, we get the same conclusion by considering $-G$.

Next consider $\mathcal{T}_2$, where $G>0$ and $G_t<0$. We find

$$2kG^{\frac{1}{p-1}}G_t \leq G_tG_{tt}+(N-p)G_t^2 \leq \frac{k}{2}G^{\frac{1}{p-1}}G_t;$$

hence $\frac{G_t^2}{2}-\frac{k}{2}\frac{p-1}{p}G^{\frac{p}{p-1}}$is nonincreasing and tends to 0 as $t\to\infty$; thus $\frac{G_t^2}{2}\geq\frac{k}{2}\frac{p-1}{p}G^{\frac{p}{p-1}}$, consequently $G^{-\frac{p}{2(p-1)}}G_t\leq -c$, then $G^{\frac{p-2}{2(p-1)}}+ct$ is nonincreasing, hence again we get a solution u such that u_r is zero for $r\geq e^T$, that means u is constant for large r. Then $u(r)-|\mu|^{\frac{1}{q+1-p}}=C_2((R-r)^+)^{\frac{p}{p-2}}$, with $C_2>0$. The result is similar for $\mathcal{T}_1$. ∎

Remark 7.5 *Here we make a comment on the regularity of system (4.8). Let us write it under the form*

$$\left\{\begin{array}{rcl} G_t &=& f(G(t),V(t)),\\ V_t &=& g(G(t),V(t)).\end{array}\right.$$

If $p\leq 2$, the functions f,g are of class C^1, so the system has no singular point. If $p>2$, the system is singular at $G=0$, that means $u'=0$, since g is only continuous at $G=0$. However, for any given $\widetilde{V}>0$, $\widetilde{V}\neq-\mu$, $\gamma\neq 0$, the point $(0,\widetilde{V})$ is not a fixed point, even if $\gamma=0$; then there is only one trajectory passing by the point $(0,\widetilde{V})$. Indeed consider the Cauchy conditions $G(t_0)=0, V(t_0)=\widetilde{V}$. There holds $G_t(t_0)=-\mu-\widetilde{V}\neq 0$. Then one can define t as a C^1 function $t=\psi(G)$ near t_0, and (4.8) is equivalent to

$$\frac{d}{dG}(\psi,V)=\mathcal{F}(G,V)=(\frac{1}{f(G,V)},\psi(G)g(G,V))$$

where $\mathcal{F}$ is of class C^1 with respect to V, and continuous with respect to G, so the Cauchy Theorem can be applied.

In conclusion ***the unique singular point is*** $(0,|\mu|)$ ***when*** $\mu<0$ ***and*** $p>2$.

- *When $\gamma=0$, it is the fixed point $\mathbf{M}_0$, studied at Lemma 7.4.*

- *When $\gamma\neq 0$, $p>2$, then $(0,|\mu|)$ is not a fixed point. Consider a trajectory $\mathcal{T}_\mu$ passing by this point, and a solution $t\longmapsto U(t)$ passing by this point at time t_0 (there exist at least one, from the Peano theorem). In that case $G(t_0)=G_t(t_0)=0$, but f is of class C^1, then $G_{tt}(t_0)$ exists so G is C^2 and $G_{tt}(t_0)=-V_t(t_0)=-\gamma(q+1-p)\,|\mu|\neq 0$. Then $G(t)\sim_{t\longrightarrow t_0}-c\gamma(t-t_0)^2$, with $c=\frac{(q+1-p)|\mu|}{2}$ thus G has the sign of $-\gamma$; and $\mathcal{T}_\mu$ is tangent to the axis $\{G=0\}$ at this point We do not know if $\mathcal{T}_\mu$ is unique.*

7.1.2 Behaviour near the fixed points $\mathbf{A}_1$,$\mathbf{A}_2$

Next we study the fixed points A_i :

Lemma 7.6 *Suppose that $\mu\geq\mu_0$. Then the points $A_i=(G_i,0)$ are well defined (confounded when $\mu=\mu_0$), and the associated eigenvalues are*

$$\rho_i=pS_i-N+p,\qquad \eta_i=(q+1-p)(\gamma-S_i). \tag{7.4}$$

As a consequence, when $\mu>\mu_0$, if $\gamma>S_1$, then $\mathbf{A}_1$ is a source, and $\mathbf{A}_2$ is a saddle point; if $\gamma<S_2$, $\mathbf{A}_1$ is a saddle point and $\mathbf{A}_2$ is a sink. If $S_2<\gamma<S_1$, $\mathbf{A}_1$ and $\mathbf{A}_2$ are saddle points. In those cases, corresponding eigenvectors to ρ_i,η_i are

$$\overrightarrow{u_i}=(1,0),\qquad \overrightarrow{v_i}=(1,pS_i-N+p-\eta_i), \tag{7.5}$$

and the slope of $\overrightarrow{v_i}$ is $m_i=pS_i-N+p-\eta_i=(q+1)S_i-(N+\theta)$.

Proof. Setting $G=G_i+\overline{G}$, $(p-1)\left|G_i+\overline{G}\right|^{\frac{p}{p-1}}\simeq(p-1)\left|G_i\right|^{\frac{p}{p-1}}+p\left|G_i\right|^{\frac{2-p}{p-1}}G_i\overline{G}$ the linearized system at A_i (here available for any $p>1$, and any $\mu\geq\mu_0$) is

$$\left\{\begin{array}{rcccl} G_t &=& ((p\left|G_i\right|^{\frac{2-p}{p-1}}G_i-(N-p))\overline{G}-V &=& (pS_i-N+p)\overline{G}-V,\\ V_t &=& (q+1-p)(\gamma-\left|G_i\right|^{\frac{2-p}{p-1}}G_i)V &=& (q+1-p)(\gamma-S_i)V, \end{array}\right. \tag{7.6}$$

and the eigenvalues λ are given by

$$\det\begin{pmatrix} pS_i-N+p-\lambda & -1\\ 0 & (q+1-p)(\gamma-S_i)-\lambda \end{pmatrix}=0,$$

so we get (7.4) and (7.5). ∎

Remark 7.7 *A point of discussion, first encountered when $\mu = 0$ in [3] is the repartition of the trajectories in case of a source or a sink, that means two eigenvalues have the same sign. First suppose that $\rho_1 > 0$ and $\eta_1 > 0$. We get*

$$\rho_1 - \eta_1 = pS_1 - N + p - (q + 1 - p)(\gamma - S_1) = (q + 1)S_1 - (N + \theta) = m_1.$$

If $\eta_1 > \rho_1 > 0$ and $m_1 \neq 0$ we know that there exists only one trajectory tangent to the vector $\overrightarrow{v_1}$, with negative slope m_1, and all the other are tangent to $\overrightarrow{u_1} = (1, 0)$, that means to the axis $\{V = 0\}$. On the contrary, if $\rho_1 > \eta_1 > 0$, there is only one trajectory tangent to $\overrightarrow{u_1}$, and it is not admissible. It means that all the trajectories are tangent to $\overrightarrow{v_1}$, with positive slope. It is the case when $q > \widetilde{\mathbf{q}_1}$, where

$$\widetilde{\mathbf{q}_1} = \frac{N + \theta}{S_1} - 1 = \frac{\theta + p}{S_1} + \frac{N - p}{S_1} - 1 = \mathbf{q}_1 - p + \frac{N - p}{S_1} < \mathbf{q}_1.$$

In the same way, consider the case where $\rho_2 < 0$ and $\eta_2 < 0$. Then $\rho_2 - \eta_2 = m_2$. If $\eta_1 > \rho_1 > 0$,

there exists only one trajectory tangent to the vector $\overrightarrow{v_1}$, with negative slope, and all the other are tangent to $\{V = 0\}$ On the contrary, if $\rho_1 > \eta_1 > 0$, there is only one trajectory tangent to $\{V = 0\}$, and it is not admissible. It means that all the trajectories are tangent to $\overrightarrow{v_1}$, with positive slope. It is the case when $(q + 1)S_1 > N + \theta$. If $\mu < 0$ it is equivalent to $q > \widetilde{\mathbf{q}_2}$, where

$$\widetilde{\mathbf{q}_2} = \mathbf{q}_2 - p + \frac{N - p}{S_2} > \mathbf{q}_2.$$

If $\mu > 0$ it is only possible if $N + \theta < 0$, and $q < \widetilde{\mathbf{q}_2}$.

7.2 Proofs and comments

We first consider the case $(\mathcal{H}_1)$. In that case $\gamma \geq S_1 > 0$ so there always hold $\theta + p > 0$.

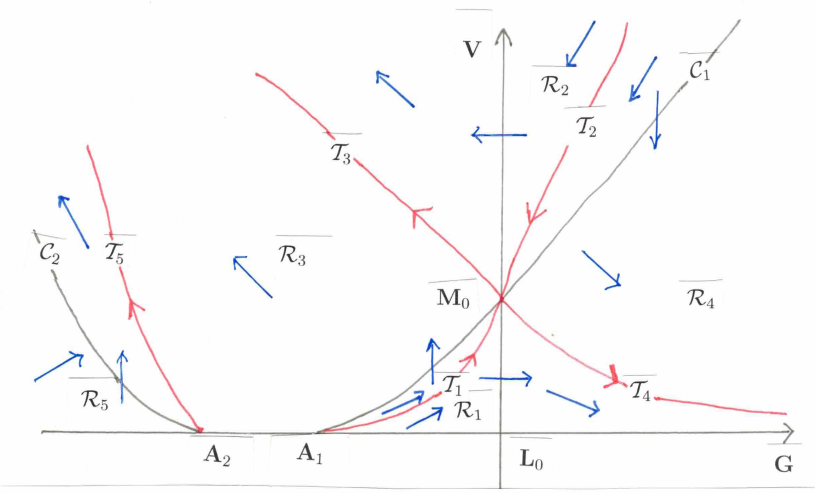

Figure 1. Case $(\mathcal{H}_1)$ (i) : $\mu > \mu_0$ and $\gamma > S_1$

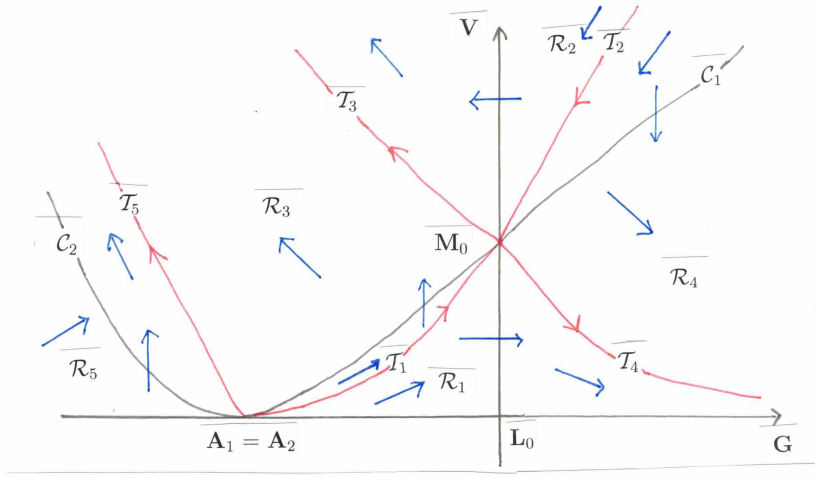

Figure 2. Case $(\mathcal{H}_1)$ (ii) : $\mu = \mu_0$ and $\gamma > S_1 = S_2$

Proof of Theorem 4.6. Case $(\mathcal{H}_1)$: $\mu \geq \mu_0$ and $\gamma \geq S_1$. In that case the point $\mathbf{M}_0$ exists, with the four trajectories $\mathcal{T}_i, i = 1, .., 4$ associated.

(i) We first assume that $\mu > \mu_0$. We define the regions (see Figure 1)

$$\begin{aligned} \mathcal{R}_1 &= \{V < F(G), G_1 < G < G_0\}, \qquad \mathcal{R}_2 = \{V > F(G), G > G_0\}, \\ \mathcal{R}_3 &= \{V > F(G), G < G_0\}, \quad \mathcal{R}_4 = \{V < F(G), G > G_0\}, \quad \mathcal{R}_5 = \{V < F(G), G < G_2\}. \end{aligned}$$

• The region $\mathcal{R}_1$ is negatively invariant. Moreover the slope of F_1 at $\mathbf{M}_0$ is

$$m = pG_0^{\frac{1}{p-1}} - N + p = p\gamma - N + p = \lambda_1 + \lambda_2 < \lambda_2,$$

then $\mathcal{T}_1$ lies in $\mathcal{R}_1$ as $t \to \infty$, then it stays in it, and necessarily converges to $\mathbf{A}_1$ (even in the case $\rho_1 = \eta_1$) So there is a, unique trajectory, namely $\mathcal{T}_1$, joining $\mathbf{A}_1$ to $\mathbf{M}_0$, then u satisfies (4.13).

• The trajectory $\mathcal{T}_3$ starts from $\mathbf{M}_0$ in the region $\mathcal{R}_3$; and $\mathcal{R}_3$ is positively invariant: indeed the vector field is entering $\mathcal{R}_3$ at any point or the curve $\mathcal{C}_2$, except possibly at the singular point $(0, |\mu|)$ when $\mu < 0$ and $p > 2$. But no trajectory with a point in $\mathcal{R}_3$ can pass by this point, since such trajectories satisfy $G < 0$, from then $\mathcal{T}_3$ stays in $\overline{\mathcal{R}_3}$.. Consider any corresponding solution $u(r) = U(t)$ defined on an maximal interval $(-\infty, T)$. Suppose that V is bounded; then $F(G)$ is bounded, so $|G|$ is bounded, then (G, V) converges to some (ℓ, Λ), then (G_t, V_t) has a finite limit. It cannot happen that $(\ell, \Lambda) = (0, -\mu)$, because we have seen that the trajectories passing by this point are contained in the set $G < 0$. Then $T = \infty$, thus (G, V) converges to a fixed point; it is impossible, since there is no other fixed point in $\overline{\mathcal{R}_3}$. Then T is finite, and $\lim_{t\longrightarrow T} V = \infty$; if $\lambda = \lim_{t\longrightarrow T} G$ is finite, and $\lim_{t\longrightarrow T} \frac{V_t}{V} = c > 0$, which contradicts $\lim_{t=T} V = \infty$. Then $\lim_{t\longrightarrow T} G = -\infty$. It implies that the function u, such that $u \sim_{r\longrightarrow 0} u^*$decreasing as $r \to 0$, has a minimum point, and then is increasing to ∞ as $r \to e^T$. Namely we obtain a solution u, satisfying (4.17), often called large solution.

Moreover all the trajectories with one point in $\mathcal{R}_3$ present the same type of behaviour, corresponding to large solutions.

• The trajectory $\mathcal{T}_4$ starts from $\mathbf{M}_0$ as $t \to -\infty$, in the region $\mathcal{R}_4$, where $V_t < 0 < G_t$ which is positively invariant, so it stays in it. Then V is bounded. Here also it is impossible that G to be bounded, since there is no other fixed point in $\overline{\mathcal{W}}$. Consider any corresponding solution $u(r) = U(t)$ defined on an maximal interval $(-\infty, T)$.Then $\lim_{t\longrightarrow T} G = \infty$; and $\lim_{t\longrightarrow T} V = l \geq 0$. Then $G_t \sim_{t\longrightarrow T} (p-1)G^{\frac{p}{p-1}}$, thus $(G^{-\frac{1}{p-1}})_t \sim_{t\longrightarrow T} -1$, which is impossible if $T = \infty$ since $\lim_{t\longrightarrow T} G^{-\frac{1}{p-1}} = 0$. Then T is finite and $G^{-\frac{1}{p-1}} = S = -\frac{U_t}{U} \sim_{t\longrightarrow T} (T-t)^{-1}$; if $l > 0$, $l \sim_{t\longrightarrow T} e^{T(\theta+p)} U^{q+1-p}$, then U has a positive limit, and this is contradictory because $(T-t)^{-1}$ is not integrable. Then $\lim_{t\longrightarrow T} V = 0$, that means U is vanishing at T, and u satisfies (4.16). All the trajectories with one point in the region $\mathcal{R}_4$ have the same type of behaviour, so they lead to solutions U which are decreasing and vanish in finite time.

• The trajectory $\mathcal{T}_2$ ends to the point $\mathbf{M}_0$, in the region $\mathcal{R}_2$. Indeed we have seen that the slope of the function F_1 at $\mathbf{M}_0$ is smaller than the slope of $\mathcal{T}_2$. This region is negatively invariant, so $\mathcal{T}_2$ stays in it. As in the case of region $\mathcal{R}_3$, we obtain that V cannot be bounded, and that the trajectory is defined in a maximal interval (T, ∞), such that T is finite, and $\lim_{t=T} V = \infty = \lim_{t\longrightarrow T} G$. The corresponding functions u are decreasing and satisfy $u \sim_{r\longrightarrow\infty} u^*$ and $\lim_{r\longrightarrow R} u = \infty$, where $R = e^T$, satisfying (4.20).

• There exists two types of trajectories with one point in $\mathcal{R}_2$:

(a) the trajectories with one point above $\mathcal{T}_3$ cross the line $\mathcal{L}$ and then pass into the region $\mathcal{R}_3$, where we still have established the behaviour. They correspond to solutions defined on a finite maximal interval (R_1, R_2), such that $\lim_{r\longrightarrow R_1} u = \infty = \lim_{r\longrightarrow R_2} u = \infty$, with a minimum point inside, satisfying

$$\lim_{r\to R_1} u = \infty, \qquad \lim_{r\to R_2} u = \infty; \tag{7.7}$$

(b) the trajectories with one point under $\mathcal{T}_3$ cross the curve $\mathcal{C}_2$ and pass into the region $\mathcal{R}_4$ where we have established the behaviour. They correspond to solutions defined on a finite maximal interval (R_1, R_2), such that $\lim_{r\longrightarrow R_1} u = \infty$ and vanishing at R_2, satisfying

$$\lim_{r\to R_1} u = \infty, \qquad \lim_{r\to R_2} u = 0. \tag{7.8}$$

• All the trajectories with one point in $\mathcal{R}_1$ above $\mathcal{T}_1$ cross the curve C_2 and enter $\mathcal{R}_3$, with the same behaviour as $\mathcal{T}_3$. The corresponding solutions u satisfy (4.17). All the trajectories with one point in $\mathcal{R}_1$ under $\mathcal{T}_1$ enter $\mathcal{R}_4$ and stay under $\mathcal{T}_4$, and present the same behaviour, and the corresponding solutions u satisfy (4.18).

• The point $\mathbf{A}_2$ is a saddle point, its eigenvalues are $\rho_2 < 0 < \eta_2$, in $\mathbb{R}\times[0,\infty)$ there is precisely one trajectory $\mathcal{T}_5$ starting from $\mathbf{A}_2$ and directed by $\overrightarrow{v_2}$, with a negative slope, and two trajectories, ending at $\mathbf{A}_2$ and directed by $\overrightarrow{u_2} = (1,0)$ not admissible, since that are contained in the set $\{V = 0\}$. The slope of F_2 at point $\mathbf{A}_2$ equal to is $M_2 = p\,|G_1|^{\frac{p-2}{p-1}}\,G_1 - N + p$ is greater than the slope m_2 of $\overrightarrow{v_2}$, thus Θ_1 starts in the region $\{(G,V) \mid V > F(G), G < G_2\}$ contained in $\mathcal{R}_3$. The corresponding solutions satisfy (4.19), and by the scaling (1.7), for any $k > 0$ there exists a unique solution u satisfying (4.19).

•The region $\mathcal{R}_5$ is negatively invariant. The trajectories with one point $(g,v) \in C_2$ pass from $\mathcal{R}_5$ to $\mathcal{R}_3$ and stay under $\mathcal{T}_5$, and the corresponding solutions u are defined on an interval (R_1,R_2) satisfy

$$\lim_{r\to R_1} u = 0, \qquad \lim_{r\to R_2} u = \infty. \tag{7.9}$$

Moreover either $g < 0$, and then $G < 0$, thus u is increasing, or $g > 0$, then u has a maximum and a minimum point. Or eventually when $p > 2$, $\mu < 0$, there can exist (possibly mulptiple) trajectories tangent to the axis $G = 0$ at the point $(0,-\mu)$, such that u is nondecreasing, with an inflexion point.

(ii) Next we suppose $\mu = \mu_0$, $\gamma > S_1$ (see Figure 2). Here a great part of the analysis of Theorem 4.6 is still available. The difference is that the point $\mathbf{A}_1$ is critical: from (7.4) the eigenvalues are $\rho_1 = pS_1 - N + p = 0$ and $\eta_1 = (q+1-p)(\gamma - S_1) > 0$ and the eigenvectors are $\overrightarrow{u_1} = (1,0)$ for ρ_1 and $\overrightarrow{v_1} = (1,-\eta_1)$, with a negative slope.

• There still exists a unique trajectory $\mathcal{T}_5$, associated to η_1, such that the correponding solutions u have the behaviour (4.17).

• There exists a global trajectory joining $\mathbf{A}_1$ to $\mathbf{M}_0$, and it is still unique: indeed $\gamma \neq 0$, then $\mathbf{M}_0$ is a saddle point, so this trajectory is precisely $\mathcal{T}_1$ from Lemma 7.3. This trajectory is on the central manifold of $\mathbf{A}_1$, it converges to $\mathbf{A}_1$ as $t \to -\infty$, and it tangent to the axis $V = 0$.

For giving a more precise behaviour, we proceed as in Lemma 4.5. Setting $G = G_1 + \overline{G}$, there holds $\lim_{t\longrightarrow-\infty} \frac{V}{\overline{G}} = 0$ and

$$\begin{cases} \overline{G}_t &= F_1(G_1+\overline{G}) - V, \\ V_t &= (q+1-p)(\gamma - (G_1+\overline{G})^{\frac{1}{p-1}})V. \end{cases}$$

Now

$$F_1(G_1+\overline{G}) = F_1(G_2) + F_1'(G_2)\overline{G} + \frac{1}{2}F_1''(G_1+\tau\overline{G})\overline{G}^2 = \frac{p}{2(p-1)}(G_1+\tau\overline{G})^{\frac{2-p}{p-1}}\overline{G}^2$$

for some $\tau \in (0,1)$, thus $F_1(G_1+\overline{G}) \sim_{t\longrightarrow-\infty} c\overline{G}^2$ with $c = \frac{1}{2}\frac{p}{p-1}S_1^{2-p}$; and

$$\overline{G}_t = c\overline{G}^2(1+O(\overline{G})) - V$$

There holds $V_t \sim_{t\longrightarrow-\infty} a_1 V$ with $a_1 = (q+1-p)(\gamma - S_1) > 0$. Thus $V = O(e^{a_1 t/2})$ as $t \to -\infty$. And $\overline{G}_t \leq 2c\overline{G}^2$, hence $\overline{G} \geq \frac{C'}{|t|}$for t small enough, thus $V = o(\overline{G}^3)$. Therefore $\overline{G}_t \sim_{t\longrightarrow-\infty} c\overline{G}^2$. We deduce that $\overline{G}(t) \sim_{t\longrightarrow-\infty} \frac{1}{c|t|}$; in turn $\overline{G}_t = c\overline{G}^2(1 + O(\frac{1}{|t|})) = c\overline{G}^2 + O(\frac{1}{|t|^3})$ and finally by integration $\overline{G}(t) = \frac{1}{c|t|} + O(\frac{\ln|t|}{|t|^2})$; then

$$S = -\frac{U_t}{U} = S_1 + \frac{S_1^{2-p}}{(p-1)c}\frac{1}{|t|} + O(\frac{\ln|t|}{|t|^2}) = S_1 - \frac{2}{p}\frac{1}{t} + O(\frac{\ln|t|}{|t|^2}),$$

then thus $\ln(Ue^{S_1 t}|t|^{-\frac{2}{p}}) = O(\frac{\ln|t|}{|t|^2})$, and $\frac{\ln|t|}{|t|^2}$ is integrable, then $Ue^{S_1 t}|t|^{-\frac{2}{p}}$ admits some limit $\ell > 0$, which is precisely (4.14); by the scaling (1.7), using the fact that $\gamma \neq S_1$, for any $\ell > 0$ there exists a unique solution u satisfying (4.14). Note that there is an infinity of trajectories on the central manifold, corresponding to solutions u satisfying (4.18). ∎

Remark 7.8 *In the description of the radial solutions, we have obtained all the global and all the local solutions near* 0 *or* ∞*, corresponding to all the trajectories converging to a fixed point, from Lemma 7.2. Moreover our decription of the phase plane is more complete: we have described the other trajectories, corresponding to maximal solutions in an interval* (R_1, R_2)*, with* $0 < R_1 < R_2 <$ 0*, showing the existence of solutions satisfying (7.7), (7.8) or (7.9).*

Next we study the case $(\mathcal{H}_2)$, where $\mu \geq \mu_0$ and $\gamma < S_2$. Here it can happen that $\gamma < 0$, that means $\theta + p < 0$. It appears in particular when $\mu > 0$, since $S_1 < 0$.

Remark 7.9 *Our idea is to deduce the case* $(\mathcal{H}_2)$ *from* $(\mathcal{H}_1)$*. In case* $p = 2$ *it follows from the Kelvin transform* $u(x) = |x|^{2-N} v(\frac{x}{|x|^2})$*. Indeed it maps equation (1.3) into*

$$-\Delta v + \mu\frac{v}{|x|^2} + |x|^{\widetilde{\theta}}\, u^q = 0$$

where $\widetilde{\theta} = (N-2)q - (N+2+\theta)$*, then* $\gamma = \frac{\theta+2}{q-1}$ *is replaced by* $\widetilde{\gamma} = \frac{\widetilde{\theta}}{q-1} = N - 2 - \gamma$*. In the radial case the inversion* $x \longmapsto \frac{x}{|x|^2}$ *corresponds to a change of* t *into* $-t$*, and* $\gamma > S_2 \Longleftrightarrow \widetilde{\gamma} < S_1$ *since* $S_1 + S_2 = N - 2$*, then the equation in* u *satisfies* $(\mathcal{H}_2)$ *if and only if the equation in* v *satisfies* $(\mathcal{H}_1)$*. When* $p \neq 2$ *we cannot use this argument, but we replace the Kelvin tranform by suitable symmetry properties of the phase plane to reduce the study.*

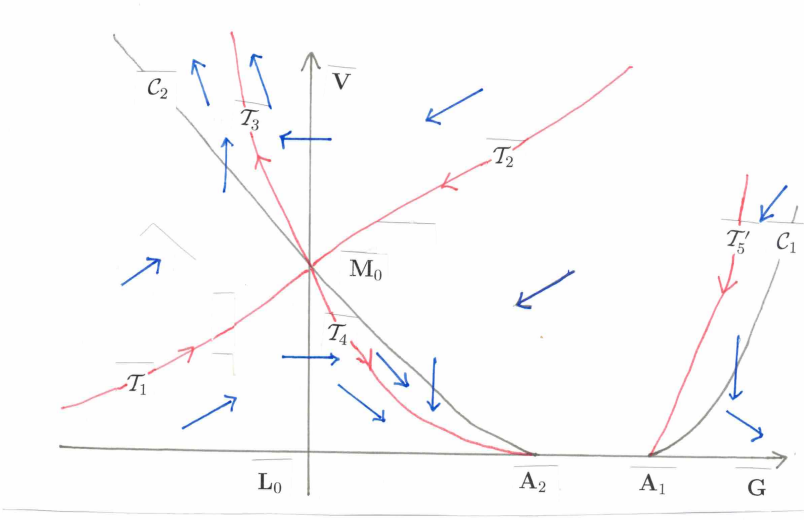

Figure 3. Case $(\mathcal{H}_2)$ (i): $\mu > \mu_0$ **and** $\gamma < S_2$

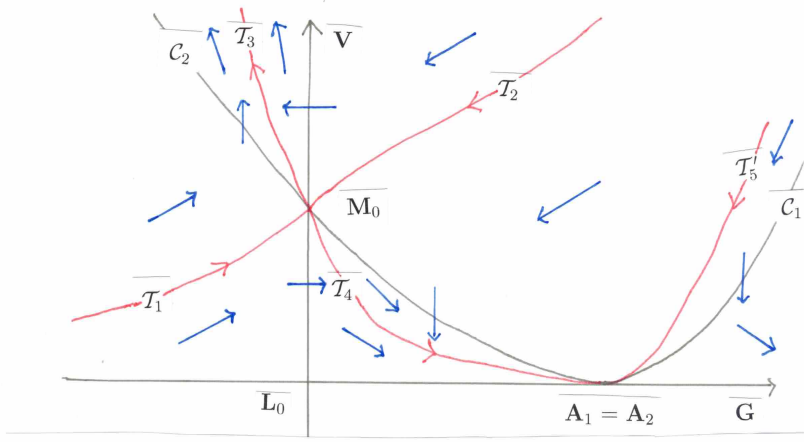

Figure 4. Case $(\mathcal{H}_2)$ (ii): $\mu = \mu_0$ **and** $\gamma < S_2 = S_1$

Proof of Theorem 4.7. Case $(\mathcal{H}_2)$: $\mu \geq \mu_0$ and $\gamma < S_2$. Here the point $\mathbf{M}_0$ exists.

• When $\gamma \neq 0$ there exist precisely four trajectories $\mathcal{T}_i$, $i = 1, .., 4$ converging to $\mathbf{M}_0$ from Lemma 7.3 . The point $\mathbf{A}_2$ is a sink (if $\rho_2 \neq \eta_2$), $\mathbf{A}_1$ is a saddle point when $\mu > \mu_0$. We note that **the phase plane has exactly the same shape** as the preceeding one, after making $t \to -t$ and a transformation $G \longmapsto \varphi(G)$ exchanging G_1 and G_2 (see Figure 3 for $\mu > \mu_0$ and figure 4 for $\mu = \mu_0$). Thus as above we obtain the behaviour of the four trajectories $\mathcal{T}_i$ and the existence of a unique trajectory $\mathcal{T}_5'$ ending at $\mathbf{A}_1$ staying in the region $\{(G, V) \mid V > F(G), G > G_1\}$. For simplicity we do not give the detail of the proofs.

• The case $\gamma = 0 < S_2$ can happen when $\mu < 0$. From Lemma 7.4. When $p \leq 2$ there still exists precisely four trajectories converging to $\mathbf{M}_0$ so we can conclude as above. When $p > 2$, there exists at least a trajectory joining $\mathbf{A}_1 = \mathbf{A}_2$ to $\mathbf{M}_0$. of (1.1) satisfying This case is delicate, because

this trajectory could not be unique, since no linearization is possible at $\mathbf{M}_0$, and the eigenvalues are $\rho_1 = 0$ and $\rho_2 = (q+1-p)(\gamma - S_2) < 0$. Such trajectory correpond to radial solutions u of equation (1.1) satisfying (4.22) with $\gamma = 0$, that is

$$\lim_{r\to 0} u = |\mu|\,, \qquad \lim_{r\to\infty} r^{\frac{N-p}{p}} \left|\ln(r)\right|^{\frac{2}{p}} u(r) = \ell > 0.$$

Here we show directly the uniqueness for given $\ell > 0$, using a main argument which will be used in the nonradial case: let u and $\widetilde{u}$ be two solutions with such a behaviour. then for any $\varepsilon > 0$, the function $(1+\varepsilon)\widetilde{u}$ is a **supersolution** of (1.1) , greater than u near 0 and near ∞. Then $(1+\varepsilon)\widetilde{u} \geq u$, from Corollary 5.3. As $\varepsilon \longrightarrow 0$ we get $\widetilde{u} \geq u$ and then $\widetilde{u} = u$. ■

Remark 7.10 *When $\gamma \neq 0$, the sign of γ is not apparent in our conclusions. However, it appears in the behaviours of the different functions u.Recall that when $\mu > 0$, then $S_2 < 0$, then $\gamma < S_2$ implies $\gamma < 0$. For example, consider the solutions satisfying (4.21): when $0 < \gamma < S_2$, implying $\mu < 0$, they are decreasing from ∞ to 0. When $\gamma < 0 < S_2$, they are increasing from 0 to a maximum point, and then decreasing to 0. When $\gamma = 0 < S_2$, they are decreasing from a^* to 0. When $\gamma < S_2 < 0$, implying $\mu > 0$, they are increasing from 0 to ∞. When $\gamma < S_2 = 0$, implying $\mu = 0$, they are increasing from 0 to a constant.*

Remark 7.11 *Under the assumptions of Theorem 4.6, As in the proof of Theorem 4.7, there also exist solutions in an interval (R_1, R_2), such that respectively $\lim_{r\to R_1} u = \infty$, $\lim_{r\to R_2} u = \infty$, or $\lim_{r\to R_1} u = 0$, $\lim_{r\to R_2} u = \infty$, or $\lim_{r\to R_1} u = \infty$, $\lim_{r\to R_2} u = 0$.*

Next we consider the case $(\mathcal{H}_3)$:

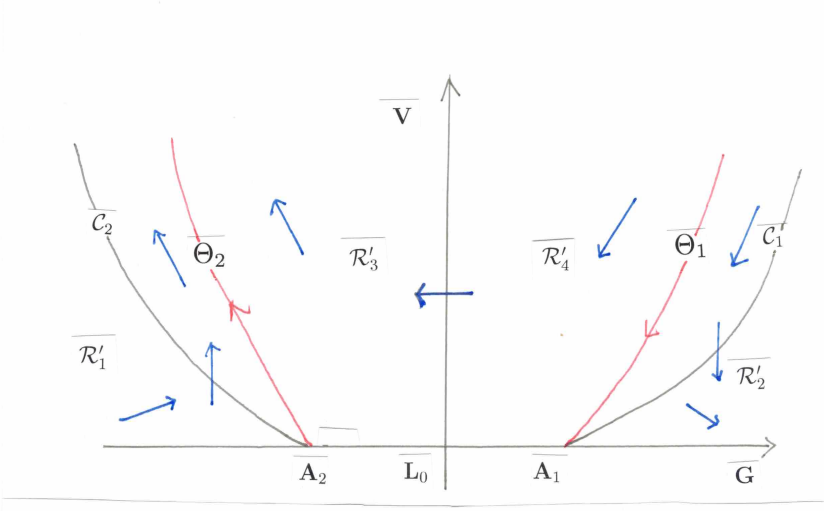

Figure 5. Case $(\mathcal{H}_3)$ (i): $\mu > \mu_0$ and $S_2 < \gamma < S_2$

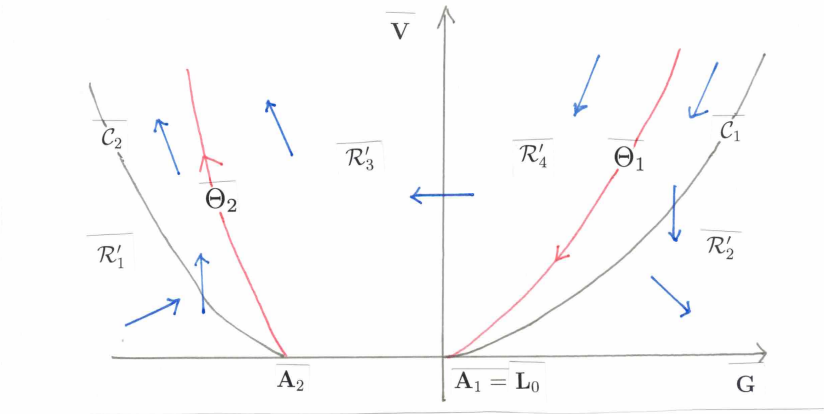

Figure 6. Case $(\mathcal{H}_3)$ (ii): $\mu > \mu_0$ and $S_2 < \gamma = S_1$

Proof of Theorem 4.8. Case $(\mathcal{H}_3)$: $\mu > \mu_0$ and $S_2 \leq \gamma \leq S_1$. Here $\mathbf{M}_0$ does not exist. We consider the regions

$$\begin{aligned} \mathcal{R}'_1 &= \{V < F(G), G < G_2\}, \qquad \mathcal{R}'_2 = \{V > F(G), G > G_1\}, \\ \mathcal{R}'_3 &= \{V > F(G), G < G_0\}, \qquad \mathcal{R}'_4 = \{V < F(G), G > G_0\}. \end{aligned}$$

(i) We first assume that $S_2 < \gamma < S_1$ (see Figure 5).

• The points $\mathbf{A}_1$ and $\mathbf{A}_2$ are saddle points. For $i = 1, 2$, there exists a unique admissible trajectory, of direction $\overrightarrow{v_i}$. the slope of $\overrightarrow{v_2}$, eigenvector for $\eta_2 > 0$ is negative and the slope of $\overrightarrow{v_1}$ eigenvector for $\eta_1 < 0$ is positive. There is precisely one trajectory Θ_2 starting from $\mathbf{A}_2$ and one trajectory Θ_1 ending in $\mathbf{A}_1$. And the slope of $\overrightarrow{v_2}$ is less than the slope of F_2 at G_2, and the slope

of $\overrightarrow{v_1}$ is positive, greater than the slope of F_1 at G_1, so the two trajectories lie in the region

$$\{V > F(G)\} = \mathcal{R}_3' \cup \mathcal{R}_4' \cup \mathcal{L},$$

where $G_t < 0$. The field on $\mathcal{C}_1$ is directed by $(0,-1)$ and the field on $\mathcal{C}_2$ is directed by $(0,1)$. In $\mathcal{R}_3'$,$\mathcal{R}_4'$, there holds respectively $V_t > 0$, and $V_t < 0$, and $\mathcal{R}_3$ is positively invariant, and $\mathcal{R}_4'$ is negatively invariant. Then Θ_2 stays in $\mathcal{R}_3'$ and Θ_1 in $\mathcal{R}_4'$. As in case $(\mathcal{H}_1)$, we get that the corresponding solutions are local, respectively on $(0,R)$ and (R',∞), and satisfying (4.29) and (4.32).

• The solutions u associated to other trajectories with one point above Θ_1 and Θ_2 satisfy

$$\lim_{r\to R_1} u = \infty, \qquad \lim_{r\to R_2} u = \infty. \tag{7.10}$$

• The trajectories with a point in $\mathcal{R}_3'$ and under Θ_2 have crossed the curve $\mathcal{C}_2$ and are issued of $\mathcal{R}_1'$, and the associated solutions u satisfy

$$\lim_{r\to R_1} u = 0, \qquad \lim_{r\to R_2} u = \infty. \tag{7.11}$$

• The trajectories with one point under $\mathcal{C}_1$ cross $\mathcal{C}_1$ and then stay in $\mathcal{R}_2'$, and the solutions u satisfy

$$\lim_{r\to R_1} u = \infty, \qquad \lim_{r\to R_2} u = 0, \tag{7.12}$$

(ii) Next we assume $\mu > \mu_0$ and $\gamma = S_1$, hence $\gamma > 0$ (see Figure 6). There still exists a unique trajectory Θ_2 starting from $\mathbf{A}_2$ as above. At the point $\mathbf{A}_1$, the eigenvalues are $\rho_1 = pS_2 - N + p > 0$, and $\eta_1 = (q+1-p)(\gamma - S_1) = 0$. There still exists a unique trajectory, corresponding to ρ_1, not admissible. There exists at least a trajectory $\mathcal{T}$ on the central manifold, directed by the eigenvector $\overrightarrow{v_1} = (1, pS_1 - N + p)$, which has a positive slope m_1; but a priori it can converge to $\mathbf{A}_1$ as $t \to \infty$ or as $t \to -\infty$. And this slope is equal to the slope of F_1, that is $M_1 = p\,|G_1|^{\frac{p-2}{p-1}}\,G_1 - N + p$. Then $\mathcal{T}$ converges to $\mathbf{A}_1$ necessarily in the region $\mathcal{R}_4'$ as $t \to \infty$. Setting $G = G_1 + \overline{G}$, system (4.8) can be written under the form

$$\begin{cases} \overline{G}_t &= (pG_1^{\frac{1}{p-1}} - N + p)\overline{G}(1 + o(1)) - V, \\ V_t &= -\frac{q+1-p}{p-1} G_1^{\frac{2-p}{p-1}} \overline{G} V (1 + o(1)), \end{cases}$$

and $\frac{V}{\overline{G}} = m_1(1 + o(1))$; then

$$V_t \sim_{t\longrightarrow\infty} -\frac{q+1-p}{p-1}\frac{G_1^{\frac{2-p}{p-1}}}{m_1} V^2 = -V^2/b_1,$$

with $b_1 = \frac{(p-1)(pS_1 - N + p)S_1^{p-2}}{q+1-p}$. By integration we get that $V = r^{\theta+p}u^{q+1-p} \sim_{t\longrightarrow\infty} \frac{b_1}{t}$, that means

$$u \sim_{r\longrightarrow\infty} r^{-S_1}(b_1/\ln r)^{\frac{1}{q+1-p}},$$

so we get (4.33) in any case. Our description of the solutions shows that there is no global solution.

(iii) Next assume $\mu > \mu_0$ and $\gamma = S_2 \neq 0$, then $\gamma > 0$ if $\mu < 0$, or $\gamma < 0$ if $\mu > 0$. Then G_1 is replaced by G_2 and t by $-t$. Since $G_2 \neq 0$, the result is similar and we get (4.30).

(iv) Assume $\gamma = S_2 = 0$, that means $\mu = 0$, and $p + \theta = 0$. As we mentioned in the proof of Lemma 7.6, the linearization (7.6) at $\mathbf{A}_2 = (0, 0)$ is still valid with $\gamma = 0$, even if $p > 2$, and the eigenvalues given by $\rho_2 = pS_2 - N + p < 0$ and $\eta_2 = 0$. The trajectory corresponding to ρ_2 is still nonadmissible. There exists a trajectory $\mathcal{T}$ on the central manifold, directed by the eigenvector $\overrightarrow{v_2} = (1, -(N-p))$, then $\frac{V}{G} \sim -(N-p)$. And $\mathcal{T}$ converges to $\mathbf{A}_2$ necessarily in the region $\mathcal{R}'_3$ as $t \to -\infty$; moreover

$$V_t = -(q+1-p)V\,|G|^{\frac{2-p}{p-1}}\,G = (q+1-p)V\,|G|^{\frac{1}{p-1}} \sim_{t\longrightarrow-\infty} (q+1-p)(N-p)^{-\frac{1}{p-1}}V^{\frac{p}{p-1}}.$$

By integration, we obtain

$$V = u^{q+1-p} \sim_{t\longrightarrow-\infty} (-\frac{(q+1-p)(N-p)^{-\frac{1}{p-1}}}{p-1}t)^{1-p},$$

thus (4.31) follows. ∎

Finally we study the doubly critical case ($\mathcal{H}_4$), and the case ($\mathcal{H}_5$) where $\mu < \mu_0$.

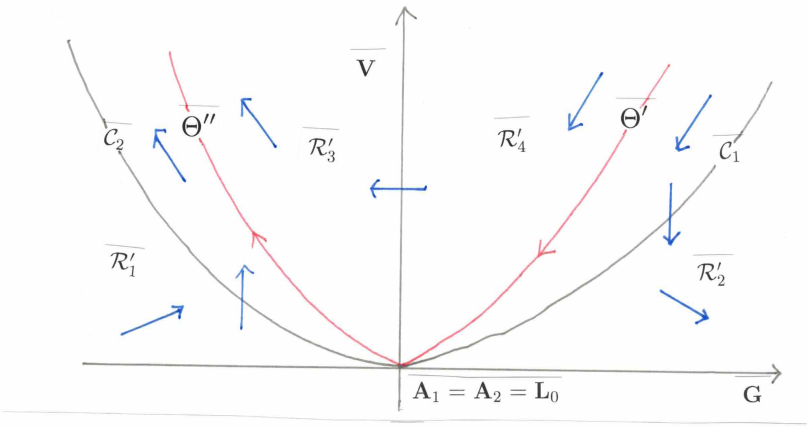

Figure 7. Case $(\mathcal{H}_4)$: $\mu = \mu_0$ **and** $\gamma = S_1 = S_2$

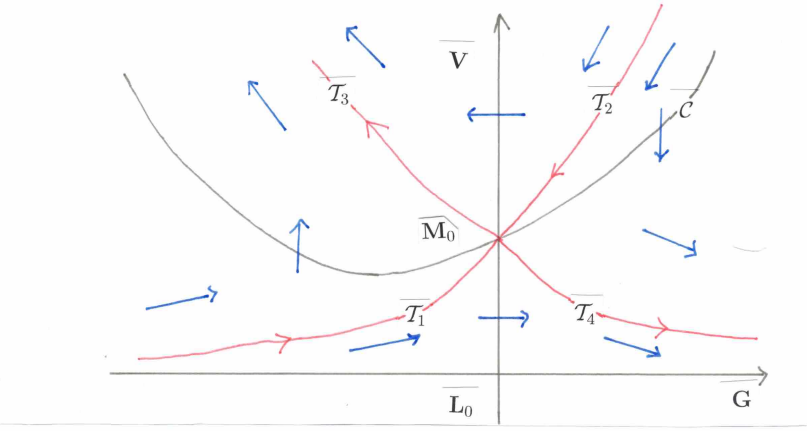

Figure 8. Case $(\mathcal{H}_5)$**:** $\mu < \mu_0$

Proof of Theorem 4.9. Case $(\mathcal{H}_4)$: $\mu = \mu_0$ and $\gamma = \frac{N-p}{p}$. Here $\mathbf{M}_0$ does not exist, and $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{L}_0 = ((\frac{N-p}{p})^{p-1}, 0)$ (see Figure 7).

- The region

$$\mathcal{R}'_4 = \{(G, V) \mid V > F(G), G > G_0\}$$

is negatively invariant, then any trajectory defined on a maximal interval (T_1, T_2) (finite or not) with one point in this region stays in it as $t \to T_1$, and necessarily T_1 is finite. Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of points P of $\mathcal{R}'_4$ such that the trajectory passing by P cuts the curve $\mathcal{C}_2$ (resp. the line $\mathcal{L}$); then $\mathcal{U}$ (resp. $\mathcal{V}$) is an open set in $\mathcal{R}'_4$, since the intersections are transverse. Then $\mathcal{U} \cup \mathcal{V} \neq \mathcal{R}'_4$. Then there exists at least one trajectory Θ' starting in $\mathcal{R}'_4$ and converging to $\mathbf{L}_0 = (G_0, 0)$

as $t \to \infty$. Similarly there exists at least one trajectory Θ'' in region

$$\mathcal{R}_3' = \{(G,V) \mid V > F(G), G < G_0\}$$

converging to $\mathbf{L}_0 = (G_0, 0)$ as $t \to -\infty$.

• Here for more precision we still use the energy function defined at (4.5) with $\gamma = \frac{N-p}{p}$, equivalently $q = \mathbf{q}_s$:

$$\mathcal{E} = V^{\frac{p}{q+1-p}}(\frac{F(G)}{p} - \frac{V}{q+1}) = V^{\frac{p}{q+1-p}}(\frac{G_t+V}{p} - \frac{V}{q+1}) = \frac{1}{p}V^{\frac{p}{q+1-p}}(G_t + \frac{(q+1-p)V}{q+1})$$

satisfies $\mathcal{E}_t = 0$, since then $D = N - p - p\gamma = 0$, so that $\mathcal{E}$ is constant. Consider any trajectory converging to $\mathbf{A}_1 = \mathbf{A}_2 = ((\frac{N-p}{p})^{p-1}, 0)$ as $t \to -\infty$ (or as $t \to \infty$), we get that $\mathcal{E}$ tends to 0. So on such a trajectory, $\mathcal{E} \equiv 0$. This gives a **first integral** satisfied by such solution: there holds

$$\begin{cases} G_t &= -\frac{(q+1-p)V}{q+1}, \\ V_t &= (q+1-p)V(\gamma - |G|^{\frac{2-p}{p-1}}G), \end{cases}$$

then $G_t < 0$, and $V_t + (q+1)(\gamma - |G|^{\frac{2-p}{p-1}}G)G_t = 0$. By integration,

$$\frac{V}{q+1} + \gamma G - \frac{p-1}{p}|G|^{\frac{p}{p-1}} = C,$$

$$G_t + \frac{(q+1-p)V}{q+1} = 0 = G_t + (q+1-p)(-\gamma G + \frac{p-1}{p}|G|^{\frac{p}{p-1}} + C;$$

and $C = (q+1-p)(\frac{N-p}{p}G_0 - \frac{p-1}{p}G_0^{\frac{p}{p-1}}) = \frac{1}{p}(q+1-p)\gamma^p$, so that finally

$$G_t + \frac{q+1-p}{p}F(G) = 0. \tag{7.13}$$

• In the case $p = 2$ we find

$$G_t + \frac{q-1}{2}(G - \frac{N-2}{2})^2 = 0,$$

then $G = \frac{N-2}{2} + \frac{2}{(q-1)(t+C)}$, thus $\frac{V_t}{V} = -\frac{2}{t+C}$, implying $V = \frac{C_1}{(t+C)^2}$, and

$$\frac{(q-1)V}{q+1} = \frac{q-1}{q+1}\frac{C_1}{(t+C)^2} = -G_t = \frac{2}{q-1}\frac{1}{(t+C)^2},$$

then $C_1 = \frac{2(q+1)}{(q-1)^2}$and $V^{\frac{1}{q-1}} = r^{\frac{N-2}{2}}u$, so u satisfies (4.36).

• In the general case $p > 1$, we can solve the equation (7.13) by quadrature, and get

$$-\frac{q+1-p}{p}(t+C) = \int_{G_0}^{G}\frac{dg}{F(g)}.$$

Setting $G = G_0 + \overline{G}$, there holds $F(G) \sim_{G \longrightarrow G_0} c\overline{G}^2$, with $c = \frac{pG_0^{\frac{2-p}{p-1}}}{2}$; then $\overline{G} \sim -\frac{p}{c(q+1-p)}\frac{1}{t}$; and

$$\frac{(q+1-p)V}{q+1} = -G_t = \frac{q+1-p}{p}F(G) \sim_{t \longrightarrow \pm\infty} \frac{1}{c}\frac{p}{q+1-p}\frac{1}{t^2},$$

so that $u(r) = r^{-\frac{N-p}{p}}V^{\frac{1}{q+1-p}}$satisfies (4.34) or (4.35). ∎

Proof of Theorem 4.10. Case ($\mathcal{H}_5$): $\mu < \mu_0$. Here the point $\mathbf{M}_0$ still exists (independently of the value of γ) and it is the unique fixed point (see Figure 8). Any trajectory of global solution of the system must join some fixed points, from Lemma 7.2, then it is unique, reduced to the point $\mathbf{M}_0$, so $u^\star$ is the unique global solution. There exists trajectories $\mathcal{T}_i$ $(i = 1, .., 4)$ as before, and the corresponding solutions u satisfy respectively (4.37),(4.38),(4.39),(4.40). And all the trajectories converging to $\mathbf{M}_0$ present one of these types. ∎

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