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Correlation analysis of degrading systems based on bivariate Wiener processes under imperfect maintenance

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This paper focuses on the correlation between the degradation levels of the two components that form a system. The degradation evolution of each component is modelled using Wiener processes. Both components are dependent and this dependence is described using the trivariate reduction method. To reduce the degradation and extend the system lifetime, preventive maintenance actions are periodically performed. These preventive maintenance actions are imperfect and they are modelled by using an Arithmetic Reduction of Degradation of infinite order model with a determined maintenance efficiency parameter. The evolution of the maintained system is analysed by assessing the expectation and variance of both degradation processes at successive maintenance times. The novelty of this work is the analysis of the Pearson correlation coefficient between the degradation levels of the two components. Different properties of the monotonicity of the Pearson correlation coefficient between the two degradation paths are obtained by considering equal maintenance efficiency and equal general time scales functions for the two Wiener degradation processes associated to each degrading component.

Keywords: Pearson correlation coefficient, trivariate reduction method, imperfect preventive maintenance, maintenance efficiency, Wiener process

1 | INTRODUCTION

In recent years, research in system deterioration modelling based on stochastic processes has predominantly focused on univariate processes. Gamma processes ([3], [21], [19], [6]), Wiener processes ([4], [26], [24]) or inverse Gaussian processes ([15], [5], [7]) have been extensively used to model the deterioration in the univariate case. However, the growing complexity of industrial systems, characterized by interrelated components, renders the exclusive reliance on univariate processes unrealistic for modelling degradation. Consequently, there is a pressing need to employ more sophisticated approaches incorporating dependencies.

Under the multi-components approach, an important aspect is how to describe the dependence between components. In this sense, several models have been proposed in the literature to analyse dependent degradation processes. According to [8] and [24], they can be divided into copula-based models, degradation rate interaction methods and multivariate distribution models. Copulas have been widely employed in the recent literature since they provide a flexible dependence structure that links the marginal distributions (see [22], [14] or [15], for citing a few). Degradation rate interaction models (DRI) assume that the deterioration of one component can affect the state of other components state [1], [16]. Multivariate distribution models extend from univariate models to multivariate versions by using a joint probability distribution to model the dependence of the degradation paths. Under this multivariate approach, a way to model the dependence between components is through the superposition of independent stochastic processes [11]. For example, in a two-components system, the trivariate reduction method constructs bivariate distributions by adding a common process (that captures the functional dependence between components) to two different degradation processes that describe the intrinsic deterioration of each component. Usually, the stochastic processes come from the same family of processes and with probability distributions closed under convolution. So, the degradation of each component is modelled as the sum of two independent processes, where one represents the common effect shared by the two components and the other represents the intrinsic degradation of the component. If the three stochastic processes come from the same family with probability distributions closed under convolution, the marginal processes of the bivariate joint process belong to the same family as well. The dependence between the processes is given in the covariance matrix of the multivariate process.

Following this approach, univariate gamma processes have been used to construct multivariate gamma processes by using the trivariate reduction method. For example, in [10], from three independent gamma processes and by using the trivariate reduction method, two performance dependent indicators of the railway track degradation are modelled. Later, Zhou *et al.* [29] used the trivariate reduction method to build the bivariate inverse Gaussian to model the dependence in the growth of different defects in pressurized pipelines. In the case of Wiener process, Zhai and Zhi-Yeng [25] used the trivariate reduction method to model the dependence between components by assuming that the drift of the process is random. In [18], covariates are incorporated in the multivariate Wiener model. The bivariate case is expanded in [20] to integrate the dependencies in the multivariate Wiener case. Although the results for the first hitting time of the bivariate process through boundaries are still scarce [17], there is an expression of the first hitting time to reach a failure threshold considering a two-components series system whose lifetime is described by a bivariate Wiener process built by using the trivariate reduction method and subject to repairs. This result is developed in [2].

Other important aspect of the multi-component dependent system is how to measure the degree of dependence

between components, and how this dependence measures evolves with time. A simple measure of this dependence is the Pearson correlation coefficient between the degradation levels of the components. Although simple, its analysis in the context of a multi-component system is useful for the following reasons:

- Components whose degradation processes show a stronger correlation are more likely to fail together. Hence a failure in a component is an alarm signal to inspect the state of the other component.
- In an imperfect inspection context, a stronger correlation between the degradation process of the two components would imply a smaller probability of a false negative [12]. Hence, the analysis of the correlation coefficient would help to reduce the probability of an incorrect inspection.
- By using predictive maintenance, the correlation coefficient is used to detect anomalies in the data since this coefficient changes abruptly when an anomaly appears [27].

For any bivariate Lévy process, the Pearson correlation coefficient is time-invariant. In [10], the Pearson correlation coefficient of the bivariate gamma process built by using the trivariate reduction method is computed and, since the bivariate gamma process is a Lévy process, it is time-invariant. However, for non stationary increments processes, the correlation coefficient is not constant with respect to time. For example, in [24], the Pearson correlation coefficient is not time invariant. In [23], a bivariate Wiener process is built by using the trivariate reduction method and incorporating random effects. The Pearson correlation coefficient is also computed and it is shown to increase with time.

Usually, the expected degradation of degrading components increases with time. To mitigate the effect of the deterioration, some maintenance actions are performed on the system. For example, in [4], preventive maintenance actions and corrective maintenance actions are performed on a system whose lifetime is described by a bivariate Wiener process. These preventive maintenance actions imply the replacement of the system by a new one ("*as-good-as-new*" state) where the degradation levels and the age of each component are reset to zero. However, it is well known that there are many cases in which a maintenance action is far from be perfect. In [28], imperfect maintenance actions are performed to balance the degradation between components. Models based on the efficiency of preventive and corrective actions were first described in the univariate case in [13] for degrading systems. They proposed the so-called ARD *(Arithmetic Reduction of Degradation)* models to reduce the system degradation by an amount proportional to the degradation state. This reduction could be proportional to the degradation state just before the maintenance action (ARD of infinite order, denoted by ARD(∞)), or proportional to the degradation accumulated by the system from the last maintenance action (ARD of order 1, or ARD(1)). Statistical inference study was carried out by [9] in a ARD(1) univariate Wiener model under different observation schemes. In [2], a bivariate Wiener process was analyzed by integrating imperfect maintenance actions modelled as an ARD(∞). Expanding the number of performance indicators of the system, [18] applied an ARD(∞) to the multivariate Wiener process. However, these works do not focus on the correlation between degradation levels under the imperfect maintenance strategy. In general, the literature on imperfect maintenance models in multi-component degrading systems is scarce. Furthermore, to our knowledge, there are not many works that study the evolution of the correlation between degradation levels. This paper seeks to bridge the gap studying the correlation coefficient in a two component system subject to imperfect maintenance.

In this work, a bivariate Wiener degradation model built by using the trivariate reduction method is studied. The system is subject to imperfect maintenance actions modelled as an ARD(∞) with a maintenance efficiency for component i equal to ρ_i. The expectation and the variance of the degradation levels of the maintained components are computed and their time evolution is studied. The Pearson correlation coefficient between the degradation levels of the two components is computed and some properties of the monotonicity of this coefficient are obtained. This

work is inspired by [2] by considering different general time scales functions and different maintenance efficiencies. The main differences between [2] and this paper are the following:

1. This work deals with a system consisting of two heterogeneous components that exhibit different degradation trends. The model discussed in this work considers the model in [2] as a special case. It allows us to assess how the different degradation trends affect the Pearson correlation coefficient. We assume that the study covers a period during which the components have not yet failed. Therefore, only the correlation of their degradation levels is of interest and the system configuration (series or parallel) does not play any role.
2. This work also considers different maintenance efficiencies for the two components (in [2], the same maintenance efficiency is considered for the two components). It also allows us to compare how the maintenance efficiency affects the evolution of the degradation of the two components and the correlation between the degradation levels.
3. While [2] mainly focuses on the computation of the first hitting time to reach a predefined threshold, this paper focuses on analysing the evolution of the Pearson correlation coefficient between the degradation levels of the two components over time and the different monotonicity properties of this coefficient.
4. Another novelty in this paper is the analysis of the jumps in the deterioration due to the maintenance actions. This analysis includes the study of the expected length of the jumps and how this expectation evolves with the number of maintenance actions. The influence of the maintenance efficiency on the jumps is also studied.

The paper is structured as follows. In Section 2, the bivariate Wiener degradation process is presented and the Pearson correlation coefficient between the two degradation paths of this bivariate process is computed. The evolution of the degradation of the system subject to maintenance actions and its theoretical development are studied in Section 3. The correlation analysis is carried out in Section 4, considering equal maintenance efficiency (Section 4.1) and equal general time scales (Section 4.2). Finally, Section 5 concludes this work.

2 | BIVARIATE DEGRADATION MODELLING

The bivariate Wiener process considered in this paper can represent a system made up of two components which are related by its correlation coefficient or, on the other hand, it can also represent two performance characteristics (PCs) of a system, whose deterioration state is provided by two correlated processes.

In the following, we will use the framework of two degrading components. For $i = 1, 2$, let $\{X_i(t), t \geq 0\}$ be the deterioration process of component i. Starting from three independent univariate Wiener processes $\{W_i(t), t \geq 0\}$, for $i = 0, 1, 2$, and following the trivariate reduction technique [11], the evolution of the degradation of both components in absence of maintenance is modelled as:

$$X_i(t) = W_i(t) + W_0(t), \qquad i = 1, 2.$$

$$W_i(t) = \mu_i \Lambda_i(t) + \sigma_i B_i(\Lambda_i(t)) \qquad W_0(t) = \sigma_0 B_0(\Lambda_0(t)), \qquad t \geq 0,$$

where B_0, B_1 and B_2 denote independent standard Brownian motions with $\mu_i > 0$, $\sigma_i > 0$ for all i. Hence, the evolution

of the deterioration of the two components is given by

$$X_1(t) = \mu_1\Lambda_1(t) + \sigma_1 B_1(\Lambda_1(t)) + \sigma_0 B_0(\Lambda_0(t)), \quad (1)$$

$$X_2(t) = \mu_2\Lambda_2(t) + \sigma_2 B_2(\Lambda_2(t)) + \sigma_0 B_0(\Lambda_0(t)). \quad (2)$$

The parameter μ_i is known as drift parameter which indicates the degradation rate of $W_i(t)$. The parameter σ_i denotes the diffusion coefficient of $W_i(t)$. Functions $\Lambda_i(t) > 0$ for $i = 1, 2$ and $\Lambda_0(t) > 0$, for $t > 0$, are called general time scales representing the non-linearity of the degradation paths. These functions are assumed to be differentiable and non-decreasing in t.

From (1) and (2), the degradation at time t of component i follows a normal distribution with expectation and variance equal to

$$\mathbb{E}[X_i(t)] = \mu_i\Lambda_i(t), \qquad \mathrm{Var}(X_i(t)) = \sigma_0^2\Lambda_0(t) + \sigma_i^2\Lambda_i(t), \qquad t \geq 0.$$

Since $\Lambda_i(t)$ and $\Lambda_0(t)$ are increasing functions in t, the expectation and the variance of the degradation of component i increase with time.

The common process $\sigma_0 B_0(\cdot)$ introduces dependence among the degradation processes. The covariance between $X_1(t)$ and $X_2(t)$ is

$$\mathrm{Cov}(X_1(t), X_2(t)) = \sigma_0^2\Lambda_0(t),$$

since B_0, B_1 and B_2 are independent Brownian processes. Consequently, at each given t, the Pearson correlation coefficient between $X_1(t)$ and $X_2(t)$ is,

$$\begin{aligned}\theta(t) &= \frac{\mathrm{Cov}(X_1(t), X_2(t))}{\sqrt{\mathrm{Var}(X_1(t))}\sqrt{\mathrm{Var}(X_2(t))}} \\ &= \frac{\sigma_0^2\Lambda_0(t)}{\sqrt{\sigma_0^2\Lambda_0(t) + \sigma_1^2\Lambda_i(t)}\sqrt{\sigma_0^2\Lambda_0(t) + \sigma_2^2\Lambda_i(t)}} \\ &= \frac{1}{\sqrt{1 + \sigma_1^2/\sigma_0^2 h_1(t)}\sqrt{1 + \sigma_2^2/\sigma_0^2 h_2(t)}}, \quad t \geq 0, \end{aligned} \quad (3)$$

where $h_i(t) = \Lambda_i(t)/\Lambda_0(t)$, for $i = 1, 2$, denotes the ratio of the two general time scales. Hence, if $\sigma_0 \neq 0$, there exists a correlation between the two degradation processes. Intuitively, since the process $\sigma_0 B_0(\cdot)$ applies to the two components and $\sigma_0 > 0$, the correlation between $X_1(t)$ and $X_2(t)$ is always positive. Different general time scales lead to dynamic changes in the correlation: $\theta(t)$ changes over time if $\Lambda_i(t) \neq \Lambda_0(t)$. Some results about the monotonicity of the Pearson correlation function given by (3) are shown below:

- When both $h_i(t)$ increase in t, $\theta(t)$ decreases in t. This can be interpreted as follows: if $h_i(t)$ increases, then $\Lambda_i(t)$ is more dominant with respect to $\Lambda_0(t)$ over time. Hence, the Pearson correlation decreases.
- When σ_0^2 increases, $\theta(t)$ increases. This is evident since σ_0 is related to the common part of the two degrading components.
- When both σ_i^2 for $i = 1, 2$ increase, $\theta(t)$ decreases. This is clear since σ_i is related to the independent part of the two degrading components.

- If $\Lambda_i(t) = \Lambda_0(t)$, for $i = 1, 2$, the Pearson correlation coefficient is

$$\theta(t) = \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}\sqrt{\sigma_0^2 + \sigma_2^2}}. \tag{4}$$

This correlation coefficient is time invariant since $h_i(t) = \Lambda_i(t)/\Lambda_0(t)$ is equal to one. It cannot reach 1 unless $\sigma_1^2 = \sigma_2^2 = 0$. When the common noise is dominant ($\sigma_0^2 >> \sigma_i^2$), the correlation between processes $X_1(t)$ and $X_2(t)$ is stronger. When the common noise is negligible, the two Wiener processes evolve almost independently.

Example 1. Figure 1 shows a realization of $X_1(t)$ and $X_2(t)$ given by (1) and (2) with parameters $\mu_1 = \mu_2 = 1$, $\sigma_0 = \sigma_1 = \sigma_2 = 2$, and $\Lambda_1(t) = \Lambda_2(t) = t^{2.1}$ and $\Lambda_0(t) = t^{1.1}$, for $t \geq 0$. With these parameters, the Pearson correlation coefficient $\theta(t)$ is given by

$$\theta(t) = \frac{1}{1+t}.$$

As $h_1(t) = h_2(t) = t$ increases with respect to t, then the correlation between the processes decreases with time.

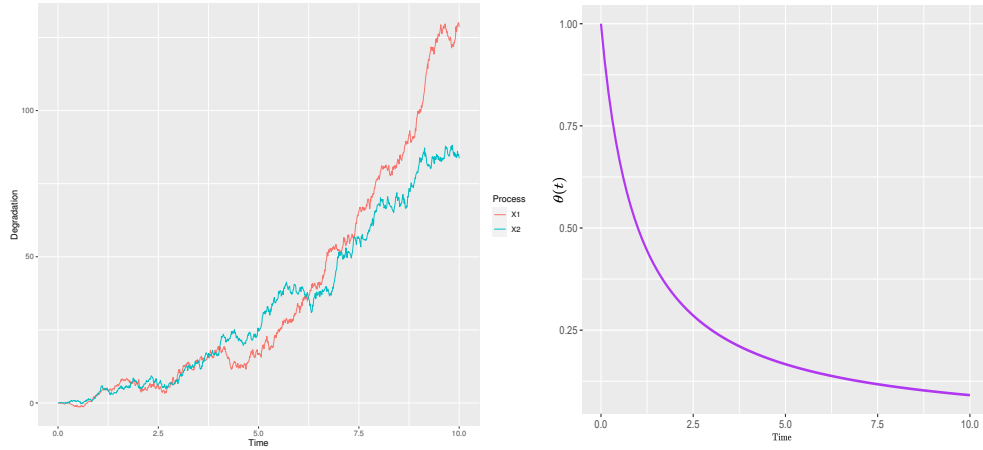

FIGURE 1 Realization of processes $X_1(t)$ and $X_2(t)$, and their corresponding Pearson correlation coefficient $\theta(t)$ in Example 1.

A preventive maintenance policy is imposed since the expectation of the degradation of the two components increases over time. This policy is described below and its effects on the Pearson correlation coefficient are investigated.

3 | EVOLUTION OF THE MAINTAINED SYSTEM

The system is preventively maintained under an imperfect maintenance strategy following the *Arithmetic Reduction of Degradation* of order infinite, denoted by ARD(∞), proposed in [13] with periodic maintenance actions each T time

units. The effect of maintenance is to reduce the degradation level of a component in a quantity which is proportional to its value. It is assumed that the time to perform the preventive maintenance action is negligible.

In this section, we focus on the modelling of the degradation evolution of the two maintained components and on the properties of the jumps in the degradation of each component due to the maintainence actions

3.1 | Degradation of the maintained system

Let $\mathbf{Y}(t) = (Y_1(t), Y_2(t))$ be the degradation levels of the two maintained components at time t where $\{Y_i(t), t \geq 0\}$ is the process that describes the deterioration level of the maintained component i, for $i = 1, 2$. Since $X_i(t)$ has independent increments, the piece-wise evolution of $\{Y_i(t), t \geq 0\}$ is as follows.

- Starting at time t, with $0 \leq t < T$, the deterioration level of component i is given by

$$Y_i(t) = X_i(t), \qquad \text{for} \quad i = 1, 2.$$

The first preventive maintenance action is performed at time T. Just after the first maintenance, the degradation of component i is reduced by $100\rho_i\%$, with $i = 1, 2$. Denoting by T^+ the instant of time just after this first maintenance action

$$Y_i(T^+) \quad = \quad (1 - \rho_i) X_i(T), \qquad \text{for} \quad i = 1, 2. \tag{5}$$

- At time t, with $T \leq t < 2T$, the evolution of the deterioration of the maintained components is given by

$$Y_i(t) = Y_i(T^+) + X_i(t) - X_i(T) = X_i(t) - \rho_i X_i(T);$$

hence $Y_i(t)$ is also a Wiener process with expectation

$$\mathbb{E}[Y_i(t)] \quad = \quad \mu_i \Lambda_i(T)(1 - \rho_i) + \mu_i \left(\Lambda_i(t) - \Lambda_i(T)\right)$$

and variance

$$\begin{aligned} \text{Var}(Y_i(t)) \quad &= \quad \sigma_0^2 (1 - \rho_i)^2 \Lambda_0(T) + \sigma_0^2 \left(\Lambda_0(t) - \Lambda_0(T)\right) \\ &+ \quad \sigma_i^2 (1 - \rho_i)^2 \Lambda_i(T) + \sigma_i^2 \left(\Lambda_i(t) - \Lambda_i(T)\right). \end{aligned}$$

Just before the second imperfect maintenance action, at time $2T^-$, the deterioration of the maintained component i is given by

$$Y_i(2T^-) = Y_i(T^+) + X_i(2T) - X_i(T).$$

At time $2T$, the second imperfect preventive maintenance is performed. It means that the overall degradation of

component i is reduced by $100\rho_i\%$. Hence

$$\begin{aligned} Y_i(2T^+) &= (1-\rho_i)Y_i(2T^-) \\ &= (1-\rho_i)^2 X_i(T) + (1-\rho_i)(X_i(2T) - X_i(T)). \end{aligned}$$

At time $2T^+$, the degradation of component i follows a normal distribution with parameters

$$\mathbb{E}[Y_i(2T^+)] = \mu_i(1-\rho_i)^2\Lambda_i(T) + \mu_i(1-\rho_i)(\Lambda_i(2T) - \Lambda_i(T))$$

and

$$\begin{aligned} \mathrm{Var}(Y_i(2T^+)) &= \sigma_0^2\left((1-\rho_i)^4\Lambda_0(T) + (1-\rho_i)^2(\Lambda_0(2T) - \Lambda_0(T))\right) \\ &+ \sigma_i^2\left((1-\rho_i)^4\Lambda_i(T) + (1-\rho_i)^2(\Lambda_i(2T) - \Lambda_i(T))\right). \end{aligned}$$

- In a general setting, for t fulfilling $nT \le t < (n+1)T$, we get that

$$\begin{aligned} Y_i(t) &= Y_i(nT^+) + (X_i(t) - X_i(nT)) \\ &= \sum_{j=1}^{n}(1-\rho_i)^{n-j+1}\left(X_i(jT) - X_i((j-1)T)\right) + (X_i(t) - X_i(nT)). \end{aligned} \tag{6}$$

The random variable $Y_i(t)$ follows a normal distribution with expectation

$$\mathbb{E}(Y_i(t)) = \mu_i \sum_{j=1}^{n}\Delta\Lambda_i(jT)(1-\rho_i)^{n-j+1} + \mu_i\left(\Lambda_i(t) - \Lambda_i(nT)\right)$$

and variance

$$\mathrm{Var}(Y_i(t)) = \sigma_0^2 g(t,\rho_i,\rho_i,\Lambda_0) + \sigma_i^2 g(t,\rho_i,\rho_i,\Lambda_i), \tag{7}$$

where

$$\Delta\Lambda_i(jT) = \Lambda_i(jT) - \Lambda_i((j-1)T) \tag{8}$$

denotes the increments of the function $\Lambda_i(\cdot)$, and

$$g(t,\rho_1,\rho_2,\Lambda_i) = \sum_{j=1}^{n}\Delta\Lambda_i(jT)(1-\rho_1)^{n-j+1}(1-\rho_2)^{n-j+1} + \Lambda_i(t) - \Lambda_i(nT),$$

with $nT \le t < (n+1)T$, where $n = [t/T]$ stands for the floor function of t/T.

- Similarly, after the n-th imperfect preventive maintenance action, the degradation of component i, given by (6),

is distributed as a normal with expectation

$$\mathbb{E}[Y_i(nT^+)] \quad = \quad \mu_i \sum_{j=1}^{n} \Delta\Lambda_i(jT)(1-\rho_i)^{n-j+1}$$

and variance

$$\text{Var}(Y_i(nT^+)) = \sigma_0^2 g(nT, \rho_i, \rho_i, \Lambda_0) + \sigma_i^2 g(nT, \rho_i, \rho_i, \Lambda_i).$$

Example 2. Figure 2 shows realizations of $Y_1(t)$ and $Y_2(t)$ with the parameters presented in Example 1. Imperfect preventive maintenance actions are performed each $T = 3$ time units with maintenance efficiency equals to $\rho_1 = 0.9$ and $\rho_2 = 0.1$ respectively. These values have been chosen such that the maintenance effect is nearly maximal for component 1 and minimal for component 2.

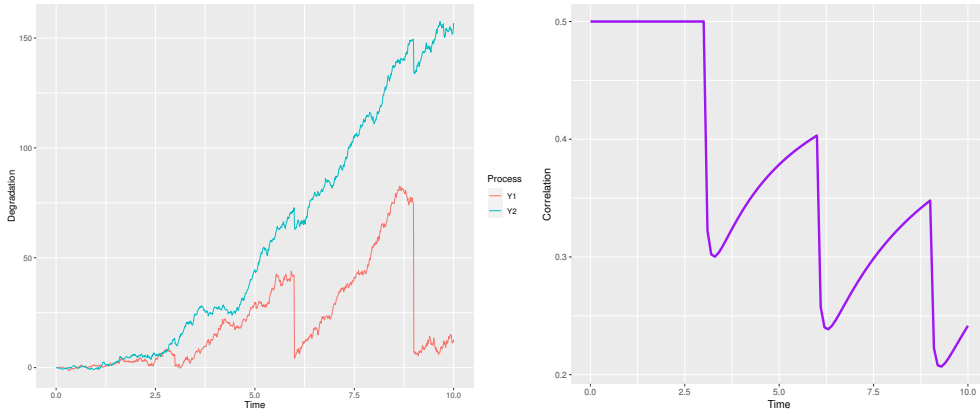

FIGURE 2 Realization of processes $Y_1(t)$ and $Y_2(t)$, and their corresponding Pearson correlation coefficient in Example 2.

Proposition 1 gives results about the monotonicity with respect to n for fixed T for the expectation and the variance of $Y_i(nT^+)$, with $i = 1, 2$. To prove this result, it is assumed that $\Lambda_i(t)$ and $\Lambda_0(t)$ have increasing increments. The definition of a function with increasing increments is reviewed below.

Definition 1. A function $\Lambda(\cdot)$ has increasing increments if

$$\Lambda(t_1 + s) - \Lambda(t_1) \leq \Lambda(t_2 + s) - \Lambda(t_2),$$

whenever $s > 0$ and $t_1 \leq t_2$.

Remark 1. If $\Lambda(\cdot)$ is convex, then it has increasing increments. Commonly used forms for $\Lambda(t)$ include the power law function $\Lambda(t) = \alpha t^{\beta}$, with $\beta \geq 1$, and the log-linear form $\Lambda(t) = \exp(\alpha t) - 1$ with $\alpha > 1$. Both functions have increasing increments.

Proposition 1. *For fixed T and $\Lambda_i(\cdot)$ having increasing increments, it holds that*

$$\mathbb{E}[Y_i(nT^+)] \leq \mathbb{E}[Y_i((n+1)T^+)],$$

for $n \geq 1$. If, furthermore, $\Lambda_0(\cdot)$ has increasing increments,

$$Var(Y_i(nT^+)) \leq Var(Y_i((n+1)T^+)).$$

Proof. Evaluating the difference,

$$\begin{aligned}\mathbb{E}[Y_i((n+1)T^+)] - \mathbb{E}[Y_i((n)T^+)] &= \sum_{j=1}^{n} \mu_i(1-\rho_i)^{n-j+1}[\Delta\Lambda_i((j+1)T) - \Delta\Lambda_i(jT)] \\ &+ \mu_i \Delta\Lambda_i(T)(1-\rho_i)^{n+1},\end{aligned}$$

where $\Delta\Lambda_i(\cdot)$ denotes the increments of $\Lambda_i(\cdot)$ given by (8). If $\Lambda_i(\cdot)$ has increasing increments, then

$$\Delta\Lambda_i((j+1)T) - \Delta\Lambda_i(jT) \geq 0$$

and, in consequence,

$$\mathbb{E}[Y_i((n+1)T^+)] \geq \mathbb{E}[Y_i((nT^+)].$$

In the case of the variances, we get that

$$\begin{aligned}&\mathrm{Var}(Y_i((n+1)T^+)) - \mathrm{Var}(Y_i(nT^+)) = \\ &\sigma_0^2\left(\Delta\Lambda_0(T)(1-\rho_i)^{2(n+1)} + \sum_{j=1}^{n}(\Delta\Lambda_0((j+1)T) - \Delta\Lambda_0(jT))(1-\rho_i)^{2(n-j+1)}\right) \\ &+\sigma_i^2\left(\Delta\Lambda_i(T)(1-\rho_i)^{2(n+1)} + \sum_{j=1}^{n}(\Delta\Lambda_i((j+1)T) - \Delta\Lambda_i(jT))(1-\rho_i)^{2(n-j+1)}\right),\end{aligned}$$

where $\Delta\Lambda_0(\cdot)$ denotes the increments of $\Lambda_0(\cdot)$ given by (8). Therefore, if $\Lambda_i(\cdot)$ and $\Lambda_0(\cdot)$ have increasing increments, then

$$\mathrm{Var}(Y_i((n+1)T^+)) \geq \mathrm{Var}(Y_i(nT^+))$$

and the result holds. ∎

Proposition 1 means that if the general time scale functions $\Lambda_i(\cdot)$ and $\Lambda_0(\cdot)$ have increasing increments, in spite of performing imperfect preventive maintenance actions each T time units, the expectation and variance of processes $Y_1(t)$ and $Y_2(t)$ after a preventive maintenance action increase with respect to the expectation and the variance after the previous maintenance action, whatever the value of ρ_i is, for $i = 1, 2$.

Example 3. Figure 3 shows the expected degradation of component i after the n-th imperfect preventive maintenance action with respect to n. The parameters used are $\mu_i = 1$, $\rho_i = 0.75$ and $\Lambda_i(t) = t^\beta$, with $\beta > 1$. Since $\Lambda_i(t)$ is convex,

it has increasing increments. Hence, the expected degradation just after the maintenance increases with the number of maintenance actions performed on the system.

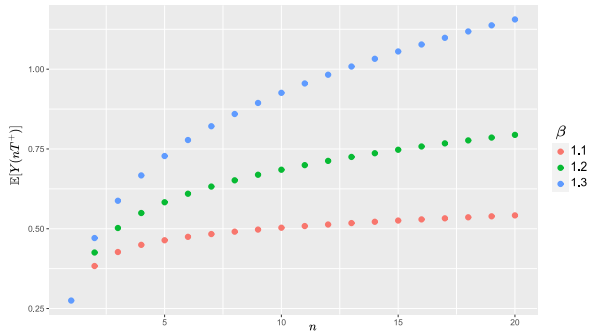

FIGURE 3 Expected degradation after the n-th imperfect maintenance action in Example 3.

Example 4. Figure 4 shows the variance of the degradation of component i after the n-th preventive maintenance action. The following parameters $\mu_i = 1$, $\rho_i = 0.75$, $\sigma_0 = \sigma_i = 2$, $\Lambda_i(t) = t^{\beta}$ and $\Lambda_0(t) = t^{1.1}$ are used. Since $\Lambda_i(t)$ and $\Lambda_0(t)$ are convex, they have increasing increments. Hence, the variance of the degradation level after the imperfect preventive maintenance actions increases with the number of maintenance actions.

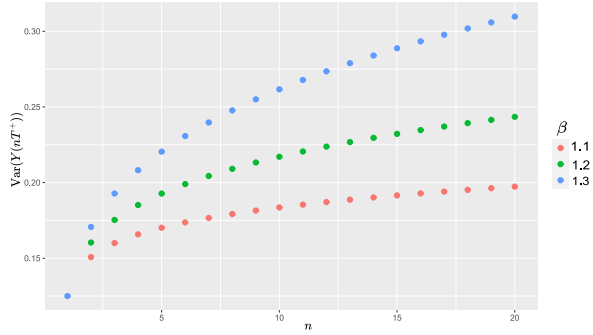

FIGURE 4 Variance after the n-th imperfect maintenance action in Example 4.

The aim of proposition 2 is to give a result analogous to proposition 1 when the degradation levels are compared at time t (with $nT < t < (n+1)T$), instead of just after maintenance.

Proposition 2. *For fixed $T > 0$, let t_1, t_2 and $t_1 < t_2$, where t_1 and t_2 fulfils*

$$t_1 - \lfloor t_1/T \rfloor T \leq t_2 - \lfloor t_2/T \rfloor T.$$

If $\Lambda_i(\cdot)$ has increasing increments then

$$\mathbb{E}[Y_i(t_1)] \leq \mathbb{E}[Y_i(t_2)].$$

If, furthermore, $\Lambda_0(\cdot)$ has increasing increments then

$$Var(Y_i(t_1)) \leq Var(Y_i(t_2)).$$

Proof. If $\lfloor t_1/T \rfloor = \lfloor t_2/T \rfloor = n$, then

$$\begin{aligned}
\mathbb{E}[Y_i(t_1)] &= \mu_i \sum_{j=1}^{n} (1-\rho_i)^{n-j+1} \Delta\Lambda_i(jT) + \mu_i(\Lambda_i(t_1) - \Lambda_i(nT)) \\
\mathbb{E}[Y_i(t_2)] &= \mu_i \sum_{j=1}^{n} (1-\rho_i)^{n-j+1} \Delta\Lambda_i(jT) + \mu_i(\Lambda_i(t_2) - \Lambda_i(nT))
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(Y_i(t_1)) &= \sigma_0^2 \sum_{j=1}^{n} \Delta\Lambda_0(jT)(1-\rho_i)^{2(n-j+1)} \\
&+ \sigma_i^2 \sum_{j=1}^{n} \Delta\Lambda_i(jT)(1-\rho_i)^{2(n-j+1)} \\
&+ \sigma_0^2(\Lambda_0(t_1) - \Lambda_0(nT)) + \sigma_i^2(\Lambda_i(t_1) - \Lambda_i(nT)),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(Y_i(t_2)) &= \sigma_0^2 \sum_{j=1}^{n} \Delta\Lambda_0(jT)(1-\rho_i)^{2(n-j+1)} \\
&+ \sigma_i^2 \sum_{j=1}^{n} \Delta\Lambda_i(jT)(1-\rho_i)^{2(n-j+1)} \\
&+ \sigma_0^2(\Lambda_0(t_2) - \Lambda_0(nT)) + \sigma_i^2(\Lambda_i(t_2) - \Lambda_i(nT)).
\end{aligned}$$

By applying that $\Lambda_i(t)$ is increasing in t, then

$$\mathbb{E}[Y_i(t_1)] \leq \mathbb{E}[Y_i(t_2)].$$

By applying that $\Lambda_i(t)$ and $\Lambda_0(t)$ are increasing in t, then $\mathrm{Var}(Y_i(t_1)) \leq \mathrm{Var}(Y_i(t_2))$.

If $\lfloor t_1/T \rfloor = n_1$ and $\lfloor t_2/T \rfloor = n_2$, being $n_1 < n_2$, then

$$\mu_i \sum_{j=1}^{n_1} (1-\rho_i)^{n_1-j+1} \Delta\Lambda_i(jT) \leq \mu_i \sum_{j=1}^{n_2} (1-\rho_i)^{n_2-j+1} \Delta\Lambda_i(jT).$$

On the other hand,

$$\Lambda_i(t_1) - \Lambda_i(n_1T) = \Lambda_i(t_1 - n_1T + n_1T) - \Lambda_i(n_1T)$$

and with the assumption of $\Lambda(\cdot)$ having increasing increments, that is,

$$\Lambda_i(t_1+T)-\Lambda_i(t_1)\leq\Lambda_i(t_2+T)-\Lambda_i(t_2),$$

it is fulfilled that

$$\begin{aligned}\Lambda_i(t_1)-\Lambda_i(n_1T) &= \Lambda_i(t_1-n_1T+n_1T)-\Lambda_i(n_1T)\\ &\leq \Lambda_i(t_1-n_1T+n_2T)-\Lambda_i(n_2T).\end{aligned}$$

Since $\Lambda_i(\cdot)$ is increasing and $t_1-n_1T\leq t_2-n_2T$, then

$$\Lambda_i(t_1-n_1T+n_2T)-\Lambda_i(n_2T)\leq\Lambda_i(t_2-n_2T+n_2T)-\Lambda_i(n_1T)$$

and the result holds.

The reasoning for the monotonicity of the variance is analogous since

$$\Lambda_0(t_1)-\Lambda_0(n_1T)+\Lambda_i(t_1)-\Lambda_i(n_1T)\leq\Lambda_0(t_2)-\Lambda_0(n_2T)+\Lambda_i(t_2)-\Lambda_i(n_2T).$$

In the next section, we focus on the jumps in the degradation due to the maintenance actions. Some results are given analysing the effect of the number of maintenance actions on the degradation reduction and the effect of the maintenance efficiency on the jump in the degradation

3.2 | Jumps in the degradation due to the maintenance actions

Let $Z_i(nT)$ be the jump in the degradation level of component i due to the maintenance action performed at time nT, i.e.,

$$Z_i(nT)=Y_i(nT^+)-Y_i(nT^-). \tag{9}$$

By using $Y_i(nT^+)=(1-\rho_i)Y_i(nT^-)$, we get that

$$\begin{aligned}Z_i(nT) &= (1-\rho_i)Y_i(nT^-)-Y_i(nT^-)\\ &= -\rho_iY_i(nT^-)\\ &= -\rho_i\left(\sum_{j=1}^{n-1}(1-\rho_i)^{n-j}\Delta X_i(jT)+\Delta X_i(nT))\right).\end{aligned} \tag{10}$$

Hence, the expected length of this jump is equal to

$$\begin{aligned}\mathbb{E}[Z_i(nT)] &= -\mu_i\rho_i\sum_{j=1}^{n-1}(1-\rho_i)^{n-j}\Delta\Lambda_i(jT)-\mu_i\rho_i\Delta\Lambda_i(nT)\\ &= -\mu_i\rho_i\sum_{j=1}^{n}(1-\rho_i)^{n-j}\Delta\Lambda_i(jT).\end{aligned}$$

Similar to the previous reasoning related to the monotonicity of the expected degradation levels, we get the following result.

Lemma 1. *If $\Lambda_i(\cdot)$ has increasing increments, then the expected length of the jump decreases with the number of maintenance actions performed on the system, i.e.,*

$$\mathbb{E}[Z_i(nT)] \leq \mathbb{E}[Z_i((n-1)T)],$$

for $n = 1, 2, \ldots$.

Lemma 1 states that, on average, the effect of the maintenance on the degradation of each component decreases with the number of maintenance actions performed on the system if $\Lambda_i(\cdot)$ has increasing increments. Furthermore, on average, the absolute values of the sizes of the jump increase with the maintenance efficiency ρ_i if $\Lambda_i(\cdot)$ has increasing increments. Proposition 3 analyzes the effect of the maintenance actions on the expected length of the jump.

Proposition 3. *If $\Lambda_i(\cdot)$ has increasing increments, then the expected length of the jump decreases with the maintenance efficiency.*

Proof. For $n = 1$, we get that

$$\mathbb{E}[Z_i(t)] = -\mu_i \rho_i \Delta\Lambda_i(T),$$

which decreases with respect to ρ_i. For $n = 2$, we get that

$$\mathbb{E}[Z_i(2T)] = -\mu_i \rho_i \Delta\Lambda_i(2T) - \mu_i \rho_i (1-\rho_i)\Delta\Lambda_i(T)$$

Derivating $\mathbb{E}[Z_i(2T)]$ with respect to ρ_i, we get that

$$\frac{\partial \mathbb{E}[Z_i(2T)]}{\partial \rho_i} = -\mu_i(\Delta\Lambda_i(2T) + \Delta\Lambda_i(T) - 2\rho_i\Delta\Lambda_i(T)) = -\mu_i(\Lambda_i(2T) - \rho_i\Lambda_i(T) + (1-\rho_i)\Lambda_i(T))$$

By applying that $\Lambda_i(\cdot)$ has increasing increments, then $\mathbb{E}[Z_i(2T)]$ is decreasing in ρ_i. Finally, when $n > 2$, then the derivative with respect to ρ_i is given by

$$\frac{\mathbb{E}[Z_i(nT)]}{\partial \rho_i} = -\mu_i \left(\Delta\Lambda_i(nT) + \sum_{j=0}^{n-2} (1-\rho_i)^j (1-(j+2)\rho_i)\Delta\Lambda_i((n-j-1)T) \right).$$

The term in (11),

$$\sum_{j=0}^{n-2} (1-\rho_i)^j (1-(j+2)\rho_i)\Delta\Lambda_i((n-j-1)T),$$

is decreasing in ρ_i, then

$$\sum_{j=0}^{n-2} (1-\rho_i)^j (1-(j+2)\rho_i)\Delta\Lambda_i((n-j)T) \leq -\Delta\Lambda_i((n-1)T).$$

Then

$$\frac{\mathbb{E}[Z_i(nT)]}{\partial \rho_i} \leq -\mu_i(\Delta\Lambda(nT) - \Delta\Lambda((n-1)T),$$

hence the jumps are decreasing with respect to ρ_i.

The following section provides an analysis of the Pearson correlation coefficient between the degradation levels of the maintained components over time.

4 | CORRELATION ANALYSIS IN THE MAINTAINED SYSTEM

With the foregoing, the covariance between processes $Y_1(t)$ and $Y_2(t)$ at time t, with $nT \leq t < (n+1)T$, is given by

$$\begin{aligned} \mathrm{Cov}(Y_1(t), Y_2(t)) &= \sigma_0^2 g_n(t, \rho_1, \rho_2, \Lambda_0) \\ &= \sigma_0^2 \sum_{j=1}^{n} (1-\rho_1)^{n-j+1}(1-\rho_2)^{n-j+1}\Delta\Lambda_0(jT) + \sigma_0^2\left(\Lambda_0(t) - \Lambda_0(nT)\right). \end{aligned} \tag{11}$$

The Pearson correlation coefficient between $Y_1(t)$ and $Y_2(t)$ is next evaluated. We get that

$$\theta(t, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2) = \frac{\mathrm{Cov}(Y_1(t), Y_2(t))}{\sqrt{\mathrm{Var}(Y_1(t))}\sqrt{\mathrm{Var}(Y_2(t))}},$$

where $\mathrm{Var}(Y_1(t))$ and $\mathrm{Var}(Y_2(t))$ are given by (7) and the covariance between the two degradation processes is given by (11). Hence, the Pearson correlation coefficient is equal to

$$\theta(t, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2) = \frac{\sigma_0^2 g(t, \rho_1, \rho_2, \Lambda_0)}{\sqrt{\sigma_0^2 g(t, \rho_1, \rho_1, \Lambda_0) + \sigma_1^2 g(t, \rho_1, \rho_1, \Lambda_1)}\sqrt{\sigma_0^2 g(t, \rho_2, \rho_2, \Lambda_0) + \sigma_2^2 g(t, \rho_2, \rho_2, \Lambda_2)}}, \tag{12}$$

where $g(t, \rho_1, \rho_2, \Lambda_i)$ is given by (9). The degradation levels of each component are positively correlated. It is intuitive since we have assumed that both degradation processes share a common part. Moreover, it is straightforward to check that

$$\theta(nT^-, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2) = \theta(nT^+, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2),$$

hence the correlation coefficient is continuous.

Figure 5 represents the Pearson correlation coefficient $\theta(t, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2)$ versus time t considering the following parameters: $\sigma_0 = 4, \sigma_1 = 4.5, \sigma_2 = 3.5, \rho_1 = 0.3, \rho_2 = 0.5, \Lambda_0(t) = \Lambda_1(t) = \Lambda_2(t) = t$ and $T = 15$. Notice that this coefficient is such that $\theta(iT, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2) \leq \theta((i-1)T, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2)$, for $i \geq 1$, but $\theta(t, \rho_1, \rho_2, \Lambda_0, \Lambda_1, \Lambda_2)$ is not monotone in t, for $nT \leq t < (n+1)T$.

Also, the jumps in the degradation levels of the two components due to the maintenance actions are correlated.

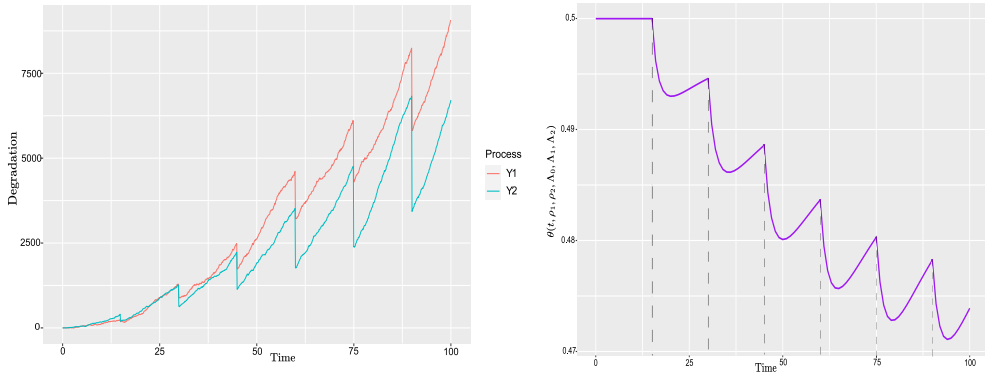

FIGURE 5 Processes $Y_1(t)$ and $Y_2(t)$, and their Pearson correlation coefficient versus time t.

Next, we analyze

$$\theta(Z_1(nT), Z_2(nT)) = \frac{Cov(Z_1(nT), Z_2(nT))}{\sqrt{Var(Z_1(nT))}\sqrt{Var(Z_2(nT))}},$$

where $Z_i(nT)$ is given by (9). By applying $Z_i(nT) = -\rho_i Y(nT^-)$, we get that

$$\begin{aligned}\theta(Z_1(nT), Z_2(nT)) &= \frac{\rho_1\rho_2 Cov(Y_1(nT^-), Y_2(nT^-))}{\rho_1\rho_2\sqrt{Var(Y_1(nT^-))}\sqrt{Var(Y_2(nT^-))}},\\ &= \theta(nT^-, \Lambda_0, \Lambda_1, \Lambda_2, \rho_1, \rho_2),\end{aligned}$$

hence the correlation in the jumps between the two components due to the maintenance actions is equal to the correlation between the degradation levels of the two components before performing the maintenance actions.

Next, we focus on the monotonicity of the Pearson correlation coefficient. This monotonicity is analyzed considering two special cases: equal maintenance efficiency $\rho_1 = \rho_2$ and equal general time scales functions $\Lambda_0(t) = \Lambda_1(t) = \Lambda_2(t)$.

4.1 | Equal maintenance efficiency

By assuming $\rho_1 = \rho_2 = \rho$, the Pearson correlation coefficient is equal to

$$\theta(t, \rho, \Lambda_0, \Lambda_1, \Lambda_2) = \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2 h^{(1)}(t,\rho)}\sqrt{\sigma_0^2 + \sigma_2^2 h^{(2)}(t,\rho)}}, \tag{13}$$

with

$$h^{(i)}(t,\rho) = \frac{\sum_{j=1}^{n}\Delta\Lambda_i(jT)(1-\rho)^{2(n-j+1)} + \Lambda_i(t) - \Lambda_i(nT)}{\sum_{j=1}^{n}\Delta\Lambda_0(jT)(1-\rho)^{2(n-j+1)} + \Lambda_0(t) - \Lambda_0(nT)}, \qquad nT \le t < (n+1)T, \tag{14}$$

where $\Delta\Lambda_i(jT)$ and $\Delta\Lambda_0(jT)$ denote the increments of functions $\Lambda_i(\cdot)$ and $\Lambda_0(\cdot)$, given by (8) for $i = 1, 2$.

If $\rho_1 = \rho_2 = \rho$ and $\Lambda_0(\cdot) = \Lambda_1(\cdot) = \Lambda_2(\cdot)$, then the Pearson correlation coefficient is equal to

$$\theta(t,\rho,\Lambda) = \frac{\sigma_0^2 g(t,\rho,\rho,\Lambda)}{g(t,\rho,\rho,\Lambda)\sqrt{\sigma_0^2+\sigma_1^2}\sqrt{\sigma_0^2+\sigma_2^2}} = \frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma_1^2}\sqrt{\sigma_0^2+\sigma_2^2}}$$

for $nT \le t < (n+1)T$. This means that, by using the same maintenance efficiency for the two components and the same general time scales, the Pearson correlation coefficient is time-invariant and equal to the coefficient obtained in (4) in absence of preventive maintenance actions (see [2]).

The particular case $\rho_1 = \rho_2 = 1$ means that the most efficient maintenance is performed on the two components. It means that, after each maintenance action, the deterioration of each component resets to zero. However, for $\Lambda_i(\cdot)$ and $\Lambda_0(\cdot)$ not homogeneous, the most efficient maintenance action does not imply that an "*as good as new*" (AGAN) maintenance (replacement) is performed on the system since the imperfect maintenance reduces the deterioration but not the system age.

Next, the Pearson correlation coefficient in the case $\rho_1 = \rho_2 = 1$ is compared to the Pearson correlation coefficient by using an "*as good as new*" maintenance. Imposing $\rho_1 = \rho_2 = 1$, the Pearson correlation coefficient is

$$\theta(t,1,\Lambda_0,\Lambda_1,\Lambda_2) = \frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma_1^2 h^{(1)}(t,1)}\sqrt{\sigma_0^2+\sigma_2^2 h^{(2)}(t,1)}}, \qquad nT \le t < (n+1)T, \tag{15}$$

where $h^{(i)}(t,1)$ is given by (14), that is,

$$h^{(i)}(t,1) = \frac{\Lambda_i(t)-\Lambda_i(nT)}{\Lambda_0(t)-\Lambda_0(nT)}, \qquad \text{for } nT \le t < (n+1)T,$$

for $i = 1, 2$. If a complete replacement of a system is performed at time nT, the Pearson correlation coefficient is given by

$$\theta_*(t-nT,\rho,\Lambda_0,\Lambda_1,\Lambda_2) = \frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma_1^2 h_*^{(1)}(t-nT)}\sqrt{\sigma_0^2+\sigma_2^2 h_*^{(2)}(t-nT)}}, \quad nT \le t < (n+1)T,$$

where

$$h_*^{(i)}(t-nT) = \frac{\Lambda_i(t-nT)}{\Lambda_0(t-nT)}, \qquad nT \le t < (n+1)T.$$

for $i = 1, 2$. The relationship between $\theta(\cdot)$ and $\theta_*(\cdot)$ is given in proposition 4.

Proposition 4. *If $h_*^{(i)}(t) = \Lambda_i(t)/\Lambda_0(t)$ is increasing (decreasing) in t, with $nT \le t < (n+1)T$, then*

$$\theta_*(t-nT,\rho,\Lambda_0,\Lambda_1,\Lambda_2) \ge (\le)\, \theta(t,1,\Lambda_0,\Lambda_1,\Lambda_2), \quad nT \le t < (n+1)T.$$

Proof. We get that, for $i = 1, 2$,

$$\frac{\Lambda_i(t)-\Lambda_i(nT)}{\Lambda_0(t)-\Lambda_0(nT)} = \frac{\Lambda_i(t)}{\Lambda_0(t)}\frac{1-\Lambda_i(nT)/\Lambda_i(t)}{1-\Lambda_0(nT)/\Lambda_0(t)} \ge \frac{\Lambda_i(t)}{\Lambda_0(t)}$$

since $h_*^{(i)}(t)$ is increasing in t and, in consequence,

$$\frac{\Lambda_i(nT)}{\Lambda_0(nT)} \leq \frac{\Lambda_i(t)}{\Lambda_0(t)}$$

for $nT \leq t$. Finally, applying again that $h_*^{(i)}(t)$ is increasing in t,

$$\frac{\Lambda_i(t)}{\Lambda_0(t)} \geq \frac{\Lambda_i(t-nT)}{\Lambda_0(t-nT)}$$

and the result holds. The proof for $h_i^*(t)$ decreasing in t is analogous. ∎

Increasing function $h_*^{(i)}(t)$ implies that the part of degradation specific to each component (expressed by Λ_i) takes more and more importance with respect to the part of degradation which is common to both components (expressed by Λ_0). Therefore, it is evident that the correlation is weaker than for an as-good-as-new (AGAN) system. Hence, if $\Lambda_i(t)/\Lambda_0(t)$ is increasing (resp. decreasing) in t for $i = 1, 2$, in spite of doing the most efficient maintenance for both components, the Pearson correlation coefficient is lower (resp. greater) than the Pearson correlation coefficient in the AGAN case by assuming non-homogeneous general time scales. In the homogeneous case, that is, if $\Lambda_i(t) = \alpha_i t$ and $\Lambda_0(t) = \alpha_0 t$, then both correlation coefficients are equal.

Propositions 2 and 3 give results about the monotonicity of $h^{(i)}(nT, \rho)$ and $h^{(i)}(t, \rho)$ in ρ. These results will be next used to analyze the monotonicity of the Pearson correlation coefficient $\theta(t, \rho, \Lambda_0, \Lambda_1, \Lambda_2)$.

Lemma 2. *For fixed $T > 0$, if the function*

$$a_i(t) = \frac{\Lambda_i(t+T) - \Lambda_i(t)}{\Lambda_0(t+T) - \Lambda_0(t)}, \qquad t > 0 \tag{16}$$

is increasing in t for $i = 1, 2$, then $h^{(i)}(nT, \rho)$ given by

$$h^{(i)}(nT, \rho) = \frac{\sum_{j=1}^{n} \Delta\Lambda_i(jT)(1-\rho)^{2(n-j+1)}}{\sum_{z=1}^{n} \Delta\Lambda_0(zT)(1-\rho)^{2(n-z+1)}} = \frac{\sum_{j=1}^{n} \Delta\Lambda_i(jT)(1-\rho)^{2(n-j)}}{\sum_{z=1}^{n} \Delta\Lambda_0(zT)(1-\rho)^{2(n-z)}}$$

is increasing in ρ.

Proof. Deriving $h^{(i)}(nT, \rho)$ with respect to ρ, we get that, for $i = 1, 2$,

$$\begin{aligned}
\frac{\partial h^{(i)}(nT, \rho)}{\partial \rho} &= \frac{-2\sum_{j=1}^{n-1}\sum_{z=1}^{n} \Delta\Lambda_i(jT)\Delta\Lambda_0(zT)(n-j)(1-\rho)^{2(n-j)+2(n-z)-1}}{\left(\sum_{z=1}^{n} \Delta\Lambda_0(zT)(1-\rho)^{2(n-z)}\right)^2} \\
&+ \frac{2\sum_{j=1}^{n}\sum_{z=1}^{n-1} \Delta\Lambda_i(jT)\Delta\Lambda_0(zT)(n-z)(1-\rho)^{2(n-j)+2(n-z)-1}}{\left(\sum_{z=1}^{n} \Delta\Lambda_0(zT)(1-\rho)^{2(n-z)}\right)^2} \\
&= \frac{2E_n}{\left(\sum_{z=1}^{n} \Delta\Lambda_0(zT)(1-\rho)^{2(n-z)}\right)^2}.
\end{aligned}$$

Hence,

$$\frac{\partial h^{(i)}(nT, \rho)}{\partial \rho} \geq 0$$

if and only if

$$E_n = \sum_{j=1}^{n}\sum_{z=1}^{n}(j-z)\pi(j,z)W(j,z) \geq 0,$$

where

$$\pi(j,z) = \Delta\Lambda_i(jT)\Delta\Lambda_0(zT) \quad \text{and} \quad W(j,z) = (1-\rho)^{2(n-j)+2(n-z)-1}.$$

Since $W(j,z) = W(z,j)$, then for $n \geq 2$

$$\begin{aligned}
E_n &= \sum_{j=1}^{n}\sum_{z=1}^{n}\pi(j,z)(j-z)W(j,z) \\
&= \sum_{j=2}^{n}\sum_{z=1}^{j-1}\pi(j,z)(j-z)W(j,z) + \sum_{j=1}^{n-1}\sum_{z=j+1}^{n}\pi(j,z)(j-z)W(j,z) \\
&= \sum_{j=2}^{n}\sum_{z=1}^{j-1}\pi(j,z)(j-z)W(j,z) + \sum_{z=2}^{n}\sum_{j=1}^{z-1}\pi(j,z)(j-z)W(j,z) \\
&= \sum_{j=2}^{n}\sum_{z=1}^{j-1}\pi(j,z)(j-z)W(j,z) + \sum_{j=2}^{n}\sum_{z=1}^{j-1}\pi(z,j)(z-j)W(z,j) \\
&= \sum_{j=2}^{n}\sum_{z=1}^{j-1}W(j,z)\left(\pi(j,z)(j-z) + \pi(z,j)(z-j)\right).
\end{aligned}$$

On the other hand, as $a_i((j-1)T) = \Delta\Lambda_i(jT)/\Delta\Lambda_0(jT)$ is increasing in j because $a(t)$ is increasing in t ((16)), then $\pi(j,z) \geq \pi(z,j)$ for $j \geq z$. Furthermore, $W(j,z) = W(z,j)$, hence $E_n \geq 0$ for $n \geq 2$. If $n = 1$, the function $h^{(i)}(nT,\rho)$ is constant in ρ and the result holds. ∎

Using the previous results, if $a_i(t)$ given by (16) is increasing in t then $h^{(i)}(nT,\rho)$ is increasing in ρ, hence

$$\lim_{\rho\to 0} h^{(i)}(nT,\rho) \leq h^{(i)}(nT,\rho) \leq \lim_{\rho\to 1} h^{(i)}(nT,\rho),$$

or, equivalently,

$$\frac{\Lambda_i(nT)}{\Lambda_0(nT)} \leq h^{(i)}(nT,\rho) \leq \frac{\Lambda_i(nT)-\Lambda_i((n-1)T)}{\Lambda_0(nT)-\Lambda_0((n-1)T)}. \tag{17}$$

The monotonicity of $h^{(i)}(t,\rho)$ is next analyzed with respect to ρ. This result is given in the following proposition.

Lemma 3. *For fixed $T > 0$, if the function $a_i(t)$ is increasing in t and*

$$\Lambda_i'(t)\Delta\Lambda_0(jT) \geq \Lambda_0'(t)\Delta\Lambda_i(jT) \qquad j = 1,2,\ldots,n, \quad nT \leq t < (n+1)T, \tag{18}$$

then $h^{(i)}(t,\rho)$ is increasing in ρ for fixed t and $nT \leq t < (n+1)T$.

Proof. Deriving $h_n^{(i)}(t,\rho)$ with respect to ρ, we have

$$\frac{\partial h^{(i)}(t,\rho)}{\partial \rho} \geq 0 \Leftrightarrow \frac{\partial h^{(i)}(nT,\rho)}{\partial \rho} + \frac{\partial}{\partial \rho}\left(\sum_{j=1}^{n} \Delta\Lambda_i(jT)(1-\rho)^{2(n-j+1)}(\Lambda_0(t)-\Lambda_0(nT)) - \sum_{j=1}^{n} \Delta\Lambda_0(jT)(1-\rho)^{2(n-j+1)}(\Lambda_i(t)-\Lambda_i(nT))\right) \geq 0.$$

From Lemma 2 it follows that

$$\frac{\partial h^{(i)}(nT,\rho)}{\partial \rho} \geq 0.$$

Hence if

$$\frac{\partial}{\partial \rho}\left(\sum_{j=1}^{n} \Delta\Lambda_i(jT)(1-\rho)^{2(n-j+1)})(\Lambda_0(t)-\Lambda_0(nT)) - \sum_{j=1}^{n} \Delta\Lambda_0(jT)(1-\rho)^{2(n-j+1)}(\Lambda_i(t)-\Lambda_i(nT))\right) \geq 0 \tag{19}$$

then

$$\frac{\partial h^{(i)}(t,\rho)}{\partial \rho} \geq 0.$$

To prove this, the term of (19) is derived with respect to t. If

$$2\sum_{j=1}^{n}(n-j+1)(1-\rho)^{2(n-j+1)-1}\left(\Lambda_i'(t)\Delta\Lambda_0(jT) - \Lambda_0'(t)\Delta\Lambda_i(jT)\right) > 0, \tag{20}$$

then (19) is increasing with respect to t. Using the assumption given by (18), (20) is fulfilled and consequently, the expression given by (19) is greater than or equal to 0. Therefore, it finally follows that

$$\frac{\partial h^{(i)}(t,\rho)}{\partial \rho} \geq 0$$

and the result holds. ∎

Proposition 5. *If $\Lambda_i(t) = t^{\beta_i}$ and $\Lambda_0(t) = t^{\beta_0}$ for $\beta_i > \beta_0 > 1$, then the condition given by (18) is fulfilled.*

Proof. By using $\Lambda_i(t) = t^{\beta_i}$ and $\Lambda_0(t) = t^{\beta_0}$ in the condition given by (18), we get that

$$\beta_i t^{\beta_i - 1}\left[(jT)^{\beta_0} - ((j-1)T)^{\beta_0}\right] \geq \beta_0 t^{\beta_0 - 1}\left[(jT)^{\beta_i} - ((j-1)T)^{\beta_i}\right],$$

that is,

$$\begin{aligned}\frac{\beta_i}{\beta_0}\frac{t^{\beta_i-1}}{t^{\beta_0-1}} &\geq \frac{(jT)^{\beta_i}}{(jT)^{\beta_0}}\frac{1-\left(\frac{(j-1)T}{jT}\right)^{\beta_i}}{1-\left(\frac{(j-1)T}{jT}\right)^{\beta_0}}\\ \frac{\beta_i}{\beta_0}\left(\frac{t}{jT}\right)^{\beta_i-\beta_0} &\geq \frac{1-u^{\beta_i}}{1-u^{\beta_0}},\end{aligned}$$

where $u = \frac{(j-1)T}{jT}$. Then, let $g(u) = \frac{1-u^{\beta_i}}{1-u^{\beta_0}}$ and deriving $g(u)$,

$$\begin{aligned}g'(u) &= -\beta_i u^{\beta_i-1}(1-u^{\beta_0}) + (1-u^{\beta_i})\beta_0 u^{\beta_0-1}\\ &= u^{\beta_0-1}\left(-\beta_i u^{\beta_i-\beta_0} + \beta_i u^{\beta_i} + \beta_0 - \beta_0 u^{\beta_i}\right).\end{aligned}$$

Now, taking the term $z(u) = -\beta_i u^{\beta_i-\beta_0} + \beta_i u^{\beta_i} + \beta_0 - \beta_0 u^{\beta_i}$, it holds that

$$\begin{aligned}z'(u) &= -\beta_i(\beta_i-\beta_0)u^{\beta_i-\beta_0-1} + \beta_i^2 u^{\beta_i-1} - \beta_0\beta_i u^{\beta_i-1}\\ &= \beta_i(\beta_i-\beta_0)\left(u^{\beta_i-1} - u^{\beta_i-\beta_0-1}\right).\end{aligned}$$

Since $u < 1$, then $u^{\beta_i-1} - u^{\beta_i-\beta_0-1} < 0$. Consequently, $z'(u) \leq 0$ and function $z(u)$ is decreasing in u. Therefore, $z(u) \geq z(1) \geq 0$ and $g(u)$ is increasing in u . Finally, $g(u) \leq g(1) = \frac{\beta_i}{\beta_0}$, and the result holds since $\left(\frac{t}{jT}\right)^{\beta_i-\beta_0} \geq 1$ for $j = 1, 2, \ldots, n$ and $t \geq jT$, given by (18). ∎

Corollary 1. *If $\Lambda_i(t) = t^{\beta_i}$ and $\Lambda_0(t) = t^{\beta_0}$ for $\beta_i > \beta_0 > 1$, then function $h^{(i)}(t,\rho)$ is increasing in ρ.*

Proposition 6. *Assuming $\Lambda_i(t) = \exp(\beta_i t) - 1$ and $\Lambda_0(t) = \exp(\beta_0 t) - 1$, if $\beta_i > \beta_0$, for $i = 1, 2$, then condition given by (18) is fulfilled.*

Proof. (18) is fulfilled if and only if,

$$\frac{\beta_i}{\beta_0}\frac{\exp(\beta_i t)}{\exp(\beta_0 t)} \geq \frac{\exp(\beta_i jT) - \exp(\beta_i(j-1)T)}{\exp(\beta_0 jT) - \exp(\beta_0(j-1)T)},$$

or, equivalently

$$\frac{\beta_i}{\beta_0}\exp((\beta_i-\beta_0)(t-jT)) \geq \frac{1-\exp(-\beta_i T)}{1-\exp(-\beta_0 T)}. \tag{21}$$

We get that

$$\frac{1-\exp(-\beta_i T)}{1-\exp(-\beta_0 T)} \leq \frac{\beta_i}{\beta_0}.$$

Hence, since $\exp((\beta_i-\beta_0)(t-jT)) \geq 1$, then condition given in (21) is fulfilled. ∎

Corollary 2. *If $\Lambda_i(t) = \exp(\beta_i t) - 1$ and $\Lambda_0(t) = \exp(\beta_0 t) - 1$, the function $h^{(i)}(t,\rho)$ is increasing in ρ.*

Lemma 4. *For fixed T and ρ, if $a_i(t)$ given by (16) is increasing in t then function $h^{(i)}(nT,\rho)$ is increasing in n.*

Proof. It holds that, for $i = 1, 2$,

$$h^{(i)}((n+1)T,\rho) - h^{(i)}(nT,\rho) \geq 0 \Leftrightarrow h^{(i)}(nT,\rho) \leq \frac{\Lambda_i((n+1)T) - \Lambda_i(nT)}{\Lambda_0((n+1)T) - \Lambda_0(nT)}.$$

As $h^{(i)}(nT,\rho)$ is increasing in ρ, we get that

$$h^{(i)}(nT,\rho) \leq \lim_{\rho\to 1} h^{(i)}(nT,\rho) = \frac{\Lambda_i(nT) - \Lambda_i((n-1)T)}{\Lambda_0(nT) - \Lambda_0((n-1)T)}.$$

Since function $a_i(t)$ given by (16) is increasing in t, then

$$\frac{\Lambda_i(nT) - \Lambda_i((n-1)T)}{\Lambda_0(nT) - \Lambda_0((n-1)T)} \leq \frac{\Lambda_i((n+1)T) - \Lambda_i(nT)}{\Lambda_0((n+1)T) - \Lambda_0(nT)}.$$

By (17), the result is fulfilled. ∎

Proposition 7. *With respect to the monotonicity of the Pearson correlation coefficient and assuming that $a_i(t)$ is increasing in t, the following results are obtained.*

1. *For fixed T and n, $\theta(nT,\rho,\Lambda_0,\Lambda_1,\Lambda_2)$ is decreasing with respect to ρ. It means that, if the maintenance efficiency increases, the correlation between the processes decreases after the maintenance action.*
2. *For fixed T and ρ, $\theta(nT,\rho,\Lambda_0,\Lambda_1,\Lambda_2)$ is decreasing with respect to n. It means that the correlation between the processes decreases with the number of maintenance actions performed on the system.*
3. *If $\Lambda_i'(t)\,(\Lambda_0(t) - \Lambda_0(nT)) \geq \Lambda_0'\,(\Lambda_i(t) - \Lambda_i(nT))$ for $i = 1, 2$, then $\theta(t,\rho,\Lambda_0,\Lambda_1,\Lambda_2)$ is decreasing with respect to ρ.*
4. *If $\Lambda_i(t) = t^{\beta_i}$ and $\Lambda_0(t) = t^{\beta_0}$, with $\beta_i > \beta_0 > 1$ for $i = 1, 2$, then $\theta(t,\rho,\Lambda_0,\Lambda_1,\Lambda_2)$ is decreasing with respect to ρ.*
5. *If $\Lambda_i(t) = \exp(\beta_i t)$ and $\Lambda_0(t) = \exp(\beta_0 t)$, with $\beta_i > \beta_0 > 1$ for $i = 1, 2$, then $\theta(t,\rho,\Lambda_0,\Lambda_1,\Lambda_2)$ is decreasing with respect to ρ.*

Proof. Points 1 and 2 are deduced from Lemmas 2 and 4, respectively. Point 3 follows from Lemma 3. Finally, points 4 and 5 result from Corollaries 1 and 2, respectively. ∎

Example 5. Figures 6 and 7 plot the Pearson correlation coefficient versus ρ and n, respectively, for different values of $\beta > 1$. These figures are plotted using the following parameters: $\Lambda_0(t) = t, \Lambda_1(t) = \Lambda_2(t) = t^{\beta}, T = 5$ and $\sigma_0^2 = 1, \sigma_1^2 = \sigma_2^2 = 0.25$. In addition, for Figure 6, $n = 3$, and for Figure 7, $\rho = 0.2$ are considered. The correlation coefficient in both cases is decreasing in ρ and n for different values of β, as it is stated in Proposition 7.

4.2 | Equal general time scales

The monotonicity of the Pearson correlation coefficient $\theta(nT,\rho_1,\rho_2,\Lambda_0,\Lambda_1,\Lambda_2)$ is analyzed in the case $\Lambda_0(t) = \Lambda_1(t) = \Lambda_2(t)$ and $\rho_1 \neq \rho_2$. For the sake of simplicity, we denote $\Lambda_i := \Lambda$, for $i = 0, 1, 2$ and $\theta(t,\rho_1,\rho_2,\Lambda)$ for the Pearson correlation coefficient. For equal general time scales, the square of the Pearson correlation coefficient is

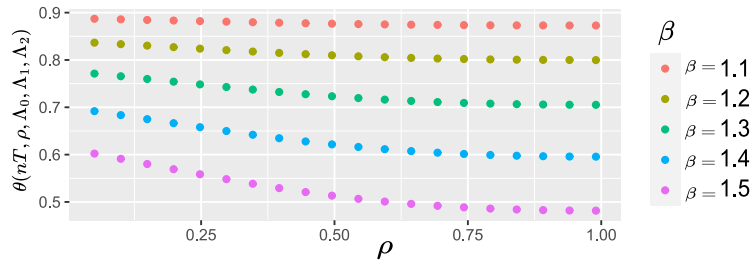

FIGURE 6 Pearson correlation coefficient versus ρ for $\Lambda_0(t) = t$ and $\Lambda_1(t) = \Lambda_2(t) = t^\beta$ in Example 5.

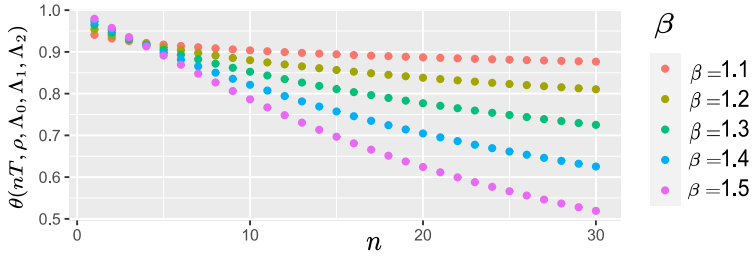

FIGURE 7 Pearson correlation coefficient versus n for $\Lambda_0(t) = t$ and $\Lambda_1(t) = \Lambda_2(t) = t^\beta$. in Example 5.

given by

$$\begin{aligned}\theta(t,\rho_1,\rho_2,\Lambda) &= \frac{\sigma_0^2 g(t,\rho_1,\rho_2,\Lambda)}{\sqrt{\sigma_0^2+\sigma_1^2}\sqrt{g(t,\rho_1,\rho_1,\Lambda)}\sqrt{\sigma_0^2+\sigma_2^2}\sqrt{g(t,\rho_2,\rho_2,\Lambda)}} \\ &= \frac{\sigma_0^2}{\sqrt{\sigma_0^2+\sigma_1^2}\sqrt{\sigma_0^2+\sigma_2^2}} \frac{g(t,\rho_1,\rho_2,\Lambda)}{\sqrt{g(t,\rho_1,\rho_1,\Lambda)}\sqrt{g(t,\rho_2,\rho_2,\Lambda)}} \end{aligned} \tag{22}$$

where $g(\cdot)$ is given by (9). The first term of (22) corresponds to the correlation coefficient in absence of maintenance actions. The second term corresponds to the influence on the correlation coefficient of the maintenance actions.

Next, the monotonicity of $\theta(t,\rho_1,\rho_2,\Lambda)$ with respect to ρ_i for n and T fixed is analyzed.

Proposition 8. *For fixed n and T, the Pearson correlation coefficient $\theta(nT,\rho_1,\rho_2,\Lambda)$ is increasing in ρ_1 if $\rho_1 \leq \rho_2$ and decreasing in ρ_1 if $\rho_2 \leq \rho_1$.*

Proof. We get that,

$$\frac{\partial\theta^2(nT,\rho_1,\rho_2,\Lambda)}{\partial\rho_1} = \frac{K}{g(nT,\rho_2,\rho_2,\Lambda)} \frac{\partial}{\partial\rho_1}\left(\frac{g^2(nT,\rho_1,\rho_2,\Lambda)}{g(nT,\rho_1,\rho_1,\Lambda)}\right),$$

where

$$K = \frac{\sigma_0^4}{(\sigma_0^2+\sigma_1^2)(\sigma_0^2+\sigma_2^2)}.$$

On the other hand,

$$\frac{\partial}{\partial\rho_1}\left(\frac{g^2(nT,\rho_1,\rho_2,\Lambda)}{g(nT,\rho_1,\rho_1,\Lambda)}\right) = 2\sum_{i=1}^{n}\Delta\Lambda(iT)(1-\rho_1)^{n-i+1}(1-\rho_2)^{n-i+1}\cdot$$
$$\cdot\quad \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta\Lambda(iT)\Delta\Lambda(jT)(i-j)(1-\rho_1)^{n-i+2(n-j+1)-1}(1-\rho_2)^{n-i+1}}{g^2(nT,\rho_1,\rho_1,\Lambda)}.$$

Hence,

$$\frac{\partial}{\partial\rho_1}\left(\frac{g^2(nT,\rho_1,\rho_2,\Lambda)}{g(nT,\rho_1,\rho_1,\Lambda)}\right) \geq 0 \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}\Pi(i,j)(i-j)(1-\rho_1)^{n-i+2(n-j+1)-1}(1-\rho_2)^{n-i+1} \geq 0,$$

where

$$\Pi(i,j) = \Delta\Lambda(iT)\Delta\Lambda(jT).$$

Furthermore, the term can be expanded as

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\Pi(i,j)(i-j)(1-\rho_1)^{n-i+2(n-j+1)-1}(1-\rho_2)^{n-i+1}\\
&=\sum_{i=2}^{n}\sum_{j=1}^{i-1}\Pi(i,j)(i-j)(1-\rho_1)^{n-i+2(n-j+1)-1}(1-\rho_2)^{n-i+1}\\
&+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Pi(i,j)(i-j)(1-\rho_1)^{n-i+2(n-j+1)-1}(1-\rho_2)^{n-i+1}\\
&=\sum_{i=2}^{n}\sum_{j=1}^{i-1}\Pi(i,j)(i-j)(1-\rho_1)^{n-i+2(n-j+1)-1}(1-\rho_2)^{n-i+1}\\
&+\sum_{i=2}^{n-1}\sum_{j=1}^{i-1}\Pi(i,j)(j-i)(1-\rho_1)^{n-j+2(n-i+1)-1}(1-\rho_2)^{n-j+1}\\
&=\sum_{i=2}^{n}\sum_{j=1}^{i-1}\Pi(i,j)(i-j)(1-\rho_2)^{n-i+1}(1-\rho_1)^{n-j+2(n-i+1)-1}\left((1-\rho_1)^{i-j}-(1-\rho_2)^{i-j}\right).
\end{aligned}$$

In the case $\rho_1 \leq \rho_2$, then

$$\frac{\partial}{\partial\rho_1}\left(\frac{g^2(nT,\rho_1,\rho_2,\Lambda)}{g(nT,\rho_1,\rho_1,\Lambda)}\right) \geq 0.$$

Similarly, if $\rho_1 \geq \rho_2$, then

$$\frac{\partial}{\partial\rho_1}\left(\frac{g^2(nT,\rho_1,\rho_2,\Lambda)}{g(nT,\rho_1,\rho_1,\Lambda)}\right) \leq 0$$

and the result holds. ∎

Corollary 3. *For fixed n and T, the Pearson correlation coefficient $\theta(nT,\rho_1,\rho_2,\Lambda)$ is increasing in ρ_2 if $\rho_2 \leq \rho_1$ and de-*

creasing if $\rho_2 \geq \rho_1$.

Corollary 4. *If $\rho_1 = \rho_2$, then the correlation $\theta(nT, \rho_1, \rho_2, \Lambda)$ reaches the maximum and it is equal to*

$$\sqrt{K} = \theta(t, \rho, \Lambda) = \frac{\sigma_0^2}{\sqrt{\sigma_0^2 + \sigma_1^2}\sqrt{\sigma_0^2 + \sigma_2^2}}, \quad t \geq 0.$$

As an interpretation of Proposition 8 and Corollary 3, the equality $\rho_1 = \rho_2$ means that the maintenance process is the same for both components, so the resulting degradation levels are linked. If ρ_1 and ρ_2 move apart, the maintenance process diverge, so the correlation decreases.

Proof. Using proposition 8, if $0 < \rho_1 \leq \rho_2$, it follows that

$$\theta_n^2(nT, 0, \rho_2, \Lambda) \leq \theta_n^2(nT, \rho_1, \rho_2, \Lambda) \leq \theta_n^2(nT, \rho_2, \rho_2, \Lambda).$$

Hence, the Pearson correlation coefficient fulfils

$$\theta_n^2(nT, \rho_1, \rho_2, \Lambda) \leq K,$$

where

$$K = \frac{\sigma_0^4}{(\sigma_0^2 + \sigma_1^2)(\sigma_0^2 + \sigma_2^2)}.$$

On the other hand, if $\rho_2 \leq \rho_1 < 1$,

$$\theta_n^2(nT, 1, \rho_2, \Lambda) \leq \theta_n^2(nT, \rho_1, \rho_2, \Lambda) \leq \theta_n^2(nT, \rho_2, \rho_2, \Lambda),$$

and the Pearson correlation coefficient fulfils

$$\theta_n^2(nT, \rho_1, \rho_2, \Lambda) \leq K.$$

∎

Example 6. Figure 8 plots the Pearson correlation coefficient $\theta(nT, \rho_1, \rho_2, \Lambda)$ versus ρ_1 for $n = 30$, $T = 5$, $\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = 1$, $\rho_2 = 0.5$ and $\Lambda(t) = t^\beta$. As can be seen, this coefficient is increasing in ρ_1 for $\rho_1 \leq \rho_2$ and decreasing in ρ_1 for $\rho_1 \geq \rho_2$, where the maximum is reached for $\rho_1 = \rho_2$.

5 | CONCLUSIONS AND FURTHER WORKS

The trivariate reduction method is proposed to model the dependence between two degrading processes with one common factor between the two components. Imperfect preventive maintenance actions are performed periodically on the system, implementing an ARD model of infinite order for describing such maintenance actions. The correlation

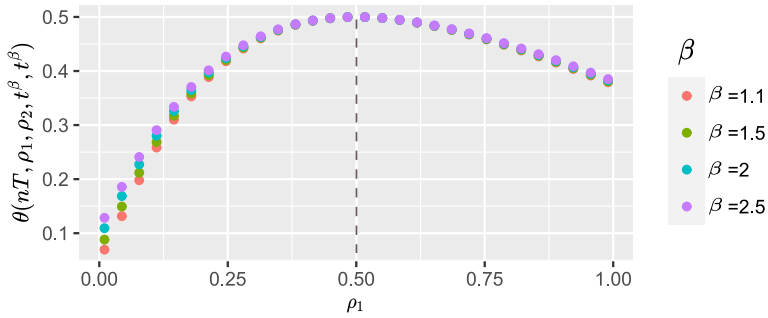

FIGURE 8 Pearson correlation coefficient $\theta(nT, \rho_1, \rho_2, t^\beta, t^\beta)$ versus ρ_1 for different β in Example 6.

analysis between degradation levels of the components is performed evaluating the Pearson's correlation coefficient. Different properties of the monotonicity of the Pearson correlation coefficient are studied.

To model the dependence between the two components, just a single common process is considered. In some practical cases, it could be insufficient and the number of common stochastic processes should be extended to two or more [20]. On the other hand, to assess the correlation between components, the Pearson correlation coefficient is used. However, this coefficient has some shortcomings in measuring the associated relationship of variables when the dependence relationship is non-linear.

As future work, we intend to analyse the correlation from a different perspective than using Pearson's correlation coefficient, for instance, with the use of the Kendall rank correlation coefficient or copula functions. Another perspective to consider would be the estimation of the model parameters in different maintenance situations.
Finally, predicting the lifetime (or the RUL) of the two-component system, taking into account both the series or parallel configuration and the correlation is another interesting prospect of this study.

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