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Abstract

We present a study of some theoretical aspects for a model describing a nonisothermal adsorption column without axial diffusion, with a single gaseous species and adiabatic operation. In this model, the velocity of the gas is part of the unknown (sorption effect). We obtain an existence and uniqueness result in a variational framework. We then consider the particular case of instantaneous equilibrium between the gaseous and solid phases.

Key words: gas chromatography, nonlinear chromatography, nonisothermal adsorption, systems of conservation laws, boundary conditions.
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1 Introduction

Pressure Swing Adsorption (PSA) is a technology that is used to separate some species from a gas under pressure according to these species' molecular characteristics and affinity for an adsorbent material (activated carbon, silica gel, alumina, resin, zeolite...). One of the primary applications of PSA is in the removal of carbon dioxide (CO2) as the final step in the large-scale commercial synthesis of hydrogen (H2) for use in oil refineries and in the production of ammonia (NH3). PSA is used extensively in the production and purification of oxygen for medical uses. Another application of PSA is the separation of carbon dioxide from biogas to increase the methane (CH4) ratio. A typical PSA system involves a cyclic process where a number of connected vessels containing adsorbent material undergo successive pressurization and depressurization steps in order to produce a continuous stream of purified product gas.

In previous papers by some of the present authors on this topic, we focused on a model describing a step of the cyclic process, restricted to isothermal behavior [2] and, in a particular situation leading to a hyperbolic system, two species [4, 5, 7, 8, 9, 10] (see also [14]). Working with two species is of course restrictive in view of industrial applications but remains relevant because this corresponds to an important field of applications: for example separation of an He-CO_2 mixture [21], of CO2-N2, CO2-CH4 or CO2-H2 mixtures. As examples of more complex situations, the reader will find in [1] a numerical approach to the oxygen enrichment of a mixture of three components (main constituents of air) and in [13] a theoretical and experimental study for the bulk separation of a ternary mixture H_2-CH_4-CO_2.

In this paper, we are interested in some theoretical aspects for a model describing a nonisothermal adsorption column, with only one component and adiabatic operation. For a description of the general model and a classification of the various systems with respect to their complexity, we refer to Ruthwen ([20] chapter 9). In general fixed bed chromatography, each of the N components simultaneously exists under two phases, a gaseous and movable one with concentration $C_i(t,x)$ and velocity u or a solid (adsorbed) other with concentration $q_i(t,x)$, $1 \leq i \leq N$.

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The system describing there evolution, in a simplified and adimensional form, may be written as follows

$$\partial_t C_i + \partial_t q_i + \partial_x(u\,C_i) = \alpha\,\partial^2_{xx} C_i, \quad 1 \le i \le N, \tag{1}$$

$$\partial_t q_i + A_i\,q_i = A_i\,q_i^*(\underline{C}), \quad 1 \le i \le N, \tag{2}$$

$$\partial_t T - \beta\,\partial^2_{xx} T = -k\,p(t)\,\partial_x u - \sum_{i=1}^{N} \gamma_i\,\partial_t\,q_i - k\,p'(t) \tag{3}$$

$$\rho\,T = p(t), \tag{4}$$

where $\underline{C} = (C_1, \cdots, C_N)$, $\rho = \sum_{i=1}^N C_i$. T is the temperature of the gaseous mixture as well as that of the solid part (the adsorbent bed and the adsorbed components), see Remark 2.1. The given constants α, k, A_i, γ_i are non negative.
The so-called isotherms q_i^* quantify the amount of adsorbate i on the adsorbent as a function of the concentrations. A given species i is said at equilibrium when it satisfies $q_i = q_i^*(\underline{C})$. The precise form of the isotherms is usually unknown and is experimentally obtained, but there exists empirical models as the linear isotherm due to Henry [22] given by $q_i^* = K_i\,c_i$, with $K_i \ge 0$, semi-empirical models as the Langmuir isotherm [17] $q_i^* = \frac{Q_i K_i c_i}{1+\sum_{j=1}^N K_j c_j}$, with $K_i \ge 0$, $Q_i > 0$ (see also [14, 18]) and theoretical models as the Brunauer-Emmett-Teller (BET) isotherm [12] which writes, for a single adsorbate with concentration c, $q^*(c) = \frac{Q\,K\,c}{(1+(K-\frac{1}{c_s})\,c)\,(1-\frac{1}{c_s})}$ with Q, $K > 0$ and $0 < c_s < 1$.
Notice that the BET theory is an extension of the Langmuir theory, which is a theory for monolayer molecular adsorption, to multilayer adsorption: gas molecules physically adsorb on a solid in layers infinitely, only interact with adjacent layers and the Langmuir theory can be applied to each layer. This model is mainly suitable for materials with pores larger than a few nanometers.

In each of these models, the isotherms are smooth and increasing with respect to the concentration of the corresponding adsorbate, thus we assume that q_i^* is defined on $(\mathbb{R}^+)^N$ and satisfies

$$q_i^*(\underline{C} - \underline{D}) \le C_q\,|\underline{C} - \underline{D}| \text{ with } C_q > 0,\ |\underline{C} - \underline{D}| = \sum_{i=1}^{N} |C_i - D_i| \tag{5}$$

$$q_i^*(C_1, \cdots, C_{i-1}, 0, C_{i+1}, \cdots, C_N) = 0, \quad \frac{\partial q_i^*}{\partial c_i} \ge 0, \quad i \in \{1, \cdots, N\}. \tag{6}$$

The function p, in Equation (4), is the given total pressure, depending only upon t. The unknown are $\underline{C}$, $\underline{q}$, u and T. In gas chromatography, velocity variations accompany changes in gas composition, especially in the case of high concentration solute: it is known as the sorption effect. This effect is taken into account through Equation (4), which comes from the law of perfect gas assumed to be valid here. The reader can refer for instance to [14].
The isothermal model consists of the set of Equations (1), (2) and (4) which means that ρ is a given function depending only upon t. It was studied by one of the authors [2, 3] from both theoretical and numerical points of view (without axial dispersion, i.e. $\alpha = 0$). Notice that in this model, the case $N = 1$ is of poor interest because $\underline{C}(t,x) = C_1(t,x) = \rho(t)$ is a given function and the system is solved analytically. On the contrary, in the model studied in this paper, the case of a single species is interesting because it presents most of the difficulties with simpler notations.

The paper is organized as follows: in a first section we present the physical model. Next we show how to build the dimensionless model (1)-(2)-(3)-(4). In Section 4, we focus on the resolution of a simplified case with only one species and no diffusion. Next, in section 5, we consider the case of the instantaneous equilibrium between the gaseous and solid phases and show how to adapt the previous case. Finally we evoke certain variants of the problem, in particular with regard to the initial and boundary conditions, and the general case of $N \ge 1$ species.

2 A general model for a nonisothermal multicomponent adsorption column

We present here the model described by Ruthwen ([20] chapter 9) and which is the starting point of our study. It consists of a set of fluid phase mass balance for each of N species

$$\partial_\tau C_i + \partial_z(u\,C_i) + \frac{1-\varepsilon}{\varepsilon}\partial_\tau q_i = D_L \partial^2_{zz} C_i, \quad 1 \le i \le N, \quad \tau > 0, \quad 0 < z < L \tag{7}$$

a set of N o.d.e's describing the exchanges of mass between the gaseous phase ads the solid (or adsorbed) phase

$$\partial_\tau q_i = f(\underline{q}, \underline{C}), \quad 1 \le i \le N, \quad \tau > 0, \quad 0 < z < L \tag{8}$$

where $\underline{q} = (q_1, \cdots, q_N)$, $\underline{C} = (C_1, \cdots, C_N)$, and a set of two equations governing the temperature T_g of the gaseous mixture, the temperature T_s of the solid phase (the temperature T_w of the wall is known)

$$\begin{aligned} \partial_\tau \left[\varepsilon C_g T_g + (1-\varepsilon) C_s T_s\right] + \varepsilon \partial_z (C_g u T_g) - \varepsilon \lambda_L \partial^2_{zz} T_g \quad = \\ (1-\varepsilon)\sum_i (-\Delta H_i)\partial_\tau q_i - \frac{4h_w}{d}(T_g - T_w), \quad \tau > 0, \quad 0 < z < L, \end{aligned} \tag{9}$$

$$C_s \partial_\tau T_s - \frac{3h}{R_p}(T_g - T_s) = (-\Delta H_i)\partial_\tau q_i, \quad \tau > 0, \quad 0 < z < L. \tag{10}$$

In this study we use a classical form of Equation (8) with $f(\underline{q}, \underline{C}) = A_i\,(q_i^*(\underline{C}) - q_i)$ where $A_i > 0$ is a constant, q_i^* satisfying (5)-(6).

We assume also that the perfect gas law is valid for the mixture, namely

$$P(t) = \rho\, R\, T \tag{11}$$

with a pressure P depending only upon t and $\rho = \sum_{i=1}^N C_i$ (total concentration in fluid phase).

In appendix A the various unknown, functions and constants are given with the corresponding physical units (Tables 1, 2).

Remark 2.1 *In the sequel we choose to focus on the case of an adiabatic process, so that in* (9) *we have $T_g = T_w$. Next we assume that the temperatures of the gas and the adsorbent are the same, i.e. $T_g = T_s$ and the common value is noted T in the sequel. Notice that these assumptions in no way modify the difficulty of the problem but make it possible to lighten the notations.*

3 The dimensionless model

Let C, T^*, u^* be respectively a characteristic concentration, a characteristic temperature and a characteristic velocity. We set

$$\tilde{C}_i = \frac{C_i}{C} \text{ and } \tilde{q}_i = \frac{1-\varepsilon}{\varepsilon\, C} q_i \text{ for } i : 1, \cdots, N,$$

$$\tilde{\rho} = \frac{\rho}{C} = \sum_{i=1}^N \tilde{C}_i,\ \tilde{T} = \frac{T}{T^*},\ \tilde{u} = \frac{u}{u^*} \text{ and } \tau^* = \frac{L}{u^*}.$$

We have then $\tilde{\rho}\,\tilde{T} = \dfrac{\rho\, T}{C\, T^*} = \dfrac{P(t)}{RCT^*} = \dfrac{P(t)}{P^*} = p(t)$ where $P^* = RCT^*$ is a characteristic pressure. Dividing (7) by C, (8) by $\dfrac{\varepsilon\, C}{1-\varepsilon}$, (9) by $(1-\varepsilon)C_s T^*$ and using the change of variables $t = \dfrac{\tau}{\tau^*}$, $x = \dfrac{z}{L}$ we get the system under its adimensional form

$$\begin{aligned} \partial_t \tilde{C}_i + \partial_t \tilde{q}_i + \partial_x(\tilde{u}\,\tilde{C}_i) &= \alpha\, \partial^2_{xx} \tilde{C}_i, \\ \\ \partial_t \tilde{q}_i + \tilde{A}_i \tilde{q}_i &= \tilde{A}_i\, \tilde{q^*}_i(\underline{\tilde{C}}), \\ \partial_t \tilde{T} - \beta\, \partial^2_{xx} \tilde{T} &= -k\, p(t)\, \partial_x u - \sum_{i=1}^N \gamma_i\, \partial_t \tilde{q}_i - k\, p'(t), \\ \tilde{\rho}\,\tilde{T} &= p(t) \end{aligned}$$

where we have set

$$\alpha = \frac{D_L}{Lu^*},\ \beta = \frac{\varepsilon\,\lambda_L}{(1-\varepsilon)C_sLu^*},\ \gamma_i = \frac{\varepsilon\,C}{(1-\varepsilon)C_sT^*}\Delta H_i,\ k = \frac{\varepsilon\,C\,C_{pg}}{(1-\varepsilon)C_s},\ \tilde{A}_i = A_i\tau^*$$

and

$$\tilde{q^*}_i(\underline{\tilde{C}}) = \frac{1-\varepsilon}{\varepsilon\,C}q_i^*(C\,\underline{\tilde{C}})$$

with

$$\underline{\tilde{C}} = (\tilde{C_1}, \cdots, \tilde{C_N}).$$

Notice that $\tilde{q^*}_i$ is lipschitz continuous with constant $\tilde{C}_q = \dfrac{1-\varepsilon}{\varepsilon}C_q$.

Omiting the $\sim$ over the various unknown, functions and constants we obtain the dimensionless model (1)-(2)-(3)-(4).

4 A single species without axial diffusion

In this section we focus on the case of a single species, i.e. $N = 1$, with gaseous concentration ρ, solid concentration q and neglect the axial diffusion i.e. $\alpha = 0$. Moreover, for the sake of simplicity, we assume that the total pressure is a constant $p_0 > 0$ but we could easily obtain the same results assuming

$$p \in H^1(0,+\infty) \text{ with } p(t) \geq p_0 > 0.$$

We write the system in the following form for $(t,x) \in (0,\tau)\times(0,1)$ where $\tau > 0$, in the sequel, is no longer a variable but the fixed final time:

$$\begin{aligned}
\partial_t\rho + \partial_x(u\,\rho) &= A\,(q - q^*(\rho)), &(12)\\
\partial_t q &= -A\,(q - q^*(\rho)), &(13)\\
\partial_t T - \beta\,\partial^2_{xx}T &= -k\,p_0\,\partial_x u + \gamma\,\partial_t q &(14)
\end{aligned}$$

with A, k, $\gamma > 0$ and the constraint

$$\rho\,T = p_0, \quad 0 \leq t \leq \tau, \quad 0 \leq x < 1. \tag{15}$$

In view of (5)-(6) the function q^* is C_q-lipschitzian and satisfies

$$q^*(0) = 0, \quad (q^*)' \geq 0.$$

Setting $B = A\gamma$, we can also replace the term $\gamma\,\partial_t q$ in (14) by $-B\,(q - q^*(\rho))$.

The initial and boundary conditions are

$$(IBC) \quad \begin{cases} \forall t \in (0,\tau),\ T(t,0) = T_{in}(t), \\ \forall t \in (0,\tau),\ \partial_x T(t,1) = 0, \\ \forall t \in (0,\tau),\ u(t,0) = u_{in}(t) > 0, \\ \forall x \in (0,1),\ T(0,x) = T_0(x), \\ \forall x \in (0,1),\ q(0,x) = q_0(x). \end{cases}$$

The Neumann condition (4) for T at the outlet is convenient because (see the next section) we seek a solution such that $u(t,1) > 0$, which is not a priori garanteed whithout some various constraints on the initial and boundary data in order to get sufficient regularity on the temperature and the velocity.

In all the sequel L^2, resp. L^∞, stands for $L^2((0,\tau)\times(0,1))$, resp. $L^\infty((0,\tau)\times(0,1))$ and for all $t \in (0,\tau]$ we note L^2_t the space $L^2((0,t)\times(0,1))$. We introduce also the Hilbert space

$$\mathcal{H} = \Big\{\phi \in H^1(0,1)\,;\,\phi(0) = 0,\ \frac{\partial\phi}{\partial x}(1) = 0\Big\}.$$

The main result is

Theorem 4.1 *Let $\tau > 0$. We assume that $T_0 > 0$ is constant on $(0,1)$, $q_0 \in H^1(0,1)$, $u_{in} \in W^\infty_1(0,\tau)$ and $T_{in} \in H^2(0,\tau)$ with $T_{in}(0) = T_0$.*
In addition we assume that the function

$$g_u = \left(-B - \frac{kp_0}{1 + kp_0/T_0}\left(\frac{AT_0}{p_0} - \frac{B}{T_0}\right)\right)(q_0(x) - q^*(p_0/T_0)) - T'_{in}(0)$$

belongs to $\mathcal{H}$.

Then there exists $T_m > 0$, $p_m > 0$ *and* $u_m > 0$ *such that if* $T_{in} \geq T_m$, $p_0 \leq p_m$ *and* $u_{in} \geq u_m$, *the problem* (12)-(13)-(14)-(15)-*(IBC) has a unique global solution* $(u\,,\,q\,,\,\rho\,,\,T)$ *with*

$$u \in L^\infty(0,\tau;L^2(0,1)), \tag{16}$$
$$\partial_x u \in L^\infty(0,\tau\,;\,L^2(0,1)) \cap L^2(0,\tau\,;\,H^1(0,1)), \tag{17}$$
$$\partial_t\partial_x u \in L^\infty(0,\tau;L^2(0,1)), \tag{18}$$
$$\partial_t T,\ \partial_x T \in L^\infty, \tag{19}$$
$$\partial_t\partial_x T,\ \partial^2_{xx} T \in L^\infty(0,\tau\,;\,L^2(0,1)), \tag{20}$$
$$\partial^2_{tt} T \in L^2 \tag{21}$$

and there exists two real numbers a *and* b *such that* $0 < a \leq T \leq b$.

Comments. The conditions $g_u(0) = 0$ and $g'_u(1) = 0$ ($g_u \in \mathcal{H}$), together with the choice of the initial and boundary conditions ensure the validity of the compatibility conditions intended to get sufficient regularity to obtain the outlet condition $u(t,1) > 0$. They are for instance realized if $T'_{in}(0) = 0$ (or even $T_{in} = T_0$ on $(0,\tau)$) and $q_0(x) = q^*(p_0/T_0)$ on $(0,1)$ (equlibrium at $t = 0$), which is physically relevant in an experimental setting.

In order to prove Theorem 4.1 we need an intermediate result

Proposition 4.1 *Let* $\tau > 0$. *Let* u *such that*

$$(H_u1) \quad \begin{cases} u \in L^\infty(0,\tau;L^2(0,1)), \\ \partial_x u \in L^\infty(0,\tau;L^2(0,1)), \\ \partial_t\partial_x u \in L^2, \\ u(t,1) > 0. \end{cases}$$

Then there exists a real number $T_m > 0$ *such that if* $T_{in} \geq T_m$ *then there exists constants* $0 < a < b$ *and a unique couple* (q,T) *solution of*

$$\partial_t q + A\,q = A\,q^*(p_0/T), \tag{22}$$
$$\partial_t T - \beta\,\partial^2_{xx} T = -k\,p_0\,\partial_x u - B(q - q^*(p_0/T)), \tag{23}$$

$$q(0,\cdot) = q_0 \in H^1(0,1),$$
$$T(\cdot,0) = T_{in} \in H^2(0,\tau),$$
$$T(0,\cdot) = T_0 = T_{in}(0)$$

such that

$$0 < a \leq T \leq b$$

and

$$T \in L^\infty(0,\tau;H^1(0,1)) \cap L^2(0,\tau;H^2(0,1)),$$
$$\partial_t T \in L^2.$$

Moreover, assuming

$$(H_u2) \quad \partial_t\partial_x u \in L^\infty(0,\tau;L^2(0,1)),$$

if p_0 *is small enough and if the following compatibility condition is fulfilled*

$$(CC) \quad -k\,p_0\,\partial_x u(0,\cdot) - B\,(q_0 - q^*(p_0/T_0)) - T'_{in}(0) \in \mathcal{H}$$

then

$$\partial_t T,\ \partial_x T \in L^\infty,$$
$$\partial_t\partial_x T,\ \partial^2_{xx} T \in L^\infty(0,\tau;L^2(0,1)),$$
$$\partial^2_{tt} T \in L^2.$$

To prove the main result (Theorem 4.1) as well as Proposition 4.1, we will have to use some classical estimates in the domain of parabolic PDE that we recall in the next subsection.

4.1 Classical estimates

We recall the following classic result adapted to the context of our study (see for instance [16], [11]).

Proposition 4.2 *Let $\tau > 0$, $\beta > 0$, $D_0 \in \mathcal{H}$ and $f \in L^2([0,\tau) \times [0,1])$. Let D the unique weak solution of*

$$\partial_t D - \beta\, \partial^2_{xx} D = f,$$
$$D(t,0) = 0,\ t \geq 0,$$
$$\partial_x D(t,1) = 0,\ t \geq 0,$$
$$D(0,x) = D_0(x),\ 0 \leq x \leq 1$$

then

$$||D||^2_{L^2} \leq \tau\, e^{\tau}\, (||f||^2_{L^2} + ||D_0||^2_{L^2(0,1)}), \tag{24}$$

$$||\partial_t D||^2_{L^2} \leq ||f||^2_{L^2} + \beta ||D_0'||^2_{L^2(0,1)}, \tag{25}$$

$$||\partial^2_{xx} D||^2_{L^2} \leq \frac{1}{\beta^2} ||f||^2_{L^2} + \frac{1}{\beta} ||D_0'||^2_{L^2(0,1)}, \tag{26}$$

$$\sup_{t\in(0,\tau)} ||\partial_x D||^2_{L^2(0,1)}(t) \leq \frac{1}{\beta} ||f||^2_{L^2} + ||D_0'||^2_{L^2(0,1)}. \tag{27}$$

In order to get more regularity fot T we will use the following result

Proposition 4.3 *Let $\tau > 0$, $\beta > 0$, $G_0 \in \mathcal{H} \cap H^2(0,1)$ and $h \in H^1((0,\tau)\,;\, L^2(0,1))$. Let G the unique weak solution of*

$$\partial_t G - \beta\, \partial^2_{xx} G = h,$$
$$G(t,0) = 0,\ t \geq 0,$$
$$\partial_x G(t,1) = 0,\ t \geq 0,$$
$$G(0,x) = G_0(x),\ ,0 \leq x \leq 1.$$

If the following compatibility condition is fulfilled

$$g := h(0,\cdot) + \beta G_0'' \in \mathcal{H} \tag{28}$$

then

$$||\partial_t G||^2_{L^2} \leq ||h||^2_{L^2} + \beta ||G_0'||^2_{L^2(0,1)}, \tag{29}$$

$$||\partial^2_{tt} G||^2_{L^2} \leq 2||\partial_t h||^2_{L^2} + 2\beta ||g'||^2_{L^2(0,1)}, \tag{30}$$

$$||\partial_t \partial^2_{xx} G||^2_{L^2} \leq \frac{1}{\beta^2} ||\partial_t h||^2_{L^2} + \frac{1}{\beta} ||g'||^2_{L^2(0,1)}, \tag{31}$$

$$\sup_{t\in(0,\tau)} ||\partial_t \partial_x G||^2_{L^2(0,1)}(t) \leq \frac{1}{\beta} ||\partial_t h||^2_{L^2} + ||g'|^2_{L^2(0,1)}. \tag{32}$$

4.2 Proof of Proposition 4.1

We use an argument of fixed point on T in a ball of $L^2(0,\tau^*; H^1(0,1))$ with $0 < \tau^* \leq \tau$ small enough and we show that it is possible to iterate this process until reaching time τ. Let E_τ be the closed convex set defined by

$$E_\tau = \{\phi \in L^2(0,\tau; H^1(0,1)); 0 < a \leq \phi \leq b,\ \phi(0,x) = T_0,\ \phi(t,0) = T_{in}(t), \partial_x \phi(t,1) = 0\}$$

where the reals a and b are defined further. The space $L^2(0,\tau; H^1(0,1))$ is equipped with the classical norm

$$|||f||| = ||f||_{L^2} + ||\partial_x f||_{L^2}.$$

We introduce the mapping

$$\Phi\ :\ E_\tau \to L^\infty(0,\tau; H^1(0,1)) \cap L^2(0,\tau; H^2(0,1))$$

by setting $\Phi(S) = T$ where T is the unique weak solution of

$$\begin{aligned}\partial_t T - \beta\partial^2_{xx}T &= -k\,p_0\,\partial_x u - B\,(q - q^*(p_0/S)),\\ T(t,0) &= T_{in}(t) > 0,\ t\in(0,\tau),\\ T(0,x) &= T_0,\ x\in(0,1),\\ \partial_x T(t,1) &= 0,\ t\in(0,\tau)\end{aligned} \tag{33}$$

with q solution of

$$\begin{aligned}\partial_t q + A\,q &= A\,q^*(p_0/S),\\ q(0,x) &= q_0(x),\ x\in(0,1).\end{aligned} \tag{34}$$

Using Proposition 4.2 with $D = T - T_{in}$ (notice that $D_0 = T_0 - T_{in}(0) = 0$) we get the following estimates on T

$$||T - T_{in}||^2_{L^2} \le \tau\, e^{\tau}||f||^2_{L^2}$$

where we have set

$$f = -k\,p_0\,\partial_x u - B\,(q - q^*(p_0/S)) - T'_{in} \tag{35}$$

and thus

$$||T||^2_{L^2} \le 2\,\tau\, e^{\tau}||f||^2_{L^2} + 2\,||T_{in}||^2_{L^2(0,\tau)}, \tag{36}$$

$$||\partial_t T||^2_{L^2} \le 2||f||^2_{L^2} + 2\,||T'_{in}||^2_{L^2(0,\tau)}, \tag{37}$$

$$||\partial^2_{xx}T||^2_{L^2} \le \frac{1}{\beta^2}||f||^2_{L^2}, \tag{38}$$

$$\sup_{t\in(0,\tau)} ||\partial_x T||^2_{L^2(0,1)}(t) \le \frac{1}{\beta}||f||^2_{L^2}. \tag{39}$$

We also need L^2 estimates for q and q^*.

Lemma 4.1 *Let $\tau > 0$, $A > 0$, S a measurable function satisfying $S \ge a > 0$, $q_0 \in L^2(0,1)$ and q a solution of*

$$\partial_t q + A\,q = A\,q^*(p_0/S)$$

with $q(0,x) = q_0(x)$ and q^* C_q-lipschitzian then for all $t \in (0,\tau]$:

$$\left|\left|q^*\left(\frac{p_0}{S}\right)\right|\right|^2_{L^2_t} \le t\frac{C_q^2 p_0^2}{a^2}, \tag{40}$$

$$||q||^2_{L^2_t} \le t\left(||q_0||^2_{L^2(0,1)} + \tau\,\frac{A\,C_q^2 p_0^2}{2\,a^2}\right). \tag{41}$$

Proof: We have

$$\partial_t q + Aq = Aq^*(p_0/S)$$

with $q(0,x) = q_0(x)$. We multiply by $2q$ and integrate on $(0,1)$ then

$$\frac{d}{dt}\int_0^1 q^2\,dx + 2A\int_0^1 q^2\,dx = 2A\int_0^1 q\,q^*(p_0/S)\,dx.$$

We integrate with respect to time on $(0,\sigma)$ where $\sigma \in (0,t)$, then

$$\int_0^1 q^2(\sigma,x)dx + 2A\int_0^\sigma\int_0^1 q^2(s,x)\,dx\,ds = 2A\int_0^\sigma\int_0^1 q\,q^*(p_0/S)\,dx\,ds + \int_0^1 q_0^2(x)\,dx.$$

But $2\,A\,q\,q^* \le 2\,A\,q^2 + \dfrac{A}{2}(q^*)^2$ then

$$\begin{aligned}||q||^2_{L^2(0,1)}(\sigma) &= \int_0^1 q^2(\sigma,x)dx\\ &\le \int_0^1 q_0^2\,dx + \frac{A}{2}||q^*(p_0/S)||^2_{L^2_t}.\end{aligned}$$

q^* is C_q-lipschitzian then

$$\begin{aligned} ||q^*(p_0/S)||^2_{L^2_t} &= \int_0^t \int_0^1 (q^*)^2(p_0/S)\, dx\, ds \\ &\leq t\, C_q^2\, \frac{p_0^2}{a^2} \end{aligned}$$

namely (40). Finally

$$||q||^2_{L^2(0,1)}(\sigma) \leq \int_0^1 q_0^2\, dx + \tau \frac{AC_q^2 p_0^2}{2a^2}$$

and

$$||q||^2_{L^2_t} \leq t \left(||q_0||^2_{L^2(0,1)} + \tau\, \frac{A\, C_q^2 p_0^2}{2\, a^2} \right)$$

namely (41). □

We will prove that for T_{in} large enough and a convenient choice of a and b we have $\phi(E_\tau) \subset E_\tau$. Next we will show that, for a time $\tau^* \leq \tau$, ϕ is stictly contracting on E_{τ^*}. Then, thanks to Schauder's theorem, there exists a unique fixed point T_{τ^*} for ϕ in E_{τ^*} and this fixed point is solution of

$$\partial_t T - \beta\, \partial^2_{xx} T = -k\, p_0\, \partial_x u - B\, (q - q^*(\rho))$$

on $[0, \tau^*] \times [0, 1]$ with $\rho = \dfrac{p_0}{T}$. In a first step we have to find a, b such that if $0 < a \leq S \leq b$ then $a \leq T = \Phi(S) \leq b$. For this, we are looking for L^∞ estimates on T. First, we have

$$\begin{aligned} |T(t,x) - T_{in}(t)| &= |T(t,x) - T(t,0)| \\ &= \left| \int_0^x (\partial_x T)(t,s)\, ds \right| \\ &\leq ||\partial_x T||_{L^2(0,1)}(t). \end{aligned} \tag{42}$$

Next, using (27) with $D = T - T_{in}$ and f defined in (35) we have

$$\begin{aligned} ||\partial_x T||^2_{L^2(0,1)}(t) &\leq \underbrace{||T_0'||^2_{L^2}}_{0} + \frac{1}{\beta}||f||^2_{L^2} = \frac{1}{\beta}||f||^2_{L^2} \\ &\leq \frac{4}{\beta}\left(k^2 p_0^2 ||\partial_x u||^2_{L^2} + B^2\, ||q||^2_{L^2} + B^2 ||q^*(p_0/S))||^2_{L^2} + ||T_{in}'||^2_{L^2(0,\tau)} \right) \\ &\leq \frac{4}{\beta}\Big(k^2 p_0^2 ||\partial_x u||^2_{L^2} + B^2 \tau (||q_0||^2_{L^2} + \frac{C_q^2 p_0^2}{a^2}(1 + \frac{A\tau}{2})) \\ &\quad + ||T_{in}'||^2_{L^2(0,\tau)} \Big) \end{aligned} \tag{43}$$

In the sequel, the total pressure p_0 will have to be adjusted in order to ensure the contracting property of Φ and we assume that $p_0 \leq p_M$ where $p_M > 0$ is some arbitrary constant.

Setting

$$\delta(a) = \left[\frac{4}{\beta} \left(k^2 p_M^2\, ||\partial_x u||^2_{L^2} + B^2 \tau ||q_0||^2_{L^2} + \frac{C_q^2 p_M^2}{a^2}(1 + \frac{A\tau}{2})) + ||T_{in}'||^2_{L^2(0,\tau)} \right) \right]^{1/2}$$

we get

$$T_{in}(t) - \delta(a) \leq T(t,x) \leq \delta(a) + T_{in}(t).$$

The real number $a > 0$ has to be defined such that $T_{in}(t) \geq a$ and $T_{in}(t) - \delta(a) \geq a$ for all $t \in (0, \tau)$, i.e.

$$T_{in}(t) \geq a \quad \text{ and } \quad (T_{in}(t) - a)^2 - C - \frac{d}{a^2} \geq 0$$

where we have set

$$C = \frac{4}{\beta}\left(k^2 ||p_M\, \partial_x u||^2_{L^2} + \tau\, B^2\, ||q_0||^2_{L^2} + ||T_{in}'||^2_{L^2(0,\tau)} \right) > 0$$

and

$$d = \frac{4B^2 \tau\, C_q^2 p_M^2}{\beta} \left(1 + \tau\, \frac{A}{2} \right) > 0.$$

Lemma 4.2 *There exists a real number $T_m > 0$ which depends on the various constants of the problem (except p_0) and the intial-boundary data such that if $T_{in} \geq T_m$ then there exists a real number $a > 0$ independant of p_0 such that*

$$T_{in} \geq a \text{ and } T_{in} - \delta(a) \geq a.$$

Proof: we search a under the form $a = \lambda T_m, \lambda \in]0,1[$ where T_m has to satisfy

$$(T_m - a)^2 - C - \frac{d}{a^2} \geq 0$$

i.e. $\varphi(\lambda) \geq 0$ where we have set $\varphi(\lambda) = T_m^4 \lambda^2 (1-\lambda)^2 - C T_m^2 \lambda^2 - d$.
We have $\varphi'(\lambda) = 2\lambda T_m^4 P(\lambda)$ where $P(\lambda) = 2\lambda^2 - 3\lambda + 1 - \dfrac{C}{T_m^2} = 2(\lambda - \lambda_1)(\lambda - \lambda_2)$ has two real roots λ_1, λ_2 such that, provided $1 - \dfrac{C}{T_m^2} > 0$:

$$0 < \lambda_1 = \frac{1}{4}(3 - \sqrt{\Delta}) < \frac{1}{2} < 1 < \lambda_2 = \frac{1}{4}(3 + \sqrt{\Delta})$$

with $\Delta = 1 + \dfrac{8C}{T_m^2} > 1$. We have $\varphi(0) = -d < 0$, $\varphi(1) = -CT_m^2 - d < 0$ and φ reaches its maximum on $]0,1[$ for $\lambda = \lambda_1$, thus a exists if and only if $\varphi(\lambda_1) \geq 0$.

In order to get a lower bound $\lambda_1^* > 0$ for λ_1 we choose T_m such that $T_m^2 - \mu C \geq 0$ for some fixed $\mu > 1$ and we set

$$T_m^{(1)} = \sqrt{\mu C},$$
$$\lambda_1^* = \frac{1}{4}\left(3 - \sqrt{1 + \frac{8}{\mu}}\right) > 0.$$

Now, using $P(\lambda_1) = 0$ and $\lambda_1^* \leq \lambda_1 < \dfrac{1}{2}$ we have

$$\begin{aligned}
\varphi(\lambda_1) &= T_m^4 \lambda_1^2 \left(\lambda_1^2 - 2\lambda_1 + 1 - \frac{C}{T_m^2}\right) - d \\
&= T_m^4 \lambda_1^2 \left(\lambda_1^2 - 2\lambda_1 - 2\lambda_1^2 + 3\lambda_1\right) - d \\
&= T_m^4 \lambda_1^3 (1 - \lambda_1) - d \\
&> T_m^4 \frac{(\lambda_1^*)^3}{2} - d.
\end{aligned}$$

Thus $\varphi(\lambda_1) > 0$ as soon as $T_m \geq T_m^{(2*)}$ where $(T_m^{(2*)})^4 = \dfrac{2d}{(\lambda_1^*)^3}$.
Finally we set $T_m = \max(T_m^{(1)}, T_m^{(2*)})$ and $a = \lambda_1 T_m$. □

It remains to define b by setting for instance $b = ||T_{in}||_\infty + \delta(a)$. Notice that a, b, T_m are independant of p_0. With $T_{in} \geq T_m$ defined in Lemma 4.2 and the corresponding choices of a, b we get $\Phi(E_\tau) \subset E_\tau$.

Lemma 4.3 *There exists a time $\tau^* \leq \tau$ such that the application ϕ is contracting on E_{τ^*}.*

Proof: let $S_1 \in E_{\tau^*}$, $S_2 \in E_{\tau^*}, T_1 = \phi(S_1), T_2 = \phi(S_2)$. Let q_1 be associated with T_1. Let q_2 be associated with T_2. We put $D = T_2 - T_1$ and $Q = q_2 - q_1$. Then D, Q satisfy

$$\partial_t D - \beta \partial_{xx}^2 D = -B\,Q - B(q^*(p_0/S_2) - q^*(p_0/S_1)),$$

$$\partial_t Q + A\,Q = A(q^*(p_0/S_2) - q^*(p_0/S_1)) \tag{44}$$

with

$$D(t,0) = 0,\ t \in (0,\tau),$$
$$\partial_x D(t,1) = 0,\ t \in (0,\tau),$$
$$D(0,x) = 0, 0 \leq x \leq 1,$$
$$Q(0,x) = 0, 0 \leq x \leq 1.$$

In order to estimate $||Q||^2_{L^2}$ we mutliply Equation (44) by $2\,Q$ and we integrate on $(0,1)$. We obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 Q^2 dx + 2\,A\int_0^1 Q^2\,dx &= 2\,A\int_0^1 Q(q^*(p_0/S_2) - q^*(p_0/S_1))\,dx \\ &\leq 2\,A\int_0^1 Q^2\,dx + \frac{A}{2}\int_0^1 (q^*(p_0/S_2) - q^*(p_0/S_1))^2\,dx\end{aligned}$$

then

$$\frac{d}{dt}\int_0^1 Q^2 dx \leq \frac{A}{2}\int_0^1 (q^*(p_0/S_2) - q^*(p_0/S_1))^2\,dx.$$

We integrate in time then

$$\int_0^1 Q^2 dx \leq \frac{A}{2}||q^*(p_0/S_2) - q^*(p_0/S_1)||^2_{L^2(0,1)}.$$

But q^* is C_q-lipschitzian then

$$\begin{aligned}|q^*(p_0/S_2) - q^*(p_0/S_1)| &\leq C_q p_0\left|\frac{1}{S_2} - \frac{1}{S_1}\right| \\ &\leq \frac{C_q\,p_0}{a^2}|S_2 - S_1|\end{aligned}$$

because $a^2 \leq |S_1 S_2|$. Then

$$||q^*(p_0/S_2) - q^*(p_0/S_1)||^2_{L^2(0,1)} \leq \frac{C_q^2}{a^4}||p_0(S_2 - S_1)||^2_{L^2} \tag{45}$$

and then

$$\int_0^1 Q^2 dx \leq \frac{A}{2}\frac{C_q^2}{a^4}||p_0(S_2 - S_1)||^2_{L^2}.$$

Finally

$$\forall t \in (0,1], \quad ||Q||^2_{L^2_t} \leq t\,\frac{A}{2}\,\frac{C_q^2}{a^4}||p_0(S_2 - S_1)||^2_{L^2}. \tag{46}$$

Using Estimate (27) with $D_0 = 0$ and

$$f = -B\,Q + B(q^*(p_0/S_2) - q^*(p_0/S_1)) \tag{47}$$

we obtain, using (45) and (46)

$$\begin{aligned}||\partial_x D||^2_{L^2_t} &\leq \frac{t}{\beta}||h||^2_{L^2} \\ &\leq t\,\frac{2B^2}{\beta}\left(||Q||^2_{L^2} + ||q^*(p_0/S_2) - q^*(p_0/S_1)||^2_{L^2}\right) \\ &\leq t\frac{2B^2}{\beta\,a^4}\left(\tau\,\frac{A}{2} + 1\right)C_q^2||p_0(S_2 - S_1)||^2_{L^2} \\ &\leq t\frac{2B^2}{\beta\,a^4}\left(\tau\,\frac{A}{2} + 1\right)C_q^2||p_0(\partial_x(S_2 - S_1))||^2_{L^2}\end{aligned}$$

and we have also $||D||^2_{L^2} \leq ||\partial_x D||^2_{L^2}$. It remains to choose $t \leq \tau$ such that

$$t\,\frac{2B^2}{\beta\,a^4}\left(\tau\,\frac{A}{2} + 1\right)C_q^2 < 1.$$

Then there exists a time $\tau^* \leq \tau$ such that Φ is strictly contracting on E_{τ^*} and thus there exists a unique fixed point for Φ, denoted T_{τ^*} which is the unique solution of (14) where $\rho = \dfrac{q}{T}$, q satisfying (13). □

To obtain a global solution, it remains to show that we can iterate the above process on $[\tau^*, 2\tau^*]$ and so on until reaching the final time τ.

Let $\overline{S} \in L^2((\tau^*, \tau\,); H^1(0,1))$ be a function satisfying the boundary conditions of the problem, the initial condition $\overline{S}(\tau^*, \cdot) = T(\tau^*, \cdot)$ on $(0,1)$ and $a \leq \overline{S} \leq b$. Let $\overline{T}$ the solution of (33) on $(\tau^*, \tau) \times (0,1)$ with $\overline{S}$ instead of S, with the same initial and boundary conditions

as $\overline{S}$.
Let S, resp. T, the function equal to T_{τ^*} for $t \in [0, \tau^*)$ and equal to $\overline{S}$, resp. $\overline{T}$, for $t \in [\tau^*, \tau)$. Let q be the solution of (34) with $q(0, \cdot) = q_0$. Then S, T, q satisfy (33), (34) for $t \in (0, \tau)$ with the initial and boundary conditions of the problem and the estimate (43) is valid on $(0, \tau)$, thus on (τ^*, τ). This allows us to get $a \leq \overline{T} \leq b$, to iterate the process on $(\tau^*, \max(2\tau^*, \tau))$ and finally to get a solution on $(0, \tau)$.

In order to obtain the estimates of additional regularity under the assumptions $(H_u 2)$ and (CC), we set, using the notations of Proposition 4.3

$$h = -k\, p_0\, \partial_x u - B(q - q^*(p_0/T)) - T'_{in}(t), \tag{48}$$

$$G = T - T_{in},$$
$$G_0 = T_0 - T_{in}(0) = 0,$$
$$g = h(0, .) = -k\, p_0\, \partial_x u(0, \cdot) - B(q_0 - q^*(p_0/T_0)) - T'_{in}(0)$$

where (q, T) is the solution of (22)-(23). Then we have juste to apply Proposition (4.3) and we get

$$||\partial_t T||^2_{L^2} \leq 2||h||^2_{L^2} + 2||T'_{in}||^2_{L^2(0,\tau)}, \tag{49}$$

$$||\partial^2_{tt} T||^2_{L^2} \leq 2\left(||\partial_t h||^2_{L^2} + \beta||g'||^2_{L^2(0,1)} + ||T''_{in}||^2_{L^2(0,\tau)}\right), \tag{50}$$

$$||\partial_t \partial^2_{xx} T||^2_{L^2} \leq \frac{1}{\beta^2}||\partial_t h||^2_{L^2} + \frac{1}{\beta}||g'||^2_{L^2(0,1)}, \tag{51}$$

$$\sup_{t\in(0,\tau)} ||\partial_t \partial_x T||^2_{L^2(0,1)}(t) \leq \frac{1}{\beta}||\partial_t h||^2_{L^2} + ||g'||^2_{L^2(0,1)}. \tag{52}$$

In the sequel, C denotes various constants depending only on initial and boundary data. First we have, with (48)

$$\begin{aligned} ||h||^2_{L^2} &\leq 3\left(k^2\, ||p_0\, \partial_x u||^2_{L^2} + B^2||q - q^*(p_0/T)||^2_{L^2} + ||T'_{in}||^2_{L^2(0,\tau)}\right) \\ &\leq C\,(1 + ||p_0\, \partial_x u||^2_{L^2}) \end{aligned}$$

or, as well

$$||h||^2_{L^2} \leq C\,(1 + ||p_0\, \partial_t \partial_x u||^2_{L^2}),$$

and we get

$$||\partial_t T||^2_{L^2} \leq C\,(1 + ||p_0\, \partial_t \partial_x u||^2_{L^2}). \tag{53}$$

Next we have

$$\partial_t h = -k p_0 \partial_t \partial_x u + AB(q - q^*(p_0/T)) - B(q^*)'(p_0/T)\frac{p_0}{T^2}\partial_t T - T''_{in}(t)$$

and thus

$$||\partial_t h||^2_{L^2} \leq 4\left(k^2\, ||p_0\, \partial_t \partial_x u||^2_{L^2} + A^2B^2||q - q^*(p_0/T)||^2_{L^2} + p_0^2\frac{B^2C_q^2}{a^2}||\partial_t T||^2_{L^2} + ||T''_{in}||^2_{L^2(0,\tau)}\right).$$

Finally, with (53)

$$||\partial_t h||^2_{L^2} \leq C\,(1 + ||p_0\, \partial_t \partial_x u||^2_{L^2}). \tag{54}$$

Using these estimates in (50), (51) and (52) we get similarly

$$||\partial^2_{tt} T||^2_{L^2} \leq C\,(1 + ||p_0\, \partial_t \partial_x u||^2_{L^2}), \tag{55}$$

$$||\partial_t \partial^2_{xx} T||^2_{L^2} \leq C\,(1 + ||p_0\, \partial_t \partial_x u||^2_{L^2}), \tag{56}$$

$$\sup_{t\in(0,\tau)} ||\partial_t \partial_x T||^2_{L^2(0,1)} \leq C\,(1 + ||p_0\, \partial_t \partial_x u||^2_{L^2}). \tag{57}$$

The L^∞ bounds for $\partial_t T$ and $\partial_x T$ follow easily. First

$$\begin{aligned} |\partial_t T(t,x)| &\leq ||T'_{in}||_\infty + \int_0^1 |\partial_t \partial_x T(t,\xi)|\, d\xi \\ &\leq ||T'_{in}||_\infty + \sup_t ||\partial_t \partial_x T||_{L^2(0,1)}(t) \\ &\leq C\,(1 + ||p_0\,\partial_t \partial_x u||^2_{L^2})^{1/2}. \end{aligned} \tag{58}$$

Next we have, with T_0 constant

$$\partial^2_{xx} T = \int_0^\tau \partial_t \partial^2_{xx} T(s,x)\, ds,$$

$$(\partial^2_{xx} T)^2 \leq \tau\, ||\partial_t \partial^2_{xx} T||^2_{L^2_{(0,\tau)}}(x)$$

and thus, integrating over $(0,1)$ with respect to x we get, thanks to (56), $\partial^2_{xx} T \in L^\infty(0,\tau\,;\, L^2(0,1))$ with the estimate

$$\sup_{t\in(0,\tau)} ||\partial^2_{xx} T||^2_{L^2(0,1)}(t) \leq C\,\tau\,(1 + ||p_0\,\partial_t \partial_x u||^2_{L^2}).$$

Finally we write

$$\begin{aligned} |\partial_x T(t,x)| &\leq |\partial_x T(t,1)| + \int_1^x |\partial^2_{xx} T(t,\xi)|\, d\xi \\ &\leq \int_0^1 |\partial^2_{xx} T(t,\xi)|\, d\xi \\ &\leq \sup_t ||\partial^2_{xx} T||_{L^2(0,1)}(t) \\ &\leq \tau^{1/2}\, ||\partial_t \partial^2_{xx} T||^2_{L^2} \\ &\leq C\,\tau^{1/2}\,(1 + ||p_0\,\partial_t \partial_x u||^2_{L^2})^{1/2}. \end{aligned} \tag{59}$$

This concludes the proof of Proposition 4.1. □

Remark 4.1 *Under the assumptions $(H_u 2)$ and (CC), using the L^∞ estimate (58) we get immediately new estimates for $\partial_t T$ and $\partial_x T$*

$$||\partial_t T||_{L^2} \leq C\,\tau\,(1 + ||p_0\,\partial_t \partial_x u||^2_{L^2}), \tag{60}$$

$$\sup_{t\in(0,\tau)} ||\partial_x T||^2_{L^2(0,1)}(t) \leq C\,\tau\,(1 + ||p_0\,\partial_t \partial_x u||^2_{L^2}). \tag{61}$$

4.3 Proof of Theorem 4.1

The proof is based on a fixed point method for the velocity u, using Equation (12). More precisely, for a velocity u given in a closed convex set $\mathcal{U}_\tau$ defined below, with q, T the solutions of (13)-(14) and $\rho = p_0/T$, we define $v = \psi(u)$ as the solution of

$$\partial_x(\rho\, v) = -\partial_t \rho + A(q - q^*(\rho)),$$

$$v(t,0) = u_{in}(t)$$

and we show that, for p_0 small enough, ψ is contracting for an appropriate norm and for a small final time. Then we iterate the process as in the proof of Proposition 4.1.
The previous equation gives succesively

$$\begin{aligned} p_0\, v(t,x) &= \rho\, T(t,x)\, v(t,x) \\ &= T(t,x)\,(\rho\, v)(t,0) + T(t,x) \int_0^x \partial_x(\rho\, v(t,\xi))\, d\xi \\ &= T(t,x)\,(\rho_{in}\, u_{in})(t) + T(t,x) \int_0^x (-\partial_t \rho + A(q - q^*(\rho)))\, d\xi \\ &= T(t,x)\,(\rho_{in}\, u_{in})(t) + T(t,x) \int_0^x (-\partial_t (p_0/T) + A(q - q^*(p_0/T)))\, d\xi \\ &= T(t,x)\,(\rho_{in}\, u_{in})(t) + T(t,x) \left(\int_0^x p_0 \frac{\partial_t T}{T^2}(t,\xi) d\xi + \int_0^x (A(q - q^*(p_0/T))(t,\xi)\, d\xi \right) \end{aligned}$$

and

$$p_0\,\partial_x v(t,x) = (\rho_{in}\,u_{in})(t)\,\partial_x T(t,x) + \partial_x T(t,x)\int_0^x \left[A(q-q^*(p_0/T)) + p_0\frac{\partial_t T}{T^2}\right](t,\xi)\,d\xi$$
$$+T(t,x)\,A(q-q^*(p_0/T))(t,x) + p_0\frac{\partial_t T}{T}(t,x). \tag{62}$$

Assuming $T(0,\cdot) = T_0$ constant on $(0,1)$ we have

$$p_0\,\partial_x v(0,x) = T_0\,A(q_0(x) - q^*(p_0/T_0)) + p_0\frac{\partial_t T(0,x)}{T_0}.$$

Moreover, the compatibility conditions being fulfilled, we have

$$\begin{aligned}\partial_t T(0,x) &= \beta T_0'' - kp_0\,\partial_x u(0,x) - B\,(q_0(x) - q^*(p_0/T_0))\\ &= -kp_0\,\partial_x u(0,x) - B\,(q_0(x) - q^*(p_0/T_0))\end{aligned}$$

and finally

$$\partial_x v(0,x) = \left(\frac{AT_0}{p_0} - \frac{B}{T_0}\right)(q_0(x) - q^*(p_0/T_0)) - \frac{kp_0}{T_0}\,\partial_x u(0,x), \tag{63}$$

thus the fixed point u, if it exists, satisfies

$$\partial_x u(0,x) = \frac{\left(\frac{AT_0}{p_0} - \frac{B}{T_0}\right)}{1+\frac{kp_0}{T_0}}(q_0(x) - q^*(p_0/T_0)). \tag{64}$$

Using this expression of $\partial_x u(0,x)$ we get, with the notations of Proposition 4.3

$$g = g_u = \left(-B - \frac{kp_0}{1+kp_0/T_0}\left(\frac{AT_0}{p_0} - \frac{B}{T_0}\right)\right)(q_0(x) - q^*(p_0/T_0)) - T_{in}'(0)$$

and the asumptions of Theorem 4.1 ensure the compatility conditions. Of course, with this choice, $\partial_x v(0,x) = \partial_x \psi(u)(0,x) = \partial_x u(0,x)$ and the compatibility conditions remain valid with v.

Now, let $M > 0$, $M_1 > 0$, $u_m > 0$ and

$$\mathcal{U}_\tau = \Big\{u \in L^\infty(0,\tau;H^1(0,1)),\ \partial_t\partial_x u \in L^2,\ u(t,0) = u_{in}(t) \geq u_m > 0,$$

$\partial_x u(0,x)$ satifying (64) on $(0,1)$, $\sup\limits_{t\in(0,\tau)} ||p_0\partial_x u||^2_{L^2(0,1)}(t) \leq M$, $\sup\limits_{t\in(0,\tau)} ||p_0\,\partial_t\partial_x u||^2_{L^2(0,1)}(t) \leq M_1\Big\}$.

The constants T_m, a, b introduced in the proof of Proposition 4.1 are now associated to the constant M in place of $||p_0 M\partial_x u||^2_{L^2(0,1)}$and thus no longer depend on the velocity $u \in \mathcal{U}_\tau$.

We recall that for all T associated to some $u \in \mathcal{U}_\tau$ we have $0 < a \leq T < b$ as soon as $T_{in} \geq T_m$.

Moreover the various estimates stated in the proof of Proposition 4.1 are used in the sequel with $||p_0\partial_x u||^2_{L^2}$ and $||p_0\,\partial_t\partial_x u||^2_{L^2}$ respectively replaced by M and M_1. The minimum input velocity u_m must be defined such as $u(t,1) > 0$, but we have

$$\begin{aligned}u(t,1) &= u_{in}(t) + \int_0^1 \partial_x u(t,\xi)\,d\xi\\ &\geq u_m - \int_0^1 |\partial_x u(t,\xi)|\,d\xi\\ &\geq u_m - \sup_t ||\partial_x u||_{L^2(0,1)}(t)\\ &\geq u_m - \frac{\sqrt{M}}{p_0}\end{aligned}$$

thus we choose u_m such that

$$u_m > \frac{\sqrt{M}}{p_0}. \tag{65}$$

In all the sequel, C denotes various constants depending only on the constants of the problem, including τ, M, M_1 and on the initial and boundary data via appropriate norms.

Lemma 4.4 *If p_0 is small enough then there exists $\tau_1 \in (0,\tau]$ such that for all $u \in \mathcal{U}_{\tau_1}$ we have $||p_0 \partial_x v||^2_{L^2_{\tau_1}} \leq M$.*

Proof: using (62) we can write

$$p_0\, \partial_x v = A_1 + A_2 + A_3 + A_4$$

with

$$A_1 = (\rho_{in}\, u_{in})(t)\, \partial_x T,$$
$$A_2 = \partial_x T \int_0^x r(t, \xi\, d\xi$$

where

$$r = A(q - q^*(p_0/T)) + p_0 \frac{\partial_t T}{T^2},$$
$$A_3 = T\, A(q - q^*(p_0/T)),$$
$$A_4 = p_0 \frac{\partial_t T}{T}.$$

Then

$$\forall t \in (0,\tau], \quad ||p_0\, \partial_x v||^2_{L^2_t} \leq 4(||A_1||^2_{L^2_t} + ||A_2||^2_{L^2_t} + ||A_3||^2_{L^2_t} + ||A_4||^2_{L^2_t}).$$

We have first, thanks to Estimate (39)

$$\begin{aligned} ||A_1||^2_{L^2_t} &= ||u_{in}\, \rho_{in}\, \partial_x T||^2_{L^2_t} \\ &\leq ||u_{in}\, \rho_{in}||^2_\infty ||\partial_x T||^2_{L^2} \\ &\leq C\, t \end{aligned}$$

Next we have

$$\begin{aligned} ||A_2||^2_{L^2_t} &= ||\partial_x T \int_0^x r||^2_{L^2_t} \\ &= \int_0^t \int_0^1 (\partial_x T)^2 \left(\int_0^x r \right)^2 \\ &\leq \sup_{s\in(0,\tau)} ||\partial_x T||^2_{L^2(0,1)}(s) ||r||^2_{L^2(0,1)}(t) \end{aligned}$$

then, using (40) and (41)

$$\begin{aligned} ||A_2||^2_{L^2_t} &\leq 3 \sup_{s\in(0,\tau)} ||\partial_x T||^2_{L^2(0,1)}(s) \left(A^2\, ||q||^2_{L^2_t} + A^2 ||q^*(p_0/T)||^2_{L^2_t} + \frac{p_0^2}{a^4} ||\partial_t T||^2_{L^2} \right) \\ &\leq C(t + p_0^2). \end{aligned}$$

Next, with (40) and (41), we have

$$\begin{aligned} ||A_3||^2_{L^2_t} &= ||T\, A(q - q^*(p_0/T))||^2_{L^2_t} \\ &\leq 2\, b^2\, A^2 \left(||q||^2_{L^2_t} + ||q^*(p_0/T))||^2_{L^2_t} \right) \\ &\leq C\, t. \end{aligned}$$

Finally, with (37)

$$\begin{aligned} ||A_4||^2_{L^2_t} &= ||p_0 \frac{\partial_t T}{T}||^2_{L^2_t} \\ &\leq \frac{p_0^2}{a^2} ||\partial_t T||^2_{L^2} \\ &\leq C p_0^2 \end{aligned}$$

and we can write finally

$$||p_0\, \partial_x v||^2_{L^2_t} \leq C\, (t + p_0^2).$$

If, for instance, $p_0 \leq \dfrac{M}{2C}$, we set $\tau_1 = \dfrac{M}{2C}$ and we get finally $||p_0 \partial_x v||^2_{L^2_{\tau_1}} \leq M$ for $u \in \mathcal{U}_{\tau_1}$. □

Lemma 4.5 *If p_0 is small enough then there exists $\tau_2 \in (0,\tau]$ such that for all $u \in \mathcal{U}_{\tau_2}$ we have $||p_0\partial_t\partial_x v||^2_{L^2_{\tau_2}} \leq M_1$.*

Proof: we have

$$p_0\partial_t\partial_x v = \sum_{i=1}^{8} B_i \tag{66}$$

where

$$B_1 = (u_{in}\,\rho_{in})'(t)\partial_x T,$$
$$B_2 = (u_{in}\,\rho_{in})(t)\partial_t\partial_x T,$$
$$B_3 = \partial_t\partial_x T \int_0^x r$$

where again

$$r = A(q - q^*(p_0/T)) + p_0\frac{\partial_t T}{T^2},$$
$$B_4 = \partial_x T \int_0^x \partial_t r,$$
$$B_5 = A(q - q^*(p_0/T))\,\partial_t T,$$
$$B_6 = T\partial_t r,$$
$$B_7 = \frac{p_0}{T}\,\partial^2_{tt} T,$$
$$B_8 = -\frac{p_0}{T^2}\,(\partial_t T)^2.$$

Under the asumptions of Theorem 4.1, the compatibility conditions are satisfied and we will have to use the various estimates (53) to (61) with $||p_0\partial_t\partial_x u||^2_{L^2((0,\tau^*]\times(0,1))} \leq M_1$.

With (59) we get first

$$||B_1||^2_{L^2_t} \leq ||(u_{in}\,\rho_{in})'||^2_\infty\,||\partial_x T||^2_{L^2_t} \leq C\,t.$$

Next, with (57)

$$||B_2||^2_{L^2_t} \leq ||u_{in}\,\rho_{in}||^2_\infty\,||\partial_t\partial_x T||^2_{L^2_t} \leq C\,t.$$

For B_3 we proceed as above for A_2 and we get

$$||B_3||^2_{L^2_t} \leq \sup_{s\in(0,\tau)} ||\partial_t\partial_x T||^2_{L^2(0,1)}(s)\,||r||^2_{L^2_t}.$$

Using (40), (41) and (60) we have finally

$$||B_3||^2_{L^2_t} \leq C\,t.$$

Similar estimates are also obtained easily for $||B_4||^2_{L^2}$, $||B_5||^2_{L^2}$ and $||B_6||^2_{L^2}$.
For B_7, using (55)

$$||B_7||^2_{L^2_t} \leq C\,p_0^2.$$

For B_8, using (58)

$$||B_8||^2_{L^2_t} \leq C\,t\,p_0^2.$$

Finally

$$||p_0\partial_t\partial_x v||^2_{L^2_t} \leq C\,(t + p_0^2)$$

and we conclude as for $||p_0\partial_x v||^2_{L^2_t}$ in Lemma 4.4. □

From Lemma 4.4 and Lemma 4.5 we deduce that for p_0 small enough there exists $\tau' = \min(\tau_1,\tau_2) \leq \tau$ such that $\psi(\mathcal{U}_{\tau'}) \subset \mathcal{U}_{\tau'}$. Now we show that for ψ is contracting on a subset of $\mathcal{U}_{\tau'}$ under some conditions. More precisely

Lemma 4.6 *For p_0 small enough there exists a time $\tau^* \leq \tau$ such that the application ψ is contracting on $\mathcal{U}_{\tau^*}$ for the norm*

$$||u|| = ||u(t,0)||_{L^2(0,\tau^*)} + ||p_0\partial_x u||_{L^2_{\tau^*}}.$$

Proof: Let $u_1, u_2 \in \mathcal{U}_{\tau'}$, let T_1, T_2, q_1, q_2 associated to u_1, u_2 respectiveley, with the same boundary conditions and $u_1(0,.) = u_2(0,.)$. We set

$$\rho_1 = \frac{p_0}{T_1}, \rho_2 = \frac{p_0}{T_2},\ \psi(u_1) = v_1, \psi(u_2) = v_2.$$

As $v_1(t,0) = v_2(t,0) = u_{in}(t)$, we have just to control $||p_0\,\partial_x(v_2 - v_1)||^2_{L^2_t}$ for $0 < t \leq \tau'$.

Setting

$$r_1 = A(q_1 - q^*(p_0/T_1)) + p_0\frac{\partial_t T_1}{T_1^2},$$

$$r_2 = A(q_2 - q^*(p_0/T_2)) + p_0\frac{\partial_t T_2}{T_2^2},$$

$$D = T_2 - T_1$$

we have

$$p_0\,\partial_x(v_2 - v_1) = A_1 + A_2 + A_3 + A_4 + A_5$$

$$A_1 = u_0(t)\,\rho_0\,\partial_x D,$$

$$A_2 = \partial_x D \int_0^x r_2,$$

$$A_3 = \partial_x T_1 \left(\int_0^x (r_2 - r_1)\right),$$

$$A_4 = D r_1,$$

$$A_5 = T_2(r_2 - r_1)$$

and $D = T_2 - T_1$ satisfies

$$k\,\partial_t D - \beta\,\partial^2_{xx} D = p_0\,(\partial_x u_2 - \partial_x u_1) + B\,(q_2 - q_1) - B\,(q^*(p_0/T_2) - q^*(p_0/T_1)),$$

$$D(t,0) = 0, t \in (0,\tau),$$

$$\partial_x D(t,1) = 0, t \in (0,\tau),$$

$$D(0,x) = 0, 0 < x < 1.$$

Setting

$$h = p_0\,(\partial_x u_2 - \partial_x u_1) + B\,(q_2 - q_1) - B\,(q^*(p_0/T_2) - q^*(p_0/T_1))$$

thanks to Proposition 4.2 and Equation (43) we have the following estimates

$$||\partial_t D||^2_{L^2_t} \leq ||h||^2_{L^2_t}, \tag{67}$$

$$\sup_{s\in(0,t)} ||\partial_x D||^2_{L^2(0,1)}(s) \leq \frac{1}{\beta}||h||^2_{L^2_t} \tag{68}$$

and the estimates (60)-(61) are valid for T_1 and T_2.

Then

$$||A_1||^2_{L^2_t} \leq \frac{t}{\beta}||u_0\rho_0||^2_\infty ||h||^2_{L^2},$$

$$||A_2||^2_{L^2_t} \leq \frac{t}{\beta}||r_2||^2_{L^2}||h||^2_{L^2},$$

$$\begin{aligned} ||A_3||^2_{L^2_t} &= ||\partial_x T_1 \left(\int_0^x (r_2 - r_1)\right)||_{L^2_t} \\ &\leq \sup_{s\in(0,\tau)} ||\partial_x T_1||_{L^2(0,1)}(s)\,||r_2 - r_1||^2_{L^2_t} \\ &\leq C\,||r_2 - r_1||^2_{L^2_t}, \end{aligned}$$

$$
\begin{aligned}
||A_4||^2_{L^2_t} &= ||Dr_1||^2_{L^2_t} \\
&= \int_0^t \int_0^1 D^2 r_1^2 \\
&\leq \int_0^t \int_0^1 \left(\int_0^x \partial_x D \right)^2 r_1^2 \\
&\leq \int_0^t \left(\int_0^1 (\partial_x D)^2 \right) \int_0^1 r_1^2 \\
&\leq \sup_{0\leq s\leq t} ||\partial_x D||^2_{L^2(0,1)}(s) ||r_1||^2_{L^2_t} \\
&\leq \frac{1}{\beta} ||h||^2_{L^2_t} ||r_1||^2_{L^2_t},
\end{aligned}
$$

$$
||A_5||^2_{L^2} \leq C\, ||r_2 - r_1||^2_{L^2}.
$$

Now, we have to estimate $||r_1||^2_{L^2_t}, ||r_2||^2_{L^2_t}, ||r_2 - r_1||^2_{L^2_t}$ and $||h||^2_{L^2_t}$.
As seen before (proof of Lemma 4.4), we have for $i = 1, 2$

$$
||r_i||^2_{L^2_t} \leq C\,(t + p_0^2).
$$

Next,

$$
r_2 - r_1 = A(q_2 - q_1) - A(q^*(p_0/T_1) - q^*(p_0/T_2)) + p_0 \left(\frac{1}{T_2^2} - \frac{1}{T_1^2} \right) \partial_t T_1 - \frac{\partial_t D}{T_2^2} p_0.
$$

With Equation (46) we have

$$
\begin{aligned}
||q_2 - q_1||^2_{L^2_t} &\leq t\, p_0^2\, \frac{AC_q^2}{2a^4} ||D||^2_{L^2} \\
&\leq t^2\, p_0^2\, \frac{AC_q^2}{2a^4\beta} ||h||^2_{L^2_t}
\end{aligned} \tag{69}
$$

and with (45), where $S_i = T_i$

$$
\begin{aligned}
||q^*(p_0/T_1) - q^*(p_0/T_2)||^2_{L^2_t} &\leq \frac{C_q^2}{a^4} ||p_0\, D||^2_{L^2_t} \\
&\leq t\, p_0^2\, \frac{C_q^2}{a^4\,\beta} ||h||^2_{L^2_t}.
\end{aligned} \tag{70}
$$

For the third term

$$
\begin{aligned}
\left| p_0 \left(\frac{1}{T_2^2} - \frac{1}{T_1^2} \right) \partial_t T_1 \right| &= p_0 \frac{|T_1 + T_2|}{T_1^2 T_2^2} |D||\partial_t T_1| \\
&\leq p_0 \frac{2\,b}{a^4} |D||\partial_t T_1|
\end{aligned}
$$

then, thanks to Equations (68) and (60)

$$
\begin{aligned}
\left|\left| p_0 \left(\frac{1}{T_2^2} - \frac{1}{T_1^2} \right) \partial_t T_1 \right|\right|^2_{L^2_t} &\leq p_0^2 \frac{4b^2}{a^{*8}} \int_0^t \int_0^1 D^2 (\partial_t T_1)^2 \\
&\leq p_0^2\, \frac{4b^2}{a^{*8}} \sup_{s\in(0,\tau)} ||\partial_x D||^2_{L^2(0,1)}(s) ||\partial_t T_1||^2_{L^2_t} \\
&\leq C\, p_0^2\, ||h||^2_{L^2_t}
\end{aligned}
$$

Finally, with Equation (67), we have

$$
||r_2 - r_1||^2_{L^2_t} \leq C\, p_0^2\, ||h||^2_{L^2_t}.
$$

Using again Equations (69) and (70), we have

$$
\begin{aligned}
||h||^2_{L^2_t} &\leq 3 \left(||p_0\, \partial_x (u_2 - u_1)||^2_{L^2_t} + B^2 ||q_2 - q_1||^2_{L^2_t} + B^2\, ||q^*(p_0/T_2) - q^*(p_0/T_1)||^2_{L^2_t} \right) \\
&\leq 3 \left(||p_0\, \partial_x (u_2 - u_1)||^2_{L^2_t} + B^2 \left(t^2\, p_0^2\, \frac{AC_q^2}{2a^4\beta} ||h||^2_{L^2_t} \right) + B^2 \left(t\, p_0^2\, \frac{C_q^2}{a^4\,\beta} ||h||^2_{L^2_t} \right) \right) \\
&\leq 3 \left(||p_0\, (\partial_x (u_2 - \ u_1)||^2_{L^2_t} + K\, p_0^2\, ||h||^2_{L^2_t} \right)
\end{aligned}
$$

with

$$K = \tau\, B^2 \frac{C_q^2}{\beta a^4} \left(1 + \frac{A}{2}\tau\right).$$

Thus, assuming p_0 small enough such that $3Kp_0^2 < 1$, we get

$$||h||^2_{L^2_t} \leq C||p_0\,(\partial_x u_2 - \partial_x u_1)||^2_{L^2_t}$$

and

$$||r_2 - r_1||^2_{L^2_t} \leq C\, p_0^2\, ||p_0\,(\partial_x u_2 - \partial_x u_1)||^2_{L^2_t}.$$

Finally, by gathering the previous estimates we obtain

$$||p_0\,\partial_x(v_2 - v_1)||^2_{L^2_t} \leq C\,(p_0^2 + t)\,||p_0\,(\partial_x u_2 - \partial_x u_1)||^2_{L^2_t}$$

Now, we fix p_0 satisfying all the previous constraints and being such that $C\,p_0^2 < 1$, then there exists a time $\tau^* \leq \tau'$ satisfying $C\,(p_0^2 + \tau^*) < 1$, so that ψ is contracting on $\mathcal{U}_{\tau^*}$.
Let $u \in \mathcal{U}_{\tau^*}$ be the unique fixed point of ψ, let (q, T) be the corresponding solution of (22)-(23) and let be $\rho = \dfrac{p_0}{T}$, then we easily get that $(u\,,\,q\,,\,\rho\,,\,T)$ is the unique solution of the problem (12)–(4) on $(0, \tau^*) \times (0, 1)$. It remains to iterate the process on $(\tau^*, 2\tau^*)$ and so on, until reaching the final time τ. This is achieved in the same spirit than the end of the proof of Proposition 4.1. The important point is that it is possible to use the same estimates as on $(0, \tau^*)$, using only the initial data (it means $t = 0$ and not $t = \tau^*$) and boundary data of the problem. □

5 The case of instantaneous equilibrium

We consider now, as in references [4] to [10], the case of instantaneous equilibrium : formally, when $A \to \infty$, Equation (13) reduces to $q = q^*(\rho)$ and the system of Equations (12)-(13)-(14)-(15) becomes

$$\begin{aligned}
\partial_t(\rho + q^*(\rho)) + \partial_x(u\,\rho) &= 0, && (71)\\
\partial_t T - \beta\,\partial^2_{xx} T &= -k\,p_0\,\partial_x u + \gamma\,\partial_t q^*(\rho), && (72)\\
\rho\, T &= p_0, && (73)
\end{aligned}$$

with the same initial-boundary conditions (IBC), without the one concerning q.
Notice that the problem of the convergence of the solution of System (12)-(13)-(14) towards the solution of System (71)-(72)-(73) is open. We can observe that the key L^2 and H^1 estimates obtained in Section 4 blow up as $A \to \infty$. The same difficulty would arise in [2] in the BV framework due to the same parameter. The corresponding system was sudied in a serie of papers ([4, 5, 7, 8, 9, 10]) but the analogous problem of convergence remains also open.

In the sequel we assume that $q^* \in W^{2,\infty}(\mathbb{R}^+)$.
We will get an existence and uniqueness result for this problem using globally the same procedure as in the previous case but the regularity needed to ensure $u(t, 1) \geq 0$ (in particular $\partial^2_{tt} T \in L^2$) cannot be obtained as in the proof of Proposition 4.1 because the function h in (48) would become

$$\begin{aligned}
h &= -k\,p_0\,\partial_x u + \gamma\,\partial_t q^*(\rho) - T'_{in}(t)\\
&= -k\,p_0\,\partial_x u - \gamma p_0 (q^*)'(p_0/T)\frac{1}{T^2}\partial_t T - T'_{in}(t) && (74)
\end{aligned}$$

and $\partial_t h$ involves $\partial^2_{tt} T$ and $(\partial_t T)^2$. We must therefore take into account this regularity in the space used for the fixed point.
Notice that, subject to sufficient regularity, Equation (72) implies

$$\partial_t T(0, x) = \underbrace{\beta \partial^2_{xx} T_0}_{0} - k\,p_0\,\partial_x u(0, x) - \gamma p_0 (q^*)'(p_0/T_0)\frac{1}{T_0^2}\partial_t T(0, x)$$

and thus

$$\partial_t T(0, x) = \frac{-k\,p_0}{1 + \gamma p_0 (q^*)'(p_0/T_0)\frac{1}{T_0^2}}\,\partial_x u(0, x). \qquad (75)$$

On the other hand, the formula (62) becomes

$$\begin{aligned} p_0\,\partial_x v(t,x) &= (\rho_{in}\,u_{in})(t)\,\partial_x T(t,x) \\ &\quad +\partial_x T(t,x)\int_0^x \left[(q^*)'(p_0/T)\frac{1}{T^2}\partial_t T + p_0\frac{\partial_t T}{T^2}\right](t,\xi)\,d\xi \\ &\quad +p_0\,(1+(q^*)'(p_0/T)\frac{1}{T})\,\partial_t T \end{aligned} \tag{76}$$

thus, for the solution (u,T) we get

$$p_0\,\partial_x u(0,x) = p_0\,(1+(q^*)'(p_0/T_0)\frac{1}{T_0})\,\partial_t T(0,x)$$

that is also

$$\partial_t T(0,x) = \frac{T_0}{1+(q^*)'(p_0/T_0)}\,\partial_x u(0,x). \tag{77}$$

From (75) and (77) we deduce that necessarily $\partial_x u(0,x)=0$ and thus $\partial_t T(0,x)=0$. In addition we get easily $u(0,\cdot)=u_{in}(0)$ on $(0,1)$.
The main result for this problem is

Theorem 5.1 *Let $\tau>0$. We assume $T_0>0$ constant on $(0,1)$, $u_{in}\in W_1^\infty(0,\tau)$ and $T_{in}\in H^2(0,\tau)$ with $T_{in}(0)=T_0$ and $T'_{in}(0)=0$.*

Then there exists some real numbers $T_m>0$, p_m and $u_m>0$ such that if $T_{in}\geq T_m$, $p_0\leq p_m$ and $u_{in}\geq u_m$ then the problem (71)-(72)-(73)-(IBC) has a unique global solution $(u,\,\rho,\,T)$ satisfying

$$u\in L^\infty(0,\tau;L^2(0,1)), \tag{78}$$
$$\partial_x u\in L^\infty(0,\tau\,;\,L^2(0,1))\cap L^2(0,\tau\,;\,H^1(0,1)), \tag{79}$$
$$\partial_t\partial_x u\in L^\infty(0,\tau;L^2(0,1)), \tag{80}$$
$$\partial_t T,\,\partial_x T\in L^\infty, \tag{81}$$
$$\partial_t\partial_x T,\,\partial^2_{xx}T\in L^\infty(0,\tau\,;\,L^2(0,1)), \tag{82}$$
$$\partial^2_{tt}T\in L^2. \tag{83}$$

This solution satisfies $u(0,\cdot)=u_{in}(0)$, $\partial T(0,\cdot)=0$ on $(0,1)$ and there exists two reals numbers a, b such that $0<a\leq T\leq b$.

As a first step we show the following result

Proposition 5.1 *Let $\tau>0$. Assume that $T_0>0$ is constant on $(0,1)$, $T_{in}\in H^2(0,\tau)$ with $T_{in}(0)=T_0$ and $T'_{in}(0)=0$.*
Let u such that

$$(H_u1)\quad \begin{cases} u\in L^\infty(0,\tau;L^2(0,1)), \\ \partial_t\partial_x u\in L^\infty(0,\tau;L^2(0,1)), \\ \partial_x u(0,\cdot)=0 \text{ on } (0,1), \\ u(\cdot,1)>0 \text{ on } (0,\tau). \end{cases}$$

If p_0 is small enough, then there exists a real number $T_m>0$ such that if $T_{in}\geq T_m$ then there exists constants $0<a<b$ and a unique function $T\in L^\infty(0,\tau;H^1(0,1))\cap L^2(0,\tau;H^2(0,1))$ solution of Equation (72) with

$$T(0,\cdot)=T_0 \text{ on } (0,1),$$
$$T(\cdot,0)=T_{in},\ \partial_t T(\cdot,1)=0 \text{ on } (0,\tau),$$

such that $0<a\leq T\leq b$ and

$$\partial_t T,\,\partial_x T\in L^\infty,$$
$$\partial_t\partial_x T,\,\partial^2_{xx}T\in L^\infty(0,\tau;L^2(0,1)),$$
$$\partial^2_{tt}T\in L^2.$$

5.1 Proof of Proposition 5.1

We will start by giving some a priori estimates necessary for the definition of the workspace. If T solves (71)-(72)-(IBC) and satisfies $T \geq a > 0$ where a is some constant independant of p_0, then applying Proposition 4.2 to $D = T - T_{in}$ we get, with h defined in (74)

$$\begin{aligned} ||\partial_t T||^2_{L^2} &\leq 2||h||^2_{L^2} + 2||T'_{in}||^2_{L^2(0,\tau)} \\ &\leq 6\left(k^2||p_0\partial_x u||^2_{L^2} + \frac{p_0^2\gamma^2 C_q^2}{a^4}\,||\partial_t T||^2_{L^2} + ||T'_{in}||^2_{L^2(0,\tau)}\right) + 2||T'_{in}||^2_{L^2(0,\tau)} \end{aligned}$$

where $C_q = ||(q^*)'||_{L^\infty}$. Thus, assuming $p_0 \leq p_M$ where $p_M > 0$ is some arbitrary constant, and for instance

$$\frac{6p_M^2\gamma^2 C_q^2}{a^4} \leq \frac{1}{2}, \tag{84}$$

we get

$$||\partial_t T||^2_{L^2} \leq N_1, \tag{85}$$

where we have set

$$N_1 = 12k^2||p_M\partial_x u||^2_{L^2} + 16\,||T'_{in}||^2_{L^2(0,\tau)}.$$

Let E_τ the convex subset of $H^1((0,\tau)\times(0,1))$ defined by

$$E_\tau = \Big\{\phi \in H^1((0,\tau)\times(0,1))\,;\, \partial_t\phi \in H^1((0,\tau)\times(0,1)),\, 0 < a \leq \phi \leq b,$$

$$\phi(0,x) = T_0,\, \partial_t\phi(0,x) = 0,\, \phi(t,0) = T_{in}(t), \partial_x\phi(t,1) = 0,\, ||\partial_t\phi||^2_{L^2} \leq N_1\,, ||\partial_t\phi||^2_{H^1} \leq N_2\Big\}$$

where the real numbers N_2, a and b are defined further. The space $H^1((0,\tau)\times(0,1))$ is equipped with the classical norm

$$|||\phi||| = ||\phi||_{L^2} + ||\partial_t\phi||_{L^2} + ||\partial_x\phi||_{L^2}.$$

We introduce the mapping

$$\Phi\,:\, E_\tau \to H^1((0,\tau)\times(0,1))$$

by setting $\Phi(S) = T$ where T is the unique weak solution of

$$\begin{aligned} \partial_t T - \beta\partial^2_{xx}T = -k\,p_0\,\partial_x u - \gamma p_0(q^*)'(p_0/S)\frac{1}{S^2}\partial_t S, \\ T(t,0) = T_{in}(t) > 0,\, t \in (0,\tau), \\ T(0,x) = T_0,\, x \in (0,1), \\ \partial_x T(t,1) = 0,\, t \in (0,\tau). \end{aligned} \tag{86}$$

In order to define a, b such that $a \leq T \leq b$ we proceed as in Section 4. First, with $S \in E_\tau$ and

$$f = -k\,p_0\,\partial_x u - \gamma p_0(q^*)'(p_0/S)\frac{1}{S^2}\partial_t S - T'_{in}$$

we have

$$\begin{aligned} \sup_{0\leq t\leq\tau} ||\partial_x T||^2_{L^2} &\leq \frac{1}{\beta}||f||^2_{L^2} \\ &\leq \frac{3}{\beta}\left(k^2||p_M\partial_x u||^2_{L^2} + ||T'_{in}||^2_{L^2(0,\tau)} + \frac{p_M^2\gamma^2 C_q^2}{a^4}N_1\right) = C + \frac{d}{a^4} \end{aligned}$$

After similar computations as in the proof of Proposition 4.1 we are led to set

$$T_m = \max(\sqrt{\mu C}, \left(\frac{6d}{(\lambda_1^*)^5}\right)^{1/8}),\, a = \lambda_1 T_m,\, b = ||T_{in}||_\infty + \delta$$

where $\mu > 1$ is some arbitrary constant, $\delta = \sqrt{C + \dfrac{d}{a^4}}$ and

$$\lambda_1^* = \frac{5 - \sqrt{1 + 24/\mu}}{6}, \quad \lambda_1 = \frac{5 - \sqrt{1 + 24C/T_m^2}}{6}.$$

With these values of a, b we have $a \leq S \leq b \implies a \leq T \leq b$ and this value of a is used in (84).
Next, with (84) we get immediately

$$\begin{aligned} ||\partial_t T||_{L^2}^2 &\leq 2||f||_{L^2}^2 + 2||T'_{in}||_{L^2}^2 \\ &\leq 6\left(k^2||p_M \partial_x u||_{L^2}^2 + \frac{p_0^2\gamma^2 C_q^2}{a^4} N_1 + ||T'_{in}||_{L^2(0,\tau)}^2\right) + 2||T'_{in}||_{L^2(0,\tau)}^2 \\ &\leq N_1. \end{aligned}$$

Now we redefine h as $h = -k\, p_0\, \partial_x u - \gamma p_0 (q^*)'(p_0/S)\dfrac{1}{S^2}\partial_t S - T'_{in}(t)$. We have $h \in L^2$ and also $\partial_t h \in L^2$, indeed:

$$\partial_t h = -k p_0 \partial_t \partial_x u + \frac{\gamma p_0}{S^2}\left((q^*)'(p_0/S) + \frac{p_0}{S}(q^*)''(p_0/S)\right)(\partial_t S)^2 \tag{87}$$

$$-\frac{\gamma p_0}{S}\,\partial_{tt}^2 S - T''_{in}(t) \tag{88}$$

and $\partial_t \phi \in H^1((0,\tau)\times(0,1)) \implies \forall p \geq 2,\ \partial_t \phi \in L^p$, thus $(\partial_t S)^2 \in L^2$ and

$$||(\partial_t S)^2||_{L^2}^2 = ||\partial_t S||_{L^4}^4 \leq c\,||\partial_t S||_{H^1}^4 \leq c\, N_2^2$$

where $c > 0$ is a true constant. With the assumptions of Proposition 5.1 and $\partial_t S(0,x) = 0$ we have $h(0,x) = 0$ and the compatibility condition needed to apply Proposition 4.3 to $D = T - T_{in}$ is obviously fulfilled and we get in particular

$$||\partial_t T||_{H^1}^2 \leq C\left(||k p_M \partial_t \partial_x u||_{L^2}^2 + p_M N_1 + 1 + p_0^2 ||\partial_t S||_{H^1}^2\right)$$

where C is some constant depending only on the various constants of the problem, except p_0, and the initial-boundary data. Choosing N_2 such that $N_2 \geq 2C\left(||k p_M \partial_t \partial_x u||_{L^2}^2 + p_M N_1 + 1\right)$, we have $||\partial_t T||_{H^1}^2 \leq N_2$ provided $p_0^2 \leq \dfrac{||k p_M \partial_t \partial_x u||_{L^2}^2 + p_M N_1 + 1}{N_2}$.
Moreover, with Equation (86) we get $\partial T(0,\cdot) = 0$ on $(0,1)$ and with the various constaints on p_0 and a, b, N_2 previously defined we hace $\Phi(E_\tau) \subset E_\tau$. Next, we can proof that Φ is contracting on E_{τ^*} for some $\tau^* \leq \tau$ as in the proof of Proposition 4.1, but we have to impose a supplementary constraint on p_0 due to the term $||\partial_t S_2 - \partial_t S_1||_{L^2}^2$. The unique fixed point lies in E_{τ^*} and we finally iterate the process until we reach the final time τ. □

5.2 Short proof of Theorem 5.1

The proof is very similar to that of Theorem 4.1.With the same notations we work here with

$$\mathcal{U}_\tau = \Big\{ u \in L^\infty(0,\tau; H^1(0,1)),\ \partial_t \partial_x u \in L^2,\ u(t,0) = u_{in}(t) \geq u_m > 0,$$

$$\partial_x u(0,x) = 0 \text{ on } (0,1),\ \sup_{t\in(0,\tau)} ||p_0 \partial_x u||_{L^2(0,1)}^2 \leq M,\ \sup_{t\in(0,\tau)} ||p_0\, \partial_t \partial_x u||_{L^2(0,1)}^2 \leq M_1 \Big\}$$

and $v = \psi(u)$ satisfies

$$\begin{aligned} p_0\, \partial_x v(t,x) &= (\rho_{in}\, u_{in})(t)\, \partial_x T(t,x) + \partial_x T(t,x) \int_0^x \left[(1 + (q^*)'(p_0/T))\, \frac{p_0}{T^2}\partial_t T\right](t,\xi)\, d\xi \\ &\quad + (1 + (q^*)'(p_0/T))\, \frac{p_0}{T}\partial_t T. \end{aligned}$$

With $T(0,x) = T_0$ and $\partial_t T(0,x) = 0$ we get $\partial_x v(0,x) = 0$. the rest of the proof does not involve any additional difficulty. □

6 About some variants

The model studied in this paper admits multiple variants depending on the type of phenomenon in which the experimenter is interested. One can take into account the axial diffusion in the mass conservation equations ($\alpha \neq 0$ in Equation (1)).

We could also take into account the pressure drop along the column according to the Ergun Equation ([15])

$$\frac{\partial P}{\partial z} = \frac{150\mu}{(2R_p)^2}\,\frac{(1-\epsilon)^2}{\epsilon^3}\,u + \frac{1.75\,\rho_g}{2R_p}\,\frac{1-\epsilon}{\epsilon^3}\,u^2.$$

The choice of boundary conditions is not always very explicit in the articles based on experimentation, but the velocity is often considered constant or, depending on the type of phenomenon considered, its inlet or outlet sign is known a priori. This is the reason why we have chosen to take a positive velocity at the column inlet and to set sufficient regularity conditions to ensure a positive velocity at the outlet, which justifies the type of simple boundary conditions envisaged. For other conditions one may consult for instance Ruthwen [20], or [19] (Danckwerts boundary conditions).
To be closer to real conditions we could for example, with a inlet positive speed u_{in}, adopt the following Danckwerts type conditions for the temperature in a dimensionless form

$$T(t,0) = T_{in}, \qquad \frac{\partial T}{\partial x}(t,1) = -\delta\, p_e\, u^-(t,1)\,(T(t,1) - T_e), \tag{89}$$

where δ is a constant, p_e is the external pressure, T_e is the external temperature at the outlet (assumed constant) and $u^- = -\min(0,u)$. Of course we retrive the outlet boundary condition used in this study when $u(t,1) \geq 0$. The main difficult here is of course the fact that u, thus its sign, is part of the unknown. Note that from a numerical point of view this is not an obstacle because by using for example a splitting method (as in [3] for instance) it is possible to predict the sign of the outlet speed then to apply the corresponding condition. The case of N species, $N \geq 2$, and its numerical treatment with conditions of the type (91) is the subject of a work in preparation by the same authors.

A Variables and constants

C_i $(mole \cdot m^{-3})$	concentration in fluid phase for species i
q_i $(mole \cdot m^{-3})$	concentration in solid phase for species i
u $(m \cdot s^{-1})$	velocity of the gaseous mixture
T_g (K)	temperature of the gaseous mixture
T_s (K)	temperature of the solid phase
$q_i^*(C_1, \cdots, C_N)$ $(mole \cdot m^{-3})$	i^{th} isotherm: concentration in solid phase at equilibrium for species i
$P(t)$ (Pa)	time-dependant pressure in the column

Table 1: Unknown and functions

Notice also that $C_g = C_{pg}\rho$.

A_i (s^{-1})	mass transfer rate for species i
C_g $(J \cdot m^{-3} \cdot K^{-1})$	volumetric heat capacity of fluid phase
C_s $(J \cdot m^{-3} \cdot K^{-1})$	volumetric heat capacity of solid phase
C_{pg} $(J \cdot mole^{-1} \cdot K^{-1})$	heat capacity per unit volume of fluid phase
C_{ps} $(J \cdot kg^{-1} \cdot K^{-1})$	heat capacity per unit volume of solid phase
D_L $(m \cdot s^{-1})$	axial dispersion coefficient
L (m)	lenght of the column
N	number of species
R $(J \cdot mole^{-1} \cdot K^{-1})$	molar gas constant
R_p (m)	adsorbent pellet radius
T_w (K)	wall temperature
d (m)	internal diameter of adsorbent bed
h $(W \cdot m^{-2} \cdot K^{-1})$	overall heat transfer coefficient between adsorbent particle and ambient fluid
$h_w (W \cdot m^{-2} \cdot K^{-1})$	overall heat transfer coefficient at column wall
z (m)	distance measured from inlet
$-\Delta H_i$ $(J \cdot mole^{-1})$	heat of adsorption for species i
ε (adim)	voidage of adsorbent bed
λ_L $(W \cdot m^{-1} \cdot K^{-1})$	effective axial thermal conductivity
μ $(kg \cdot s^{-1} \cdot m^{-1})$	dynamic viscosity of a gas mixture
ρ $(mole \cdot m^{-3})$	total concentration in fluid phase
ρ_b $(kg \cdot m^{-3})$	density of adsorbent
τ (s)	time

Table 2: Constants

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