Proofs on Inductive Predicates in Why3

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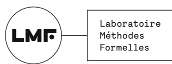
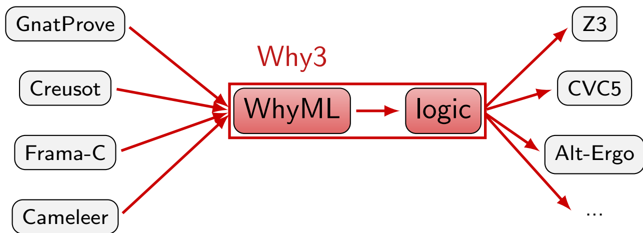
Proofs on Inductive Predicates in Why3

Jean-Christophe Filliâtre, Andrei Paskevich, Henri Saudubray

Isaac Newton Institute
Big Specification
Cambridge, October 8, 2024




GnatProve
Creusot
Frama-C
Cameleer
Why3
WhyML
logic
Z3
CVC5
Alt-Ergo
...

1. the logic of Why3
2. proofs by induction
3. a proposal: `match inductive`
4. an application

this is a logic of compromise:

- expressiveness is good for users
- simplicity is good for SMT solvers (and for users, too)

we end up with an extension of first-order logic of our own

One logic to use them all [JCF, CADE 13]

A Formalization of Core Why3 in Coq [Cohen, Johnson-Freyd, POPL 24]

- type polymorphism
- algebraic data types

```
type list 'a = Nil | Cons 'a (list 'a)
```

```
constant alist: list int =
  Cons 34 (Cons 55 (Cons 89 Nil))
```

- type polymorphism • algebraic data types • pattern matching
- recursive functions and predicates (with termination check)

```
function length (l: list 'a) : int
= match l with
  | Nil      -> 0
  | Cons _ t -> 1 + length t
  end
```

```
lemma L:
  forall l: list 'a. length l >= 0
```

- type polymorphism • algebraic data types • pattern matching
- recursive functions and predicates (with termination check)
- inductive predicates

```
inductive nim_step int int =
| RemoveOne: forall n. n >= 0 -> nim_step (n+1) n
| RemoveTwo: forall n. n >= 0 -> nim_step (n+2) n

inductive nim int int =
| Step: forall x y. nim_step x y -> nim x y
| Path: forall x y z. nim x y -> nim y z -> nim x z
```

```
lemma L:
  forall x y. nim x y -> 0 <= y < x
```

using SMT solvers

when it comes to calling SMT solvers, the logic of Why3 is translated to first-order logic

which means encoding

- polymorphism
- algebraic data types
- pattern matching

possibly losing completeness in the process

an algebraic data type such as

```
type list 'a = Nil | Cons 'a (list 'a)
```

is sent to SMT solvers as

- an uninterpreted type `list`
- a constant `Nil` and a function `Cons`
- injectivity + inversion principle
- but no induction principle

similarly, an inductive predicate such as

```
inductive nim int int =
| Step: forall x y. nim_step x y -> nim x y
| Path: forall x y z. nim x y -> nim y z -> nim x z
```

is sent to SMT solvers as

- an undefined predicate `nim`
- two axioms `Step` and `Path`
- an inversion principle
- but no induction principle

2

proofs by induction

```
lemma L:
  forall l: list 'a. length l >= 0
```

can't be proved by SMT solvers *

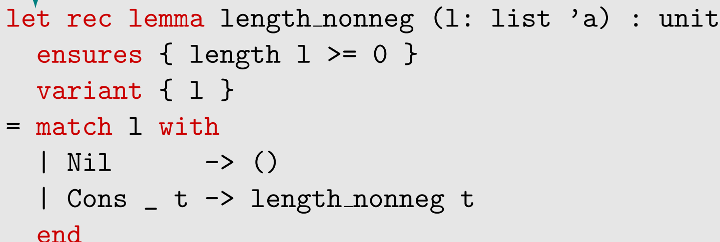
we resort to lemma functions, which are ghost program functions whose sole purpose is to build proofs
(see for instance Program Proofs [Leino, 2023])

```
let rec lemma length_nonneg (l: list 'a) : unit
  ensures { length l >= 0 }
  variant { l }
= match l with
  | Nil      -> ()
  | Cons _ t -> length_nonneg t
  end
```

lemma function

we resort to lemma functions, which are ghost program functions whose sole purpose is to build proofs
(see for instance Program Proofs [Leino, 2023])

program function



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program function *ghost*

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program function *ghost* *returns nothing*

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  end
```

lemma function

we resort to lemma functions, which are ghost program functions whose sole purpose is to build proofs
(see for instance Program Proofs [Leino, 2023])

program function — *ghost* — *returns nothing*

```
let rec lemma length_nonneg (l: list 'a) : unit
  ensures { length l >= 0 }      ← postcondition
  variant { l }      ← must terminate
= match l with
  | Nil       -> ()
  | Cons _ t -> length_nonneg t
  end
```

we resort to lemma functions, which are ghost program functions whose sole purpose is to build proofs
(see for instance Program Proofs [Leino, 2023])

program function *ghost* *returns nothing*

```
let rec lemma length_nonneg (l: list 'a) : unit
  ensures { length l >= 0 }        ← postcondition
  variant { l }        ← must terminate
= match l with
  | Nil       -> ()
  | Cons _ t -> length_nonneg t        ← recursive call
  end
```

1. first, the lemma function is verified, like any program function

2. then it generates a hypothesis in the context

```
axiom length_nonneg:
  forall l: list 'a. length l >= 0
```

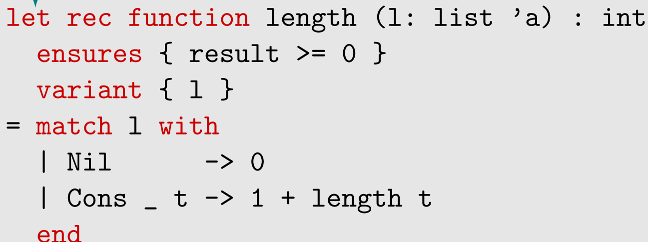
3. besides, it can be called like any ghost function

```
  ...
  let n = length mylist in
  length_nonneg mylist;
  ...
```

alternatively, we can define and prove at the same time

```
let rec function length (l: list 'a) : int
  ensures { result >= 0 }
  variant { l }
= match l with
  | Nil      -> 0
  | Cons _ t -> 1 + length t
  end
```

alternatively, we can define and prove at the same time

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let rec function length (l: list 'a) : int
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= match l with
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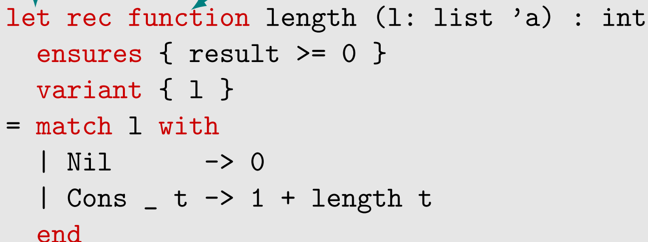


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  ensures { result >= 0 }
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= match l with
  | Nil      -> 0
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  end
```

alternatively, we can define and prove at the same time

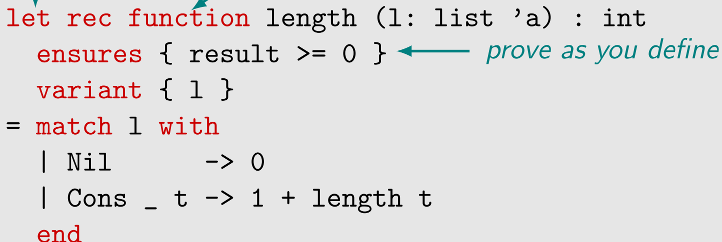
program function *and logic function*

```
let rec function length (l: list 'a) : int
  ensures { result >= 0 }
  variant { l }
= match l with
  | Nil      -> 0
  | Cons _ t -> 1 + length t
  end
```

prove as you define

alternatively, we can define and prove at the same time

program function *and logic function*




```
let rec function length (l: list 'a) : int
  ensures { result >= 0 }
  variant { l }
= match l with
  | Nil       -> 0
  | Cons _ t -> 1 + length t
  end
```

```
axiom length_def : forall l. length l = ...
axiom length_spec: forall l. length l >= 0
```

when a lemma involves an inductive predicate and requires induction, *e.g.*,

```
lemma L:
  forall x y. nim x y -> 0 <= y < x
```

we would like to prove it with a lemma function as well

3

a proposal: `match inductive`

rationale

- stick to the usual practice in Why3
  - lemma functions
  - user-guided induction via explicit recursive calls
- do not pollute the logical context
  - if possible, do not introduce any type/function/axiom

- a new construct `match inductive` to perform case analysis on inductive proofs
- a new kind of `variant` to ensure that recursive calls are legal

```
let rec lemma nim_bounds (x z: int)
  requires H { nim x z    }
  ensures    { 0 <= z < x }
= match inductive H with
  | Step _ _ _     -> ()
  | Path _ y _ _ _ -> nim_bounds x y; nim_bounds y z
  end
```

named hypothesis

```
let rec lemma nim_bounds (x z: int)
  requires H { nim x z    }
  ensures    { 0 <= z < x }
= match inductive H with
  | Step _ _ _     -> ()
  | Path _ y _ _ _ -> nim_bounds x y; nim_bounds y z
  end
```

case analysis on H

```
let rec lemma nim_bounds (x z: int)
  requires { nim x z }
  ensures  { 0 <= z < x }
= let ghost Step (x1 y1: int)
    requires { nim_step x1 y1 }
    requires { x = x1 /\ z = x1 }
    ensures  { 0 <= z < x } = ()
  in
  let ghost Path (x1 y z1)
    requires { nim x1 y }
    requires { nim y z1 }
    requires { x = x1 /\ z = z1 }
    ensures  { 0 <= z < x } =
    nim_bounds x y;
    nim_bounds y z
  in
  assume { false }; absurd
```

```
let rec lemma nim_bounds (x z: int)
  requires { nim x z }
  ensures  { 0 <= z < x }
= let ghost Step (x1 y1: int)      <------- first case
    requires { nim_step x1 y1 }
    requires { x = x1 /\ z = x1 }
    ensures  { 0 <= z < x } = ()
  in
  let ghost Path (x1 y z1)      <------- second case
    requires { nim x1 y }
    requires { nim y z1 }
    requires { x = x1 /\ z = z1 }
    ensures  { 0 <= z < x } =
    nim_bounds x y;
    nim_bounds y z
  in
  assume { false }; absurd
```

```
let rec lemma nim_bounds (x z: int)
  requires { nim x z }
  ensures  { 0 <= z < x }
= let ghost Step (x1 y1: int)
    requires { nim_step x1 y1 }
    requires { x = x1 /\ z = x1 }
    ensures  { 0 <= z < x } = ()
  in
  let ghost Path (x1 y z1)
    requires { nim x1 y }
    requires { nim y z1 }
    requires { x = x1 /\ z = z1 }
    ensures  { 0 <= z < x } =
    nim_bounds x y;
    nim_bounds y z
  in
  assume { false }; absurd
```

```
let rec lemma nim_bounds (x z: int)
  requires { nim x z }
  ensures  { 0 <= z < x }
= let ghost Step (x1 y1: int)
    requires { nim_step x1 y1 }
    requires { x = x1 /\ z = x1 }
    ensures  { 0 <= z < x } = ()
  in
  let ghost Path (x1 y z1)
    requires { nim x1 y }
    requires { nim y z1 }
    requires { x = x1 /\ z = z1 }
    ensures  { 0 <= z < x } =
    nim_bounds x y;
    nim_bounds y z
  in
  assume { false }; absurd
```

```
let rec lemma nim_bounds (x z: int)
  requires { nim x z }
  ensures  { 0 <= z < x }
= let ghost Step (x1 y1: int)       <------  first case
    requires { nim_step x1 y1 }
    requires { x = x1 /\ z = x1 }   <------  unification
    ensures  { 0 <= z < x } = ()    user code
  in
  let ghost Path (x1 y z1)          <------  second case
    requires { nim x1 y }
    requires { nim y z1 }
    requires { x = x1 /\ z = z1 }   <------  unification
    ensures  { 0 <= z < x } =
    nim_bounds x y;
    nim_bounds y z                  user code
  in
  assume { false }; absurd          <------  nothing else to prove
```

since postconditions for the cases are taken from the surrounding function, the `match inductive` must be in a tail position

```
match inductive H with ... end; ...
```

```
File "test.mlw", line 42, ...
match inductive should be in tail position
```

ghost code must terminate, and lemma functions make no exception

we extend Why3 with a new kind of variant

```
variant { inductive H }
```

meaning: on a recursive call, we must be able to match the parameters of smaller instances of the predicate collected so far (possibly transitively)

```
let rec lemma nim_bounds (x z: int)
  requires H { nim x z     }
  ensures    { 0 <= z < x  }
  variant    { inductive H }
= match inductive H with
  | Step _ _ _     -> ()
  | Path _ y _ _ _ -> nim_bounds x y; nim_bounds y z
  end
```

```
let rec lemma nim_bounds (x z: int)
  requires H { nim x z     }
  ensures    { 0 <= z < x  }
  variant    { inductive H }
= match inductive H with
  | Step _ _ _      -> ()
  | Path _ y _ _ _ -> nim_bounds x y; nim_bounds y z
  end
```

user-provided variant

```
let rec lemma nim_bounds (x z: int)
  requires H { nim x z     }
  ensures    { 0 <= z < x  }
  variant    { inductive H }
= match inductive H with
  | Step _ _ _       -> ()
  | Path _ y _ _ _ -> nim_bounds x y; nim_bounds y z
  end
```

user-provided variant

collected instances

```
let rec lemma nim_bounds (x z: int)
  requires H { nim x z      }
  ensures    { 0 <= z < x   }
  variant    { inductive H }
= match inductive H with
  | Step _ _ _      -> ()
  | Path _ y _ _ _ -> nim_bounds x y; nim_bounds y z
  end
```

user-provided variant

collected instances

$$x = x \wedge y = y \vee x = y \wedge y = z$$

```
let rec lemma nim_bounds (x z: int)
  requires H { nim x z      }
  ensures    { 0 <= z < x   }
  variant    { inductive H }
= match inductive H with
  | Step _ _ _       -> ()
  | Path _ y _ _ _ -> nim_bounds x y; nim_bounds y z
  end
```

user-provided variant

collected instances

$x = x \wedge y = y \vee x = y \wedge y = z$

$y = x \wedge z = y \vee y = y \wedge z = z$

as for any variant, an inductive variant can be part of a lexicographic tuple variant

e.g.,

```
variant { n, inductive H, t }
```

mutually inductive predicates

```
inductive even int = ...
with      odd  int = ...
```

induce mutually recursive lemma functions, with suitable variants,

```
let rec lemma evenP (n: int)
  requires E { even n      }
  variant    { inductive E }
  ...
with lemma oddP (n: int }
  requires O { odd n       }
  variant    { inductive O }
  ...
```

4

an application

λ-calculus

Constructive Computation Theory [Huet, 1988]

- course notes on λ-calculus
- OCaml implementation (de Bruijn indices, parsing, printing, computation strategies, Church numerals, lists, etc.)
- Prism Theorem, Standardisation, Separability
- 3 kloc

Residual theory in λ-calculus: a formal development [Huet, 1994]

- a Coq formalization of the above, up to the Prism Theorem and its corollaries
- 2 kloc

a good candidate for us, as it contains

- 2 algebraic data types
- 11 inductive predicates
- 78 proofs by induction

Coq

```
Lemma red_abs:
  forall M M': lambda,
  red M M' → red (Abs M) (Abs M').
Proof.
  simple induction 1; intros.
  - apply step; apply abs; trivial.
  - apply refl.
  - apply trans with (Abs N); trivial.
Qed.
```

Why3

```
let rec lemma red_abs (m m': lambda)
  requires r { red m m' }
  ensures    { red (Abs m) (Abs m') }
  variant    { inductive r }
= match inductive r with
  | step  _ _ _       -> ()
  | refl  _           -> ()
  | trans m n m' _ _ -> red_abs m n; red_abs n m'
  end
```

the why3 proof is roughly two times smaller than the Coq proof

Coq	Why3
81 kb	42 kb
78 `induction`	72 `match inductive`

SMT solvers are useful, even for such kind of proofs!

note: Why3 has no `nat` type and suitable preconditions were added at various places

Why3's ghost code is now extended with

- a new construct `match inductive` to perform case analysis on inductive proofs
- a new kind of `variant` to ensure that recursive calls are legal

future work: support infinite branches (*e.g.* accessibility predicate)