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# The Strong $(2, 2)$ -Conjecture for more classes of graphs

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## Abstract

The Strong  $(2, 2)$ -Conjecture asks whether, for all connected graphs different from  $K_2$  and  $K_3$ , we can assign red and blue labels with value 1 or 2 so that no two adjacent vertices have the same sum of incident red labels or the same sum of incident blue labels. This conjecture, which can be perceived as a generalisation of the so-called 1-2-3 Conjecture, as, thus far, been proved only for a handful number of graph classes. In this work, we prove the Strong  $(2, 2)$ -Conjecture holds for more classes of graphs. In particular, we prove the conjecture for cacti, subcubic outerplanar graphs, graphs with maximum average degree less than  $\frac{9}{4}$ , and Halin graphs, among others.

*Keywords:* Strong  $(2, 2)$ -Conjecture; proper labelling; 3-colourable graph.

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## 1. Introduction

In this work, we essentially prove the so-called **Strong  $(2, 2)$ -Conjecture** for new classes of graphs. So that the extent of our contribution is clear, we first start by recalling the context of this conjecture.

Let  $G$  be a graph, and  $l, k \geq 1$  be two integers. An  $(l, k)$ -colouring of  $G$  is an assignment  $\ell : E(G) \rightarrow \{1, \dots, l\} \times \{1, \dots, k\}$  of *coloured labels* to the edges, where each edge  $e$  gets assigned a *label*  $\ell(x, y)$  by  $\ell$ , with *colour*  $x$  and *value*  $y$ . Now, for every vertex  $v$  of  $G$  and any  $i \in \{1, \dots, l\}$ , one can compute the  *$i$ -sum*  $\sigma_\ell^i(v)$  of  $v$  by  $\ell$ , which is, essentially, the sum of labels with colour  $i$  assigned to the edges incident to  $v$ . Now, depending on certain parameters, we consider that two adjacent vertices of  $G$  are *distinguished* by  $\ell$ , or *conflicting* otherwise. More precisely, we are mainly interested in three distinction conventions, introduced in [3]:

- $\ell$  is *weak* if, for any two adjacent vertices  $u$  and  $v$ , there is  $i \in \{1, \dots, l\}$  such that  $\sigma_\ell^i(u) \neq \sigma_\ell^i(v)$ ;
- $\ell$  is *standard* if, for any two adjacent vertices  $u$  and  $v$  such that  $uv$  is assigned a label with colour  $i \in \{1, \dots, l\}$  by  $\ell$ , we have  $\sigma_\ell^i(u) \neq \sigma_\ell^i(v)$ ;
- $\ell$  is *strong* if, for any two adjacent vertices  $u$  and  $v$ , we have  $\sigma_\ell^i(u) \neq \sigma_\ell^i(v)$  for all  $i \in \{1, \dots, l\}$  such that  $\sigma_\ell^i(u), \sigma_\ell^i(v) \neq 0$ .

Obviously, a strong colouring is standard and a standard colouring is weak, but the contrary is not always true. These three types of colourings were mostly introduced, in [3], because they generalise a number of other types of distinguishing labellings, such as labellings distinguishing adjacent vertices by their incident sums (standard  $(1, k)$ -colourings – notions lying behind the 1-2-3 Conjecture [14]), labellings distinguishing adjacent vertices by their incident multisets (weak  $(1, k)$ -colourings – notions lying behind the multiset version of the 1-2-3 Conjecture [1]), and edge-colourings distinguishing adjacent

vertices by their coloured degrees (standard  $(l, 1)$ -colourings – notions lying behind the Local Irregularity Conjecture [4]).

In the line of the related conjectures and problems, the main goal, given a graph  $G$ , is to figure out whether  $G$  admits weak, standard, and strong  $(l, k)$ -colourings with  $l$  and/or  $k$  as small as possible. First off, under the assumption that  $l$  and  $k$  can be anything, it can be observed that weak, standard, and strong  $(l, k)$ -colourings exist for all connected graphs different from  $K_2$  (for instance, the 1-2-3 Conjecture, which holds for all such graphs [15], implies they all admit strong  $(1, 3)$ -colourings). The main topic of this paper is when  $l, k \leq 2$ . For the weak convention, we say a graph is *nice* if it does not contain  $K_2$  as a connected component, while for the standard and strong conventions we say a graph is *nice* if it does not contain  $K_2$  or  $K_3$  as a connected component. We can now state the three main conjectures of interest here, all raised in [3]:

**Weak  $(2, 2)$ -Conjecture.** *Every nice graph admits weak  $(2, 2)$ -colourings.*

**Standard  $(2, 2)$ -Conjecture.** *Every nice graph admits standard  $(2, 2)$ -colourings.*

**Strong  $(2, 2)$ -Conjecture.** *Every nice graph admits strong  $(2, 2)$ -colourings.*

Out of the three conjectures, only the weakest one has been proved to date. Namely, it was observed in [5] that the 1-2-3 Conjecture, if proved true, would imply the Weak  $(2, 2)$ -Conjecture; since the 1-2-3 Conjecture was recently proved [15], the Weak  $(2, 2)$ -Conjecture thus also holds. This apart, the other two stronger conjectures are wide open in general, despite a few works on the topic [3, 5, 8, 17, 18]. In brief, the Standard  $(2, 2)$ -Conjecture was mainly proved for 2-degenerate graphs, subcubic graphs, and graphs with large minimum degree. Regarding the Strong  $(2, 2)$ -Conjecture, it was mostly proved for complete graphs and bipartite graphs (thus also the Standard  $(2, 2)$ -Conjecture holds for these).

Our main goal in this work is to prove the Strong  $(2, 2)$ -Conjecture for more classes of graphs of interest. More precisely, we focus on classes of graphs that are either dense, or 3-colourable (w.r.t. the chromatic number). Regarding the latter graphs, this is mainly because this is a common way to proceed in the field. Indeed, partitioning the vertices of a graph into a very restricted number of independent sets provides a layout for designing distinguishing labellings, in the sense that we can build upon sets of vertices that will never be in conflict (the best illustration being that, in nice 3-colourable graphs, we can design strong  $(1, 3)$ -colourings where adjacent vertices have distinct sums modulo 3, following a proper 3-colouring serving as a layout [14]). Since the Strong  $(2, 2)$ -Conjecture was proved for nice bipartite graphs in [3], the next natural step to make is thus to focus on 3-colourable graphs. Regarding the former graphs, namely dense graphs, this is mainly because they tend to have unbounded chromatic number, which allows us to attack the Strong  $(2, 2)$ -Conjecture on another angle. Also, dense graphs behave in some expected way structure-wise, allowing for an easier design of distinguishing labellings.

Throughout this work, we thus focus on proving the Strong  $(2, 2)$ -Conjecture for more classes of graphs, focusing on graphs with low chromatic number or large density. Namely, we prove the conjecture for cacti in Section 2, for subcubic outerplanar graphs in Section 3, for graphs with maximum average degree less than  $\frac{9}{4}$  in Section 4, for Halin graphs in Section 5, and for planar graphs with girth at least 16 in Section 6, which, roughly speaking, are classes of graphs with chromatic number at most 3. We also prove the Strong  $(2, 2)$ -Conjecture for classes of graphs with unbounded chromatic number, namely powers of cycles in Section 7, and complete  $k$ -partite graphs in Section 8. We finish off with a few concluding words in Section 9.

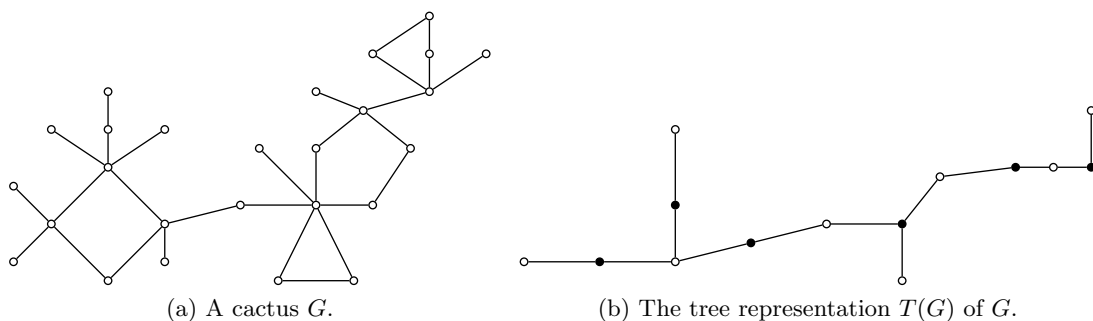


Figure 1: An example of a cactus  $G$  (a), with the corresponding tree representation  $T(G)$  (b) as introduced in the proof of Theorem 1. Black vertices are vertex nodes of  $T(G)$ , while white vertices are subgraph nodes. This example is taken from [6].

### Terminology

Note that, in the context of the Strong  $(2, 2)$ -Conjecture, we are only interested in strong  $(l, k)$ -colourings with  $l = k = 2$ . For this reason, it is more convenient to see  $(2, 2)$ -colourings as labellings assigning, say, red and blue labels, with value 1 or 2. Thus, in any graph, by any  $(2, 2)$ -colouring  $\ell$ , for every vertex  $v$  we derive a *red sum*, which we denote by  $\sigma_\ell^r(v)$ , and a *blue sum*, which we denote by  $\sigma_\ell^b(v)$ . Under that terminology, the Strong  $(2, 2)$ -Conjecture asks whether every nice graph admits a labelling  $\ell$  assigning red and blue 1's and 2's so that, for every two adjacent vertices  $u$  and  $v$ , we have  $\sigma_\ell^r(u) \neq \sigma_\ell^r(v)$  and  $\sigma_\ell^b(u) \neq \sigma_\ell^b(v)$ , unless  $0 \in \{\sigma_\ell^r(u), \sigma_\ell^r(v)\}$  or  $0 \in \{\sigma_\ell^b(u), \sigma_\ell^b(v)\}$ , respectively.

## 2. Cacti

Recall that a *cactus* is a connected graph in which no two cycles share an edge. That is, cacti generalise trees in that, when contracting cycles to vertices, we essentially get a tree. Let us remind that cacti are 2-degenerate, and thus 3-colourable. Also, it is worth recalling that nice trees verify the Strong  $(2, 2)$ -Conjecture, essentially because they all admit strong  $(1, 2)$ -colourings (see *e.g.* [16]).

In the next result, we prove the Strong  $(2, 2)$ -Conjecture for nice cacti. In the following, for  $n \geq 0$ , we denote by  $K_n$  the complete graph of order  $n$ , and by  $C_n$  the cycle on  $n$  vertices. Let us bring to the attention of the readers that, in the upcoming proof and in later ones as well, several configurations will be introduced through the notation “ $C_k$ ”, which should not be confused with “ $C_k$ ”.

**Theorem 1.** *If  $G$  is a nice cactus, then  $G$  is strongly  $(2, 2)$ -colourable.*

*Proof.* Before starting, we claim that every cactus  $G$  has a *tree representation*  $T(G)$  where:

- all nodes of  $T(G)$  are either *vertex nodes*, associated to a vertex of  $G$ , or *subgraph nodes*, associated to an induced tree or to an induced cycle of  $G$ ;
- all leaves of  $T(G)$  are subgraph nodes;
- the set of vertex nodes, as well as the set of subgraph nodes, are both independent sets;
- two subgraph nodes associated to  $H_1, H_2$  are adjacent to a vertex node  $u$  iff  $V(H_1) \cap V(H_2) = \{u\}$ .

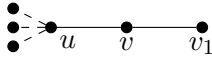


Figure 2: A particular case of Configuration C1, with only one degree-1 vertex. In this figure and in later ones as well, dotted edges are incident edges in  $G'$ .

Such a tree representation is very close to the block representation of cacti (see e.g. [6]) with additional nodes representing connected components that are not cycles. It can be computed by adding a subgraph node associated to every maximal biconnected component, and a tree node associated to every remaining edge (not in a biconnected component). We complete it by adding a vertex node associated to every vertex shared by two or more subgraph nodes. Altogether, the representation is indeed a tree, as otherwise one edge would be either shared by two cycles, or a tree node would belong to some cycle, which is not possible. An example is given in Figure 1.

Let us consider  $G$  a minimal counterexample (w.r.t. the number of edges) to Theorem 1. Assuming  $T(G)$  is rooted at some vertex, we call *terminal cycle* (resp. *terminal tree*) of  $G$  an induced cycle (resp. tree) represented by a leaf of maximal depth in  $T(G)$ . Our goal is to prove that  $G$  cannot contain a certain number of configurations (C1 to C6c below), from which we can eventually contradict the existence of  $G$ . Remark that since, below, we are proving that some configurations do not appear in a minimal counterexample, we are free to consider any way to root  $T(G)$  as long as we do not assume anything on the position of a node w.r.t. the root.

Let us thus first prove that  $G$ , due to its minimality, cannot contain some configurations.

- (C1) Let us first assume that  $T(G)$  has a leaf  $T$  associated to a tree of depth greater or equal to 2. In particular, we can assume that, up to seeing  $T$  as two trees sharing a vertex, the depth of  $T$  is exactly 2. More intuitively, this configuration contains an induced tree of depth greater or equal to 2 where at most one vertex is adjacent to vertices outside the tree (see Figure 2). We denote by  $u$  the parent vertex in  $T$ , and by  $v$  the unique neighbour of  $u$  in  $T$ . Since  $T$  is of depth at least 2, let us denote by  $v_1, \dots, v_k$  the children of  $v$  in  $T$ , and consider  $G' = G - \{v_1, \dots, v_k\}$ . Note that either  $G'$  is nice, or  $G$  is a star in which case, clearly, strong  $(2, 2)$ -colourings exist. Since  $G'$  is a cacti, and  $G$  is a minimal counterexample to Theorem 1, graph  $G'$  is strongly  $(2, 2)$ -colourable; we denote by  $\ell'$  a  $(2, 2)$ -colouring. From now on, and in several instances throughout this work, our goal is to *extend*  $\ell'$  by assigning labels to the edges of  $E(G) \setminus E(G')$  so that the resulting  $(2, 2)$ -colouring of  $G$  is strong (thereby contradicting that  $G$  is a counterexample, and that  $G$  cannot contain the current configuration). In the present case, assuming w.l.o.g. that  $\sigma_{\ell'}^b(v) > 0$ , it suffices to assign blue label 1 to  $vv_2, \dots, vv_k$ , and to assign to  $vv_1$  whichever blue label makes  $\sigma_{\ell'}^b(v)$  differ from  $\sigma_{\ell'}^b(u)$ .
- (C2) Assume  $T(G)$  has a leaf  $C$  associated to a  $C_3$ , with a parent vertex  $v$ , where the vertex node  $v$  is also adjacent to an induced tree  $T$  where  $v$  is a leaf of  $T$ . We further assume that the degree of  $v$  in  $T(G)$  is exactly 2, and denote by  $u$  the only neighbour of  $v$  in  $T$ . More intuitively, this configuration corresponds to a triangle hanging at the end of an edge (see Figure 3). We denote by  $v_1, v_2$  the two vertices of  $C$  other than  $v$ , and consider  $G' = G - \{v_1, v_2\}$ . Unless  $V(G) = \{u, v, v_1, v_2\}$  (in which case it is easy to notice that  $G$  is not a counterexample to Theorem 1),  $G'$  is nice and thus strongly  $(2, 2)$ -colourable, and we consider  $\ell'$  strong  $(2, 2)$ -colouring. Assuming w.l.o.g. that  $\sigma_{\ell'}^b(v) > 0$ , we assign to  $vv_1$  whichever blue label makes  $\sigma_{\ell'}^b(v)$  differ from  $\sigma_{\ell'}^b(u)$ . We then assign to  $vv_2$  whichever red label makes  $\sigma_{\ell'}^r(v)$  differ from  $\sigma_{\ell'}^r(u)$ , and to  $v_1v_2$  whichever red label makes  $\sigma_{\ell'}^r(v_1)$  differ from  $\sigma_{\ell'}^r(v)$  (which happens to be the value we did not pick for  $vv_2$ ).

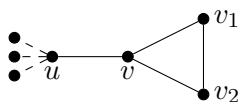


Figure 3: Configuration C2.

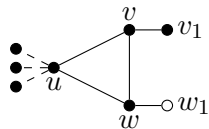


Figure 4: A particular case of Configuration C3, where each of  $v$  and  $w$  is adjacent to only one degree-1 vertex. In this figure and in later ones as well, white vertices are vertices that might not exist.

(C3) Assume there exists a subgraph node  $C$  of  $T(G)$  associated to a  $C_3$  with vertices  $u, v, w$ , where, if the vertex nodes  $v, w$  exist, then they are adjacent to  $C$  and to subgraph nodes associated with trees (that can be assumed to be of depth 1, as otherwise  $G$  would contain Configuration 1). We further assume there is at least one such tree. More intuitively, this configuration corresponds to a  $C_3$  with at least one degree-1 vertex attached (see Figure 4). We denote by  $u$  the parent vertex node of  $C$ , and by  $v, w$  the other two vertices of  $C$ . We also denote by  $v_1, \dots, v_k$  (resp.  $w_1, \dots, w_l$ ) the degree-1 vertices of the children of  $v$  (resp.  $w$ ). Unless  $u$  is of degree 2 in  $G$  (in which case  $G$  is completely revealed and it is easy to find a strong  $(2, 2)$ -colouring of  $G$ ), graph  $G' = G - \{v_1, \dots, v_k, w_1, \dots, w_l\}$  is strongly  $(2, 2)$ -colourable; so let  $\ell'$  be a strong  $(2, 2)$ -colouring of  $G'$ . Note that  $v$  and  $w$  are completely interchangeable in  $G'$ , and note also that  $vw$  must have the same colour (say blue, w.l.o.g.) by  $\ell'$  as one of its adjacent edges, say  $uv$ .

There are two cases to consider.

- Assume  $uw$  is also blue by  $\ell'$ . Up to swapping  $v$  and  $w$  in  $G'$ , we can assume  $\sigma_{\ell'}^b(v) > \sigma_{\ell'}^b(w)$  and  $k \geq l$ . If  $l = 0$ , then we assign to  $vw$  whichever red label makes  $\sigma_{\ell'}^r(w)$  and  $\sigma_{\ell'}^r(u)$  differ (note that with the colour change we operated,  $v$  and  $w$  still have distinct blue sums). We finish by assigning red label 1 to  $vv_2, \dots, vv_k$ , and assigning to  $vv_1$  whichever red label makes  $\sigma_{\ell'}^r(v)$  and  $\sigma_{\ell'}^r(u)$  differ. It can be checked that this results in a strong  $(2, 2)$ -colouring of  $G$ . We can now assume  $l, k \geq 1$ . If  $\sigma_{\ell'}^b(u) = \sigma_{\ell'}^b(v) + k$  (that is, assigning blue label 1 to every  $vv_i$  for  $1 \leq i \leq k$  yields a colouring that is not strong), then we assign blue label 1 to every  $ww_j$  with  $1 \leq j \leq l$ , blue label 1 to every  $vv_i$  with  $2 \leq i \leq k$ , and blue label 2 to  $vv_1$ . This way,  $\sigma_{\ell'}^b(u) = \sigma_{\ell'}^b(v) + k = \sigma_{\ell'}^b(v) - 1$ , and  $\sigma_{\ell'}^b(w) \leq \sigma_{\ell'}^b(v) - 2$ . Conversely, if  $\sigma_{\ell'}^b(u) = \sigma_{\ell'}^b(w) + l$ , then we assign blue label 2 to  $vv_1$  and  $ww_1$ , and blue label 1 to all other edges. By similar arguments, this results in a strong  $(2, 2)$ -colouring of  $G$ .
- We assume now that  $uw$  is assigned a red label by  $\ell'$ . Up to swapping  $v$  and  $w$  in  $G'$ , we can also assume that  $k \geq l$ . If  $l = 0$ , then we swap the labels assigned to  $uv$  and  $uw$  by  $\ell'$  so that  $uv$  is now coloured red. Then, we assign red label 1 to  $vv_2, \dots, vv_k$ , and we assign to  $vv_1$  whichever red label makes  $\sigma_{\ell'}^r(v)$  and  $\sigma_{\ell'}^r(u)$  differ; this results in a strong  $(2, 2)$ -colouring of  $G$ . We can now assume  $k, l \geq 1$ . By similar arguments as earlier, we assign blue (resp. red) label 1 to  $vv_2, \dots, vv_k$  (resp.  $ww_2, \dots, ww_l$ ), and we assign to  $vv_1$  (resp.  $ww_1$ ) whichever blue (resp. red) makes the blue (resp. red) sums of  $u$  and  $v$  (resp.  $w$ ) differ. Again, it can be checked that this results in a desired strong  $(2, 2)$ -colouring of  $G$ .

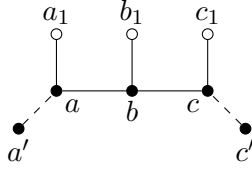


Figure 5: A particular case of Configuration C4, where each of  $a, b, c$  is adjacent to one degree-1 vertex.

(C4) Assume there exists  $C$ , a node of  $T(G)$  associated to a cycle with three consecutive vertices  $a, b, c$  of  $C$  such that none of them is shared by another node of  $T(G)$  other than maybe a leaf tree node (see Figure 5). If they exist, we denote the vertices (other than  $a, b, c$ ) of those trees by  $a_1, b_1, c_1$ , respectively. Note that  $G' = G - \{b\}$  is always nice, so let  $\ell'$  be a strong  $(2, 2)$ -colouring of  $G'$ . We denote by  $a'$  (resp.  $c'$ ) the only neighbour of  $a$  (resp.  $c$ ) in  $G'$ .

- Assume first that  $aa'$  and  $cc'$  have the same colour by  $\ell'$ , say blue. We assign to  $ab$  (resp.  $bc$ ) whichever red label makes  $\sigma_{\ell'}^r(a)$  and  $\sigma_{\ell'}^r(a')$  differ (resp.  $c, c'$ ). If  $a_1$  or  $c_1$  exists, then we assign to their incident edges whichever blue label makes  $\sigma_{\ell'}^b(a)$  and  $\sigma_{\ell'}^b(a')$  differ (resp.  $c, c'$ ). If  $b_1$  exists, then we assign red label 1 to  $bb_1$ .
- Now assume, say, that  $aa'$  is coloured red and  $cc'$  blue by  $\ell'$ . We assign, if it exists, red (resp. blue) label 1 to  $aa_1$  (resp.  $cc_1$ ). We then assign to  $ab$  whichever red label makes  $\sigma_{\ell'}^r(a)$  and  $\sigma_{\ell'}^r(a')$  differ, and to  $bc$  whichever blue label makes  $\sigma_{\ell'}^b(c)$  and  $\sigma_{\ell'}^b(c')$  differ. If  $b_1$  exists, then we also assign to  $bb_1$  whichever red label makes  $\sigma_{\ell'}^r(b)$  differ from  $\sigma_{\ell'}^r(a)$ .

Both cases, it can be checked that we obtain a strong  $(2, 2)$ -colouring of  $G$ .

(C5) Assume there exists a vertex node  $u$  such that exactly one node of  $T(G)$  adjacent to  $u$  is not a leaf of  $T(G)$ , and all other nodes of  $T(G)$  adjacent to  $u$  are associated with either a tree or a  $C_3$ . Intuitively, this configuration encapsulates the cases where there are  $k$  trees and  $l$  triangles hanging from a single vertex (see Figure 6). We denote by  $w_1, \dots, w_k$  the  $k \geq 0$  vertices of the trees other than  $u$  (remember they are of depths at most 1), and by  $v_1, v'_1, \dots, v_l, v'_l$  the  $l \geq 0$  vertices of the cycles other than  $u$  (where  $v_i v'_i$  is an edge of some triangle, for every  $i \in \{1, \dots, l\}$ ). Let us consider  $G' = G - \{v_1, \dots, v'_l, w_1, \dots, w_k\}$ , where  $k + l \geq 2$  (the case where  $k + l \leq 1$  will be handled later, in Configurations 6a to 6c). Note that the degree of  $u$  in  $G'$  is 2. Since there exists a non-leaf node of  $T(G)$  adjacent to  $u$  in  $T(G)$ , necessarily  $G'$  is nice.

- Assume first  $k + l \geq 4$ . We can assume w.l.o.g. that  $\sigma_{\ell'}^b(u) > 0$ . We pick  $A$ , a set of four vertices  $a_1, a_2, a_3, a_4$  in  $G - V(G')$  such that if, for some  $1 \leq i \leq l$ , we have  $v_i \in A$ , then  $v'_i \in A$ . More intuitively, this means we pick arbitrarily either four degree-1 vertices, a cycle and two degree-1 vertices, or two cycles. For  $1 \leq i \leq k$ , if  $w_i \notin A$ , then we assign blue label 1 to  $uw_i$ . For  $1 \leq j \leq l$ , if  $v_j \notin A$  (hence  $v'_j \notin A$  as well), then we assign blue label 1 to  $uv_j$ , red label 2 to  $uv'_j$ , and red label 1 to  $v_j v'_j$ . W.l.o.g., we can assume that  $a_1, a_3$  do not share a cycle. We assign to  $ua_1, ua_3$  (resp.  $ua_2, ua_4$ ) whichever choice of red (resp. blue) labels makes  $\sigma_{\ell'}^r(u)$  (resp.  $\sigma_{\ell'}^b(u)$ ) differs from the red (resp. blue) sums of both its neighbours in  $G'$ . Finally, if two vertices of  $A$  are adjacent, then we assign blue label 1 to the edge incident to both of them. Let us prove that this results in a strong  $(2, 2)$ -colouring of  $G$ . It is easy to check that  $u$  is distinguished in both colours from its neighbours in  $G'$ . Since  $a_1$  and  $a_3$  received incident blue labels and  $a_2, a_4$  received incident red labels,  $u$  is distinguished from

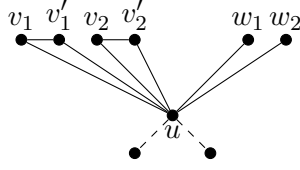


Figure 6: A particular case of Configuration C5, where  $k = l = 2$ .

every of its degree-1 neighbours. Let us now consider  $v_i$  and  $v'_i$  for  $1 \leq i \leq l$ . If  $v_i \in A$ , then, w.l.o.g.,  $\sigma_\ell^b(v_i) = 2$ ,  $\sigma_\ell^b(v'_i) = 1$ ,  $\sigma_\ell^r(v_i) = 0$ , and  $\sigma_\ell^b(v'_i) = 1$ . Note then that  $v_i$  and  $v'_i$  do not have the same red sum. We are thus left to prove that  $\sigma_\ell^b(u) > 2$ . This holds because  $\sigma_{\ell'}^b(u) > 0$ , and at least two edges of  $E(G) \setminus E(G')$  incident to  $u$  were assigned blue labels.

- Assume now  $k \geq 2$  and  $l = 0$ . We denote by  $v_1, \dots, v_k$  the children of  $u$  in  $T$ , and consider  $G' = G - \{v_1, \dots, v_k\}$ . Note that the degree of  $u$  in  $G'$  is 2. Unless  $G'$  is a  $C_3$  (in which case it is easy to produce a strong  $(2, 2)$ -colouring of  $G$ ),  $G'$  is nice; let thus  $\ell'$  be a strong  $(2, 2)$ -colouring of  $G'$ . We assign to  $uv_3, \dots, uv_k$  blue label 1, then assign blue labels to  $uv_1$  and  $uv_2$  so that  $\sigma_\ell^b(u)$  differs from the blue sums of both its neighbours in  $G'$ , which is always possible since we can alter the blue sum of  $u$  in three distinct ways.

Note that this case, along with Configuration 1, prevents any vertex of a minimal counterexample to Theorem 1 from being adjacent to multiple tree subgraph nodes that are leaves.

- Assume next that  $k + l = 3$ .
  - \* If  $k = 2$  and  $l = 1$ , then we assign blue label 2 to  $uv_1$ , blue label 1 to  $uv'_1$  and  $v_1v'_1$ , and then assign blue labels to  $uw_1$  and  $uw_2$  so that  $\sigma_\ell^b(u)$  differs from the blue sums of both its neighbours in  $G'$ .
  - \* If  $k = 1$  and  $l = 2$ , then we assign red label 2 for  $uw_1$ , blue (resp. red) labels to  $uv_1$  and  $uv_2$  (resp.  $uv'_1, uv_2$ ) so that  $\sigma_\ell^b(u)$  (resp.  $\sigma_\ell^r(u)$ ) differs from the blue sums of both its neighbours in  $G'$ , and red label 1 to  $v_1v'_1, v_2v'_2$ .
  - \* If  $l = 3$ , then we assign blue label 1 to  $v_3v'_3$ , red label 2 to  $uv_3$ , and blue label 2 to  $uv'_3$ . We apply the exact same process as in the last case for all other edges.

In all cases, we obtain a strong  $(2, 2)$ -colouring of  $G$ .

- Assume finally that  $k + l = 2$  and  $l \neq 0$ .
  - \* Assume first  $k = 0$ . We assign blue label 1 to  $v_1v'_1, v_2v'_2$ , and blue (resp. red) labels to  $uv_1$  and  $uv_2$  (resp.  $uv'_1, uv_2$ ) so that  $\sigma_\ell^b(u)$  (resp.  $\sigma_\ell^r(u)$ ) differs from the blue sums of both its neighbours in  $G'$ . W.l.o.g., we can assume  $\sigma_{\ell'}^b(u) > 0$ , so we obtain  $\sigma_\ell^b(u) > 2$ .
  - \* Assume now  $k = l = 1$ . We denote by  $a, b$  the two neighbours of  $u$  in  $G'$ . We first handle the case where  $\{\sigma_{\ell'}^r(a), \sigma_{\ell'}^r(b)\} = \{\sigma_{\ell'}^r(u) + 4, \sigma_{\ell'}^r(u) + 5\}$  and  $\{\sigma_{\ell'}^b(a), \sigma_{\ell'}^b(b)\} = \{\sigma_{\ell'}^b(u) + 4, \sigma_{\ell'}^b(u) + 5\}$ . In this case, we assign blue label 1 to  $uv_1$  and  $uw_1$ , red label 2 to  $uv'_1$ , and we assign to  $v_1v'_1$  whichever red label makes  $\sigma_\ell^r(v_1)$  and  $\sigma_\ell^r(u)$  differ. We now consider cases where, for instance, one of  $\sigma_{\ell'}^r(a), \sigma_{\ell'}^r(b)$  is different from both  $\sigma_{\ell'}^r(u) + 4$  and  $\sigma_{\ell'}^r(u) + 5$ . Here, we assign red label 1 to  $uv_1$  and  $v_1v'_1$ , red label 2 to  $uv'_1$ , and to  $uw_1$  whichever red label makes  $\sigma_\ell^r(u)$  differ from the red sums of its neighbours in  $G'$ , which always exists because one of  $\sigma_{\ell'}^r(a), \sigma_{\ell'}^r(b)$  is different from both  $\sigma_{\ell'}^r(u) + 4$  and  $\sigma_{\ell'}^r(u) + 5$ .

In all cases, we again obtain a strong  $(2, 2)$ -colouring of  $G$ .



(C6a) Assume there exist two leaf nodes  $C, C'$  associated to two  $C_3$ 's, where  $C$  (resp.  $C'$ ) is adjacent to a vertex node  $u_1$  (resp.  $u_2$ ), and  $u_1$  and  $u_2$  are adjacent in  $G$ . We further assume that,  $C, C'$  apart, nodes  $u_1, u_2$  are only adjacent to some subgraph node  $N$  of  $T(G)$ . This configuration is illustrated in Figure 7. In particular,  $d(u_1) = d(u_2) = 4$  (as otherwise we would have Configuration 2). We denote by  $v_1, v'_1$  (resp.  $v_2, v'_2$ ) the other two vertices of  $C$  (resp.  $C'$ ), and consider  $G' = G - \{v_1, v'_1, v_2, v'_2, u_1u_2\}$ . We additionally denote by  $w_1$  (resp.  $w_2$ ) the only neighbour of  $u_1$  (resp.  $u_2$ ) in  $G'$ . Since  $G'$  is nice (as otherwise  $G$  would contain Configuration 5), there is  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ . We split our arguments into two main cases.

- Assume first that  $w_1u_1$  and  $w_2u_2$  are assigned the same colour by  $\ell'$ , say blue. W.l.o.g., we can assume  $\sigma_{\ell'}^b(u_1) \geq \sigma_{\ell'}^b(u_2)$ . We assign blue label 1 to  $u_1v_1$  and  $v'_1v_1$ , and blue label 2 to  $u_1v'_1$ . We then assign to  $u_2v_2$  whichever red label makes  $\sigma_{\ell'}^r(u_2)$  and  $\sigma_{\ell'}^r(w_2)$  differ. We assign to  $v_2v'_2$  whichever red label makes  $\sigma_{\ell'}^r(v'_2)$  and  $\sigma_{\ell'}^r(u_2)$  differ. We assign to  $u_1u_2$  whichever blue label makes  $\sigma_{\ell'}^b(u_1)$  and  $\sigma_{\ell'}^b(w_1)$  differ. Finally, we assign to  $u_2v'_2$  whichever blue label makes  $\sigma_{\ell'}^b(u_2)$  and  $\sigma_{\ell'}^b(w_2)$  differ. Since  $\sigma_{\ell'}^b(u_1) \geq \sigma_{\ell'}^b(u_2)$ , note that  $\sigma_{\ell'}^b(u_1) - \sigma_{\ell'}^b(u_2) = \sigma_{\ell'}^b(u_1) + 3 - (\sigma_{\ell'}^b(u_2) + \ell(u_2v'_2)) > 0$ ; hence, we obtain a strong  $(2, 2)$ -colouring of  $G$ .
- Assume now that  $w_1u_1$  is assigned a blue label by  $\ell'$ , while  $w_2u_2$  is assigned a red label. If  $\sigma_{\ell'}^r(w_2) \neq \sigma_{\ell'}^r(u_2) + 3$ , then we assign red label 1 to  $u_2v_2$  and  $v_2v'_2$ , and red label 2 to  $u_2v'_2$ . Otherwise, we assign blue label 1 to  $u_2v_2$ , and to  $u_2v'_2$  and  $v_2v'_2$  the same red label, with the value being the one that is not chosen by  $\ell'$  for  $u_2w_2$ . Then, we assign to  $u_1u_2$  whichever blue label makes  $\sigma_{\ell'}^b(u_2)$  and  $\sigma_{\ell'}^b(w_2)$  differ. We assign to  $u_1v_1$  whichever red label makes  $\sigma_{\ell'}^r(u_1)$  and  $\sigma_{\ell'}^r(w_1)$  differ, and to  $v_1v'_1$  whichever red label makes  $\sigma_{\ell'}^r(v'_1)$  and  $\sigma_{\ell'}^r(u_1)$  differ. Finally, we assign to  $u_1v'_1$  whichever blue label makes  $\sigma_{\ell'}^b(u_1)$  and  $\sigma_{\ell'}^b(w_1)$  differ.

Let us comment on the fact that this does result in a strong  $(2, 2)$ -colouring of  $G$ . In particular, let us check that  $u_1, u_2, v_2, v'_2$  are not in conflict. Since  $u_2v'_2$  was assigned either blue label 1 or a red label, and  $u_1$  has two incident edges distinct from  $u_1u_2$  assigned a blue label, vertices  $u_1$  and  $u_2$  indeed have distinct blue sums. Moreover,  $\sigma_{\ell'}^r(u_1) \leq 2$  while  $\sigma_{\ell'}^r(u_2) \geq 3$ ; hence,  $u_1$  and  $u_2$  are distinguished in both colours. All pairs involving  $u_2, v_2, v'_2$  are trivially distinguished in blue, while  $\sigma_{\ell'}^r(v'_2) \leq 2$  from which we deduce  $u_2$  and  $v'_2$  are distinguished in red. Now, either  $u_2v'_2$  was assigned a red label and it is easy to check that  $\ell$  is as desired, or  $u_2v'_2$  was assigned blue label 1. In that case, since we assigned the same red label to  $u_2v_2$  and  $v_2v'_2$ , note that  $\sigma_{\ell'}^r(v_2)$  and  $\sigma_{\ell'}^r(v'_2)$  are both even, while, since we assigned to both the value that was not assigned to  $u_2w_2$ , we have that  $\sigma_{\ell'}^r(u_2)$  is odd, guaranteeing another distinction. From these arguments, it is not too complicated to be convinced that  $\ell$  has the desired properties.

(C6b) Assume there exist two leaf nodes  $C, T$  associated to a  $C_3$  and a tree, respectively, where  $C$  (resp.  $T$ ) is adjacent to a vertex node  $u_1$  (resp.  $u_2$ ), and  $u_1$  and  $u_2$  are adjacent in  $G$ . We further assume that,  $C$  and  $T$  apart,  $u_1, u_2$  are only adjacent to some subgraph node  $N$  of  $T(G)$  (see Figure 7). We denote by  $v_1, v'_1$  (resp.  $v_2$ ) the other vertices of  $C_1$  (resp.  $T$ ), and consider  $G' = G - \{v_1, v'_1, v_2, u_1u_2\}$ . Also, we denote by  $w_1$  (resp.  $w_2$ ) the only neighbour of  $u_1$  (resp.  $u_2$ ) in  $G'$ . Since  $G'$  is nice (as otherwise  $G$  would contain one of Configurations 1 and 5), we consider  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ . Assuming w.l.o.g. that  $w_1u_1$  is assigned a blue label by  $\ell'$ , we consider two main cases.

- Assume first  $\sigma_{\ell'}^b(u_2) > \sigma_{\ell'}^b(u_1)$  (hence that  $w_2u_2$  is assigned a blue label by  $\ell'$ , and  $\ell'(w_1u_1) = 1$  while  $\ell'(w_2u_2) = 2$ ). We assign red label 1 to  $u_1v_1$  and  $v_1v'_1$ , and red label 2 to  $u_1v'_1$ . We then

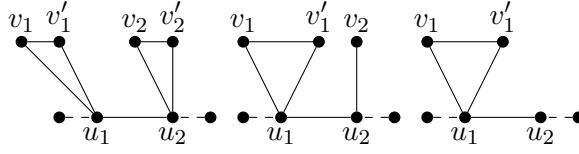


Figure 7: Configurations C6a to C6c.

assign to  $u_1u_2$  whichever red label makes  $\sigma_\ell^r(u_1)$  and  $\sigma_\ell^r(w_1)$  differ, and to  $u_2v_2$  whichever red label makes  $\sigma_\ell^r(u_2)$  and  $\sigma_\ell^r(w_2)$  differ.

- Assume second  $\sigma_{\ell'}^b(u_2) \leq \sigma_{\ell'}^b(u_1)$ . We here assign blue label 1 to  $u_1v_1$  and  $v_1v'_1$ , and blue label 2 to  $u_1v'_1$ . We then assign to  $u_1u_2$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(w_1)$  differ, and to  $u_2v_2$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(w_2)$  differ.

In both cases, it is easy to check that this results in a strong  $(2, 2)$ -colouring of  $G$ .

(C6c) Assume there exists a leaf node  $C$  associated to a  $C_3$ , where  $C$  is adjacent to a vertex node  $u_1$  adjacent to a vertex  $u_2$  in  $G$  where  $u_2$  is not a vertex node of  $T(G)$ . We further assume that,  $C$  apart, vertex  $u_1$  is only adjacent to some subgraph node  $N$  of  $T(G)$  (see Figure 7 for an illustration). In particular, remark that  $u_2$  must be a vertex of  $N$ . We can further assume  $u_2$  is of degree 2 in  $G$  (as otherwise we would fall back into Configuration 6b). We consider  $G' = G - \{u_1u_2, v_1, v'_1\}$ , where  $v_1$  and  $v'_1$  are the vertices of  $C$ . We additionally denote by  $w_1, w_2$  the respective neighbour of  $u_1, u_2$  in  $G'$ . Since  $G'$  is nice (as otherwise  $G$  would contain one of Configurations 1 and 5), we can consider  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ . Again, we focus on two main cases.

- Assume first  $u_1w_1$  is assigned a blue label while  $u_2w_2$  is assigned a red label by  $\ell'$ . We assign to  $u_1u_2$  whichever red label makes  $\sigma_{\ell'}^r(u_2)$  and  $\sigma_{\ell'}^r(w_2)$  differ. Then, if  $\sigma_{\ell'}^r(w_1) = 3$ , then we assign red label 1 to  $u_1v_1, v_1v'_1$  and red label 2 to  $u_1v'_1$ . Otherwise, we assign to  $u_1v_1$  whichever blue label makes  $\sigma_{\ell'}^b(u_1)$  and  $\sigma_{\ell'}^b(w_1)$  differ, and to  $u_1v'_1, v_1v'_1$  the red label with the value that we did not assign to  $u_1u_2$ . By argument similar as earlier, this yields the desired properties.
- Assume now  $u_1w_1, u_2w_2$  are assigned a blue label by  $\ell'$ . We consider  $k \in \{1, 2\}$ , the value such that assigning blue label  $k$  to  $u_1u_2$  makes  $\sigma_{\ell'}^b(u_2)$  and  $\sigma_{\ell'}^b(w_2)$  differ. If  $\sigma_{\ell'}^b(u_1) + k + 3 \neq \sigma_{\ell'}^b(w_1)$ , then we assign blue label  $k$  to  $u_1u_2$ , blue label 1 to  $u_1v_1$  and  $v_1v'_1$ , and blue label 2 to  $u_1v'_1$ . Otherwise, we assign to  $u_1u_2$  whichever red label makes  $\sigma_{\ell'}^r(u_2)$  and  $\sigma_{\ell'}^r(w_2)$  differ. We then assign to  $u_1v_1$  whichever red label makes  $\sigma_{\ell'}^r(u_1)$  and  $\sigma_{\ell'}^r(w_1)$  differ. Finally, we assign to  $u_1v'_1, v_1v'_1$  the same blue label, with the value being different from the label assigned to  $u_1w_1$ . By arguments similar as before, this results in a strong  $(2, 2)$ -colouring of  $G$ .

We are now ready to prove that there is no cactus other than  $K_2$  and  $K_3$  that does not contain any of those configurations, which implies Theorem 1. Let us consider  $G$ , any cactus (different from  $K_2$  and  $K_3$ ) with a tree decomposition  $T(G)$ , and root  $T(G)$  arbitrarily at any subgraph node. Let  $L$  be a deepest leaf of  $T(G)$ . Since  $G$  does not contain Configuration 4,  $L$  cannot be associated to a cycle of length more than 4. Let us denote  $P$  the parent node of  $L$  in  $T(G)$ . If  $L$  is a tree, then, since  $G$  does not contain Configuration 1,  $L$  is of depth at most 1. We can assume  $L$  is either a tree or a  $C_3$ . If  $P$  is a tree, then w.l.o.g.  $L$  is a  $C_3$ , and either  $u$  is of degree 3 and  $G$  would contain Configuration 2, or  $u$  has degree greater than 4. Since  $L$  is a deepest leaf, any subgraph node adjacent to  $u$  is also a leaf node,

thus associated to either a  $C_3$  or a tree, and  $G$  would contain Configuration 5. Since  $G$  does not contain Configurations 3 and 6c,  $P$  cannot be associated to a  $C_3$ , hence  $P$  is associated to a cycle of length at least 4. Let us consider the neighbours of  $u$ , the vertex shared by  $P$  and  $L$ . Because  $L$  is a deepest leaf, at least one neighbour of  $u$  in  $G$  is not shared by  $P'$ , the parent node of  $P$  in  $T(G)$ ; let us denote it by  $v$ . Finally, let us denote by  $w$  the other neighbour of  $v$  in  $P$ . Since  $P$  is of length at least 4, vertices  $u, v, w$  are pairwise distinct, and every node containing any of them is a leaf. Since  $G$  does not contain Configuration 4, none of them is shared by a cycle of length more than 4 other than  $P$ . If one of them is shared by a  $C_3$ , then at least one of its neighbours in  $\{u, v, w\}$  is either shared by another  $C_3$  (and  $G$  would contain Configuration 6a), a tree (and  $G$  would contain Configuration 6b) or none (and  $G$  would contain Configuration 6c). Hence, none of them is shared by a  $C_3$  and  $G$  would contain Configuration 4. Hence,  $L$  must be the only node of  $T(G)$ . If  $L$  is a cycle of length more than 4, we can conclude; so,  $G$  is either  $K_2$  or  $K_3$ .  $\square$

### 3. Subcubic outerplanar graphs

Recall that a graph  $G$  is *subcubic* if  $\Delta(G) \leq 3$  (where  $\Delta(G)$  denotes the maximum degree of  $G$ ), and *outerplanar* if  $G$  admits an embedding on the plane such that all vertices lie on the outerface. It is well known that outerplanar graphs are 2-degenerate, and thus 3-colourable.

In the next result, we prove the Strong  $(2, 2)$ -Conjecture for nice subcubic outerplanar graphs.

**Theorem 2.** *If  $G$  is a nice subcubic outerplanar graph, then  $G$  is strongly  $(2, 2)$ -colourable.*

*Proof.* The structure of the proof is the same as the proof of Theorem 1. First off, similarly, we claim outerplanar graphs  $G$  have a tree representation  $T(G)$  where:

- all nodes are either *vertex nodes*, associated to a vertex of  $G$ , *edge nodes*, associated to an edge of  $G$ , or *subgraph nodes*, associated to an induced tree or to an induced cycle of  $G$ ;
- all leaves of  $T(G)$  are subgraph nodes;
- the set of vertex and edge nodes, as well as the set of subgraph nodes, are both independent sets;
- two subgraph nodes associated to  $H_1, H_2$  are adjacent to a vertex node  $u$  iff  $V(H_1) \cap V(H_2) = \{u\}$ ;
- two subgraph nodes associated to cycles  $H_1, H_2$  are adjacent to an edge node  $uv$  iff  $V(H_1) \cap V(H_2) = \{u, v\}$ .

To reach such a tree representation  $T(G)$  for any outerplanar graph  $G$ , one can consider any outerplanar embedding of  $G$ , and add to  $T(G)$  an edge node  $e_{uv}$  for every edge  $uv$  not on the outerface. Note that  $uv$  is part of exactly two induced cycles without chords (as otherwise either  $uv$  would lie on the outerface, or  $G$  would not be outerplanar). Then add to  $T(G)$  a subgraph node associated to every chordless cycle of  $G$ , and, for each node  $e_{uv}$ , add to  $T(G)$  an edge between  $e_{uv}$  and the two subgraph nodes associated to the chordless cycles containing  $uv$ . Then, for all bridges  $uv$  of  $G$ , add to  $T(G)$  a subgraph node  $s_{uv}$  associated to a tree on the two vertices  $u, v$ . Finally, if two subgraph nodes  $S_1, S_2$  share a vertex  $v$  (but not an edge), then add to  $T(G)$  a vertex node  $V$  associated to  $v$  and add an edge between  $V$  and  $S_1$ , as well as an edge between  $V$  and  $S_2$ . Because of the adjacency rules we provided, edge nodes and vertex node together induce an independent set, and the same goes for the set of subgraph nodes. Note also that if  $T(G)$  contains a cycle  $C$ , then no subgraph node of this cycle

is a tree (as otherwise the associated tree would not be a bridge). If a vertex of  $C$  is an edge node, then one vertex of  $G$  is not on the outerface, and hence  $C$  is made of vertex nodes and cycle subgraph nodes. Moreover, let us consider  $U, V$ , two vertex nodes adjacent to a subgraph node  $S$ , where all tree of them are vertices of  $C$ . We claim that  $u, v$  are adjacent in  $G$ . Assume this is not the case. Since  $S$  is associated to a cycle, then there exists  $w$ , a vertex of  $S$  not on the outerface, which is a contradiction. In particular, all the vertices of  $C$  form a chordless cycle of  $G$  sharing an edge  $uv$  with  $S$ , which is a contradiction since  $C$  does not contain edge nodes.

In particular, note that all configurations treated along the proof of Theorem 1 still apply, as cacti are outerplanar. Configurations 1 to 6c actually deal with subgraph nodes adjacent to vertex nodes. To essentially extend the proof to subcubic outerplanar graphs, we must thus deal with cases covering subgraph nodes adjacent to edge nodes.

(C7) Assume there exists a subgraph node  $C$  associated to a  $C_3$  adjacent to an edge node  $u_1u_2$  where, if we denote by  $v$  the last vertex of  $C$ , then  $v$  is adjacent to a leaf of  $T(G)$ . In particular, since  $G$  is subcubic, then it must be that  $v$  is of degree 2 in  $T(G)$ . Moreover, we assume the tree subgraph adjacent to  $v$  is of depth 1, and we denote by  $w$  its only vertex other than  $v$ . Finally, we can assume  $d(u_1) = d(u_2) = 3$  (as otherwise  $G$  would contain Configuration 3). Additionally, we denote by  $u'_1$  (resp.  $u'_2$ ) the other neighbour of  $u_1$  (resp.  $u_2$ ). Intuitively, this configuration corresponds to a triangle with a pendant edge incident to one vertex, and connected to the rest of the graph by its other two vertices (see Figure 8). We consider  $G' = G - \{u_1u_2, v, w\}$ ; since  $G'$  is nice (as otherwise  $G$  would contain Configuration 3), we can consider  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ . W.l.o.g. we can assume  $u'_1u_1$  is assigned a blue label by  $\ell'$ .

- Assume  $u'_2u_2$  is assigned a red label by  $\ell'$ . We assign to  $u_1u_2$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(u'_2)$  differ. We assign to  $u_1v$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(u'_1)$  differ. We assign to  $u_2v$  whichever red label makes  $\sigma_\ell^r(u_2)$  and  $\sigma_\ell^r(u'_2)$  differ. If  $\ell(u_1u_2) = 2$ , then we assign to  $vw$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(v)$  differ. Note that  $\sigma_\ell^b(u_1) \geq \ell(u_1v) + 3$ ; hence, no matter the label we assign to  $vw$ , vertices  $v$  and  $u_1$  must be distinguished. Otherwise, we assign to  $vw$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(v)$  differ. Since  $\sigma_\ell^b(u_2) = 1$ , then we obtain a strong  $(2, 2)$ -colouring of  $G$ .

From now on, we can thus assume  $u'_2u_2$  is assigned a blue label by  $\ell'$ .

- Assume first  $\ell'(u'_1u_1) = \ell'(u'_2u_2) = 1$ . Assume that  $\sigma_{\ell'}^r(u'_1) \neq 3$ . We assign to  $u_1u_2$  whichever red label makes  $\sigma_\ell^r(u_2)$  and  $\sigma_\ell^r(u'_2)$  differ, and to  $u_1v$  a red label with the value in  $\{1, 2\}$  we did not assign to  $u_1u_2$ . This way,  $\sigma_\ell^r(u_1) = 3$ . We then assign to  $u_2v$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(u'_2)$  differ. We assign to  $vw$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(v)$  differ. We can now assume  $\sigma_{\ell'}^r(u'_1) = \sigma_{\ell'}^r(u'_2) = 3$ . We assign red label 1 to  $vw$  and  $u_1v$ . We then assign to  $u_1u_2$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(u'_1)$  differ, and to  $vu_2$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(u'_1)$  differ. Unless  $\ell(u_1u_2) = 1$  and  $\ell(vu_2) = 2$ , this results in a strong  $(2, 2)$ -colouring of  $G$ . However, that exceptional case occurs only if  $\sigma_{\ell'}^b(u'_2) = 4$  and  $\sigma_{\ell'}^b(u'_1) = 3$ , in which case we can just swap  $u_1$  and  $u_2$  in the choices made above to be done the same way.
- Assume second  $\ell'(u'_1u_1) = 2$  and  $\ell'(u'_2u_2) = 1$ . Assume first  $\sigma_{\ell'}^r(u'_2) \neq 3$ . We assign to  $u_1u_2$  whichever red label makes  $\sigma_\ell^r(u_1)$  and  $\sigma_\ell^r(u'_1)$  differ, and the red label with the other value to  $u_2v$ . We then assign to  $u_1v$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(u'_1)$  differ, and to  $vw$

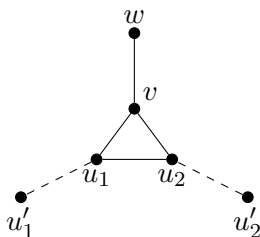


Figure 8: Configuration C7.

whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(v)$  differ. We can now assume  $\sigma_{\ell'}^r(u'_2) = 3$ . Assume first  $\sigma_{\ell'}^r(u'_1) \neq 3$ . We assign red label 1 to  $u_1u_2, u_2v, vw$ , and red label 2 to  $u_1v$ , to be done. Assume now that  $\sigma_{\ell'}^r(u'_1) = \sigma_{\ell'}^r(u'_2) = 3$ . If  $\sigma_\ell^b(u'_1) = 5$  and  $\sigma_\ell^b(u'_2) = 3$ , then we assign blue label 2 to  $u_1u_2, u_1v, vw$ , and blue label 1 to  $u_2v$ . Otherwise, we assign red label 1 to  $u_1v$  and  $vw$ , to  $u_1u_2$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(u'_2)$  differ, and to  $u_1v$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(u'_1)$  differ. This results in a strong  $(2, 2)$ -colouring of  $G$ , unless  $\sigma_\ell^b(v) = \sigma_\ell^b(u_2)$ ; in that case, we had to choose  $\ell(u_1u_2)$  to have value 1 and  $\ell(u_1v)$  to have value 2, a case we already covered.

- Assume third  $\ell'(u'_1u_1) = \ell'(u'_2u_2) = 2$ . Assume first that  $\sigma_{\ell'}^r(u'_1) \neq 3$ . We assign to  $u_1u_2$  whichever red label makes  $\sigma_\ell^r(u_2)$  and  $\sigma_\ell^r(u'_2)$  differ, and to  $u_1v$  the red label with the value we did not assign to  $u_1u_2$ . This way,  $\sigma_\ell^r(u_1) = 3$ . We then assign to  $u_2v$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(u'_2)$  differ. If  $\ell(u_2v) = 1$ , then we assign red label 2 to  $vw$ . Otherwise, we assign blue label 1 to  $vw$ . We can now assume that  $\sigma_{\ell'}^r(u'_1) = \sigma_{\ell'}^r(u'_2) = 3$ . We assign to  $u_1u_2$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(u'_1)$  differ, and to  $u_2v$  whichever blue label makes  $\sigma_\ell^b(u_2)$  and  $\sigma_\ell^b(u'_2)$  differ. Here, we finally assign red label 1 to  $u_1v$  and  $vw$ . In all cases, it can be checked that we eventually obtain a strong  $(2, 2)$ -colouring of  $G$ .

(C8) Assume there exists a node  $C$  associated to a  $C_4$  such that  $C$  is adjacent to some edge node  $u_1u_2$ . We denote by  $v_1$  and  $v_2$  the other two vertices of  $C$ . We further assume that if  $T(G)$  contains either  $v_1, v_2$ , or  $v_1v_2$  as a node, then all nodes adjacent to it other than  $C$  are leaves. More intuitively, this configuration corresponds to a  $C_4$  where two vertices are adjacent to trees of depth 1 or to a  $C_3$  (see Figure 9). Additionally, if it exists, then we denote by  $w_1$  (resp.  $w_2$ ) the child of the vertex node  $v_1$  (resp.  $v_2$ ), where possibly  $w_1 = w_2$ . We here consider  $G' = G - \{u_1u_2, v_1, v_2\}$ , as well as  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$  (note that  $G'$ , indeed, is nice). We assume w.l.o.g. that  $\sigma_{\ell'}^b(u_1) \geq \sigma_{\ell'}^b(u_2)$  and that  $u_1u'_1$  was assigned a blue label by  $\ell'$ . We assign to  $u_1u_2$  whichever blue label makes  $\sigma_\ell^b(u'_2)$  and  $\sigma_\ell^b(u_2)$  differ. We then assign to  $v_1u_1$  whichever blue label makes  $\sigma_\ell^b(u'_1)$  and  $\sigma_\ell^b(u_1)$  differ, to  $v_1w_1$  (if it exists) whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(v_1)$  differ, and to  $u_2v_2$  whichever red label makes  $\sigma_\ell^r(u'_2)$  and  $\sigma_\ell^r(u_2)$  differ. We also assign to  $v_1v_2$  whichever red label makes  $\sigma_\ell^r(v_2)$  and  $\sigma_\ell^r(u_2)$  differ. If  $v_2w_2$  exists, then we also assign to  $v_2w_2$  the red label with the value we did not assign to  $v_1v_2$ . This way, in the case where  $w_2$  exists and  $w_2$  and  $v_1$  are adjacent, they have distinct red sums. It is also easy to check that any pair of adjacent vertices are distinguished in both colours by the provided colouring. Hence, we obtain a strong  $(2, 2)$ -colouring of  $G$ .

(C9) Assume there exists an edge node  $u_1u_2$  adjacent to two subgraph nodes both associated to a  $C_3$ . Further assume that one of those subgraph nodes is a leaf of  $T(G)$  (see Figure 10). We here

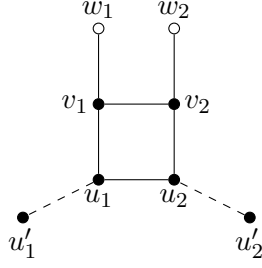


Figure 9: A particular case of Configuration C8, where  $w_1$  and  $w_2$  exist but are distinct.

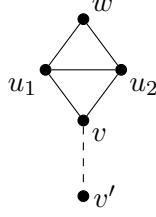


Figure 10: Configuration C9.

denote by  $w$  the last vertex of the leaf node, and by  $v$  the last vertex of the other  $C_3$ . We can also assume that  $d(v) = 3$ , that  $v'$  denotes its last neighbour, and that  $G' = G - \{u_1, u_2, w\}$  is nice, and consider  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ . W.l.o.g. assume  $vv'$  is assigned a blue label by  $\ell'$ . We assign to  $vu_1$  whichever blue label makes  $\sigma_\ell^b(v)$  and  $\sigma_\ell^b(v')$  differ, and to  $u_1w$  whichever blue label makes  $\sigma_\ell^b(v)$  and  $\sigma_\ell^b(u_1)$  differ. We then assign to  $vu_2$  whichever red label makes  $\sigma_\ell^r(v)$  and  $\sigma_\ell^r(v')$  differ, the same red label to  $u_2w$ , and to  $u_1u_2$  the red label with the value we did not assign to  $u_2w$ , so that the red sums of  $u_2$  and  $w$  are indeed distinct. It can then be checked that this results in a strong  $(2, 2)$ -colouring of  $G$ .

(C10) Assume there exist two subgraph nodes  $C, C'$  both associated to a  $C_3$ , such that  $C$  contains a vertex  $u_1$  and  $C'$  contains a vertex  $u_2$ , where  $u_1$  and  $u_2$  are adjacent (see Figure 11). We denote by  $v_1$  and  $w_1$  (resp.  $v_2$  and  $w_2$ ) the other two vertices of  $C$  (resp.  $C'$ ) and further assume that  $d(w_1) = d(w_2) = 2$ . Since  $G$  does not contain Configuration 2, it must be that  $d(v_1) = d(v_2) = 3$ , and we denote by  $v'_1$  (resp.  $v'_2$ ) the other neighbour of  $v_1$  (resp.  $v_2$ ). Also, since  $G$  does not contain Configuration 3,  $G' = G - \{u_1, u_2, w_1, w_2\}$  is nice and we can consider  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ . We, again, consider two main cases.

- First assume  $v_1v'_1$  and  $v_2v'_2$  are assigned labels with the same colour by  $\ell'$ , say blue. We assign to  $w_2v_2$  whichever blue label makes  $\sigma_\ell^b(v_2)$  and  $\sigma_\ell^b(v'_2)$  differ, and to  $v_1u_1$  whichever blue label makes  $\sigma_\ell^b(v_1)$  and  $\sigma_\ell^b(v'_1)$  differ. We assign to  $v_1w_1$  whichever red label makes  $\sigma_\ell^r(v_1)$  and  $\sigma_\ell^r(v'_1)$  differ, and to  $u_1u_2$  the red label with the other value. We also assign to  $u_1w_1$  whichever red label makes  $\sigma_\ell^r(u_1)$  and  $\sigma_\ell^r(v_1)$  differ, and to  $u_2v_2$  whichever red label makes  $\sigma_\ell^r(v_2)$  and  $\sigma_\ell^r(v'_2)$  differ. We last assign to  $u_2w_2$  whichever red label makes  $\sigma_\ell^r(v_2)$  and  $\sigma_\ell^r(w_2)$  differ, which is the value we did not assign to  $u_2v_2$ . This way, no matter what labels we assigned to  $u_1w_1$ , vertices  $u_1$  and  $u_2$  are distinguished, so we get a strong  $(2, 2)$ -colouring.
- We now assume  $v_1v'_1$  is assigned a blue label, while  $v_2v'_2$  is assigned a red one. Further assume that either  $\ell'(v_1v'_1) = 2$  or  $\sigma_\ell^b(v'_2) \neq 3$ . We assign to  $v_1w_1$  whichever blue label makes  $\sigma_\ell^b(v_1)$  and  $\sigma_\ell^b(v'_1)$  differ. The condition above implies that either we assigned the label with value

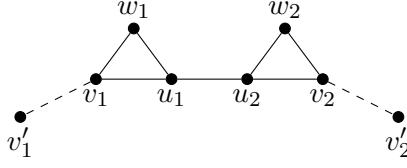


Figure 11: Configuration C10.

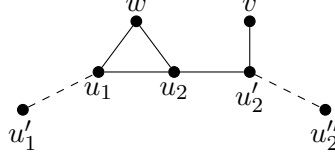


Figure 12: Configuration C11.

2, or  $\sigma_\ell^b(v_1) \neq 2$ . We then assign to  $u_1v_1$  whichever red label makes  $\sigma_\ell^r(v_2)$  and  $\sigma_\ell^r(v'_2)$  differ, and to  $u_1w_1$  the red label with the other value. We then assign to  $u_1u_2$  whichever blue label does not create any conflict between  $u_1$  and neither  $w_1$  nor  $v_1$ , which is always possible since either  $\ell(v_1w_1) = 2$  (and we can assign blue label 1 to  $u_1u_2$ ) or  $\sigma_\ell^b(v_1) = 3$  (and we can assign blue label 2 to  $u_1u_2$ ). We also assign to  $v_2w_2$  blue label 1, to  $u_2v_2$  whichever blue label makes  $\sigma_\ell^b(v_2)$  and  $\sigma_\ell^b(v'_2)$  differ, as well as the blue label of the other value to  $u_2w_2$ . It can be checked that we obtain a strong  $(2, 2)$ -colouring of  $G$ .

We can now assume that  $\sigma_\ell^b(v'_2) = 3$  and  $\ell'(v_1v'_1) = 1$ . We assign blue label 2 to all of  $u_1v_1, v_1w_1, u_1w_1, u_1u_2$ . We then assign blue label 1 to  $v_2w_2$ . We assign to  $u_2v_2$  whichever blue label makes  $\sigma_\ell^b(v_2)$  and  $\sigma_\ell^b(v'_2)$  differ, as well as the blue label of the other value to  $u_2w_2$ . It can be checked that doing this makes the blue sums of  $v_2$  and  $w_2$  differ; in particular, again, we obtain a strong  $(2, 2)$ -colouring of  $G$ .

Either way, we are thus done.

- (C11) Assume there exists a subgraph node  $C$  associated to a  $C_3$  adjacent to either two vertex nodes  $u_1, u_2$  or to an edge node  $u_1u_2$  (see Figure 12). We denote by  $w$  the last vertex of  $C$ , by  $u'_1$  the other neighbour of  $u_1$ , and by  $u'_2$  the other neighbour of  $u_2$ . Further assume that if  $d(u'_2) = 3$ , then one of its neighbours is a degree-1 vertex, denoted  $v$ . Either way, we denote by  $u''_2$  the other neighbour of  $u'_2$ . More intuitively, this configuration corresponds to a triangle adjacent either to a degree-2 vertex or to a degree-3 vertex with an incident pendant edge. Since  $G$  does not contain Configuration 1 or 3, graph  $G' = G - \{w, u_2, v\}$  is nice; let us denote by  $\ell'$  a strong  $(2, 2)$ -colouring of  $G'$ . We can assume w.l.o.g. that  $u_1u'_1$  is assigned a blue label by  $\ell'$ . We assign to  $u_1w$  whichever blue label makes  $\sigma_\ell^b(u_1)$  and  $\sigma_\ell^b(u'_1)$  differ, to  $u_1u_2$  whichever red label makes  $\sigma_\ell^r(u_1)$  and  $\sigma_\ell^r(u'_1)$  differ, and to  $u_2w$  the red label of the other value. We also assign to  $u'_2v$ , if  $v$  exists, whichever red label results in  $\sigma_\ell^r(u'_2) + \ell(u'_2v) \neq 3$ . We last assign to  $u_2u'_2$  whichever red label makes  $\sigma_\ell^r(u'_2)$  and  $\sigma_\ell^r(u''_2)$  differ. One can check that if  $v$  does not exist, then  $\sigma_\ell^r(u'_2) < \sigma_\ell^r(u_2)$ . Otherwise,  $u_2$  and  $u'_2$  are distinguished due to the red label we assigned to  $u'_2v$ . Either way, we thus end up with a strong  $(2, 2)$ -colouring of  $G$ .

We are now ready to prove that there are no minimal counterexamples to Theorem 2. Consider  $G$  such a counterexample, and consider  $T(G)$ , the tree representation of  $G$ , rooted arbitrarily. First, assume

there exists  $T$ , a deepest leaf of  $T(G)$  associated to a tree. Since  $G$  does not contain Configuration 1, then  $T$  is associated to a  $K_2$ . We denote by  $u$  its father in  $T(G)$ . Since  $G$  is subcubic,  $u$  has exactly one neighbour other than  $T$ , which we denote by  $N$ . Because  $G$  does not contain Configuration 7,  $N$  cannot be a  $C_3$ . If  $N$  is a  $C_4$ , then either  $u$  is adjacent to a vertex of degree 2 and  $G$  contains one of Configurations 4 and 8, or both neighbours of  $u$  are of degree 3. In particular, since  $G$  is outerplanar, one of them, denote it  $u'$ , is also part of a leaf of  $T(G)$  other than  $T$ , denote it  $L$ . Since  $G$  does not contain Configuration 8,  $L$  must be a cycle. Now, since  $G$  contains neither Configuration 4 nor 8,  $L$  cannot be a cycle of length more than 3, and hence  $G$  contains Configuration 11.

So,  $T(G)$  cannot contain any deepest leaf associated to a tree. Assume now there exists  $C$ , a deepest leaf associated to a cycle. Since  $G$  contains neither Configuration 4 nor 8,  $C$  must be associated to a  $C_3$ . Since  $G$  does not contain Configuration 9, there must exist a vertex  $u'$  adjacent to a vertex of  $C$  shared by another deepest leaf of  $T(G)$ . Using arguments similar to those we used by end of the proof of Theorem 1, we deduce that a vertex of  $C$  is either adjacent to a vertex of degree 2 (resulting in Configuration 11), to another  $C_3$  (resulting in Configuration 10), to a  $C_4$  (resulting in Configuration 8), or to a larger cycle (resulting in Configuration 4). So,  $G$  cannot exist, and the claim holds.  $\square$

#### 4. Graphs of maximum average degree less than $\frac{9}{4}$

Recall that the *average degree*  $\text{ad}(G)$  of a graph  $G$  is defined as  $\frac{2|E(G)|}{|V(G)|}$ , while the *maximum average degree*  $\text{mad}(G)$  of  $G$  is the maximum value of the average degree  $\text{ad}(H)$  over all subgraphs  $H$  of  $G$ . Below, we prove the Strong (2, 2)-Conjecture for nice graphs with maximum average degree less than  $\frac{9}{4}$ . Note that such graphs are 2-degenerate, and thus 3-colourable.

**Theorem 3.** *If  $G$  is a graph with  $\text{mad}(G) < \frac{9}{4}$ , then either  $G$  is strongly (2, 2)-colourable, or  $G$  has a connected component isomorphic to  $K_2$  or  $K_3$ .*

*Proof.* We use the *discharging method*, where we first prove that a minimal (w.r.t. the number of edges) counterexample to Theorem 3 cannot contain some sparse configurations in the same fashion as the proofs of Theorems 1 and 2. We then reach a contradiction on the sparseness of the minimum counterexample through the use of charge functions. Although this method is well established, we refer the reader to the first two paragraphs of the proof of Theorem 4.1 in [9] for a more detailed explanation. We enhance the discharging method with the so-called *ghost vertices method* (introduced formally in [11]), which, in the context of graphs with bounded maximum average degree, provides a way, through some discharging process, to establish that a graph has large maximum average degree.

**Theorem 4** (see e.g. [10]). *Let  $G$  be a graph,  $m$  be some value, and  $(V_1, V_2)$  be any partition of  $V(G)$ . Let also  $\omega$  be a charge function where  $\omega(v) = d(v) - m$  for every  $v \in V(G)$ . If there is a discharging process resulting in a charge function  $\omega^*$  where*

- $\omega^*(v) \geq 0$  for every  $v \in V_1$ , and
- $\omega^*(v) \geq \omega(v) + d_{V_1}(v)$  for every  $v \in V_2$ ,

then  $\text{mad}(G) \geq m$ .

We now consider  $G$ , a minimal counterexample to Theorem 3. We call a vertex  $u \in V(G)$  a *k-vertex* (resp.  $k^+$ ,  $k^-$ ) if  $d(u) = k$  (resp.  $d(u) \geq k$ ,  $d(u) \leq k$ ). We call a 3-vertex *weak* if it is adjacent to a 1-vertex.



*Claim 4.1.*  $G$  does not contain any of the following configurations:

- (C1) a  $3^-$ -vertex adjacent to two 1-vertices;
- (C2) a  $d$ -vertex adjacent to more than  $\lfloor \frac{d+1}{2} \rfloor$  1-vertices, for any  $d \geq 4$ ;
- (C3) two adjacent weak 3-vertices;
- (C4) a  $d$ -vertex adjacent to  $d - 2$  1-vertices and one weak 3-vertex, for any  $d \geq 4$ ;
- (C5) a 2-vertex adjacent to two 2-vertices;
- (C6) a 2-vertex adjacent to a 1-vertex;
- (C7) a weak 3-vertex adjacent to a 2-vertex.

*Proof of the claim.* For each configuration  $H$ , let us assume  $G$  contains  $H$  as an induced subgraph. We will consider  $G'$ , obtained from  $G$  by deleting edges, and whenever it is possible, obtain  $\ell'$  a strong  $(2, 2)$ -colouring of  $G'$  by the minimality of  $G$ . If we are able to extend  $\ell'$  over the deleted edges so that the resulting colouring  $\ell$  is a strong  $(2, 2)$ -colouring of  $G$ , then we reach a contradiction. Note that the extension of  $\ell'$  into  $\ell$  may only alter the sums of vertices incident to the deleted edges; hence, we only have to locally check that  $\ell$  is a strong  $(2, 2)$ -colouring.

In what follows, the order in which we deal with Configurations C1 to C7 is with respect to the complexity of our proof arguments.

- (C1) Assume there exists  $u \in V(G)$ , a  $3^-$ -vertex adjacent to two 1-vertices  $v_0, v_1$ . If  $u$  is a 2-vertex, then  $G$  is  $P_3$  and the result obviously holds. Hence, we denote by  $v_2$  the last neighbour of  $u$ , and consider  $G' = G - \{uv_0, uv_1\}$ . Note that since the degree of  $u$  in  $G'$  must be 1, then either  $G'$  is isomorphic to  $K_2$  (in which case  $G$  is strongly  $(2, 2)$ -colourable since it is isomorphic to the star on three leaves) or  $G'$  is strongly  $(2, 2)$ -colourable and we denote by  $\ell'$  such a colouring of the edges of  $G'$ . We further assume w.l.o.g. that  $uv_2$  is assigned a red label. We assign blue label 1 to  $uv_0$  and assign to  $uv_1$  whichever blue label makes  $\sigma_{\ell'}^b(u)$  distinct from  $\sigma_{\ell'}^b(v_2)$  (which is equal to  $\sigma_{\ell'}^b(v_2)$ ). It is easy to check that this results in a strong  $(2, 2)$ -colouring of  $G$ .
- (C6) Assume there exists  $u \in V(G)$ , a 2-vertex adjacent to a 1-vertex  $v_0$ . We denote by  $v_1$  the other neighbour of  $u$ , and consider  $G' = G - \{uv_0\}$ . Note that since the degree of  $u$  in  $G'$  must be 1, then either  $G'$  is isomorphic to  $K_2$  (in which case  $G$  is strongly  $(2, 2)$ -colourable since it is isomorphic to  $P_3$ ) or  $G'$  is strongly  $(2, 2)$ -colourable and we denote by  $\ell'$  such a colouring of the edges of  $G'$ . We further assume w.l.o.g. that  $uv_1$  is assigned a red label. We assign to  $uv_0$  whichever red label makes  $\sigma_{\ell'}^r(u)$  distinct from  $\sigma_{\ell'}^r(v_1)$  (which is equal to  $\sigma_{\ell'}^r(v_1)$ ). Again, we obtain a strong  $(2, 2)$ -colouring of  $G$ .
- (C5) Assume there exists  $u \in V(G)$ , a 2-vertex adjacent to two 2-vertices  $v_0, v_1$ . Notice that  $v_0$  and  $v_1$  cannot be adjacent, as otherwise  $G$  would be isomorphic to  $K_3$ ; so we can denote by  $v'_0, v'_1$  their respective other neighbour. We consider  $G' = G - \{uv_0, uv_1\}$ . Note that  $G'$  can have a connected component isomorphic to neither  $K_2$  (as otherwise  $G$  would contain Configuration 6) nor  $K_3$  (because both connected components have a 1-vertex). There is thus a strong  $(2, 2)$ -colouring  $\ell'$  of  $G'$ .
  - Assume  $v_0v'_0$  and  $v_1v'_1$  have the same colour by  $\ell'$  (w.l.o.g. blue). We assign to  $uv_0$  (resp.  $uv_1$ ) whichever red label makes  $\sigma_{\ell'}^r(v_0)$  and  $\sigma_{\ell'}^r(v'_0)$  (resp.  $\sigma_{\ell'}^r(v_1)$  and  $\sigma_{\ell'}^r(v'_1)$ ) differ.

- Assume  $v_0v'_0$  and  $v_1v'_1$  have different colours by  $\ell'$ , say  $v_0v'_0$  has colour blue by  $\ell'$  w.l.o.g. We assign to  $uv_0$  (resp.  $uv_1$ ) whichever blue (resp. red) label makes the blue (resp. red) sums of  $v_0$  and  $v'_0$  (resp.  $v_1$  and  $v'_1$ ) differ.

It is easy to check that, either way, we end up with a strong  $(2, 2)$ -colouring of  $G$ .

- (C3) Assume there exist  $u_0, u_1 \in V(G)$ , two adjacent 3-vertices, and  $v_0, v_1$ , two 1-vertices with  $u_0$  adjacent to  $v_0$  and  $u_1$  adjacent to  $v_1$ . We also denote by  $w_0$  (resp.  $w_1$ ) the other neighbour of  $u_0$  (resp.  $u_1$ ), and consider  $G' = G - \{u_0v_0, u_0u_1, u_1v_1\}$ . As in the previous case,  $G'$  cannot have a connected component isomorphic to  $K_2$ , as otherwise  $G$  would contain Configuration 1. Each connected component of  $G'$  has at least one 1-vertex; hence, no connected component of  $G'$  can be isomorphic to  $K_3$ , and we can consider  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ .

- Assume  $u_0w_0$  and  $u_1w_1$  have the same colour by  $\ell'$  (w.l.o.g. blue). We can also further assume that  $\sigma_{\ell'}^b(u_0) \geq \sigma_{\ell'}^b(u_1)$ . We then assign to  $u_0v_0$  whichever blue label makes  $\sigma_{\ell'}^b(u_0)$  and  $\sigma_{\ell'}^b(w_0)$  differ. We then assign to  $u_0u_1$  whichever red label makes  $\sigma_{\ell'}^r(u_0)$  and  $\sigma_{\ell'}^r(w_0)$  differ. We complete the construction of  $\ell$  by assigning to  $u_1v_1$  whichever red label makes  $\sigma_{\ell'}^r(u_1)$  and  $\sigma_{\ell'}^r(w_1)$  differ. Since  $\sigma_{\ell'}^b(u_1) = \sigma_{\ell'}^b(u_1)$ , we only need to check that  $u_0$  and  $u_1$  have different blue and red sums. Remember we assumed that  $\sigma_{\ell'}^b(u_0) \geq \sigma_{\ell'}^b(u_1)$ . Since  $\sigma_{\ell'}^b(u_0) > \sigma_{\ell'}^b(u_0)$ , and because the only red edge incident to  $u_0$  is  $u_0u_1$ , clearly  $\ell$  is a strong  $(2, 2)$ -colouring of  $G$ .
- Assume  $u_0w_0$  and  $u_1w_1$  have different colours by  $\ell'$ , say  $u_0w_0$  is blue w.l.o.g. Again, through similar arguments, we assign to  $u_0v_0$  whichever blue label makes  $\sigma_{\ell'}^b(u_0)$  and  $\sigma_{\ell'}^b(w_0)$  differ, to  $u_0u_1$  whichever red label makes  $\sigma_{\ell'}^r(u_0)$  and  $\sigma_{\ell'}^r(w_0)$  differ, and finally to  $u_1v_1$  whichever red label makes  $\sigma_{\ell'}^r(u_1)$  and  $\sigma_{\ell'}^r(w_1)$  differ. It is then easy to check that we end up with a strong  $(2, 2)$ -colouring of  $G$ .

- (C2) Assume there exist  $u$ , a  $d$ -vertex for  $d \geq 4$ , and  $v_1, \dots, v_k$ , some 1-vertices adjacent to  $u$ , where  $k = \lfloor \frac{d+1}{2} \rfloor$  (note that there can be more than  $k$  vertices of degree 1 adjacent to  $u$ , but we only take into consideration  $k$  of them). We consider  $G' = G - \{uv_1, \dots, uv_k\}$ . Note that the degree of  $u$  in  $G'$  is still at least 2; hence, if  $G'$  is  $K_3$ , then  $d = 4$  and  $G$  is completely revealed, and it can be checked that  $G$  admits strong  $(2, 2)$ -colourings. We can thus assume there exists,  $\ell'$  a strong  $(2, 2)$ -colouring of  $G'$ . We assume w.l.o.g. that the majority of the edges incident to  $u$  in  $G'$  are assigned a red label by  $\ell'$ . We assign blue labels to all the edges in  $\{uv_1, \dots, uv_k\}$ , so that  $\sigma_{\ell'}^r(u)$  remains equal to  $\sigma_{\ell'}^r(u)$ . Through assigning blue labels to the edges in  $\{uv_1, \dots, uv_k\}$ , we can alter the blue sum of  $u$  to any value between  $\sigma_{\ell'}^b(u) + k$  and  $\sigma_{\ell'}^b(u) + 2k$ , hence to  $k + 1$  distinct possible values. Since  $u$  has at most  $k$  neighbours in  $G'$ , there exists at least one open choice of  $\sigma_{\ell'}^b(u)$ , resulting in a strong  $(2, 2)$ -colouring of  $G$ .

- (C4) Note that this is covered by Configuration 2.

- (C7) Assume there exists  $u_0$ , a weak 3-vertex adjacent to  $u_1$ , a 2-vertex, and to  $v_0$ , a 1-vertex. We denote by  $u'_0$  and  $u'_1$  the other neighbour of  $u_0$  and  $u_1$ , respectively. We consider  $G' = G - \{u_0u_1, u_0v_0\}$ , and note that, by arguments similar as earlier, no connected component of  $G'$  can be isomorphic to  $K_3$  nor  $K_2$ , as otherwise  $G$  would contain Configuration 1 or 6. We can thus consider  $\ell'$ , a strong  $(2, 2)$ -colouring of  $G'$ . W.l.o.g, we assume  $u_0u'_0$  is assigned a blue label. We consider three cases:

- Assume  $\sigma_{\ell'}^b(u_0) = \sigma_{\ell'}^b(u_1)$ . We assign to  $u_0u_1$  whichever blue label makes  $\sigma_{\ell'}^b(u_1)$  differ from  $\sigma_{\ell'}^b(u'_1)$ , and then assign to  $u_0v_0$  whichever blue label makes  $\sigma_{\ell'}^b(u_0)$  differ from  $\sigma_{\ell'}^b(u'_0)$ . Since  $\sigma_{\ell'}^b(u_0) = \sigma_{\ell'}^b(u_1)$ , and because we assigned a blue label to  $u_0v_0$ , we have  $\sigma_{\ell'}^b(u_0) > \sigma_{\ell'}^b(u_1)$ .
- Otherwise, assume  $u_1u'_1$  is assigned a blue label by  $\ell'$ . We assign to  $u_0u_1$  whichever red label makes  $\sigma_{\ell'}^r(u_1)$  differ from  $\sigma_{\ell'}^r(u'_1)$ . We then assign to  $u_0v_0$  whichever red label makes  $\sigma_{\ell'}^r(u_0)$  differ from  $\sigma_{\ell'}^r(u'_0)$ .
- Finally, assume  $u_1u'_1$  is assigned a red label by  $\ell'$ . We assign to  $u_0u_1$  whichever blue label makes  $\sigma_{\ell'}^b(u_1)$  differ from  $\sigma_{\ell'}^b(u'_1)$ . We then assign to  $u_0v_0$  whichever blue label makes  $\sigma_{\ell'}^b(u_0)$  differ from  $\sigma_{\ell'}^b(u'_0)$ .

In each case, it can be check that the resulting  $(2, 2)$ -colouring of  $G$  is strong.

This concludes the proof of the claim. ◇

We now define a charge function  $\omega$  on  $G$ , where, for every  $u \in V(G)$ , we set  $\omega(u) = d(u) - \frac{9}{4}$ . We also consider a partition of the vertices of  $V(G)$  into  $V_2$ , the set of 1-vertices of  $G$ , and  $V_1$ , the set of all the  $2^+$ -vertices. Note that since  $G$  is not isomorphic to  $K_2$ , no 1-vertex is adjacent to another 1-vertex, and hence for every  $v \in V_2$  we have  $d_{V_1}(v) = d(v) = 1$ . We then consider the following discharging rules:

- (R1) Every  $3^+$ -vertex sends  $\frac{1}{4}$  to each adjacent 2-vertex.
- (R2) Every  $3^+$ -vertex sends  $\frac{1}{4}$  to each adjacent weak 3-vertex.
- (R3) Every  $3^+$ -vertex sends 1 to each adjacent 1-vertex.

We denote by  $\omega^*$  the resulting charge function. Our goal is to get a contradiction through Theorem 4, which requires, for any vertex  $u \in V(G)$ , that if  $d(u) \geq 2$  then  $\omega^*(u) \geq 0$ ; and otherwise that  $\omega^*(u) \geq \omega(u) + 1$ . We prove this is true for every vertex  $u \in V(G)$ :

- Assume  $d(u) = 1$ . Since  $G$  is not  $K_2$ , and because  $G$  does not contain Configuration 6, vertex  $u$  is adjacent to a  $3^+$ -vertex, and received 1 from it by Rule 3. Meanwhile,  $u$  did not send any charge through Rules 1, 2, and 3. The final charge is thus  $\omega^*(u) = \omega(u) + 1$ .
- Assume  $d(u) = 2$ . Again,  $u$  did not send any charge through Rules 1, 2, and 3. Now, since  $G$  contains neither Configuration 5 nor Configuration 6, then  $u$  is adjacent to a  $3^+$ -vertex, and received  $\frac{1}{4}$  from it by Rule 2. The final charge is thus  $\omega^*(u) \geq 0$ .
- Assume  $d(u) = 3$ .
  - Assume first  $u$  is weak. Since  $G$  does not contain Configuration 7, vertex  $u$  is not adjacent to any 2-vertex. Since  $G$  does not contain Configuration 1, vertex  $u$  is also adjacent to at most one 1-vertex, and, since  $G$  does not contain Configuration 3, vertex  $u$  is not adjacent to any weak 3-vertex. In particular,  $u$  is adjacent to a  $3^+$ -vertex, from which it received  $\frac{1}{4}$  by Rule 2, and sent 1 by Rule 1. The final charge is thus  $\omega^*(u) \geq \frac{3}{4} - 1 + \frac{1}{4} = 0$ .
  - Assume now  $u$  is not weak. For every neighbour  $v$  of  $u$ , either  $v$  is a weak 3-vertex and  $u$  sent  $\frac{1}{4}$  to  $v$  by Rule 2, or  $v$  is a 2-vertex and  $u$  sent  $\frac{1}{4}$  to  $v$  by Rule 1, or  $u$  did not send any charge to  $v$  (in particular, since  $u$  is not weak,  $v$  cannot be a 1-vertex). Either way, the final charge is  $\omega^*(u) \geq \frac{3}{4} - 3 \times \frac{1}{4} = 0$ .

- Assume  $d(u) \geq 4$ . We denote by  $k_1$  the number of 1-vertices adjacent to  $u$ , and by  $k_2$  the number of 2-vertices or weak 3-vertices adjacent to  $u$ . By Rules 1, 2 and 3,  $u$  sent 1 to  $k_1$  vertices and  $\frac{1}{4}$  to  $k_2$  vertices. The final charge is thus  $\omega^*(u) = \omega(u) - k_1 - k_2 \frac{1}{4}$ . Since  $k_1 + k_2 \leq d(u)$ , the value  $\omega^*(u)$  is minimal when  $k_1$  is maximal. Because  $G$  does not contain configuration  $C4$ , we can thus assume

$$\omega^*(u) \geq d(u) - \frac{9}{4} - \left\lfloor \frac{d(u)+1}{2} \right\rfloor + 1 - \left( d(u) - \left\lfloor \frac{d(u)+1}{2} \right\rfloor + 1 \right) \frac{1}{4}.$$

Note that the right-hand side of this inequality can be seen as a function of  $d(u)$ , and is trivially increasing. We can then check that it is positive when  $d(u) = 4$ , concluding the case.

Thus, by Theorem 4, we have  $\text{mad}(G) \geq \frac{9}{4}$ , which is a contradiction. The result thus holds.  $\square$

## 5. Halin graphs

A *Halin graph*  $G$  is a planar graph obtained from a tree  $T$  on at least four vertices and having no vertex of degree 2, by connecting, through a cycle  $C$ , the leaves of  $T$  in the cyclic ordering defined by a plane embedding of  $T$ . We call the edges of  $T$  the *tree edges* of  $G$ , and the edges of  $C$  its *cycle edges*.

It can be noticed that Halin graphs are 3-degenerate; they are thus all 4-colourable, which is tight since wheels with even order are 4-chromatic Halin graphs (while all others are 3-chromatic, see *e.g.* [19]). In what follows, we prove that the Strong  $(2, 2)$ -Conjecture holds for Halin graphs.

**Theorem 5.** *If  $G$  is a Halin graph, then  $G$  is strongly  $(2, 2)$ -colourable.*

*Proof.* We denote by  $T$  and  $C$  the tree and cycle, respectively, partitioning the edges of  $G$ . We also choose  $r$ , any vertex of  $G$  such that  $r$  is adjacent to at least two leaves in  $T$ , and root  $T$  at  $r$ . We also (virtually) orient  $C$  in any of the two natural ways, thereby defining an ordering over the consecutive vertices of  $C$ ; in particular, if any vertex of  $T$  is adjacent to leaves, then these leaves appear consecutively along  $C$ . We also let  $\phi$  be any proper  $\{1, 2\}$ -vertex-colouring of the non-leaf vertices of  $T$ . Free to permute colours by  $\phi$ , we can assume  $\phi(r) = 2$ . Last, for every vertex  $v$  of  $T$  being adjacent to leaves, as mentioned we have that the  $d \geq 1$  leaves  $u_1, \dots, u_d$  adjacent to  $v$  (in  $T$ ) appear consecutively along  $C$ , say they appear following the sequence  $(u_1, \dots, u_d)$ . We define  $vu_1$  as the *special edge* incident to  $v$ .

We now construct a strong  $(2, 2)$ -colouring  $\ell$  of  $G$ . We start by labelling the edges of  $T$  in the following way:

- We consider the non-leaf vertices of  $T$  one by one in order, as they are encountered during any BFS algorithm performed from  $r$ .
- For every non-leaf vertex  $v$  of  $T$  considered this way, let  $w_1, \dots, w_d$  denote the  $d \geq 2$  children of  $v$  in  $T$ . Due to an earlier step of the process, at this point only  $uv$ , where  $u$  is the unique parent of  $v$  (unless  $u = r$ ), can be assigned a label by  $\ell$ , and, in that case, it is assigned a red label. Now:
  - If, say,  $w_d$  is not a leaf of  $T$ , then we proceed as follows. In case  $v$  is adjacent to a leaf, then, assuming  $vw_1$  is the special edge incident to  $v$  w.l.o.g., we assign blue label  $\phi(v)$  to  $vw_1$ . In any case, we then assign red labels with alternating values  $1, 2, 1, 2, \dots$  to all remaining  $vw_i$ 's but  $vw_d$ , in such a way that any two consecutive  $w_i$ 's along  $C$  do not get the same red sum. Eventually, we label  $vw_d$  with any red label so that  $\sigma_\ell^r(v) \equiv \phi(v) \pmod{2}$ .

- If, now, all  $w_i$ 's are leaves of  $T$ , then we essentially proceed the same way. That is, assuming  $vw_1$  is the special edge incident to  $v$ , we assign blue label  $\phi(v)$  to  $vw_1$ . Then, assuming the other  $w_i$ 's appear following the sequence  $(w_2, \dots, w_d)$  along  $C$ , we assign red labels  $1, 2, 1, 2, \dots$  or  $2, 1, 2, 1, \dots$  to  $vw_2, \dots, vw_d$ , whichever way, taking  $\ell(uv)$  into account, results in  $\sigma_\ell^r(v) \equiv \phi(v) \pmod{2}$ . One of this two ways of doing achieves this, since these two sums of  $d - 1$  labels do not have the same parity.

Once this process achieves, every non-leaf vertex of  $T$  gets all its incident (tree) edges labelled by  $\ell$ . In particular, if  $v$  is a non-leaf vertex of  $T$ , then  $\sigma_\ell^r(v) \equiv \phi(v) \pmod{2}$ ; regarding blue sums, either  $v$  is not adjacent to leaves and  $\sigma_\ell^b(v) = 0$ , or  $\sigma_\ell^b(v) = \phi(v)$ . In particular, already at this point, by the definition of  $\phi$ , if  $u$  and  $v$  are two adjacent non-leaf vertices of  $T$ , then  $\sigma_\ell^r(u) \neq \sigma_\ell^r(v)$ , and  $0 \in \{\sigma_\ell^b(u), \sigma_\ell^b(v)\}$  or  $\sigma_\ell^b(u) \neq \sigma_\ell^b(v)$ . Thus, to be done, it remains to label the cycle edges (of  $C$ ) so that leaves of  $T$  are not involved in conflicts (in  $G$ ).

All cycle edges will actually be assigned blue labels, which means that the red sums of the leaves of  $T$  will not be modified further. Before continuing, let us thus notice right away that any two leaves  $u$  and  $v$  of  $T$  that are adjacent in  $G$  cannot have the same red sum, due to earlier choices. First off, we can assume that neither  $u$  nor  $v$  is incident to a special edge, as otherwise we would have  $\sigma_\ell^r(u) = 0$  or  $\sigma_\ell^r(v) = 0$ , thus no conflict. So,  $\sigma_\ell^r(u), \sigma_\ell^r(v) \in \{1, 2\}$ . Now, for  $u$  and  $v$  to be adjacent in  $G$ , they must be consecutive along  $C$ ; then either  $u$  and  $v$  have the same parent  $w$ , or not. In the former case, recall that we assigned red labels 1 and 2 to the leaf edges incident to  $w$  alternately following  $C$ ; thus, we must have  $\{\sigma_\ell^r(u), \sigma_\ell^r(v)\} = \{1, 2\}$ , and no conflict between  $u$  and  $v$  here. The latter case cannot occur, as it would require that  $u$  or  $v$  was considered as the first child of its parent, and is thus incident to a special edge; in other words, we must have  $0 \in \{\sigma_\ell^r(u), \sigma_\ell^r(v)\}$  here.

Set now  $C = u_0 \dots u_{k-1} u_0$ . For every  $i \in \{0, \dots, k-1\}$ , note that there might be a unique edge  $u'_i u_i$  being assigned a blue label by  $\ell$ , and, in case  $u'_i$  exists, we have  $\sigma_\ell^b(u'_i) \in \{1, 2\}$ . Upon labelling the edges of  $C$  with blue labels, note that we cannot alter the blue sums of the  $u'_i$ 's; in particular, this means that, eventually, any  $u'_i$  cannot be in conflict (w.r.t. blue sums) with  $u_i$ . By our choice of  $r$ , we can assume  $u_0$  and  $u_1$  are adjacent to  $r$ , and  $u_0 r$  is the special edge incident to  $r$ ; this means  $u'_1$  does not exist. Also, by our choice of  $\phi$ , we have  $\ell(u_0 u_1) = 2$ ; so, eventually, we will necessarily have (upon assigning blue labels to the edges of  $C$ )  $\sigma_\ell^b(u_0) > 3 + \ell(u_0 u_1) > 2 + \ell(u_0 u_1) \geq \sigma_\ell^b(u_1)$ . In other words, we do not have to worry about a possible conflict between  $u_0$  and  $u_1$  w.r.t. blue sums.

To show we can label the edges of  $C$  as desired, we model the situation via a polynomial, and then apply an algebraic tool to get our conclusion. For every  $i \in \{0, \dots, k-1\}$ , we denote by  $e_i$  the edge  $u_i u_{i+1}$  (where, here and further, operations over the subscripts are modulo  $k$ ), and we define  $n_i$  as the current blue sum of  $u_i$  (inherited from the blue label assigned to  $u_i u'_i$  in case  $u'_i$  exists). For every  $i \in \{0, \dots, k-1\}$ , we also let  $X_i \in \{1, 2\}$  be a variable modelling whether  $e_i$  is assigned blue label 1 or 2 in an extension of  $\ell$  to the edges of  $C$ . We now represent the constraints we have to take into account, modelled by the polynomial

$$P(X_0, \dots, X_{k-1}) = \prod_{i=0}^{k-1} (X_{i-1} + n_i - X_{i+1} - n_{i+1}).$$

Now, if there are values  $x_0, \dots, x_{k-1}$  in  $\{1, 2\}$  for  $X_0, \dots, X_{k-1}$ , respectively, such that  $P(x_0, \dots, x_{k-1}) \neq 0$ , then we can derive a correct extension of  $\ell$  to the edges of  $C$ . Essentially, Alon's Combinatorial Nullstellensatz provides conditions, in terms of coefficients of monomials of maximum degree in the

polynomial's expansion, for such values to exist (see [2]). According to Lemma 2.3 in [7], we can now label the edges of  $C$  as desired, by the following arguments:

- If  $k \neq 4$ , then first assign blue label 2 to  $u_0u_1$  and update  $n_0$  and  $n_1$  accordingly. By earlier arguments, recall that, since  $u_0u'_0$  is assigned blue label 2 while  $u'_1$  does not exist, we cannot get a conflict w.r.t. blue sums between  $u_0$  and  $u_1$ , regardless how the other edges of  $C$  are labelled (with blue labels). The rest of the edges of  $C$  then form a path of length different from 3; by Lemma 2.3 in [7], we can apply the polynomial method to finish.
- If  $k = 4$ , then  $C$  is a cycle with length multiple of 4, the second case covered by Lemma 2.3 in [7]. Again, that lemma tells a desired extension of  $\ell$  to the edges of  $C$  can be achieved.

Thus in all cases we can extend  $\ell$  to the edges of  $C$  with blue labels, and in such a way that adjacent vertices along  $C$  have distinct blue sums. By previous arguments, the resulting labelling  $\ell$  of  $G$  is thus a strong  $(2, 2)$ -colouring. This concludes the proof.  $\square$

## 6. Planar graphs with girth at least 16

Recall that a *planar graph* is a graph admitting an embedding on the plane where no two edges cross, and that the *girth* of a graph is the length of its shortest cycles. It is well known that if  $G$  is a planar graph with girth at least  $g$ , then  $\text{mad}(G) < \frac{2g}{g-2}$ . Now, if  $g \geq 18$ , then  $\text{mad}(G) < \frac{9}{4}$  and previous Theorem 3 applies. So, to go beyond Theorem 3, it makes sense to wonder about planar graphs with smaller girth, which have higher  $\text{mad}$ . Through the next result, we prove the Strong  $(2, 2)$ -Conjecture for nice planar graphs with girth at least 16.

**Theorem 6.** *If  $G$  is a nice planar graph with girth at least 16, then  $G$  is strongly  $(2, 2)$ -colourable.*

*Proof.* The proof is by induction on  $|V(G)| + |E(G)|$ . Since the claim can easily be verified for the base cases, we focus on proving the general case. An important result we will employ is the following. Recall that, in any graph, for some  $k \geq 1$ , a  $k$ -*thread* is a path  $u_1, \dots, u_{k+2}$  where the  $k$  inner vertices  $u_2, \dots, u_{k+1}$  have degree 2 (in the whole graph).

**Lemma 1** (see *e.g.* [12]). For any integer  $k \geq 1$ , every planar graph with minimum degree at least 2 and girth at least  $5k + 1$  contains a  $k$ -thread.

Now consider  $G$ . By induction, we may assume  $G$  is connected. First off, if  $G$  is a tree, then the claim holds according *e.g.* to Theorem 1. Assume now  $G$  has cycles, all of length at least 16. We consider  $G^-$ , the graph obtained from  $G$  by repeatedly contracting degree-1 vertices (that is, we just repeatedly delete degree-1 vertices from the graph, as long as there are some). Since  $G$  has cycles, note that, once the process ends, it cannot be that  $G^-$  is empty. Also, in case  $\delta(G) \geq 2$ , note that  $G^- = G$ .

Now, since  $G^-$  is a planar graph of minimum degree at least 2 and of girth at least 16, by Lemma 1 there must be a 3-thread  $P = xuvw$  in  $G^-$ . Let us consider  $P$  back in  $G$ . By construction of  $G^-$ , each of  $u$ ,  $v$ , and  $w$  is the root of a pendant tree  $S_u$ ,  $S_v$ , and  $S_w$ , respectively, in  $G$ . In particular, each of these three pendant trees might be reduced to only one vertex (thus of degree 2 in  $G$ ), in case it has no edge. Actually, if any of  $S_u$ ,  $S_v$ , or  $S_w$  is a tree with at least two edges, then we can remove some of its edges, apply induction (note that, due to cycles of length at least 16 in  $G$ , removing pendant trees from  $G$  indeed results in a nice graph), and extend a strong  $(2, 2)$ -colouring obtained by induction to the whole of  $G$ , using either arguments we used to deal with Configuration 1 in the proof of Theorem 1

(case of a pendant tree with depth at least 2), or arguments we used to deal with Configuration 2 in the proof of Theorem 3 (case of a star on at least two edges). Thus,  $S_u$ ,  $S_v$ , and  $S_w$  have at most one edge each, and the same arguments as for Configuration 4 in the proof of Theorem 1 apply.  $\square$

## 7. Powers of cycles

For any  $k \geq 1$ , the  $k$ -th power  $G^k$  of a graph  $G$  is obtained from  $G$  by adding an edge between any two vertices that are (originally) at distance at most  $k$ .

Obviously,  $k$ -th powers of graphs tend to have larger cliques as  $k$  grows, and thus increasing chromatic number. Still, exploiting several previous ideas and results, in particular from [13], we can derive that the Strong  $(2, 2)$ -Conjecture holds for  $k$ -th powers of nice cycles.

**Theorem 7.** *If  $G$  is a nice cycle, then  $G^k$  is strongly  $(2, 2)$ -colourable for all  $k \geq 1$ .*

*Proof.* We will mostly use a result from [13], stating that any  $k$ th power of a cycle of length  $n$  admits standard  $(2, 1)$ -colourings<sup>1</sup> for all  $k \geq 2$  and  $n \geq 2k + 2$ , and an observation from [3], asserting that, in any regular graph, weak, standard, and strong  $(2, 1)$ -colourings are equivalent notions.

Let  $G$  be any nice cycle (thus of length  $n$  at least 4), and consider some  $k \geq 1$ . If  $k = 1$ , then  $G^k$  is nothing but a cycle; in particular,  $G^k$  is a nice cactus, and the result follows *e.g.* from Theorem 1. Likewise, we can assume  $G^k$  is not complete (which occurs when  $k \geq \lfloor n/2 \rfloor$ ), since the Strong  $(2, 2)$ -Conjecture was proved to hold for nice complete graphs [3]. Thus, we can assume  $k \geq 2$  and  $k < \lfloor n/2 \rfloor$ ; by the aforementioned result from [13], there are thus standard  $(2, 1)$ -colourings of  $G^k$ . Now, since  $G^k$  is regular, any such  $(2, 1)$ -colouring is also strong, by the observation from [3].  $\square$

## 8. Complete $k$ -partite graphs

Recall that a *complete  $k$ -partite graph* on  $n$  vertices is a graph with all the possible edges between  $k$  independent sets of size  $n_1, \dots, n_k \geq 1$  such that  $n_1 + \dots + n_k = n$  (that is, there are all possible edges joining pairs of vertices in distinct parts). We denote the corresponding graph by  $K(n_1, \dots, n_k)$ .

In this section, we prove the Strong  $(2, 2)$ -Conjecture for all nice complete  $k$ -partite graphs, through a proof scheme inspired from [13].

**Theorem 8.** *For any  $n \geq 1$  and  $n_1, \dots, n_k \geq 1$  such that  $n_1 + \dots + n_k = n$ , either  $n \leq 3$  and  $k = n$ , or  $K(n_1, \dots, n_k)$  is strongly  $(2, 2)$ -colourable.*

*Proof.* Note that if  $n \leq 3$  and  $k = n$ , then  $K(n_1, \dots, n_k)$  is not nice and we are done. W.l.o.g., we can assume  $n_1 \geq \dots \geq n_k$ . If  $k \leq 2$ , then  $K(n_1, \dots, n_k)$  is a bipartite graph and the result is already known (recall earlier results from [3]). So, we can assume  $k \geq 3$ .

We first treat the case  $k = 3$  by providing an explicit assignment of red and blue labels (with value 1 or 2), such that all vertices have a non-zero blue sum and no two adjacent vertices have red sum equal to 0. We denote by  $I_1$  (resp.  $I_2, I_3$ ) the independent set of size  $n_1$  (resp.  $n_2, n_3$ ).

- Assume first  $n_1 > n_2 \geq n_3$ . We assign red label 2 to all edges joining a vertex of  $I_1$  and one of  $I_2$ , blue label 2 to all edges joining a vertex of  $I_1$  and one of  $I_3$ , and finally blue label 1 to all edges joining a vertex of  $I_2$  and one of  $I_3$ . It should be clear that, by the resulting  $(2, 2)$ -colouring, no

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<sup>1</sup>Standard  $(2, 1)$ -colourings correspond exactly to locally irregular 2-edge-colourings; that is, 2-edge-colourings  $\phi$  such that, for any edge  $uv$ , we have  $d_i(u) \neq d_i(v)$  assuming  $\phi(uv) = i$ .

two adjacent vertices share the same red sum. Also, for any  $u_1 \in I_1$ , we have  $\sigma^b(u_1) = 2n_3$ ; for any  $u_2 \in I_2$ , we have  $\sigma^b(u_2) = n_3$ ; and for any  $u_3 \in I_3$ , we have  $\sigma^b(u_3) = 2n_1 + n_2$ . Now, since  $n_3 < n_2 < n_1$ , all those three values are indeed pairwise distinct.

- Assume now  $n_1 = n_2$  and  $n_2 > n_3$ . We assign red label 2 to all edges joining a vertex of  $I_1$  and one of  $I_3$ , blue label 2 to all edges joining a vertex of  $I_1$  and one of  $I_2$ , and finally blue label 1 to all edges joining a vertex of  $I_2$  and one of  $I_3$ . Since  $n_1 > n_3$ , no two adjacent vertices share the same red sum. Also, for any  $u_1 \in I_1$ , we have  $\sigma^b(u_1) = 2n_2$ ; for any  $u_2 \in I_2$ , we have  $\sigma^b(u_2) = 2n_1 + n_3 = 2n_2 + n_3$ ; and for any  $u_3 \in I_3$ , we have  $\sigma^b(u_3) = n_2$ . Since  $n_3 < n_2 = n_1$ , all those three values are again pairwise distinct.
- Finally assume  $n_1 = n_2 = n_3$ . Since  $n_1 = n_3$ , we can consider  $E$ , a perfect matching between  $I_1$  and  $I_3$ . We assign red label 2 to all edges joining a vertex of  $I_2$  and one of  $I_3$ , red label 2 to the edges of  $E$ , and blue label 2 to all other edges. This way, for any  $u_1 \in I_1$ , we have  $\sigma^b(u_1) = 4n_1 - 2$ ; for any  $u_2 \in I_2$ , we have  $\sigma^b(u_2) = 2n_1$ ; and for any  $u_3 \in I_3$ , we have  $\sigma^b(u_3) = 2n_1 - 2$ . Similarly, for any  $u_1 \in I_1$ , we have  $\sigma^r(u_1) = 2$ ; for any  $u_2 \in I_2$ , we have  $\sigma^r(u_2) = 2n_1$ ; and for any  $u_3 \in I_3$ , we have  $\sigma^r(u_3) = 2n_1 + 2$ . Unless  $n_1 = 1$  (in which case the graph is  $C_3$ , and is thus not nice), we thus obtain a strong  $(2, 2)$ -colouring.

We can now assume that  $k \geq 4$ . We proceed by induction on  $k$ . Set  $G = K(n_1, \dots, n_k)$ , and let us denote by  $I_1, \dots, I_k$  the parts of size  $n_1, \dots, n_k$ , respectively, partitioning  $V(G)$ . In what follows, we prove a stronger statement, namely that if  $k$  is odd (resp. even), then there exists a strong  $(2, 2)$ -colouring of  $G$  such that:

- all vertices have non-zero blue (resp. red) sum;
- no two adjacent vertices have red (resp. blue) sum 0.

We proved already that this is true for  $k = 3$ . So assume now  $k \geq 4$  is even (resp. odd), and consider a strong  $(2, 2)$ -colouring of  $K(n_1, \dots, n_{k-1})$  satisfying the induction hypothesis. We extend it to  $G$  by just assigning red (resp. blue) label 2 to all edges incident to  $I_k$ . The resulting  $(2, 2)$ -colouring still has the property that if  $k$  is odd (resp. even), then all vertices have non-zero blue (resp. red) sums, as well as the property that no two adjacent vertices have sum 0 in any of the two colours. Since no two adjacent vertices had red sum or blue sum 0, and all the sums increased by the same amount, conflicts can only arise between a vertex of  $I_1 \cup \dots \cup I_{k-1}$  and a vertex of  $I_k$ .

Assume  $k$  is even (the odd case being similar). Let us consider  $u_k \in I_k$  and  $v \in I_j$  for some  $j \in \{1, \dots, k-1\}$ . Since all edges incident to  $u_k$  were assigned blue label 2, we have  $\sigma^b(u_k) = 2(n_1 + \dots + n_{k-1})$ . However, since all vertices of  $I_1 \cup \dots \cup I_{k-1}$  have non-zero red sum, we have  $\sigma^b(v) < 2n_k + 2(n_1 + \dots + n_{k-1} - n_i)$ . Hence, the resulting  $(2, 2)$ -colouring of  $G$  is strong.  $\square$

## 9. Conclusion

In this work, we provided further support to the Strong  $(2, 2)$ -Conjecture, by essentially proving it for more classes of graphs. Guided by a common research direction of the field, we focused mainly on graphs with low chromatic number. Since, prior to this work, the Strong  $(2, 2)$ -Conjecture was mostly proved to hold for bipartite graphs, we considered several classes of 2-degenerate graphs, thus 3-colourable graphs. As a result, we proved the Strong  $(2, 2)$ -Conjecture for cacti, subcubic outerplanar



graphs, graphs with low maximum average degree, and Halin graphs (which are 3-colourable for the most part).

As a first general direction for further work on the topic, we would be interested in more or less strong generalisations of some of these results. An ultimate goal would of course be to prove the Strong  $(2, 2)$ -Conjecture for all 3-colourable graphs, or at least for all 2-degenerate graphs only. Perhaps an intermediate step would be to generalise some of our results. We wonder, for instance, about proofs for all subcubic graphs, all outerplanar graphs, and for graphs with larger maximum average degree. One could also consider other classes of 2-degenerate graphs, such as series-parallel graphs.

In this work, we also considered the opposite approach, being to prove the Strong  $(2, 2)$ -Conjecture for graphs with larger, unbounded chromatic number. One reason for doing so, is that, prior to this work, the conjecture was known to hold for complete graphs. We proved, mostly adapting and using earlier results, that the Strong  $(2, 2)$ -Conjecture holds for powers of cycles and complete  $k$ -partite graphs. In that line, we wonder about other classes of graphs with unbounded chromatic number, such as split graphs and cographs, among others.

From a more global perspective, the Strong  $(2, 2)$ -Conjecture remains mostly open to date, and it would thus be of prime interest to provide more general results supporting it might hold true. Maybe an interesting step could be to establish that, for some  $k \geq 2$ , all nice graphs admit strong  $(k, 2)$ -colourings. Let us remind that, on the other hand, a recent proof of the 1-2-3 Conjecture implies that all nice graphs admit strong  $(1, 3)$ -colourings.

## References

- [1] L. Addario-Berry, R. Aldred, K. Dalal, and B. Reed. Vertex colouring edge partitions. *Journal of Combinatorial Theory, Series B*, 94(2):237–244, 2005.
- [2] N. Alon. Combinatorial nullstellensatz. *Combinatorics, Probability and Computing*, 8(1–2):7–29, 1999.
- [3] O. Baudon, J. Bensmail, T. Davot, H. Hocquard, J. Przybyło, M. Senhaji, E. Sopena, and M. Woźniak. A general decomposition theory for the 1-2-3 conjecture and locally irregular decompositions. *Discrete Mathematics & Theoretical Computer Science*, 21, 2019.
- [4] O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak. On decomposing regular graphs into locally irregular subgraphs. *European Journal of Combinatorics*, 49:90–104, 2015.
- [5] J. Bensmail. On a graph labelling conjecture involving coloured labels. *Discussiones Mathematicae Graph Theory*, 44:231–244, 2024.
- [6] J. Bensmail, F. Fioravantes, and F. Mc Inerney. On the role of 3s for the 1-2-3 conjecture. *Theoretical Computer Science*, 892:238–257, 2021.
- [7] J. Bensmail, H. Hocquard, and C. Marcille. The Weak  $(2, 2)$ -Labelling Problem for graphs with forbidden induced structures. Research report, Université côte-d’Azur ; Université de Bordeaux, LaBRI, UMR 5800, France, 2022.
- [8] J. Bensmail, H. Hocquard, and C. Marcille. The weak  $(2, 2)$ -labelling problem for graphs with forbidden induced structures. In A. Bagchi and R. Muthu, editors, *Algorithms and Discrete Applied Mathematics*, pages 204–215, Cham, 2023. Springer International Publishing.

- [9] J. Bensmail, H. Hocquard, and C. Marcille. On inducing degenerate sums through 2-labellings. *Graphs and Combinatorics*, 40(2):23, 2024.
- [10] M. Bonamy. *Global discharging methods for coloring problems in graphs*. PhD thesis, Université de Montpellier, 2015.
- [11] M. Bonamy, N. Bousquet, and H. Hocquard. Adjacent vertex-distinguishing edge coloring of graphs. In J. Nešetřil and M. Pellegrini, editors, *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, pages 313–318, Pisa, 2013. Scuola Normale Superiore.
- [12] G. Chang and G.-H. Duh. On the precise value of the strong chromatic index of a planar graph with a large girth. *Discrete Applied Mathematics*, 247:389–397, 2018.
- [13] I. Grzelec, T. Madaras, A. Onderko, and R. Soták. On a new problem about the local irregularity of graphs, 2024.
- [14] M. Karoński, T. Łuczak, and A. Thomason. Edge weights and vertex colours. *Journal of Combinatorial Theory, Series B*, 91(1):151–157, 2004.
- [15] R. Keusch. A solution to the 1-2-3 conjecture. *Journal of Combinatorial Theory, Series B*, 166:183–202, 2024.
- [16] H. Lu, Q. Yu, and C.-Q. Zhang. Vertex-coloring 2-edge-weighting of graphs. *European Journal of Combinatorics*, 32(1):21–27, 2011.
- [17] J. Przybyło. A note on the weak (2,2)-conjecture. *Discrete Mathematics*, 342(2):498–504, 2019.
- [18] J. Przybyło. On the standard (2,2)-conjecture. *European Journal of Combinatorics*, 94:103305, 2021.
- [19] W.-F. Wang and K.-W. Lih. List coloring halin graphs. *Ars Combinatoria*, 77:53 – 63, 2005.