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► **To cite this version:**

Jérôme Lelong, Véronique Maume-Deschamps, William Thevenot. Sample Average Approximation for Portfolio Optimization under CVaR constraint in an (re)insurance context. 2024. hal-04733015

HAL Id: hal-04733015

<https://hal.science/hal-04733015v1>

Preprint submitted on 11 Oct 2024

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Sample Average Approximation for Portfolio Optimization under CVaR constraint in an (re)insurance context

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October 11, 2024

Abstract

We consider optimal allocation problems with Conditional Value-At-Risk (CVaR) constraint. We prove, under very mild assumptions, the convergence of the Sample Average Approximation method (SAA) applied to this problem, and we also exhibit a convergence rate and discuss the uniqueness of the solution. These results give (re)insurers a practical solution to portfolio optimization under market regulatory constraints, i.e. a certain level of risk.

Keywords: Value-At-Risk, Conditional Value-At-Risk, Expected shortfall, Sample average approximation, Portfolio optimization, Insurance, Reinsurance, Uniform strong large law of numbers, Central limit theorem.

1 Introduction

(Re)insurers are constantly looking for opportunities to develop their business, increase their incomes and improve their profitability. However, in every line of business, portfolio growth often leads to increased incomes and increased risk accumulation. The company's aim is to maximize profitability by achieving an optimal risk/reward ratio between exposure to losses and expected profits. Although, the assessment of individual risks is important, getting the right mix of risks is just as crucial.

In addition, on the European market, an insurer must meet the requirements of Solvency II regulations, in particular it must have an amount of own funds at least equal to the Solvency Capital Requirement (SCR). The SCR is the capital required to ensure that the (re)insurance company will be able to meet its obligations over the next 12 months with a probability greater than 99.5%. Formally, it is modeled with the Value-At-Risk (VaR) at the level $\alpha = 0.995$.

Other risk measures can be used to model the overall risk of a (re)insurance company, the most widely used alternative being the Conditional Value-At-Risk ($CVaR$), also called Tail Value-At-Risk (TVaR) or expected shortfall (ES) for continuous distributions. CVaR is usually preferred to VaR because it has better properties such as sub-additivity and its coherent in the sense of

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Artzner et al.[1]. It is in the company's interest to reduce risk through diversification, in order to achieve the best risk/return ratio.

The classic approach to portfolio optimization was introduced by Markowitz in 1952 [3]. It consists in the maximization of the expectation under the constraint of maximum variance or, equivalently, minimizing the variance of the portfolio, for a fixed return, this problem is called the mean-variance optimization. Its equivalent for the conditional value-at-risk (CVaR) is the mean-CVaR optimization.

We model the (re)insurance asset market with business lines represented by the random vector \mathbf{X} of asset returns, taking values in a subset $\mathcal{R}_{\mathbf{X}}$ of \mathbb{R}^d . We assume that $\mathbb{E}(|\mathbf{X}|) < +\infty$. A (re)insurance portfolio is defined by a vector $\boldsymbol{\gamma} \in \mathbb{R}^d$ representing the quantity held in each business line by the (re)insurer.

Let us fix some notations, with $\alpha \in]0, 1[$:

$$\begin{aligned} V_{\alpha}(\boldsymbol{\gamma}) &= VaR_{\alpha}(-\boldsymbol{\gamma}^T \mathbf{X}) = \min \left\{ M \in \mathbb{R} : \mathbb{P} \left(-\boldsymbol{\gamma}^T \mathbf{X} \leq M \right) \geq \alpha \right\}, \\ C_{\alpha}(\boldsymbol{\gamma}) &= CVaR_{\alpha}(-\boldsymbol{\gamma}^T \mathbf{X}) = \mathbb{E} \left(-\boldsymbol{\gamma}^T \mathbf{X} \mid -\boldsymbol{\gamma}^T \mathbf{X} \geq V_{\alpha}(\boldsymbol{\gamma}) \right). \end{aligned}$$

Our original goal is to solve the following equation with a fixed $\alpha \in]0, 1[$ and constraints on the weights and a capital requirement limit $K > 0$. It is quite common that L_0 depends on $C_{\alpha}(\boldsymbol{\gamma})$. Even if $L_0(\boldsymbol{\gamma}, \mathbf{X})$ could already take the dependency on $C_{\alpha}(\boldsymbol{\gamma})$ into account, it is convenient to emphasize on this dependency explicitly for future analysis.

$$\begin{aligned} v^* &:= \inf_{\boldsymbol{\gamma} \in \mathbb{R}_+^d} \mathbb{E}(L_0(\boldsymbol{\gamma}, C_{\alpha}(\boldsymbol{\gamma}), \mathbf{X})) \\ \text{s.t.} \quad &\gamma_i^{low} \leq \gamma_i \leq \gamma_i^{up} \quad \forall i \in \{1, \dots, d\} \\ \text{s.t.} \quad &C_{\alpha}(\boldsymbol{\gamma}) \leq K. \end{aligned} \tag{1.1}$$

A new approach was introduced by R.T. Rockafellar and S. Uryasev in 2000 [4] and was later extended by Krokmal P., Jonas Palmquist J., Uryasev S. (2002) [2] who proposed an embedding technique to reformulate the CVaR.

We aim to maximize a return function or minimize a loss function of a portfolio under CVaR constraints, because this approach is well adapted to the needs of (re)insurance companies. In [2], it is solved using linear programming, but this resolution can be very time-consuming. In this work, we prefer to use *Sample Average Approximation* (SAA), see Rubinstein and Shapiro [5].

For this formulation with explicit constraints, no convergence or convergence speed results with the SAA method has been published as far as we know, the closest result to our work is [8]. In this last one, the function to be minimized does not depend on the data sample. Under convexity, continuity, integrability assumptions, we prove a.s. the convergence and find a rate of convergence for the SAA version in the case where the function to be minimized depends on the data sample as do the constraint. Moreover, if the CVaR appears in the function to be minimized, we show that for the optimization, under monotonic assumption, it can be replaced by the auxiliary function introduced in [2] and [4]. We also propose a sufficient condition to obtain the uniqueness of the solution.

The paper is organized as follows. Section 2 describes the context and states the problem. Section 2.2 introduces the SAA method and presents our main results on convergence (Theorem 3.8), uniqueness (Theorems 3.11 and 3.10) and convergence rate (Theorem 3.19). Section 4 is dedicated to some numerical studies on two different cases, where we show how these results can be used concretely in the context of a (re)insurer seeking to balance its various exposures.

2 Optimization with CVaR constraint

This section presents the transformation of the original problem and the SAA method applied to this equivalent problem.

2.1 An equivalent setting

Note that the Conditional Value-At-Risk can be rewritten using the Value-At-Risk as follows,

$$C_\alpha(\boldsymbol{\gamma}) = V_\alpha(\boldsymbol{\gamma}) + (1 - \alpha)^{-1} \mathbb{E} \left((-\boldsymbol{\gamma}^T \mathbf{X} - V_\alpha(\boldsymbol{\gamma}))^+ \right).$$

We introduce $g(\boldsymbol{\gamma}, \zeta) = \zeta + (1 - \alpha)^{-1} \mathbb{E} \left((-\boldsymbol{\gamma}^T \mathbf{X} - \zeta)^+ \right)$ for any $\zeta \in \mathbb{R}$.

Rockafellar, R. T., Uryasev, S. (2000) [4] showed that $g(\boldsymbol{\gamma}, \cdot)$ is convex, continuously differentiable and that for any $\boldsymbol{\gamma}$, $C_\alpha(\boldsymbol{\gamma})$ can be determined by minimising $g(\boldsymbol{\gamma}, \cdot)$:

$$\begin{aligned} C_\alpha(\boldsymbol{\gamma}) &= \min_{\zeta \in \mathbb{R}} g(\boldsymbol{\gamma}, \zeta), \quad A_\alpha(\boldsymbol{\gamma}) = \operatorname{argmin}_{\zeta \in \mathbb{R}} g(\boldsymbol{\gamma}, \zeta), \\ V_\alpha(\boldsymbol{\gamma}) &= \min(\{x \in A_\alpha(\boldsymbol{\gamma})\}), \quad C_\alpha(\boldsymbol{\gamma}) = g(\boldsymbol{\gamma}, V_\alpha(\boldsymbol{\gamma})). \end{aligned}$$

Krokhmal P., Jonas Palmquist J., Uryasev S. (2002) [2] showed the equivalence of the following two optimization problems in the sense that their objectives functions achieve the same minimum values. (Proof in Annex 1)

Proposition 2.1. *For any function $R : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ such that for any $\boldsymbol{\gamma} \in \mathbb{R}^d$, $R(\boldsymbol{\gamma}, \cdot)$ is not increasing then the following two problems are equivalent*

$$\inf_{\boldsymbol{\gamma} \in \Xi} -R(\boldsymbol{\gamma}, C_\alpha(\boldsymbol{\gamma})) \quad \text{s.t.} \quad C_\alpha(\boldsymbol{\gamma}) \leq K \quad (2.1)$$

and

$$\inf_{(\boldsymbol{\gamma}, \zeta) \in \Xi \times \mathbb{R}} -R(\boldsymbol{\gamma}, g(\boldsymbol{\gamma}, \zeta)) \quad \text{s.t.} \quad g(\boldsymbol{\gamma}, \zeta) \leq K. \quad (2.2)$$

Moreover, if the C_α constraint is active in (2.1), $(\boldsymbol{\gamma}^*, \zeta^*)$ achieves the minimum of (2.2) solution if and only if $\boldsymbol{\gamma}^*$ achieves the minimum of (2.1) and $\zeta^* \in A_\alpha(\boldsymbol{\gamma}^*)$.

It is convenient to write $L(\boldsymbol{\gamma}, \zeta, \mathbf{X}) = L_0(\boldsymbol{\gamma}, g(\boldsymbol{\gamma}, \zeta), \mathbf{X})$ for future analysis.

Note that g is also continuous and convex.

This result is fundamental in transforming our problem into a simpler one where it is no longer necessary to compute the value of the function $C_\alpha(\boldsymbol{\gamma})$, which is long and complex because we need to compute $V_\alpha(\boldsymbol{\gamma})$ first. Instead of that, we compute $g(\boldsymbol{\gamma}, \zeta)$ which is no longer the result of an optimisation problem. Using the equivalence between (2.1) and (2.2), (1.1) may be written as:

$$\begin{aligned} v^* &:= \inf_{\boldsymbol{\gamma}, \zeta \in \mathbb{R}_+^d \times \mathbb{R}} \mathbb{E}(L(\boldsymbol{\gamma}, \zeta, \mathbf{X})) \\ \text{s.t.} & \quad \gamma_i^{low} \leq \gamma_i \leq \gamma_i^{up} \quad \forall i \\ \text{s.t.} & \quad \zeta^{low} \leq \zeta \leq \zeta^{up} \quad \forall i \\ \text{s.t.} & \quad g(\boldsymbol{\gamma}, \zeta) \leq K \end{aligned} \quad (2.3)$$

where $\zeta^{low} \leq \min_{\gamma_i^{low} \leq \gamma_i \leq \gamma_i^{up} \quad \forall i} \operatorname{VaR}_\alpha(-\boldsymbol{\gamma}^T \mathbf{X})$ and $\zeta^{up} \geq \max_{\gamma_i^{low} \leq \gamma_i \leq \gamma_i^{up} \quad \forall i} \operatorname{VaR}_\alpha(-\boldsymbol{\gamma}^T \mathbf{X})$.

Remark 2.2. The most commonly used loss function is $L_0(\boldsymbol{\gamma}, \mathbf{X}) = -\boldsymbol{\gamma}^T \mathbf{X}$. But in order to represent the cost of capital in insurance, it is usual to penalise the function with a term proportional to the capital requirement i.e. the CVaR. So the loss could become $L_0(\boldsymbol{\gamma}, \mathbf{X}) = -\boldsymbol{\gamma}^T \mathbf{X} + c \cdot C_\alpha(\boldsymbol{\gamma}) = L_0(\boldsymbol{\gamma}, C_\alpha(\boldsymbol{\gamma}), \mathbf{X})$ with $c \in]0, 1[$ usually 0.05. In this last case, L becomes $L(\boldsymbol{\gamma}, \zeta, \mathbf{X}) = -\boldsymbol{\gamma}^T \mathbf{X} + c \cdot g(\boldsymbol{\gamma}, \zeta)$. in the equivalent Problem (2.3)

2.2 Sample Average Approximation

Usually we do not have access to $\mathbb{E}(L(\boldsymbol{\gamma}, \zeta, \mathbf{X}))$, so we use a real or simulated sample of \mathbf{X} (SAA: Sample Average Approximation [6]). Let $(\mathbf{X}^{(N)})_{N \in \mathbb{N}^*}$ be a sequence of iid random variables with the same distribution as \mathbf{X} , for $N \geq 1$. We shall solve an approximated version of Problem (2.3). Consider

$$g_N(\boldsymbol{\gamma}, \zeta) := \zeta + \frac{(1-\alpha)^{-1}}{N} \sum_{j=1}^N (-\boldsymbol{\gamma}^T \mathbf{X}^{(j)} - \zeta)^+$$

and

$$\ell_N(\boldsymbol{\gamma}, \zeta) := \frac{1}{N} \sum_{j=1}^N L(\boldsymbol{\gamma}, \zeta, \mathbf{X}^{(j)}).$$

The sample average approximation of (2.3) is given by

$$\begin{aligned} v_N &:= \inf_{\boldsymbol{\gamma}, \zeta \in \mathbb{R}^d \times \mathbb{R}} \ell_N(\boldsymbol{\gamma}, \zeta) \\ \text{s.t.} \quad &0 \leq \gamma_i^{low} \leq \gamma_i \leq \gamma_i^{up} \quad \forall i \\ &\zeta^{low} \leq \zeta \leq \zeta^{up} \\ &g_N(\boldsymbol{\gamma}, \zeta) \leq K. \end{aligned} \tag{2.4}$$

Let us introduce $\Gamma := \{\boldsymbol{\gamma} : 0 \leq \gamma_i^{low} \leq \gamma_i \leq \gamma_i^{up} \forall i\}$ and $\mathcal{U} := \Gamma \times \{\zeta : \zeta^{low} \leq \zeta \leq \zeta^{up}\}$.

Define $\ell : (\boldsymbol{\gamma}, \zeta) \mapsto \mathbb{E}(L(\boldsymbol{\gamma}, \zeta, \mathbf{X}))$, we assume the following properties

(P1) $\mathbb{P}((\boldsymbol{\gamma}, \zeta) \rightarrow L(\boldsymbol{\gamma}, \zeta, \mathbf{X}) \text{ is continuous and convex}) = 1$.

(P2) $\mathbb{E} \left(\sup_{(\boldsymbol{\gamma}, \zeta) \in \mathcal{U}} |L(\boldsymbol{\gamma}, \zeta, \mathbf{X})| \right) < +\infty$.

The functions ℓ_N and ℓ are continuous and convex by (P1) and (P2). Note that g_N and g are convex and continuous.

Now we define the feasibility sets of the original problem and of its SAA counterpart; $\mathcal{U}^K := \{(\boldsymbol{\gamma}, \zeta) : g(\boldsymbol{\gamma}, \zeta) \leq K\} \cap \mathcal{U}$ and $\mathcal{U}_N^K := \{(\boldsymbol{\gamma}, \zeta) : g_N(\boldsymbol{\gamma}, \zeta) \leq K\} \cap \mathcal{U}$. We assume that $\mathcal{U}, \mathcal{U}^K$ have non-empty interiors.

We also define the solution sets of the original problem and of its SAA counterpart.

$$S := \operatorname{argmin}_{(\boldsymbol{\gamma}, \zeta) \in \mathcal{U}^K} \ell(\boldsymbol{\gamma}, \zeta) ; S_N := \operatorname{argmin}_{(\boldsymbol{\gamma}, \zeta) \in \mathcal{U}_N^K} \ell_N(\boldsymbol{\gamma}, \zeta),$$

Our goal is to prove that v_N converges a.s. to v^* and $\mathbb{D}(S_N, S) \xrightarrow[N \rightarrow \infty]{} 0$ a.s. with $\mathbb{D}(A, B) := \sup_{x \in A} \inf_{x' \in B} \|x - x'\|$.

3 Main results

3.1 Convergence of the SAA method

We shall use two results from Shapiro, A., Dentcheva, D., Ruszczyński, A. (2009) [7], in order to obtain a.s. convergence of our SAA Method (Theorem 3.4/ Theorem 3.8).

Theorem 3.1 (Theorem 7.48 p375 in [7]).

Consider a random function $H : \Xi \times \mathcal{R}_X \rightarrow \mathbb{R}$, where Ξ is a non-empty compact subset of \mathbb{R}^d . Define $h(\boldsymbol{\xi}) := \mathbb{E}(H(\boldsymbol{\xi}, \mathbf{X}))$ and $h_N(\boldsymbol{\xi}) := N^{-1} \sum_{j=1}^N H(\boldsymbol{\xi}, \mathbf{X}^{(j)})$. Suppose that:

(i) $\mathbb{P}(\xi \rightarrow H(\xi, \mathbf{X}) \text{ is continuous}) = 1.$

(ii) $\mathbb{E} \left(\sup_{\xi \in \Xi} |H(\xi, \mathbf{X})| \right) < +\infty.$

Then, the function h is continuous on Ξ , and $(h_N)_{N \in \mathbb{N}^*}$ converges to h a.s. uniformly on Ξ i.e.

$$\sup_{\xi \in \Xi} |h_N(\xi) - h(\xi)| \rightarrow 0 \text{ a.s. as } N \rightarrow \infty.$$

We use Theorem 3.1 with the functions L and ℓ_N on the compact set Γ using (P1) and (P2) to obtain the following proposition.

Proposition 3.2. *If (P1) and (P2) hold, then, the function ℓ is continuous and the sequence $(\ell_N)_{N \in \mathbb{N}^*}$ converges to ℓ a.s. uniformly on Γ .*

And we use Theorem 3.1 again with $G((\gamma, \zeta), \mathbf{X}) = \zeta + (1 - \alpha)^{-1}(-\gamma^T \mathbf{X} - \zeta)^+$ and g_N on the compact set \mathcal{U} to obtain the following proposition.

Proposition 3.3. *The sequence $(g_N)_{N \in \mathbb{N}^*}$ converges to g a.s. uniformly on \mathcal{U} .*

The set \mathcal{U} is compact and \mathcal{U}^K is non-empty by assumption, so neither is S . The function ℓ is continuous and convex, by (P1) and (P2), and $(\ell_N)_{N \in \mathbb{N}^*}$ converges to ℓ a.s. uniformly on \mathcal{U} from Proposition 3.2 with (P1) and (P2). Thanks to these results, we can state a modified version of Theorem 5.5 p160 in [7] adapted to our framework.

Theorem 3.4 (Restatement of Theorem 5.5 p160 in [7]).

If the following conditions hold:

- (a) *Let $(u_N)_N$ be a sequence taking values in \mathcal{U} and such that u_N converges \bar{u} a.s. If $u_N \in \mathcal{U}_N^K$ for all N , then $\bar{u} \in \mathcal{U}^K$.*
- (b) *For $u \in S$, there exists a sequence $(u_N)_{N \in \mathbb{N}^*}$ such that $u_N \in \mathcal{U}_N^K$ for all N and $u_N \xrightarrow{N \rightarrow \infty} u$ a.s.*
- (c) *A.s. for N large enough the set S_N is nonempty and $S_N \subset \mathcal{U}$.*

Then $v_N \rightarrow v^$ a.s. and $\mathbb{D}(S_N, S) \rightarrow 0$ a.s. when $N \rightarrow \infty$.*

Proposition 3.5. *Let $(u_N)_N$ be a sequence taking values in \mathcal{U} and such that u_N converges \bar{u} a.s. If $u_N \in \mathcal{U}_N^K$ for all N , then $\bar{u} \in \mathcal{U}^K$.*

Proof. Let $(u_N)_{N \in \mathbb{N}^*}$ be a sequence taking values in \mathcal{U} such that u_N converges \bar{u} a.s. and such that $u_N \in \mathcal{U}_N^K$ for all N . We have

$$g_N(u_N) \leq K. \tag{3.1}$$

By Proposition 3.3, $(g_N)_{N \in \mathbb{N}^*}$ converges to g uniformly on \mathcal{U} , so $(g_N(u_N))_{N \in \mathbb{N}^*}$ converges a.s. to $g(\bar{u})$ and we deduce from (3.1), that $g(\bar{u}) \leq K$ so $(\bar{u}) \in \mathcal{U}^K$. ■

Lemma 3.6. *If the interior of \mathcal{U}^K is non-empty, there exists $\bar{u} \in \mathcal{U}^K$ and $N' \in \mathbb{N}^*$ such that $g_N(\bar{u}) < K$ for all $N > N'$.*

Proof. By assumption, \mathcal{U}^K has a non-empty interior, so that there exists \bar{u} such that $g(\bar{u}) < K$. We apply Proposition 3.3 to prove the a.s. uniform convergence of $(g_N)_{N \in \mathbb{N}^*}$ to g on \mathcal{U} . Then, for $\varepsilon = K - g(\bar{u}) > 0$, there exists N' such that $|g_N(\bar{u}) - g(\bar{u})| < \varepsilon$ for all $N > N'$. Therefore, $g_N(\bar{u}) < K$. ■

Proposition 3.7. *For any $u \in S$, there exists a sequence $(u_N)_{N \in \mathbb{N}^*}$ such that $(u_N) \in \mathcal{U}_N^K$ for all $N \in \mathbb{N}^*$ and $u_N \xrightarrow{N \rightarrow \infty} u$ a.s.*

Proof. Let us show this result for all $u \in \mathcal{U}^K$ which will give the desired result since $S \subset \mathcal{U}^K$. Let $u^* \in \mathcal{U}^K$. We construct a sequence $u_N \in \mathcal{U}_N^K$ which converges a.s. to u^* . Since the interior of \mathcal{U}^K is non-empty, by Lemma 3.6, there exists $\bar{u} = (\bar{\gamma}, \bar{\zeta}) \in \bigcap_{N > N'} \mathcal{U}_N^K$ for some $N' > 0$. For $\lambda \in [0, 1]$, we define $\theta_N(\lambda) := g_N(\lambda \bar{u} + (1 - \lambda)u^*)$ and $\theta(\lambda) := g(\lambda \bar{u} + (1 - \lambda)u^*)$. Note that $\theta_N(1) = g_N(\bar{u}) < K$ and $g(u^*) \leq K$.

Let us define $\lambda_N = \min\{\lambda \in [0, 1]; \theta_N(\lambda) \leq K\}$ and $u_N = \lambda_N \bar{u} + (1 - \lambda_N)u^*$, $u_N \in \mathcal{U}_N^K$.

Case 1: $g(u^*) < K$: For any $\varepsilon > 0$, by Proposition 3.3 there exists N' such that for all $N > N'$ we have $\sup_{\mathcal{U}} |g_N - g| \leq \varepsilon$ so $\theta_N(0) = g_N(u^*) \leq K - \varepsilon$. We deduce that for N large enough $\lambda_N = 0$ i.e. $u_N = u^*$ so that $u_N \rightarrow u^*$ a.s.

Case 2: $g(u^*) = K$: Note that the function θ_N is convex because g_N is convex. So that $\theta_N(\lambda_N) \leq \frac{\theta_N(0) + \lambda_N(\theta_N(1) - \theta_N(0))}{\lambda_N}$. As λ_N is bounded, we may extract a convergent sub-sequence $\lambda_{\phi(N)} \rightarrow \bar{\lambda}$ a.s. We assume that $\bar{\lambda} > 0$ and we use a proof by contradiction. We have:

$$\theta_{\phi(N)}(\lambda_{\phi(N)}) \leq \theta_{\phi(N)}(0) + \lambda_N(\theta_{\phi(N)}(1) - \theta_{\phi(N)}(0))$$

taking the limit leads to

$$\theta(\bar{\lambda}) - \theta(0) \leq \bar{\lambda}(\theta(1) - \theta(0)).$$

Now $\theta(0) = K$ and $\theta(1) < K$. Therefore $\theta(\bar{\lambda}) - K < 0$ which we rewrite as $\theta(\bar{\lambda}) \leq K - 3\varepsilon$ for some $\varepsilon > 0$. By continuity of θ , there exists $\delta > 0$ such that $\theta(\bar{\lambda} - \delta) \leq K - 2\varepsilon$. Note that $\sup_{\lambda} |\theta_N(\lambda) - \theta(\lambda)| \xrightarrow[N \rightarrow \infty]{} 0$ a.s. because of Proposition 3.3. Then, for N large enough $|\theta_N(\lambda) - \theta(\lambda)| \leq \varepsilon$ for all $\lambda \in [0, 1]$, so that $\theta_N(\bar{\lambda} - \delta) \leq K - \varepsilon$. Thus, $\lambda_N \leq \bar{\lambda} - \delta$ which contradicts that $\lambda_{\phi(N)} \rightarrow \bar{\lambda}$ a.s. So that, $(\lambda_N)_{N \in \mathbb{N}^*}$ admits as unique limit 0 and we conclude that $\lambda_N \rightarrow 0$ a.s., hence $u_N \rightarrow u^*$ a.s. which completes the proof. \blacksquare

Theorem 3.8. Let v_N define by (1.1) and v^* define by (2.4), if (P1) and (P2) hold, then

$$\begin{aligned} v_N &\xrightarrow[N \rightarrow \infty]{a.s.} v^* \\ \mathbb{D}(S_N, S) &\xrightarrow[N \rightarrow \infty]{a.s.} 0. \end{aligned} \tag{3.2}$$

Proof. We want to apply Theorem 3.4.

Assumption (a) holds thanks to Proposition 3.5 and Assumption (b) holds thanks to Proposition 3.7.

\mathcal{U}^K has a non-empty interior, we can apply Lemma 3.6 and find $\bar{u} \in \mathcal{U}^K$ and $N' \in \mathbb{N}^*$ such that $g_N(\bar{u}) < K$ for all $N > N'$. So that for $N > N'$, $\bar{u} \in \mathcal{U}_N^K$, then \mathcal{U}_N^K is non-empty. Finally, v_N is well defined and because \mathcal{U}_N^K is closed, S_N is included in $\mathcal{U}_N^K \subset \mathcal{U}$. Hence Assumption (c) holds. We can therefore apply Theorem 3.4 and conclude the proof. \blacksquare

3.2 Uniqueness Condition

Let us focus on the uniqueness of the solution to (1.1) and therefore (2.3).

Let us define $\ell_0 : \boldsymbol{\gamma} \mapsto \mathbb{E}(L_0(\boldsymbol{\gamma}, \mathbf{X}))$. We shall make the following assumptions:

(P3) \mathbf{X} has a density on \mathbb{R}^d and so for all $\boldsymbol{\gamma} \in \mathbb{R}_*^d$, $\boldsymbol{\gamma}^T \mathbf{X}$ has also a density on \mathbb{R}^d .

(P4) $C_\alpha(\boldsymbol{\gamma}) > K, \forall \boldsymbol{\gamma} \in \arg \min_{\Gamma} \ell_0$.

(P5) For all $\boldsymbol{\gamma} \in \Gamma \setminus \{0_d\}$, $\text{supp}(\boldsymbol{\gamma}^T \mathbf{X})$ is connected i.e. an interval.

Remark 3.9. This is realistic since the capital is naturally a limiting constraint. Moreover, $\text{supp}(\boldsymbol{\gamma}^T \mathbf{X})$ is an interval because all the business line's loss usually have a connected support. The few risks inside business lines that have unconnected support, if it exists, like some natural disasters loss, are erased by the others.

Note that we exclude the case where 0_d is an optimal solution because we choose a positive capital requirement limit K .

Assumption (P2) implies $\mathbb{E} \left(\sup_{\boldsymbol{\gamma} \in \Gamma} |L_0(\boldsymbol{\gamma}, \mathbf{X})| \right) < +\infty$ and assumption (P1) implies

$\mathbb{P}(\boldsymbol{\gamma} \rightarrow L_0(\boldsymbol{\gamma}, \mathbf{X}) \text{ is continuous and convex}) = 1$ since $L_0(\boldsymbol{\gamma}, \mathbf{X}) = L(\boldsymbol{\gamma}, \tilde{\boldsymbol{\zeta}}, \mathbf{X})$ with $\tilde{\boldsymbol{\zeta}} \in A_\alpha(\boldsymbol{\gamma})$. Therefore, function ℓ_0 is continuous and convex thanks to (P1) and (P2).

Theorem 3.10 provides the uniqueness of the solution to the original problem (1.1) and Theorem 3.11 provide the uniqueness to the solution (2.3).

We also present a sufficient condition for capital constraint to be achieved with Lemma 3.12, (P1),(P2) and (P4), this allows us to use the problem equivalence.

We will not prove Theorem 3.10 and Theorem 3.11 directly as they actually derive from intermediate results.

Theorem 3.10. *If (P1) to (P4) hold, then Problem (1.1) has a unique solution $\boldsymbol{\gamma}^*$.*

Theorem 3.11. *If (P1) to (P5) hold, then Problem (2.3) has a unique solution $(\boldsymbol{\gamma}^*, \zeta^*)$ and in addition $\boldsymbol{\gamma}^*$ is the unique solution to Problem (1.1).*

We will now present the intermediate results.

Lemma 3.12. *Let $\boldsymbol{\gamma}^*$ be an optimal solution to (1.1), if (P1),(P2) and (P4) hold, then $C_\alpha(\boldsymbol{\gamma}^*) = K$.*

Proof. Let us take $\boldsymbol{\gamma}^*$ an optimal solution to (1.1). We proceed by contradiction and assume $C_\alpha(\boldsymbol{\gamma}^*) < K$. Let $\tilde{\boldsymbol{\gamma}} \in \arg \min_{\Gamma} \ell_0$.

Let us define $\tilde{\lambda} = \max\{\lambda \in [0, 1]; C_\alpha(\lambda \tilde{\boldsymbol{\gamma}} + (1 - \lambda)\boldsymbol{\gamma}^*) \leq K\}$, so $\tilde{\lambda} < 1$ because C_α is continuous, $C_\alpha(\tilde{\boldsymbol{\gamma}}) > K$ by (P4) and $C_\alpha(\boldsymbol{\gamma}^*) < K$ by assumption. Consequently

$$\ell_0(\tilde{\lambda} \tilde{\boldsymbol{\gamma}} + (1 - \tilde{\lambda})\boldsymbol{\gamma}^*) \leq \tilde{\lambda} \ell_0(\tilde{\boldsymbol{\gamma}}) + (1 - \tilde{\lambda}) \ell_0(\boldsymbol{\gamma}^*) < \ell_0(\boldsymbol{\gamma}^*),$$

because ℓ_0 is convex by (P1) and (P2), and $\boldsymbol{\gamma}^* \notin \arg \min_{\Gamma} \ell_0$ by (P4). This contradicts the optimality of the solution $\boldsymbol{\gamma}^*$. Therefore $C_\alpha(\boldsymbol{\gamma}^*) = K$. ■

Proposition 3.13. *Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ be two optimal solutions of (1.1), if (P1),(P2) and (P4) holds, then any convex combinations of these two solutions is optimal.*

Proof. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ be two optimal solutions of (1.1). By Lemma 3.12 the capital constraint is achieved for each of them:

$$v^* = \ell_0(\boldsymbol{\gamma}_1) = \ell_0(\boldsymbol{\gamma}_2) \tag{3.3}$$

$$K = C_\alpha(\boldsymbol{\gamma}_1) = C_\alpha(\boldsymbol{\gamma}_2). \tag{3.4}$$

Remember that C_α is convex because *CVaR* is sub-additive. For any convex combinations $\boldsymbol{\gamma}_c$ we have $C_\alpha(\boldsymbol{\gamma}_c) \leq K$ by convexity of C_α . The function ℓ_0 is convex by (P1) and (P2) so that $\ell_0(\boldsymbol{\gamma}_c) \leq v^*$. If $\ell_0(\boldsymbol{\gamma}_c) < v^*$, it contradicts the optimality of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ and therefore $\ell_0(\boldsymbol{\gamma}_c) = v^*$. This means that $\boldsymbol{\gamma}_c$ is optimal. Once more, we apply Lemma 3.12 to obtain $C_\alpha(\boldsymbol{\gamma}_c) = K$. ■

Lemma 3.14. Let $\alpha \in]0, 1[$ and Y (resp. Z) be a real valued random variable with a density on \mathbb{R} and cumulative distribution function F_Y (resp. F_Z). We define $A_Y := \mathbb{1}_{\{U_Y \geq \alpha\}}$ with $U_Y := F_Y(Y)$ (resp. $A_Z := \mathbb{1}_{\{U_Z \geq \alpha\}}$ with $U_Z := F_Z(Z)$). If $\mathbb{E}(YA_Y) + \mathbb{E}(ZA_Z) - \mathbb{E}((Y+Z)A_{Y+Z}) = 0$, then $\{U_Y \geq \alpha\} = \{U_{Y+Z} \geq \alpha\} = \{U_Z \geq \alpha\}$.

Proof. Note that A_Y and A_{Y+Z} have the same distribution thus $\mathbb{E}(A_Y - A_{Y+Z}) = 0$. Then, for all $m \in \mathbb{R}$ we have $\mathbb{E}(Y(A_Y - A_{Y+Z})) = \mathbb{E}((Y - m)(A_Y - A_{Y+Z}))$. Take $m = F_Y^{-1}(\alpha)$, with F^{-1} the cdf general inverse; we have $(Y - m)(A_Y - A_{Y+Z}) \geq 0$. As a consequence $\mathbb{E}(Y(A_Y - A_{Y+Z})) \geq 0$ and $\mathbb{E}(Z(A_Z - A_{Y+Z})) \geq 0$. The assumption can be rewritten as $\mathbb{E}(Y(A_Y - A_{Y+Z})) + \mathbb{E}(Z(A_Z - A_{Y+Z})) = 0$. We conclude that $\mathbb{E}(Y(A_Y - A_{Y+Z})) = \mathbb{E}(Z(A_Z - A_{Y+Z})) = 0$. In addition $\mathbb{E}((Y - m)(A_Y - A_{Y+Z})) = 0$ if and only if $A_Y = A_{Y+Z}$ a.s, similarly $A_Z = A_{Y+Z}$ a.s. because $(Y - m) \neq 0$ a.s. on \mathbb{R} . Finally, $\{U_Y \geq \alpha\} = \{U_{Y+Z} \geq \alpha\} = \{U_Z \geq \alpha\}$. ■

Lemma 3.15. let \mathcal{R} be a d -dimensional connected subset of \mathbb{R}^d , ϕ_1, ϕ_2 two non-zero \mathbb{R}^d -affine forms. If $\phi_1 \phi_2 \geq 0$ on \mathcal{R} and $\mathcal{R} \cap \ker(\phi_i) \neq \emptyset$, $i = 1, 2$ then $\phi_1 = a\phi_2$ with $a > 0$.

Proof. Suppose $\ker(\phi_1) \neq \ker(\phi_2)$ which are two $(d-1)$ -dimensional affine hyperplanes. There exists $x \in \mathcal{R} \cap (\ker(\phi_1) \setminus \ker(\phi_2))$, so $\phi_1(x) = 0$ and $\phi_2(x) \neq 0$. Let us take the case where $\phi_2(x) < 0$, there exists a neighbourhood $v(x) \subset \mathcal{R}$, such that $\forall y \in v(x) \phi_2(y) < 0$. $v(x)$ is an open set in \mathbb{R}^d and $\ker(\phi_1)$ is $(d-1)$ -dimensional, thus there exists $y' \in v(x)$, such that $\phi_1(y') > 0$ and therefore $\phi_1(y')\phi_2(y') < 0$ with $y' \in \mathcal{R}$. Symmetrically, it's the same with $\phi_2(x) > 0$. Finally $\ker(\phi_1) = \ker(\phi_2)$ and so $\phi_1(x) = a\phi_2(x)$ with $a > 0$. ■

Now, we are going to prove Theorem 3.10.

Proof of Theorem 3.10. Using Proposition 3.13, the solution set of (1.1) is convex and C_α equals K , on this set. Let γ_1, γ_2 be two optimal solutions. For all $\mu \in]0, 1[$ we have the following equality:

$$C_\alpha(\mu\gamma_1) + C_\alpha((1-\mu)\gamma_2) = C_\alpha(\mu\gamma_1 + (1-\mu)\gamma_2). \quad (3.5)$$

We use notations from Lemma 3.14 and recall that

$$\begin{aligned} C_\alpha(\gamma) &= \mathbb{E}\left(-\gamma^T \mathbf{X} \mid -\gamma^T \mathbf{X} \geq V_\alpha(\gamma)\right) = \mathbb{E}\left(-\gamma^T \mathbf{X} \mid U_{-\gamma^T \mathbf{X}} \geq \alpha\right) \\ &= (1-\alpha)^{-1} \mathbb{E}\left(-\gamma^T \mathbf{X} \mathbb{1}_{\{U_{-\gamma^T \mathbf{X}} \geq \alpha\}}\right). \end{aligned}$$

Take $Y := -\mu\gamma_1^T \mathbf{X}$, $Z := -(1-\mu)\gamma_2^T \mathbf{X}$. We can rewrite (3.5) as $\mathbb{E}(YA_Y) + \mathbb{E}(ZA_Z) - \mathbb{E}((Y+Z)A_{Y+Z}) = 0$ and apply Lemma 3.14 with (P3) to obtain $\{U_Y \geq \alpha\} = \{U_{Y+Z} \geq \alpha\} = \{U_Z \geq \alpha\}$.

These events can be rewritten as:

$$\{U_Y \geq \alpha\} = \{Y \geq VaR_\alpha(Y)\} = \{-\mu\gamma_1^T \mathbf{X} \geq VaR_\alpha(-\mu\gamma_1^T \mathbf{X})\} = \{-\gamma_1^T \mathbf{X} \geq VaR_\alpha(-\gamma_1^T \mathbf{X})\}.$$

The same holds for Z . Recall that \mathbf{X} has a density by (P3). Take $t_i = VaR_\alpha(-\gamma_i^T \mathbf{X})$ the last result becomes $\{-\gamma_1^T \mathbf{X} \geq t_1\} = \{-\gamma_2^T \mathbf{X} \geq t_2\}$ i.e.

$$\mathbb{P}\left((-\gamma_1^T \mathbf{X} - t_1)(-\gamma_2^T \mathbf{X} - t_2) \geq 0\right) = 1. \quad (3.6)$$

Consider $\phi_1(\mathbf{x}) = -\gamma_1^T \mathbf{x} - t_1$, $\phi_2(\mathbf{x}) = -\gamma_2^T \mathbf{x} - t_2$, $\phi_1 \phi_2 \geq 0$ on $\mathcal{R}_\mathbf{X}$ thanks to (3.6). Thanks to the definition of t_i , t_i belongs to $\text{supp}(-\gamma^T \mathbf{X})$ and $\mathcal{R}_\mathbf{X} \cap \ker(\phi_i) \neq \emptyset$, $i = 1, 2$. Therefore, by Lemma 3.15 $\phi_1 = a\phi_2$, with $a > 0$, so $\gamma_1 = a\gamma_2$ and $a = 1$ because C_α is positive homogeneous and $C_\alpha(\gamma_1) = C_\alpha(\gamma_2)$. We obtain the uniqueness of Problem (1.1). ■

Proposition 3.16. For any fixed $\gamma \neq 0_d$, if (P5) holds, the function $\zeta \mapsto g(\gamma, \zeta)$ is strictly convex on $\text{supp}(-\gamma^T \mathbf{X})$.

Proof. Consider the derivative $\frac{\partial g}{\partial \zeta}(\boldsymbol{\gamma}, \zeta) = 1 - (1 - \alpha)^{-1} \mathbb{E}(-\mathbb{1}_{\{-\boldsymbol{\gamma}^T \mathbf{X} - \zeta > 0\}}) = 1 - (1 - \alpha)^{-1} \mathbb{P}(-\boldsymbol{\gamma}^T \mathbf{X} > \zeta)$. Let us $\zeta_1 < \zeta_2 \in \text{supp}(-\boldsymbol{\gamma}^T \mathbf{X})$

$$\begin{aligned} \frac{\partial g}{\partial \zeta}(\boldsymbol{\gamma}, \zeta_2) - \frac{\partial g}{\partial \zeta}(\boldsymbol{\gamma}, \zeta_1) &= (1 - \alpha)^{-1} \left(\mathbb{P}(-\boldsymbol{\gamma}^T \mathbf{X} > \zeta_1) - \mathbb{P}(-\boldsymbol{\gamma}^T \mathbf{X} > \zeta_2) \right) \\ &= (1 - \alpha)^{-1} \mathbb{P}(\zeta_1 < -\boldsymbol{\gamma}^T \mathbf{X} < \zeta_2) \geq 0 \end{aligned}$$

this quantity is non negative on \mathbb{R} and positive on $\text{supp}(-\boldsymbol{\gamma}^T \mathbf{X})$ because $-\boldsymbol{\gamma}^T \mathbf{X}$ has a density by (P3) and $\text{supp}(-\boldsymbol{\gamma}^T \mathbf{X})$ is an interval thanks to (P5). ■

Finally, we are going to prove Theorem 3.11.

Proof of Theorem 3.11. Using Theorem 3.10, let $\boldsymbol{\gamma}^*$ be the unique solution to Problem (1.1). By equivalence of Problems (2.1) and (2.2), any $(\tilde{\boldsymbol{\gamma}}, \tilde{\zeta})$ solution to (2.3) satisfies $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}^*$.

Moreover $C_\alpha(\boldsymbol{\gamma}^*) = \min_{\zeta \in \mathbb{R}} g(\boldsymbol{\gamma}^*, \zeta)$ and thanks to Proposition 3.16, $A_\alpha(\boldsymbol{\gamma}^*) = \{\zeta^*\}$. Therefore $(\boldsymbol{\gamma}^*, \zeta^*)$ is the unique solution of Problem (2.3). ■

3.3 Convergence rate

In Shapiro, A., Dentcheva, D., Ruszczyński, A. (2009) [7] the rate of convergence is also obtained. We state a version (Theorem 3.17), adapted to our setting. It is used to get the convergence rate in Theorem 3.19.

Consider some additional notations and assumption:

(P6) $\mathbb{E}(|\mathbf{X}|^2) < +\infty$.

(P7) $\mathbb{E} \left(\sup_{(\boldsymbol{\gamma}, \zeta) \in \mathcal{U}} |L(\boldsymbol{\gamma}, \zeta, \mathbf{X})|^2 \right) < +\infty$.

(P8) There exists a measurable function $C_L : X \rightarrow \mathbf{R}^+$ such that $\mathbb{E}(C_L(\mathbf{X})^2) < \infty$ and $\forall u, u' \in \mathcal{U}, |L(u, \mathbf{X}) - L(u', \mathbf{X})| \leq C_L(\mathbf{X})|u - u'|$ almost surely.

(P9) $(\boldsymbol{\gamma}, \zeta) \rightarrow \ell(\boldsymbol{\gamma}, \zeta)$ is differentiable.

Theorem [5.11 p173 [7]] deals with problems of the following form:

$$\min_{u \in \mathcal{U}} \ell_N(u) \quad \text{s.t.} \quad g_{iN}(u) \leq 0, \quad i = 1, \dots, p. \quad (3.7)$$

For Problem (2.3) there are two kinds of constraints, one on the parameters $\boldsymbol{\gamma}, \zeta$ and one with the g function. Consider $g_1 : (\boldsymbol{\gamma}, \zeta) \mapsto g(\boldsymbol{\gamma}, \zeta) - K$, $g_2 : \boldsymbol{\gamma} \mapsto \boldsymbol{\gamma} - \boldsymbol{\gamma}^{up}$ and $g_3 : \boldsymbol{\gamma} \mapsto \boldsymbol{\gamma}^{low} - \boldsymbol{\gamma}$. We have $G_1 : ((\boldsymbol{\gamma}, \zeta), \mathbf{X}) \mapsto G((\boldsymbol{\gamma}, \zeta), \mathbf{X}) - K = \zeta + (1 - \alpha)^{-1}(-\boldsymbol{\gamma}^T \mathbf{X} - \zeta)^+ - K$, $G_2 = g_2$ and $G_3 = g_3$. They are defined such that for any $u \in \mathcal{U}$ we have $g_i(u) = \mathbb{E}(G_i(u, \mathbf{X}))$. We denote by g_{iN} the approximated version of g_i .

Let $u \in \mathcal{U}$, then the empirical mean estimator $\ell_N(u)$ of $\ell(u)$ is unbiased and has a variance of $\sigma^2(u) := \text{Var}(L(u, \mathbf{X}))$ which is assumed to be finite. Moreover the standard Central Limit Theorem gives:

$$N^{1/2} (\ell_N(u) - \ell(u)) \xrightarrow[N \rightarrow +\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2(u))$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution. Let $Y(u)$ be a random variable such that $Y(u) \sim \mathcal{N}(0, \text{Var}(L(u, \mathbf{X})))$. Let $Y_i(u)$ be a random variable such that $Y_i(u) \sim \mathcal{N}(0, \text{Var}(G_i(u, \mathbf{X})))$, $i = 1, 2, 3$.

Let us introduce the Lagrangian of Problem (3.7):

$$\mathcal{L}(u, \lambda) := \ell(u) + \sum_{i=1}^p \lambda_i g_i(u),$$

which can be rewritten as

$$\mathcal{L}(\boldsymbol{\gamma}, \zeta, (\lambda, \underline{\mu}, \underline{\mu})) := \ell(\boldsymbol{\gamma}, \zeta) + \lambda(g(\boldsymbol{\gamma}, \zeta) - K) + \underline{\mu} \cdot (\boldsymbol{\gamma} - \boldsymbol{\gamma}^{up}) - \underline{\mu} \cdot (\boldsymbol{\gamma} - \boldsymbol{\gamma}^{low}). \quad (3.8)$$

In our context, Problem (2.3) is convex because ℓ is convex. In addition the set S of optimal solutions to the original problem (1.1) is non-empty and bounded. Moreover ℓ and g_i are bounded on \mathcal{U} and in effect in a neighbourhood of S . Thanks to these results, we can state a modified version of Theorem 5.11 p173 in [7] adapted to our framework.

Theorem 3.17 (Restatement of Theorem 5.11 p173 in [7]). *Let v_N be the optimal value of the above problem (3.7). Assume that the following assumptions are satisfied:*

- (i) *There exists $\bar{u} \in \mathcal{U}$ such that $g_i(\bar{u}) < 0$, $i = 1, 2, 3$. (Slater's condition).*
- (ii) *The following two assumptions are satisfied for $H = L, G_i, i = 1, 2, 3$:*

(A₁) *There exists $\tilde{u} \in \mathcal{U}$ such that $\mathbb{E}(H(u, \mathbf{X})^2) < +\infty$.*

(A₂) *There exists a measurable function $C : X \rightarrow \mathbf{R}^+$ such that $\mathbb{E}(C(\mathbf{X})^2) < \infty$ and $\forall u, u' \in \mathcal{U}$, $|H(u, \mathbf{X}) - H(u', \mathbf{X})| \leq C(\mathbf{X})|u - u'|$ almost surely.*

Then

$$N^{1/2} (v_N - v^*) \xrightarrow{\mathcal{D}} \inf_{u \in S} \sup_{\lambda \in \Lambda} \left[Y(u) + \sum_{i=1}^p \lambda_i Y_i(u) \right]. \quad (3.9)$$

A set Λ of Lagrange vectors $\lambda = (\lambda_1, \dots, \lambda_p)$ is associated to each optimal solution $u^* \in S$ which satisfy,

$$u^* \in \arg \min_{u \in \mathcal{U}} \mathcal{L}(u, \lambda), \quad \lambda_i \leq 0 \text{ and } \lambda_i g_i(u^*) = 0, \quad i = 1, \dots, p.$$

The set Λ coincides with the set of optimal solutions to the dual of the original problem and is therefore the same for any optimal solution $u^* \in S$.

Furthermore, if $S = \{u^*\}$ and $\Lambda = \{\lambda^*\}$ are singletons, then

$$N^{1/2} (v_N - v^*) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

with

$$\sigma^2 := \mathbb{V}ar \left(L(u^*, \mathbf{X}) + \sum_{i=1}^p \lambda_i^* G_i(u^*, \mathbf{X}) \right).$$

Lemma 3.18. *Given a set of constraints $\boldsymbol{\gamma}^{up}, \boldsymbol{\gamma}^{low}$ and $\boldsymbol{\gamma} \in \Gamma$, if (P1), (P2) hold, then there exists at most a single $K \leq \min_{\boldsymbol{\gamma} \in \arg \min_{\Gamma} \ell_0} C_\alpha(\boldsymbol{\gamma})$ such that $\boldsymbol{\gamma}$ solves (1.1) with K as the capital limit.*

Proof. This is a corollary of Lemma 3.12, because if $\boldsymbol{\gamma}$ is solution of (1.1) with $K' \leq C_\alpha(\arg \min_{\Gamma} \ell_0)$, then $K' = C_\alpha(\boldsymbol{\gamma})$ and so there can only be one K' . ■

Theorem 3.19. *If (P1) to (P9) hold, denote by $u^* = (\boldsymbol{\gamma}^*, \zeta^*)$ the unique solution to Problem (2.3), then*

$$N^{1/2} (v_N - v^*) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

with

$$\sigma^2 := \mathbb{V}ar (L(u^*, \mathbf{X}) + \lambda^* (G(u^*, \mathbf{X}) - K))$$

and λ^* is the maximum value of the Lagrange multiplier λ associated with the capital constraint.

Proof of Theorem 3.19. The uniqueness of u^* is given by (P1) to (P5) using Theorem 3.11. Let us look at the conditions for applying Theorem 3.17. Condition (i) holds because we assumed that the interior of \mathcal{U}^K is non-empty which means that Slater's condition is satisfied for g_1, g_2, g_3 . The functions G_2, G_3 are deterministic and linear functions of $\boldsymbol{\gamma}$ so (A₁) and (A₂) hold. Conditions (A₁) and (A₂) hold for L by (P7) and (P8) and for G_1 it is easy to prove with (P6).

We can apply Theorem 3.17 to obtain (3.9). The functions G_2 and G_3 are deterministic so $Y_2 = Y_3 = 0$, therefore $\sum_{i=1}^p \lambda_i Y_i((\boldsymbol{\gamma}, \zeta)) = \lambda_1 Y_1((\boldsymbol{\gamma}, \zeta))$. Thanks to the uniqueness of u^* , we only need the maximum value of λ_1 or the uniqueness of λ_1 to apply the second part of Theorem 3.17.

Consider the Lagrange multiplier (3.8) and recall that $g(\boldsymbol{\gamma}, \zeta) = \zeta + (1 - \alpha)^{-1} \mathbb{E} [(-\boldsymbol{\gamma}^T \mathbf{X} - \zeta)^+]$.

The set Λ of Lagrange multipliers vectors $(\lambda, \underline{\boldsymbol{\mu}}, \underline{\boldsymbol{\mu}})$ is satisfying the optimality conditions. The optimal solution $(\boldsymbol{\gamma}^*, \zeta^*)$ satisfies $\nabla_{\boldsymbol{\gamma}, \zeta} \mathcal{L}(\boldsymbol{\gamma}^*, \zeta^*) = 0$ and the following optimality conditions $\lambda(g(\boldsymbol{\gamma}^*, \zeta^*) - K) = 0$, $\underline{\boldsymbol{\mu}}(\boldsymbol{\gamma}^* - \boldsymbol{\gamma}^{up}) = 0$ and $\underline{\boldsymbol{\mu}}(\boldsymbol{\gamma}^* - \boldsymbol{\gamma}^{low}) = 0$. Note that $\mathbb{P}(-\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^* \geq 0) = 1 - \alpha$ because $\zeta^* \in A_\alpha(\boldsymbol{\gamma}^*)$.

Consider

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \zeta}(\boldsymbol{\gamma}^*, \zeta^*) &= \frac{\partial \ell}{\partial \zeta}(\boldsymbol{\gamma}^*, \zeta^*) + \lambda \left(1 + (1 - \alpha)^{-1} \mathbb{E}(-\mathbb{1}_{\{-\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^* \geq 0\}}) \right) \\ &= \frac{\partial \ell}{\partial \zeta}(\boldsymbol{\gamma}^*, \zeta^*) + \lambda \left(1 - (1 - \alpha)^{-1} \mathbb{P}(-\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^* \geq 0) \right) \\ &= \frac{\partial \ell}{\partial \zeta}(\boldsymbol{\gamma}^*, \zeta^*). \end{aligned}$$

Thus,

$$\frac{\partial \ell}{\partial \zeta}(\boldsymbol{\gamma}^*, \zeta^*) = 0. \quad (3.10)$$

Consider

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}}(\boldsymbol{\gamma}^*, \zeta^*) &= \frac{\partial \ell}{\partial \boldsymbol{\gamma}}(\boldsymbol{\gamma}^*, \zeta^*) + \lambda(1 - \alpha)^{-1} \mathbb{E}(-\mathbf{X} \cdot \mathbb{1}_{\{-\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^* \geq 0\}}) + \underline{\boldsymbol{\mu}} - \underline{\boldsymbol{\mu}} \\ &= \frac{\partial \ell}{\partial \boldsymbol{\gamma}}(\boldsymbol{\gamma}^*, \zeta^*) + \lambda(1 - \alpha)^{-1} \mathbb{E}(-\mathbf{X} | -\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^* \geq 0) \mathbb{P}(-\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^* \geq 0) + \underline{\boldsymbol{\mu}} - \underline{\boldsymbol{\mu}}. \end{aligned}$$

So for each $i = 1, \dots, d$:

$$\frac{\partial \ell}{\partial \boldsymbol{\gamma}_i}(\boldsymbol{\gamma}^*, \zeta^*) + \lambda \mathbb{E}(-\mathbf{X}_i | -\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^* \geq 0) + \underline{\mu}_i - \underline{\mu}_i = 0. \quad (3.11)$$

We want to find $j \in \{1, \dots, d\}$ such that $\frac{\partial \ell}{\partial \boldsymbol{\gamma}_j}(\boldsymbol{\gamma}^*, \zeta^*) \neq 0$ and $\underline{\mu}_j = \underline{\mu}_j = 0$. Let $H \subset \{1, \dots, d\}$ such that $\forall j \in H, \frac{\partial \ell}{\partial \boldsymbol{\gamma}_j}(\boldsymbol{\gamma}^*, \zeta^*) \neq 0$.

Let $\tilde{\boldsymbol{\gamma}} \in \arg \min_{\Gamma} \ell_0$ and $\tilde{\zeta} \in V_\alpha(\tilde{\boldsymbol{\gamma}})$, so $g(\tilde{\boldsymbol{u}}) = C_\alpha(\tilde{\boldsymbol{\gamma}}) > K$ by (P4), with $\tilde{\boldsymbol{u}} = (\tilde{\boldsymbol{\gamma}}, \tilde{\zeta})$. Hence $\nabla \ell(u^*) \cdot (\tilde{\boldsymbol{u}} - u^*) < 0$ thanks to the convexity of ℓ by (P1) and (P2). We use (3.10) to obtain

$$\nabla \ell(u^*) \cdot (\tilde{\boldsymbol{u}} - u^*) = \nabla \ell(\boldsymbol{\gamma}^*, \zeta^*) \cdot (\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*) < 0.$$

Thus, there exists $j \in \{1, \dots, d\}$ such that $(\tilde{\gamma}_j - \gamma_j^*) \frac{\partial \ell}{\partial \gamma_j}(\gamma^*, \zeta^*) < 0$ and so $H \neq \emptyset$.

There are three possible cases, with the optimality conditions, for $j \in H$:

1. $\gamma_j = \gamma_j^{up}$ implies $\underline{\mu}_j = 0$ and $\bar{\mu}_j \geq 0$;
2. $\gamma_j = \gamma_j^{low}$ implies $\underline{\mu}_j \geq 0$ and $\bar{\mu}_j = 0$;
3. $0 < \gamma_j^{low} < \gamma_j < \gamma_j^{up}$ implies $\underline{\mu}_j = \bar{\mu}_j = 0$.

If for all $j \in H$, $\gamma_j = \gamma_j^{up}$ or $\gamma_j = \gamma_j^{low}$ then we cannot conclude but we decided to exclude this case because it is unlikely in a sense that for a sufficiently small $\epsilon > 0$, for all K' such that $|K - K'| < \epsilon$, the corresponding optimal solution $\gamma_{K'}$ is not in this case. Indeed, thanks to 3.18 the optimal solution cannot be the same as for K , and for all $j \notin H$, $\frac{\partial \ell}{\partial \gamma_j}(\gamma^*, \zeta^*) = 0$. Therefore, it is a γ_i with an $i \in H$, so in case 1 or 2, which must move, which places it directly in case 3.

So there exists $j \in H$ in the case 3, knowing that $\frac{\partial \ell}{\partial \gamma_j}(\gamma^*, \zeta^*) \neq 0$, $\bar{\mu}_j = \underline{\mu}_j = 0$ and thanks to (3.11), $\mathbb{E}(-\mathbf{X}_j | -\gamma^{*T} \mathbf{X} - \zeta^* \geq 0) \neq 0$. Thus $\lambda = -\frac{\partial \ell}{\partial \gamma_j}(\gamma^*, \zeta^*) \mathbb{E}(-\mathbf{X}_j | -\gamma^{*T} \mathbf{X} - \zeta^* \geq 0)^{-1}$. Therefore the Lagrange multiplier λ is unique.

Moreover, denoting this Lagrange multiplier by λ^* , $\mathbb{V}ar(L(u^*, \mathbf{X}) + \sum_{i=1}^p \lambda_i^* G_i(u^*, \mathbf{X}))$ becomes $\mathbb{V}ar(L(u^*, \mathbf{X}) + \lambda^*(G(u^*, \mathbf{X}) - K))$. \blacksquare

4 Numerical study

This section illustrates the results of section 3, in particular convergence and convergence rate. When it was possible to compare the results of the SAA method with an explicit and computational result. We first consider the Gaussian case for which an explicit solution is known for the mean-CVaR, where all the asset returns follow a Gaussian distribution. Second, we consider a more realistic case where all the asset returns do not follow the same distribution. This last case is the closest to what can be found in (re)insurance.

All the examples are solved within a few seconds and are carried out using Python libraries (mainly Scipy, Numpy).

4.1 Context

In what follows, we consider $L_0(\gamma, \mathbf{X}) = -\gamma^T \mathbf{X}$ with a risk level $\alpha = 0.99$.

Note that L_0 is convex and continuous so (P1) hold.

Let us rewrite $L(\gamma, \zeta, \mathbf{X}) = -\gamma^T \mathbf{X}$ and $\ell_N(\gamma, \zeta) := -\frac{\gamma^T}{N} \sum_{j=1}^N \mathbf{X}^{(j)}$.

Once we have chosen the asset distributions and other parameter for the problem $(K, \gamma^{up}, \gamma^{low})$, for a given N , we run k optimizations, i.e. solve the SAA problem (2.4), on k different \mathbf{X} i.i.d. sample of size N to obtain a sample of size k of γ_N, v_N .

4.2 Gaussian case

Let us consider the Gaussian case, $\mathbf{X} \sim \mathcal{N}_d(\mathbf{1}_d, \Sigma)$, with Σ a positive definite matrix. Note that $\gamma^T \mathbf{X} \sim \mathcal{N}(\gamma^T \mathbf{1}_d, \gamma^T \Sigma \gamma)$ and $CVaR_\alpha(-\gamma^T \mathbf{X}) = -\gamma^T \mathbf{1}_d + T_Z \sqrt{\gamma^T \Sigma \gamma}$, with $T_Z = CVaR_\alpha(\mathcal{N}(0, 1))$.

If $\tilde{\mathbf{X}} \sim \mathcal{N}(\tilde{\mu}, \tilde{\Sigma})$, then $\gamma^T \tilde{\mathbf{X}} = (\gamma * \tilde{\mu})^T \mathbf{X}$ and with $\mathbf{X} \sim \mathcal{N}(\mathbf{1}_d, \Sigma)$ and $\Sigma = \tilde{\mu}^T \tilde{\Sigma} \tilde{\mu}$. Hence considering $\mathbb{E}(X) = \mathbf{1}_d$ is sufficiently general.

Denote $\boldsymbol{\gamma}^*$ the optimal solution to (1.1), if $\gamma_i^{up} < \gamma_i^* < \gamma_i^{low}$ for $i = 1, \dots, d$, then:

$$\boldsymbol{\gamma}^* = K \left(\frac{1}{T_Z - \sqrt{\mathbf{1}_d^T \Sigma^{-1} \mathbf{1}_d}} \right) \frac{\Sigma^{-1} \mathbf{1}_d}{\sqrt{\mathbf{1}_d^T \Sigma^{-1} \mathbf{1}_d}}, \quad (4.1)$$

See Appendix 2 for a proof.

Properties (P2), (P5) and (P6) hold immediately. We choose the parameters bounds for (P4) to hold: $C_\alpha(\boldsymbol{\gamma}) > K, \forall \boldsymbol{\gamma} \in \arg \min_{\boldsymbol{\gamma} \in \Gamma} \ell_0(\boldsymbol{\gamma})$. If not the capital limit is not reached. Note that (P7) to (P9) hold too.

We set $d = 5, \alpha = 0.99, K = 100, k = 5000$ runs and covariance matrix Σ picked at random.

$$\Sigma = \begin{pmatrix} 4.490 & -0.377 & 0.059 & 0.585 & -1.709 \\ -0.377 & 6.109 & -1.300 & 0.229 & 1.380 \\ 0.059 & -1.300 & 7.059 & -1.401 & 0.210 \\ 0.585 & 0.229 & -1.401 & 8.400 & -1.250 \\ -1.709 & 1.380 & 0.210 & -1.250 & 19.934 \end{pmatrix}.$$

4.2.1 Without bounds

This case is without bounds in the sense that we set $\gamma_i^{up}, \gamma_i^{low}$ to have $\gamma_i^{up} < \gamma_i^* < \gamma_i^{low}$, in other words in such a way that they are never reached. So, we use (4.1) to compute $\boldsymbol{\gamma}^* = (15.113, 12.576, 12.610, 8.702, 3.958)$ and $v^* = -52.960$.

We also compute the standard estimator of $\sigma^2 = \text{Var}(-\boldsymbol{\gamma}^{*T} \mathbf{X} + \lambda^*(\zeta^* + \frac{1}{1-\alpha}(-\boldsymbol{\gamma}^{*T} \mathbf{X} - \zeta^*)^+ - K))$ from Theorem 3.19 and we obtain $\hat{\sigma} = 149.860$.

Figures 1a to 1e and 4a illustrate Theorem 3.8, showing that the solution to the SAA formulation (2.4) converges to the original solution found with (4.1). Indeed, the convergence of $\boldsymbol{\gamma}_N$ and v_N is observed and a sample size of 10^5 is sufficient to obtain an acceptable error, here 2% on each components of $\boldsymbol{\gamma}^*$ and 1% on v^* .

Figure 4b illustrates a convergence rate of \sqrt{N} seen in Theorem 3.19 since the slope of each curve is -0.5 . We see empirically the same results for the parameters with figure 2a to 2e.

The histogram of Figure 5 has a Gaussian shape with parameters $(\mu, \sigma) = (-9.15, 160.33)$. This illustrates Theorem 3.19 with $\hat{\sigma} = 149.860$. From Figure 3, we may infer the the parameters have an asymptotic gaussian behavior.

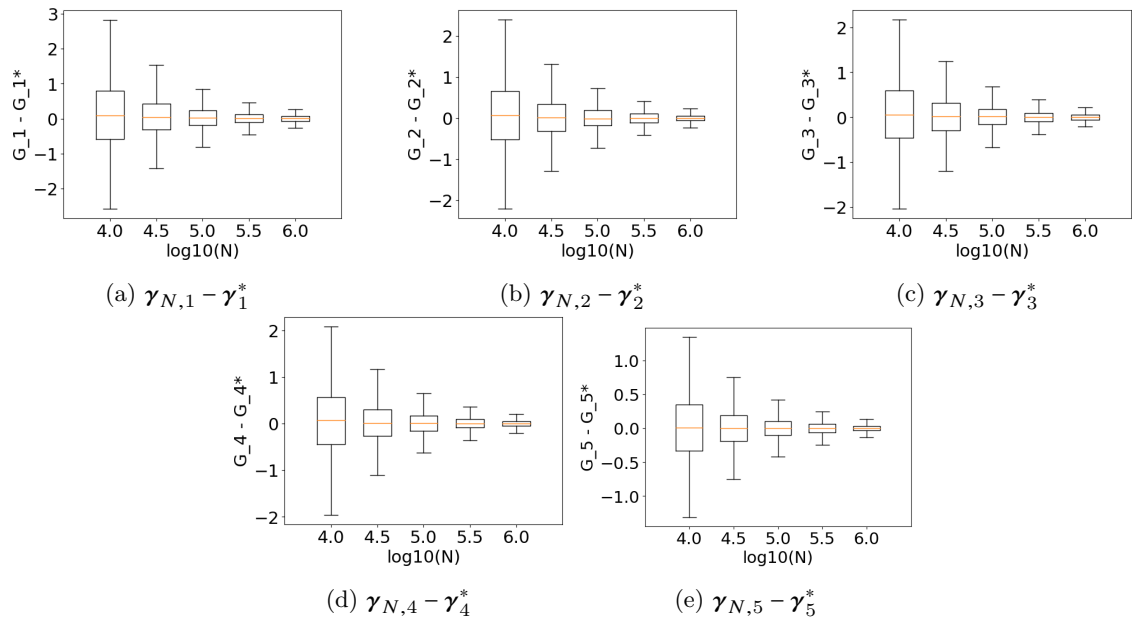


Figure 1: Component-wise Boxplots of $\gamma_N - \gamma^*$. (boxplot standard parameters: Q1-1.5IQR, Q1, median, Q3, Q3+1.5IQR)

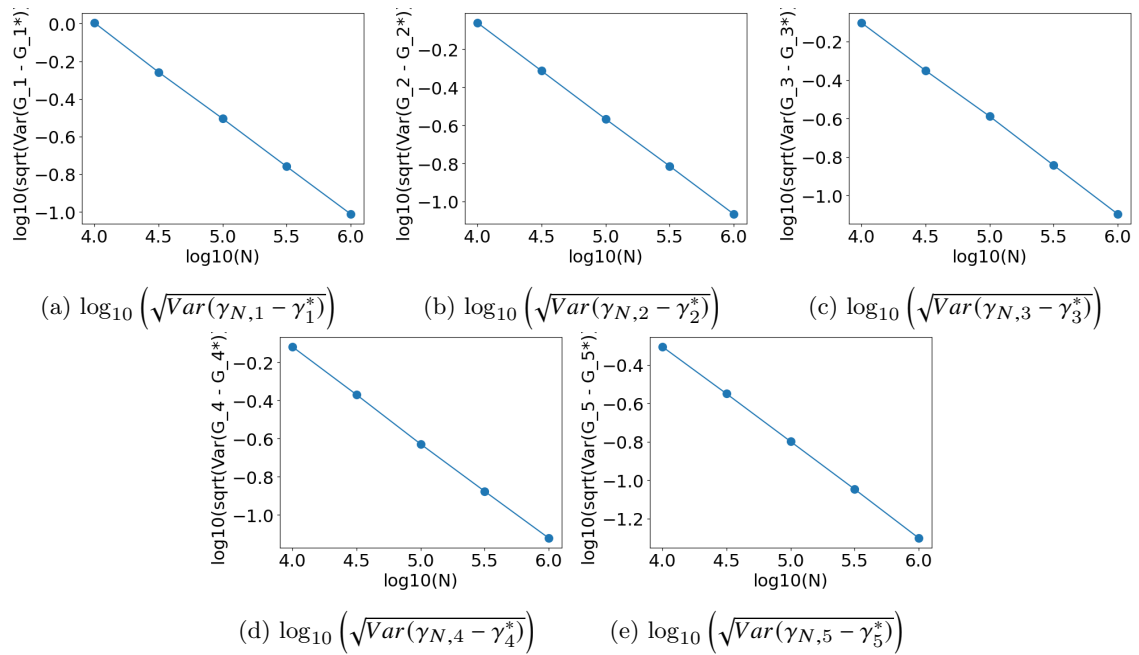


Figure 2: Log plot of $\log_{10}(\sqrt{\text{Var}(\gamma_N - \gamma^*)})$ components-wise.

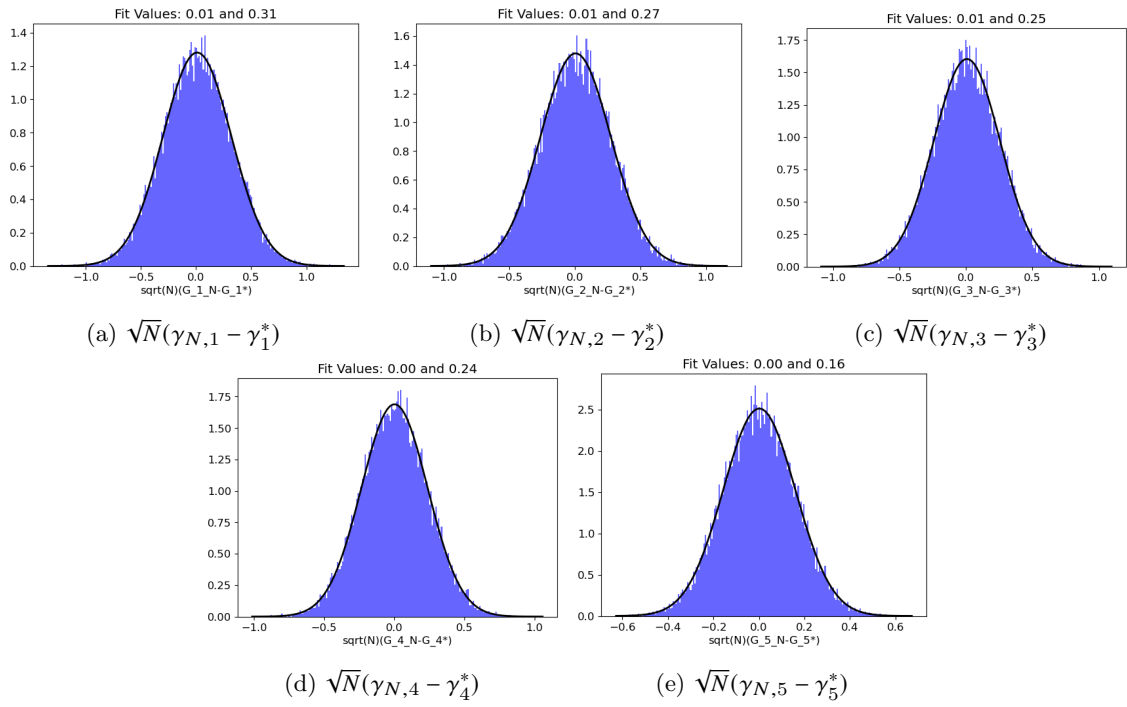


Figure 3: Components-wise histograms of $\sqrt{N}(\gamma_N - \gamma^*)$ for $N = 10^5$ and plot of a fitted normal distribution.

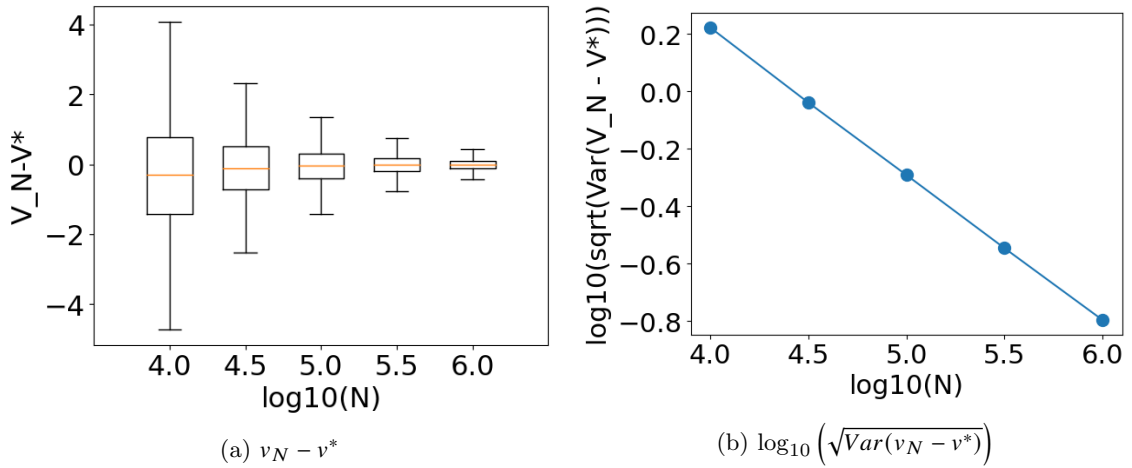


Figure 4: Boxplots of $v_N - v^*$ and \log_{10} plot of $\log_{10}(\sqrt{\text{Var}(v_N - v^*)})$.

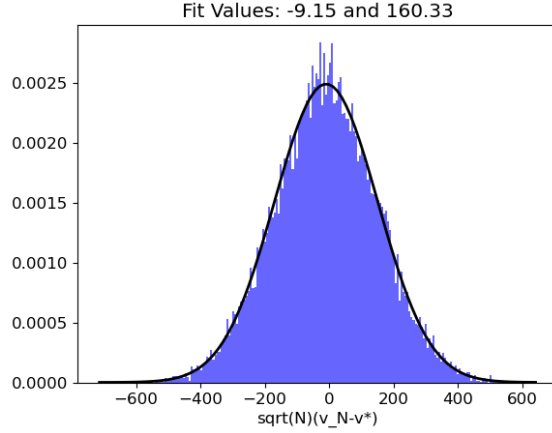


Figure 5: Histogram of $\sqrt{N}(v_N - v^*)$ for $N = 10^5$ and plot of a fitted normal distribution.

4.2.2 With bounds

Now, in the case where the optimal parameters reach at least one bound, we cannot compute the theoretical values for γ^* and v^* .

Looking at the unconstrained optimal values (15.113, 12.576, 12.610, 8.702, 3.958), from subsection 4.2.1, helps us choose the limits $\gamma^{low} = (0, 0, 15, 0, 6)$ and $\gamma^{up} = (10, 30, 30, 30, 30)$ such that they will be reached. We expect parameter 1 to hit its higher limit and parameters 3 and 5 to hit their lower limits. These claims are confirmed by a simulation with $N = 10^6$ where $\gamma_{10^6} = (10, 9.747, 15, 7.867, 6)$ and $v_{10^6} = -48.970$, which is expected more than the unconstrained case -52.960 .

Figure 6 and 8 show the convergence to a solution γ^* which has its components 1, 3, 5 bounded (6a, 6c, 6e) as expected. Figure 9 shows $N^{1/2}(v_N - v^*) \sim \mathcal{N}(0, \sigma^2)$ thanks to Theorem 3.19. We approximate v^* by $\mathbb{E}(v_N)$ and γ^* by $\mathbb{E}(\gamma_N)$ on k runs. Here we do not have access to a theoretical σ value, we estimate it by $\tilde{\sigma} = 226.99$ with the gaussian shape estimation. Figure 7a shows an asymptotic Gaussian behavior, while Figure 7b is less well approximated by a Gaussian.

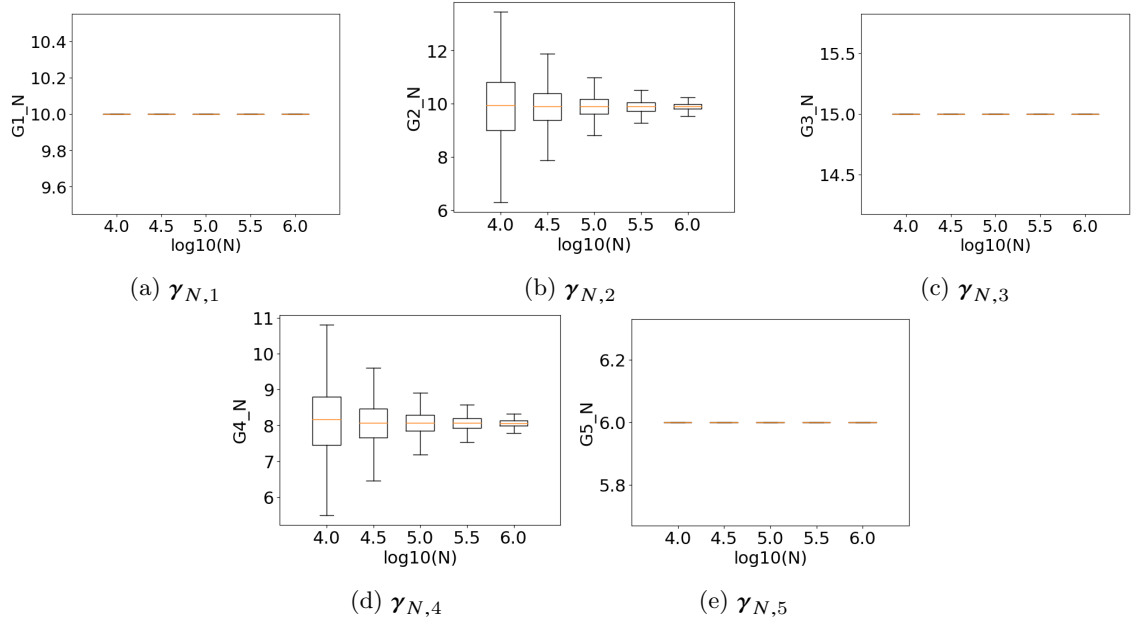


Figure 6: Component-wise Boxplots of γ_N . (boxplot standard parameters: Q1-1.5IQR, Q1, median, Q3, Q3+1.5IQR)

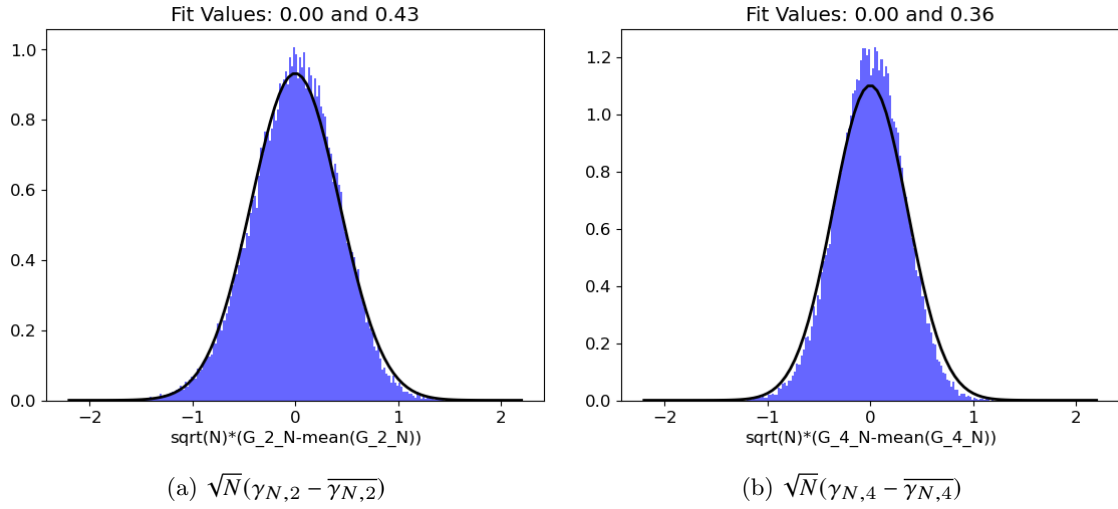


Figure 7: Histograms of $\sqrt{N}(\gamma_N - \gamma^*)$ for components 2 and 4, for $N = 10^5$ and plot of a fitted normal distribution on each.

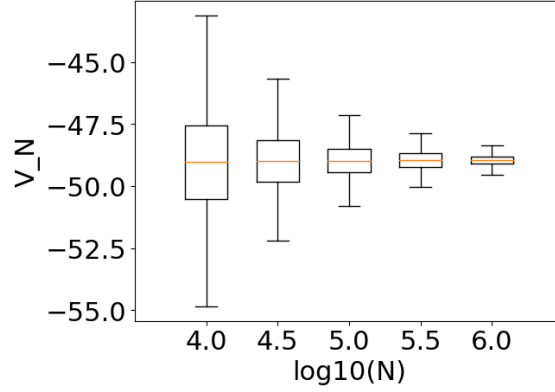


Figure 8: Boxplots of v_N . (boxplot standard parameters: Q1-1.5IQR, Q1, median, Q3, Q3+1.5IQR)

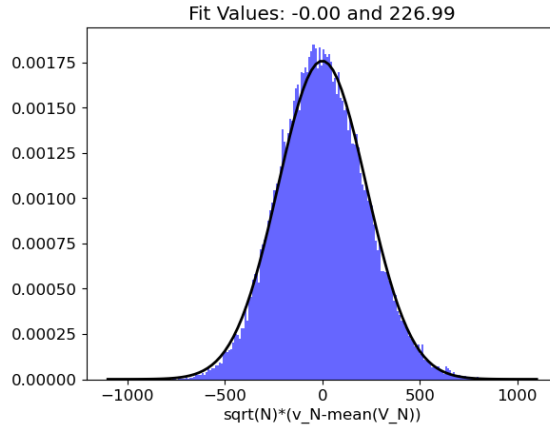


Figure 9: Histogram of $\sqrt{N}(v_N - \bar{v}_N)$ for $N = 10^5$ and plot of a fitted normal distribution.

4.3 Multi laws setting

In practice, we only have access to a sample of data for each Line of Business (LoB) returns. This sample may come from complex modeling and may not fit any known law in practice. All LoBs have different distributions with very different risk profiles.

We choose to have 5 LoBs and for each, we choose a distribution and shifted it to obtain a mean of 1 like in the Gaussian case. Let us consider $\tilde{\mathbf{X}}_i, i = 1, \dots, 5$, five random variables. $\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2$ follow a generalized Pareto distribution with α_{pareto} parameter 0.45 and 0.25 (as it is defined on the scipy python package), $\tilde{\mathbf{X}}_3, \tilde{\mathbf{X}}_4$ follow a log normal distribution with parameters $(\mu, \sigma) = (1.7, 1)$ and $(\mu, \sigma) = (1.3, 1)$. Finally, $\tilde{\mathbf{X}}_5$ follows a normal distribution with $(\mu, \sigma) = (2, 6)$.

Let us define $\mathbf{X}_i = 1 - (\tilde{\mathbf{X}}_i - \bar{\tilde{\mathbf{X}}}_i)$ for $i = 1, \dots, 5$. In such a setting, the standalone Var and $CVaR$ can be calculated analytically.

We choose three different dependency settings: a Gaussian copula, a clayton copula and the independent case. For the clayton case we use $\alpha_{clayton} = 2$ and in the Gaussian case we use the following correlation matrix picked at random.

i	$\mu(\mathbf{X}_i)$	$\sigma(\mathbf{X}_i)$	$VaR_{0.99}(-\mathbf{X}_i)$	$CVaR_{0.99}(-\mathbf{X}_i)$
1	1	5.75	12.61	27.05
2	1	1.89	6.32	10.53
3	1	17.49	46.94	80.55
4	1	4.89	17.25	53.73
5	1	6	12.96	15.99

Table 1: distribution description for the multi laws case

$$\Sigma = \begin{pmatrix} 1 & -0.1285 & 0.3979 & -0.4731 & 0.3879 \\ -0.1285 & 1 & -0.0574 & -0.2253 & -0.3532 \\ 0.3979 & -0.0574 & 1 & -0.5363 & 0.12 \\ -0.4731 & -0.2253 & -0.5363 & 1 & 0.0999 \\ 0.3879 & -0.3532 & 0.12 & 0.0999 & 1 \end{pmatrix}.$$

We run the optimization with the following parameters $K = 100$, without bounds and with bounds $\boldsymbol{\gamma}^{low} = (2, 0, 0, 1, 0)$ and $\boldsymbol{\gamma}^{up} = (10, 5, 10, 10, 10)$ and $N = 10^6$.

$N = 10^6$	γ_1	γ_2	γ_3	γ_4	γ_5	v
without bound						
independent	1.407	7.596	0.101	1.196	4.571	-14.922
Gaussian	0.575	10.386	0.0628	1.131	5.498	-17.655
clayton	0.7521	6.535	0.0178	0.656	3.794	-11.763
with bounds						
independent	2	5	0.146	1.501	5.343	-13.987
Gaussian	2	5	0.0333	1.577	4.839	-13.404
clayton	2	5	2.968	1	2.969	-10.959

Table 2: multi laws results

At optimum and for all cases, we expect that the LoBs with the lowest $CVaR$, like 2 and 5 are prioritized over those with the highest $CVaR$, like 3. The case without bounds leads us to expect that LoB 1 and 2 will reach their bounds, and that is what we are seeing. Also, the bounded cases, as expected, slow the diversification effects and then the return are worse than without.

In the independent case, γ_i is approximately inversely proportional to $CVaR_{0.99}(-\mathbf{X}_i)$, except for the bounded LoBs. The Gaussian dependency case without bound gives a better return thanks to a diversification with the chosen correlation matrix, like LoB 2 with LoBs 4 and 5 because of $\Sigma_{2,4} = -0.2253, \Sigma_{2,5} = -0.3532$. However, this diversification benefit cannot be exploited in the bounded case, mainly due to LoB 2 which is limited to 5. The Clayton case intensifies the queue positive dependency and weakens the $CVaR$ diversification, so the return is worse.

5 Appendix

There are two parts to this appendix, the first is the proof of the Equivalent Theorem used in 2.1 and the second part is the proof of the optimal value formula for the Gaussian case in 4.2 when the bounds are not reached.

5.1 Equivalent Theorem proof

We use in section 2.1 a result from Krokmal P., Jonas Palmquist J., Uryasev S. (2002) [2] on the equivalence of two optimization problems but this particular property does not appear explicitly in the article so we give a proof here.

The proof is based on the Karush-Kuhn-Tucker necessary and sufficient conditions.

Proposition 2.1. *For any function $R : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ such that for any $\gamma \in \mathbb{R}^d$, $R(\gamma, \cdot)$ is not increasing then the following two problems are equivalent*

$$\inf_{\gamma \in \Xi} -R(\gamma, C_\alpha(\gamma)) \quad \text{s.t.} \quad C_\alpha(\gamma) \leq K \quad (2.1)$$

and

$$\inf_{(\gamma, \zeta) \in \Xi \times \mathbb{R}} -R(\gamma, g(\gamma, \zeta)) \quad \text{s.t.} \quad g(\gamma, \zeta) \leq K. \quad (2.2)$$

Moreover, if the C_α constraint is active in (2.1), (γ^*, ζ^*) achieves the minimum of (2.2) solution if and only if γ^* achieves the minimum of (2.1) and $\zeta^* \in A_\alpha(\gamma^*)$.

Proof. Let us write down the necessary and sufficient Karush-Kuhn-Tucker conditions for problems (2.1) and (2.2).

For $R(\cdot, C_\alpha(\cdot))$ to achieve its minimum at $\gamma^* \in \Gamma$, it must exist a constant μ_1 such that, for all $\gamma \in \Gamma$:

$$\begin{aligned} -R(\gamma^*, C_\alpha(\gamma^*)) + \mu_1 C_\alpha(\gamma^*) &\leq -R(\gamma, C_\alpha(\gamma)) + \mu_1 C_\alpha(\gamma), \\ \mu_1 (C_\alpha(\gamma^*) - K) &= 0. \end{aligned} \quad (\text{KKT-(2.1)})$$

If $\mu_1 \geq 0$, then the conditions are also sufficient.

We have the same kind of conditions for (2.2). For $(\gamma, \zeta) \in \mathcal{U}$:

$$\begin{aligned} -R(\gamma^*, g(\gamma^*, \zeta^*)) + \mu_2 g(\gamma^*, \zeta^*) &\leq -R(\gamma, g(\gamma, \zeta)) + \mu_2 g(\gamma, \zeta), \\ \mu_2 (g(\gamma^*, \zeta^*) - K) &= 0, \quad \mu_2 \geq 0. \end{aligned} \quad (\text{KKT-(2.2)})$$

First, suppose that γ^* is a solution to (2.1) and $\zeta^* \in A_\alpha(\gamma^*)$. Let us show that (γ^*, ζ^*) is a solution to (2.2). Using necessary and sufficient conditions (KKT-(2.1))

$$\begin{aligned} -R(\gamma^*, g(\gamma^*, \zeta^*)) + \mu_1 g(\gamma^*, \zeta^*) &= -R(\gamma^*, C_\alpha(\gamma^*)) + \mu_2 C_\alpha(\gamma^*) \\ &\leq -R(\gamma, C_\alpha(\gamma)) + \mu_1 C_\alpha(\gamma) = -R(\gamma, \min_{\zeta} g(\gamma, \zeta)) + \mu_1 \min_{\zeta} g(\gamma, \zeta), \\ &\leq -R(\gamma, g(\gamma, \zeta)) + \mu_1 g(\gamma, \zeta) \quad \text{because } R(\gamma, \cdot) \text{ is not increasing,} \end{aligned}$$

and

$$\mu_2 (g(\gamma^*, \zeta^*) - K) = \mu_1 (C_\alpha(\gamma^*) - K) = 0, \quad \mu_1 \geq 0, (\gamma, \zeta) \in \mathcal{U}.$$

Thus, (KKT-(2.2)) conditions are satisfied and (γ^*, ζ^*) is a solution to (2.2).

Now let us suppose that (γ^*, ζ^*) achieves the minimum of (2.2) and $\mu_2 > 0$. For a fixed γ^* , ζ^* minimizes the function $\zeta \mapsto -R(\gamma^*, g(\gamma^*, \zeta)) + \mu_2 g(\gamma^*, \zeta)$, and, consequently, the function $\zeta \mapsto g(\gamma^*, \zeta)$ because $R(\gamma, \cdot)$ is not increasing. Then, it implies that $\zeta^* \in A_\alpha(\gamma^*)$. Thus

$$\begin{aligned} -R(\gamma^*, C_\alpha(\gamma^*)) + \mu_2 C_\alpha(\gamma^*) &= -R(\gamma^*, g(\gamma^*, \zeta^*)) + \mu_2 g(\gamma^*, \zeta^*) \\ &\leq -R(\gamma, g(\gamma, V_\alpha(\gamma))) + \mu_2 g(\gamma, V_\alpha(\gamma)) = -R(\gamma, C_\alpha(\gamma)) + \mu_2 C_\alpha(\gamma) \end{aligned}$$

and

$$\mu_2 (C_\alpha(\gamma^*) - K) = \mu_2 (g(\gamma^*, \zeta^*) - K) = 0, \quad \mu_2 \geq 0, \gamma \in \Gamma.$$

We proved that (KKT-(2.2)) are satisfied, i.e. γ^* is a solution to (2.1) which completes the proof. \blacksquare

5.2 The Gaussian case resolution

We use in section 4.2 some theoretical results, we give a proof here.

The Lagrangian of (1.1) in the Gaussian case can be written as:

$$\begin{aligned} \mathcal{L}(\boldsymbol{\gamma}, (\lambda, \underline{\bar{\boldsymbol{\mu}}}, \underline{\boldsymbol{\mu}})) &:= -\boldsymbol{\gamma}^T \mathbf{1}_d + \lambda(-\boldsymbol{\gamma}^T \mathbf{1}_d + T_Z \sqrt{\boldsymbol{\gamma}^T \boldsymbol{\Sigma} \boldsymbol{\gamma}} - K) \\ &\quad + \underline{\bar{\boldsymbol{\mu}}}(\boldsymbol{\gamma} - \boldsymbol{\gamma}^{up}) - \underline{\boldsymbol{\mu}}(\boldsymbol{\gamma} - \boldsymbol{\gamma}^{low}). \end{aligned} \quad (5.1)$$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ implies

$$K = -\boldsymbol{\gamma}^T \mathbf{1}_d + T_Z \sqrt{\boldsymbol{\gamma}^T \boldsymbol{\Sigma} \boldsymbol{\gamma}}, \quad (5.2)$$

and $\frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}} = 0$ implies

$$\left(\frac{1 - (\underline{\bar{\boldsymbol{\mu}}} - \underline{\boldsymbol{\mu}})}{\lambda} + 1 \right) \mathbf{1}_d = T_Z \frac{\boldsymbol{\Sigma} \boldsymbol{\gamma}}{\sqrt{\boldsymbol{\gamma}^T \boldsymbol{\Sigma} \boldsymbol{\gamma}}}. \quad (5.3)$$

If the optimal solution is not constrained by the bounds, then $\underline{\bar{\boldsymbol{\mu}}} = \underline{\boldsymbol{\mu}} = 0$. Denote $y = \frac{1}{\lambda} + 1$ and $\sigma_S = \sqrt{\boldsymbol{\gamma}^T \boldsymbol{\Sigma} \boldsymbol{\gamma}}$

We multiply (5.3) on the left by $\boldsymbol{\gamma}$,

$$\frac{y}{T_Z} \boldsymbol{\gamma}^T \mathbf{1}_d = \frac{\boldsymbol{\gamma}^T \boldsymbol{\Sigma} \boldsymbol{\gamma}}{\sqrt{\boldsymbol{\gamma}^T \boldsymbol{\Sigma} \boldsymbol{\gamma}}},$$

so

$$\frac{y}{T_Z} \boldsymbol{\gamma}^T \mathbf{1}_d = \sigma_S. \quad (5.4)$$

We multiply (5.3) on the left by $\mathbf{1}_d^T \boldsymbol{\Sigma}^{-1}$,

$$y(\mathbf{1}_d^T \boldsymbol{\Sigma}^{-1} \mathbf{1}_d) = \frac{T_Z}{\sigma_S} \mathbf{1}_d^T \boldsymbol{\gamma}.$$

Now we replace $\mathbf{1}_d^T \boldsymbol{\gamma}$ by $\frac{T_Z}{y} \sigma_S$ with (5.4) and isolate y ,

$$y = \frac{T_Z}{\sqrt{\mathbf{1}_d^T \boldsymbol{\Sigma}^{-1} \mathbf{1}_d}}. \quad (5.5)$$

In the same way, we replace $\mathbf{1}_d^T \boldsymbol{\gamma}$ by $\frac{T_Z}{y} \sigma_S$ with (5.4) in (5.2),

$$K = -\frac{T_Z}{y} \sigma_S + T_Z \sigma_S,$$

and replace $\frac{T_Z}{y}$ by $\sqrt{\mathbf{1}_d^T \boldsymbol{\Sigma}^{-1} \mathbf{1}_d}$ with (5.5) and isolate σ_S ,

$$\sigma_S = K \left(\frac{1}{T_Z - \sqrt{\mathbf{1}_d^T \boldsymbol{\Sigma}^{-1} \mathbf{1}_d}} \right). \quad (5.6)$$

Let us now replace y and σ_S in (5.3),

$$\frac{T_Z}{\sqrt{\mathbf{1}_d^T \boldsymbol{\Sigma}^{-1} \mathbf{1}_d}} \mathbf{1}_d = T_Z \frac{\boldsymbol{\Sigma} \boldsymbol{\gamma}}{K \left(\frac{1}{T_Z - \sqrt{\mathbf{1}_d^T \boldsymbol{\Sigma}^{-1} \mathbf{1}_d}} \right)},$$

finally

$$\gamma = K \left(\frac{1}{T_Z - \sqrt{\mathbf{1}_d^T \Sigma^{-1} \mathbf{1}_d}} \right) \frac{\Sigma^{-1} \mathbf{1}_d}{\sqrt{\mathbf{1}_d^T \Sigma^{-1} \mathbf{1}_d}}. \quad (5.7)$$

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