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Macroscopic orientation of inertial flows in porous media

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Abstract

Macroscopic models of inertial flows in porous media have many practical applications where direct numerical simulations are not feasible. The Forchheimer equation describes macroscopic momentum transport accounting for inertial effects at the pore scale through a non-linear correction tensor \mathbf{F}_β . Using up-scaling approaches such as the volume averaging method, \mathbf{F}_β can be determined through the resolution of a closure problem. However this later is non-linear and still depends on the intensity and orientation of the macroscopic flow, complicating its resolution. Moreover, while the influence of the pore Reynolds number on the Forchheimer correction has been studied in details, its dependency on the flow orientation has drawn much less attention. In this work, a system of linearized closure problem is proposed and solved, allowing to determine \mathbf{F}_β without the need to solve the full closure problems for each orientation of the macroscopic pressure gradient. The validity of this approach is assessed for various rectangular unit cells against numerical solutions of the corresponding non-linear problem, showing excellent agreement for pore Reynolds number up to unity. Then macroscopic simulations are performed to evaluate the importance of varying flow orientation on the macroscopic inertial flow. Numerical results of the general non-linear macroscopic model obtained by the volume averaging method highlight the necessity to account for extra-diagonal terms as well as macroscopic gradient orientation in the determination of the Forchheimer tensor.

1 Introduction

Inertial laminar flows in porous media are present in a wide range of environmental or industrial systems such as atmospheric flows in canopies [38, 46], geological flows [3, 14], and heat exchangers [4, 29, 32]. Although direct numerical simulation (DNS) of such flows can be performed at the pore scale, the complexity of the local geometry as well as the length scale separation between the pore and the macroscopic system require prohibitively massive computing resources for most applications.

This has motivated the use of macroscopic models of transport phenomena in porous media, where macroscopic conservation equations and effective parameters can be obtained by averaging

36 the local (pore-scale) equations. For incompressible non-inertial flows, averaging Stokes equation
 37 results in Darcy’s law where the filtration velocity is linearly dependent on the pressure gradient
 38 through the intrinsic permeability tensor \mathbf{K}_β^* [15, 19, 55]. This linearity holds for pore Reynolds
 39 number up to one [54]. Above this value, it is the Navier-Stokes equation that must be homogenized,
 40 resulting in a macroscopic equation analogous to Darcy’s law but with an additional non-linear
 41 term called the Forchheimer tensor \mathbf{F}_β^* that accounts for inertial effects [19, 56]. In this case, the
 42 macroscopic equation that relates the seepage velocity $\langle \mathbf{v}_\beta^* \rangle$ to the macroscopic pressure gradient
 43 $\nabla \langle p_\beta^* \rangle^\beta$ is known as the Forchheimer equation, and can be written in the absence of gravity as

$$44 \quad \langle \mathbf{v}_\beta^* \rangle = -\frac{\mathbf{K}_\beta^*}{\mu_\beta} \cdot \nabla \langle p_\beta^* \rangle^\beta - \mathbf{F}_\beta \cdot \langle \mathbf{v}_\beta^* \rangle = -\frac{\mathbf{H}_\beta^*}{\mu_\beta} \cdot \nabla \langle p_\beta^* \rangle^\beta \quad (1)$$

45 where $\mathbf{H}_\beta^* = (\mathbf{I} + \mathbf{F}_\beta^*)^{-1} \cdot \mathbf{K}_\beta^*$ is the global permeability tensor and μ_β is the dynamic viscosity of
 46 the fluid. Here and in the rest of this article, the subscript β designates quantities defined in the
 47 fluid phase β , and the * exponent indicates dimensional quantities.

48 Recent studies identified distinct inertial flow regimes corresponding to the onset of inertia
 49 effects on the pore scale flow structure [1, 2, 23, 24, 31, 52]. The dependency of the Forchheimer
 50 correction with the Reynolds number have been largely discussed in the literature, as detailed for
 51 instance in Lasseux et al. 2019 [25]. Briefly, for pore Reynolds number slightly greater than one,
 52 a *weak inertia regime* characterized by a cubic correction of Darcy’s law was identified. Then
 53 for larger pore Reynolds numbers up to hundred, a *strong inertia regime* showing a quadratic
 54 dependency on the pore Reynolds number was found to occur for various geometries of periodic
 55 unit cells. Other transition regimes were observed between unsteady laminar and turbulent flows
 56 at pore Reynolds number around hundred. In this case, the transient nature of the flow requires a
 57 time-averaging procedure that assumes a linear Forchheimer correction [12, 17, 28, 47, 59]. Although
 58 this hypothesis is in agreement with experimental observations, its theoretical justification remains
 59 unclear.

60 Forchheimer’s equation (Eq. (1)) has been derived by Whitaker using the volume averaging
 61 method [56] to formally upscale Navier-Stokes from the pore to the macroscopic scale. Note that
 62 the form shown in Eq. (1) neglects additional terms arising from the up-scaling procedure, in
 63 particular the viscous diffusion term also known as the Brinkman correction. In addition to the
 64 macroscopic transport equation, the volume averaging method provides a series of closure problems
 65 which solutions allow to determine the effective transport properties involved in the global perme-
 66 ability tensor. The closure problem associated with the intrinsic permeability is linear while the
 67 one associated with the Forchheimer correction is non-linear and still depends on a local velocity
 68 field [24–27, 39]. To solve this non-linear problem, it has been proposed to first solve an associated
 69 local flow problem on a periodic unit cell for each values of the pore Reynolds number [1, 2, 24, 52]
 70 and macroscopic pressure gradient orientation [11, 24, 37, 53]. As a result, the Forchheimer tensor
 71 is an implicit function of these two dimensionless parameters.

72 Although the dependency of the Forchheimer tensor with the pore Reynolds number has been
 73 the subject of several studies leading to the classification of weak inertial flow regimes described
 74 above [1, 2, 23, 24, 31, 39], the influence of the macroscopic pressure gradient orientation has drawn
 75 much less attention. Importantly, extra-diagonal components of the tensor are not null when the
 76 macroscopic flow is not aligned with the elementary geometry. This aspect can become essential

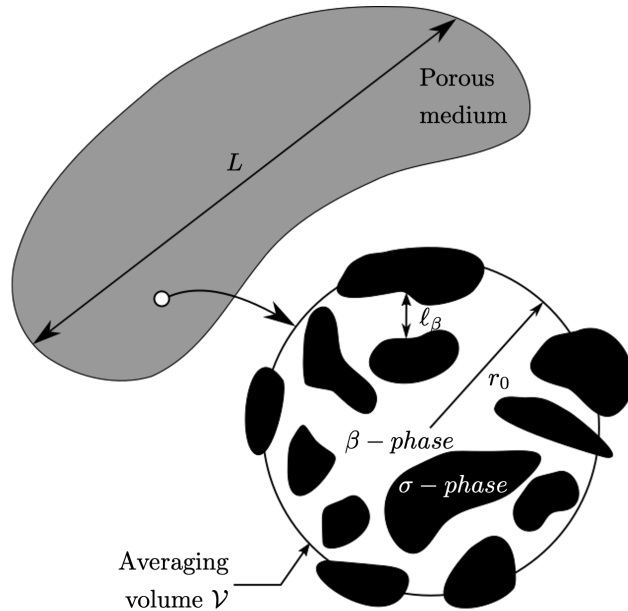


Figure 1: Schematic of a representative elementary volume (REV) with the definitions of the pore-scale phases and length scales.

77 in cases where the orientation of macroscopic flow varies with respect to the porous arrangement,
 78 such as in elbows or curved channels produced by additive manufacturing.

79 In this work, the effect of the macroscopic flow orientation on the Forchheimer correction tensor
 80 is studied in two ways. First, the local inertial flow is up-scaled using the volume averaging
 81 method in Section 2. The resulting general closure problem is then linearized in Section 3 by
 82 generalizing previous work for arbitrary macroscopic pressure gradient orientation and small pore
 83 Reynolds number [7, 31, 39]. The ability to predict the Forchheimer tensor of this linearized model
 84 is assessed on rectangular unit cells through comparisons with the classical non-linear model [24,
 85 56] to determine its range of validity in Reynolds number and orientation of the macroscopic
 86 pressure gradient. Finally, in Section 4, simulations of macroscopic inertial flow with varying
 87 pressure gradient orientation are performed. The prediction of two macroscopic models are assessed
 88 against direct numerical simulation. It is shown that, in contrast to diagonal-tensor models often
 89 encountered in CFD softwares [11, 37, 53], the non-linear macroscopic model obtained with the
 90 volume averaging method is able to correctly capture inertial flow structures and pressure losses in
 91 an elbow channel filled with a square porous medium.

92 2 Up-scaling inertial flow in porous media

93 At the pore scale, two phases are considered: a Newtonian fluid (β -phase) and an immobile non-
 94 permeable solid (σ -phase). The flow is assumed incompressible, laminar and steady. The velocity
 95 field \mathbf{v}_β^* and pressure p_β^* in the fluid β verify the incompressible Navier-Stokes equations with no-
 96 slip boundary conditions at the fluid-solid interface $A_{\beta\sigma}$. The boundary value problem at the pore

97 scale is therefore

$$98 \quad \nabla \cdot \mathbf{v}_\beta^* = 0 \quad \text{in } V_\beta \quad (2a)$$

$$99 \quad \rho_\beta \mathbf{v}_\beta^* \cdot \nabla \mathbf{v}_\beta^* = -\nabla p_\beta^* + \mu_\beta \nabla^2 \mathbf{v}_\beta^* \quad \text{in } V_\beta \quad (2b)$$

$$100 \quad \mathbf{v}_\beta^* = 0 \quad \text{at } A_{\beta\sigma} \quad (2c)$$

101 Here ρ_β and μ_β are the fluid density and dynamic viscosity, while V_β is the volume occupied by the β -
 102 phase (see Fig. 1). Here and along the document, the * exponent denotes dimensional variables. For
 103 readability purpose, the notations of the differential operators, volumes and areas, and characteristic
 104 lengths are kept without exponents weather they are dimensional or dimensionless.

105 To proceed to up-scaling by volume averaging, the general methodology [58] is to construct
 106 an averaging volume \mathcal{V} of characteristic length r_0 that verifies the length-scales separation $l_\beta \ll$
 107 $r_0 \ll L$ where l_β and L are the characteristics lengths of the pores and the macroscopic domain
 108 respectively (see Fig. 1). The superficial and intrinsic averages on the averaging volume \mathcal{V} are
 109 defined respectively as

$$110 \quad \langle \psi_\beta \rangle = \frac{1}{\mathcal{V}} \int_{V_\beta} \psi_\beta dV_\beta \quad \text{and} \quad \langle \psi_\beta \rangle^\beta = \frac{1}{V_\beta} \int_{V_\beta} \psi_\beta dV_\beta \quad (3)$$

111 where ψ_β is an arbitrary variable defined in the β -phase. These two definitions are related through
 112 the volume fraction of the β -phase $\varepsilon_\beta = V_\beta/\mathcal{V}$ by $\langle \psi_\beta \rangle = \varepsilon_\beta \langle \psi_\beta \rangle^\beta$. The averages of differential
 113 operators can be expressed using the following averaging theorems [58]

$$114 \quad \langle \nabla \psi_\beta \rangle = \nabla \langle \psi_\beta \rangle + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \psi_\beta dA \quad (4a)$$

$$115 \quad \langle \nabla \cdot \mathbf{a}_\beta \rangle = \nabla \cdot \langle \mathbf{a}_\beta \rangle + \frac{1}{\mathcal{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot \mathbf{a}_\beta dA \quad (4b)$$

116 where \mathbf{a}_β is an arbitrary tensor field of order $n \geq 1$. These expressions involve surface integrals
 117 on the fluid-solid interface $A_{\beta\sigma}$ contained in \mathcal{V} and thus allow to take into account the influence
 118 pore-scale boundary conditions in the averaged pore-scale equations. Moreover the presence of the
 119 local field such as ψ_β and \mathbf{a}_β in these integrals motivates the introduction of a decomposition of
 120 the pore-scale fields using their intrinsic average and a corresponding spatial deviation [21]

$$121 \quad \psi_\beta = \langle \psi_\beta \rangle^\beta + \tilde{\psi}_\beta \quad (5)$$

122 Making use of these definitions and following the volume averaging method [57, 58, 60, 61], one
 123 can upscale the local flow description Eqs. (2) to obtain the non-closed macroscopic transport
 124 equations

$$125 \quad \nabla \cdot \langle \mathbf{v}_\beta^* \rangle^\beta = 0 \quad (6a)$$

$$127 \quad \rho_\beta \langle \mathbf{v}_\beta^* \rangle^\beta \cdot \nabla \langle \mathbf{v}_\beta^* \rangle^\beta + \rho_\beta \nabla \cdot \langle \tilde{\mathbf{v}}_\beta^* \tilde{\mathbf{v}}_\beta^* \rangle^\beta =$$

$$128 \quad -\nabla \langle p_\beta^* \rangle^\beta + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta^* \rangle^\beta + \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_\beta^* + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta^*) dA \quad (6b)$$

129 The validity of the above equations relies on the assumption of homogeneous porous medium
 130 and of the scale separation ($l_\beta \ll r_0 \ll L$)¹. Equation (6b) is not closed since the local deviations
 131 of the pressure and velocity still appear in the inertial term and in the area integral that represents
 132 the friction forces exerted by the solid phase σ on the fluid β .

133 In order to obtain a closed form of the averaged equations, one can write a boundary-value
 134 problem verified by the spatial deviations $\tilde{\mathbf{v}}_\beta^*$ and \tilde{p}_β^* by subtracting the averaged problem (6)
 135 to the pore-scale equations (2), and expand the local fields using the decomposition in spatial
 136 deviations Eq. (5) [21]. After some simplifications based on the above length-scale constraints, one
 137 obtains the local deviation problem [9, 24, 56, 57]

$$138 \quad \nabla \cdot \tilde{\mathbf{v}}_\beta^* = 0 \quad \text{in } V_\beta \quad (7a)$$

$$139 \quad \rho_\beta \mathbf{v}_\beta^* \cdot \nabla \tilde{\mathbf{v}}_\beta^* = -\nabla \tilde{p}_\beta^* + \mu_\beta \nabla^2 \tilde{\mathbf{v}}_\beta^* - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_\beta^* + \mu_\beta \nabla \tilde{\mathbf{v}}_\beta^*) dA \quad \text{in } V_\beta \quad (7b)$$

$$140 \quad \tilde{\mathbf{v}}_\beta^* = -\langle \mathbf{v}_\beta^* \rangle^\beta \quad \text{on } A_{\beta\sigma} \quad (7c)$$

$$141 \quad \tilde{\mathbf{v}}_\beta^*(\mathbf{r} + \mathbf{l}_i) = \tilde{\mathbf{v}}_\beta^*(\mathbf{r}) \quad ; \quad \tilde{p}_\beta^*(\mathbf{r} + \mathbf{l}_i) = \tilde{p}_\beta^*(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (7d)$$

$$142 \quad \langle \tilde{\mathbf{v}}_\beta^* \rangle^\beta = 0 \quad ; \quad \langle \tilde{p}_\beta^* \rangle^\beta = 0 \quad (7e)$$

143 where \mathbf{l}_i are the periodic vector in the \mathbf{e}_i direction of the periodic unit cell. Here the deviations are
 144 supposed to satisfy periodic boundary conditions. Note that this does not suppose that the porous
 145 medium is periodic, only that the local problem is solved on a representative unit cell that satisfies
 146 the length scale constraints [20, 41, 42, 58].

147 Since the deviation problem is defined within the averaging volume V_β , it is convenient to
 148 express the system in terms of the pore Reynolds number

$$149 \quad Re_p = \frac{\rho_\beta v_{ref} \ell}{\mu_\beta} \quad (8)$$

150 where v_{ref} is a reference velocity constant over V_β , and ℓ is a characteristic length of the pore scale.
 151 In accordance, the dimensionless local variables and their deviations are defined as

$$152 \quad \mathbf{v}_\beta = \frac{\mathbf{v}_\beta^*}{v_{ref}} \quad ; \quad p_\beta = \frac{p_\beta^* \ell}{\mu_\beta v_{ref}} \quad ; \quad \tilde{\mathbf{v}}_\beta = \frac{\tilde{\mathbf{v}}_\beta^*}{v_{ref}} \quad ; \quad \tilde{p}_\beta = \frac{\tilde{p}_\beta^* \ell}{\mu_\beta v_{ref}} \quad (9)$$

153 In the rest of the document, all lengths are normalized with respect to the arbitrary pore-scale
 154 length ℓ except stated otherwise. Therefore the dimensionless deviation problem takes the form

$$155 \quad \nabla \cdot \tilde{\mathbf{v}}_\beta = 0 \quad \text{in } V_\beta \quad (10a)$$

$$156 \quad Re_p \mathbf{v}_\beta \cdot \nabla \tilde{\mathbf{v}}_\beta = -\nabla \tilde{p}_\beta + \nabla^2 \tilde{\mathbf{v}}_\beta - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_\beta + \nabla \tilde{\mathbf{v}}_\beta) dA \quad \text{in } V_\beta \quad (10b)$$

$$157 \quad \tilde{\mathbf{v}}_\beta = -\langle \mathbf{v}_\beta \rangle^\beta \quad \text{on } A_{\beta\sigma} \quad (10c)$$

¹In fact, the scale separation should be written $l_\beta \ll r_0 \ll (L_v, L_{v2}, L_{p1})$ where L_v , L_{v2} and L_{p1} are characteristic length-scales of variation of the fields $\langle \mathbf{v}_\beta \rangle^\beta$, $\nabla \nabla \langle \mathbf{v}_\beta \rangle^\beta$ and $\nabla \langle p_\beta \rangle^\beta$ respectively. Because these length-scales are of the same order of magnitude as the characteristic length of the macroscopic domain L , assuming the separation of length-scales $l_\beta \ll r_0 \ll L$ is usually equivalent to those constraints.

$$158 \quad \tilde{\mathbf{v}}_\beta(\mathbf{r} + \mathbf{l}_i) = \tilde{\mathbf{v}}_\beta(\mathbf{r}) \quad ; \quad \tilde{p}_\beta(\mathbf{r} + \mathbf{l}_i) = \tilde{p}_\beta(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (10d)$$

$$159 \quad \langle \tilde{\mathbf{v}}_\beta \rangle^\beta = 0 \quad ; \quad \langle \tilde{p}_\beta \rangle^\beta = 0 \quad (10e)$$

160 Due to the inertial effects introduced by the term on the left hand side of Eq. (10b), the system
 161 is non-linear and non-intrinsic in the sense that it still depends on the value of the flow field. Two
 162 approaches are proposed in the next section to deal with this difficulty.

163 3 Determination of the Forchheimer tensor

164 In this section, two approaches are proposed to account for the non-linearity in the deviation
 165 problem (10) and to determine the Forchheimer tensor. First a general approach based on [24]
 166 relying on the preliminary computation of a local problem for each Reynolds and macroscopic
 167 flow orientation is detailed. Then a methodology is proposed to linearize the closure problem,
 168 eliminating the need for preliminary local computations. Finally, the value of the Forchheimer
 169 tensor computed with the two methods are compared to determine the domain of validity of the
 170 linearized approach.

171 3.1 The general non-linear closure problem

172 By analogy with the up-scaling procedure of Stokes flow where the deviation problem is linear, only
 173 the macroscopic source term in Eq. (10c) is considered to map the deviations of the velocity and
 174 pressure $\tilde{\mathbf{v}}_\beta$ and \tilde{p}_β as [56]

$$175 \quad \tilde{\mathbf{v}}_\beta = \mathbf{D} \cdot \langle \mathbf{v}_\beta \rangle^\beta \quad (11a)$$

$$176 \quad \tilde{p}_\beta = \mathbf{d} \cdot \langle \mathbf{v}_\beta \rangle^\beta \quad (11b)$$

177 where \mathbf{D} and \mathbf{d} are dimensionless closure variables. Implicitly, Eqs. (11) assumes that the convec-
 178 tive velocity is known in the inertial term of Eq. (10b). Although this might seem like a strong
 179 simplification of the non-linear closure problem, this practical approximation is widely made in the
 180 literature [1, 3, 9, 24, 36, 40, 47, 51, 52, 56, 59], resulting in good predictions of macroscopic models
 181 against DNS [10, 30, 47, 48, 50, 59] and consistent with experiments [18].

182 Introducing Eqs. (11) into Eqs. (6), the closed macroscopic equations describing inertial flow in
 183 a porous medium is obtained [24, 40, 56]

$$184 \quad \nabla \cdot \langle \mathbf{v}_\beta^* \rangle^\beta = 0 \quad (12a)$$

$$185 \quad \rho_\beta \langle \mathbf{v}_\beta^* \rangle^\beta \cdot \nabla \langle \mathbf{v}_\beta^* \rangle^\beta = -\nabla \langle p_\beta^* \rangle^\beta + \mu_\beta \nabla^2 \langle \mathbf{v}_\beta^* \rangle^\beta - \mu_\beta \mathbf{H}_\beta^{*-1} \cdot \langle \mathbf{v}_\beta^* \rangle^\beta \quad (12b)$$

186 The global permeability tensor $\mathbf{H}_\beta^* = \mathbf{H}_\beta \ell^2$ appears as an effective property. It is defined in its
 187 dimensionless form as

$$188 \quad \varepsilon_\beta \mathbf{H}_\beta^{-1} = -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{Id} + \nabla \mathbf{D}) dA \quad (13)$$

189 and can be related to the Forchheimer tensor $\mathbf{F}_\beta^* = \mathbf{F}_\beta$ and the intrinsic permeability tensor
 190 $\mathbf{K}_\beta^* = \mathbf{K}_\beta \ell^2$ through $\mathbf{H}_\beta^{-1} = \mathbf{K}_\beta^{-1} \cdot (\mathbf{I} + \mathbf{F}_\beta)$. Note that when the viscous diffusion term is neglected
 191 in Eq. (12b), the well known Darcy-Forchheimer equation (1) is recovered.

192 Introducing the closure variable Eqs. (11) into the deviation problem (10), the closure problem
 193 is obtained [56]

$$194 \quad \nabla \cdot \mathbf{D} = 0 \quad \text{in } V_\beta \quad (14a)$$

$$195 \quad Re_p \mathbf{v}_\beta \cdot \nabla \mathbf{D} = -\nabla \mathbf{d} + \nabla^2 \mathbf{D} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \mathbf{d} + \nabla \mathbf{D}) dA \quad \text{in } V_\beta \quad (14b)$$

$$196 \quad \mathbf{D} = -\mathbf{I} \quad \text{on } A_{\beta\sigma} \quad (14c)$$

$$197 \quad \mathbf{D}(\mathbf{r} + \mathbf{l}_i) = \mathbf{D}(\mathbf{r}) \quad ; \quad \mathbf{d}(\mathbf{r} + \mathbf{l}_i) = \mathbf{d}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (14d)$$

$$198 \quad \langle \mathbf{D} \rangle^\beta = 0 \quad ; \quad \langle \mathbf{d} \rangle^\beta = 0 \quad (14e)$$

199 This boundary-value problem is integro-differential. To facilitate its numerical resolution, it is
 200 convenient to introduce the following change of variables

$$201 \quad \mathbf{D}' = \varepsilon_\beta^{-1} (\mathbf{D} + \mathbf{I}) \cdot \mathbf{H}_\beta \quad ; \quad \mathbf{d}' = \varepsilon_\beta^{-1} \mathbf{d} \cdot \mathbf{H}_\beta \quad (15)$$

202 to obtain a more practical form of the closure problem

$$203 \quad \nabla \cdot \mathbf{D}' = 0 \quad \text{in } V_\beta \quad (16a)$$

$$204 \quad Re_p \mathbf{v}_\beta \cdot \nabla \mathbf{D}' = -\nabla \mathbf{d}' + \nabla^2 \mathbf{D}' + \mathbf{I} \quad \text{in } V_\beta \quad (16b)$$

$$205 \quad \mathbf{D}' = \mathbf{0} \quad \text{on } A_{\beta\sigma} \quad (16c)$$

$$206 \quad \mathbf{D}'(\mathbf{r} + \mathbf{l}_i) = \mathbf{D}'(\mathbf{r}) \quad ; \quad \mathbf{d}'(\mathbf{r} + \mathbf{l}_i) = \mathbf{d}'(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (16d)$$

$$207 \quad \langle \mathbf{D}' \rangle^\beta = \varepsilon_\beta^{-1} \mathbf{H}_\beta \quad ; \quad \langle \mathbf{d}' \rangle^\beta = 0 \quad (16e)$$

208 The closure problem (16) is non-linear and still depend on the local dimensionless velocity
 209 \mathbf{v}_β . Lasseux et al. [24] proposed to deal with this non-linearity by introducing the decomposition
 210 $p_\beta = \langle p_\beta \rangle^\beta + \tilde{p}_\beta$ in the local problem (2), and consider the solution of the system in a representative
 211 periodic unit cell

$$212 \quad \nabla \cdot \mathbf{v}_\beta = 0 \quad \text{in } V_\beta \quad (17a)$$

$$213 \quad Re_p \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta = -\nabla \tilde{p}_\beta + \nabla^2 \mathbf{v}_\beta - \nabla \langle p_\beta \rangle^\beta \quad \text{in } V_\beta \quad (17b)$$

$$214 \quad \mathbf{v}_\beta = \mathbf{0} \quad \text{on } A_{\beta\sigma} \quad (17c)$$

$$215 \quad \mathbf{v}_\beta(\mathbf{r} + \mathbf{l}_i) = \mathbf{v}_\beta(\mathbf{r}) \quad ; \quad \tilde{p}_\beta(\mathbf{r} + \mathbf{l}_i) = \tilde{p}_\beta(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (17d)$$

$$216 \quad \langle \tilde{p}_\beta \rangle^\beta = 0 \quad (17e)$$

217 Decomposing once more the macroscopic pressure gradient $\nabla \langle p_\beta \rangle^\beta = |\nabla \langle p_\beta \rangle^\beta| \mathbf{e}_{\nabla p}$, it is clear that
 218 this form of the local problem is dependent on three dimensionless parameters: the magnitude
 219 $|\nabla \langle p_\beta \rangle^\beta|$ and the orientation $\mathbf{e}_{\nabla p}$ of the macroscopic pressure gradient, as well as the pore Reynolds
 220 number $Re_p = \rho_\beta v_{ref} \ell / \mu_\beta$. To reduce this dependency to two parameters, the reference velocity
 221 appearing in the pore Reynolds number and the dimensionless variables is taken so that $|\nabla \langle p_\beta \rangle^\beta| =$
 222 1, that is to say $v_{ref} = |\nabla \langle p_\beta^* \rangle^\beta| \ell^2 / \mu_\beta$ and so $Re_p = \rho_\beta |\nabla \langle p_\beta^* \rangle^\beta| \ell^3 / \mu_\beta^2$. It follows that Eq. (17b)
 223 simplifies to

$$224 \quad Re_p \mathbf{v}_\beta \cdot \nabla \mathbf{v}_\beta = -\nabla \tilde{p}_\beta + \nabla^2 \mathbf{v}_\beta - \mathbf{e}_{\nabla p} \quad \text{in } V_\beta \quad (18)$$

225 where the only source terms left are the pore Reynolds number and the flow orientation.

226 Knowing \mathbf{v}_β from the solution of Eqs. (17) with (18) on a representative elementary volume,
 227 one can solve the closure problem (16b) and obtain the global permeability tensor. Note that the
 228 dimensionless local velocity field \mathbf{v}_β depends on both the orientation of the macroscopic pressure
 229 gradient $\mathbf{e}_{\nabla p}$ appearing as a source term, and in Re_p linked to the flow intensity. Therefore the global
 230 permeability tensor also depends on these macroscopic dimensionless quantities $\mathbf{H}_\beta(Re_p, \mathbf{e}_{\nabla p})$. The
 231 procedure to solve the general non-linear closure problem will be detailed in Section 3.3.

232 3.2 The linearized closure problem

233 In order to deal with the non-linear inertial term in the closure problem and avoid having to solve
 234 an associated local problem as presented above, an alternative approach leading to a linearized
 235 closure problem is proposed using regular perturbation theory [7, 31, 39].

236 Let us consider the dimensionless deviation problem (10). In the case where $Re_p = 0$, the lower
 237 limit of a purely viscous Stokes flow is recovered. This suggests to linearize the deviation problem
 238 around that special case which is already linear and self-consistent. Using regular perturbation
 239 theory, any field ψ_β in the deviation equations can be expanded as a power series around Re_p

$$240 \quad \psi_\beta = \sum_{k=0}^{+\infty} \psi_{\beta k} Re_p^k \quad (19)$$

241 Using decomposition (19) for the spatial deviations $\tilde{\mathbf{v}}_\beta$ and \tilde{p}_β and the local velocity \mathbf{v}_β , it is
 242 possible to convert the non-linear deviation problem (10) into countable series of linearized deviation
 243 problems for each order of the Reynolds number Re_p . The details of this linearization procedure
 244 are given in Appendix A up to order 2, although the methodology can be applied to any higher
 245 order. The main results are summarized here.

246 The decompositions of the velocity and deviations are introduced in the deviation problem
 247 (10), and the terms in the different orders of Reynolds number are gathered. Starting at order
 248 0, the terms in Re_p^0 form a system of equations identical to the one obtained for a purely viscous
 249 Stokes flow. The deviations at order 0 can then be decomposed in closure variables (\mathbf{B}, \mathbf{b}) that
 250 are solution of the same closure system as the one for a Stokes flow. This allows to compute the
 251 intrinsic permeability tensor \mathbf{K}_β . Then at order 1, collecting the terms in Re_p^1 leads to a deviation
 252 problem which structure is similar to an inertial flow, but where the non-linear term only involves
 253 terms at order 0. Therefore it can be expressed in terms of the closure variables at order 0 and
 254 be treated as a source term. One can then decompose the deviations at order 1 in terms of new
 255 closure variables $(\mathcal{C}, \mathbf{C})$ that are the solution of a closure problem at order 1 (Eqs. (A.9)). A new
 256 effective tensor \mathcal{H}_β can be identified as a correction of the permeability tensor at order 1. The
 257 same procedure is carried at order 2, where the non-linear term only depends on variables at lower
 258 orders and therefore can be treated as a source term. It is shown that the closure variables at order
 259 2, defined as $(\mathcal{E}, \mathcal{E})$, are solution of a closure problem (Eqs. (A.20)), where a new effective tensor
 260 \mathcal{J}_β correcting the permeability at order 2 is identified.

261 Introducing the definition of the closure variables at different orders in the power series decom-
 262 position of the velocity and pressure deviations (Eq. (19)), they can be written as expansions of
 263 the macroscopic source terms

$$264 \quad \tilde{\mathbf{v}}_\beta = \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathcal{C} : Re_p \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \mathcal{E} : Re_p^2 \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \quad (20)$$

265
$$\tilde{\boldsymbol{\rho}}_\beta = \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{C} : Re_p \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \mathcal{E} : Re_p^2 \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \quad (21)$$

266 Inserting these decompositions in the non-closed macroscopic equation (6b), and identifying the
 267 corrective effective tensors at each order, the linearized form of the global permeability tensor up
 268 to second order can be written as

269
$$\mathbf{H}_\beta^{-1} = \mathbf{K}_\beta^{-1} \cdot \left(\mathbf{I} + Re_p \mathcal{H}_\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + Re_p^2 \mathcal{J}_\beta : \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \right). \quad (22)$$

270 In this expression, the Forchheimer correction tensor is directly identified as

271
$$\mathbf{F}_\beta = Re_p \mathcal{H}_\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + Re_p^2 \mathcal{J}_\beta : \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \quad (23)$$

272 This procedure can easily be carried at higher orders, although each additional order requires
 273 to solve a new closure problem with each time a higher dimension closure variable. It should
 274 be emphasized that, in contrast to the general non-linear approach, each of the effective tensors
 275 appearing in the definition above are obtained from the solution of closure problems that are
 276 independent of the Reynolds number and pressure gradient orientation, and thus do not require
 277 the *a priori* solution of a local flow field velocity. Therefore at the macroscopic scale, the above
 278 expression can be directly used in the average equation to determine the velocity and pressure field.

279 3.3 Influence of the flow intensity and orientation on the Forchheimer tensor

280 3.3.1 Resolution of the closure problems

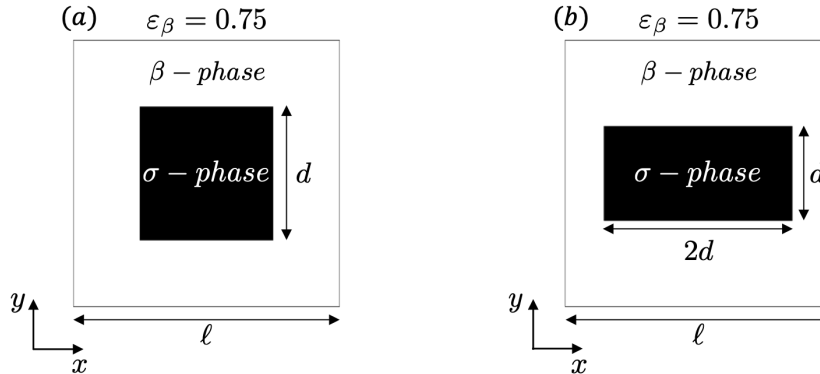


Figure 2: Example of geometries of the 2D unit cells on which the closure problems were solved (here $\varepsilon_\beta = 0.75$).

281 The non-linear closure problem Eqs. (16) was solved for various values of the pore Reynolds
 282 number Re_p and pressure gradient orientations $\mathbf{e}_{\nabla p}$ on the two-dimensional unit cells shown in
 283 Fig. 2. Two obstacle geometries were considered: squares and rectangles of aspect ratio 1:2. For
 284 each geometry, three volume fractions were examined ($\varepsilon_\beta = 0.25, 0.5$, and 0.75).

285 Concerning the linearized approach, each linearized closure problem was solved numerically in
 286 an incremental manner, starting from order 0, allowing to solve order 1, and then order 2. In
 287 this work, closure problems up to order 2 were solved in order to evaluate the global permeability

288 tensor from Eq. (23). Preliminary computations at orders up to 6 were performed but did not show
 289 significant improvement on the prediction of the permeability tensor within the range of validity of
 290 the linearized closure problem (data not shown).

291 The closure problems were solved using COMSOL Multiphysics software. The systems having
 292 the same structure as the Navier-Stokes equations, the “Laminar Flow” module was used with
 293 custom source terms added using the “Volume Force” node. For the linearized closure problem,
 294 the same mesh for the three closure problems was used in order to avoid mesh interpolations when
 295 computing the source terms.

296 To facilitate the comparison between the results from the two approaches, a definition of the
 297 Reynolds number based on the intrinsic permeability Re_k was used [39]

$$298 \quad Re_k = \frac{\rho_\beta |\langle \mathbf{v}_\beta^* \rangle| \sqrt{k^*}}{\mu_\beta} = Re_p |\langle \mathbf{v}_\beta \rangle| \sqrt{k} \quad (24)$$

299 where $k = k^*/\ell^2$ is the mean of the diagonal components of the intrinsic permeability tensor
 300 $k = (K_{xx} + K_{yy})/2$, and $|\langle \mathbf{v}_\beta^* \rangle| = (\langle \mathbf{v}_\beta^* \rangle \cdot \langle \mathbf{v}_\beta^* \rangle)^{1/2} = |\langle \mathbf{v}_\beta \rangle| v_{ref}$. The flow orientation angle θ is
 301 defined such as $\mathbf{e}_{\nabla p} = \cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y$. In all computations, dimensionless values are given with
 302 respect to the characteristic length ℓ representing the size of a unit cell (see Fig. 2).

303 **3.3.2 Computation of the Forchheimer tensor: comparison between non-linear and** 304 **linearized closure problems**

305 The values of the xx component of the Forchheimer tensor obtained from the numerical solution
 306 of the non-linear and linearized closure problems are shown in Fig. 3 as a function of the Reynolds
 307 number and pressure gradient orientation. For Reynolds number Re_k up to about 0.1, results from
 308 the linearized closure problem are in excellent agreement with the one obtained from the non-linear
 309 problem. The component F_{xx} of the Forchheimer correction tensor shows a quadratic dependency
 310 on the Reynolds number $\mathbf{F}_\beta \sim \mathbf{A} Re_k^2$ as previously reported in the weak inertia regime [1, 24].
 311 Notably, the linearized Forchheimer tensor (Eq. (23)) allows to express the proportionality tensor
 312 as $\mathbf{A} = \varepsilon_\beta^{-2} k^{-1} \mathcal{J}_\beta : \mathbf{e}_v \mathbf{e}_v$ where $\mathbf{e}_v = \langle \mathbf{v}_\beta \rangle^\beta / |\langle \mathbf{v}_\beta \rangle^\beta|$ is the unit vector giving the direction of the
 313 averaged velocity on the studied unit cell. This expression assumes that \mathcal{H}_β the linearized correction
 314 tensor at order 1 is negligible, which is the case for symmetric and periodic unit cells [7, 31, 39] and
 315 is numerically verified here.

316 For Reynolds number between 0.1 and 1, the system enters a stronger inertia regime, and
 317 the Forcheimer tensor transitions to a linear function of the Reynolds number. Results from the
 318 linearized closure problem truncated at order 2 start to deviate from the general non-linear case
 319 in a manner that depend on the porosity and flow orientation [39]. Including higher orders for
 320 the linearization only improves marginally the range of validity compared to the non-linear closure
 321 problem (data not shown).

322 Fig. 3(d-f) represents the evolution of the component F_{xx} of the Forchheimer correction tensor
 323 as a function of the angle θ of the macroscopic pressure gradient for different porosities and fixed
 324 Reynolds number Re_k . The agreement between the linearized and non-linear closure problem
 325 solutions are confirmed for all flow orientations in the weak inertial regime. With increasing pore
 326 Reynolds number above 0.1 and for large porosities, discrepancies between the two approaches are
 327 observed.

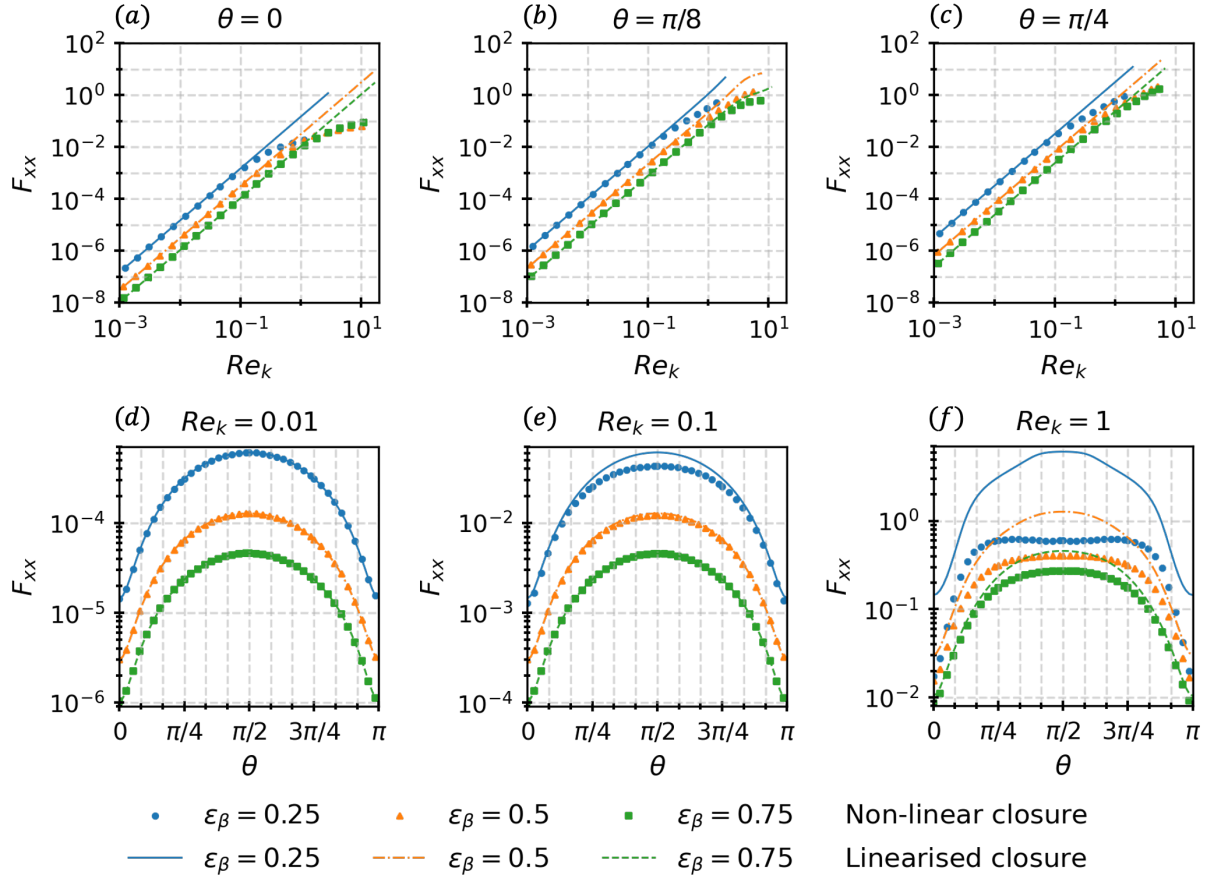


Figure 3: Comparison between non-linear and linearized component F_{xx} of the Forchheimer correction tensor for a square solid as a function of the pore Reynolds number (a-c), and as a function of the pressure gradient angle (d-f).

328 For completeness, the closure problems were solved on lower symmetry unit cell, namely rect-
329 angular obstacles with aspect ratio 1:2. The xx component of the Forchheimer tensor F_{xx} are
330 shown in 4 for three different porosities as a function of the pore Reynolds number Re_k and flow
331 orientation θ . Similarly to the square unit cell, the linearized closure problem approach shows a
332 good agreement with the non-linear closure problem in the weak inertia regime up to $Re_k \sim 0.3$,
333 but fails to capture the linear dependency on the Reynolds number in the strong inertia regime.

334 Interestingly, for low porosities ε_β , the Forchheimer correction shows a sharp maximum at
335 pressure gradient angle θ equal to $\pi/2$. This behavior can be explained in the limit where the side
336 of the rectangular obstacles reach the vertical boundaries of the unit cell, effectively producing a
337 series of impermeable channels.

338 A comparison of the computational cost of solving the non-linear versus linear closure problems
339 is presented in Appendix B. Briefly, the computational time required to solve the linear closure
340 problem at order 2 is about nine time larger than for the non-linear problem. However, a single run
341 of the linear problem gives values of the Forchheimer correction for all pore Reynolds number Re_k

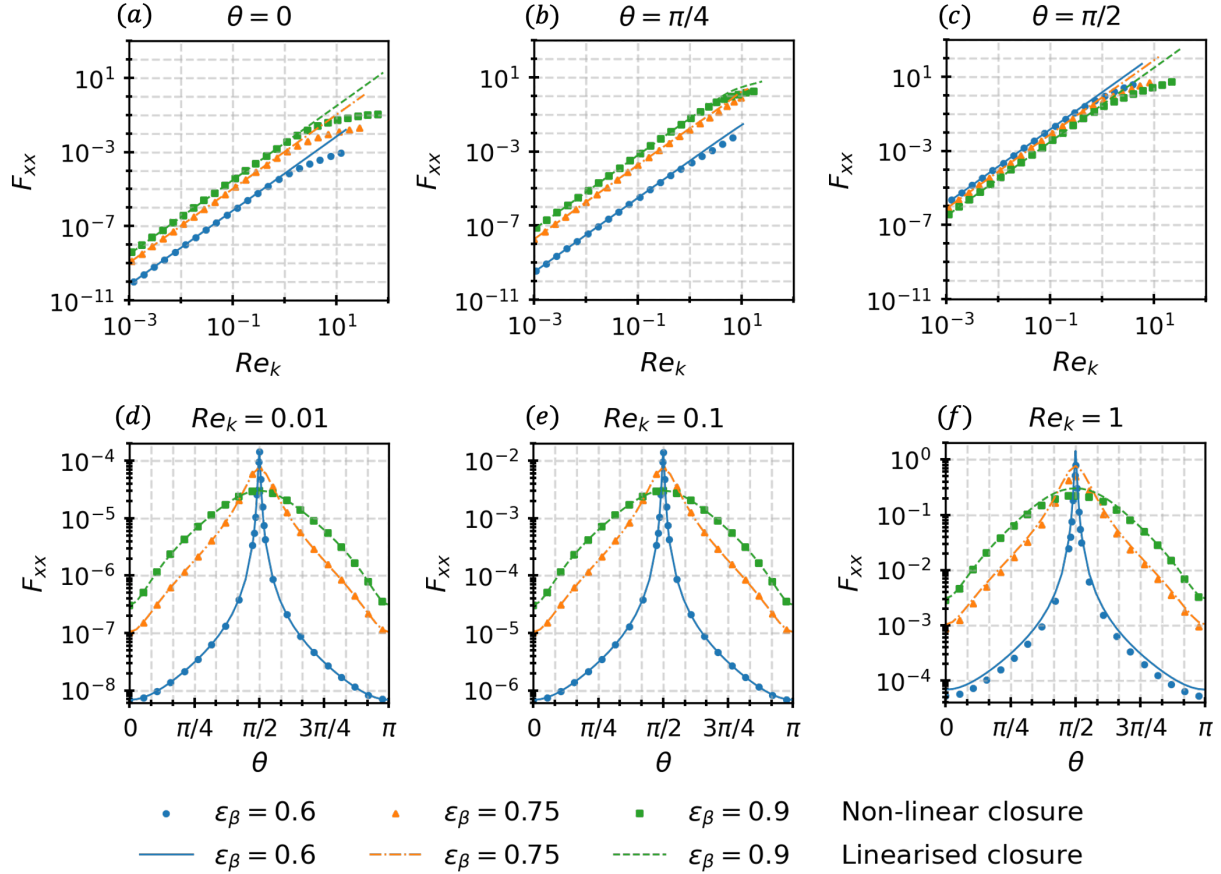


Figure 4: Comparison between non-linear and linearized component F_{xx} of the Forchheimer correction tensor for a square rectangular as a function of the pore Reynolds number (a-c), and as a function of the pressure gradient angle (d-f).

342 and flow orientation θ within the domain of validity. This is not the case for the non-linear problem
343 which requires to be solved for each values of the couple (Re_k, θ) . Therefore the linear closure
344 problem is computationally advantageous when a tabulation of at least ten values of (Re_k, θ) is
345 needed.

346 4 Effect of flow orientation on macroscopic inertial flow

347 In the previous section, the inertial correction to the permeability was computed as a function of the
348 flow intensity and orientation. Although the linearized approach gave accurate results for a limited
349 range in Reynolds, both approaches highlighted a non-trivial dependency of the Forchheimer tensor
350 with the orientation of the pressure gradient. In order to explore this relationship, macroscopic sim-
351 ulations of the non-linear general model were performed and compared with two methods commonly
352 used in engineering studies: direct numerical simulations (DNS) and diagonal-tensor macroscopic
353 models.

354 4.1 Implementation of the macroscopic models

355 4.1.1 General non-linear macroscopic model

356 For convenience, the macroscopic transport equations (Eqs. (12a) and (12b)) are normalized with
 357 respect to a global macroscopic velocity U , representing in our case the bulk velocity imposed at
 358 the inlet of the macroscopic system. Written in their superficial form, the closed averaged equations
 359 are

$$360 \quad \nabla \cdot \langle \mathbf{v}_\beta^+ \rangle = 0 \quad (25a)$$

$$361 \quad \varepsilon_\beta^{-2} Re_U \langle \mathbf{v}_\beta^+ \rangle \cdot \nabla \langle \mathbf{v}_\beta^+ \rangle = -\nabla \langle p_\beta^+ \rangle^\beta + \varepsilon_\beta^{-1} \nabla^2 \langle \mathbf{v}_\beta^+ \rangle - \mathbf{H}_\beta^{-1} \cdot \langle \mathbf{v}_\beta^+ \rangle \quad (25b)$$

363 where the global Reynolds number is $Re_U = \rho_\beta U \ell / \mu_\beta$, $\mathbf{v}_\beta^+ = \mathbf{v}_\beta^* / U$, and $p_\beta^+ = p_\beta^* \ell / (\mu_\beta U)$. Note that
 364 because ℓ is used as the reference length scale in both the pore and global Reynolds numbers Re_p
 365 and Re_U respectively, the definition of the dimensionless effective tensor \mathbf{H}_β remains unchanged.
 366 Additionally, it is worth noticing that the global Reynolds number Re_U reflects the effect of the
 367 macroscopic boundary conditions on the system and is constant within the whole domain, while
 368 the pore Reynolds number Re_p reflects the flow intensity and orientation at every point of the
 369 continuous macroscopic porous medium, and is therefore space dependent. This later affects the
 370 values of the effective tensor. For this reason the global permeability tensor \mathbf{H}_β was tabulated
 371 over a range of pore Reynolds numbers and pressure gradient orientations by solving the non-
 372 linear closure problem on a two dimensional square unit cell (see Section 3.1 for details). The
 373 macroscopic transport equations (Eqs. (25)) with a space-dependent permeability tensor were then
 374 solved numerically on the macroscopic domain.

375 As shown in Fig. 5, three macroscopic geometries were considered: a two-dimensional straight
 376 pipe, an elbow and a diverging pipe of 1:3 expansion ratio, all partially filled with a porous medium
 377 of constant porosity $\varepsilon_{\beta 0} = 0.75$ constituted of an array of squares. A symmetry boundary condition
 378 is imposed at the top wall of the straight and diverging pipes, so that the total height of the channel
 379 in both geometries is 20ℓ at the inlet. The system is constituted of fluid domains at the inlet and
 380 outlet, and a porous domain in the center of the pipes. A fully developed parabolic flow is imposed
 381 at the entrance, while a zero pressure is imposed at the outlet, and a no-slip boundary condition is
 382 imposed at the pipe walls

$$383 \quad \langle \mathbf{v}_\beta^+ \rangle = \frac{3}{2} \left(\frac{2y}{h} - \frac{y^2}{h^2} \right) \mathbf{e}_x \quad \text{at inlet} \quad (26a)$$

$$384 \quad \langle p_\beta^+ \rangle^\beta = 0 \quad \text{at outlet} \quad (26b)$$

$$385 \quad \langle \mathbf{v}_\beta^+ \rangle = 0 \quad \text{at pipe walls} \quad (26c)$$

386 with $h = 10$ the half height of the channel (see Fig. 5). As a reminder, all lengths are normalized
 387 with the pore length scale ℓ representing the size of a unit cell (see Fig. 2). Note that since the
 388 inlet and outlet are fully fluid domains, the averages are equivalent to the point fluid velocity and
 389 pressures.

390 In order to avoid to deal with boundary conditions at the fluid-porous interfaces, a one-domain
 391 modeling approach is adopted [10, 13, 22]. The principle is based on the observation that in the
 392 limit where the porosity tends to 1 ($\varepsilon_\beta \rightarrow 1$), the inverse of the permeability tensor tends to 0

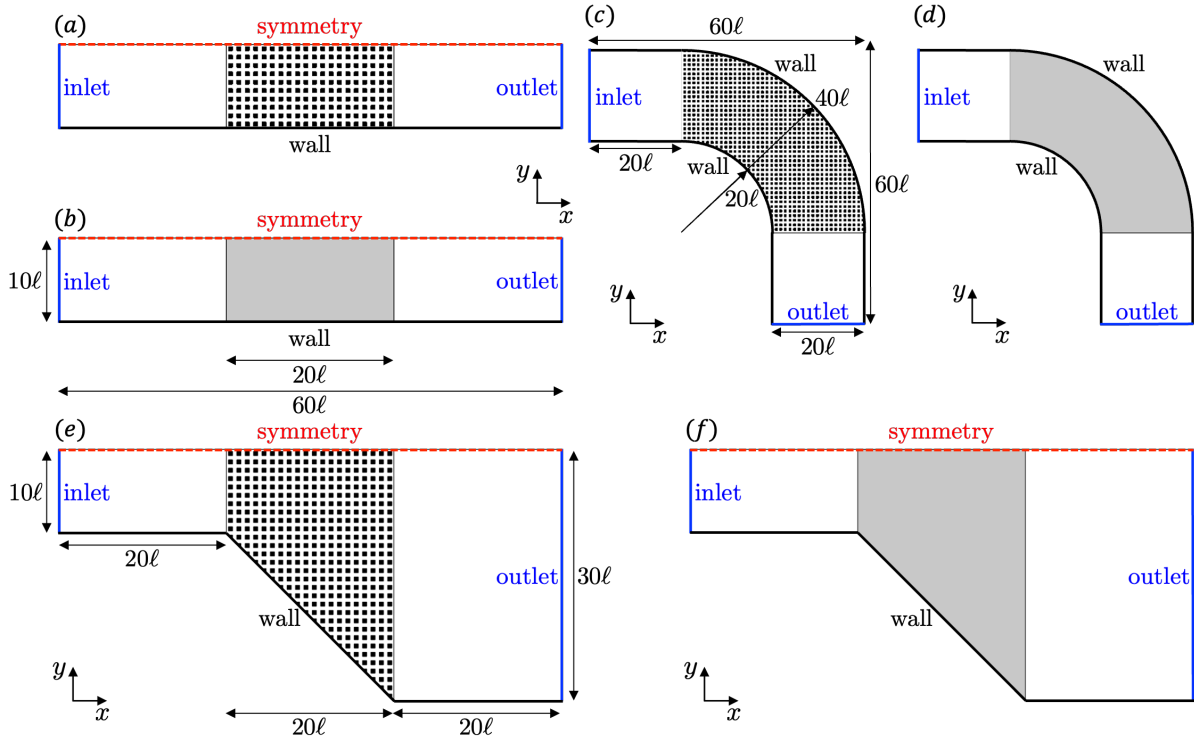


Figure 5: Geometries used for the macroscopic simulations. (a) Two-dimensional pipe partially filled with resolved square obstacles used for DNS. (b) Two-dimensional pipe partially filled with an effective continuous porous region. (c) Two-dimensional elbow partially filled with resolved square obstacles used for DNS. (d) Two-dimensional elbow partially filled with an effective continuous porous region. (e) Two-dimensional diverging pipe with expansion ratio 1:3 partially filled with resolved square obstacles used for DNS. (f) Two-dimensional diverging pipe with expansion ratio 1:3 partially filled with an effective continuous porous region.

393 $(\mathbf{H}_\beta^{-1} \rightarrow 0)$, so that the macroscopic equations (12a) and (12b) tend to the incompressible Navier-
 394 Stokes equations. Therefore they can be solved on the entire domain, provided that the porosity
 395 depends on space such as $\varepsilon_\beta(\mathbf{x}) = 1$ in the fluid region, and $\varepsilon_\beta(\mathbf{x}) = \varepsilon_{\beta 0} = 0.75$ in the porous region.
 396 To facilitate the transition from the fluid to porous regions, a smooth function of the porosity was
 397 built based on resolved fluid porous-interfaces used for direct numerical simulations (DNS) (see
 398 Fig. 5). Briefly, the space dependent porosity was defined as the cellular average of the β -phase
 399 indicator function [30, 41, 42] leading to a smooth transition of the porosity with a linear slope.
 400 Finally, to ease the evaluation of the effective permeability tensor at the fluid-porous interfacial
 401 region without having to compute it for the whole range of porosity, its values in the transition

402 regions was assumed to follow

$$403 \quad \mathbf{H}_\beta^{-1}(\mathbf{x}) = \gamma_{tr}(\mathbf{x}) \mathbf{H}_\beta^{-1}(Re_p, \theta) \quad (27)$$

404 where the region-indicator function was defined as $\gamma_{tr}(\mathbf{x}) = [1 - \varepsilon_\beta(\mathbf{x})] / [1 - \varepsilon_{\beta 0}]$. Note that this
405 approach can be applied only to the fluid-porous interface, but not to the solid-porous interface.
406 To avoid singular values at the solid-porous interface, values of the region-indicator functions were
407 computed in the bulk of the pipes/elbow and were then extrapolated using constant extrapolation
408 through the cross-section.

409 4.1.2 Direct numerical simulations and diagonal tensor model

410 For validation, results from the general non-linear model were compared against two common
411 approaches to model inertial flows in porous media: direct numerical simulations (DNS), and
412 macroscopic momentum transport models with a diagonal global permeability tensor.

413 **Direct numerical simulations (DNS)** As a reference case against which to validate the results
414 from the macroscopic models, DNS were performed. In this case, the geometric details of the porous
415 medium are resolved (see Fig. 5a & c), and the incompressible Navier-Stokes equation are solved
416 in the whole fluid domain including in the pore space. The same boundary conditions Eqs. (26)
417 were applied at the inlet, outlet, and pipe walls with the averages equal to the point velocities and
418 pressures. Additionally, a no-slip boundary condition was set at the fluid-solid interface between the
419 fluid and the square obstacles constituting the porous medium. In order to facilitate comparisons
420 with the homogenized macroscopic models, the velocity and pressure fields were averaged using the
421 cellular average [16, 41–45].

422 **Diagonal tensor model simulations** As a complementary comparison, the non-linear general
423 model was compared to a commonly used macroscopic version where the global permeability tensor
424 is diagonal and independent of the macroscopic flow orientation [11, 37]. Such model is routinely
425 used in state of the art CFD softwares, where instead of solving a closure problem to determine the
426 components of the permeability tensor, an incompressible flow at a given Reynolds number is solved
427 on a representative volume, and the diagonal components of the permeability tensor are obtained
428 by relating the mass flow rate to the pressure drop through a simplified Darcy-Forchheimer law
429 $\langle \mathbf{v}_\beta^+ \rangle = -\mathbf{H}_\beta(Re_p) \cdot \nabla \langle p_\beta^+ \rangle^\beta$.

430 4.1.3 Numerical implementation

431 All models were solved numerically using the finite element commercial software COMSOL Multi-
432 physics 6.1 with the “Laminar Flow” module. The additional macroscopic terms were implemented
433 as volume forces in order to obtain their values iteratively in the domain. For the general non-linear
434 macroscopic model, the values of the inertial permeability tensor were tabulated *a priori* using the
435 results of the non-linear general closure problem. The tabulated values were interpolated linearly
436 with respect to the pore Reynolds number and the angle of the superficial averaged velocity.

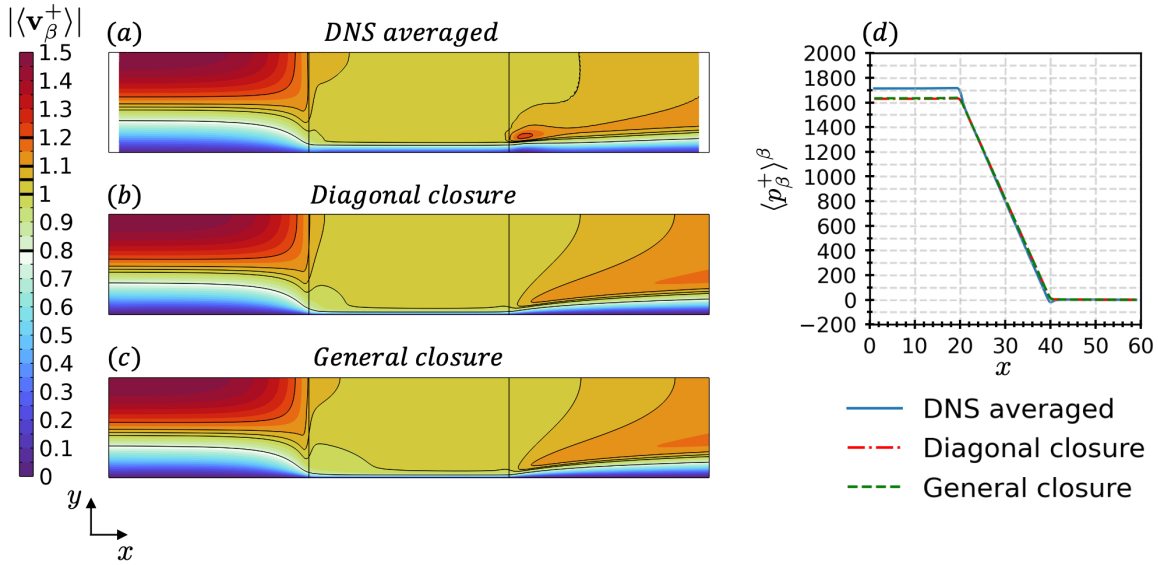


Figure 6: Comparison of macroscopic results of an inertial flow at global Reynolds number $Re_U = 30$ in a pipe partially filled with a porous medium constituted of square obstacles of porosity $\varepsilon_\beta = 0.75$. (a) Magnitude of the cellular average of the velocity field obtained from DNS. (b) Magnitude of the superficial average of the velocity field obtained from the diagonal tensor model. (c) Magnitude of the superficial average of the velocity field obtained from the general non-linear model. (d) Cross section average pressure (cellular for DNS, intrinsic for homogenized models) as a function of x .

4.2 Macroscopic simulation results

Macroscopic simulations of inertial flows in conduits filled with porous media were first performed for validation in a straight pipe filled with a porous medium. Then, to assess the influence of the extra-diagonal terms and of the macroscopic pressure gradient orientation on the global permeability tensor, an elbow filled with porous medium was considered. In all cases, values of the global Reynolds number $Re_U = \rho_\beta U \ell / \mu_\beta$ were tested for 1, 10, 30, and 50. Finally, a diverging pipe geometry was considered.

4.2.1 Inertial flow in a straight pipe partially filled with a porous medium

Fig. 6 compares the simulations results from the three models for of an inertial flow with global Reynolds number $Re_U = 30$ in a two-dimensional pipe partially filled with a porous medium. The averaged DNS simulations hold as the reference against which the homogenized models, namely the diagonal tensor model and the general non-linear model, are compared. As seen in Fig. 6, very little difference is observed between the models. All three models show a very good agreement in the inlet fluid region, validating the use of the one-domain approach to describe the fluid part. In the central porous region, the flow becomes uniform. It can be noticed that the two homogenized models slightly under-predict the thickness of the wall boundary layer at the porous-solid interface compared to DNS. Finally, in the outlet fluid region, although the homogenized models capture the overall flow behavior, they fail to predict the small recirculation zone at the corner between

Table 1: Dimensionless pressure loss between inlet and outlet of a pipe partially filled with a porous medium as a function for different values of the global Reynolds numbers for DNS and the two homogenized models.

Re_U	$\Delta\langle p_\beta^+ \rangle^\beta$			Relative difference to DNS (%)	
	DNS	Diag. tensor	Gen. non-linear	Diag. tensor	Gen. non-linear
0	1618	1566	1566	3.2	3.2
1	1618	1566	1566	3.2	3.2
10	1641	1583	1583	3.5	3.5
30	1714	1630	1634	4.9	4.7
50	1771	1657	1668	6.5	5.8

455 the fluid-porous interface and the wall, as predicted by DNS. These limitations of the homogenized
 456 models are due to the difficulty to define a representative elementary volume that satisfies the
 457 length-scale separation close to a wall or a fluid interface. Although some jump conditions have
 458 been proposed for two-domain approaches [5, 6, 8, 33–35], these are generally limited to non-inertial
 459 flows. More recently a methodology to account for the influence of porous media macroscopic
 460 boundaries on inertial flows was proposed [50]. Such model refinements are out of the scope of the
 461 current study.

462 As seen in Fig. 6d, the evolution of the cross section average of the pressure as a function of the
 463 distance from the entrance also show very similar results between the three models, although both
 464 homogenized models slightly under-predict the inlet pressure by less than 5% compared to DNS.
 465 Given that the outlet pressure is set to zero, these results point to the fact that the homogenized
 466 models underestimate the singular head losses at the fluid-porous interfaces and the distributed
 467 head-losses at the porous-wall interfaces, leading to an overall lower pressure loss between the inlet
 468 and outlet.

469 Simulations for various global Reynolds number up to $Re_U = 50$ were performed, giving the
 470 same qualitative behavior as for the case $Re_U = 30$ (data not shown). For completeness, the total
 471 pressure drop and their relative difference with the DNS results are summarized in Table 1. Relative
 472 differences in pressure losses of the two homogenized models compared to DNS remain lower than
 473 7% for all the cases considered, confirming the relevance of these models. The general non-linear
 474 model and diagonal tensor models are equivalent at low global Reynolds number, with a relative
 475 difference in pressure loss compared to DNS remaining below 3.5% for $Re_U = 10$. This relative
 476 difference in pressure loss increases with larger values of the Reynolds number, with the diagonal
 477 tensor model having larger differences than the general non-linear model. This discrepancy can be
 478 explained by the appearance of recirculation zones with increasing Reynolds numbers, leading to
 479 local streamlines having a non-zero angle with the x-axis. Although the influence of flow orientation
 480 in the unit cell is accounted for in the general non-linear model during the determination of the
 481 effective permeability tensor, this is not the case in the diagonal tensor model.

482 To investigate further the effect of flow orientation on inertial flow in porous media, an elbow
 483 partially filled with a porous medium is considered in the next section.

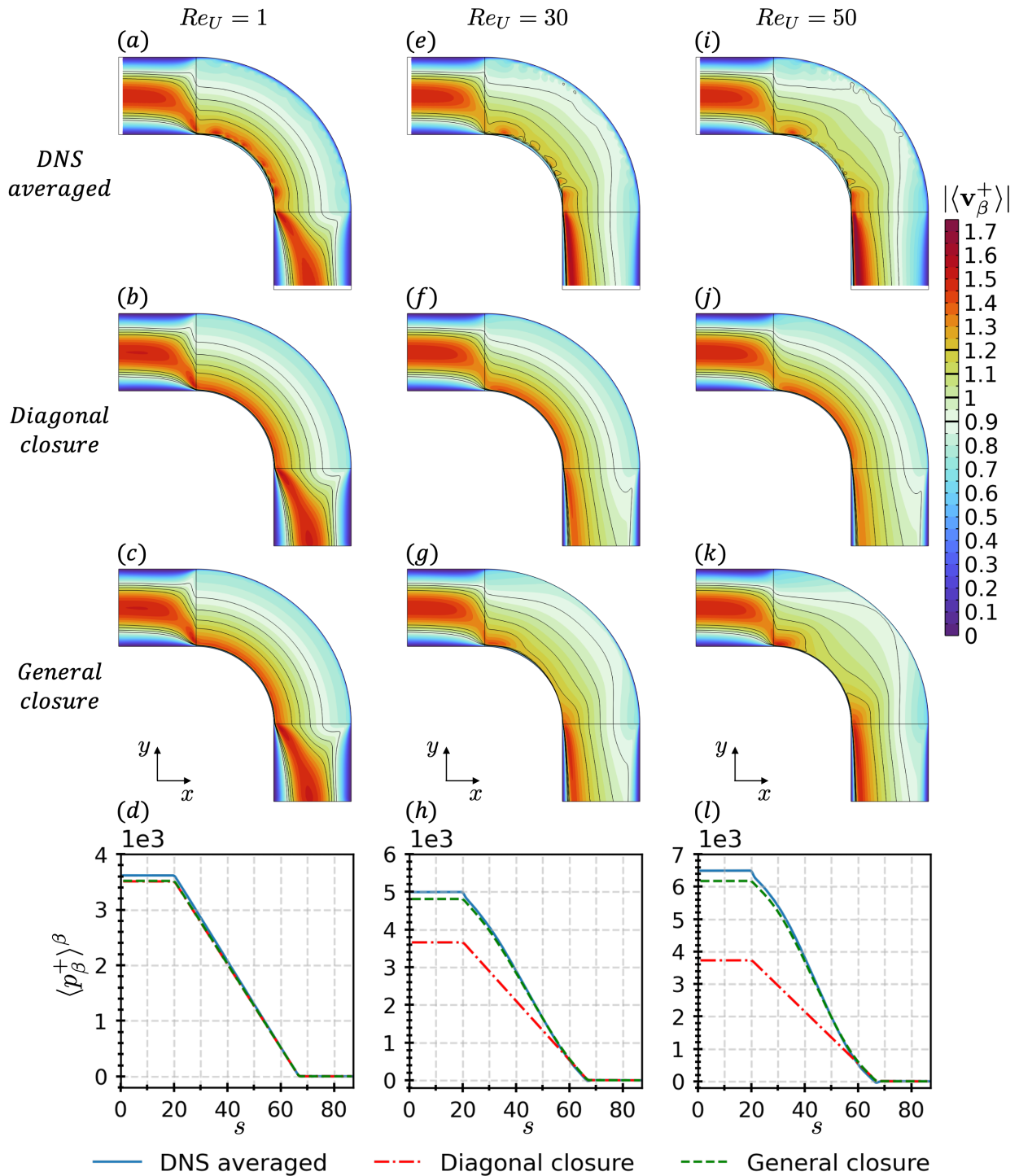


Figure 7: Comparison of macroscopic results of inertial flows in a 90° elbow partially filled with a porous medium constituted of square obstacles of porosity $\varepsilon_\beta = 0.75$. Results are shown for three global Reynolds numbers: (a, d, g, j) $Re_U = 1$, (b, e, h, k) $Re_U = 30$, (c, f, i, l) $Re_U = 50$. (a-c) Magnitude of the cellular average of the velocity field obtained from DNS. (d-f) Magnitude of the superficial average of the velocity field obtained from the diagonal tensor model. (g-i) Magnitude of the superficial average of the velocity field obtained from the general non-linear model. (j-l) Cross section average pressure (cellular for DNS, superficial for homogenized models) as a function of the arclength s .

484 4.2.2 Inertial flow in an elbow partially filled with a porous medium

485 In order to test the importance of the pressure gradient orientation on the macroscopic inertial
486 flow in a porous medium, the three models were solved in a 90° elbow partially filled with a porous
487 medium (see Fig. 5c & d). Importantly, the porous medium is the same as the one used in the
488 straight pipe, namely square obstacles of porosity $\varepsilon_\beta = 0.75$ where the unit cell geometry is not
489 rotated to follow the curvature of the domain. As a consequence, the angle between the average
490 flow and the x -axis of the square obstacle varies from 0° at the entrance of the porous medium to
491 -90° at its exit.

492 Comparison between the results obtained by DNS, the diagonal-tensor macroscopic model, and
493 the general non-linear macroscopic model are presented in Fig. 7 for global Reynolds number of
494 $Re_U = 1, 30,$ and 50 . At $Re_U = 1$, the velocity profiles obtained for the macroscopic simulations are
495 very similar to the one obtained from averaged DNS. Although averaged DNS results show some
496 oscillations in average velocity field close to the walls due the contact of some solid obstacles with
497 the wall, overall the three simulations predict a constant averaged velocity profile along the length
498 of the porous region. For larger global Reynolds numbers of $Re_U = 30$ and 50 , inertial effects induce
499 variations of the average velocity profile along the length of the elbow, as shown by the averaged
500 DNS (Fig. 7e & i). This behavior is not captured by the diagonal tensor model which predicts a
501 constant average velocity profile in each section of the porous region (Fig. 7f & j). In contrast, the
502 general non-linear model does reproduce well the non-uniform velocity profile predicted by DNS
503 (Fig. 7g & k). The difference between the predictions of the two homogenized model increases with
504 the global Reynolds number as inertial effects become more and more present. Interestingly, it is
505 clear from the streamlines at $Re_U = 30$ and 50 that the inertial flow follows preferential paths at
506 angles $\theta = 0, \pi/4,$ and $\pi/2$ as shown in the DNS and general closure model. This is due to the
507 choice of the unit cell geometry with the regular arrangement of square obstacles.

508 The difference of predictions between the homogenized models is even more evident from the
509 values of the averaged intrinsic pressures. The evolution of the cross-section averaged pressure
510 along the length of the elbow is shown in Fig. 7d, h, & l, and the values of the pressure losses
511 and relative difference to DNS are given in Table 2. As inertial effects increase, both DNS and the
512 general non-linear model predict a significantly larger pressure loss between the inlet and outlet
513 (more than a 40% increase in pressure at $Re_U = 50$ compared to $Re_U = 1$), as well as a deviation
514 from a linear pressure evolution in the porous region. In contrast, the diagonal tensor model only
515 predicts a mild increase in pressure (about 5% at $Re_U = 50$ compared to $Re_U = 1$) and maintains
516 a linear decrease in pressure along the elbow. Importantly, in the fully inertial regime, the general
517 non-linear model remains within the 5% of the pressure drop predicted by DNS, while the diagonal
518 tensor model shows differences up to 42.6% at $Re_U = 50$.

519 The ability of the general non-linear model to capture realistically the variation of average
520 velocity field along the elbow is due to the dependency of the global permeability tensor on the
521 pressure gradient orientation. Indeed, in the elbow the flow is diverted by a 90° angle while the
522 unit cell geometry remains with the same orientation. As a consequence, the global permeability
523 tensor displays non-negligible extra-diagonal terms and a dependency of its components to the flow
524 orientation compared to the unit cell geometry. While these two features are accounted for in the
525 general non-linear model, they are not captured in the diagonal tensor model which fails to account
526 for the change of tortuosity and permeability (including inertial corrections) induced by the change

Table 2: Pressure loss between inlet and outlet of a 90° elbow partially filled with a porous medium as a function for different values of the global Reynolds numbers for DNS and the two homogenized models.

Re_U	$\Delta\langle p_\beta^+ \rangle^\beta$			Relative difference to DNS (%)	
	DNS	Diag. tensor	Gen. non-linear	Diag. tensor	Gen. non-linear
0	3611	3508	3508	2.8	2.8
1	3617	3510	3519	3.0	2.7
10	3846	3550	3729	7.7	3.0
30	4995	3659	4811	26.8	3.7
50	6491	3727	6173	42.6	4.9

527 of flow orientation relative to the pore unit geometry.

528 4.2.3 Inertial flow in a diverging pipe partially filled with a porous medium

529 Finally, a diverging pipe filled with a porous medium is considered (see Fig. 5e-f). A Reynolds
530 number of $Re_U = 50$ is imposed at the entrance to ensure a strong inertial but laminar flow regime.
531 As seen in Fig. 8, the general observations on the different macroscopic models are similar to the
532 ones from the elbow geometry: the diagonal model fails to capture the change of average velocity
533 orientation and predicts a more uniform flow field than the one obtained with the general closure
534 model which is much closer to DNS results. Some discrepancies between the general model and
535 DNS remain, particularly near the pipe walls due to the macroscopic boundary layer and at the
536 fluid/porous interfaces. Once again, the evolution of the section average pressure along the channel
537 shows that the diagonal model largely under-predicts the head loss (19.5% difference with DNS)
538 while the general model successfully predicts the head loss within 3.05% of the DNS values.

539 5 Conclusion

540 Incompressible inertial flows in porous media are well described by the Forchheimer equation, where
541 inertial effects are accounted for through a corrective tensor \mathbf{F}_β to the intrinsic permeability. Using
542 up-scaling techniques such as the volume averaging method [24, 56], one can compute the effective
543 tensor from the solution of a non-linear closure problem on a representative unit cell. However
544 this closure problem requires that, for each flow orientation and for each pore Reynolds number, an
545 associated local problem must be solved before solving the non-linear closure problem. To deal with
546 this difficulty, in this work, a linearized approach where the resulting linear closure problems are
547 independent of the flow orientation was proposed and assessed. Then, macroscopic simulations were
548 carried to study the importance of the macroscopic flow orientation, highlighting the importance
549 of extra-diagonal components in the Forchheimer correction tensor.

550 The non-linear closure problem obtained with the volume averaging method [24] was linearized
551 into a series of linear problems independent of the macroscopic flow orientation. This procedure is
552 a generalization of previous works for weakly inertial flows [7, 31, 39] leading to linear and intrinsic
553 closure problems independent of the macroscopic flow orientation. Developments were pushed up to
554 order 2 with respect to the pore Reynolds number, resulting in a Forchheimer correction expressed
555 as a quadratic polynomial of Re_k . The limit of validity of this approach was assessed by comparison

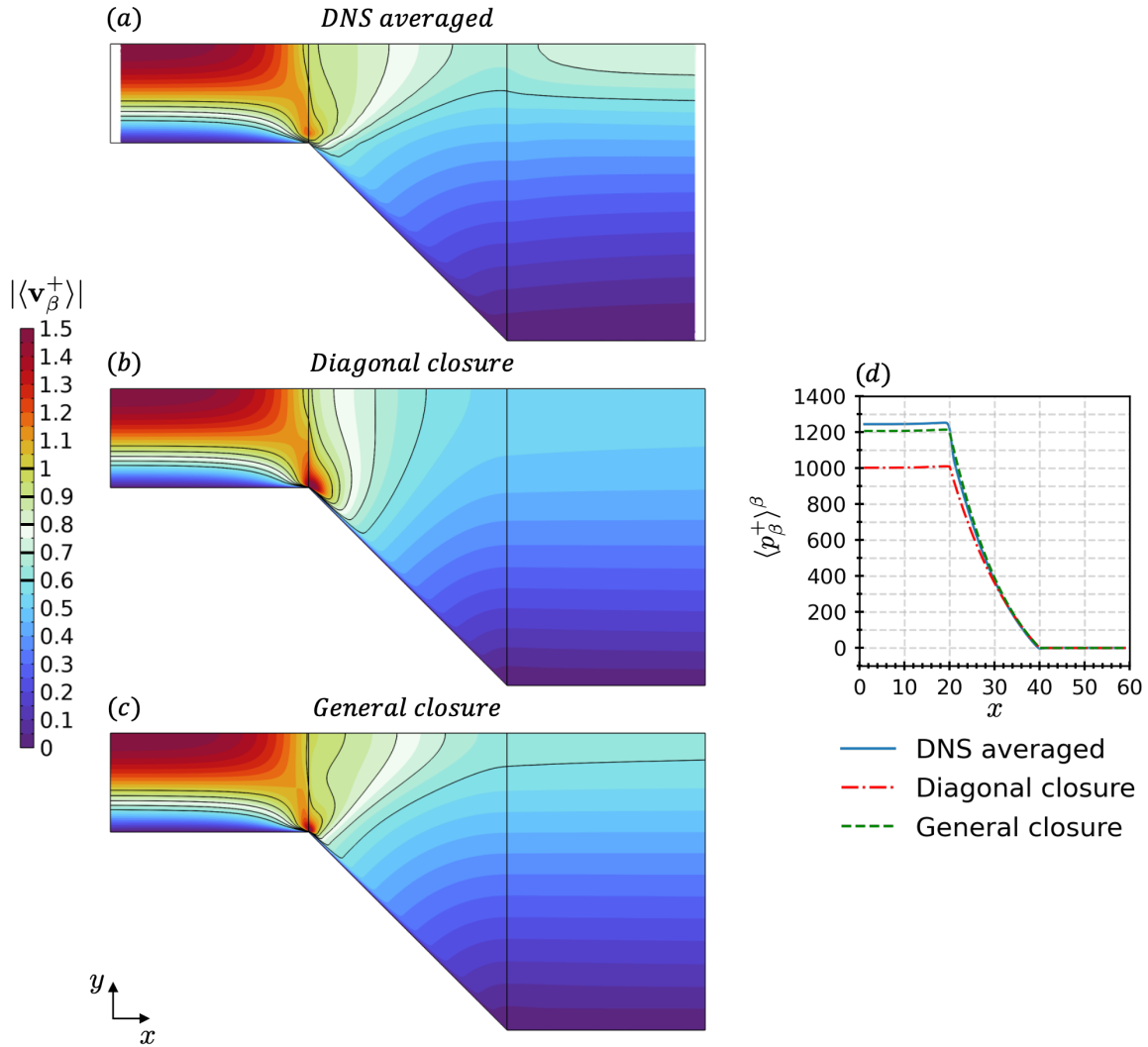


Figure 8: Comparison of macroscopic results of an inertial flow at global Reynolds number $Re_U = 50$ in a diverging pipe with expansion ratio 1:3, partially filled with a porous medium constituted of square obstacles of porosity $\varepsilon_\beta = 0.75$. (a) Magnitude of the cellular average of the velocity field obtained from DNS. (b) Magnitude of the superficial average of the velocity field obtained from the diagonal tensor model. (c) Magnitude of the superficial average of the velocity field obtained from the general non-linear model. (d) Cross section average pressure (cellular for DNS, intrinsic for homogenized models) as a function of x .

556 with a general non-linear closure problem. The effect of flow orientation and intensity was studied
 557 for different square and rectangle unit cell geometries, showing a good match between the linearized

558 and general non-linear models up to pore Reynolds number Re_k of the order one. The linearized
559 quadratic model was able to accurately predict the weak inertia regime [2, 23, 24, 39] for different
560 flow orientations. The generality of the presented method allows to expand to higher orders, but
561 at the cost of increasing tensor order leading to exponential numerical difficulties.

562 To assess the effect of flow orientation on the extra-diagonal terms of the correction tensor,
563 macroscopic simulations were conducted for Reynolds number up to 50 in straight and 90° elbow
564 conduits filled with porous media. Results from the macroscopic general non-linear model were
565 compared against a diagonal macroscopic model commonly used in engineering applications [11,
566 37, 53]. In the straight conduit, due to the flow orientation being constant, no significant difference
567 were observed between diagonal and general non-linear models. However, in the case where the
568 flow orientation varies along the elbow conduit, the diagonal model was unable to predict the non-
569 homogeneous velocity field along the channel's length, leading to a under-prediction of the pressure
570 loss by more than 40% at $Re_U = 50$ compared to DNS. In contrast, the general diagonal model
571 captured well the average velocity field, and predicted a pressure loss within 5% of the values
572 obtained by DNS. Qualitatively similar conclusions were obtained on a diverging pipe reproducing
573 a progressive expansion flow.

574 All together, this work highlights the importance of macroscopic pressure orientation on the
575 global permeability tensor in inertial flow in porous media. Further developments to improve the
576 prediction of the macroscopic inertial flow include introducing accurate jump conditions [5, 6, 8]
577 or profile transition at fluid-porous and solid-porous interfaces [10, 22, 33, 48–50]. Concerning the
578 proposed linearization approach, future applications to transfers where the critical phenomenon
579 occurs with a dimensionless number below one might be particularly relevant, such as in weakly
580 compressible and dilatable flows.

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732 **Statements and declarations**

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735 All authors contributed to the study conception and design. Numerical simulations were per-
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737 authors commented on previous versions of the manuscript. All authors read and approved the
738 final manuscript.

739 **APPENDIX**

740 **A Linearization of the closure problem associated with inertial**
 741 **flow**

742 Based on regular perturbation theory, any field ψ_β in the deviation problem (10) is expanded as a
 743 power series around Re_p

$$744 \quad \psi_\beta = \sum_{k=0}^{+\infty} \psi_{\beta k} Re_p^k \quad (\text{A.1})$$

745 This leads to a self-sufficient series of linear deviation problems for each order of the Reynolds
 746 number Re_p . Here the details of their derivation and the procedure to obtain the corresponding
 747 closure problem are detailed up to order 2, although the methodology can be easily applied to
 748 higher orders.

749 **Order $k = 0$**

750 First, the closure problem at order 0 is developed as the reference case of the linearization. This
 751 corresponds to the limit case where $Re_p = 0$

$$752 \quad \nabla \cdot \tilde{\mathbf{v}}_{\beta 0} = 0 \quad \text{in } V_\beta \quad (\text{A.2a})$$

$$753 \quad 0 = -\nabla \tilde{p}_{\beta 0} + \nabla^2 \tilde{\mathbf{v}}_{\beta 0} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_{\beta 0} + \nabla \tilde{\mathbf{v}}_{\beta 0}) dA \quad \text{in } V_\beta \quad (\text{A.2b})$$

$$754 \quad \tilde{\mathbf{v}}_{\beta 0} = -\langle \mathbf{v}_{\beta 0} \rangle^\beta \quad \text{on } A_{\beta\sigma} \quad (\text{A.2c})$$

$$755 \quad \tilde{\mathbf{v}}_{\beta 0}(\mathbf{r} + \mathbf{l}_i) = \tilde{\mathbf{v}}_{\beta 0}(\mathbf{r}) \quad ; \quad \tilde{p}_{\beta 0}(\mathbf{r} + \mathbf{l}_i) = \tilde{p}_{\beta 0}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.2d})$$

$$756 \quad \langle \tilde{\mathbf{v}}_{\beta 0} \rangle^\beta = 0 \quad ; \quad \langle \tilde{p}_{\beta 0} \rangle^\beta = 0 \quad (\text{A.2e})$$

757 which is exactly the spatial deviation problem for a Stokes flow at the pore-scale. Because it is
 758 linear and self-consistent, it is possible to use the superposition principle to express the spatial
 759 deviations $\tilde{\mathbf{v}}_{\beta 0}$ and $\tilde{p}_{\beta 0}$ as a function of the only macroscopic source term $\langle \mathbf{v}_{\beta 0} \rangle^\beta$

$$760 \quad \tilde{\mathbf{v}}_{\beta 0} = \mathbf{B} \cdot \langle \mathbf{v}_{\beta 0} \rangle^\beta \quad (\text{A.3a})$$

$$761 \quad \tilde{p}_{\beta 0} = \mathbf{b} \cdot \langle \mathbf{v}_{\beta 0} \rangle^\beta \quad (\text{A.3b})$$

762 Here \mathbf{B} and \mathbf{b} are closure variable that are solution of the closure problem at order 0

$$763 \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } V_\beta \quad (\text{A.4a})$$

$$764 \quad 0 = -\nabla \mathbf{b} + \nabla^2 \mathbf{B} + \varepsilon_\beta \mathbf{K}_\beta^{-1} \quad \text{in } V_\beta \quad (\text{A.4b})$$

$$765 \quad \mathbf{B} = -\mathbf{I} \quad \text{on } A_{\beta\sigma} \quad (\text{A.4c})$$

$$766 \quad \mathbf{B}(\mathbf{r} + \mathbf{l}_i) = \mathbf{B}(\mathbf{r}) \quad ; \quad \mathbf{b}(\mathbf{r} + \mathbf{l}_i) = \mathbf{b}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.4d})$$

$$767 \quad \langle \mathbf{B} \rangle^\beta = 0 \quad ; \quad \langle \mathbf{b} \rangle^\beta = 0 \quad (\text{A.4e})$$

768 where the intrinsic permeability is defined as

$$769 \quad \varepsilon_\beta \mathbf{K}_\beta^{-1} = -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\mathbf{b} + \nabla\mathbf{B}) dA \quad (\text{A.5})$$

770 Similarly to the closure problem (14), the above system can easily be written in the form of an
771 incompressible Stokes flow with no-slip boundary conditions using the following change of variable
772 [58]: $\mathbf{B}' = \varepsilon_\beta^{-1} (\mathbf{B} + \mathbf{I}) \cdot \mathbf{K}_\beta$ and $\mathbf{b}' = \varepsilon_\beta^{-1} \mathbf{b} \cdot \mathbf{K}_\beta$

773 The solution of the closure problem at order 0 on a unit cell gives the value of the intrinsic
774 permeability tensor \mathbf{K}_β . Moreover, the values of the closure variables \mathbf{B} and \mathbf{b} will be needed to
775 solve the linearized closure problems at higher order. These properties is linked to the linearization
776 method and motivates the need to solve the closure problems in a recursive way in order to linearized
777 the results for high orders of the Reynolds number.

778 Order $\mathbf{k} = 1$

779 Identifying the terms in Re^1 in the power-series decomposition of the deviation problem, one gets
780 the linearized spatial deviation problem at order 1

$$781 \quad \nabla \cdot \tilde{\mathbf{v}}_{\beta 1} = 0 \quad \text{in } V_\beta \quad (\text{A.6a})$$

$$782 \quad \mathbf{v}_{\beta 0} \cdot \nabla \tilde{\mathbf{v}}_{\beta 0} = -\nabla \tilde{p}_{\beta 1} + \nabla^2 \tilde{\mathbf{v}}_{\beta 1} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\tilde{p}_{\beta 1} + \nabla \tilde{\mathbf{v}}_{\beta 1}) dA \quad \text{in } V_\beta \quad (\text{A.6b})$$

$$783 \quad \tilde{\mathbf{v}}_{\beta 1} = -\langle \mathbf{v}_{\beta 1} \rangle^\beta \quad \text{on } A_{\beta\sigma} \quad (\text{A.6c})$$

$$784 \quad \tilde{\mathbf{v}}_{\beta 1}(\mathbf{r} + \mathbf{e}_i) = \tilde{\mathbf{v}}_{\beta 1}(\mathbf{r}) \quad ; \quad \tilde{p}_{\beta 1}(\mathbf{r} + \mathbf{e}_i) = \tilde{p}_{\beta 1}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.6d})$$

$$785 \quad \langle \tilde{\mathbf{v}}_{\beta 1} \rangle^\beta = 0 \quad ; \quad \langle \tilde{p}_{\beta 1} \rangle^\beta = 0 \quad (\text{A.6e})$$

786 Note that the term $\mathbf{v}_{\beta 0} \cdot \nabla \tilde{\mathbf{v}}_{\beta 0}$ is at order 0. Thanks to the spacial decomposition of the velocity
787 and (A.3a), it can be expressed using the zero order closure variables as

$$788 \quad \mathbf{v}_{\beta 0} \cdot \nabla \tilde{\mathbf{v}}_{\beta 0} = \left[(\mathbf{B} + \mathbf{I}) \cdot \langle \mathbf{v}_{\beta 0} \rangle^\beta \right] \cdot \left(\nabla \mathbf{B} \cdot \langle \mathbf{v}_{\beta 0} \rangle^\beta \right) \\ 789 \quad = \left[(\nabla \mathbf{B})^{T(123)} \cdot (\mathbf{B} + \mathbf{I}) \right] : \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta$$

790 where the transpose operator is defined as $\mathcal{A}^{T(123)} = (\mathcal{A}_{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k)^{T(123)} = \mathcal{A}_{kij} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k$ for a third
791 order tensor \mathcal{A} . Using this expression, the spatial deviation problem can be written in a more
792 explicit form that exhibits different macroscopic source terms

$$793 \quad \nabla \cdot \tilde{\mathbf{v}}_{\beta 1} = 0 \quad \text{in } V_\beta \quad (\text{A.7a})$$

$$794 \quad \left[(\nabla \mathbf{B})^{T(123)} \cdot (\mathbf{B} + \mathbf{I}) \right] : \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \\ = -\nabla \tilde{p}_{\beta 1} + \nabla^2 \tilde{\mathbf{v}}_{\beta 1} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I}\tilde{p}_{\beta 1} + \nabla \tilde{\mathbf{v}}_{\beta 1}) dA \quad \text{in } V_\beta \quad (\text{A.7b})$$

$$795 \quad \tilde{\mathbf{v}}_{\beta 1} = -\langle \mathbf{v}_{\beta 1} \rangle^\beta \quad \text{on } A_{\beta\sigma} \quad (\text{A.7c})$$

$$796 \quad \tilde{\mathbf{v}}_{\beta 1}(\mathbf{r} + \mathbf{l}_i) = \tilde{\mathbf{v}}_{\beta 1}(\mathbf{r}) \quad ; \quad \tilde{p}_{\beta 1}(\mathbf{r} + \mathbf{l}_i) = \tilde{p}_{\beta 1}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.7d})$$

$$797 \quad \langle \tilde{\mathbf{v}}_{\beta 1} \rangle^\beta = 0 \quad ; \quad \langle \tilde{p}_{\beta 1} \rangle^\beta = 0 \quad (A.7e)$$

798 This last problem is linear and self-consistent because the closure variables \mathbf{B} and \mathbf{b} have been
 799 solved previously at order 0. Moreover, two macroscopic source terms are identified. The first one
 800 $\langle \mathbf{v}_{\beta 1} \rangle^\beta$ appears in the boundary condition and is very similar to the macroscopic source term at
 801 order 0. The second one $\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta$ is new and arises in the momentum balance equation at
 802 order 1 due to the coupling between order 0 and 1. Using the superposition principle, the spatial
 803 deviations $\tilde{\mathbf{v}}_{\beta 1}$ and $\tilde{p}_{\beta 1}$ can be mapped using the macroscopic source terms

$$804 \quad \tilde{\mathbf{v}}_{\beta 1} = \mathbf{B} \cdot \langle \mathbf{v}_{\beta 1} \rangle^\beta + \mathcal{C} : \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \quad (A.8a)$$

$$805 \quad \tilde{p}_{\beta 1} = \mathbf{b} \cdot \langle \mathbf{v}_{\beta 1} \rangle^\beta + \mathbf{C} : \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \quad (A.8b)$$

806 where \mathbf{B} and \mathbf{b} are the closure variables that solve the same closure problem as previously introduced
 807 at order 0, and \mathcal{C} and \mathbf{C} are two new closure variables introduced for the second source term. They
 808 verify the following closure problem at order 1

$$809 \quad \nabla \cdot \mathcal{C} = 0 \quad \text{in } V_\beta \quad (A.9a)$$

$$810 \quad \left[(\nabla \mathbf{B})^{T(123)} \cdot (\mathbf{B} + \mathbf{I}) \right] = -\nabla \mathcal{C} + \nabla^2 \mathcal{C} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \mathbf{C} + \nabla \mathcal{C}) dA \quad \text{in } V_\beta \quad (A.9b)$$

$$811 \quad \mathcal{C} = 0 \quad \text{on } A_{\beta\sigma} \quad (A.9c)$$

$$812 \quad \mathcal{C}(\mathbf{r} + \mathbf{l}_i) = \mathcal{C}(\mathbf{r}) \quad ; \quad \mathbf{C}(\mathbf{r} + \mathbf{l}_i) = \mathbf{C}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (A.9d)$$

$$813 \quad \langle \mathcal{C} \rangle^\beta = 0 \quad ; \quad \langle \mathbf{C} \rangle^\beta = 0 \quad (A.9e)$$

814 The solution of this problem only depends on the geometry of the unit cell, and not on the Reynolds
 815 number nor the macroscopic pressure gradient orientation. Once again, to deal with the integro-
 816 differential nature of this problem, it is convenient to define

$$817 \quad \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{H}_\beta = -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \mathbf{C} + \nabla \mathcal{C}) dA \quad (A.10)$$

818 where \mathcal{H}_β can be interpreted as a correction to the intrinsic permeability to account for inertial
 819 effects contributing to the global permeability tensor \mathbf{H}_β at order 1. The surface integral being
 820 constant within a given unit cell, $\varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{H}_\beta$ can be seen as a constant source term. This suggests
 821 to proceed once more to a change of variable

$$822 \quad \mathcal{C} = \mathcal{C}' + \mathcal{S} : \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{H}_\beta \quad ; \quad \mathbf{C} = \mathbf{C}' + \mathcal{S} : \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{H}_\beta \quad (A.11)$$

823 so that the integro-differential closure problem can be decomposed into two purely differential
 824 problems

$$825 \quad \nabla \cdot \mathcal{C}' = 0 \quad \text{in } V_\beta \quad (A.12a)$$

$$826 \quad \left[(\nabla \mathbf{B})^{T(123)} \cdot (\mathbf{B} + \mathbf{I}) \right] = -\nabla \mathcal{C}' + \nabla^2 \mathcal{C}' \quad \text{in } V_\beta \quad (A.12b)$$

$$827 \quad \mathcal{C}' = 0 \quad \text{on } A_{\beta\sigma} \quad (A.12c)$$

$$828 \quad \mathbf{C}'(\mathbf{r} + \mathbf{l}_i) = \mathbf{C}'(\mathbf{r}) \quad ; \quad \mathbf{C}'(\mathbf{r} + \mathbf{l}_i) = \mathbf{C}'(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.12d})$$

$$829 \quad \langle \mathbf{C}' \rangle^\beta = 0 \quad (\text{A.12e})$$

830 and

$$831 \quad \nabla \cdot \mathcal{S} = 0 \quad \text{in } V_\beta \quad (\text{A.13a})$$

$$832 \quad 0 = -\nabla \mathcal{S} + \nabla^2 \mathcal{S} + \mathcal{S} \quad \text{in } V_\beta \quad (\text{A.13b})$$

$$833 \quad \mathcal{S} = 0 \quad \text{on } A_{\beta\sigma} \quad (\text{A.13c})$$

$$834 \quad \mathcal{S}(\mathbf{r} + \mathbf{l}_i) = \mathcal{S}(\mathbf{r}) \quad ; \quad \mathcal{S}(\mathbf{r} + \mathbf{l}_i) = \mathcal{S}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.13d})$$

$$835 \quad \langle \mathcal{S} \rangle^\beta = 0 \quad (\text{A.13e})$$

836 Problem (A.13) is very similar to the practical form of the Stokes closure problem but needs to be
837 solved for a fourth order tensor \mathcal{S} and a source term $\mathcal{S} = \delta_{il}\delta_{jk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$.

838 The superficial averages of the closure variables \mathcal{C} being null, one gets

$$839 \quad 0 = \langle \mathcal{C} \rangle^\beta = \langle \mathbf{C}' \rangle^\beta + \langle \mathcal{S} \rangle^\beta : \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{H}_\beta \quad (\text{A.14})$$

840 so that

$$841 \quad \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{H}_\beta = - \left(\langle \mathcal{S} \rangle^\beta \right)^{-1} : \langle \mathbf{C}' \rangle^\beta \quad (\text{A.15})$$

842 After making use of some tensor algebra and the properties of problem (A.13), the following ex-
843 pression for the correction tensor at order 0 is obtained

$$844 \quad \mathcal{H}_\beta = - \langle \mathbf{C}' \rangle^\beta \quad (\text{A.16})$$

845 Order $\mathbf{k} = 2$

846 Applying the same procedure as above, coefficients in Re^2 are identified in the expanded closure
847 problem, giving the deviation problem at order 2

$$848 \quad \nabla \cdot \tilde{\mathbf{v}}_{\beta 2} = 0 \quad \text{in } V_\beta \quad (\text{A.17a})$$

$$849 \quad \mathbf{v}_{\beta 0} \cdot \nabla \tilde{\mathbf{v}}_{\beta 1} + \mathbf{v}_{\beta 1} \cdot \nabla \tilde{\mathbf{v}}_{\beta 0} = -\nabla \tilde{p}_{\beta 2} + \nabla^2 \tilde{\mathbf{v}}_{\beta 2} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_{\beta 2} + \nabla \tilde{\mathbf{v}}_{\beta 2}) dA \quad (\text{A.17b})$$

$$850 \quad \tilde{\mathbf{v}}_{\beta 2} = - \langle \mathbf{v}_{\beta 2} \rangle^\beta \quad \text{on } A_{\beta\sigma} \quad (\text{A.17c})$$

$$851 \quad \tilde{\mathbf{v}}_{\beta 2}(\mathbf{r} + \mathbf{e}_i) = \tilde{\mathbf{v}}_{\beta 2}(\mathbf{r}) \quad ; \quad \tilde{p}_{\beta 2}(\mathbf{r} + \mathbf{e}_i) = \tilde{p}_{\beta 2}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.17d})$$

$$852 \quad \langle \tilde{\mathbf{v}}_{\beta 2} \rangle^\beta = 0 \quad ; \quad \langle \tilde{p}_{\beta 2} \rangle^\beta = 0 \quad (\text{A.17e})$$

853 Once again the non-linear terms only involve variables at lower orders. After some rearrangements,
854 the problem can be expressed in a more explicit form that exhibits all the different macroscopic
855 source terms

$$856 \quad \nabla \cdot \tilde{\mathbf{v}}_{\beta 2} = 0 \quad \text{in } V_\beta \quad (\text{A.18a})$$

$$\begin{aligned}
& \left[(\nabla \mathbf{B})^{T(123)} \cdot (\mathbf{B} + \mathbf{I}) \right] : \left[\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 1} \rangle^\beta + \langle \mathbf{v}_{\beta 1} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \right] \\
& + \left[(\nabla \mathcal{C})^{T(1234)} \cdot (\mathbf{B} + \mathbf{I}) + (\nabla \mathbf{B})^{T(123)} \cdot \mathcal{C} \right] : \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta
\end{aligned} \tag{A.18b}$$

$$= -\nabla \tilde{p}_{\beta 2} + \nabla^2 \tilde{\mathbf{v}}_{\beta 2} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \tilde{p}_{\beta 2} + \nabla \tilde{\mathbf{v}}_{\beta 2}) dA \quad \text{in } V_\beta$$

$$\tilde{\mathbf{v}}_{\beta 2} = -\langle \mathbf{v}_{\beta 2} \rangle^\beta \quad \text{on } A_{\beta\sigma} \tag{A.18c}$$

$$\tilde{\mathbf{v}}_{\beta 2}(\mathbf{r} + \mathbf{l}_i) = \tilde{\mathbf{v}}_{\beta 2}(\mathbf{r}) \quad ; \quad \tilde{p}_{\beta 2}(\mathbf{r} + \mathbf{l}_i) = \tilde{p}_{\beta 2}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \tag{A.18d}$$

$$\langle \tilde{\mathbf{v}}_{\beta 2} \rangle^\beta = 0 \quad ; \quad \langle \tilde{p}_{\beta 2} \rangle^\beta = 0 \tag{A.18e}$$

where the notation $\mathcal{A}^{T(1234)} = (\mathcal{A}_{ijkl} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l)^{T(1234)} = \mathcal{A}_{lkji} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l$ has been used for the transpose of a fourth order tensor \mathcal{A} . In this problem, three independent source terms are identified. The first one $\langle \mathbf{v}_{\beta 2} \rangle^\beta$ appears in the boundary condition at $A_{\beta\sigma}$ just as in the case of order 0 and 1. The second one $[\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 1} \rangle^\beta + \langle \mathbf{v}_{\beta 1} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta]$ is similar to the second source term of order 1. The third one $\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta$ is a new source term that appears in the momentum balance equation. This terms couple order 0 and 1 with the solution of the linearized deviations at order 2. Based on these observations, the following decomposition is introduced

$$\tilde{\mathbf{v}}_{\beta 2} = \mathbf{B} \cdot \langle \mathbf{v}_{\beta 2} \rangle^\beta + \mathcal{C} : \left(\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 1} \rangle^\beta + \langle \mathbf{v}_{\beta 1} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \right) + \mathcal{E} : \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \tag{A.19a}$$

$$\tilde{p}_{\beta 2} = \mathbf{b} \cdot \langle \mathbf{v}_{\beta 2} \rangle^\beta + \mathbf{C} : \left(\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 1} \rangle^\beta + \langle \mathbf{v}_{\beta 1} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \right) + \mathcal{E} : \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \tag{A.19b}$$

where \mathbf{B} , \mathbf{b} and \mathcal{C} and \mathbf{C} are the closure variables at order 0 and order 1 respectively. The new closure variables \mathcal{E} and \mathcal{E} verify the following closure problem

$$\nabla \cdot \mathcal{E} = 0 \quad \text{in } V_\beta \tag{A.20a}$$

$$\begin{aligned}
& \left[(\nabla \mathcal{C})^{T(1234)} \cdot (\mathbf{B} + \mathbf{I}) + (\nabla \mathbf{B})^{T(123)} \cdot \mathcal{C} \right] = \\
& -\nabla \mathcal{E} + \nabla^2 \mathcal{E} - \frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \mathcal{E} + \nabla \mathcal{E}) dA \quad \text{in } V_\beta
\end{aligned} \tag{A.20b}$$

$$\mathcal{E} = 0 \quad \text{on } A_{\beta\sigma} \tag{A.20c}$$

$$\mathcal{E}(\mathbf{r} + \mathbf{l}_i) = \mathcal{E}(\mathbf{r}) \quad ; \quad \mathcal{E}(\mathbf{r} + \mathbf{l}_i) = \mathcal{E}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \tag{A.20d}$$

$$\langle \mathcal{E} \rangle^\beta = 0 \quad ; \quad \langle \mathcal{E} \rangle^\beta = 0 \tag{A.20e}$$

The integral term is defined as

$$\varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{I}_\beta = -\frac{1}{V_\beta} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (-\mathbf{I} \mathcal{E} + \nabla \mathcal{E}) dA \tag{A.21}$$

Once again, \mathcal{I}_β can be interpreted as a correction tensor of the intrinsic permeability \mathbf{H}_β at order 2.

To solve this integro-differential problem, the following decomposition is introduced based on the superposition principle

$$\mathcal{E} = \mathcal{E}^I + \mathcal{T} : \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{I}_\beta \tag{A.22a}$$

$$884 \quad \mathcal{E} = \mathcal{E}' + \mathcal{T} : \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{J}_\beta \quad (\text{A.22b})$$

885 It follows that the closure problem can be decomposed into two purely differential closure problems

$$886 \quad \nabla \cdot \mathcal{E}' = 0 \quad \text{in } V_\beta \quad (\text{A.23a})$$

$$887 \quad \left[(\nabla \mathcal{C})^{T(1234)} \cdot (\mathbf{B} + \mathbf{I}) + (\nabla \mathbf{B})^{T(123)} \cdot \mathcal{C} \right] = -\nabla \mathcal{E}' + \nabla^2 \mathcal{E}' \quad \text{in } V_\beta \quad (\text{A.23b})$$

$$888 \quad \mathcal{E}' = 0 \quad \text{on } A_{\beta\sigma} \quad (\text{A.23c})$$

$$889 \quad \mathcal{E}'(\mathbf{r} + \mathbf{l}_i) = \mathcal{E}'(\mathbf{r}) \quad ; \quad \mathcal{E}'(\mathbf{r} + \mathbf{l}_i) = \mathcal{E}'(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.23d})$$

$$890 \quad \langle \mathcal{E}' \rangle^\beta = 0 \quad (\text{A.23e})$$

891 and

$$892 \quad \nabla \cdot \mathcal{T} = 0 \quad \text{in } V_\beta \quad (\text{A.24a})$$

$$893 \quad 0 = -\nabla \mathcal{T} + \nabla^2 \mathcal{T} + \mathcal{T} \quad \text{in } V_\beta \quad (\text{A.24b})$$

$$894 \quad \mathcal{T} = 0 \quad \text{on } A_{\beta\sigma} \quad (\text{A.24c})$$

$$895 \quad \mathcal{T}(\mathbf{r} + \mathbf{l}_i) = \mathcal{T}(\mathbf{r}) \quad ; \quad \mathcal{T}(\mathbf{r} + \mathbf{l}_i) = \mathcal{T}(\mathbf{r}) \quad ; \quad i = 1, 2, 3 \quad (\text{A.24d})$$

$$896 \quad \langle \mathcal{T} \rangle^\beta = 0 \quad (\text{A.24e})$$

897 Note that \mathcal{T} is solution of the exact same closure problem as Eqs. (A.13), so that $\mathcal{T} = \mathcal{S}$.

898 To compute the correction tensor at order 2, a similar strategy as at order 1 is used. By
899 definition of the intrinsic average (A.20e), one can write the equality

$$\begin{aligned} 900 \quad \langle \mathcal{E} \rangle^\beta = 0 &= \langle \mathcal{E}' \rangle^\beta + \langle \mathcal{T} \rangle^\beta : \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{J}_\beta \\ 901 &= \langle \mathcal{E}' \rangle^\beta + \varepsilon_\beta^{-1} \mathbf{K}_\beta \cdot \varepsilon_\beta \mathbf{K}_\beta^{-1} \cdot \mathcal{J}_\beta \\ 902 &= \langle \mathcal{E}' \rangle^\beta + \mathcal{J}_\beta \end{aligned} \quad (\text{A.25})$$

903 so that the value of \mathcal{J}_β is obtained from $\mathcal{J}_\beta = -\langle \mathcal{E}' \rangle^\beta$.

904 Total deviations

905 The total deviations $\tilde{\mathbf{v}}_\beta$ and \tilde{p}_β can be reconstructed from Eq. (A.1) using the mappings (A.3),
906 (A.8), and (A.19), giving up to order 2

$$\begin{aligned} \tilde{\mathbf{v}}_\beta &= \tilde{\mathbf{v}}_{\beta 0} + \tilde{\mathbf{v}}_{\beta 1} Re_p + \tilde{\mathbf{v}}_{\beta 2} Re_p^2 + \dots \\ &= \mathbf{B} \cdot \left[\langle \mathbf{v}_{\beta 0} \rangle^\beta + Re_p \langle \mathbf{v}_{\beta 1} \rangle^\beta + Re_p^2 \langle \mathbf{v}_{\beta 2} \rangle^\beta + \dots \right] \\ 907 \quad &+ \mathcal{C} : Re_p \left[\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta + Re_p \left(\langle \mathbf{v}_{\beta 1} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta + \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 1} \rangle^\beta \right) + \dots \right] \\ &+ \mathcal{E} : Re_p^2 \left[\langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta \langle \mathbf{v}_{\beta 0} \rangle^\beta + \dots \right] + \dots \end{aligned} \quad (\text{A.26})$$

908 and similarly for the pressure deviation. Higher order developments can show that the total devi-
909 ations are recursively recovered. Therefore, it is possible to simply group all these terms to obtain
910 a second order expansion of the total deviations with the total macroscopic source terms as follows

$$911 \quad \tilde{\mathbf{v}}_\beta = \mathbf{B} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathcal{C} : Re_p \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \mathcal{E} : Re_p^2 \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \quad (\text{A.27a})$$

$$\tilde{\rho}_\beta = \mathbf{b} \cdot \langle \mathbf{v}_\beta \rangle^\beta + \mathbf{C} : Re_p \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \mathcal{E} : Re_p^2 \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \quad (\text{A.27b})$$

Similarly to the non-linear case, it is possible to close the averaged equation (6b) by introducing the above linearized mapping of the spatial deviations and identifying the corrective effective tensors at each order. Identical closed averaged equations are obtained but the effective parameters are linearized around the Reynolds number. Up to order 2, the linearized global permeability tensor is

$$\mathbf{H}_\beta^{-1} = \mathbf{K}_\beta^{-1} \cdot \left(\mathbf{I} + Re_p \mathcal{H}_\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + Re_p^2 \mathcal{J}_\beta : \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \right) \quad (\text{A.28})$$

where the Forchheimer correction tensor can be directly identified as

$$\mathbf{F}_\beta = Re_p \mathcal{H}_\beta \cdot \langle \mathbf{v}_\beta \rangle^\beta + Re_p^2 \mathcal{J}_\beta : \langle \mathbf{v}_\beta \rangle^\beta \langle \mathbf{v}_\beta \rangle^\beta + \dots \quad (\text{A.29})$$

B Comparison of computation times of the closure problems

Here the computational costs of solving the linearized and non-linear closure problems are compared in the weak inertia regime where the linearized closure problem is valid ($Re_k \lesssim 1$). In two dimensions, the general non-linear closure requires to solve one non-linear vectorial problem (Eqs. (17)), plus $2^{2-1} = 2$ linear vector problems (Eqs. (16)) over a unit cell *for each value of the pore Reynolds number Re_p and the macroscopic pressure gradient orientation θ* . On the other hand, the linearized closure truncated at order 2 in Reynolds number introduces 3 linear tensor closure problems of increasing orders: at order 0 a second-order tensor problem (Eqs. (A.4)), at order 1 a third-order tensor problem (Eqs. (A.9)), and at order 2 a fourth-order tensor problem (Eqs. (A.20)). In total, in two dimensions, the numerical resolution of the linearized closure up to order 2 consists of solving $2^{2-1} + 2^{3-1} + 2^{4-1} = 14$ vector problems with Stokes-like structure over the same unit cell but without any dependency on the local macroscopic flow.

Computation times for numerical simulations of the non-linear and linearized closure problems up to order 2 were compared on the symmetric unit cell presented in Fig. 2a. The same mesh composed of 230,000 elements was used for all the simulations in order to ensure mesh independence on the numerical results, limit the propagation of numerical errors through the incremental linearized methodology and measure computation times for comparable numerical resources. Computations were performed in Comsol Multiphysics as described in the main text.

Numerical simulations of the general non-linear closure problem take approximately 200 seconds for each value of pore Reynolds numbers Re_p and macroscopic pressure gradient orientation θ . In contrast, solving the 14 vector problems of the linearized closure truncated at order 2 take approximately 1800 seconds. However these are valid for any (Re_p, θ) within the domain of validity of the linear problem. Therefore solving the linearized closure problem truncated at order 2 costs as much as solving the non-linear problem for $1880\text{s}/200\text{s} = 9$ values of the couples of parameter (Re_p, θ) .

The use of the linearized closure problem is interesting when more than 10 points are needed for the tabulation of the Forchheimer correction tensor. For instance, the results presented in Fig. 3 for a unit cell with solid square obstacle and porosity $\varepsilon_\beta = 0.75$ required approximately 50 computations of the non-linear closure problem with different values of (Re_p, θ) satisfying $Re_k < 0.1$, thus making the linearized closure much more cost efficient in this special case.