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Atsnaita Yasrina, Aziz Albar Rofi Uddarojad, Arief Hermanto, Muhammad Farchani Rosyid

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ATSNAITA YASRINA

Department of Physics, Universitas Gadjah Mada Sekip Utara Bulaksumur, Yogyakarta 55281, Indonesia
Department of Physics, Universitas Negeri Malang Jalan Semarang No. 05, Malang 65145, Indonesia
atsnaita.yasrina.fmipa@mail.ugm.ac.id

AZIZ ALBAR ROFI UDDAROJAD

Department of Physics, Universitas Gadjah Mada Sekip Utara Bulaksumur, Yogyakarta 55281, Indonesia
azizrofiuddarojad@gmail.com

ARIEF HERMANTO

Department of Physics, Universitas Gadjah Mada Sekip Utara Bulaksumur, Yogyakarta 55281, Indonesia
arief_hermanto@ugm.ac.id

MUHAMMAD FARCHANI ROSYID

Department of Physics, Universitas Gadjah Mada Sekip Utara Bulaksumur, Yogyakarta 55281, Indonesia
farchani@ugm.ac.id

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The covariant and non-covariant field equations involving cosmological constant and electromagnetic fields in Teleparallel Equivalent of General Relativity (TEGR) have been derived. The solution of the field equation in the presence of magnetic charge has been obtained. The covariant formulation of teleparallel gravity for the magnetic and electric black hole have been obtained. The singularities of the solution in covariant formulation have been analyzed using some scalars obtained from the torsion and the curvature tensor associated to the solution. The gravitoelectromagnetic fields of both magnetic and electric black hole have been calculated and discussed.

Keywords: Teleparallel gravity; covariant formulation; gravitoelectromagnetism; magnetic and electric black hole.

1. Introduction

Some alternative formulations for the theory of gravity as well as modifications to the general theory of relativity have been proposed [1–6]. The motivations to propose alternative formulations of gravity and to modify general relativity include the attempts to explain various discourses and observations in gravity, such as quantum gravity, cosmological dark energy, and cosmological dark matter, which cannot be explained by using general relativity. One alternative theory of gravity which is paradigmatically different from Einstein's general relativity is the teleparallel theory of gravity. In the theory, the gravity is not regarded as the manifestation of spacetime geometry, but it is regarded as a field of force. In this theory, tetrad fields instead of metric tensors play the role of fundamental fields with the assumption of absolute parallelism [7–11]. The theory involves the Weitzenböck connection. In the so-called pure tetrad approach, the curvature of the connection is zero, but not its torsion. The torsion of the connection describes the gravitational field strength as a substitute for curvature in the general theory of relativity [12]. Teleparallel gravity provides a new framework or perspective for observing and interpreting gravitational phenomena that are different from the way of observing and interpreting gravitational phenomena according to general relativity. Teleparallel gravity is phenomenologically and conceptually consistent [13].

For a specific choice of parameters, teleparallel gravity is equivalent to Einstein's general relativity [6]. This version of teleparallel gravity is referred to as the Teleparallel Equivalent of General Relativity (TEGR) [14]. However, despite being equivalent, the two approaches are conceptually very different [6]. In general relativity, the Levi-Civita connection includes both gravitational and inertial effects. From the point of view of TEGR, gravitational field is regarded as translational gauge field and the effects of inertia are represented by the spin connections which depends on the frame of reference. If we restrict our observation to the inertial frames, then the spin connection must vanish [6]. The implication of ignoring the spin connection is that the formulation is not covariant under local Lorentz transformations. If the spin connection is assumed to be zero, then the tetrad appears as the only field variable. Such approach is called pure tetrad teleparallel gravity or non-covariant teleparallel gravity [6].

Using a suitable choice of spin connestions, we obtaind a fully covariant theory of teleparallel gravity which is covariant under coordinate and local Lorentz transformations [6]. Such formulation for instance is proposed by Krššák and is called the covariant teleparallel gravity. The advantage of the covariant formulation is that study of gravity is not restricted to a particular frame of reference. In the procedure proposed by Krššák, this purely inertial spin connection is obtained from a so-called reference tetrad in which the gravity is "switched off", for instance by setting the universal constant G of gravitation equal to zero. However, there are some criticisms against the method and shortcomings in some cases. For example, in the case of the closed Friedmann–Lemaître–Robertson–Walker (FRLW) cosmology, the

curvature components are not equal to the curvature components yielded in inertial spin connection [15]. Another example is that for the same metric tensor, different tetrads (especially non-diagonal tetrads) produce the same reference tetrad so that the spurious inertial effect is lost not only in one proper frame [16]. However, the method proposed by Krššák for other cases is still acceptable.

While, the general theory of relativity can explain various natural phenomena, in particular around astrophysical objects which are classified as compact objects in astrophysics, TEGR should also be able to explain such natural phenomena. Studies on compact objects especially neutron stars and black holes in the contexts of TEGR have been widely carried out [17–26]. Such studies for instance are to obtain solutions and energy condition as well as to understand the physical properties and singularities. Among the works mentioned above the solutions of the magnetic and electric black hole have gained great attentions, for instance in [24] and [27]. However, the approach taken in the works [24] and [27] are still in non-covariant formulation of TEGR. Thus, the applications of the results obtained there are restricted only to inertial frames of reference. In order to ensure that the application of the results obtained are not limited to the inertial frame of reference, it is necessary to study the problems in the covariance approach. Therefore, the aims of our work here is to study the possibility of formulating the problems proposed and solved in [24] and [27] in the covariant fashion of TEGR. We will investigate physical systems (magnetic and electric black holes) studied for instance in [24] and [27], but in the covariant formulation. We derive the covariant field equations for such systems, find their solutions, and analyze the singularities of the magnetic and electric black holes.

A Black hole is described by their mass, angular momentum and charge (electric and/or magnetic) [28], which is khown as the no-hair theorem [29]. Due to the small charge of a black hole, most researchers ignore the charge and emphasise the mass and angular momentum of black hole [30, 31]. However, the small charge of a black hole can significantly affect the motion of charged particles, especially cosmic rays around the black hole [30]. In addition, it is important not to overlook the charge of a black hole, as they can absorb surrounding matter, including charged matter [32]. Charged black holes also provide an opportunity for new physics findings with gravitational waves [31]. They are also considered as the regular black hole models [33, 34] use to explain the concept of singularity [35]. In the context of Einstein-Maxwell equations charged black holes play some important role [24], since they provide easy example.

The similarity between the Coloumb force in electromagnetism and the gravitational force gave rise to the idea of gravitoelectromagnetism. In the context of the general theory of relativity, Einstein and several physicists have discussed the derivation of gravitoelectromagnetism through a review of perturbations in Minkowski spacetime [36]. Gravitoelectromagnetism is also formulated in the context of TEGR [37, 38] as a gauge theory based on the strong component of the gauge field [37]. As in electromagnetism where Maxwell's equations consist of electric and

magnetic fields, in gravitoelectromagnetism, there are Maxwell-like equations consist of gravitoelectric and gravitomagnetic fields (gravitational electric and magnetic fields) [37, 38]. Spaniol and Andrade (2010) have derived two equations which are similar to Gauss' law and Ampere's law in electromagnetism [37]. While the complete Maxwell-like equations have been obtained by Ming at all [38]. They identified the spatial components of the torsion tensor fields in Cartesian coordinate system as gravitoelectric and gravitomagnetic fields based on the similarity between the kinetic part of the Lagrangian density of the teleparallel gravity and the kinetic part of the Lagrangian density of electromagnetism as well as the similarity between the expression of the torsion and of electromagnetic strength field [38]. They investigated gravitoelectromagnetism in some astrophysical casess [37], such as the gravitoelectromagnetic fields corresponding to the Schwarzschild metric [37, 38], slowly rotating spherical shells [37], and gravitational waves [38] in non-covariant formulations. Here, we will also investigate gravitoelectromagnetism corresponding to the magnetic and electric black holes studied in [24, 27].

This article contains five sections. In Sect. 2 we will review TEGR and formulate the covariant field equation concerning magnetic black hole using the method proposed by Krššák. In Sects. 3 and 4 we will analyze the solutions of our field equation, discuss the singularities of the solutions, and formulate the gravitoelectromagnetism for the electric and magnetic black holes.

2. The Formulation of the Covariant Field Equation in Teleparallel Gravity

H. Weyl (1918), Th. Kaluza and O. Klein (1920) and Einstein (ten years later) sought to unify gravity and electromagnetics [13]. Although these efforts were unsuccessful, the ideas of unification remain essential today. Einstein proposed the idea of unification based on the mathematical structure of teleparallelism by introducing the tetrad fields. A tetrad is an orthonormal basis field in the tangent space at each point of four-dimensional spacetime. Later, Cartan, with the Eintein-Cartan theory, developed a modification of the general theory of relativity with spacetime having curvature and torsion. Hayashi and Shirafuji also proposed the so-called new general relativity.

Describing gravitational interactions requires curvature and torsion which corresponds to a connection. In general relativity the connection involved in the theory has vanishing torsion. In teleparallel gravity, the connection involved has vanishing curvature. General relativity and teleparallel gravity are conceptually different but produce equivalent descriptions of gravitational interactions. A direct consequence of this equivalence is that curvature and torsion are alternative ways of describing the gravitational field [13].

Here will use the Greek alphabets $\mu, \nu, \rho, ... = 0, 1, 2, 3$ as spacetime indices, while for the tangent space indices we will use the Latin alphabets $a, b, c, ... = 0, 1, 2, 3$. Here, a quantities with the symbol "$\circ$" over them are related to general

relativity, and all quantities with the symbol "$\bullet$" over them to teleparallel gravity. In teleparallel gravity, the fundamnetal field is a tetrad field ($h^a{}_\mu$) describing gravitational field of the form

$$h^a{}_\mu \equiv \partial_\mu x^a + B^a{}_\mu, \tag{1}$$

with

$$B_\mu = B^a{}_\mu P_a \tag{2}$$

is the gauge potential and $P_a = \frac{\partial}{\partial x^a}$ is the translational generators.

Every point in spacetime with a metric $g_{\mu\nu}$ has a tangent space attached to it. The tangent space is defined as Minkowski spacetime with the tangent space metric being $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$. The tetrad field connects the metrics $g_{\mu\nu}$ and η_{ab} according to

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu. \tag{3}$$

The connection in teleparallel gravity is the Weitzenböck connection ($\overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu}$) given by

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu} = h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho \overset{\bullet}{\omega}{}^a{}_{b\mu} h^b{}_\nu, \tag{4}$$

where $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ is the inertial spin connection. The torsion of the Weitzenböck connection is formally given by

$$\overset{\bullet}{T}{}^\rho{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu} - \overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu}. \tag{5}$$

The torsion of the Weitzenböck connection in the *non-covariant* formulation is expressed in term of tetrad field as [12]

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} \equiv \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu. \tag{6}$$

It is well known fact that the Weitzenböck and Levi-Civita connections are related by

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu} - \overset{\bullet}{K}{}^\rho{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu}, \tag{7}$$

where $\overset{\bullet}{K}{}^\rho{}_{\mu\nu}$ is the contortion of the connection defined by

$$\overset{\bullet}{K}{}^\rho{}_{\mu\nu} = -\frac{1}{2}\left(\overset{\bullet}{T}{}^{\mu\nu}{}_\alpha - \overset{\bullet}{T}{}^{\nu\mu}{}_\alpha - \overset{\bullet}{T}{}_\alpha{}^{\mu\nu}\right). \tag{8}$$

Furthermore, the superpotentials of the Weitzenböck connection is defined by

$$\overset{\bullet}{S}{}_\alpha{}^{\mu\nu} = \frac{1}{2}\left(\overset{\bullet}{K}{}^{\mu\nu}{}_\alpha + \delta^\mu_\alpha \overset{\bullet}{T}{}^{\beta\nu}{}_\beta - \delta^\nu_\alpha \overset{\bullet}{T}{}^{\beta\mu}{}_\beta\right), \tag{9}$$

and the curvature of the Weitzenböck connection by

$$\overset{\bullet}{R}{}^\rho{}_{\theta\mu\nu} = \partial_\mu \overset{\bullet}{\Gamma}{}^\rho{}_{\theta\nu} - \partial_\nu \overset{\bullet}{\Gamma}{}^\rho{}_{\theta\mu} + \overset{\bullet}{\Gamma}{}^\rho{}_{\sigma\mu} \overset{\bullet}{\Gamma}{}^\sigma{}_{\theta\nu} - \overset{\bullet}{\Gamma}{}^\rho{}_{\sigma\nu} \overset{\bullet}{\Gamma}{}^\rho{}_{\theta\mu} \tag{10a}$$

$$= \partial_\nu \overset{\bullet}{\omega}{}^a{}_{b\mu} - \partial_\mu \overset{\bullet}{\omega}{}^a{}_{b\nu} + \overset{\bullet}{\omega}{}^a{}_{e\nu} \overset{\bullet}{\omega}{}^e{}_{b\mu} - \overset{\bullet}{\omega}{}^a{}_{b\mu} \overset{\bullet}{\omega}{}^e{}_{b\nu}. \tag{10b}$$

From equation (4), if the spin connection $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$ vanishes then the curvature $\overset{\bullet}{R}{}^{\rho}{}_{\theta\mu\nu}$ also vanishes. The Weitzenböck connection $(\overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu})$ is a connection with non-zero torsion and zero curvature. It implies the condition of absolute parallelism where the Weitzenböck covariant derivative of tetrad fields vanishes,

$$\overset{\bullet}{\nabla} h^{a}{}_{\mu} \equiv \partial_{\nu} h^{a}{}_{\mu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} h^{a}{}_{\rho} = 0. \tag{11}$$

If the expression of Levi-Civita conection $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}$ in Equation (7) is subtituted in the Riemannian curvature tensor

$$\overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu} = \partial_{\mu}\overset{\circ}{\Gamma}{}^{\rho}{}_{\theta\nu} - \partial_{\nu}\overset{\circ}{\Gamma}{}^{\rho}{}_{\theta\mu} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\mu}\overset{\circ}{\Gamma}{}^{\sigma}{}_{\theta\nu} - \overset{\circ}{\Gamma}{}^{\rho}{}_{\sigma\nu}\overset{\circ}{\Gamma}{}^{\rho}{}_{\theta\mu} \tag{12}$$

then we obtain the following equation [13]

$$\overset{\bullet}{R}{}^{\rho}{}_{\theta\mu\nu} = \overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu} + \overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu}, \tag{13}$$

where

$$\begin{aligned}\overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu} = {} & \partial_{\mu}\overset{\bullet}{K}{}^{\rho}{}_{\theta\nu} - \partial_{\nu}\overset{\bullet}{K}{}^{\rho}{}_{\theta\mu} + \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\mu}\overset{\bullet}{K}{}^{\sigma}{}_{\theta\nu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{K}{}^{\rho}{}_{\theta\mu} - \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\mu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu} \\ & + \overset{\bullet}{\Gamma}{}^{\sigma}{}_{\theta\mu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{K}{}^{\rho}{}_{\sigma\nu}\overset{\bullet}{K}{}^{\sigma}{}_{\theta\mu} - \overset{\bullet}{K}{}^{\rho}{}_{\sigma\mu}\overset{\bullet}{K}{}^{\sigma}{}_{\theta\nu}.\end{aligned} \tag{14}$$

Whenever the curvature $\overset{\bullet}{R}{}^{\rho}{}_{\theta\mu\nu}$ of the Weitzenböck connection vanishes, then we have

$$-\overset{\circ}{R}{}^{\rho}{}_{\theta\mu\nu} = \overset{\bullet}{Q}{}^{\rho}{}_{\theta\mu\nu}. \tag{15}$$

In this case, from equation (14) and equation (15), the scalar $\overset{\bullet}{Q}$ therefore is given by [13]

$$\overset{\bullet}{Q} = -\overset{\circ}{R} = \overset{\bullet}{K}{}^{\mu\nu\rho}\overset{\bullet}{K}{}_{\rho\nu\mu} - \overset{\bullet}{K}{}^{\mu\rho}{}_{\mu}\overset{\bullet}{K}{}^{\nu\rho}{}_{\nu} + \frac{2}{h}\partial_{\mu}\left(h\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right). \tag{16}$$

2.1. *Covariant formulation of teleparallel gravity*

A nontrivial field or anholonomic tetrad is of the form [6]

$$h^{a}{}_{\mu} = \partial_{\mu} x^{a} + \overset{\bullet}{\omega}{}^{a}{}_{b\mu} x^{b} + B^{a}{}_{\mu}. \tag{17}$$

The difference between equation (17) and equation (1) is that there is a second term $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}x^{b}$, where $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$ is the inertial spin connection. In general relativity, the spin connection $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$ includes both gravitational and inertial effects. However, in teleparallel gravity $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$ represents only inertial effects, while the contorsion $\overset{\bullet}{K}{}^{a}{}_{b\mu}$ represents only gravity [39] with

$$\overset{\bullet}{\omega}{}^{a}{}_{b\mu} - \overset{\bullet}{K}{}^{a}{}_{b\mu} = \overset{\circ}{\omega}{}^{a}{}_{b\mu}. \tag{18}$$

The torsion corresponding to the tetrad appearing (17) is given by

$$\overset{\bullet}{T}{}^{a}{}_{\mu\nu} \equiv \partial_\mu h^{a}{}_{\nu} - \partial_\nu h^{a}{}_{\mu} + \overset{\bullet}{\omega}{}^{a}{}_{b\mu} h^{b}{}_{\nu} - \overset{\bullet}{\omega}{}^{a}{}_{b\nu} h^{b}{}_{\mu}. \tag{19}$$

Equation (19) shows that the teleparallel gravitational field strength is just the torsion tensor.

In teleparallel gravity, each tetrad is associated with a spin connection. Therefore, there exists a frame of reference characterized by pairs of tetrads and spin connections $(h^{a}{}_{\mu}, \overset{\bullet}{\omega}{}^{a}{}_{b\mu})$. One of the methods to obtain the spin connection of a tetrad was proposed by Krššák [6]. The methods involves finding the reference tetrad $({}^{(\mathrm{r})}h^{a}{}_{\mu})$ in which the spin connection is given by

$$\overset{\bullet}{\omega}{}^{a}{}_{b\mu} = \frac{1}{2}{}^{(\mathrm{r})}h^{a}{}_{\mu}\left(f_{b}{}^{a}{}_{c}({}^{(\mathrm{r})}h) + f_{c}{}^{a}{}_{b}({}^{(\mathrm{r})}h) - f^{a}{}_{bc}({}^{(\mathrm{r})}h)\right), \tag{20}$$

where the reference tetrad $({}^{(\mathrm{r})}h^{a}{}_{\mu})$ is the tetrad obtained by equating the gravitational constant G to zero (or turning the gravity off)

$${}^{(\mathrm{r})}h^{a}{}_{\mu} \equiv h^{a}{}_{\mu}|_{G\longrightarrow 0}, \tag{21}$$

and $f^{c}{}_{ab}({}^{(\mathrm{r})}h)$ is the coefficients of anholonomy

$$f^{a}{}_{bc}({}^{(\mathrm{r})}h) = {}^{(\mathrm{r})}h^{a}{}_{\mu}\ {}^{(\mathrm{r})}h^{b}{}_{\nu}\left(\partial^{(\mathrm{r})}_{\nu} h^{c}{}_{\mu} - \partial^{(\mathrm{r})}_{\mu} h^{c}{}_{\nu}\right). \tag{22}$$

2.2. *The Formulation of Covariant Field Equations*

The action of teleparallel gravity is given by

$$\begin{aligned}\overset{\bullet}{\mathcal{I}} &= \frac{1}{16\pi G}\int \mathrm{Tr}\left(\overset{\bullet}{T}\wedge\star\overset{\bullet}{T}\right)\\ &= \frac{h}{16\pi G}\int\left(\frac{1}{4}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\lambda\mu\nu} + \frac{1}{2}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\mu\lambda\nu} - \overset{\bullet}{T}^{\nu}{}_{\mu\nu}\overset{\bullet}{T}^{\lambda\mu}{}_{\lambda}\right) h d^4x,\end{aligned} \tag{23}$$

where G is the universal gravitational constant (in the unit system with $c = 1$), and h the determinant of the tetrad $h^{a}{}_{\mu}$. Therefore, the Lagrangian density of the action is of the form

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{16\pi G}\left(\frac{1}{4}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\lambda\mu\nu} + \frac{1}{2}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\mu\lambda\nu} - \overset{\bullet}{T}^{\nu}{}_{\mu\nu}\overset{\bullet}{T}^{\lambda\mu}{}_{\lambda}\right). \tag{24}$$

Equation (24) can be written in torsion scalars by using the definitions of the contortion tensor (equation (8)) and superpotential (equation (9)) as follows

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{16\pi G}\left(\overset{\bullet}{T}{}^{\alpha}{}_{\mu\nu}\overset{\bullet}{S}_{\alpha}{}^{\mu\nu}\right) = \frac{h}{16\pi G}\overset{\bullet}{T}, \tag{25}$$

where the torsion scalar $\overset{\bullet}{T}$ is defined as [13]

$$\overset{\bullet}{T} = \frac{1}{4}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\lambda\mu\nu} + \frac{1}{2}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\mu\lambda\nu} - \overset{\bullet}{T}^{\nu}{}_{\mu\nu}\overset{\bullet}{T}^{\lambda\mu}{}_{\lambda}. \tag{26}$$

If the physical effect related to the cosmological constant and electromagnetic field are taken into account the action in teleparallel gravity becomes [24]

$$\begin{aligned}\overset{\bullet}{\mathcal{I}}(h_{i\mu}) &= \int \left(\frac{1}{16\pi G}\left(\overset{\bullet}{T}-2\Lambda\right)-\frac{1}{2}F\wedge\star F\right) h\, d^4x \\ &= \int \left(\frac{1}{16\pi G}\left(\overset{\bullet}{T}-2\Lambda\right)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) h\, d^4x, \end{aligned} \tag{27}$$

where $\mathcal{L}_{em} = -\frac{1}{2}F\wedge\star F$ is the Lagrangian of the electromagnetic field. The corresponding Lagrangian density for the action $\overset{\bullet}{\mathcal{I}}(h_{i\mu})$ equation (27) is given by

$$\begin{aligned}\overset{\bullet}{\mathcal{L}} &= \frac{h}{16\pi G}\left(\overset{\bullet}{T}-2\Lambda\right)-\frac{h}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{h}{16\pi G}\left(\frac{1}{4}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\lambda\mu\nu}+\frac{1}{2}\overset{\bullet}{T}_{\lambda\mu\nu}\overset{\bullet}{T}^{\mu\lambda\nu}-\overset{\bullet}{T}^{\nu}{}_{\mu\nu}\overset{\bullet}{T}^{\lambda\mu}{}_{\lambda}\right). \end{aligned} \tag{28}$$

Teleparallel gravitational field equations are derived through the Euler-Lagrange equations

$$\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho}-\partial_\sigma\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial\left(\partial_\sigma h^a{}_\rho\right)}=0. \tag{29}$$

The superpotential $(\overset{\bullet}{S}_a{}^{\rho\sigma})$ given in equation (9) can be derived from the Lagrangian density through

$$\overset{\bullet}{S}_a{}^{\rho\sigma}=-\overset{\bullet}{S}_a{}^{\sigma\rho}\equiv\frac{k}{h}\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial\left(\partial_\sigma h^a{}_\rho\right)}. \tag{30}$$

Furthermore, the Noether gravitational energy-momentum flow $(\overset{\bullet}{J}_a{}^{\rho})$ is defined as

$$\overset{\bullet}{J}_a{}^{\rho}\equiv-\frac{1}{h}\frac{\partial\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho}. \tag{31}$$

The formulation of the field equations requires explicit expressions of the superpotential $\overset{\bullet}{S}_a{}^{\rho\sigma}$ (equation (30)) and Noether gravitational energy-momentum current $\overset{\bullet}{J}_a{}^{\rho}$ (equation (31)). We require also the expression of the torsion given in equation (19). We obtain the momentum flow of Noether gravitational energy as follows

$$\begin{aligned}\overset{\bullet}{J}_a{}^{\rho} =& \frac{1}{2k}\left(4\overset{\bullet}{S}_c{}^{\rho\sigma}\overset{\bullet}{\omega}^c{}_{a\sigma}-4\overset{\bullet}{h}_a{}^{\lambda}\overset{\bullet}{S}_c{}^{\rho\nu}\overset{\bullet}{T}^c{}_{\nu\lambda}\right)-\frac{\overset{\bullet}{h}_a{}^{\rho}}{2k}\left(\overset{\bullet}{T}-2\Lambda\right) \\ &-\overset{\bullet}{h}_a{}^{\mu}g_{\rho\sigma}F^{\mu\rho}F_\nu{}^{\sigma}+\frac{1}{4}h_a{}^{\nu}g^{\lambda\rho}g^{\epsilon\sigma}F_{\lambda\epsilon}F_{\rho\sigma}. \end{aligned} \tag{32}$$

Then, equations (9) and (32) are subsumed intothe Euler Lagrange equation (29) to obtain the covariant field equations involving the cosmological constants and the

electromagnetic fields, namely

$$I_{\nu\mu} = h^{-1}h^{i}{}_{\mu}\partial_{\rho}\left(hh_{i}{}^{\alpha}\overset{\bullet}{S}{}_{\alpha}{}^{\rho\nu}\right) - \overset{\bullet}{S}{}_{\alpha}{}^{\rho\nu}\overset{\bullet}{\omega}{}^{\alpha}{}_{\mu\rho} - \overset{\bullet}{T}{}^{\alpha}{}_{\lambda\mu}\overset{\bullet}{S}{}_{\alpha}{}^{\nu\lambda} - \frac{1}{4}\delta^{\nu}_{\mu}\left(\overset{\bullet}{T} - 2\Lambda\right) + 4\pi G\left(-F^{\mu\rho}F_{\nu\rho} + \frac{1}{4}\delta^{\nu}_{\mu}F^{\rho\sigma}F_{\rho\sigma}\right) = 0, \quad (33)$$

If the spin connection $\overset{\bullet}{\omega}{}^{\alpha}{}_{\mu\rho}$ vanishes, the term $\overset{\bullet}{S}{}_{\alpha}{}^{\rho\nu}\overset{\bullet}{\omega}{}^{\alpha}{}_{\mu\rho}$ also vanishes and we obtained the non-covariant field equation

$$I_{\nu\mu} = h^{-1}h^{i}{}_{\mu}\partial_{\rho}\left(hh_{i}{}^{\alpha}\overset{\bullet}{S}{}_{\alpha}{}^{\rho\nu}\right) - \overset{\bullet}{T}{}^{\alpha}{}_{\lambda\mu}\overset{\bullet}{S}{}_{\alpha}{}^{\nu\lambda} - \frac{1}{4}\delta^{\nu}_{\mu}\left(\overset{\bullet}{T} - 2\Lambda\right) + 4\pi G\left(-F^{\mu\rho}F_{\nu\rho} + \frac{1}{4}\delta^{\nu}_{\mu}F^{\rho\sigma}F_{\rho\sigma}\right) = 0. \quad (34)$$

2.3. *Gravitoelectromagnetism in teleparallel gravity*

The first term of the vacuum Lagrangian density is of the same form of the Lagrangian density of electromagnetic field except the summation in internal (tangential) space, where the torsion tensor fields $\overset{\bullet}{T}{}^{a}{}_{\mu\nu}$ and electromagnetic field tensor $F_{\mu\nu}$ are in the same place. From Eq. (5), the tetrad fields $h^{a}{}_{\mu}$ can be regarded as the potentials of the theory and the torsion fields $\overset{\bullet}{T}{}^{a}{}_{\mu\nu}$ as the strength of field. Ming, et.al. [38] have proposed the following form of the torsion field

$$\overset{\bullet}{T}{}^{a}{}_{\mu\nu} = \begin{pmatrix} 0 & E^{a}{}_{x} & E^{a}{}_{y} & E^{a}{}_{z} \\ -E^{a}{}_{x} & 0 & -B^{a}{}_{z} & B^{a}{}_{y} \\ -E^{a}{}_{y} & B^{a}{}_{z} & 0 & -B^{a}{}_{x} \\ -E^{a}{}_{z} & -B^{a}{}_{y} & B^{a}{}_{x} & 0 \end{pmatrix}, \quad (35)$$

where with $E^{a}{}_{i}$ and $B^{a}{}_{i}$ $(a = 0, 1, 2, 3)$ and $(i = 1, 2, 3)$ are the gravitoelectric and gravitomagnetic fields respectively. In equation (35) the gravitoelectric and gravitomagnetic field components are written in Cartesian coordinates.

3. Magnetic Black Holes

3.1. *Exact solutions*

The way to understand the general theory of relativity and TEGR is to obtain the solution of equations concerning black holes, especially magnetic black holes [24]. A magnetic black hole with cylindrical spacetime symmetry describes a black hole with a flat horizon. The metric used to describe flat horizon spacetime in the cylindrical coordinate system is of the form

$$ds^2 = A(r)\, dt^2 - \frac{1}{A_1(r)}dr^2 - r^2 d\phi^2 - r^2 dz^2, \quad (36)$$

where $A(r)$ and $A_1(r)$ are functions to be determined. The tetrad chosen here is the diagonal, namely

$$h_{a\mu} = \text{diag}\left(\sqrt{A}, \frac{1}{\sqrt{A_1}}, r, r\right), \tag{37}$$

with the various index variations being

$$h^a{}_\mu = \text{diag}\left(\sqrt{A}, -\frac{1}{\sqrt{A_1}}, -r, -r\right), \tag{38a}$$

and

$$h_a{}^\mu = \text{diag}\left(\frac{1}{\sqrt{A}}, -\sqrt{A_1}, -\frac{1}{r}, -\frac{1}{r}\right). \tag{38b}$$

The tetrad determinant of equation (37) is clearly

$$\det h_{a\mu} = h = r^2\sqrt{\frac{A}{A_1}}. \tag{39}$$

The field equations involving cosmological constant and electromagnetic field in non-covariant formulation are obtained by subtituting the expression of the components of the torsion tensor $\overset{\bullet}{T}{}^\alpha{}_{\mu\nu}$, the superpotential tensor $\overset{\bullet}{S}{}^\alpha{}_{\mu\nu}$, the torsion scalar $\overset{\bullet}{T}$, and of the electromagnetic tensor $F_{\mu\nu}$ into equation (34). The components of torsion $\overset{\bullet}{T}{}^a{}_{\mu\nu}$ that are not equal to zero are obtained using equation (6), namely

$$\overset{\bullet}{T}{}^0{}_{01} = -\overset{\bullet}{T}{}^0{}_{10} = -\frac{A_{,r}}{2\sqrt{A}}, \tag{40a}$$

and

$$\overset{\bullet}{T}{}^2{}_{12} = -\overset{\bullet}{T}{}^2{}_{21} = \overset{\bullet}{T}{}^3{}_{13} = -\overset{\bullet}{T}{}^3{}_{31} = -1. \tag{40b}$$

The nonzero torsion components $\overset{\bullet}{T}{}^\alpha{}_{\mu\nu}$ are

$$\overset{\bullet}{T}{}^0{}_{01} = -\overset{\bullet}{T}{}^0{}_{10} = -\frac{A_{,r}}{2A}, \tag{41a}$$

and

$$\overset{\bullet}{T}{}^2{}_{12} = -\overset{\bullet}{T}{}^2{}_{21} = \overset{\bullet}{T}{}^3{}_{13} = -\overset{\bullet}{T}{}^3{}_{31} = -\frac{1}{r}. \tag{41b}$$

The components of the superpotential $\overset{\bullet}{S}{}^\alpha{}_{\mu\nu}$ that do not vanish are obtained using equation (9), namely

$$\overset{\bullet}{S}{}^0{}_{01} = -\overset{\bullet}{S}{}^0{}_{10} = -\frac{A_1}{r}, \tag{42a}$$

and

$$\overset{\bullet}{S}{}^2{}_{12} = -\overset{\bullet}{S}{}^2{}_{21} = \overset{\bullet}{S}{}^3{}_{13} = -\overset{\bullet}{S}{}^3{}_{31} = \frac{A_1}{4rA}\left(rA_{,r} + 2A\right). \tag{42b}$$

Furthemore, the torsion scalar $\overset{\bullet}{T}$ obtained using equation (25) is given by

$$\overset{\bullet}{T} = \frac{2A_1}{r^2 A}\left(rA_{,r} + A\right). \tag{43}$$

Here, we adopt the generalized gauge potential for a magnetic black hole with cylindrical symmetry from [24], i.e.

$$v = \left(a(\phi) + a_1(z)\right)dr + \left(b(z) + b_1(r)\right)d\phi + \left(s(\phi) + s_1(r)\right)dz. \tag{44}$$

Therefore the nonzero components of the electromagnetic field strength tensor $F_{\mu\nu}$ and $F^{\mu\nu}$ are given by

$$F_{12} = F_{21} = (b_1(r))_{,r} - (a(\phi))_{,\phi}\,, \tag{45a}$$
$$F_{13} = F_{31} = (s_1(r))_{,r} - (a_1(z))_{,z}\,, \tag{45b}$$
$$F_{23} = F_{32} = (s(\phi))_{,\phi} - (b(z))_{,z}\,, \tag{45c}$$
$$F^{12} = F^{21} = \frac{A_1}{r^2}(b_1(r))_{,r} - (a(\phi))_{,\phi}\,, \tag{45d}$$
$$F^{13} = F^{31} = \frac{A_1}{r^2}(s_1(r))_{,r} - (a_1(z))_{,z}\,, \tag{45e}$$

and

$$F_{23} = F_{32} = \frac{1}{r^4}(s(\phi))_{,\phi} - (b(z))_{,z}\,. \tag{45f}$$

The non-covariant field components are obtained by subsuming the torsion $\overset{\bullet}{T}{}^{\alpha}{}_{\mu\nu}$, the superpotential $\overset{\bullet}{S}{}^{\alpha}{}_{\mu\nu}$, the electromagnetic field tensor $F_{\mu\nu}$ and $F^{\mu\nu}$, and the

torsion scalar $\overset{\bullet}{T}$ to equation (34). Therefore, we obtain

$$I_{12} = I_{21} = \frac{4\pi G}{r^2}\left(s(\phi)_{,\phi} - b(z)_{,z}\right)\left(s_1(r)_{,r} - a_1(z), z\right) = 0, \tag{46a}$$

$$I_{13} = I_{31} = \frac{4\pi G}{r^2}\left(s(\phi)_{,\phi} - b(z)_{,z}\right)\left(b_1(r)_{,r} - a(\phi)_{,\phi}\right) = 0, \tag{46b}$$

$$I_{23} = I_{31} = 4\pi G A_1\left(s_1(r)_{,r} - a_1(z)_{,z}\right)\left(b_1(r)_{,r} - a(\phi)_{,\phi}\right) = 0, \tag{46c}$$

$$\begin{aligned} I_{00} =& \frac{A_{1,r}}{2r} + \frac{A_1}{2r^2} + \frac{\Lambda}{2} + \frac{2\pi G}{r^2}\Big[A_1\left(b_1(r)_{,r} - a(\phi)_{,\phi}\right)^2 + A_1\left(s_1(r)_{,r}\right. \\ & \left. - a_1(z)_{,z}\right)^2 + \frac{1}{r^2}\left(s(\phi)_{,\phi} - b(z)_{,z}\right)^2\Big] = 0, \end{aligned} \tag{46d}$$

$$\begin{aligned} I_{11} =& -\frac{A_{,r}}{2rA} - \frac{1}{2r^2} - \frac{\Lambda}{2A_1} + \frac{2\pi G}{r^2}\Big[\left(b_1(r)_{,r} - a(\phi)_{,\phi}\right)^2 + \left(s_1(r)_{,r}\right. \\ & \left. - a_1(z)_{,z}\right)^2 - \frac{1}{r^2 A_1}\left(s(\phi)_{,\phi} - b(z)_{,z}\right)^2\Big] = 0, \end{aligned} \tag{46e}$$

$$\begin{aligned} I_{22} =& -\frac{r^2 A_1}{4A}A_{,rr} + \frac{r^2 A_1}{8A^2}(A_{,}r)^2 - \frac{rA_1}{4A}A_{,r} - \frac{r^2}{8A}A_{1,r}\,A_{,r} - \frac{r}{4}A_{1,r} - \frac{r^2\Lambda}{2} \\ & + 2\pi G r^2\Big[\frac{A_1}{r^2}\left(b_1(r)_{,r} - a(\phi)_{\phi}\right)^2 - \frac{A_1}{r^2}\left(s_1(r)_{,r} - a_1(z)_{,z}\right)^2 \\ & + \frac{1}{r^2}\left(s(\phi)_{,\phi} - b(z)_{,z}\right)^2\Big] = 0, \end{aligned} \tag{46f}$$

and

$$\begin{aligned} I_{33} =& -\frac{r^2 A_1}{4A}A_{,rr} + \frac{r^2 A_1}{8A^2}(A_{,}r)^2 - \frac{rA_1}{4A}A_{,r} - \frac{r^2}{8A}A_{1,r}\,A_{,r} - \frac{r}{4}A_{1,r} - \frac{r^2\Lambda}{2} \\ & + 2\pi G r^2\Big[-\frac{A_1}{r^2}\left(b_1(r)_{,r} - a(\phi)_{\phi}\right)^2 + \frac{A_1}{r^2}\left(s_1(r)_{,r} - a_1(z)_{,z}\right)^2 \\ & + \frac{1}{r^2}\left(s(\phi)_{,\phi} - b(z)_{,z}\right)^2\Big] = 0. \end{aligned} \tag{46g}$$

The value of each term containing the magnetic field strength $(s_1(r)_{,r} - a_1(z)_{,z})$, $(b_1(r)_{,r} - a(\phi)_{\phi})$, and $(s(\phi)_{,\phi} - b(z)_{,z})$ can be obtained from the field equation components I_{12}, I_{13} and I_{23}. The possible values of the three terms are as follows

(a) All terms $(s_1(r)_{,r} - a_1(z)_{,z})$, $(b_1(r)_{,r} - a(\phi)_{,\phi})$, and $(s(\phi)_{,\phi} - b(z)_{,z})$ vanish. However the field equations obtained with this choice do not contain the gravitational constant G needed to determine the reference tetrad.

(b) Only one of the terms is not equal to zero.

  (i) The only nonzero term is $(s_1(r)_{,r} - a_1(z)_{,z})$. This choice results in differential equations being insufficient to generate the solutions $A(r)$ and $A_1(r)$ explicitly.

  (ii) The only term is $(b_1(r)_{,r} - a(\phi)_{,\phi}) \neq 0$. This choice results also in differential equations being insufficient to generate the solutions $A(r)$ and $A_1(r)$ explicitly.

(iii) The only term is $(s(\phi)_{,\phi} - b(z)_{,z}) \neq 0$. This choice yields deferential equations which are sufficient to generate the solutions $A(r)$ and $A_1(r)$ explicitly.

Therefore, in order to obtain the solutions $A(r)$ and $A_1(r)$ explicitly which contain the gravitational constant G, we choose the last possibility (b.iii) where the only nonzero term is $s(\phi)_{,\phi} - b(z)_{,z}$. In this case the solutions for $a_1(z)$, $b_1(r)$, and $s_1(r)$ are

$$a_1(z) = \alpha_1 z,\ b_1(r) = \beta_1 r,\ s_1(r) = \gamma_1 r, \tag{47}$$

where α_1, β_1, and γ_1 are the integration constants. For the solution $b(z)$ we choose

$$b(z) = \beta z, \tag{48}$$

where β is the integration constant.

Since the system under consideration is a black hole in cylindrical symmetry, the solutions for $a(\phi)$ and $s(\phi)$ are

$$a(\phi) = \alpha,\ s(\phi) = \gamma, \tag{49}$$

where α and γ are the integration constants. The solutions $A(r)$ and $A_1(r)$ of the field equations (46d-46g) therefore are given by

$$A_1(r) = c_1 A(r), \tag{50}$$

and

$$A(r) = c_3 - \frac{c_2}{r} + \frac{4\pi G\beta^2}{c_1 r^2} - \frac{\Lambda r^2}{3c_1}, \tag{51}$$

where $c_i, i = 1, 2, 3$, are constants of integration. If $c_1 = c_3 = 1$ and $c_2 = 2GM$ are chosen, the solutions $A(r)$ and $A_1(r)$ are given respectively by

$$A_1(r) = A(r), \tag{52}$$

and

$$A(r) = 1 - \frac{2GM}{r} + \frac{4\pi G\beta^2}{r^2} - \frac{\Lambda r^2}{3}. \tag{53}$$

The metric corresponding to the solution (52)-(53) is

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{4\pi G\beta^2}{r^2} - \frac{\Lambda r^2}{3}\right) dt^2 - \left(1 - \frac{2GM}{r} + \frac{4\pi G\beta^2}{r^2} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\phi^2 - r^2 dz^2. \tag{54}$$

These results (equation (54)) are in concommitance with the Reissner-Nordström de-Sitter metric discussed for instance in [39–43] in which the term

$$\frac{4\pi G\beta^2}{r^2}$$

indicates the presence of the magnetic field sources. Our solution obtained here are different from [24]. The differences are due to the fact that our field equations which are derived via the variational principle have different sign in the terms corresponding to electromagnetic field.

3.2. *Covariant formulation*

Here we will formulate our solution in covariant fashion by using the method proposed by Krššák through the reference tetrad [6]. Now we will use Krššák procedure by computing firstly the reference tetrad. Afterwards, we calculate the anholonomic coefficients, the spin connections, the torsion, the superpotensial, the torsion scalar, the electromagnetic field tensor, the contortion, and curvature of the spin connection. In our study, we choose the diagonal tetrad

$$ {}^{(\mathrm{r})}h_{a\mu} = \mathrm{diag}\left(\sqrt{A_G(r)}, \frac{1}{\sqrt{c_1 A_G(r)}}, r, r\right), \tag{55} $$

as reference tetrad, where $(A_G(r))$ and $(A_{1G}(r) = c_1 A_G(r))$ are the values of $(A(r))$ and $(A_1(r))$ respectively when the universal gravitational constant G is set to be zero. The reference tetrad with varying indices is therefore given by

$$ {}^{(\mathrm{r})}h^{a}{}_{\mu} = \mathrm{diag}\left(\sqrt{A_G}, -\frac{1}{\sqrt{c_1 A_G}}, -r, -r\right), \tag{56a} $$

and

$$ {}^{(\mathrm{r})}h_{a}{}^{\mu} = \mathrm{diag}\left(\frac{1}{\sqrt{A_G}}, -\sqrt{c_1 A_G}, -\frac{1}{r}, -\frac{1}{r}\right). \tag{56b} $$

To calculate the spin connection appearing in equation (20) we require to calculate the anholonomic coefficients $(f^{a}{}_{bc}({}^{(\mathrm{r})}h))$ associated with the reference tetrad, namely

$$ f^{a}{}_{bc}({}^{(\mathrm{r})}h) = {}^{(\mathrm{r})}h_{a}{}^{\mu}\, {}^{(\mathrm{r})}h_{b}{}^{\nu}\left(\partial_\nu\, {}^{(\mathrm{r})}h^{c}{}_{\mu} - \partial_\mu\, {}^{(\mathrm{r})}h^{c}{}_{\mu}\right). \tag{57} $$

Then, nonzero anholonomic coefficients $f^{a}{}_{bc}({}^{(\mathrm{r})}h)$ and $f_{a}{}^{b}{}_{c}({}^{(\mathrm{r})}h)$ are

$$ f^{0}{}_{01}({}^{(\mathrm{r})}h) = -f^{0}{}_{10}({}^{(\mathrm{r})}h) = -\frac{1}{2}\sqrt{\frac{c_1}{A_G}}A_{G,r}\,, \tag{58a} $$

$$ f^{2}{}_{12}({}^{(\mathrm{r})}h) = -f^{2}{}_{21}({}^{(\mathrm{r})}h) = f^{3}{}_{13}({}^{(\mathrm{r})}h) = -f^{3}{}_{31}({}^{(\mathrm{r})}h) = \frac{\sqrt{c_1 A_G}}{r}, \tag{58b} $$

$$ f_{0}{}^{0}{}_{1}({}^{(\mathrm{r})}h) = f_{0}{}^{1}{}_{0}({}^{(\mathrm{r})}h) = -\frac{1}{2}\sqrt{\frac{c_1}{A_G}}A_{G,r}\,, \tag{58c} $$

and

$$ f_{2}{}^{1}{}_{2}({}^{(\mathrm{r})}h) = -f_{2}{}^{2}{}_{1}({}^{(\mathrm{r})}h) = f_{3}{}^{1}{}_{3}({}^{(\mathrm{r})}h) = -f_{3}{}^{3}{}_{1}({}^{(\mathrm{r})}h) = \frac{\sqrt{c_1 A_G}}{r}. \tag{58d} $$

The components of the spin connection $\overset{\bullet}{\omega}{}^{c}{}_{a\mu}$ that do not vanish can be obtained from equation (20), namely

$$ \overset{\bullet}{\omega}{}^{0}{}_{10} = \overset{\bullet}{\omega}{}^{1}{}_{00} = -\frac{\sqrt{c_1}}{2}A_{G,r}\,, \tag{59a} $$

$$ \overset{\bullet}{\omega}{}^{1}{}_{22} = \overset{\bullet}{\omega}{}^{1}{}_{33} - \sqrt{c_1 A_G}, \tag{59b} $$

and

$$\overset{\bullet}{\omega}{}^{2}{}_{12} = \overset{\bullet}{\omega}{}^{3}{}_{13} = \sqrt{c_1 A_G}. \tag{59c}$$

Then, the torsion components $\overset{\bullet}{T}{}^{c}{}_{\mu\nu}$ in the covariance formulation are calculated using equation (19). The torsion components $\overset{\bullet}{T}{}^{c}{}_{\mu\nu}$ that are not equal to zero are given by

$$\overset{\bullet}{T}{}^{0}{}_{01} = -\overset{\bullet}{T}{}^{0}{}_{10} = \frac{1}{2\sqrt{A}}\left(-A_{,r} + A_{G,r}\right), \tag{60a}$$

and

$$\overset{\bullet}{T}{}^{2}{}_{12} = -\overset{\bullet}{T}{}^{2}{}_{21} = \overset{\bullet}{T}{}^{3}{}_{13} = -\overset{\bullet}{T}{}^{3}{}_{31} = \frac{1}{\sqrt{A}}\left(-\sqrt{A} + \sqrt{A_G}\right). \tag{60b}$$

While the torsion components $\overset{\bullet}{T}{}^{\alpha}{}_{\mu\nu}$, $\overset{\bullet}{T}{}_{\alpha}{}^{\mu\nu}$, and $\overset{\bullet}{T}{}^{\mu\nu}{}_{\alpha}$ that are not equal to zero are given by

$$\overset{\bullet}{T}{}^{0}{}_{01} = -\overset{\bullet}{T}{}^{0}{}_{10} = \frac{1}{2A}\left(-A_{,r} + A_{G,r}\right), \tag{61a}$$

$$\overset{\bullet}{T}{}^{2}{}_{12} = -\overset{\bullet}{T}{}^{2}{}_{21} = \overset{\bullet}{T}{}^{3}{}_{13} = -\overset{\bullet}{T}{}^{3}{}_{31} = -\frac{1}{r\sqrt{A}}\left(-\sqrt{A} + \sqrt{A_G}\right), \tag{61b}$$

$$\overset{\bullet}{T}{}_{0}{}^{01} = -\overset{\bullet}{T}{}_{0}{}^{10} = \frac{c_1}{2}\left(-A_{,r} + A_{G,r}\right), \tag{61c}$$

$$\overset{\bullet}{T}{}_{2}{}^{12} = -\overset{\bullet}{T}{}_{2}{}^{21} = \overset{\bullet}{T}{}_{3}{}^{13} = -\overset{\bullet}{T}{}_{3}{}^{31} = \frac{c_1\sqrt{A}}{r}\left(-\sqrt{A} + \sqrt{A_G}\right), \tag{61d}$$

$$\overset{\bullet}{T}{}^{00}{}_{1} = \frac{1}{2A^2}\left(-A_{,r} + A_{G,r}\right), \tag{61e}$$

$$\overset{\bullet}{T}{}^{01}{}_{0} = \frac{c_1}{2}\left(-A_{,r} + A_{G,r}\right), \tag{61f}$$

$$\overset{\bullet}{T}{}^{21}{}_{2} = \overset{\bullet}{T}{}^{31}{}_{3} = \frac{c_1\sqrt{A}}{r}\left(-\sqrt{A} + \sqrt{A_G}\right), \tag{61g}$$

and

$$\overset{\bullet}{T}{}^{22}{}_{1} = \overset{\bullet}{T}{}^{33}{}_{1} = \frac{1}{r^3\sqrt{A}}\left(-\sqrt{A} + \sqrt{A_G}\right). \tag{61h}$$

From the above expressions of the torsion, we can compute the superpotential according to

$$\overset{\bullet}{S}{}_{\alpha}{}^{\mu\nu} = \frac{1}{2}\left(\frac{1}{2}\overset{\bullet}{T}{}^{\mu\nu}{}_{\alpha} + \frac{1}{2}\overset{\bullet}{T}{}^{\nu\mu}{}_{\alpha} + \frac{1}{2}\overset{\bullet}{T}{}_{\alpha}{}^{\mu\nu} + \delta^{\mu}_{\alpha}\overset{\bullet}{T}{}^{\beta\nu}{}_{\beta} - \delta^{\nu}_{\alpha}\overset{\bullet}{T}{}^{\beta\mu}{}_{\beta}\right). \tag{62}$$

The nonzero superpotential components $\overset{\bullet}{S}{}_{\alpha}{}^{\mu\nu}$ are given by

$$\overset{\bullet}{S}{}_{0}{}^{01} = -\overset{\bullet}{S}{}_{0}{}^{10} = \frac{c_1\sqrt{A}}{r}\left(-\sqrt{A} + \sqrt{A_G}\right), \tag{63a}$$

and

$$\overset{\bullet}{S}_2{}^{12} = -\overset{\bullet}{S}_2{}^{21} = \overset{\bullet}{S}_3{}^{13} = \overset{\bullet}{S}_3{}^{31} = -\frac{c_1}{4r}\left(r\left(-\sqrt{A}+\sqrt{A_G}\right)\right.$$
$$\left.+2\sqrt{A}\left(-\sqrt{A}+\sqrt{A_G}\right)\right). \tag{63b}$$

According to equation (25), the torsion scalar $\overset{\bullet}{T}$ is

$$\overset{\bullet}{T} = -\frac{2c_1}{r^2\sqrt{A}}\left(-\sqrt{A}+\sqrt{A_G}\right)\left(r\left(-A_{,r}+A_{G,r}\right)+\sqrt{A}\left(-\sqrt{A}+\sqrt{A_G}\right)\right). \tag{64}$$

Furthemore, the components of the contortion tensor $\overset{\bullet}{K}{}^{\mu\nu}{}_{\alpha}$ and $\overset{\bullet}{K}{}^{\mu}{}_{\nu\alpha}$ are calculated using equation (8) and the nonvanishing components of contortion are

$$\overset{\bullet}{K}{}^{01}{}_{0} = -\overset{\bullet}{K}{}^{10}{}_{0} = -\frac{c_1}{2}\left(-A_{,r}+A_{G,r}\right), \tag{65a}$$

$$\overset{\bullet}{K}{}^{12}{}_{2} = -\overset{\bullet}{K}{}^{21}{}_{2} = \overset{\bullet}{K}{}^{13}{}_{3} = -\overset{\bullet}{K}{}^{31}{}_{3} = \frac{c_1\sqrt{A}}{r}\left(-\sqrt{A}+\sqrt{A_G}\right), \tag{65b}$$

$$\overset{\bullet}{K}{}^{0}{}_{10} = \frac{1}{2A}\left(-A_{,r}+A_{G,r}\right), \tag{65c}$$

$$\overset{\bullet}{K}{}^{1}{}_{00} = \frac{c_1 A}{2}\left(-A_{,r}+A_{G,r}\right), \tag{65d}$$

$$\overset{\bullet}{K}{}^{1}{}_{22} = \overset{\bullet}{K}{}^{1}{}_{33} = -c_1 r\sqrt{A}\left(-\sqrt{A}+\sqrt{A_G}\right), \tag{65e}$$

and

$$\overset{\bullet}{K}{}^{2}{}_{12} = \overset{\bullet}{K}{}^{3}{}_{13} = \frac{1}{r\sqrt{A}}\left(-\sqrt{A}+\sqrt{A_G}\right). \tag{65f}$$

In this case, the curvature $\overset{\bullet}{R}{}^{a}{}_{b\nu\mu}$ of the spin connection does not vanish and its non zero components are given by

$$\overset{\bullet}{R}{}^{0}{}_{110} = -\overset{\bullet}{R}{}^{0}{}_{101} = \overset{\bullet}{R}{}^{1}{}_{010} = -\overset{\bullet}{R}{}^{1}{}_{001} = -\frac{\sqrt{c_1}}{2}A_{G,rr}, \tag{66a}$$

$$\overset{\bullet}{R}{}^{0}{}_{220} = -\overset{\bullet}{R}{}^{0}{}_{202} = \overset{\bullet}{R}{}^{0}{}_{303} = -\overset{\bullet}{R}{}^{0}{}_{330} = \overset{\bullet}{R}{}^{2}{}_{020} = -\overset{\bullet}{R}{}^{2}{}_{002} = \overset{\bullet}{R}{}^{3}{}_{030} = -\overset{\bullet}{R}{}^{3}{}_{003} \tag{66b}$$

$$= -\frac{c_1}{2}\sqrt{A_G}A_{G,r}, \tag{66c}$$

$$\overset{\bullet}{R}{}^{1}{}_{212} = -\overset{\bullet}{R}{}^{1}{}_{221} = \overset{\bullet}{R}{}^{1}{}_{313} = -\overset{\bullet}{R}{}^{1}{}_{331} = \overset{\bullet}{R}{}^{2}{}_{121} = -\overset{\bullet}{R}{}^{2}{}_{112} = \overset{\bullet}{R}{}^{3}{}_{131} = -\overset{\bullet}{R}{}^{3}{}_{113} \tag{66d}$$

$$= -\frac{1}{2}\frac{A_{G,r}}{\sqrt{c_1 A_G}}, \tag{66e}$$

$$\overset{\bullet}{R}{}^{2}{}_{323} = -\overset{\bullet}{R}{}^{2}{}_{332} = \overset{\bullet}{R}{}^{3}{}_{232} = -\overset{\bullet}{R}{}^{3}{}_{223} = -c_1 A_G. \tag{66f}$$

That the curvature of the spin connection $\overset{\bullet}{R}{}^{a}{}_{b\nu\mu}$ does not vanish can be explained through the following spin connection transformation [44]

$$\overset{\bullet}{\omega}{}^{a}{}_{b\mu} \mapsto \overset{\bullet}{\omega}{}'^{a}{}_{b\mu} = \Lambda^{a}{}_{c}(x)\overset{\bullet}{\omega}{}^{c}{}_{d\mu}\Lambda_{b}{}^{d}(x) + \Lambda^{a}{}_{c}(x)\partial_{\mu}\Lambda_{b}{}^{c}(x). \tag{67}$$

Tetrad and spin connection variables come together with a Lorentz gauge symmetry. The Lorentz gauge freedom is of particular relevance for metric teleparallelism, with a flat, metric compatible spin connection [44]. In this case it is always possible to choose the Weitzenböck gauge (locally) such that the spin connection vanishes identically, $\overset{\bullet}{\omega}{}^{a}{}_{b\mu} = 0$. It further follows from Equation (67) that in other gauge the spin connection takes the simple form

$$\overset{\bullet}{\omega}{}^{'a}{}_{b\mu} = \Lambda^{a}{}_{c}(x)\partial_{\mu}\Lambda_{b}{}^{c}(x). \tag{68}$$

It is straightforwards to compute so that the curvature $\overset{\bullet}{R}{}^{'a}{}_{b\nu\mu}$ of the spin connection $\overset{\bullet}{\omega}{}^{'a}{}_{b\mu}$ given in equation (68) vanishes [6].

However, other choice of gauges may lead to a non zero spin connection $\overset{\bullet}{\omega}{}^{a}{}_{b\mu}$ which transforms to other spin connection $\overset{\bullet}{\omega}{}^{'a}{}_{b\mu}$ according to Equation (67) whose curvature does not vanish. That the curvature of the spin connection $\overset{\bullet}{\omega}{}^{'a}{}_{b\mu}$ does not vanish indicates that the geometry involved in this consideration is Riemann-Cartan geometry instead of Weitzenböck geometry.

3.3. *Singularities*

Now, we discuss the singularities of our solution. It is well-known that the analysis of singularities in Einstein's general theory of relativity is made by involving the Kretschmann scalar [45] which is calculated from the Riemannian curvature tensor as given in equation (12). Therefore we need to calculate the Christoffel symbol of Levi-Civita connection. The non vanishing Christoffel symbol of the Levi-Civita connection are

$$\overset{\circ}{\Gamma}{}^{0}{}_{01} = \overset{\circ}{\Gamma}{}^{0}{}_{10} = \frac{A_{,r}}{2A}, \tag{69a}$$

$$\overset{\circ}{\Gamma}{}^{1}{}_{00} = \frac{c_1 A}{2} A_{,r}\,, \tag{69b}$$

$$\overset{\circ}{\Gamma}{}^{1}{}_{11} = -\frac{A_{,r}}{2A}, \tag{69c}$$

$$\overset{\circ}{\Gamma}{}^{1}{}_{22} = \overset{\circ}{\Gamma}{}^{1}{}_{33} = -c_1 r A, \tag{69d}$$

and

$$\overset{\circ}{\Gamma}{}^{2}{}_{12} = \overset{\circ}{\Gamma}{}^{2}{}_{21} = \overset{\circ}{\Gamma}{}^{3}{}_{13} = \overset{\circ}{\Gamma}{}^{3}{}_{31} = \frac{1}{r}. \tag{69e}$$

While the Riemannian tensor components $\overset{\circ}{R}{}^{\rho}{}_{\lambda\nu\mu}$ and $\overset{\circ}{R}_{\rho\lambda\nu\mu}$ are calculated using the Levi-Civita conection given in equation (69) whose non vanishing components are given in equation (A.1). From the Riemannian curvature tensor given in equation (A.1), we obtain three important scalars, namely

(a) the Kretschmann scalar

$$\overset{\circ}{R}{}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu} = \frac{c_1}{r^4}\left(r^4 A_{,rr} + 4r^2 (A_{,r})^2 + 4A^2\right), \tag{70}$$

(b) the scalar $\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu}$

$$\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu} = \frac{{c_1}^2}{2r^4}\left(r^2\left(rA_{,rr}+2A_{,r}\right)^2+4\left(rA_{,r}+A\right)^2\right), \tag{71}$$

and

(c) the Ricci scalar

$$\overset{\circ}{R} = \frac{c_1}{r^2}\left(r^2A_{rr}+4rA_{,r}+2A\right). \tag{72}$$

Equation (13) shows that the Riemannian curvature tensor can be obtained from tetrads instead of from the metric. Therefore in the analysis of singularities in teleparallel gravity, we can still apply the standard procedure of analysis of singularities involving Kretschmann scalar

$$\overset{\circ}{R}^{\rho\theta\mu\nu}\overset{\circ}{R}_{\rho\theta\mu\nu} = \overset{\bullet}{R}^{\rho\theta\mu\nu}\overset{\bullet}{R}_{\rho\theta\mu\nu} - 2\overset{\bullet}{R}_{\rho\theta\mu\nu}\overset{\bullet}{Q}_{\rho\theta\mu\nu} + \overset{\bullet}{Q}^{\rho\theta\mu\nu}\overset{\bullet}{Q}_{\rho\theta\mu\nu}, \tag{73}$$

without calculating the Riemannian curvatur tensor.

The analysis of singularities corresponding to our general solutions appearing in equation (53) faces some very difficult calculations. Therefore, to analyze the singularities of our solutions, we will firstly consider some special cases and then use graphical methods.

3.3.1. *Analysis with Kretshmann scalar*

Firstly, we analyze the singularities of our solutions with Kretshmann scalars.

(1) Case 1: Schwarzschild-like black hole, i.e. $A_1(r) = A(r)$ and $A(r) = 1 - \frac{2GM}{r}$

In this case, $A_G(r) = 1$. Therefore, the Kretschmann scalar (70), scalar $\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu}$ (71), and the Ricci scalar (72) are given respectively by

$$\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu} = \frac{1}{r^6}\left(12\left(GM\right)^2+4\left(r-2GM\right)^2\right), \tag{74}$$

$$\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu} = \frac{2}{r^4}, \tag{75}$$

and

$$\overset{\circ}{R} = \frac{2}{r^2}. \tag{76}$$

The Kretschmann scalar (74), the scalar $\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu}$ (75), and the Ricci scalar (75) at $r = 0$ are clearly infinite. Whereas, at the horizon ($r_h = 2GM$), the Kretschmann scalar, the scalar $\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu}$, and the Ricci scalar are finite, namely

$$\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu} = \frac{1}{2(GM)^4}, \tag{77}$$

$$\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu} = \frac{8}{(GM)^4}, \tag{78}$$

and

$$\overset{\circ}{R} = \frac{1}{(GM)^2}. \tag{79}$$

Those the singularity at $r = 0$ is physical singularity and the singularity at $r = r_h$ is a pseudo-singularity. This result remind us to the case of Schwarzschild black holes with $A_1(\varrho) = A(\varrho) = 1 - \frac{2GM}{\varrho^2}$, where ϱ is the distance from the origin in spherical coordinate system [46].

(2) Case 2: Schwarzschild-de-Sitter-like black hole, i.e. $A_1(r) = A(r)$ and $A(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}$

In this case, $A_G(r) = 1 - \frac{\Lambda r^2}{3}$ so that the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$, the scalar $\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu}$, and the Ricci scalar $\overset{\circ}{R}$ are given respectively by

$$\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu} = \frac{4}{3r^6}\left(2\Lambda^2 r^6 - 2\lambda r^4 + 3r^2 - 12GMr + 36\left(GM\right)^2\right), \tag{80}$$

$$\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu} = \frac{2}{r^6}\left(2\Lambda^2 r^6 - 2\Lambda r^4 + r^2\right), \tag{81}$$

and

$$\overset{\circ}{R} = \frac{2}{r^2}\left(-2\Lambda r^2 + 1\right). \tag{82}$$

It is clear that the three scalars appearing in Equation (80-82) at $r = 0$ are infinite so that the singularity is physical. The other singularity is found at $r = r_h$, where r_h is given by

$$r_h = -\frac{\left(\Lambda + \left(3GM\Lambda^2 + \sqrt{-\Lambda^3 + 9\left(GM\Lambda^2\right)}\right)^{2/3}\right)}{\Lambda\left(3GM\Lambda^2 + \sqrt{-\Lambda^3 + 9\left(GM\Lambda^2\right)^2}\right)^{1/3}}. \tag{83}$$

It is straightforward to show that the values of the three scalars appearing in equation (80-82) at $r = r_h$ are finite, so that it is a pseudo-singularity.

(3) Case 3: Reissner-Nordström-like black hole, i.e. $A_1(r) = A(r)$ dan $A(r) = 1 - \frac{2GM}{r} + \frac{4\pi G\beta^2}{r^2}$

Here, $A_G = 1$ so that the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$, the scalar $\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu}$, and the Ricci scalar $\overset{\circ}{R}$ are

$$\begin{aligned}\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu} = &\frac{1}{r^8}\left(\left(-4GMr + 24\pi G\beta^2\right)^2 + 4\left(2GMr - 8\pi G\beta^2\right)^2\right.\\ &\left.+4\left(r^2 - 2GMr + 4\pi G\beta^2\right)^2\right),\end{aligned} \tag{84}$$

$$\overset{\circ}{R}^{\mu\nu}\overset{\circ}{R}_{\mu\nu} = \frac{1}{2r^8}\left(\left(8\pi G\beta^2\right)^2 + 4\left(r^2 - 4\pi G\beta^2\right)^2\right), \tag{85}$$

and

$$\overset{\circ}{R} = \frac{2}{r^4}\left(r^2 + 4\pi G\beta^2\right) \tag{86}$$

respectively. The singularity at $r = 0$ is physical. The other singularities are found at $r = r_{h_1}$ and at $r = r_{h_2}$, where $r = r_{h_1}$ and $r = r_{h_2}$ are given respectively by

$$r_{h_1} = GM - \sqrt{(GM)^2 - 4\pi G\beta^2} \tag{87}$$

and

$$r_{h_2} = GM + \sqrt{(GM)^2 - 4\pi G\beta^2}. \tag{88}$$

It is also straightforward to show that the values of the three scalars appearing in equation (84-86) at $r = r_{h_1}$ and at $r = r_{h_2}$ are finite, so that they are clearly pseudo-singularities.

3.3.2. *Analysis with the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, the scalar $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$, and the torsion scalar $\overset{\bullet}{T}$*

Some researchers [24, 27, 47] involved the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, the scalar $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$, and the torsion scalar $\overset{\bullet}{T}$ which can be obtained from equation (61) and (61b) to analyze singularities in teleparallel gravity. Here the scalars $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$ and $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$ are

$$\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda} = -\frac{c_1}{2r^2A}\left(r^2\left(-A_{,r} + A_{G,r}\right)^2 + 8A\left(\sqrt{A} + \sqrt{A_G}\right)^2\right) \tag{89}$$

and

$$\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu} = -\frac{c_1}{4r^2A}\left(r\left(-A_{,r} + A_{G,r}\right) + 4\sqrt{A}\left(-\sqrt{A} + \sqrt{A_G}\right)\right), \tag{90}$$

whereas the scalar torsion $\overset{\bullet}{T}$ is given in equation (130). In this subsubsection, we will calculate the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, the scalar $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$, and the torsion scalar $\overset{\bullet}{T}$ at singular points for each case and then compare them to the results of analysis of singularities obtained from the Kretschmann scalars.

(1) Case 1: Schwarzschild-like black hole i.e. $A_1(r) = A(r)$ dan $A(r) = 1 - \frac{2GM}{r^2}$

For this case, the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$ (89), the scalar $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$ (90), and the

torsion scalar $\overset{\bullet}{T}$ (64) are given respectively by

$$\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda} = -\frac{1}{r^2\left(2GM-r\right)}\left(2\left(GM\right)^2+4r\left(r-2GM\right)\right.$$
$$\left.\times\left(-\sqrt{1-\frac{2GM}{r}}+1\right)\right), \tag{91}$$

$$\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu} = -\frac{1}{4r^5\left(r-2GM\right)}\left(4\left(GM\right)^2+4r^3\sqrt{1-\frac{2GM}{r}}\right.$$
$$\left.\times\left(-\sqrt{1-\frac{2GM}{r}}+1\right)\right), \tag{92}$$

and

$$\overset{\bullet}{T} = -\frac{2}{r^3\sqrt{1-\frac{2GM}{r}}}\left(-\sqrt{1-\frac{2GM}{r}}+1\right)\left(-2GMr\right.$$
$$\left.+r\sqrt{1-\frac{2GM}{r}}\left(-\sqrt{1-\frac{2GM}{r}}+1\right)\right). \tag{93}$$

The scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$ (91), scalar $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$ (92), and scalar $\overset{\bullet}{T}$ (93) at $r = 0$ are infinite. Likewise, at the horizon ($r = r_h$), all three values are infinite. These results differ from those obtained from analyses (77-79).

(2) Case 2: Schwarzschild-de Sitter-like black hole, i.e. $A_1(r) = A(r)$ dan $A(r) = 1 - \frac{2GM}{r^2} - \frac{\Lambda r^2}{3}$

In this case, the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, the scalar $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$, and the torsion scalar $\overset{\bullet}{T}$, are

$$\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda} = -\frac{1}{6r^3\left(3r-6GM-\lambda r^3\right)}\left(36\left(GM\right)^2+8\left(3r-6GM-\lambda r^3\right)\right.$$
$$\left.\times\left(-\sqrt{3r-6GM-\Lambda r^3}+\sqrt{3r-\Lambda r^3}\right)^2\right), \tag{94}$$

$$\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu} = -\frac{1}{4r^2\left(3r-6GM-\Lambda r^3\right)}\left(-6GM+4\sqrt{3r-6GM-\Lambda r^3}\right.$$
$$\left.\times\left(-\sqrt{3r-6GM-\Lambda r^3}+\sqrt{3r-\Lambda r^3}\right)\right), \tag{95}$$

$$\overset{\bullet}{T} = \frac{2}{3r^3\sqrt{3r-6GM-\Lambda r^3}}\left(-\sqrt{3r-6GM-\Lambda r^3}+\sqrt{3r-\Lambda r^3}\right)$$
$$\times\left(-6GM+\sqrt{3r-6GM-\Lambda r^3}\left(-\sqrt{3r-6GM-\Lambda r^3}\right.\right.$$
$$\left.\left.+\sqrt{3r-\Lambda r^3}\right)\right). \tag{96}$$

All such scalars are infinite at $r = 0$ and at the horizon $r = r_h$. These results are different from the analysis with the Kretschmann scalar.

(3) Case 3: Reissner-Nordström-like black hole, i.e. $A_1(r) = A(r)$ dan $A(r) = 1 - \frac{2GM}{r^2} - \frac{4\pi G\beta^2}{r^2}$

In this case, the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$, and the torsion scalar $\overset{\bullet}{T}$ are given respectively by

$$\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda} = \frac{1}{r^4\left(r^2-2GMr+4\pi G\beta^2\right)}\left(\left(2GMr-8\pi G\beta^2\right)^2+8\left(r^2-2GMr+4\pi G\beta^2\right)\left(\sqrt{r^2-2GMr+4\pi G\beta^2}+r\right)^2\right), \tag{97}$$

$$\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu} = -\frac{1}{4r^2\left(r^2-2GMr+4\pi G\beta^2\right)}\left(-\left(2GMr-8\pi G\beta^2\right)+\sqrt{r^2-2GMr+4\pi G\beta^2}\left(-\sqrt{r^2-2GMr+4\pi G\beta^2}+r\right)\right), \tag{98}$$

$$\overset{\bullet}{T} = \frac{2\left(\sqrt{r^2-2GMr+4\pi G\beta^2}+r\right)}{r^4\sqrt{r^2-2GMr+4\pi G\beta^2}}\left(-\left(2GMr-8\pi G\beta^2\right)+\sqrt{r^2-2GMr+4\pi G\beta^2}\left(-\sqrt{r^2-2GMr+4\pi G\beta^2}+r\right)\right). \tag{99}$$

The three scalars are infinite at $r = 0$ and at both horizons r_{h_1} and r_{h_2}. These results also differ from results of the analysis with the Kretschmann scalar.

3.3.3. *Analysis with the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{Q}$*

From equations (13) and (15), it is clear that the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ is equal to $-\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$ whenever the curvature of the Weitzenböck connection vanishes. Therefore, the analysis of the singularities by using both the scalars will yield the same conclusions. However, although the curvature of the Weitzenböck connection in our cases do not vanish we will show here that the values of the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ in our cases are consistent with the values of the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$, in the sense that the value of scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ at a point is infinite (or finite) if the value of Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$ is infinite (or finite) at that point. To calculate the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ we need to determine firstly the Weitzenböck connection ($\overset{\bullet}{\Gamma}^{\rho}{}_{\nu\mu}$). Therefore, from equation (7) and equations (65c)-(65f) we obtain

$$\overset{\bullet}{\Gamma}{}^{0}{}_{01} = \frac{A_{,r}}{2A}, \tag{100a}$$

$$\overset{\bullet}{\Gamma}{}^{0}{}_{10} = \frac{A_{G,r}}{2A}, \tag{100b}$$

$$\overset{\bullet}{\Gamma}{}^{1}{}_{00} = \frac{c_1 A}{2} A_{G,r}\,, \tag{100c}$$

$$\overset{\bullet}{\Gamma}{}^{1}{}_{11} = -\frac{A_{,r}}{2A}, \tag{100d}$$

$$\overset{\bullet}{\Gamma}{}^{1}{}_{22} = \overset{\bullet}{\Gamma}{}^{1}{}_{33} = -c_1 r\sqrt{A_G A}, \tag{100e}$$

$$\overset{\bullet}{\Gamma}{}^{2}{}_{12} = \overset{\bullet}{\Gamma}{}^{3}{}_{13} = \frac{\sqrt{A_G}}{r\sqrt{A}}, \tag{100f}$$

and

$$\overset{\bullet}{\Gamma}{}^{2}{}_{21} = \overset{\bullet}{\Gamma}{}^{3}{}_{31} = \frac{1}{r}, \tag{100g}$$

whereas the other components vanish.

Using equation (14), equations (65c)-(65f), and equation (100), we have non vanishing components of the tensors $\overset{\bullet}{Q}{}^{\rho}{}_{\lambda\nu\mu}$ and $\overset{\bullet}{Q}{}_{\rho\lambda\nu\mu}$ given in equation (B.1) From the components of the tensor $\overset{\bullet}{Q}{}^{\rho}{}_{\lambda\nu\mu}$ given in equation (B.1) we obtain

(a) the scalar $\overset{\bullet}{Q}{}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}{}_{\rho\lambda\nu\mu}$

$$\begin{aligned}\overset{\bullet}{Q}{}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}{}_{\rho\lambda\nu\mu} =& \frac{{c_1}^2}{4r^2 A_G A}\Big[r^4 A_G A\left(-A_{,rr}+A_{G,rr}\right)^2 + 2r^2 A_G\left(-\sqrt{A}A_{,r}\right.\\ &\left.+\sqrt{A_G}A_{G,r}\right)^2 + 2r^2 A\left(-\sqrt{A_G}A_{,r}+\sqrt{A}A_{G,r}\right)^2\\ &+ 4A_G A\left(A-A_G\right)^2\Big],\end{aligned} \tag{101}$$

and

(b) the scalar $\overset{\bullet}{Q}$

$$\begin{aligned}\overset{\bullet}{Q} =& \frac{c_1}{r^2\sqrt{A_G A}}\Big[r^2\sqrt{A_G A}\left(-A_{,rr}+A_{G,rr}\right) + 2r\left(-2\sqrt{A_G A}A_{,r}\right.\\ &\left.+\left(A_G+A\right)A_{G,r}\right) - 2\sqrt{A_G A}(A-A_G)\Big].\end{aligned} \tag{102}$$

Now, we consider every case discussed in the previous part by using the scalar $\overset{\bullet}{Q}{}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}{}_{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{Q}$.

(1) Case 1: Schwarzschild-like black hole, i.e. $A_1(r) = A(r)$ dan $A(r) = 1 - \frac{2GM}{r^2}$
In this case, equation (101) and equation (102) yields

$$\overset{\bullet}{Q}{}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}{}_{\rho\lambda\nu\mu} = \frac{48\left(GM\right)^2}{r^6}, \tag{103}$$

and

$$\overset{\bullet}{Q} = \frac{4GM}{r^3}. \tag{104}$$

At $r = 0$ where the Kretschmann scalars is infinite, the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{Q}$ are clearly also infinite. While at the horizon ($r = r_h$), where the Kretschmann scalars is finite, the two scalars are finite and given by

$$\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} = \frac{3}{4\left(GM\right)^4}, \tag{105}$$

and

$$\overset{\bullet}{Q} = \frac{1}{2\left(GM\right)^2}, \tag{106}$$

respectively.

(2) Case 2: Schwarzschild-de-Sitter-like black hole, i.e. $A_1(r) = A(r)$ dan $A(r) = 1 - \frac{2GM}{r^2} - \frac{\Lambda r^2}{3}$

In the same way, the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{Q}$ in this case are given respectively by

$$\begin{aligned}\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} =& \frac{2}{9r^6\left(3r - 6GM - \Lambda r^3\right)\left(3 - \Lambda r^2\right)}\left(216\left(3r - 6GM - \Lambda r^3\right)\right.\\ &\times\left(3 - \Lambda r^2\right) + \left(3r - \Lambda r^3\right)\left(-\sqrt{3r - 6GM - \Lambda r^3}\left(6GM - 2\Lambda r^3\right)\right.\\ &\left.-2\Lambda r^{7/2}\sqrt{3 - \Lambda r^2}\right)^2 + \left(3r - 6GM - \Lambda r^3\right)\left(-\sqrt{3 - \Lambda r^2}\right.\\ &\left.\left.\times\left(6GM - 2\Lambda r^3\right) - 2\Lambda r^{5/2}\sqrt{3r - 6GM - \Lambda r^3}\right)^2\right),\end{aligned} \tag{107}$$

and

$$\begin{aligned}\overset{\bullet}{Q} =& \frac{4}{3R^3\sqrt{3r - 6GM - \Lambda r^3}\sqrt{3 - \Lambda r^2}}\left(6GM\sqrt{3r - 6GM - \Lambda r^3}\right.\\ &\times\sqrt{3 - \Lambda r^2} - \left(\sqrt{3r - 6GM - \Lambda r^3}\sqrt{3 - \Lambda r^2}\left(6GM - 2\Lambda r^3\right)\right.\\ &\left.\left.-2r^{5/2}\left(3r - 3GM - \Lambda r^3\right)\right)\right).\end{aligned} \tag{108}$$

Both scalars are infinite at $r = 0$, and finite at the horizon $r = r_h$. These results are consistent with that obtained from singularities analysis using the Kretschmann scalars.

(3) Case 3: Reissner-Nordström-like black hole, i.e. $A_1(r) = A(r)$ and $A(r) = 1 - \frac{2GM}{r^2} + \frac{4\pi G\beta^2}{r^2}$

The scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{Q}$ in this case are given respectively by

$$\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} = \frac{1}{r^8}\left(48\left(GM\right)^2 r^2 - 384\pi G^2 M\beta^2 r + 896\left(\pi G\beta^2\right)^2\right), \tag{109}$$

and

$$\overset{\bullet}{Q} = 0. \tag{110}$$

The scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ is infinite at $r = 0$ and finite at both horizons r_{h_1} and r_{h_2}. These results are also consistent with the values of the Kretschmann scalar whose values are infinite at $r = 0$ and finite at both horizons r_{h_1} and r_{h_2}. While the scalar $\overset{\bullet}{Q}$ vanishes at $r = 0$ and at both horizons r_{h_1} and r_{h_2}.

The analysis of singularities using the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$ shows that the physical singularity exists only at $r = 0$. The values of the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ calculated above lead to the same conclusion. However, when we apply the scalars derived from the torsion tensor to analyze the singularities as in [24, 27, 47] we obtain the different results. For the three cases the values of the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$ and $\overset{\bullet}{T}$ are infinite at $r = 0$ as well as at horizon $r = r_h$. The author in [24] have analyzed the singularities of their solution using such scalars obtained in non-covariant formulation in which the scalars $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$ and $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$ are infinite at the horizon. They argued that the values of the scalars at $r = r_h$ are due to the fact that the scalars are not covariant under local Lorentz transformations. However in our covariant formulation, the values of the three scalars remains infinite at the horizon. Therefore, it can be concluded that the analysis of singularities in teleparallel gravity using the scalars $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$, and $\overset{\bullet}{T}$ is not appropriate.

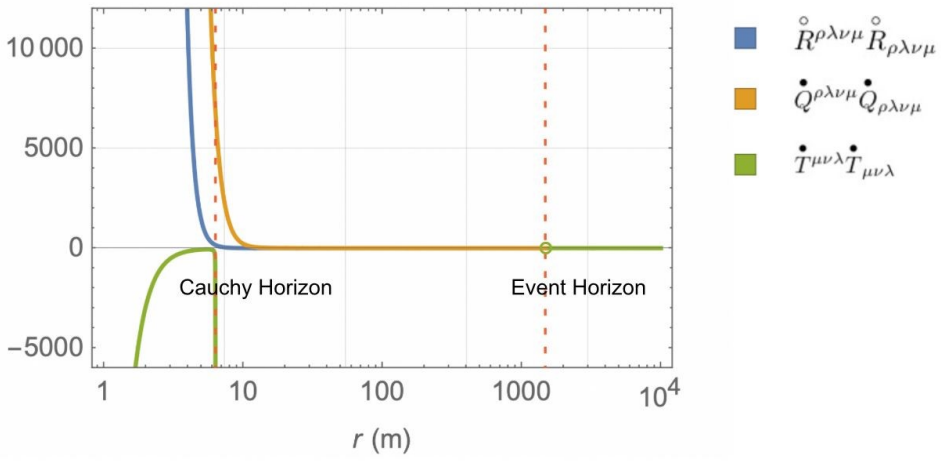

Fig. 1. The scalar $\overset{\bullet}{R}^{\rho\lambda\nu\mu}\overset{\bullet}{R}_{\rho\lambda\nu\mu}$, the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, and the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ with respect to r, with $M = 10^{30}$ kg and $\beta = 10^{15}$.

Now, consider again the case were $A_1(r) = A(r) = 1 - \frac{2GM}{r} + \frac{4\pi G\beta^2}{r^2} - \frac{\Lambda r^2}{3}$. In this case, the values of the Kretschmann scalar $\overset{\bullet}{R}^{\rho\lambda\nu\mu}\overset{\bullet}{R}_{\rho\lambda\nu\mu}$, the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, and the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ as functions of r are calculated through computations and their graphs are depicted in Fig. 1. It is clear that the Kretschmann scalar $\overset{\bullet}{R}^{\rho\lambda\nu\mu}\overset{\bullet}{R}_{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ are infinite at $r = 0$. While at all three horizons (namely the cosmological horizon r_{h1}, the event horizon r_{h2}, and the Cauchy horizon r_{h3}), both scalars are finite. Furthermore, the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$ is infinite at Cauchy horizon r_{h3} and finite at cosmological horizon r_{h1}. It is however undefined between Cauchy horizon r_{h3} and event horizon r_{h2}. Therefore, the result of the analysis with the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$ is incompatible with the analysis of the Kretschmann scalar $\overset{\bullet}{R}^{\rho\lambda\nu\mu}\overset{\bullet}{R}_{\rho\lambda\nu\mu}$ especially between the Cauchy horizon r_{h3} and the event horizon r_{h2}.

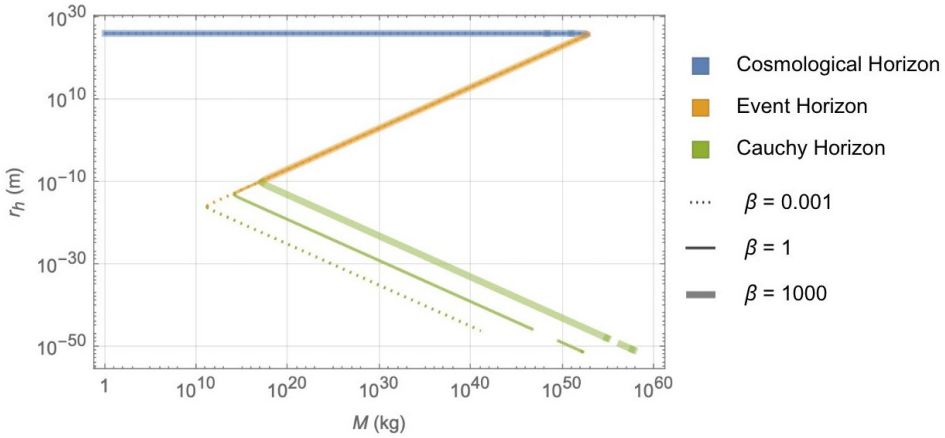

Fig. 2. The effect of the magnetic charge value β on the black hole mass and the cosmological horizon, the horizon event, and the Cauchy horizon.

The dependence of the three horizons on the magnetic charge β and on the black hole mass are shown in Fig. 2. The variation of the value of the magnetic charge β as well as the variation of the black hole mass effects only very small variation in the cosmological horizon, so that the cosmological horizon can be said to be constant around $\approx 10^{26}$ m. While, the event and Cauchy horizons varies with respect to the black hole mass variation. For the same mass, the event horizon increases as the magnetic charge β increases. However, for the same mass, the increase of the magnetic charge β, leads to the decrease of the Cauchy horizon. The value of magnetic charge β also affects the minimum allowable mass value for black holes. The smaller the value of magnetic charge β, the smaller the minimum mass

limit a black hole can have, and vice versa. This minimum black hole mass limit is indicated by the point where the Cauchy horizon meets the equatorial horizon.

3.4. *Gravitoelectromagnetism*

Now, we calculate here the gravitoelectric and gravitomagnetic fields associated with the tetrad (37) according to equation (35). Therefore, we need to obtain the expression of the torsion tensor $\overset{\bullet}{T'}{}^{a}{}_{\mu\nu}$ in Cartesian coordinate system (τ, x, y, z') through the following coordinate transformation, namely

$$\overset{\bullet}{T'}{}^{a}{}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \overset{\bullet}{T}{}^{a}{}_{\mu\nu} \frac{\partial x^{\nu}}{\partial x'^{\beta}} = A^{T}{}_{\alpha\mu} \overset{\bullet}{T}{}^{a}{}_{\mu\nu} A_{\nu\beta}, \tag{111}$$

where $\overset{\bullet}{T}{}^{a}{}_{\mu\nu}$ are the components of the torsion tensor in cylinder coordinate system (t, r, ϕ, z) and $A_{\mu\nu}$ is the coordinate transformation matrix given by

$$A_{\mu\nu} = \begin{pmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z'} \\ \frac{\partial r}{\partial \tau} & \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z'} \\ \frac{\partial \phi}{\partial \tau} & \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z'} \\ \frac{\partial z}{\partial \tau} & \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & -\frac{z\cos\phi}{r} & -\frac{z\sin\phi}{r} & \frac{1}{r} \end{pmatrix}. \tag{112}$$

The torsion tensor $\overset{\bullet}{T'}{}^{a}{}_{\mu\nu}$ is related to gravitoelectric and gravitomagnetic fields trough equation (35) so that

$$\begin{aligned} \overset{\bullet}{T'}{}^{0}{}_{\alpha\beta} &= \begin{pmatrix} 0 & \frac{(-A_{,r}+A_{G,r})}{2\sqrt{A}}\cos\phi & \frac{(-A_{,r}+A_{G,r})}{2\sqrt{A}}\sin\phi & 0 \\ -\frac{(-A_{,r}+A_{G,r})}{2\sqrt{A}}\cos\phi & 0 & 0 & 0 \\ -\frac{(-A_{,r}+A_{G,r})}{2\sqrt{A}}\sin\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E^{0}{}_{x} & E^{0}{}_{y} & E^{0}{}_{z} \\ -E^{0}{}_{x} & 0 & -B^{0}{}_{z} & B^{0}{}_{y} \\ -E^{0}{}_{y} & B^{0}{}_{z} & 0 & -B^{0}{}_{x} \\ -E^{0}{}_{z} & -B^{0}{}_{y} & B^{0}{}_{x} & 0 \end{pmatrix}, \\ \overset{\bullet}{T'}{}^{1}{}_{\alpha\beta} &= \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix} = \begin{pmatrix} 0 & E^{1}{}_{x} & E^{1}{}_{y} & E^{1}{}_{z} \\ -E^{1}{}_{x} & 0 & -B^{1}{}_{z} & B^{1}{}_{y} \\ -E^{1}{}_{y} & B^{1}{}_{z} & 0 & -B^{1}{}_{x} \\ -E^{1}{}_{z} & -B^{1}{}_{y} & B^{1}{}_{x} & 0 \end{pmatrix}, \end{aligned} \tag{113a}$$

$$\overset{\bullet}{T'}{}^{2}{}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}} & 0 \\ 0 & -\frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & E^{2}{}_{x} & E^{2}{}_{y} & E^{2}{}_{z} \\ -E^{2}{}_{x} & 0 & -B^{2}{}_{z} & B^{2}{}_{y} \\ -E^{2}{}_{y} & B^{2}{}_{z} & 0 & -B^{2}{}_{x} \\ -E^{2}{}_{z} & -B^{2}{}_{y} & B^{2}{}_{x} & 0 \end{pmatrix}, \tag{113b}$$

and

$$\overset{\bullet}{T'^{3}}{}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}\ r}\cos\phi \\ 0 & 0 & 0 & \frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}\ r}\sin\phi \\ 0 & -\frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}\ r}\cos\phi & -\frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}\ r}\sin\phi & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E^3{}_x & E^3{}_y & E^3{}_z \\ -E^3{}_x & 0 & -B^3{}_z & B^3{}_y \\ -E^3{}_y & B^3{}_z & 0 & -B^3{}_x \\ -E^3{}_z & -B^3{}_y & B^3{}_x & 0 \end{pmatrix}. \tag{113c}$$

From equations (113) we conclude that

$$E^0{}_x = \frac{(-A_{,r}+A_{G,r}\,)}{2\sqrt{A}}\cos\phi, \tag{114a}$$

$$E^0{}_y = \frac{(-A_{,r}+A_{G,r}\,)}{2\sqrt{A}}\sin\phi, \tag{114b}$$

$$B^2{}_{z'} = -\frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}}, \tag{114c}$$

$$B^3{}_x = -\frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}\ r}\sin\phi, \tag{114d}$$

and

$$B^3{}_y = \frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}\ r}\cos\phi. \tag{114e}$$

Whereas the other component of the gravitoelectric and the gravitomagnetic fields vanish. Thus, the gravitoelectric and the gravitomagnetic fields are given by

$$\vec{E}^0 = \frac{(-A_{,r}+A_{G,r}\,)}{2\sqrt{A}}\ \hat{r}, \tag{115a}$$

$$\vec{E}^1 = \vec{E}^2 = \vec{E}^3 = \vec{0}, \tag{115b}$$

$$\vec{B}^0 = \vec{B}^1 = \vec{0}, \tag{115c}$$

$$\vec{B}^2 = -\frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}}\ \hat{k}, \tag{115d}$$

and

$$\vec{B}^3 = \frac{\left(-\sqrt{A}+\sqrt{A_G}\right)}{\sqrt{A}\ r}\ \hat{\phi}. \tag{115e}$$

Therefore, the gravitoelectric field that does not vanish is only the gravitoelectric field in sector $a = 0$, namely $\vec{E}^0$. The gravitoelectric field $\vec{E}^0$ is radial and its

magnitude depends only on r. At $r = 0$, the magnitude of $\vec{E}^0$ tends to infinity. There are two sectors, namely $a = 2$ and $a = 3$ in which the gravitomagnetic fields do not vanish. The gravitomagnetic field in sector $a = 2$ is parallel to the z-axis, whose magnitude also depends on r. At $r = 0$, the gravitomagnetic field $\vec{B}^2$ is finite. Whereas the gravitomagnetic field lines in sector $a = 3$ form concentric circles around the z-axis. The magnitude of the gravitomagnetic field in this sector depends on r. At $r = 0$, the gravitomagnetic field $\vec{B}^3$ is infinite. From the equations (115), we see that the gravitoelectric and gravitomagnetic fields depend with the mass and the magnetic charge of the black hole as well as the cosmological constant.

4. Electric Black Hole

In this section, we consider a charged black hole. In General Relativity, the black hole is describe by Reissner–Nordström metric [27],

$$ds^2 = Y(r)\ dt^2 - \frac{1}{Y(r)}\ dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{116}$$

with

$$Y(r) = k - \frac{2GM}{r} + \frac{G}{4\pi}\frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2, \tag{117}$$

where $k = -1, 0, 1$ are the curvature of the spherical, hyperbolic, or plane cross sections respectively. Furthemore Q is the electric charge of the black hole. The tetrad chosen is the diagonal tetrad

$$h_{a\mu} = \text{diag}\left(\sqrt{Y(r)}, \frac{1}{\sqrt{Y(r)}}, r, r\ \sin\theta\right), \tag{118}$$

with the various index variations being

$$h^a{}_\mu = \text{diag}\left(\sqrt{Y(r)}, -\frac{1}{\sqrt{Y(r)}}, -r, -r\ \sin\theta\right), \tag{119a}$$

and

$$h_a{}^\mu = \text{diag}\left(-\frac{1}{\sqrt{Y(r)}}, \sqrt{Y(r)}, -\frac{1}{r}, -\frac{1}{r\ \sin\theta}\right). \tag{119b}$$

The tetrad determinant of equation (118) is clearly

$$\det h_{a\mu} = \sqrt{-g} = r^2 \sin\theta. \tag{120}$$

4.1. *The covariant formulation*

Now we will apply the Krššák procedure to obtain the covariant formulation. The first step of the covariant formulation is to determine the reference tetrad from the tetrad given in equation (118). A reference tetrad is a tetrad in which the parameter

governing the gravitational field strength, namely the universal gravitational constant, is set to be equal to zero ($G = 0$). Therefore, the reference tetrad associated to the tetrad given in equation (118) is

$$^{(\mathrm{r})}h_{a\mu} = diag\left(\sqrt{Y_G}, \frac{1}{\sqrt{Y_G}}, r, r\sin\theta\right), \tag{121}$$

with

$$Y_G = \sqrt{k - \frac{\Lambda}{3}r^2}. \tag{122}$$

The references tetrad with varying indices are

$$^{(\mathrm{r})}h^{a}{}_{\mu} = \mathrm{diag}\left(\sqrt{Y_G}, -\frac{1}{\sqrt{Y_G}}, -r, -r\sin\theta\right), \tag{123a}$$

$$^{(\mathrm{r})}h^{a\mu} = \mathrm{diag}\left(\frac{1}{\sqrt{Y_G}}, \sqrt{Y_G}, \frac{1}{r\sin\theta}\right), \tag{123b}$$

and

$$^{(\mathrm{r})}h_{a}{}^{\mu} = \mathrm{diag}\left(\frac{1}{\sqrt{Y_G}}, -\sqrt{Y_G}, -\frac{1}{r}, -\frac{1}{r\sin\theta}\right). \tag{123c}$$

From by equations (123)-(123c), the anholonomic coefficients $f^{a}{}_{bc}(^{(\mathrm{r})}h)$ that are not equal to zero are explicitly given by

$$f^{0}{}_{01}(^{(\mathrm{r})}h) = -f^{0}{}_{10}(^{(\mathrm{r})}h) = -\frac{1}{2\sqrt{Y_G}}Y_{G,r}\,, \tag{124a}$$

$$f^{2}{}_{12}(^{(\mathrm{r})}h) = -f^{2}{}_{21}(^{(\mathrm{r})}h) = \frac{\sqrt{Y_G}}{r}, \tag{124b}$$

$$f^{3}{}_{13}(^{(\mathrm{r})}h) = -f^{3}{}_{31}(^{(\mathrm{r})}h) = \frac{\sqrt{Y_G}}{r}, \tag{124c}$$

and

$$f^{3}{}_{23}(^{(\mathrm{r})}h) = -f^{3}{}_{32}(^{(\mathrm{r})}h) = \frac{1}{r}\cot\theta. \tag{124d}$$

Furthemore, the nonvanishing anholonomic coefficients $f_{a}{}^{b}{}_{c}(^{(\mathrm{r})}h)$ are

$$f_{0}{}^{0}{}_{1}(^{(\mathrm{r})}h) = f_{0}{}^{1}{}_{0}(^{(\mathrm{r})}h) = -\frac{Y_{G,r}}{2\sqrt{Y_G}}, \tag{125a}$$

$$f_{2}{}^{1}{}_{2}(^{(\mathrm{r})}h) = -f_{2}{}^{2}{}_{1}(^{(\mathrm{r})}h) = f_{3}{}^{1}{}_{3}(^{(\mathrm{r})}h) = -f_{3}{}^{3}{}_{1}(^{(\mathrm{r})}h) = \frac{\sqrt{Y_G}}{r}, \tag{125b}$$

and

$$f_{3}{}^{2}{}_{3}(^{(\mathrm{r})}h) = -f_{3}{}^{3}{}_{2}(^{(\mathrm{r})}h) = \frac{1}{r}\cot\theta. \tag{125c}$$

Therefore, from the above obtained anholonomic coefficients, the components of the spin connection $\overset{\bullet}{\omega}{}^{c}{}_{a\mu}$ that are not equal to zero are

$$\overset{\bullet}{\omega}{}^{0}{}_{10} = \overset{\bullet}{\omega}{}^{1}{}_{00} = -\frac{Y_{G,r}}{2}, \tag{126a}$$

$$\overset{\bullet}{\omega}{}^{1}{}_{22} = -\overset{\bullet}{\omega}{}^{2}{}_{12} = -\sqrt{Y_G}, \tag{126b}$$

$$\overset{\bullet}{\omega}{}^{1}{}_{33} = -\sqrt{Y_G}\sin\theta, \tag{126c}$$

$$\overset{\bullet}{\omega}{}^{2}{}_{33} = -\cos\theta, \tag{126d}$$

$$\overset{\bullet}{\omega}{}^{3}{}_{13} = \sin\theta\sqrt{Y_G}, \tag{126e}$$

and

$$\overset{\bullet}{\omega}{}^{3}{}_{23} = \cos\theta. \tag{126f}$$

Furthemore, the torsion components $\overset{\bullet}{T}{}^{c}{}_{\mu\nu}$ that are not equal to zero are

$$\overset{\bullet}{T}{}^{0}{}_{01} = -\overset{\bullet}{T}{}^{0}{}_{10} = \frac{-Y,r + Y_{G,r}}{2\sqrt{Y}}, \tag{127a}$$

$$\overset{\bullet}{T}{}^{2}{}_{12} = -\overset{\bullet}{T}{}^{2}{}_{21} = \frac{-\sqrt{Y} + \sqrt{Y_G}}{\sqrt{Y}}, \tag{127b}$$

and

$$\overset{\bullet}{T}{}^{3}{}_{13} = -\overset{\bullet}{T}{}^{3}{}_{31} = \frac{\sin\theta}{\sqrt{Y}}\left(-\sqrt{Y} + \sqrt{Y_G}\right). \tag{127c}$$

Whereas, the nonzero torsion components $\overset{\bullet}{T}{}^{\alpha}{}_{\mu\nu}$ are

$$\overset{\bullet}{T}{}^{0}{}_{01} = -\overset{\bullet}{T}{}^{0}{}_{10} = \frac{-Y,_r + Y_{G,r}}{2Y}, \tag{128a}$$

and

$$\overset{\bullet}{T}{}^{2}{}_{12} = -\overset{\bullet}{T}{}^{2}{}_{21} = \overset{\bullet}{T}{}^{3}{}_{13} = -\overset{\bullet}{T}{}^{3}{}_{31} = -\frac{-\sqrt{Y} + \sqrt{Y_G}}{r\sqrt{Y}}, \tag{128b}$$

The superpotential components $\overset{\bullet}{S}{}_{\alpha}{}^{\mu\nu}$ are calculated using equation (62). The nonvanishing superpotential components $\overset{\bullet}{S}{}_{\alpha}{}^{\mu\nu}$ are

$$\overset{\bullet}{S}{}_{0}{}^{01} = \overset{\bullet}{S}{}_{0}{}^{10} = \frac{-Y + \sqrt{Y_G Y}}{r}, \tag{129a}$$

and

$$\overset{\bullet}{S}{}_{2}{}^{12} = \overset{\bullet}{S}{}_{2}{}^{21} = -\frac{1}{4r}\left(r\left(-Y,_r + Y_{G,r}\right) + 2\sqrt{Y}\left(-\sqrt{Y} + \sqrt{Y_G}\right)\right). \tag{129b}$$

According to equation (25), the scalar torsion $\overset{\bullet}{T}$ is given by

$$\overset{\bullet}{T} = \frac{2}{r^2\sqrt{Y}}\left(-\sqrt{Y}+\sqrt{Y_G}\right)\left(r\left(-Y_{,r}+Y_{G,r}\right)+\sqrt{Y}\left(-\sqrt{Y}+\sqrt{Y_G}\right)\right). \tag{130}$$

The electric field produced by electrical charged of the black hole is radial. The electric field is of the form

$$F = E_r\ e^1 \wedge e^0. \tag{131}$$

where E_r is the radial component of electric field. Therefore, the component of the electromagnetic field that is not equal to zero ($F_{\mu\nu}$) is

$$F_{10} = -F_{01} = E_r. \tag{132}$$

The components of the contortion tensor $\overset{\bullet}{K}{}^{\mu\nu}{}_{\alpha}$ and $\overset{\bullet}{K}{}^{\mu}{}_{\nu\alpha}$ are calculated using equation (8) and the nonvanishing components of contortion are

$$\overset{\bullet}{K}{}^{01}{}_{0} = -\overset{\bullet}{K}{}^{10}{}_{0} = -\frac{1}{2}\left(-Y_{,r}+Y_{G,r}\right), \tag{133a}$$

$$\overset{\bullet}{K}{}^{12}{}_{2} = -\overset{\bullet}{K}{}^{21}{}_{2} = \overset{\bullet}{K}{}^{13}{}_{3} = -\overset{\bullet}{K}{}^{31}{}_{3} = \frac{\sqrt{Y}}{r}\left(-\sqrt{Y}+\sqrt{Y_G}\right), \tag{133b}$$

$$\overset{\bullet}{K}{}^{0}{}_{10} = \frac{1}{2Y}\left(-Y_{,r}+Y_{G,r}\right), \tag{133c}$$

$$\overset{\bullet}{K}{}^{1}{}_{00} = \frac{A}{2}\left(-A_{,r}+A_{G,r}\right), \tag{133d}$$

$$\overset{\bullet}{K}{}^{1}{}_{22} = -r\sqrt{Y}\left(-\sqrt{Y}+\sqrt{Y_G}\right), \tag{133e}$$

$$\overset{\bullet}{K}{}^{1}{}_{33} = -r\sin^2\theta\sqrt{Y}\left(-\sqrt{Y}+\sqrt{Y_G}\right) \tag{133f}$$

and

$$\overset{\bullet}{K}{}^{2}{}_{12} = \overset{\bullet}{K}{}^{3}{}_{13} = \frac{1}{r\sqrt{Y}}\left(-\sqrt{Y}+\sqrt{Y_G}\right). \tag{133g}$$

In this case, the curvature $\overset{\bullet}{R}{}^{a}{}_{b\nu\mu}$ also does not vanish. Where the non-vanishing

components of the curvature $\overset{\bullet}{R}{}^{a}{}_{b\nu\mu}$ are given by

$$\overset{\bullet}{R}{}^{0}{}_{110} = -\overset{\bullet}{R}{}^{0}{}_{101} = \overset{\bullet}{R}{}^{1}{}_{010} = -\overset{\bullet}{R}{}^{1}{}_{001} = -\frac{\sqrt{Y_{G,rr}}}{2}, \tag{134a}$$

$$\overset{\bullet}{R}{}^{0}{}_{220} = -\overset{\bullet}{R}{}^{0}{}_{202} = \overset{\bullet}{R}{}^{2}{}_{020} = -\overset{\bullet}{R}{}^{2}{}_{002} = -\frac{\sqrt{Y_G}}{2}Y_{G,r}, \tag{134b}$$

$$\overset{\bullet}{R}{}^{0}{}_{303} = -\overset{\bullet}{R}{}^{0}{}_{330} = \overset{\bullet}{R}{}^{3}{}_{030} = -\overset{\bullet}{R}{}^{3}{}_{003} = -\frac{\sin\theta}{2}\sqrt{Y_G}Y_{G,r}, \tag{134c}$$

$$\overset{\bullet}{R}{}^{1}{}_{212} = -\overset{\bullet}{R}{}^{1}{}_{221} = \overset{\bullet}{R}{}^{2}{}_{121} = -\overset{\bullet}{R}{}^{2}{}_{112} = -\frac{Y_{G,r}}{2\sqrt{Y_G}}, \tag{134d}$$

$$\overset{\bullet}{R}{}^{1}{}_{313} = -\overset{\bullet}{R}{}^{1}{}_{331} = \overset{\bullet}{R}{}^{3}{}_{131} = -\overset{\bullet}{R}{}^{3}{}_{113} = -\frac{\sin\theta}{2\sqrt{Y_G}}Y_{G,r}, \tag{134e}$$

$$\overset{\bullet}{R}{}^{2}{}_{323} = -\overset{\bullet}{R}{}^{2}{}_{332} = \overset{\bullet}{R}{}^{3}{}_{232} = -\overset{\bullet}{R}{}^{3}{}_{223} = -\sin\theta\left(-1+Y_G\right). \tag{134f}$$

Since, the curvature $\overset{\bullet}{R}{}^{a}{}_{b\nu\mu}$ does not vanish the geometry of the space time involved here are also the Riemann-Cartan geometry instead of the Weitzenböck geometry.

4.2. *Singularities*

Capozziello et. al. have analyzed the electric black hole singularities using two approaches. In their first approach, they used the scalar torsion $\overset{\bullet}{T}$ calculated from the tetrad and Weitzenböck connections. In their second approach, they used the Ricci scalar $\overset{\circ}{R}$ and the Kretschmann scalar $\overset{\circ}{R}{}^{\rho\lambda\nu\mu}\overset{\circ}{R}{}_{\rho\lambda\nu\mu}$ corresponding to the Levi-Civita connection. The metric considered by Capozziello et. al. is more general than the metric given in equation (116), i.e. $g_{00} \neq g_{11}$ but their singularity analysis can be applied to special case, where $g_{00} = g_{11}$. In the first case ($Y(r) = k - \frac{2GM}{r}$), the Kretschmann scalar and the torsion $\overset{\bullet}{T}$ are infinite only at $r = 0$ [27]. Therefore, the singularity at $r = 0$ is the only physical singularity. In the first case, there is a horizon $r_h = 2GM$ where the Kretschmann scalar is finite. Now, we calculate the scalar $\overset{\bullet}{T}{}^{\mu\nu\lambda}\overset{\bullet}{T}{}_{\mu\nu\lambda}$, the scalar $\overset{\bullet}{T}{}^{\mu}\overset{\bullet}{T}{}_{\mu}$, and the scalar $\overset{\bullet}{T}$ respectively as follows

$$\begin{aligned}\overset{\bullet}{T}{}^{\mu\nu\lambda}\overset{\bullet}{T}{}_{\mu\nu\lambda} = &-\frac{1}{r^3\left(r-2GM\right)}\left(4\left(GM\right)^2+8\left(r-2GM\right)\right.\\ &\left.\times\left(-\sqrt{r-2GM}+\sqrt{r}\right)^2\right),\end{aligned} \tag{135}$$

$$\begin{aligned}\overset{\bullet}{T}{}^{\mu}\overset{\bullet}{T}{}_{\mu} = &-\frac{1}{4r^2\left(r-2GM\right)}\left(-2GM+4\sqrt{r-2GM}\right.\\ &\left.\times\left(-\sqrt{r-2GM}+\sqrt{r}\right)\right)^2,\end{aligned} \tag{136}$$

and

$$\overset{\bullet}{T} = -\frac{2}{r^3\sqrt{r-2GM}}\left(2r\sqrt{r-2GM}+2GM\sqrt{r}-\sqrt{r}r\right), \tag{137}$$

with $k = 1$. It is clear that all three scalars are infinite at the horizon $r = r_h$. Since the physical singularity exists only at $r = 0$, therefore, the singularity analysis using the three scalars is not in accordance with the analysis using the Kretschmann scalar.

Likewise for the second case ($Y(r) = k - \frac{2GM}{r} - \frac{\Lambda r^2}{3}$), the physical singularity exists only at $r = 0$. Both the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{T}$ are infinite at $= 0$ [27]. The Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$ is finite at the horizon.

$$r_h = -\frac{\Lambda + \left(3GM\Lambda^2 + \sqrt{-\Lambda^3 + (3GM\Lambda^2)^2}\right)^{\frac{2}{3}}}{\Lambda\left(3GM\Lambda^2 + \sqrt{-\Lambda^3 + (3GM\Lambda^2)^2}\right)^{\frac{1}{3}}}.$$

In this second case, the scalar $\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda}$, the scalar $\overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu}$, and the torsion scalar $\overset{\bullet}{T}$ are given respectively by

$$\begin{aligned}\overset{\bullet}{T}^{\mu\nu\lambda}\overset{\bullet}{T}_{\mu\nu\lambda} =& \frac{1}{2r^3\left(r - 2GMr - \frac{\Lambda r^3}{3}\right)}\left(36\left(GM\right)^2 + \frac{16}{9}\lambda^2 r^6 + \frac{44}{3}GM\Lambda r^3\right.\\ &\left. -48GMr - \frac{32}{3}\Lambda r^4 + 16r^2\right), \qquad (138)\\ \overset{\bullet}{T}^{\mu}\overset{\bullet}{T}_{\mu} =& -\frac{1}{4r^2\left(r - 2GM - \frac{\Lambda r^3}{3}\right)}\left(6GM - 4r + \frac{4}{3}\Lambda r^3 + 4\sqrt{1 - \frac{\Lambda r^3}{3}}\right.\\ &\left.\times\left(r - 2GM - \frac{\Lambda r^3}{3}\right)\right)^2, \qquad (139)\end{aligned}$$

and

$$\begin{aligned}\overset{\bullet}{T} =& \frac{2}{r^3\sqrt{r - 2GM - \frac{\Lambda r^3}{3}}}\left(\sqrt{r - 2GM - \frac{\Lambda r^3}{3}} + \sqrt{r - \frac{\Lambda r^3}{3}}\right)\\ &\times\left(-2GM + \sqrt{r - 2GM - \frac{\Lambda r^3}{3}}\left(\sqrt{r - 2GM - \frac{\Lambda r^3}{3}} + \sqrt{r - \frac{\Lambda r^3}{3}}\right)\right), \qquad (140)\end{aligned}$$

with $k = 1$. All three values are infinite at horizon $r = r_h$, where the Kretschmann scalar is finite. Therefore, the singularity analysis using the three scalars obtained from the torsion is also not in accordance with the analysis using the Kretschmann scalar.

In the third case ($Y(r) = k - \frac{2GM}{r} + \frac{GQ^2}{4\pi}$), the physical singularity exists only at $r = 0$. But at horizon $r = r_h$, the Kretschmann scalar is finite, while the torsion scalar $\overset{\bullet}{T}$ is infinite [27]. In the fourth case, the singularity analysis using both approaches yield the same conclusion that the only physical singularity is at $r = 0$ [27].

Now, as in the case of magnetic black hole, we calculate here also the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ from the Weitzenbock connection $\overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu}$. The non-vanishing components of the Weitzenbock connection are given by

$$\overset{\bullet}{\Gamma}{}^{0}{}_{01} = \frac{Y_{,r}}{2Y}, \tag{141a}$$

$$\overset{\bullet}{\Gamma}{}^{0}{}_{10} = \frac{Y_{G,r}}{2Y}, \tag{141b}$$

$$\overset{\bullet}{\Gamma}{}^{1}{}_{00} = \frac{Y}{2}Y_{G,r}\,, \tag{141c}$$

$$\overset{\bullet}{\Gamma}{}^{1}{}_{11} = -\frac{Y_{,r}}{2Y}, \tag{141d}$$

$$\overset{\bullet}{\Gamma}{}^{1}{}_{22} = -r\sqrt{Y_G Y}, \tag{141e}$$

$$\overset{\bullet}{\Gamma}{}^{1}{}_{33} = -r\sin^2\theta\sqrt{Y_G Y}, \tag{141f}$$

$$\overset{\bullet}{\Gamma}{}^{2}{}_{12} = \overset{\bullet}{\Gamma}{}^{3}{}_{13} = \frac{\sqrt{Y_G}}{r\sqrt{Y}}, \tag{141g}$$

and

$$\overset{\bullet}{\Gamma}{}^{2}{}_{21} = \overset{\bullet}{\Gamma}{}^{3}{}_{31} = \frac{1}{r}. \tag{141h}$$

From equation (141) we obtain

(a) the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$:

$$\begin{aligned}\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} =& \frac{1}{r^4 Y_G Y}\Big[r^4 Y_G Y^2\left(-Y_{,rr}+Y_{G,rr}\right)^2 + r^2 Y_G Y\left(-\sqrt{Y}Y_{,r}\right.\\ &\left.+\sqrt{Y_G}Y_{G,r}\right)^2 + 2r^2Y^2\left(-\sqrt{Y_G}Y_{,r}+\sqrt{Y}Y_{G,r}\right)^2\\ &+r^2 Y_G\left(YY_{,r}-\sqrt{Y_G Y}Y_{G,r}\right)^2 + 4Y_G Y^2\left(Y-Y_G\right)^2\Big].\end{aligned} \tag{142}$$

and

(b) the scalar $\overset{\bullet}{Q}$:

$$\begin{aligned}\overset{\bullet}{Q} =& \frac{1}{r^2\sqrt{Y_G Y}}\Big[r^2\sqrt{Y_G Y}\left(-Y_{,rr}+Y_{G,rr}\right) - 4r\sqrt{Y_G Y}Y_{,}r + 2r\left(Y+Y_G\right)Y_{G,r}\\ &+2\sqrt{Y_G Y}\left(-Y+Y_G\right)\Big].\end{aligned} \tag{143}$$

For the first case, the scalars $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ and $\overset{\bullet}{Q}$ are

$$\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} = \frac{48\left(GM\right)^4}{r^6}, \tag{144}$$

and

$$\overset{\bullet}{Q} = 0. \tag{145}$$

The scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ is infinite at $r = 0$, and finite at $r = r_h$ which is consistent with the values of the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$. However, the scalar $\overset{\bullet}{Q}$ is constant for any r.

For the second case, the scalars $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ and $\overset{\bullet}{Q}$ are

$$\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} = \frac{1}{r^6}\left(44\,(GM)^2 + \frac{20}{9}\Lambda^2 r^6 - \frac{8}{3}GM\Lambda r^3\right), \tag{146}$$

and

$$\overset{\bullet}{Q} = \frac{1}{r^3}\left(-2GM + \frac{10}{3}\Lambda r^3\right) \tag{147}$$

respectively. Both scalars are infinite at $r = 0$ and finite at $r = r_h$. Thus the value of the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ at $r = 0$ and at $r = r_h$ is in accordance with the values of the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$.

For the third case, the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ and the scalar $\overset{\bullet}{Q}$ are given respectively by

$$\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} = \frac{1}{r^8}\left(48\,(GM)^2\,r^2 + \frac{7}{2}\left(\frac{GQ^2}{\pi}\right)^2 - \frac{24}{\pi}G^2MQ^2r\right), \tag{148}$$

and

$$\overset{\bullet}{Q} = 0. \tag{149}$$

The value of the scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ is infinite at $r = 0$ and finite at $r = r_h$. It is in accordance with the value of the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$ at the same place.

The values of the scalars $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ and $\overset{\bullet}{Q}$ in the fourth case are

$$\begin{aligned}\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu} = \frac{1}{r^8}\Bigg(&44\,(GM)^2\,r^2 + \frac{13Q^2}{4\pi^2} + \frac{16}{9}\Lambda^2 r^8 - \frac{52}{4\pi}G^2MQ^2r \\ &- \frac{8}{3}GM\Lambda r^5\Bigg),\end{aligned} \tag{150}$$

and

$$\overset{\bullet}{Q} = \frac{10\Lambda}{3} \tag{151}$$

respectively. The scalar $\overset{\bullet}{Q}^{\rho\lambda\nu\mu}\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ is infinite at $r = 0$, and finite at $r = r_h$. It is in accordance with the value the Kretschmann scalar $\overset{\circ}{R}^{\rho\lambda\nu\mu}\overset{\circ}{R}_{\rho\lambda\nu\mu}$ at $r = 0$ and at $r = r_h$.

4.3. *Gravitoelectromagnetism*

The Cartesian components of torsion $\overset{\bullet}{T'^{a}}_{\mu\nu}$ in Catersian coordinate system are obtained through coordinate transformation given by the following matrix transformation

$$A_{\mu\nu} = \begin{pmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \\ \frac{\partial r}{\partial \tau} & \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial \tau} & \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial \tau} & \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix}. \tag{152}$$

The torsion tensor $\overset{\bullet}{T'^{a}}_{\mu\nu}$ is related to gravitoelectric and gravitomagnetic fields through equation (35), namely

$$\begin{aligned} \overset{\bullet}{T'^{0}}_{\alpha\beta} &= \begin{pmatrix} 0 & \frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\sin\theta\cos\phi & \frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\sin\theta\sin\phi & \frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\cos\theta \\ -\frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\sin\theta\cos\phi & 0 & 0 & 0 \\ -\frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\sin\theta\sin\phi & 0 & 0 & 0 \\ -\frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\cos\theta & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E^{0}{}_{x} & E^{0}{}_{y} & E^{0}{}_{z} \\ -E^{0}{}_{x} & 0 & -B^{0}{}_{z} & B^{0}{}_{y} \\ -E^{0}{}_{y} & B^{0}{}_{z} & 0 & -B^{0}{}_{x} \\ -E^{0}{}_{z} & -B^{0}{}_{y} & B^{0}{}_{x} & 0 \end{pmatrix}, \end{aligned} \tag{153a}$$

$$\overset{\bullet}{T'^{1}}_{\alpha\beta} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0 & E^{1}{}_{x} & E^{1}{}_{y} & E^{1}{}_{z} \\ -E^{1}{}_{x} & 0 & -B^{1}{}_{z} & B^{1}{}_{y} \\ -E^{1}{}_{y} & B^{1}{}_{z} & 0 & -B^{1}{}_{x} \\ -E^{1}{}_{z} & -B^{1}{}_{y} & B^{1}{}_{x} & 0 \end{pmatrix}, \tag{153b}$$

$$\begin{aligned} \overset{\bullet}{T'^{2}}_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}r\cos\phi \\ 0 & 0 & 0 & -\frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}r\sin\phi \\ 0 & \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}r\cos\phi & \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}r\sin\phi & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E^{2}{}_{x} & E^{2}{}_{y} & E^{2}{}_{z} \\ -E^{2}{}_{x} & 0 & -B^{2}{}_{z} & B^{2}{}_{y} \\ -E^{2}{}_{y} & B^{2}{}_{z} & 0 & -B^{2}{}_{x} \\ -E^{2}{}_{z} & -B^{2}{}_{y} & B^{2}{}_{x} & 0 \end{pmatrix}, \end{aligned} \tag{153c}$$

and

$$\overset{\bullet}{T'^{3}}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}\sin^2\theta & \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}\sin\theta\cos\theta\sin\phi \\ 0 & -\frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}\sin^2\theta & 0 & -\frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}\sin\theta\cos\theta\cos\phi \\ 0 & -\frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}\sin\theta\cos\theta\sin\phi & \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}\sin\theta\cos\theta\cos\phi & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & E^3{}_x & E^3{}_y & E^3{}_z \\ -E^3{}_x & 0 & -B^3{}_z & B^3{}_y \\ -E^3{}_y & B^3{}_z & 0 & -B^3{}_x \\ -E^3{}_z & -B^3{}_y & B^3{}_x & 0 \end{pmatrix}. \tag{153d}$$

From equations (153) we conclude that

$$E^0{}_x = \frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\sin\theta\cos\phi, \tag{154a}$$

$$E^0{}_y = \frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\sin\theta\sin\phi, \tag{154b}$$

$$E^0{}_z = \frac{(-Y_{,r}+Y_{G,r})}{2\sqrt{Y}}\cos\theta, \tag{154c}$$

$$B^2{}_x = \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}r\sin\phi, \tag{154d}$$

$$B^2{}_y = -\frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}}r\cos\phi, \tag{154e}$$

$$B^3{}_x = \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}\ r}\sin\theta\cos\theta\cos\phi, \tag{154f}$$

$$B^3{}_y = \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}\ r}\sin\theta\cos\theta\sin\phi, \tag{154g}$$

and

$$B^3{}_z = \frac{\left(-\sqrt{Y}+\sqrt{Y_G}\right)}{\sqrt{Y}\ r}\sin^2\theta. \tag{154h}$$

Whereas the other components of the gravitoelectric and the gravitomagnetic fields vanish. Thus, the gravitoelectric and the gravitomagnetic fields are given by

$$\vec{E}^0 = \frac{(-Y_{,r} + Y_{G,r})}{2\sqrt{Y}}\ \hat{r}, \tag{155a}$$

$$\vec{E}^1 = \vec{E}^2 = \vec{E}^3 = \vec{0}, \tag{155b}$$

$$\vec{B}^0 = \vec{B}^1 = \vec{0}, \tag{155c}$$

$$\vec{B}^2 = -\frac{\left(-\sqrt{Y} + \sqrt{Y_G}\right)}{\sqrt{Y}} r\ \hat{\theta}, \tag{155d}$$

and

$$\vec{B}^3 = \frac{\left(-\sqrt{Y} + \sqrt{Y_G}\right)}{\sqrt{Y}} \sin\theta\cos\theta \left(\sin\theta\ \hat{r} + \cos\theta\ \hat{\phi}\right). \tag{155e}$$

The gravitoelectric field that does not vanish is only the gravitoelectric field in sector $a = 0$, namely $\vec{E}^0$. The gravitoelectric field $\vec{E}^0$ is radial and its magnitude depends only on r. At $r = 0$, the magnitude of $\vec{E}^0$ tends to infinity. There are two sectors, namely $a = 2$ and $a = 3$, in which the gravitomagnetic fields do not vanish. The gravitomagnetic field in sector $a = 2$ is tangent to meridian, whose magnitude also depends on r. At $r = 0$, the gravitomagnetic field $\vec{B}^2$ is finite. Furthermore, the gravitomagnetic field in sector $a = 3$ has non zero radial component as well as non zero azimutal component. The magnitude of the gravitomagnetic field in this sector depends not only on r but also depends on θ. From the equations (155), we see that the gravitoelectric and gravitomagnetic fields depend with the mass and the electric charge of the black hole as well as the cosmological constant. If we assume the black hole is a charged sphere with the cosmological constant assumed to be zero (the case of the Reissner–Nordström black hole) and the mass containing term considered to be very small, then the gravitoelectric field (equation (155a)) reduces to Coloumb's law for a charged sphere.

5. Conclusion

We have derived non-covariant and covariant field equations involving the cosmological constant and electromagnetic field in TEGR, namely equation (34) and (33), which differs by the appearance of the spin connection $\overset{\bullet}{\omega}{}^{\alpha}{}_{\mu\nu}$. The solution of the field equation in this case of magnetic black hole given equation (47-51).

The covariant formulation of teleparallel gravity for magnetic and electric black hole are reflected by the appearance of the spin connection in the expressions of the torsion, the superpotential, the contorsion, and of the curvatures. In this case, the curvature of the spin connection does not vanish. The analysis of the singularities of the solution by using the corresponding Kretschmann scalars as well as the scalars given in equation (101) and equation (142) leads to the conclusion that the singularity at $r = 0$ is physical and the singularity at horizon is a pseudo-singularity.

The analysis of the singularity by using the scalar obtained from the torsion is not consistent.

The gravitoelectric and gravitomagnetic fields are obtained from the torsion tensors after the components of the torsion tensor in Cartesian coordinate system are obtained. The gravitoelectric and gravitomagnetic fields in some sectors vanish. Those fields depend with the mass and the charge of the black hole as well as the cosmological constant.

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Appendix A. The non zero component of the Riemannian curvature tensor concerning the magnetic black hole:

$$\overset{\circ}{R}{}^{0}{}_{101} = -\overset{\circ}{R}{}^{0}{}_{110} = -\frac{A_{,rr}}{2A}, \tag{A.1a}$$

$$\overset{\circ}{R}{}^{0}{}_{202} = -\overset{\circ}{R}{}^{0}{}_{220} = \overset{\circ}{R}{}^{0}{}_{303} = -\overset{\circ}{R}{}^{0}{}_{330} = \overset{\circ}{R}{}^{1}{}_{212} = -\overset{\circ}{R}{}^{1}{}_{221} =$$
$$\overset{\circ}{R}{}^{1}{}_{313} = -\overset{\circ}{R}{}^{1}{}_{331} = \overset{\circ}{R}{}^{2}{}_{002} = -\overset{\circ}{R}{}^{2}{}_{020} = \overset{\circ}{R}{}^{3}{}_{113} = -\overset{\circ}{R}{}^{3}{}_{131} = -\frac{c_1 r}{2} A_{,r}\,, \tag{A.1b}$$

$$\overset{\circ}{R}{}^{1}{}_{001} = -\overset{\circ}{R}{}^{1}{}_{010} = \overset{\circ}{R}{}^{3}{}_{003} = -\overset{\circ}{R}{}^{3}{}_{030} = -\frac{c_1 A}{2} A_{,rr}\,, \tag{A.1c}$$

$$\overset{\circ}{R}{}^{2}{}_{112} = -\overset{\circ}{R}{}^{2}{}_{121} = -\frac{A_{,r}}{2rA}, \tag{A.1d}$$

$$\overset{\circ}{R}{}^{2}{}_{323} = -\overset{\circ}{R}{}^{2}{}_{332} = -c_1 A, \tag{A.1e}$$

$$\overset{\circ}{R}{}^{3}{}_{223} = -\overset{\circ}{R}{}^{3}{}_{232} = c_1 A, \tag{A.1f}$$

$$\overset{\circ}{R}_{0101} = -\overset{\circ}{R}_{0110} = -\frac{A_{,rr}}{2}, \tag{A.1g}$$

$$\overset{\circ}{R}_{0202} = -\overset{\circ}{R}_{0220} = \overset{\circ}{R}_{0303} = -\overset{\circ}{R}_{0330} = -\frac{-c_1 r A}{2} A_{,r}\,, \tag{A.1h}$$

$$\overset{\circ}{R}_{1001} = -\overset{\circ}{R}_{1010} = \frac{A_{,rr}}{2}, \tag{A.1i}$$

$$\overset{\circ}{R}_{1212} = -\overset{\circ}{R}_{1221} = \overset{\circ}{R}_{1313} = -\overset{\circ}{R}_{1331} = \frac{r}{2A} A_{,r}\,, \tag{A.1j}$$

$$\overset{\circ}{R}_{2002} = -\overset{\circ}{R}_{2020} = \overset{\circ}{R}_{3003} = -\overset{\circ}{R}_{3030} = \frac{-c_1 r A}{2} A_{,r}\,, \tag{A.1k}$$

$$\overset{\circ}{R}_{2112} = -\overset{\circ}{R}_{2121} = \overset{\circ}{R}_{3113} = -\overset{\circ}{R}_{3131} = -\frac{r}{2A} A_{,r}\,, \tag{A.1l}$$

$$\overset{\circ}{R}_{2323} = -\overset{\circ}{R}_{2332} = c_1 r^2 A, \tag{A.1m}$$

and

$$\overset{\circ}{R}_{3223} = -\overset{\circ}{R}_{3232} = -c_1 r^2 A. \tag{A.1n}$$

Appendix B. The non zero components $\overset{\bullet}{Q}{}^{\rho}{}_{\lambda\nu\mu}$ and $\overset{\bullet}{Q}_{\rho\lambda\nu\mu}$ concerning the magnetic black hole:

$$\overset{\bullet}{Q}{}^{0}{}_{101} = -\overset{\bullet}{Q}{}^{0}{}_{110} = -\frac{1}{2A}\left(-A_{,rr} + A_{G,rr}\right), \tag{B.1a}$$

$$\overset{\bullet}{Q}{}^{0}{}_{202} = -\overset{\bullet}{Q}{}^{0}{}_{220} = \overset{\bullet}{Q}{}^{0}{}_{303} = -\overset{\bullet}{Q}{}^{0}{}_{330} = -\frac{c_1 r}{2\sqrt{A}}\left(-\sqrt{A}A_{,r} + \sqrt{A_G}A_{G,r}\right), \tag{B.1b}$$

$$\overset{\bullet}{Q}{}^{1}{}_{001} = -\overset{\bullet}{Q}{}^{1}{}_{010} = -\frac{c_1 A}{2}\left(-A_{,rr} + A_{G,rr}\right), \tag{B.1c}$$

$$\overset{\bullet}{Q}{}^{1}{}_{212} = -\overset{\bullet}{Q}{}^{1}{}_{221} = \overset{\bullet}{Q}{}^{1}{}_{313} = -\overset{\bullet}{Q}{}^{1}{}_{331} = -\frac{c_1 r}{2\sqrt{A_G}}\left(-\sqrt{A_G}A_{,r} + \sqrt{A}A_{G,r}\right), \tag{B.1d}$$

$$\overset{\bullet}{Q}{}^{2}{}_{002} = -\overset{\bullet}{Q}{}^{2}{}_{020} = \overset{\bullet}{Q}{}^{3}{}_{003} = -\overset{\bullet}{Q}{}^{3}{}_{030} = -\frac{c_1\sqrt{A}}{2r}\left(-\sqrt{A}A_{,r} + \sqrt{A_G}A_{G,r}\right), \tag{B.1e}$$

$$\overset{\bullet}{Q}{}^{2}{}_{112} = -\overset{\bullet}{Q}{}^{2}{}_{121} = \overset{\bullet}{Q}{}^{3}{}_{113} = -\overset{\bullet}{Q}{}^{3}{}_{131} = \frac{1}{2rA\sqrt{A_G}}\left(-\sqrt{A_G}A_{,r} + \sqrt{A}A_{G,r}\right), \tag{B.1f}$$

$$\overset{\bullet}{Q}{}^{2}{}_{323} = -\overset{\bullet}{Q}{}^{2}{}_{332} = \overset{\bullet}{Q}{}^{3}{}_{232} = -\overset{\bullet}{Q}{}^{2}{}_{223} = c_1\left(A - A_G\right), \tag{B.1g}$$

$$\overset{\bullet}{Q}_{0101} = -\overset{\bullet}{Q}_{0110} = -\frac{1}{2}\left(-A_{,rr} + A_{G,rr}\right), \tag{B.1h}$$

$$\overset{\bullet}{Q}_{0202} = -\overset{\bullet}{Q}_{0220} = \overset{\bullet}{Q}_{0303} = -\overset{\bullet}{Q}_{0330} = -\frac{c_1 r}{2}\left(-AA_{,r} + \sqrt{A_G A}A_{G,r}\right), \tag{B.1i}$$

$$\overset{\bullet}{Q}_{1001} = -\overset{\bullet}{Q}_{1010} = -\frac{1}{2}\left(-A_{,rr} + A_{G,rr}\right), \tag{B.1j}$$

$$\overset{\bullet}{Q}_{1212} = -\overset{\bullet}{Q}_{1221} = \overset{\bullet}{Q}_{1313} = -\overset{\bullet}{Q}_{1331} = \frac{r}{2\sqrt{A_G}A}\left(-\sqrt{A_G}A_{,r} + \sqrt{A}A_{G,r}\right), \tag{B.1k}$$

$$\overset{\bullet}{Q}_{2002} = -\overset{\bullet}{Q}_{2020} = \overset{\bullet}{Q}_{3003} = -\overset{\bullet}{Q}_{3030} = \frac{c_1 r\sqrt{A}}{2}\left(-\sqrt{A}A_{,r} + \sqrt{A_G}A_{G,r}\right), \tag{B.1l}$$

$$\overset{\bullet}{Q}_{2112} = -\overset{\bullet}{Q}_{2121} = \overset{\bullet}{Q}_{3113} = -\overset{\bullet}{Q}_{3131} = -\frac{r}{2\sqrt{A_G}A}\left(-\sqrt{A_G}A_{,r} + \sqrt{A}A_{G,r}\right), \tag{B.1m}$$

and

$$\overset{\bullet}{Q}_{2323} = -\overset{\bullet}{Q}_{2332} = \overset{\bullet}{Q}_{3232} = -\overset{\bullet}{Q}_{2332} = -c_1 r^2\left(A - A_G\right). \tag{B.1n}$$

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