HAUSDORFF DIMENSION AND EXACT APPROXIMATION ORDER IN $\mathbb{R}^n$

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ABSTRACT. Given a non-increasing function $\psi\colon \mathbb{N} \to \mathbb{R}^+$ such that $s^{\frac{n+1}{n}}\psi(s)$ tends to zero as s goes to infinity, we show that the set of points in $\mathbb{R}^n$ that are exactly ψ-approximable is non-empty, and we compute its Hausdorff dimension. For $n \geq 2$, this answers questions of Jarník and of Beresnevich, Dickinson and Velani.

1. INTRODUCTION

Given a non-increasing function $\psi\colon \mathbb{N} \to \mathbb{R}^*_+$, one defines the set of ψ-approximable points in $\mathbb{R}^n$ as

$$W(\psi) = \left\{ x \in \mathbb{R}^n \mid \text{there exists infinitely many } \frac{p}{q} \in \mathbb{Q}^n \text{ with } \left\| x - \frac{p}{q} \right\| < \psi(q) \right\},$$

where the norm on $\mathbb{R}^n$ is given by $\|x\| = \max_{1\leq i \leq n}|x_i|$ if $x = (x_1, \ldots, x_n)$. It follows from Dirichlet's celebrated theorem that for the function

$$\psi_{\frac{n+1}{n}}(s) = s^{-\frac{n+1}{n}}$$

one has $W(\psi_{\frac{n+1}{n}}) = \mathbb{R}^n$. On the other hand, for $\tau > 0$ and $\psi_\tau(s) = s^{-\tau}$, Jarník [13] showed the following theorem.

Theorem 1 (Jarník, 1930)**.** *Let $n \geq 1$ be an integer. For every $\tau \geq \frac{n+1}{n}$, one has*

$$\dim_H W(\psi_\tau) = \frac{n+1}{\tau}.$$

This shows in particular that if $\tau' > \tau$, then the set $W(\psi_{\tau'})$ is strictly smaller than $W(\psi_\tau)$. In fact, for τ large enough Jarník was able to sharpen this result, as he observed that for every $\tau > 2$ and every $c < 1$, one even has a strict inclusion

$$W(c\psi_\tau) \subsetneq W(\psi_\tau).$$

For $n > 1$, the condition $\tau > 2$ is unnatural, and Jarník's remarks at the end of his paper suggest that the result should hold for any $\tau > \frac{n+1}{n}$. One goal of this paper is to show that this is indeed the case.

More precisely, defining the set of *exact ψ-approximable* vectors in $\mathbb{R}^n$ by

$$E(\psi) = W(\psi) \setminus \bigcup_{c<1} W(c\psi),$$

we shall prove the following result generalizing Jarník [13, Satz 6].

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Theorem 2 (Existence of exact ψ-approximable vectors)**.** *Let $n \geq 1$ be an integer. If $\psi\colon \mathbb{N} \to \mathbb{R}_+^*$ is non-increasing and satisfies $\lim_{s\to\infty} s^{\frac{n+1}{n}}\psi(s) = 0$, then $E(\psi) \neq \varnothing$.*

In the particular case $n = 1$, Yann Bugeaud [3, 4] and then Bugeaud and Moreira [5] studied the sets $E(\psi)$ from the point of view of Hausdorff dimension, and showed that $\dim_H E(\psi) = \dim_H W(\psi)$ provided $s^2\psi(s)$ tends to zero at infinity. Under the assumption that $\lim_{s\to\infty} -\frac{\log\psi(x)}{\log(x)}$ exists and is at least 2, Reynold Fregoli [12] was able to compute the Hausdorff dimension of $E(\psi)$ in the case $n \geq 3$ but as Jarník himself already observed, the condition $\psi(s) = o(s^{-2})$ is too restrictive when $n \geq 2$, and should be replaced by $\psi(s) = o(s^{-\frac{n+1}{n}})$. In [1] Bandi, Ghosh, and Nandi studied the exact approximation problem in the abstract set-up of Ahlfors regular metric spaces but again, their assumptions imply in particular that the abstract rational points satisfy a certain well-separatedness property, which the rationals in $\mathbb{R}^n$ do not satisfy for $n \geq 2$. A variant of the problem was studied by Beresnevich, Dickinson, and Velani [2] who showed that the set

$$D(\psi_1, \psi_2) = W(\psi_1) \setminus W(\psi_2)$$

satisfies $\dim_H D(\psi_1, \psi_2) = \dim_H W(\psi_1)$ under certain assumptions that imply in particular that $\frac{\psi_1(s)}{\psi_2(s)}$ tends to infinity as s goes to infinity. They observed however that their techniques completely fail if one takes $\psi_2 = c\psi_1$, and that new ideas and methods would be needed to cover this case. Our approach allows us to give a satisfactory answer to this problem, by showing that Bugeaud's result is in fact valid in any dimension. The next theorem is the main result of our paper.

Theorem 3 (Hausdorff dimension of exact approximable vectors)**.** *Let $n \geq 1$ be an integer. Assume that $\psi\colon \mathbb{N} \to \mathbb{R}_+^*$ is non-increasing and satisfies $\psi(s) = o(s^{-\frac{n+1}{n}})$. Then the set of exact ψ-approximable vectors in $\mathbb{R}^n$ satisfies*

$$\dim_H E(\psi) = \dim_H W(\psi).$$

In [2], the authors also define the set of ψ-badly approximable points

$$\mathbf{Bad}(\psi) = W(\psi) \setminus \bigcap_{c>0} W(c\psi)$$

and suggest to study the Hausdorff dimension of this set. We obtain a complete answer to that problem as an immediate corollary of Theorem 3.

Corollary 1 (Hausdorff dimension of ψ-badly approximable points)**.** *Let $\psi\colon \mathbb{N} \to \mathbb{R}_+^*$ be a non-increasing function such that $\psi(s) = o(s^{-\frac{n+1}{n}})$. Then*

$$\dim_H \mathbf{Bad}(\psi) = \dim W(\psi)$$

We note however that this corollary can be obtained more easily using the *variational principle in the parametric geometry of numbers* of Das, Fishman, Simmons and Urbański [6]. This alternative argument is sketched in paragraphs 2.2 and 2.3, as an introduction to the tools and techniques that will be further developed for the proof of Theorem 3. In the particular case of $\psi(s) = s^{-\lambda}$, Corollary 1 was obtained independently by Koivusalo, Levesley, Ward and Zhang [16] using different methods.

Theorem 3 above is new for $n \geq 2$ even in the case of the elementary function $\psi(s) = s^{-\lambda}$ for $\lambda > \frac{n+1}{n}$. In that case, the formula for the Hausdorff dimension is particularly simple: The set E_λ of points x in $\mathbb{R}^n$ for which there exist infinitely many rationals $\frac{p}{q}$ such that $\left\|x - \frac{p}{q}\right\| < q^{-\lambda}$, but only finitely many satisfying $\left\|x - \frac{p}{q}\right\| < cq^\lambda$ if $c < 1$, satisfies

$$\dim_H E_\lambda = \frac{n+1}{\lambda}.$$

More generally, one defines the *lower order at infinity* of ψ, denoted λ_ψ, to be

$$\lambda_\psi := \liminf_{s\to\infty} \frac{-\log \psi(s)}{\log s}$$

A result of Dodson [9] shows that if $\lambda_\psi \geq \frac{n+1}{n}$, the dimension of $W(\psi)$ is given by

$$\dim_H W(\psi) = \frac{n+1}{\lambda_\psi}.$$

In Theorem 3, only the lower bound $\dim_H E(\psi) \geq \dim_H W(\psi)$ requires a proof, and for that we shall construct inside $E(\psi)$ a Cantor set with the required Hausdorff dimension $\frac{n+1}{\lambda_\psi}$.

To construct that Cantor set, the general strategy is similar to the one developed by Bugeaud in [3], using balls of the form $B(y_k, \frac{\psi(H(v_k))}{k})$, where v_k is a rational point and y_k is chosen so that $d(y_k, v_k) = (1 - \frac{1}{k})\psi(H(v_k))$. It is clear that any point x lying in infinitely many such balls is ψ-approximable, but not approximated at rate $c\psi$ by the sequence (v_k) if $c < 1$.

The difficult point in the proof is to control also the quality of the approximations to x by rational points v that do not appear among the points v_k. Bugeaud's argument for that is based on continued fractions, and uses an elementary separation property for rational points on the real line: If v_1 and v_2 are two rational numbers with denominator at most q, then $d(v_1, v_2) \geq q^{-2}$. This property is of course also true for rational points in $\mathbb{R}^n$, $n \geq 2$, but one would need the stronger inequality $d(v_1, v_2) \geq q^{-\frac{n+1}{n}}$ if one wanted to use Bugeaud's approach to study $E(\psi)$ for any function ψ such that $\psi(s) = o(s^{-\frac{n+1}{n}})$. And of course, this stronger separation does not hold for $n \geq 2$.

In order to bypass this problem, the first step is to apply the celebrated *Dani correspondence*, which allows us to translate the exact approximation property of a point x in terms of the behavior of a diagonal orbit of a lattice Δ_x in $\mathbb{R}^{n+1}$ associated to x. After that, the main ideas we use are borrowed from the *parametric geometry of numbers* developed by Schmidt and Summerer [19] and Roy [17], and in particular to the remarkable preprint of Das, Fishman, Simmons and Urbański [6], in which the authors explain how to compute the Hausdorff dimension of the set of points whose associated orbits follow a given trajectory in the space of lattices. We note however that the study of exact approximation requires a precise understanding (see subsection 2.3) of the behavior of an orbit in the space of lattices, whereas the results in [6] only deal with trajectories up to a bounded error term. For that reason, we need to adapt their methods to our particular problem; what is left is that the branching of our Cantor set is best understood through a certain *template*, encoding the behavior of the diagonal orbits in the space of lattices.

2. Diagonal orbits in the space of lattices

For the construction of the Cantor set in $E(\psi)$, it will be convenient to interpret the property of exact ψ-approximability through the asymptotic behavior of a diagonal orbit in the space of lattices. This interpretation is given by the Dani correspondence [Theorem 8.5, [14]]. To make our proof self-contained, we briefly recall and prove the statement that will be used later on.

Note that given any non-increasing function $\psi\colon \mathbb{N} \to \mathbb{R}_+^*$, one may always construct a strictly decreasing function $\psi_1\colon \mathbb{N} \to \mathbb{R}_+^*$ satisfying $(1-\frac{1}{s})\psi(s) \leq \psi_1(s) \leq \psi(s)$, and then interpolate ψ_1 to obtain a decreasing function on $\mathbb{R}^+$. Then, the lower order of ψ_1 is $\lambda_{\psi_1} = \lambda_\psi$, and $E(\psi_1) \subset E(\psi)$. So, in order to prove the desired lower bound on the Hausdorff dimension of $E(\psi)$, it suffices to prove it for $E(\psi_1)$. This shows that for the proof of Theorem 3, we may assume without loss of generality that ψ extends to a decreasing (and continuous) function on $\mathbb{R}^+$. This assumption makes the statement of Dani's correspondence slightly simpler, so we shall always make it in the sequel.

2.1. **Dani's correspondence.** To any point $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ we associate the unipotent matrix

$$u_x := \begin{pmatrix} 1 & & & \\ -x_1 & 1 & & \\ \vdots & & \ddots & \\ -x_n & & & 1 \end{pmatrix}$$

and the unimodular lattice

$$\Delta_x = u_x \mathbb{Z}^{n+1} \quad \subset \quad \mathbb{R}^{n+1}.$$

The diophantine properties of x are encoded in the asymptotic behavior of the orbit of Δ_x under the diagonal semigroup $(a_t)_{t>0}$ given by

$$a_t := \begin{pmatrix} e^{-t} & & & \\ & e^{t/n} & & \\ & & \ddots & \\ & & & e^{t/n} \end{pmatrix}.$$

To state the precise correspondence, we associate to each rational point $v = (\frac{p_1}{q}, \cdots, \frac{p_n}{q})$ in $\mathbb{Q}^n$ the primitive integer vector $\mathbf{v} = (q, p_1, \cdots, p_n)$ in $\mathbb{Z}^{n+1}$, where the coordinates $(q, p_1, \dots, p_n)$ are relatively prime. We shall also use the distance on $\mathbb{R}^n$ given by

$$d(x, v) = \max_{1 \leq i \leq n} \left| x_i - \frac{p_i}{q} \right|$$

and the height on $\mathbb{Q}^n$ defined by

$$H(v) = \max \{|q|, |p_1|, \dots, |p_n|\}.$$

In the following, the space $\mathbb{R}^{n+1}$ is endowed with the norm equal to the maximal coordinate in absolute value:

$$\|\mathbf{w}\| = \max_{1 \leq i \leq n+1} |\langle e_i, \mathbf{w} \rangle|.$$

In particular, the equality $\|\mathbf{w}\| = |\langle e_1, \mathbf{w} \rangle|$ appearing in item (b) below means that the largest component of the vector $\mathbf{w}$ is along the e_1 coordinate.

Proposition 1 (Dani's correspondence). *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function and set $\Psi(s) := \psi(s)^{-\frac{n}{n+1}}$. Then,*

(a) If $d(x,v) \leq \psi(H(v))$ for some $v \in \mathbb{Q}^n \cap [0,1]^n$ and t is such that $e^t = \Psi(H(v))$, then $|\langle e_1, a_t u_x \mathbf{v}\rangle| = \|a_t u_x \mathbf{v}\|$ and $\|a_t u_x \mathbf{v}\| \leq e^{-t}\Psi^{-1}(e^t)$.

(b) If $\|a_t u_x \mathbf{v}\| \leq e^{-t}\Psi^{-1}(e^t)$ and $|\langle e_1, a_t u_x \mathbf{v}\rangle| = \|a_t u_x \mathbf{v}\|$ for some $\mathbf{v} \in \mathbb{Z}^{n+1}$, then the rational point v satisfies $H(v) \leq \Psi^{-1}(e^t)$ and $d(x,v) \leq \psi(H(v))$.

Proof. Suppose $d(x,v) \leq \psi(H(v))$ and let t be such that

$$e^t = \Psi(H(v)). \tag{1}$$

Since $v \in [0,1]$, the vector $\mathbf{v} = (q, p_1, \ldots, p_n)$ satisfies $q \geq \max_{1\leq i\leq n}|p_i|$, so $H(v) = q$ and by definition of the height and distance on $\mathbb{R}^{n+1}$,

$$\|a_t u_x \mathbf{v}\| = \max\{e^{-t}H(v), e^{t/n}H(v)d(x,v)\}$$

and from the definition of Ψ and our choice of t,

$$\begin{aligned} e^{t/n}H(v)d(x,v) &\leq e^{t/n}H(v)\psi(H(v)) \\ &\leq e^{t/n}H(v)\Psi(H(v))^{-\frac{n+1}{n}} \\ &\overset{(1)}{=} e^{-t}\Psi^{-1}(e^t). \end{aligned}$$

It follows that $|\langle e_1, a_t u_x \mathbf{v}\rangle| = e^{-t}H(v) = e^{-t}\Psi^{-1}(e^t) = \|a_t u_x \mathbf{v}\|$ and

$$\|a_t u_x \mathbf{v}\| \leq e^{-t}\Psi^{-1}(e^t).$$

For the second item of the proposition, assume that

$$\|a_t u_x \mathbf{v}\| = \max\{e^{-t}H(v), e^{t/n}H(v)d(x,v)\} \leq e^{-t}\Psi^{-1}(e^t).$$

Then clearly, $H(v) \leq \Psi^{-1}(e^t)$, and the condition $|\langle e_1, a_t u_x \mathbf{v}\rangle| = \|a_t u_x \mathbf{v}\|$ translates to

$$e^{t/n}H(v)d(x,v) \leq e^{-t}H(v)$$

whence

$$d(x,v) \leq e^{-\frac{n+1}{n}t} \leq \Psi(H(v))^{-\frac{n+1}{n}} = \psi(H(v)).$$

□

Remark. If one assumes the stronger condition that $\theta\colon H \mapsto H\psi(H)$ is non-increasing, one does not need the condition on $\langle e_1, a_t u_x \mathbf{v}\rangle$ in item (b). Indeed, in that case

$$\theta(\Psi^{-1}(e^t)) = \Psi^{-1}(e^t)e^{-\frac{n+1}{n}t} \leq \theta(H(v)) = H(v)\psi(H(v)).$$

Hence

$$d(x,v) \leq H(v)^{-1}e^{-\frac{n+1}{n}t}\Psi^{-1}(e^t) \leq \psi(H(v)).$$

The above remark in particular allows us to formulate a particularly simple corollary of Dani's correspondence, giving a necessary and sufficient condition on the orbit of the lattice Δ_x for the point x to belong to the set $E(\psi)$ of exact ψ-approximability, under the slightly more restrictive monotonicity condition on ψ.

Corollary 2. *Given $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $s \mapsto s\psi(s)$ is non-increasing and $s^{\frac{n+1}{n}}\psi(s)$ tends to zero as s goes to infinity, set $\Psi(s) = \psi(s)^{-\frac{n}{n+1}}$ and*

$$r_\psi(t) := -t + \log \Psi^{-1}(e^t).$$

Assume x in $\mathbb{R}^n$ satisfies

(a) For every $0 < c < 1$, for all $t > 0$ large enough, $\log \lambda_1(a_t u_x \mathbb{Z}^{n+1}) \geq r_{c\psi}(t)$*;*
(b) For arbitrarily large values of $t > 0$, one has $\log \lambda_1(a_t u_x \mathbb{Z}^{n+1}) \leq r_\psi(t)$*.*

Then $x \in E(\psi)$.

Proof. By the first item in Proposition 1 the first condition ensures that x does not belong to $W(c\psi)$, for any $c < 1$. The second item together with the above remark shows that x is in $W(\psi)$. □

2.2. **A template for λ_1.** In order to simplify the presentation of this introductory paragraph, we shall assume that the map $s \mapsto s\psi(s)$ is decreasing. Recall from the preceding section that $\Psi(s) = \psi(s)^{-\frac{n}{n+1}}$ and

$$r_\psi(t) = -t + \log \Psi^{-1}(e^t).$$

Together with the condition $\psi(s) = o(s^{-\frac{n+1}{n}})$, our monotonicity assumption on $s\psi(s)$ ensures that

(1) $t \mapsto r_\psi(t) - \frac{t}{n}$ is decreasing;
(2) $t \mapsto r_\psi(t) + t$ is increasing;
(3) $\lim_{t\to+\infty} r_\psi(t) = -\infty$.

Given a point x in $\mathbb{R}^n$, we define the function

$$\begin{array}{cccc} c_x\colon & \mathbb{R}^+ & \to & \mathbb{R} \\ & t & \mapsto & c_x(t) = \log \lambda_1(a_t u_x \mathbb{Z}^{n+1}). \end{array}$$

It follows from Corollary 2 that in order to prove Theorem 2 under the additional assumption that $s \mapsto s\psi(s)$ is decreasing, it suffices to construct a point x in $\mathbb{R}^n$ for which the function c_x satisfies the two conditions

$$\begin{cases} \forall c < 1,\ \forall t > 0 \text{ sufficiently large},\ c_x(t) \geq r_{c\psi}(t) \\ \exists t > 0 \text{ arbitrarily large}:\ c_x(t) = r_\psi(t). \end{cases} \tag{2}$$

The *parametric geometry of numbers*, introduced by Schmidt and Summerer [19], gives a combinatorial description of the function c_x on $\mathbb{R}^+$. It implies in particular that there exists a continuous affine by parts function $T_x\colon \mathbb{R}^+ \to \mathbb{R}^-$, with slopes in $\{-1, 0, \frac{1}{n}\}$ such that the difference $c_x - T_x$ remains bounded on $\mathbb{R}^+$. Conversely, one may start from such a *template* T and try to construct a point x in $\mathbb{R}^n$ such that c_x stays at bounded distance from T; Schmidt and Summerer gave necessary combinatorial conditions on T for the existence of such a point x, and Roy [17] showed that these conditions are also sufficient. Finally, Das, Fishman, Simmons and Urbański [6] gave a formula for the Hausdorff dimension of the set of points x in $\mathbb{R}^n$ following a given template T. Our proof of Theorems 2 and 3 is much inspired by this parametric geometry of numbers: We shall give ourselves a template T satisfying conditions (2) above and then construct points that follow closely this model trajectory. The general picture can be seen in Figure 1 below.

Observe also that the lower order at infinity λ_ψ of the function ψ can be read-off r_ψ through the formula

$$\gamma_\psi := \liminf \frac{-r_\psi(t)}{t} = \frac{n\lambda_\psi - n - 1}{n\lambda_\psi}.$$

Moreover, if q_k is an increasing sequence of denominators such that $\lambda_\psi = \lim \frac{\log 1/\psi(q_k)}{\log q_k}$, then, setting $t_k = \log \Psi(q_k)$, one has

$$\gamma_\psi = \lim \frac{-r_\psi(t_k)}{t_k}.$$

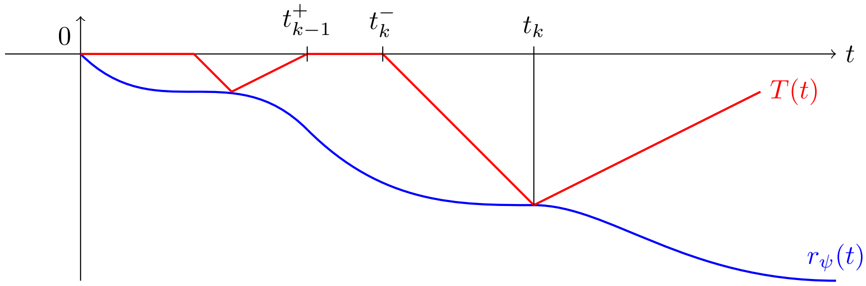

FIGURE 1. Template T above the graph of r_ψ.

We now fix such an increasing sequence $(t_k)_{k\geq 1}$ and, taking a subsequence if necessary, we shall also assume that it tends to infinity sufficiently fast. We also let

$$t_k^- = t_k + r_\psi(t_k) \quad \text{and} \quad t_k^+ = t_k - n r_\psi(t_k).$$

Provided (t_k) increases fast enough, one always has

$$0 < t_1^- < t_1 < t_1^+ < t_2^- < \dots$$

and we define a function $T\colon \mathbb{R}^+ \to \mathbb{R}^-$ with slopes in $\{-1, 0, \frac{1}{n}\}$ by $T(0) = 0$ and

$$\frac{dT}{dt}(t) = \begin{cases} 0 & \text{if} \quad t_{k-1}^+ < t < t_k^- \\ -1 & \text{if} \quad t_k^- < t < t_k \\ \frac{1}{n} & \text{if} \quad t_k < t < t_k^+. \end{cases}$$

Note that this function is continuous and satisfies $T(t_k) = r_\psi(t_k)$ for each $k \geq 1$. Moreover, it follows from the properties of r_ψ listed at the beginning of this paragraph that $T(t) \geq r_\psi(t)$ for all $t > 0$.

2.3. **The variational principle and beyond.** By Proposition 1, if x is a point in $\mathbb{R}^n$ such that c_x remains at bounded distance from the template T constructed above, then there exist constants C and $c > 0$ such that x lies in $W(C\psi)$ but not in $W(c\psi)$. By a result of Damien Roy [17, Theorem 1.3], there exists a point x in $\mathbb{R}^n$ such that $c_x(t) = T(t) + O(1)$ and this shows that the set $W(C\psi) \setminus W(c\psi)$ is non-empty if C is large and $c > 0$ small enough. Replacing the template T by $T - R_0$, where R_0 is some large positive constant, this argument can be modified slightly to show that if $c > 0$ is small enough, then

$$W(\psi) \setminus W(c\psi) \neq \varnothing.$$

In fact, the variational principle of Das, Fishman, Simmons and Urbański [6, Theorem 2.3] can be used to compute the Hausdorff dimension of the set D_T of points x in $\mathbb{R}^n$ such that c_x follows the template T up to some bounded error: If the sequence (t_k) tends to infinity fast enough, one finds

$$\dim_H D_T = n\left(1 - \lim_{k\to+\infty} \frac{-r_\psi(t_k)}{t_k}\right) = \frac{n+1}{\lambda_\psi} = \dim_H W(\psi).$$

Again, the Dani correspondence allows one to translate this into the following slightly weaker version of Corollary 1 from the introduction. Since this theorem can also be seen as an immediate consequence of Theorem 3, we do not include full details for the proof sketched above.

Corollary 3 (Hausdorff dimension of ψ-badly approximable points)**.** *Assume* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is such that* $s \mapsto s\psi(s)$ *is decreasing and* $\psi(s) = o(s^{-\frac{n+1}{n}})$*. Then, the set of badly approximable numbers in* $\mathbb{R}^n$*, defined as*

$$\mathbf{Bad}(\psi) = W(\psi) \setminus \bigcap_{c>0} W(c\psi)$$

satisfies $\dim_H \mathbf{Bad}(\psi) = \dim W(\psi)$.

Unfortunately, the available results from parametric geometry of numbers, such as the above-cited [17] or [6], only give information on the behavior of c_x up to some bounded additive constant. In contrast, to construct a point in $E(\psi)$, the Dani correspondence shows that one needs to understand $c_x(t)$ within an error term that goes to zero as t tends to infinity, at least at the times t where $c_x(t)$ approaches $r_\psi(t)$. In order to do so, we shall use a Cantor set construction, similar in spirit to the one used [6], but with better control on $c_x(t)$ when it takes large negative values.

3. A Cantor set in $E(\psi)$

Our goal is now to prove Theorem 3. Throughout this section, $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ denotes a decreasing function with lower order at infinity equal to λ_ψ, and we assume without loss of generality that $\lambda_\psi < +\infty$. We shall construct a Cantor set E_∞ inside $E(\psi)$ with Hausdorff dimension $\frac{n+1}{\lambda_\psi}$. The definition of the level sets of E_∞ is based on the behavior of the maps c_x. But before turning to the actual construction, we explain what important properties of that map we need in order to ensure that E_∞ is indeed included in $E(\psi)$.

3.1. Main properties of E_∞.

Just as in the previous section, the sequence of times (t_k) is assumed to satisfy

$$\lim_{k\to\infty} \frac{r_\psi(t_k)}{t_k} = \limsup_{t\to\infty} \frac{r_\psi(t)}{t} := -\gamma_\psi$$

where $\Psi(s) = \psi(s)^{-\frac{n}{n+1}}$ and $r_\psi(t) = -t + \log \Psi^{-1}(e^t)$.
Let

$$M_k = -\sup_{t \geq t_{k-1}} r_\psi(t).$$

Note that this definition implies that $M_k \leq -r_\psi(t_{k-1})$. We shall assume that t_k is sufficiently large compared to t_{k-1} in order to ensure that M_k is very small compared to $-r_\psi(t_k)$; this parameter M_k will then be used to define small intervals around t_k or $r_\psi(t_k)$.

In the sequel, we use three constants:

- $R_0 \geq 1$ depending only on n;
- R_1 depending on γ_ψ and R_0;
- R_2 depending on n, R_0 and R_1.

Then we define $t_k^- < t_k$ and $t_k^+ > t_k$ by

$$t_k^- = t_k + r_\psi(t_k) \quad \text{and} \quad t_k^+ = t_k + R_2 M_k.$$

We shall construct a Cantor set E_∞ of points x for which the trajectory c_x has the following two properties:

(A) For all $t \in [t^+_{k-1}, t^-_k - 4R_0M_k]$, $c_x(t) \geq -M_k$;
(B) For each k, there exists $\mathbf{v}_k = \mathbf{v}_k(x) \in \mathbb{Z}^{n+1}$ such that

$$\forall t \in [t^-_k, t^+_k], \quad c_x(t) = \log\|a_t u_x \mathbf{v}_k\|$$

and moreover, the point v_k in $\mathbb{Q}^n$ corresponding to $\mathbf{v}_k$ satisfies

$$\begin{cases} (i) & e^{t^-_k - 5R_0M_k} \leq H(v_k) \leq e^{t^-_k - 3R_0M_k} \\ (ii) & \left(1 - \frac{1}{k}\right)\psi(H(v_k)) < d(v_k, x) < \psi(H(v_k)) \\ (iii) & t_k - R_1M_k < t^x_k := -\frac{n}{n+1}\log d(v_k, x) < t_k + 1. \end{cases}$$

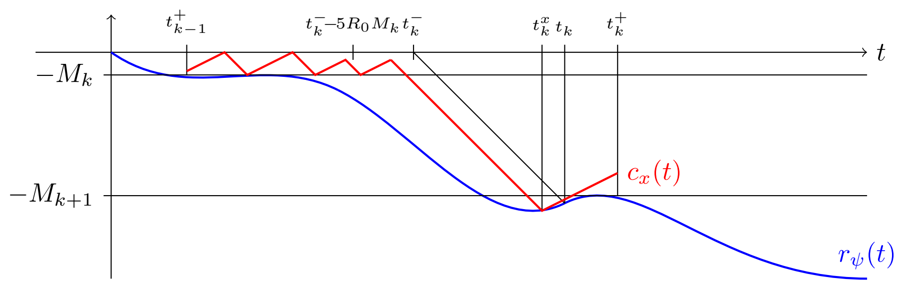

FIGURE 2. Graph of c_x on the interval $[t^-_k, t^+_k]$.

Let us first use Dani's correspondence to check that the two conditions above ensure that E_∞ will be a subset of $E(\psi)$.

Lemma 1. *Let x in $\mathbb{R}^n$ be such that c_x satisfies (A) and (B) for all $k \geq 1$. Then $x \in E(\psi)$.*

Proof. By construction, $d(v_k, x) < \psi(H(v_k))$ for all k, so x lies in $W(\psi)$. Let us show that if $c < 1$, then x does not belong to $W(c\psi)$. We use the precise form of Dani's correspondence given by Proposition 1 and show that for all $t > 0$ large enough, if $\mathbf{v} \in \mathbb{Z}^{n+1}$ satisfies $|\langle e_1, a_t u_x \mathbf{v}\rangle| = \|a_t u_x \mathbf{v}\|$, then

$$\|a_t u_x \mathbf{v}\| \geq e^{-t}\Psi_c^{-1}(e^t),$$

where $\psi_c(s) = c\psi(s)$ and $\Psi_c(s) = \psi_c(s)^{-\frac{n}{n+1}}$.

If t belongs to some interval $[t^+_{k-1}, t^-_k - 4R_0M_k]$, this follows from (A) and the definition of M_k, since

$$c_x(t) \geq -M_k \geq r_\psi(t).$$

We can then easily extend this property to a lower bound on the interval $[t^-_k - 4R_0M_k, t^-_k]$ for which one has

$$c_x(t) \geq c_x(t^-_k - 4R_0M_k) - 4R_0M_k \geq -5R_0M_k \geq r_\psi(t)$$

provided the sequence (t_k) was chosen to increase sufficiently fast in order to ensure that $\sup_{t \geq t^-_k - 4R_0M_k} r_\psi(t) < -5R_0M_k$.

Now assume $t \in [t^-_k, t^+_k]$. Note that

$$c_x(t) = -t + \log H(v_k) + \max\left(0, \tfrac{n+1}{n}t + \log d(v_k, x)\right).$$

Therefore, the condition $|\langle e_1, a_t u_x \mathbf{v}_k\rangle| = \|a_t u_x \mathbf{v}_k\|$ is met only if $t \leq -\frac{n}{n+1}\log d(v_k, x)$. For any such t, one has

$$c_x(t) = -t + \log H(v_k).$$

From $d(v_k,x)\geq(1-\frac{1}{k})\psi(H(v_k))$, we infer that $t\leq-\frac{n}{n+1}\log\big((1-\frac{1}{k})\psi(H(v_k))\big)$ i.e.

$$H(v_k)\geq\Psi^{-1}_{(1-\frac{1}{k})}(e^t).$$

Thus, we find

$$c_x(t)\geq -t+\Psi^{-1}_{(1-\frac{1}{k})}(e^t)=r_{(1-\frac{1}{k})\psi}(t).$$

If k is large enough so that $1-\frac{1}{k}\geq c$, this shows that the condition $|\langle e_1,a_tu_x\mathbf{v}_k\rangle|=\|a_tu_x\mathbf{v}_k\|$ implies $\|a_tu_x\mathbf{v}_k\|\geq e^{-t}\Psi_c^{-1}(e^t)$. Moreover, for any integer vector $\mathbf{v}$ linearly independent with $\mathbf{v}_k$, one can use Minkowski's second theorem to bound, for $t_k^-\leq t\leq t_k^x:=-\frac{n}{n+1}\log d(v_k,x)$,

$$\begin{aligned}\log\|a_tu_x\mathbf{v}\| &\geq \log\|a_{t_k^-}u_x\mathbf{v}\|+\frac{1}{n}(t-t_k^-)-O_n(1)\\ &\geq -5R_0M_k-O_n(1)\\ &\geq r_\psi(t)\geq r_{c\psi}(t).\end{aligned}$$

Then, we note that item (iii) of condition (B) and the definition of t_k^+ imply $t_k^+-t_k^x\leq(R_1+R_2)M_k$ so that for $t_k^x\leq t\leq t_k^+$,

$$\begin{aligned}\log\|a_tu_x\mathbf{v}\| &\geq \log\|a_{t_k^x}u_x\mathbf{v}\|-(t-t_k^x)\\ &\geq -5R_0M_k-O_n(1)-(R_1+R_2)M_k\\ &\geq r_\psi(t)\geq r_{c\psi}(t)\end{aligned}$$

assuming again that the (t_k) increase sufficiently fast to ensure that $-(5R_0+R_1+R_2)M_k-O_n(1)\geq r_\psi(t)$ for all $t\geq t_k-R_1M_k$.

This proves the desired inequality, and by the first part of Proposition 1, we obtain that for every $c<1$, any rational point v sufficiently close to x satisfies

$$d(v,x)\geq c\psi(H(v)).$$

□

3.2. **Construction of the Cantor set.** Having fixed some large $M>0$ such that $N=e^{\frac{n+1}{n}M}$ is an integer, the Cantor set E_∞ will be obtained as a decreasing intersection

$$E_\infty:=\bigcap_{\ell=0}^{\infty}E_\ell$$

where E_ℓ, the ℓ-th *level* of the Cantor set, is a finite union of disjoint cubes of sidelength $N^{-\ell}$. Each set E_ℓ is defined inductively so that for all x in E_ℓ, the above properties (A) and (B) are satisfied up to time $t=\ell M$. More precisely, we shall check that for arbitrarily large ℓ, for every x in E_ℓ and every $k\geq 1$, we have:

(A_ℓ) For all $t\in\{0,\dots,\ell M\}\cap[t_{k-1}^+,t_k^--4R_0M_k]$, then $c_x(t)\geq -M_k+M$;

(B_ℓ) For each k, there exists $\mathbf{v}_k=\mathbf{v}_k(x)\in\mathbb{Z}^{n+1}$ such that

$$\forall t\in[0,\ell M]\cap[t_k^-,t_k^+],\quad c_x(t)=\log\|a_tu_x\mathbf{v}_k\|$$

and moreover, the point v_k in $\mathbb{Q}^n$ corresponding to $\mathbf{v}_k$ satisfies

$$\begin{cases} (i) & e^{t_k^- -5R_0M_k}\leq H(v_k)\leq e^{t_k^- -3R_0M_k}\\ (ii) & \left(1-\frac{1}{k}\right)\psi(H(v_k))<d(v_k,x)<\psi(H(v_k))\\ (iii) & t_k-R_1M_k<t_k^x=-\frac{n}{n+1}\log d(v_k,x)<t_k+1.\end{cases}$$

Remark. In (A$_\ell$), it is enough to consider times in $\{0, \ldots, \ell M\}$, because for $kM \leq t < (k+1)M$, one has $c_x(t) \geq c_x(kM) - M$.

Let us first show that condition (iii) follows from (i) and (ii) if R_1 and the sequence (t_k) are chosen appropriately. Recall that we assumed $\lambda_\psi < +\infty$, which is equivalent to

$$\gamma := \liminf -\frac{1}{t} r_\psi(t) < 1.$$

Lemma 2. *Let $R_1 = \frac{10R_0}{1-\gamma}$ and assume that t_k is chosen so that*

$$r_\psi(t_k - R_1 M_k) \leq r_\psi(t_k) + \frac{1+\gamma}{2} R_1 M_k. \tag{3}$$

Then condition (iii) from (B$_\ell$) above is implied by (i) and (ii).

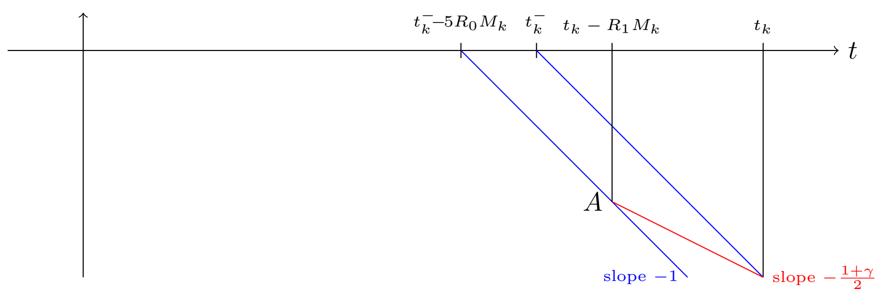

FIGURE 3. Controlling t_k^x.

Proof. First, observe that since $d(v_k, x) \geq (1 - \frac{1}{k})\psi(H(v_k))$ and $H(v_k) \leq e^{t_k^-}$, one has

$$\begin{aligned}
t_k^x &= -\frac{n}{n+1} \log d(v_k, x) \\
&\leq \log \Psi(H(v_k)) + \frac{n}{n+1} \log \frac{k}{k-1} \\
&\leq \log \Psi(e^{t_k^-}) + 1 = t_k + 1.
\end{aligned}$$

For the other inequality, we use Figure 3 above. By (3) the graph of r_ψ passes below the point $A = (t_k - R_1 M_k, r_\psi(t_k) + \frac{1+\gamma}{2} R_1 M_k)$ and therefore remains below the line of slope -1 passing through A on the interval $[0, t_k - R_1 M_k]$. On the other hand, on the interval $[t_k^-, t_k^x]$, one has

$$\log\|a_t u_x \mathbf{v}_k\| = -t + \log H(v_k)$$

so the graph of $t \mapsto \log\|a_t u_x \mathbf{v}_k\|$ follows a line of slope -1 that intersects the t-axis between $t_k^- - 5R_0 M_k$ and t_k^-, by (B$_\ell$)(i). Now $R_1 = \frac{10R_0}{1-\gamma}$ was chosen so that the line of slope -1 intersecting the t-axis at $t_k^- - 5R_0 M_k$ passes through A, so we may conclude that the graphs of r_ψ and $t \mapsto \log\|a_t u_x \mathbf{v}_k\|$ cannot meet before time $t_k - R_1 M_k$. But from $d(x, v_k) < \psi(H(v_k))$ we know that $\log\|a_{t_k^x} u_x \mathbf{v}_k\| = -t_k^x + \log H(v_k) < r_\psi(t_k^x)$ and thus $t_k^x \geq t_k - R_1 M_k$. $\square$

We now justify that it is indeed possible to choose the times t_k inductively so that (3) is always satisfied. In fact, for the proof of Theorem 3, in addition to the condition $\lim \frac{-r_\psi(t_k)}{t_k} = \gamma_\psi$, we shall need to control $r_\psi(t)$ on all times after t_k, and on times shortly before t_k as well. For this the times t_k, $k \geq 1$ are chosen inductively so that, with $-M_k = \sup_{t \geq t_{k-1}} r_\psi(t)$, one has,

(I) $-\gamma_\psi - \frac{1}{k} \leq \frac{r_\psi(t_k)}{t_k} \leq -\gamma_\psi + \frac{1}{k}$;
(II) $\forall t \geq t_k, \quad r_\psi(t) \leq r_\psi(t_k) + R_1 M_k$;
(III) $r_\psi(t_k - R_1 M_k) \leq r_\psi(t_k) + \frac{1+\gamma}{2} R_1 M_k$.

The lemma below ensures that it is indeed feasible to choose (t_k) in the desired way.

Lemma 3. *Assume $\gamma_\psi = \gamma < 1$ and t_{k-1} has been defined. Given $R > 0$ (possibly depending on t_{k-1}), we may always choose t_k arbitrarily large so that*

$$-\gamma_\psi - \frac{1}{k} \leq \frac{r_\psi(t_k)}{t_k} \leq -\gamma_\psi + \frac{1}{k} \tag{4}$$

and

$$r_\psi(t_k - R) \leq r_\psi(t_k) + \frac{1+\gamma}{2} R \tag{5}$$

and for all $t \geq t_k$,

$$r_\psi(t) \leq r_\psi(t_k) + R. \tag{6}$$

Proof. Let $0 < \varepsilon_k < \min(\frac{1}{k}, \frac{1}{8})$, and start with $t_k^{(0)}$ such that

$$\forall t \geq \frac{t_k^{(0)}}{3}, \quad \frac{r_\psi(t)}{t} < -\gamma + \varepsilon_k \quad \text{and} \quad \frac{r_\psi(t_k^{(0)})}{t_k^{(0)}} \geq -\gamma - \varepsilon_k.$$

Replacing $t_k^{(0)}$ by the largest time t for which $r_\psi(t) = r_\psi(t_k^{(0)})$ if necessary, we may also assume that

$$\forall t \geq t_k^{(0)}, \quad r_\psi(t) \leq r_\psi(t_k^{(0)}).$$

For $i \geq 1$, we define inductively $t_k^{(i)}$ in the following way. Assuming, $t_k^{(i-1)}$ has been defined, if $r_\psi(t_k^{(i-1)} - R) \leq r_\psi(t_k^{(i-1)}) + \frac{1+\gamma}{2} R$, then we stop and let $t_k = t_k^{(i-1)}$. Otherwise, let

$$t_k^{(i)} = t_k^{(i-1)} - R.$$

This procedure must stop for some $i \leq \frac{4\varepsilon_k t_k^{(0)}}{R}$, otherwise we would have, for $i = \lceil \frac{4\varepsilon_k t_k^{(0)}}{R} \rceil$,

$$\begin{aligned} r_\psi(t_k^{(i)}) &\geq \tfrac{1+\gamma}{2} iR - (\gamma + \varepsilon_k) t_k^{(0)} \\ &= (\tfrac{1+\gamma}{2} + \gamma + \varepsilon_k) iR - (\gamma + \varepsilon_k) t_k^{(i)} \\ &\geq 2\varepsilon_k t_k^{(0)} - (\gamma + \varepsilon_k) t_k^{(i)} \\ &\geq -(\gamma + \varepsilon_k) t_k^{(i)} \end{aligned}$$

while $t_k^{(i)} = t_k^{(0)} - R\lceil \frac{4\varepsilon_k t_k^{(0)}}{R} \rceil > \frac{1}{3} t_k^{(0)}$, contradicting our choice of $t_k^{(0)}$.

By construction, (5) holds for $t_k = t_k^{(i)}$ when the procedure stops, and since $t_k = t_k^{(i)} \geq \frac{t_k^{(0)}}{3}$, the right-hand side inequality in (4) also holds.

Moreover, by induction, for all $i \geq 0$,

$$\frac{r_\psi(t_k^{(i)})}{t_k^{(i)}} \geq -\gamma - \varepsilon_k.$$

Indeed,

$$\begin{aligned} r_\psi(t_k^{(i)}) &\geq r_\psi(t_k^{(i-1)}) + \frac{1+\gamma}{2}R \\ &\geq -(\gamma+\varepsilon_k)(t_k^{(i)} + R) + \frac{1+\gamma}{2}R \\ &> -(\gamma+\varepsilon_k)t_k^{(i)}. \end{aligned}$$

So (4) holds.

And by induction again,

$$\forall t \geq t_k^{(i)}, \quad r_\psi(t) \leq r_\psi(t_k^{(i)}) + R.$$

Indeed, for $t \geq t_k^{(i-1)}$, one has $r_\psi(t) \leq r_\psi(t_k^{(i-1)}) + R \leq r_\psi(t_k^{(i)}) + R$, while for t in $[t_k^{(i)}, t_k^{(i-1)}]$, since $t \mapsto t + r_\psi(t)$ is increasing, we may bound

$$r_\psi(t) \leq r_\psi(t_k^{(i-1)}) + t_k^{(i-1)} - t \leq r_\psi(t_k^{(i-1)}) + R \leq r_\psi(t_k^{(i)}) + R.$$

This proves (6). □

Now we may proceed with the definition of our Cantor set. Set $E_0 = [0,1)^n$, and assume that $E_{\ell-1}$ has been defined so that the above properties ($\mathrm{A}_{\ell-1}$) and ($\mathrm{B}_{\ell-1}$) hold. We fix a cube C in $E_{\ell-1}$, divide it into N^n subcubes of sidelength $N^{-\ell}$ and explain which among those subcubes will belong to E_ℓ. Denote by $E_\ell(C)$ the collection of these subcubes.

Letting $R_0 = \max(4n, n^2)$, $R_1 = \frac{10R_0}{1-\gamma}$ and $R_2 = 2n(R_1 + 6R_0) + 1$, we define

$$\ell_k^- = \lceil \frac{t_k^- - 4R_0M_k}{M} \rceil, \quad \text{and} \quad \ell_k^+ = \lfloor \frac{t_k + R_2M_k}{M} \rfloor.$$

We shall distinguish two cases:

Case 1: $\ell_{k-1}^+ < \ell \leq \ell_k^-$

Set $E_\ell(C)$ to be the set of subcubes $C' \subset C$ such that for all x in C',

$$c_x(\ell M) \geq -M_k + M.$$

Case 2: $\ell_k^- < \ell \leq \ell_k^+$

Let x_k denote the center of the unique cube $C_{\ell_k^-}$ of level ℓ_k^- containing C, and note that $\lambda_1(a_{\ell_k^- M} u_{x_k} \mathbb{Z}^{n+1}) \geq e^{-M_k}$. By Lemma 4 below applied at time $t = \ell_k^- M$ and with parameter $R = 2M_k$, there exists a rational point $v_k \in C_{\ell_k^-}$ such that

$$e^{\ell_k^- M - 2M_k} \leq H(v_k) \leq e^{\ell_k^- M + 4nM_k}.$$

With our choice of R_0 and the definition of ℓ_k^-, this implies

$$e^{t_k^- - 5R_0M_k} \leq H(v_k) \leq e^{t_k^- - 3R_0M_k}.$$

Pick $y_k \in C_{\ell_k^-}$ such that

$$d(v_k, y_k) = (1 - \frac{1}{2k})\psi(H(v_k)).$$

For each $\ell \in \{\ell_k^-, \dots, \ell_k^+\}$, take

$$E_\ell(C) = \{C_\ell(y_k)\}$$

where $C_\ell(y_k)$ denotes the unique cube of level ℓ containing y_k.

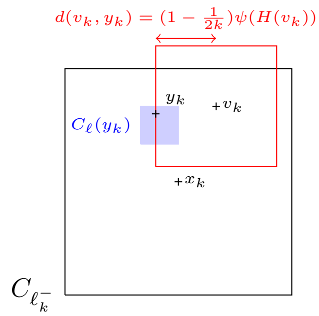

FIGURE 4. Choice of $E_\ell(C)$ for $\ell \in \{\ell_k^-, \dots, \ell_k^+\}$

Let us check by induction that if $E_\ell(C)$ is chosen as explained above, then properties (A$_\ell$) and (B$_\ell$) hold for ℓ arbitrarily large.

Case 1: Condition (A$_\ell$) is satisfied by our choice of $E_\ell(C)$, and (B$_\ell$) is satisfied because it coincides with (B$_{\ell-1}$).

Case 2: Assuming that (A$_{\ell_k^-}$) and (B$_{\ell_k^-}$) hold for C in $E_{\ell_k^-}$, let us show that (A$_{\ell_k^+}$) and (B$_{\ell_k^+}$) hold for every x in $E_{\ell_k^+}(C) = C_{\ell_k^+}(y_k)$. Let us start with $(B_{\ell_k^+})$. By construction,

$$e^{t_k^- - 5R_0M_k} < H(v_k) < e^{t_k^- - 3R_0M_k}.$$

Let $\mathbf{v}_k$ be a vector in $\mathbb{Z}^{n+1}$ corresponding to the rational point v_k in $C_{\ell_k^-}$. For x in $C_{\ell_k^+}(y_k)$, one has

$$\begin{aligned} d(x, v_k) &\leq d(x, y_k) + d(y_k, v_k) \\ &\leq e^{-\frac{n+1}{n}\ell_k^+ M} + (1 - \frac{1}{2k})\psi(H(v_k)). \end{aligned}$$

Recalling that $H(v_k) \leq e^{t_k^-} = \Psi^{-1}(e^{t_k})$, one finds $e^{-\frac{n+1}{n}t_k} \leq \psi(H(v_k))$, and since $\ell_k^+ M \geq t_k + M_k$ this yields

$$d(x, v_k) \leq e^{-M_k}\psi(H(v_k)) + (1 - \frac{1}{2k})\psi(H(v_k)) < \psi(H(v_k)),$$

provided we chose t_k large enough to ensure $e^{-M_k} < \frac{1}{2k}$. Similarly,

$$d(x, v_k) \geq (1 - \frac{1}{2k})\psi(H(v_k)) - e^{-M_k}\psi(H(v_k)) \geq (1 - \frac{1}{k})\psi(H(v_k)).$$

We have checked that conditions (i) and (ii) in (B$_{\ell_k^+}$) hold, and by Lemma 2 and our choice of the sequence (t_k), condition (iii) follows automatically.

It remains to show that $\mathbf{v}_k$ achieves the first minimum of $a_t u_x \mathbb{Z}^{n+1}$ for t in $[t_k^-, t_k^+]$. At time $T = t_k^- - 4R_0M_k$, we have $\lambda_1(a_T u_x \mathbb{Z}^{n+1}) \geq e^{-M_k}$ and therefore, by Minkowski's second theorem,

$$\lambda_{n+1}(a_T u_x \mathbb{Z}^{n+1}) \lesssim e^{nM_k}.$$

Since the largest eigenvalue of a_s is equal to $e^{\frac{s}{n}}$, we infer, for all $s \geq 0$,

$$\lambda_{n+1}(a_{T+s}u_x\mathbb{Z}^{n+1}) \lesssim e^{nM_k+\frac{s}{n}}.$$

On the other hand, for $T+s \leq t_k^x$,

$$\lambda_1(a_{T+s}u_x\mathbb{Z}^{n+1}) \leq \|a_{T+s}u_x\mathbf{v}_k\| = e^{-s}\|a_T u_x\mathbf{v}_k\| \leq e^{-s+R_0M_k}.$$

Since one always has $\lambda_2 \gtrsim \lambda_1^{-1}\lambda_{n+1}^{-n+1}$, this yields

$$\lambda_2(a_{T+s}u_x\mathbb{Z}^{n+1}) \gtrsim e^{s-R_0M_k}e^{-n(n-1)M_k-\frac{(n-1)s}{n}} \geq e^{\frac{s}{n}-2R_0M_k},$$

and if $s > 3R_0M_k$, we find

$$\lambda_2(a_{T+s}u_x\mathbb{Z}^{n+1}) \geq e^{-2R_0M_k} > e^{-s+R_0M_k} \geq \|a_{T+s}u_x\mathbf{v}_k\|.$$

This implies that for t in $[t_k^-, t_k^x]$, the vector $\mathbf{v}_k$ achieves the first minimum of $a_t u_x\mathbb{Z}^{n+1}$. Note also that at time $t_k^x = T + s_k^x$,

$$\frac{\lambda_2(a_{t_k^x}u_x\mathbb{Z}^{n+1})}{\lambda_1(a_{t_k^x}u_x\mathbb{Z}^{n+1})} \geq e^{\frac{n+1}{n}s_k^x-3R_0M_k} > e^{\frac{n+1}{n}(R_1+R_2)M_k}$$

so $\mathbf{v}_k$ continues to achieve $\lambda_1(a_t u_x\mathbb{Z}^{n+1})$ on the interval $[t_k^x, t_k^x + (R_1 + R_2)M_k]$, which contains $[t_k^x, t_k^+]$. Indeed, for $s \in [0, (R_1+R_2)M_k]$,

$$\begin{aligned}
\lambda_2(a_{t_k^x+s}u_x\mathbb{Z}^{n+1}) &\geq e^{-s}\lambda_2(a_{t_k^x}u_x\mathbb{Z}^{n+1})\\
&\geq e^{-s}e^{\frac{n+1}{n}(R_1+R_2)M_k}\lambda_1(a_{t_k^x}u_x\mathbb{Z}^{n+1})\\
&\geq e^{\frac{s}{n}}\|a_{t_k^x}u_x\mathbf{v}_k\|\\
&\geq \|a_{t_k^x+s}u_x\mathbf{v}_k\|.
\end{aligned}$$

Finally, to prove that $(\mathrm{A}_{\ell_k^+})$ holds, we only need to check that for all t in $[t_k^+, \ell_k^+ M]$, $c_x(t) > M_{k+1}$. Write, for t in $[t_k^+, \ell_k^+ M]$,

$$\begin{aligned}
c_x(t) &= c_x(t_k^x) + \frac{t-t_k^x}{n}\\
&\geq r_\psi(t_k) - 5R_0M_k + \frac{t-t_k}{n}\\
&\geq M_{k+1} - R_1M_k - 5R_0M_k + \frac{R_2M_k}{2n}\\
&> M_{k+1} + M
\end{aligned}$$

since we chose $R_2 > 2n(R_1 + 6R_0)$.

We conclude this paragraph with the lemma used in Case 2 above, to obtain a rational point of controlled height inside any cube C from $E_{\ell_k^-}$.

Lemma 4 (Rational points near badly approximable points)**.** *Given parameters $t > 0$ and $R \geq 1$, assume $x \in [0,1)^n$ is such that $\lambda_1(a_t u_x\mathbb{Z}^{n+1}) \geq e^{-R}$. Then there exists a rational point $v = v(x)$ such that:*

(1) $e^{t-R} \leq H(v) \leq e^{t+2nR}$;
(2) $d(x,v) \leq \frac{1}{2}e^{-\frac{n+1}{n}t}$.

Proof. By Minkowski's first theorem applied to the lattice $a_{t+2nR}u_x\mathbb{Z}^{n+1}$, there exists $\mathbf{v}$ in $\mathbb{Z}^{n+1}$ such that

$$\|a_{t+2nR}u_x\mathbf{v}\| \leq 1.$$

The point v in $\mathbb{R}^n$ corresponding to $\mathbf{v}$ satisfies

$$H(v) \le e^{t+2nR} \quad \text{and} \quad d(x,v) \le e^{-\frac{t}{n}-2R} H(v)^{-1}. \tag{7}$$

On the other hand, our assumption that $\lambda_1(a_t u_x \mathbb{Z}^{n+1}) \ge e^{-R}$ implies that

$$\|a_t u_x \mathbf{v}\| = H(v)\max(e^{-t}, e^{\frac{t}{n}} d(x,v)) \ge e^{-R}$$

and since $d(x,v) \le e^{-\frac{t}{n}-2R} H(v)^{-1}$, we must have

$$H(v) \ge e^{t-R}.$$

Going back to (7) this yields

$$d(x,v) \le e^{-\frac{t}{n}-2R} e^{-t+R} < \frac{1}{2} e^{-\frac{n+1}{n}t}.$$

□

3.3. **Branching and Hausdorff dimension.** It now remains to compute the Hausdorff dimension of E_∞. For that, we first need to obtain a good lower bound on the branching of the Cantor set at each level ℓ between t_{k-1}^+ and t_k^-. This will be a consequence of Lemma 5 below, whose proof uses a variant of the well-known Simplex Lemma originating in the works of Davenport and Schmidt [18, page 57].

Recall that the Cantor set E_∞ is obtained as a decreasing intersection $E_\infty = \bigcap_{\ell \ge 1} E_\ell$, where each E_ℓ is a finite union of disjoint cubes of side length $N^{-\ell}$, where N is a large integer given as

$$N = e^{\frac{(n+1)M}{n}}.$$

Lemma 5 (Large branching for $\ell_{k-1}^+ < \ell \le \ell_k^-$)**.** *There exists a constant R_3 depending on n such that for all large enough k, if $\ell_{k-1}^+ < \ell \le \ell_k^-$ and C is any cube in $E_{\ell-1}$, then*

$$\operatorname{card} E_\ell(C) \ge N^n - R_3 N^{n-\frac{1}{n+1}}.$$

Proof. Set $t = (\ell-1)M$. Since $\ell_{k-1}^+ < \ell \le \ell_k^-$ and C belongs to $E_{\ell-1}$, we have

$$\forall x \in C, \quad \lambda_1(a_t u_x \mathbb{Z}^{n+1}) > e^{-M_k+M}.$$

Let x_0 be the bottom left corner of C, and define

$$S_C := \{\mathbf{v} \in \mathbb{Z}^{n+1} : \|a_t u_{x_0} \mathbf{v}\| < e^{-M_k+3M}\}.$$

We claim that there exists some hyperplane H_C in $\mathbb{R}^{n+1}$ such that $S_C \subseteq H_C$. (This statement can be viewed as a version of the Simplex Lemma.) Otherwise, there would exist linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{Z}^{n+1}$ such that

$$\|a_t u_{x_0} \mathbf{v}_i\| < e^{-M_k+3M} \text{ for } 1 \le i \le n+1.$$

But $a_t u_{x_0} \mathbb{Z}^{n+1}$ is a lattice of covolume 1, hence it cannot have $n+1$ linearly independent vectors of norm less than 1. This yields the desired contradiction and proves our claim as soon as $M_k > 3M$.

Now let

$$A_C = \{x \in C \ ; \ d([e_1], a_t u_x H_C) \le e^{-\frac{M}{n}}\}.$$

Let x be a point in $C \setminus A_C$ and $\mathbf{v}$ any non-zero vector in $\mathbb{Z}^{n+1}$. If $\mathbf{v} \in S_C$, then $\mathbf{v} \in H_C$ so the projection of $a_t u_x \mathbf{v}$ to $e_1^\perp$ has norm at least $e^{-\frac{M}{n}} \|a_t u_x \mathbf{v}\|$, and since this coordinate is expanded by a factor $e^{\frac{s}{n}}$ under the action of a_s, we have

$$\begin{aligned}\|a_{t+M}u_x\mathbf{v}\| &\geq e^{\frac{M}{n}}e^{-\frac{M}{n}}\|a_t u_x \mathbf{v}\|\\ &\geq e^{-M_k+M}.\end{aligned}$$

On the other hand, if $\mathbf{v}\notin S_C$, we can bound

$$\|a_{t+M}u_x\mathbf{v}\| \geq e^{-M}\|a_t u_x\mathbf{v}\| \geq e^{-2M}\|a_t u_{x_0}\mathbf{v}\| \geq e^{-M_k+M}.$$

This shows that

$$\forall x\in C\setminus A_C,\quad c_x(\ell M)\geq -M_k+M.$$

To conclude the proof, we show that A_C is included in a neighborhood of size $N^{-\ell}e^{-\frac{M}{n}}$ of a hyperplane in $\mathbb{R}^n$. For x in C, we let $y=e^{\frac{n+1}{n}\ell M}(x-x_0)$ in $[0,1)^n$ so that

$$a_t u_x = u_y a_t u_{x_0}.$$

Let $\phi_{t,C}$ be a linear form of norm 1 vanishing on $H_{t,C}=a_t u_{x_0}H_C$; then

$$d([e_1],[a_t u_x H_C]) = d([e_1],[u_y H_{t,C}]) \asymp d([u_{-y}e_1],H_{t,C}) \asymp \phi_{t,C}(u_{-y}e_1).$$

Therefore $x\in A_C$ implies $\phi_{t,C}(u_{-y}e_1)\lesssim e^{-\frac{M}{n}}$, and this inequality means that y lies in a neighborhood of size $O(e^{-\frac{M}{n}})$ of the affine hyperplane of $\mathbb{R}^n$ defined by the equation $\phi_{t,C}(e_1-y_1e_2-\cdots-y_ne_{n+1})=0$. Equivalently, x lies in a neighborhood of size $O(N^{-\ell}e^{-\frac{M}{n}})$ of an affine hyperplane in $\mathbb{R}^n$. The number of subcubes of C that meet this neighborhood is bounded above by

$$\lesssim_n N^n e^{-\frac{M}{n}}$$

so

$$\operatorname{card} E_\ell(C) \geq N^n(1-O_n(e^{-\frac{M}{n}})) = N^n - O_n(N^{n-\frac{1}{n+1}}).$$

□

The lower bound on the branching given by the above lemma is all that is needed to get a good lower bound on the Hausdorff dimension of E_∞, and therefore on $E(\psi)$.

Proof of Theorem 3. To get a lower bound on the Hausdorff dimension of E_∞, we use the Mass distribution principle [10, §4.2], but first we replace E_∞ by a more regular Cantor subset, to simplify later computations.

For $\ell\geq 1$, define

$$b_\ell = \begin{cases} \lfloor N^n(1-R_3e^{-\frac{M}{n}})\rfloor & \text{if } \ell_{k-1}^+ < \ell \leq \ell_k^- \\ 1 & \text{if } \ell_k^- < \ell \leq \ell_k^+. \end{cases}$$

Removing some cubes in E_ℓ at each step, one obtains a Cantor subset $F_\infty\subset E_\infty$ given as

$$F_\infty = \bigcap_{\ell\geq 1} F_\ell,$$

where each cube C in $F_{\ell-1}$ contains exactly b_ℓ subcubes in F_ℓ. By [10, Proposition 1.7], there exists a probability measure μ supported on F_∞ such that for any level ℓ cube $C\subset F_\ell$,

$$\mu(C) = (b_1\dots b_\ell)^{-1}.$$

We claim that for $\alpha<\liminf_{\ell\to\infty}\frac{\log(b_1\dots b_\ell)}{\ell\log N}$, there exists $C=C_{n,N,\alpha}$ such that

$$\forall x\in\mathbb{R},\ \forall r>0,\quad \mu(B(x,r))\leq Cr^\alpha.$$

Indeed, picking ℓ so that $N^{-\ell} < r \leq N^{-\ell+1}$, the ball $B(x,r)$ meets at most $(3N)^n$ cubes from F_ℓ, and therefore

$$\mu(B(x,r)) \leq (3N)^n (b_1 \dots b_\ell)^{-1} \leq (3N)^n N^{-\ell\alpha} \leq (3N)^n r^\alpha$$

provided ℓ is large enough, or equivalently r small enough in terms of n, N and α. By the Mass distribution principle, this implies

$$\dim_H F_\infty \geq \alpha.$$

To conclude, observe that

$$\liminf_{\ell\to\infty} \frac{\log(b_1 \dots b_\ell)}{\ell \log N} = \lim_{k\to\infty} \frac{\log(b_1 \dots b_{\ell_k^+})}{\ell_k^+ \log N} = \lim_{k\to\infty} \frac{\log(b_1 \dots b_{\ell_k^-})}{\ell_k^+ \log N}$$

and if the sequences $(\ell_k^\pm)_{k\geq 1}$ satisfies $\ell_{k-1}^+ = o(\ell_k^-)$ — this can always be ensured by taking a sequence (t_k) increasing sufficiently fast — this limit is bounded below by

$$\begin{aligned} \lim_{k\to\infty} \frac{\ell_k^- \log\lfloor N^n - R_3 N^{n-\frac{1}{n+1}}\rfloor}{\ell_k^+ \log N} &= \frac{\log\lfloor N^n - R_3 N^{n-\frac{1}{n+1}}\rfloor}{\log N} \lim_{k\to\infty} \frac{t_k^-}{t_k} \\ &= \frac{\log\lfloor N^n - R_3 N^{n-\frac{1}{n+1}}\rfloor}{\log N} \frac{n+1}{n\lambda_\psi}. \end{aligned}$$

Thus

$$\dim_H E(\psi) \geq \dim_H F_\infty \geq \frac{\log\lfloor N^n - R_3 N^{n-\frac{1}{n+1}}\rfloor}{\log N} \frac{n+1}{n\lambda_\psi}$$

and as N (or equivalently M) tends to $+\infty$, this yields the desired result

$$\dim_H E(\psi) \geq \frac{n+1}{\lambda_\psi}.$$

$\square$

Theorem 2 is now an easy consequence of Theorem 3, and of Jarník's results. We even get the following slightly more precise result.

Theorem 4 (Existence of exact ψ-approximable vectors)**.** *Let $n \geq 1$ be an integer. If $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing and satisfies $\lim_{s\to\infty} s^{\frac{n+1}{n}}\psi(s) = 0$, then the set $E(\psi)$ is uncountable.*

Proof. If $\lambda_\psi < +\infty$, then Theorem 3 shows that $\dim_H E(\psi) = \frac{n+1}{\lambda_\psi} > 0$, so the result is clear. If $\lambda_\psi = +\infty$, then $\lim_{s\to\infty} s^2\psi(s) = 0$ and so we may use Jarník's construction to get that $E(\psi)$ is uncountable, see also [5, Theorem J]. $\square$

Conclusion

The method developed here to study exact approximation of points in $\mathbb{R}^n$ is robust, and can be adapted to study a number of similar problems.

Other norms on $\mathbb{R}^n$. In the definition of the set $W(\psi)$ of ψ-approximable points in $\mathbb{R}^n$, we used the norm on $\mathbb{R}^n$ given by $\|x\| = \max_{1\leq i\leq n}|x_i|$, because this is the standard setting for simultaneous diophantine approximation, and the one studied by Jarník in his foundational paper [13]. But one could also use the Euclidean norm on $\mathbb{R}^n$, or any other norm N, and study the corresponding notion of exact ψ-approximability. It is not difficult to check that Theorems 2 and 3 are still valid in this slightly more general context. The main difference lies in the fact that for the Dani correspondence, one should endow the space $\mathbb{R}^{n+1}$ with the norm

$$\|x'\| = \max(|x_0|, N(x)),$$

if $x' = (x_0, x)$ in the identification $\mathbb{R}^{n+1} \simeq \mathbb{R} \times \mathbb{R}^n$.

Approximation of linear forms and matrices. Given a non-increasing function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ and positive integers m and n, one defines

$$W(m,n;\psi) = \left\{ X \in M_{m\times n}(\mathbb{R}) : \begin{array}{l} \|\mathbf{q}X - \mathbf{p}\| < \psi(\|\mathbf{q}\|)\|\mathbf{q}\| \\ \text{for infinitely many}(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^n \times \mathbb{Z}^m. \end{array} \right\}$$

The matrices in $W(m,n;\psi)$ are usually called ψ-approximable $m \times n$ matrices. For a decreasing function ψ, Dodson [9] showed that $W(m,n;\psi)$ has Hausdorff dimension $(m-1)n + \frac{m+n}{\lambda_\psi}$ provided $\lambda_\psi \geq \frac{m+n}{n}$. This general setting is in fact the one used by Beresnevich, Dickinson and Velani [2] to study exact approximability. The techniques of the present paper can be used to show that if $s^{\frac{m+n}{n}}\psi(s)$ tends to zero as s goes to infinity, then the set $E(m,n;\psi)$ of $m\times n$ matrices that are exactly ψ-approximable has the same Hausdorff dimension as $W(m,n;\psi)$.

Intrinsic diophantine approximation on manifolds. Let X be an algebraic variety defined over $\mathbb{Q}$ and such that $X(\mathbb{Q})$ is dense in $X(\mathbb{R})$. Having fixed a distance on $X(\mathbb{R})$ and a height on $X(\mathbb{Q})$, one may study the quality of approximation by points in $X(\mathbb{Q})$ to points in $X(\mathbb{R})$. This problem is usually referred to as *intrinsic diophantine approximation* on X.

In a number of cases, it has been shown that Jarník's theorem holds in this context; this is so for instance when X is a quadric hypersurface [15, 11] or when X is a Grassmann variety [8] or any flag variety [7]. In all those cases, there exists some version of the Dani correspondence that interprets diophantine properties in terms of diagonal orbits in spaces of lattices, so it is natural to expect that the analog of Theorem 3 holds, and can be obtained by methods similar to the ones used here.

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