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BOUNDARY RIGIDITY OF CAT(0) CUBE COMPLEXES

JÉRÉMIE CHALOPIN AND VICTOR CHEPOI

ABSTRACT. In this note, we prove that finite CAT(0) cube complexes can be reconstructed from their boundary distances (computed in their 1-skeleta). This result was conjectured by Haslegrave, Scott, Tamitegama, and Tan (2023). The reconstruction of a finite cell complex from the boundary distances is the discrete version of the boundary rigidity problem, which is a classical problem from Riemannian geometry. In the proof, we use the bijection between CAT(0) cube complexes and median graphs, and corner peelings of median graphs.

1. INTRODUCTION

A natural question, arising in several research areas, is whether the internal structure of an object can be determined from distances between its boundary points. By a classical result in phylogeny by Buneman [6] and Zareckii [26], trees can be reconstructed from the pairwise distances between their leaves. In Riemannian geometry, the notion of reconstruction from a distance function on the boundary is well-established and related to boundary rigidity questions. A Riemannian manifold (M, g) is said to be *boundary rigid* if its metric d_g (defined on all pairs of points, including the interior) is determined up to isometry by its boundary distance function. Michel [17] conjectured that any simple compact Riemannian manifold with boundary is boundary rigid. The case of 2-dimensional Riemannian manifolds was confirmed by Pestov and Uhlmann [16]. In higher dimensions, the conjecture is wide open and has been established only in two cases by Besson, Courtois, and Gallot [4] and by Burago and Ivanov [7]. The discrete version of boundary rigidity was suggested by Benjamini (see [12, 13]) who asked if any plane triangulation in which all inner vertices have degrees at least 6 can be reconstructed from the distances between its boundary vertices. This was answered in the affirmative by Haslegrave [12], who also proved a similar result for plane quadrangulations in which all inner vertices have degrees at least 4. Haslegrave, Scott, Tamitegama, and Tan [13] generalized the second result of [12] and proved that any finite 2-dimensional CAT(0) cube complex and any finite 3-dimensional CAT(0) cube complex X with an embedding in \mathbb{R}^3 can be reconstructed from its boundary distances. They conjectured (see [13, Conjecture 20]) that all finite CAT(0) cube complexes can be reconstructed up to isomorphism from their boundary distances. In the papers [12] and [13] the boundary ∂X of X is defined combinatorially (independently of the embedding of X in some \mathbb{R}^k), and the input information is the set of all pairwise distances in the graph (1-skeleton) $G(X)$ of the complex X between all vertices of $G(X)$ belonging to ∂X . In this note, we confirm the conjecture of [13] and prove the following theorem.

Theorem 1. *Let X be a finite CAT(0) cube complex. Then X is determined up to isomorphism by the distances between the vertices of ∂X in the 1-skeleton of X . Consequently, the class of finite CAT(0) cube complexes is boundary rigid.*

In the proof, we use the bijection between CAT(0) cube complexes and median graphs and corner peelings of median graphs. CAT(0) cube complexes are central objects in geometric group theory; see Sageev [22, 23] and Wise [25]. They have been characterized by Gromov [11] as simply connected cube complexes in which the links of vertices are simplicial flag complexes. The systematic investigation and use of CAT(0) cube complexes in geometric group theory started with the paper by Sageev [22]. Soon after, it was proved by Chepoi [10] and Roller [21] that 1-skeleta of CAT(0) cube complexes are exactly the median graphs. Median graphs are central objects in metric graph theory; they were introduced by Nebeský [20] and investigated in numerous papers, see for example the papers by Mulder [18, 19] and the survey by Bandelt and Chepoi [2].

2. PRELIMINARIES

In this section, we recall the boundary rigidity problem for cell complexes. We also recall some definitions and basic facts about CAT(0) cube complexes and median graphs.

2.1. Graphs. All graphs $G = (V(G), E(G))$ considered in this paper are finite, undirected, connected, and contain neither multiple edges, nor loops. For two distinct vertices $v, w \in V(G)$ we write $v \sim w$ when there is an edge connecting v with w . The subgraph of G induced by a subset $A \subseteq V(G)$ is the graph $G[A] = (A, E')$ such that $uv \in E'$ if and only if $uv \in E(G)$. A *square* $uvwz$ is an induced 4-cycle (u, v, w, z) . Equivalently, a square is a graph that is isomorphic to the 1-skeleton of a unit Euclidean square in the plane. More generally, a *cube* Q_n of dimension n is a graph isomorphic to the 1-skeleton of the n -dimensional unit Euclidean cube. Alternatively, Q_n is a graph whose vertices can be labeled by the subsets of a set of size n and such that two vertices are adjacent if and only if the corresponding sets A, B differ in a single element, i.e., if $|A \Delta B| = 1$. The *distance* $d(u, v) = d_G(u, v)$ between two vertices u and v of a graph G is the length of a shortest (u, v) -path. An induced subgraph $H = G[A]$ of a graph G is an *isometric subgraph* of G if $d_H(u, v) = d_G(u, v)$ for any two vertices $u, v \in A$. The *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v) -paths, that is, of all vertices (metrically) *between* u and v : $I(u, v) = \{x \in V(G) : d(u, x) + d(x, v) = d(u, v)\}$. An induced subgraph of G is called *convex* if it includes the interval of G between any pair of its vertices. An induced subgraph H (or the corresponding vertex set of H) of a graph G is *gated* if for every vertex x outside H there exists a vertex x' in H (the *gate* of x) such that $x' \in I(x, y)$ for each y of H . Gated sets are convex and the intersection of two gated sets is gated.

2.2. Cube complexes and boundary rigidity. All cube complexes considered in this paper are finite cell complexes whose cells are unit Euclidean cubes and whose gluing maps are isometries (see the book of Hatcher [14]). The *dimension* $\dim(C)$ of a cell C is the dimension of the cube C . The *dimension* $\dim(X)$ of X is the largest dimension of a cell of X . If C is a cell of X of dimension k , then all cells of X contained in C and having dimension $k - 1$ are called the *facets* of C . Cells of X are called *maximal* if they are maximal by inclusion, and *non-maximal* otherwise. For a cube complex X , we denote by $X^{(k)}$ the k -skeleton of X consisting of all cubes of dimension at most k . We use the notations $V(X) = X^{(0)}$ for the set of vertices (0-cubes) of X and $G(X) := X^{(1)}$ for the 1-skeleton of X ; the graph $G(X)$ will be always endowed with the standard graph distance d_G . An *abstract simplicial complex* Δ on a finite set V is a set of nonempty subsets of V , called *simplices*, such that any nonempty subset of a simplex is also a simplex. The *clique complex* of a graph G is the abstract simplicial complex having the cliques (i.e., complete subgraphs) of G as simplices. A simplicial complex X is a *flag simplicial complex* if X is the clique complex of its 1-skeleton. In a cube complex X , the *link* $\text{Link}(x)$ of a vertex x is the simplicial complex whose vertices are the edges of X containing x and whose simplices are given by the collections of edges belonging to a common cube of X .

We will use a slightly weaker version of the definition of combinatorial boundary given by Haslegrave et al. [13]. For a finite cube complex X , the *combinatorial boundary* ∂X is the downward closure of all non-maximal cells of X such that each of them is a facet of a unique cell of X . The definition of [13] is similar but instead of maximality by inclusion the maximality by dimension is considered: in [13], the combinatorial boundary of a cube complex X of dimension k is the downward closure of all cells of X of dimension less than k belonging to at most one cell of dimension k . Obviously, the boundary of a cube complex X defined in our way is always contained in the boundary of X defined as in [13]. The two boundaries are different as soon as the cube complex X contains maximal cells of different dimensions.

The *boundary distance matrix* of a finite cube complex X is the matrix whose rows and columns are the vertices of ∂X and whose entries are the distances in $G(X)$ between the corresponding vertices of ∂X . A class \mathfrak{C} of finite cube complexes is called *boundary rigid* if for any two complexes $X, Y \in \mathfrak{C}$ that have the same boundary distance matrices up to a permutation σ of the rows and the columns (σ can be seen as an isometry between ∂X and ∂Y), then X and Y are isomorphic via an isomorphism $\tilde{\sigma}$ extending σ .

2.3. CAT(0) cube complexes and median graphs. In this subsection, we recall the definitions of CAT(0) cube complexes and median graphs, and the bijection between them.

Let (X, d) be a metric space. A *geodesic* joining two points x and y from X is a map ρ from the segment $[a, b]$ of \mathbb{R}^1 of length $|a - b| = d(x, y)$ to X such that $\rho(a) = x, \rho(b) = y$, and $d(\rho(s), \rho(t)) = |s - t|$ for all $s, t \in [a, b]$. A metric space (X, d) is *geodesic* if every pair of points in X can be joined by a geodesic. A *geodesic triangle* $\Delta = \Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) is defined by three points in X (the vertices of Δ) and three arbitrary geodesics (the sides of Δ), one between each pair of vertices x_1, x_2, x_3 . A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in the Euclidean plane \mathbb{E}^2 such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is a *CAT(0) space* [11] if all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov: *If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{E}^2}(x'_3, y')$.* A geodesic metric space (X, d) is *nonpositively curved* if it is locally CAT(0), i.e., any point has a neighborhood inside which the CAT(0) inequality holds. CAT(0) spaces can be characterized in several different natural ways and have many strong properties, see for example the book by Bridson and Haefliger [5]. In particular, a geodesic metric space (X, d) is CAT(0) if and only if (X, d) is simply connected and is nonpositively curved. For CAT(0) cube complexes, Gromov [11] gave a beautiful combinatorial analog of this result, which can be also taken as their definition:

Theorem 2 ([11]). *A cube complex X endowed with the ℓ_2 -metric is CAT(0) if and only if X is simply connected and the links of all vertices of X are flag simplicial complexes.*

A graph G is called *median* if the interval intersection $I(x, y) \cap I(y, z) \cap I(z, x)$ is a singleton for each triplet x, y, z of vertices. Median graphs are bipartite. Basic examples of median graphs are trees, hypercubes, rectangular grids, and Hasse diagrams of distributive lattices and of median semilattices [2]. Any two cubes of a median graphs are either disjoint or intersect in a cube. Therefore, replacing each cube of G by a solid unit cube of the same dimension, we obtain a cube complex $X_{\text{cube}}(G)$. We continue with the bijection between CAT(0) cube complexes and median graphs:

Theorem 3 ([10, 21]). *Median graphs are exactly the 1-skeleta of CAT(0) cube complexes.*

The proof of Theorem 3 presented in [10] in fact establishes the following local-to-global characterization of median graphs:

Theorem 4 ([10]). *A graph G is a median graph if and only if its cube complex $X_{\text{cube}}(G)$ is simply connected and G satisfies the 3-cube condition: if three squares of G pairwise intersect in an edge and all three intersect in a vertex, then they belong to a 3-cube. Furthermore, if X is a CAT(0) cube complex, then $X = X_{\text{cube}}(G(X))$, i.e., X can be retrieved from its 1-skeleton $G(X)$ by replacing each graphic cube of $G(X)$ by a solid cube.*

3. BOUNDARY RIGIDITY OF CAT(0) CUBE COMPLEXES

In this section, we describe a simple method to reconstruct a CAT(0) cube complex X from boundary distances. By Theorem 4, it suffices to reconstruct the 1-skeleton $G(X)$ of X , because we know that $X = X_{\text{cube}}(G)$. Our method of reconstructing $G(X)$ uses the fact that median graphs admit corner peelings; see the paper by Chalopin, Chepoi, Moran and Warmuth [9]. Let X be a CAT(0) cube complex, ∂X be its combinatorial boundary, and $G(X)$ be the 1-skeleton of X . For simplicity, we will denote $G(X)$ by G . We denote by $\partial G \subseteq V(X)$ the vertices (0-cubes) of X belonging to ∂X . Recall that the input of the reconstruction problem is the boundary ∂G and the distance matrix $D = (d(x, y))_{x, y \in \partial G}$, where $d(x, y)$ is computed in G . Therefore, two vertices x, y of ∂G are adjacent in G if and only if $d(x, y) = 1$.

3.1. Facts about median graphs. We recall the main properties of median graphs used in the reconstruction. These results can be found in the papers by Mulder [18, 19] and are now a

part of folklore for people working in metric graph theory. From now on, G is a finite median graph. The first property follows from the definition.

Lemma 5 (Quadrangle Condition). *For any vertices u, v, w, z of G such that v and w are adjacent to z and $d(u, v) = d(u, w) = d(u, z) - 1 = k$, there is a unique vertex x adjacent to v and w such that $d(u, x) = k - 1$ (see Fig. 1).*

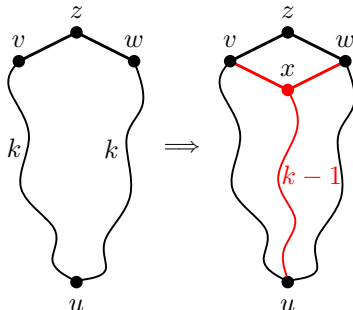


FIGURE 1. Quadrangle condition

The next lemma follows from the fact that convex subgraphs of median graphs are gated and that cubes are convex subgraphs:

Lemma 6 (Cubes are gated). *Cubes of median graphs are gated.*

Let z be a basepoint of a median graph G (z is a fixed but arbitrary vertex of G). For a vertex v , $\Lambda(v)$ consists of all neighbors of v in the interval $I(z, v)$. A graph G satisfies the *downward cube property with respect to z* if for each vertex v , all its neighbors from $\Lambda(v)$ together with v itself belong to a unique cube $C(v)$ of G . In this case, we denote by \bar{v} the unique vertex of the cube $C(v)$ opposite to v . That median graphs satisfy the downward property was proved in [18]. We provide a simple proof of this result from the paper by Bénéteau, Chalopin, Chepoi, and Vaxès [3]:

Lemma 7 ([18])(Downward Cube Property). *Let G be a median graph and fix an arbitrary basepoint z of G . Then G satisfies the downward cube property with respect to z . Moreover, for each vertex v of G , the vertex \bar{v} is the gate of z in the cube $C(v)$.*

Proof. Fix a basepoint z , pick any vertex v , and let $\Lambda(v) = \{u_1, \dots, u_d\}$. We prove that v and any subset of k vertices of $\Lambda(v)$ belong to a unique cube of dimension k . We prove this result by induction on $d(z, v)$ and on the number k of chosen neighbors. The assertion holds in the base case $d(z, v) = 2$: since median graphs are $K_{2,3}$ -free, z and v have one or two common neighbors. The assertion also holds in another base case when $k = 2$: by the quadrangle condition, the two neighbors of v in $\Lambda(v)$ have a unique common neighbor one step closer to z , yielding a 2-cube. Now pick v and the $k \geq 3$ neighbors u_1, \dots, u_k and assume that v and any $2 \leq k' < k$ neighbors of v in $\Lambda(v)$ belong to a unique k' -cube. By the quadrangle condition, for each $2 \leq i \leq k$, there exist a unique common neighbor z_i of u_1 and u_i one step closer to z . Since median graphs are $K_{2,3}$ -free, the vertices z_2, \dots, z_k are distinct neighbors of u_1 in $\Lambda(u_1)$. By induction hypothesis, u_1 and z_2, \dots, z_k belong to a unique $(k-1)$ -cube R' . Analogously, v and its neighbors $u_2, \dots, u_k \in \Lambda(v)$ belong to a unique $(k-1)$ -cube R'' . We assert that the subgraph of G induced by the vertices of $R' \cup R''$ is a k -cube of G . For each $i = 2, \dots, k$, the set $\Lambda(u_i)$ consists of the vertex z_i and the $k-2$ neighbors in R'' other than v . By induction hypothesis, u_i , z_i , and the neighbors of u_i in R'' other than v define a $(k-1)$ -cube R_i of G . This implies that there exists an isomorphism between the facet of R' containing z_i and not u_1 and the facet of R'' containing u_i and not containing v . Since R' and R'' are gated and thus convex, this defines an isomorphism between R' and R'' which maps vertices of R' to their neighbors in R'' . Hence $R' \cup R''$ defines a k -cube of G . The unicity of R' and R'' ensures that $R' \cup R''$ is the unique k -cube of G containing v and u_1, \dots, u_k , finishing the proof of the first assertion. Since

all vertices of $\Lambda(v)$ are closer to z than v and the cube $C(v)$ is gated, the gate of z in $C(v)$ is necessarily the vertex \bar{v} . \square

3.2. Corner peelings of median graphs. A *corner* of a graph G is a vertex v of G such that v and all its neighbors in G belong to a unique cube of G . A *corner peeling* of a finite graph G is a total order v_1, \dots, v_n of $V(G)$ such that v_i is a corner of the subgraph $G_i = G[v_1, \dots, v_i]$ induced by the first i vertices of this order. If v_1, \dots, v_n is a corner peeling of G , then each level-graph G_i ($i = n, \dots, 1$) is an isometric subgraph of G . A *monotone corner peeling* of G with respect to a basepoint z is a corner peeling $v_1 = z, v_2, \dots, v_n$ such that $d(z, v_1) \leq d(z, v_2) \leq \dots \leq d(z, v_n)$. That median graphs admit monotone corner peelings follows from [9, Proposition 4.9] (which was proved in a more general setting but with respect to a particular basepoint). We need a stronger version of this result:

Proposition 8. *For any finite median graph G and any basepoint z , any ordering $v_1 = z, v_2, \dots, v_n$ of $V(G)$ such that $d(z, v_1) \leq d(z, v_2) \leq \dots \leq d(z, v_n)$ is a monotone corner peeling of G . Furthermore, $C(v_i)$ is the unique cube of G_i containing v_i and the neighbors of v_i in G_i . The vertex \bar{v}_i opposite to v_i in $C(v_i)$ is the gate of z in C_i (in G and in G_i).*

Proof. Pick any vertex v_i . Since $d(z, v_1) \leq d(z, v_2) \leq \dots \leq d(z, v_i) \leq \dots \leq d(z, v_n)$ holds, for any neighbor v_j of v_i in G_i we have $d(z, v_j) \leq d(z, v_i)$. Since G is bipartite, we must have $d(z, v_j) < d(z, v_i)$. Consequently, all neighbors of v_i in G_i belong to the interval $I(z, v_i)$ and thus to the set $\Lambda(v_i)$. Conversely, for any vertex v_j of $\Lambda(v_i)$ we have $d(z, v_j) < d(z, v_i)$, thus v_j belongs to G_i . Consequently, the set of neighbors of v_i in G_i coincides with $\Lambda(v_i)$. By Lemma 7, v_i and the set $\Lambda(v_i)$ of all neighbors of v_i in G_i belong to a unique cube $C(v_i)$ of G . We assert that $C(v_i)$ is a cube of G_i . Indeed, by the second assertion of Lemma 7 the vertex \bar{v}_i opposite to v_i in $C(v_i)$ is the gate of z in $C(v_i)$. Since $C(v_i)$ is the cube induced by $I(v_i, \bar{v}_i)$ and $d(v_i, z) = d(v_i, \bar{v}_i) + d(\bar{v}_i, z)$, for any vertex $v_j \in C(v_i)$ other than v_i we obtain that $d(v_j, z) < d(v_i, z)$. Consequently, all vertices of $C(v_i)$ belong to G_i , hence v_i is a corner of G_i and $C(v_i)$ is the unique cube of G_i containing v_i and its neighbors in G_i . The last assertion now follows from the second assertion of Lemma 7. \square

Remark 9. The level-graphs G_i ($i = n - 1, \dots, 1$) are not necessarily median graphs. Notice also that since the cube complexes of graphs with a corner peeling are collapsible, this provides an alternative proof of the result of Adiprasito and Benedetti [1] that CAT(0) cube complexes are collapsible.

Let z be an arbitrary fixed vertex of ∂G and let $v_1 = z, v_2, \dots, v_n$ be a monotone corner peeling of G defined as in Proposition 8. For all $i = n, \dots, 1$ denote by ∂G_i the set of vertices belonging to the boundary of the cube complex $X_i = X_{\text{cube}}(G_i)$. Notice that $G_n = G$ and thus $\partial G_n = \partial G$. As in Proposition 8, for each vertex v_i we denote by $C_i = C(v_i)$ the unique cube of G_i containing the vertex v_i and the set $\Lambda(v_i)$ of all neighbors of v_i in G_i .

Let u_i denote the opposite vertex $u_i = \bar{v}_i$ of v_i in the cube C_i . From the second assertion of Proposition 8 we know that u_i is the gate of z in the cube C_i . Then $V(C_i) = I(v_i, u_i)$, where the interval $I(v_i, u_i)$ is considered in G or in G_i .

From the definition of a corner it immediately follows that all corners of a median graph G belong to the boundary ∂G . In fact, the following lemma shows that for any corner v of G , any vertex of $C(v)$ belongs to ∂G except possibly \bar{v} .

Lemma 10. *All vertices of the cube C_i belong to the boundary ∂G_i of G_i , except possibly u_i .*

Proof. Since in G_i all neighbors of v_i belong to the cube C_i , any facet of C_i containing v_i is not properly contained in any other cube of G_i . Since all vertices on such facets belong to ∂G_i and any vertex of C_i except u_i belong to such a facet, we conclude that every vertex of C_i different from u_i belongs to ∂G_i . \square

Set $S(G_n)$ to be the set $\partial G_n = \partial G$ and iteratively define the set $S(G_{i-1})$ to be $S(G_i) \setminus \{v_i\} \cup \{u_i\}$ ($i = n - 1, \dots, 2$). We will call $S(G_i)$ the *extended boundary* of G_i ; the name is justified by the following result:

Lemma 11. *For all $i = n, \dots, 2$, we have $\partial G_{i-1} \subseteq \partial G_i \cup \{u_i\}$ and $\partial G_i \subseteq S(G_i)$.*

Proof. We first prove that any vertex from ∂G_{i-1} belongs to $\partial G_i \cup \{u_i\}$. Suppose not and let $x \in \partial G_{i-1} \setminus \partial G_i$. Since $x \in \partial G_{i-1}$, there exists a non-maximal cube C containing x and contained as a facet in a unique cube C' of $X_{i-1} = X_{\text{cube}}(G_{i-1})$. On the other hand, since $X_{i-1} \subset X_i$ and x is not a boundary vertex of X_i , the cube C is a facet of yet another cube C'' in X_i . The cube C'' of X_i is not present in X_{i-1} , and consequently v_i is a vertex of C'' . Since all cubes of G_i containing v_i are included in C_i , we know that x is a vertex of C_i . From Lemma 10 and since $x \notin \partial G_i$, necessarily $x = u_i$. This establishes the inclusion $\partial G_{i-1} \subseteq \partial G_i \cup \{u_i\}$.

We now prove that $\partial G_i \subseteq S(G_i)$ by decreasing induction on $i = n, \dots, 1$. For $i = n$, we have $S(G_n) = \partial G_n$. Suppose now that the assertion holds for G_i and consider the graph G_{i-1} . Since $v_i \notin G_i$, by the first assertion of the lemma, and by induction hypothesis, we have $\partial G_{i-1} \subseteq \partial G_i \setminus \{v_i\} \cup \{u_i\} \subseteq S(G_i) \setminus \{v_i\} \cup \{u_i\} = S(G_{i-1})$. \square

3.3. Reconstruction via corner peeling. To reconstruct a median graph G from the pairwise distances between the vertices of the boundary ∂G we proceed in the following way. We first describe the objects that the reconstructor keeps track of during the reconstruction algorithm. It first picks an arbitrary vertex $z \in \partial G$ as a basepoint that is fixed during the whole execution of the algorithm. During the execution of the algorithm, the reconstructor knows a set S of vertices that is initially ∂G as well as the distance matrix D between the vertices of S computed in the graph G (if $u, v \in S$, then $D[u, v]$ contains the distance $d_G(u, v)$). It constructs a graph Γ that is initially isomorphic to the subgraph of G induced by the boundary ∂G and will ultimately coincide with G .

In order to analyze the algorithm, we consider the intermediate values S_i of the set S , D_i of the distance matrix D , and Γ_i of the graph Γ at the beginning of the i th step of the algorithm. In order to have similar notations as in the definition of monotone corner peelings, we denote the initial values of S , D , and Γ respectively by S_n , D_n , and Γ_n (even if the algorithm does not know n), and at each step, we decrease the values of i . For the analysis of the algorithm, we also consider the graphs G_i (unknown to the algorithm), where $G_n = G$.

We now give an outline of the reconstruction algorithm. The goal is to reconstruct the graph $G = G_n$ from the set $S_n = \partial G$ and its distance matrix D_n . The graph Γ_n induced by ∂G can be computed from D_n : two vertices u and v of S_n are adjacent in Γ_n if and only if $D_n[u, v] = 1$. At step i , the reconstructor picks a vertex v_i of S_i furthest from the basepoint z , identifies the vertex u_i opposite to v_i in the unique cube C_i of G containing v_i and its neighbors in S_i , removes v_i from S_i , and adds u_i to S_i unless it is already in S_i . The resulting set is denoted by S_{i-1} . From D_i , we compute the distance matrix D_{i-1} between the vertices of S_{i-1} ; the main point is to compute the distances from u_i to the vertices of S_{i-1} since the other distances are already known. If $u_i \in S_i$, then Γ_{i-1} is Γ_i . Otherwise, Γ_{i-1} is obtained by adding to Γ_i the vertex u_i and the edges between u_i and its neighbors in $S_{i-1} \cup \{v_i\}$. The algorithm ends when S_i becomes empty. We will show that the graph Γ_0 obtained when $S_i = \emptyset$ is isomorphic to G .

Now, we present the invariants that are maintained at each step i of the reconstruction algorithm. Let G_i be the subgraph of G obtained from G by removing the vertices v_n, \dots, v_{i+1} . Notice that the subgraph G_i is not known to the reconstructor. Furthermore, suppose that the removed vertices v_n, \dots, v_{i+1} and the possibly added vertices u_n, \dots, u_{i+1} satisfy the following inductive properties:

- (1) $d(z, v_n) \geq \dots \geq d(z, v_{i+1}) \geq d(z, v)$ for any vertex v of G_i .
- (2) Each vertex v_j with $n \geq j \geq i+1$ is a corner of the graph G_j .
- (3) For each $n \geq j \geq i+1$, either all neighbors of v_j in G_j are in S_j or u_j is the unique neighbor of v_j in G_j ; in both cases, $u_j \in S_{j-1}$.
- (4) S_i coincides with the extended boundary $S(G_i)$ of G_i , D_i is the distance matrix of $S(G_i)$ in G , and $\Gamma_i = G[\bigcup_{n \geq j \geq i} S_j]$.

The next result explains how to find a corner of the graph G_i (without knowing G_i) from the distances from z to the vertices of $S_i = S(G_i)$.

Lemma 12. *Let v_i be a vertex of S_i maximizing $d(z, v_i)$. Then $d(z, v_i) \geq d(z, v)$ for any vertex v of G_i and thus v_i is a corner of G_i .*

Proof. Suppose by way of contradiction that there exists a vertex u of G_i such that $d(z, v_i) < d(z, u)$ and assume without loss of generality that u maximizes $d(z, u)$ among all vertices in G_i . Since $d(z, v_n) \geq \dots \geq d(z, v_{i+1}) \geq d(z, v)$ for any vertex v of G_i by invariant (1), from Proposition 8 we conclude that there exists a monotone corner peeling of G starting with the vertices v_n, \dots, v_{i+1}, u . Consequently, u is a corner of G_i and therefore u belongs to ∂G_i . Since $\partial G_i \subseteq S(G_i)$ by Lemma 11 and $S(G_i) = S_i$ by invariant (4), the vertex u belongs to S_i , contradicting the definition of v_i . Consequently, v_i is a vertex of G_i maximizing $d(z, v_i)$ and hence a corner of G_i . \square

Suppose that we have performed the steps $n, \dots, i+1$ of the algorithm. Let v_i be a vertex of $S_i = S(G_i)$ maximizing $d(z, v_i)$, and assume that $v_i \neq z$. Since v_i is a corner of G_i by Lemma 12, v_i and its neighbors in G_i belong to a unique cube C_i of G_i . The dimension of C_i is the number of neighbors of v_i in G_i . Let u_i be the vertex of C_i opposite to v_i . By Lemmas 10 and 11 and invariant (4), $V(C_i) \setminus \{u_i\} \subseteq \partial G_i \subseteq S(G_i) = S_i$. If $u_i \notin S_i$, then u_i will be added to S_{i-1} .

Additionally, the distances from u_i to all other vertices x of $S_i \setminus \{v_i\}$ should be computed. If $\dim(C_i) = 1$, then u_i is an articulation point separating v_i from all other vertices of G_i , and thus $d(u_i, x) = d(v_i, x) - 1$. Now suppose that $\dim(C_i) \geq 2$ and let $N(u_i)$ be the set of neighbors of u_i in C_i . Since C_i is gated in G , we can consider the gate x' of x in C_i . We assert that $x' = u_i$ if and only if all vertices of $N(u_i)$ are at the same distance from x . Indeed, if $x' = u_i$, then all vertices of $N(u_i)$ are at distance $d(u_i, x) + 1$ from x . Conversely, if all vertices of $N(u_i)$ are at the same distance from x , then either $x' = u_i$ or $x' = v_i$. However, the second case is impossible because v_i is a corner of G_i and $x \in S_i \setminus \{v_i\} \subseteq V(G_i) \setminus \{v_i\}$. Suppose now that $x' \neq u_i$ (for an illustration, see Figure 2). By the previous assertion, there exist vertices in $N(u_i)$ that are at different distances from x . Since the vertices of $N(u_i)$ are pairwise at distance 2, we have $d(x, u_i) = \min\{d(x, u) : u \in N(u_i)\} + 1 = \max\{d(x, u) : u \in N(u_i)\} - 1$. Consequently, independently of the dimension of C_i and the position of x' in C_i , the distance $d(x, u_i)$ can be computed via the following formula:

Lemma 13. *For any $x \in S_{i-1}$, $d(x, u_i) = \max\{d(x, u) : u \in N(u_i)\} - 1$.*

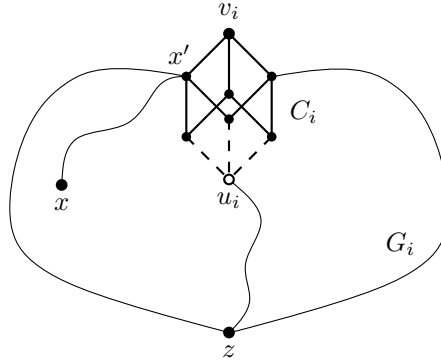


FIGURE 2. Illustration of step i of the reconstruction algorithm

We now describe step i of the algorithm. First we define the set S_{i-1} . The reconstructor picks a vertex v_i in S_i maximizing $d(z, v_i)$. If $z = v_i$, then the algorithm stops and return Γ_i . So, suppose that $v_i \neq z$. The reconstructor computes the set L_i of neighbors of v_i in S_i using the distance matrix D_i . If $|L_i| \geq 2$, then by iteratively applying the quadrangle condition, it identifies all vertices of the cube C_i except possibly u_i . Let N_i be the set of neighbors of u_i in C_i . The reconstructor can detect if the vertices of N_i have a common neighbor in S_i closer to z . If yes, then this vertex is necessarily u_i , otherwise u_i does not belong to S_i . If $|L_i| \leq 1$, then $\dim(C_i) = 1$, and $u_i \in S_i$ if and only if $|L_i| = 1$, in which case, $L_i = \{u_i\}$. In this case, let

$N_i = \{v_i\}$ (as in the previous case, N_i is the set of neighbors of u_i in C_i). If $u_i \in S_i$, then let $S_{i-1} = S_i \setminus \{v_i\}$ and if $u_i \notin S_i$, then S_{i-1} is obtained by adding a new vertex u_i to $S_i \setminus \{v_i\}$.

The reconstructor then derives the distance matrix D_{i-1} of S_{i-1} from D_i . It first removes the row and the column of D_i corresponding to v_i . Additionally, if $u_i \notin S_i$, it adds a row and a column for u_i where $D_{i-1}[u_i, x] = D_{i-1}[x, u_i] = \max\{D_i[u, x] : u \in N_i\} - 1$, for each $x \in S_i \setminus \{v_i\}$.

Finally, the graph Γ_{i-1} is Γ_i if $u_i \in S_i$. Otherwise, Γ_{i-1} is obtained from Γ_i by adding the vertex u_i , all edges in $\{u_i w : w \in S_{i-1} \text{ and } d(u_i, w) = 1\}$. If $\dim(C_i) = 1$, then the edge $u_i v_i$ is also added to Γ_{i-1} . This ends the description of step i of the algorithm: from the set S_i , the distance matrix D_i , and the graph Γ_i , the reconstructor has computed S_{i-1} , D_{i-1} , and Γ_{i-1} .

Now we prove that the invariants (1), (2), (3), (4) are satisfied after step i . Invariants (1) and (2) follow from Lemma 12 and the definition of v_i . Invariant (3) follows from the definition of u_i and Lemmas 10 and 11. Since $S_i = S(G_i)$, and by the definitions of v_i and u_i , we have $S_{i-1} = S_i \setminus \{v_i\} \cup \{u_i\} = S(G_i) \setminus \{v_i\} \cup \{u_i\} = S(G_{i-1})$. By Lemma 13, the distances from u_i to all vertices of S_{i-1} have been correctly computed at step i , and thus, by induction hypothesis, D_{i-1} is the distance matrix of $S_{i-1} = S(G_{i-1})$. If $u_i \in S_i$, then $\Gamma_{i-1} = \Gamma_i = G[\bigcup_{n \geq j \geq i} S_j] = G[\bigcup_{n \geq j \geq i-1} S_j]$. If $u_i \notin S_i$, then $V(\Gamma_{i-1}) = V(\Gamma_i) \cup \{u_i\} = \bigcup_{n \geq j \geq i-1} S_j$. Moreover, for each neighbor $w \in S_{i-1} \cup \{v_i\}$ of u_i in G , the edge $w u_i$ is in $E(\Gamma_{i-1})$. If u_i has another neighbor w in G belonging to $V(\Gamma_{i-1})$, then $w = v_j$ with $j > i$. However, since $u_i \notin S_j$, this implies by invariant (3) that $u_i \in S_{j-1}$ and thus $u_i \in S_i$, a contradiction. Therefore, Γ_{i-1} is the subgraph of G induced by $\bigcup_{n \geq j \geq i-1} S_j$. This establishes invariant (4).

Lemma 14. *The graph Γ_0 returned by the reconstructor is isomorphic to G .*

Proof. By invariant (1) of the algorithm, z is the last vertex removed from S . By Lemma 12, when z is considered by the algorithm, all vertices of G have been already processed. This implies that each vertex $x \in V(G)$ belongs to some S_i and thus to $V(\Gamma_0)$, establishing $V(\Gamma_0) = V(G)$. By invariant (4), Γ_0 is an induced subgraph of G and is thus isomorphic to G . \square

Lemma 14 and Theorem 4 imply the main result of the paper:

Theorem 1. *Let X be a finite CAT(0) cube complex. Then X is determined up to isomorphism by the distances between the vertices of ∂X in the 1-skeleton of X . Consequently, the class of finite CAT(0) cube complexes is boundary rigid.*

Remark 15. The key property of our algorithm is that the vertex v_i chosen at step i is a corner of the graph G_i , as established in Lemma 12. Note that this lemma holds for any set of vertices containing ∂G_i , in particular for any set containing S_i .

Suppose that instead of starting with $S_n = \partial G$, we start the algorithm with a set of vertices S'_n containing S_n , the distance matrix D'_n of S'_n in G , and the subgraph Γ'_n of G induced by S'_n . Then executing precisely the same algorithm and considering the intermediate values S'_i , D'_i , and Γ'_i of S , D , and Γ , one can show by induction on i that $S_i \subseteq S'_i$ holds for any step i . Thus our algorithm reconstructs G from S'_n and D'_n .

In particular, since our definition of combinatorial boundary is weaker than the one of Haslegrave et al. [13], our algorithm also reconstruct $G(X)$ starting from the combinatorial boundary of X defined as in [13].

4. FINAL REMARKS

In this note, we proved that finite CAT(0) cube complexes are boundary rigid. This generalizes the results of Haslegrave, Scott, Tamitegama, and Tan [13] and settles their main Conjecture 20. Our method uses the fact that any ordering of vertices of the 1-skeleton $G(X)$ of a CAT(0) cube complex X by decreasing the distances to a basepoint z is a corner peeling.

In our approach, it was important to assume that $G(X)$ is a median graph, i.e., that X is a CAT(0) cube complex. For example, consider the square complex consisting of three squares with a common vertex v_0 and pairwise intersecting in edges incident to this vertex. The 1-skeleton of this complex is the graph Q_3^- (the 3-cube minus one vertex). This graph is the 6-cycle $C = (v_1, v_2, v_3, v_4, v_5, v_6)$ and the vertex v_0 incident to three vertices of C . There are

two ways to connect v_0 to C , either with v_1, v_3, v_5 or with v_2, v_4, v_6 , leading to two isomorphic square complexes X' and X'' with respective 1-skeletons G' and G'' . The graph G' has v_2, v_4, v_6 as corners and the graph G'' has v_1, v_2, v_3 as corners. Both G' and G'' have the cycle C as the boundary, furthermore there is an isometry between the two boundaries $\partial G'$ and $\partial G''$. However this isometry cannot be extended to an isomorphism between the graphs G' and G'' and between the complexes X' and X'' . Therefore, any class of finite cube complexes containing the square complex $X' \simeq X''$ as a member is not boundary rigid. This example also shows that without any additional information we cannot decide which vertices of C are corners of G . The graph Q_3^- admits a monotone corner peeling if we order the vertices of Q_3^- by distance to a vertex z of degree 3. Consequently, the existence of monotone corner peelings is not sufficient to ensure boundary rigidity if we are not able to identify the corners. By Proposition 8, in median graphs, any ordering of the vertices by distance to an arbitrary vertex z gives a corner peeling. Thus, we can choose an arbitrary vertex on the boundary as a basepoint.

In order to prove that our algorithm effectively reconstructs the 1-skeleton G of a cube complex X from its boundary, it is sufficient to establish the following properties:

- (a) the cubes of G are gated;
- (b) one can find a basepoint $z \in \partial G$ such that any ordering of the vertices of G by distance to z gives a corner peeling.

Condition (a) holds for all partial cubes, i.e., graphs which can be isometrically embedded into hypercubes. In [9], Chalopin, Chepoi, Moran, and Warmuth proved that every conditional antimatroid G (conditional antimatroids form a superclass of median graphs and convex geometries and a subclass of partial cubes) admits a monotone corner peeling with respect to a specific basepoint z , corresponding to the empty set in the set-theoretical encoding of G . However, conditional antimatroids do not always satisfy condition (b). Indeed, Q_3^- is a conditional antimatroid, and thus the class of conditional antimatroids is not boundary rigid. As explained above, even if there exists a monotone corner peeling of Q_3^- with respect to a basepoint on ∂Q_3^- , one cannot identify it by knowing only the boundary of Q_3^- . Moreover, there also exist conditional antimatroids where the specific basepoint z does not belong to ∂G . However, if, in addition to the distance matrix of ∂G , the specific basepoint z of a conditional antimatroid G is given as well as the distances from z to all vertices of ∂G , then, proceeding as in the case of median graphs, we can reconstruct $X_{cube}(G)$.

Haslegrave [12] presented a plane CAT(0) triangle-square complex, that cannot be reconstructed from its boundary distance matrix. Haslegrave et al. [13] asked for conditions under which a class of finite simplicial complexes is boundary rigid. It will be interesting to investigate this question for clique complexes of bridged and Helly graphs, which are well-studied in metric graph theory [2, 8, 10]. Groups acting on them have recently been extensively studied in geometric group theory. These simplicial complexes are not CAT(0) but are considered to have combinatorial nonpositive curvature; see the papers by Januszkiewicz and Świątkowski [15] and Chalopin, Chepoi, Hirai, and Osajda [8].

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