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# The role of constant vorticity on weakly nonlinear surface gravity waves

M. A. Manna

*Université Montpellier, Laboratoire Charles Coulomb,  
UM-CNRS UMR 5221, F-34095, Montpellier, France.*

S. Noubissie

*Institute of Technology, University of Dschang, P. O. Box 134 Bandjoun, Cameroon.*

J. Touboul

*Université Toulon, Aix Marseille Université, CNRS/INSU, IRD,  
MIO UM 110, Mediterranean Institute of Oceanography, La Garde, France.*

B. Simon

*Université Toulon, Aix Marseille Université., CNRS/INSU, IRD,  
MIO UM 110, Mediterranean Institute of Oceanography, La Garde, France.*

R. A. Kraenkel

*Instituto de Física Teórica-UNESP, Universidade Estadual Paulista,  
Rua Dr. Bento Teobaldo Ferraz 271 Bloco II, 01140-070, São Paulo, Brazil*

This manuscript describes the derivation of systems of equations for weakly nonlinear gravity waves in the presence of constant vorticity. The derivation is based on a multi-layer generalization of the traditional columnar Ansatz. A perturbative development in a nonlinear parameter and a dispersive parameter allow us to obtain sets of equations able to describe propagation of weakly nonlinear and dispersive surface waves moving in water with some prescribed initial constant vorticity. We have shown that vorticity can add or remove nonlinearity, dispersion or nonlinear dispersion where before only were pure eulerian dispersion and nonlinear dispersion. An explicitly steady solution of the system corresponding to strong vorticity is obtained. It corresponds to a solitary wave. Evolution of the soliton celerity, amplitude and width for null, low, normal and strong vorticity are discussed.

## I. INTRODUCTION

When propagating in coastal waters, surface waves often encounter currents. These currents, with intensities ranging from weak to very strong are generated through various mechanisms, such as oceanic circulation, tides, wind action or wave breaking. Given these generation processes, the currents are often observed to vary with depth, and result in an underlying vorticity. Such background vorticity, included within strong tidal currents [1] or in wind driven currents [2] could be important and should be taken into account in modeling the propagation of water waves [3]. This underlying vorticity is especially observed in shallow water environments. For instance, strong currents, linearly sheared, were observed in the surf zone, in strong rip currents, in situ [4] or in laboratory experiments [5]. More recently, a similar vertical structure of the current was observed over coral reefs [6].

From a theoretical point of view the role played by constant or variable vorticity constitutes a vast and classical subject in fluid mechanics. Da Silva and Peregrini have studied [7] steady surface waves in water of finite depth with constant vorticity. In references [8–12]. Constantin et al. have developed a complete study of exact steady periodic water waves with vorticity. Castro and Lannes [13] investigated fully nonlinear long-waves under the action of vorticity with a Green-Naghdi equation. Numerical studies of this problem can be found in references Ko and Strauss [14, 15] and Milewski, Vanden-Broeck and Z. Wang [16] However very few works have been consacrated to modeling nonlinear dynamics of long surface waves in finite depth under the actions of a sheared current. The pioneering work in this domain was that of Benjamin [17] who generalized the classical solitary wave theory to surface waves under the action of vorticity. Freeman and Johnson [18] derived a Korteweg de Vries equation in a flow of arbitrary vorticity. Choi has derived the Green-Naghdi equation and the associated Boussinesq equation for small amplitude wave. In reference [19] Johnson studied the problem of two-way propagation as described by the Boussinesq equation and derived a new Boussinesq-type equation valid for constant vorticity. The solitary-wave solution for this new equation is exhibited. The Camassa–Holm equation for water waves with constant vorticity and its solitary-wave solution were described. Kharif, Abid and Touboul have studied the action of vorticity on the rogue waves dynamics [20]. More recently Kharif and Abid [21] have studied the role played by constant vorticity on surface waves using a generalized Whitham equation [22].

The main objective of this work is to derive higher order evolution equations for weakly nonlinear long surface gravity waves in the presence of constant vorticity beyond all orders in nonlinearity, dispersion and nonlinear dispersion. The employed method is based on the Johnson’s work [23]. A generalized columnar pattern Ansatz is introduced, together with the two perturbative parameters : the nonlinear parameter  $\alpha = a/h < 1$  and the dispersive parameter  $\delta = kh < 1$ , where  $h$  refers to the constant depth of the fluid while  $a$  and  $k$  are a typical amplitude and wavenumber of the wave being studied. This generalization is based on two main hypothesis. Firstly, the horizontal velocity of the fluid is represented by means of an infinite series in powers of  $z^2\delta^2$ . Secondly, the conservation of vorticity is enforced, which provides a consistent closure of the problem. This results in a generalized Serre-Green-Naghdi model with vorticity equivalent to a *multi-layer model* in place of the single-layer model (see Kim et al. [24, 25]). An interesting advantage of this approach is that it provides an inter-comparison of the terms describing frequency dispersion, and nonlinear dispersion, even when they involve vorticity. The main result in relation to previous studies is that  $\Omega$  can add or remove nonlinearity, dispersion or nonlinear dispersion, associated with the superimposed shear, where before only were pure eulerian dispersion and nonlinear dispersion. The role of vorticity can thus be discussed, considering no vorticity, weakly, normal or strong vorticity. Only the case of strong vorticity is analytically and numerically studied in this letter. Nil, weakly and normal vorticity are only studied numerically.

## II. THE COMBINATED ACTION OF VORTICITY, WEAKLY NONLINEARITY, DISPERSION, AND NONLINEAR DISPERSION ON LONG WATER WAVES

We consider the water particles to be located relative to a two-dimensional Cartesian coordinate system with origin 0 and axes  $x, z$  with  $z$  being the upward vertical direction. The governing equations are the Euler equations and the equation of mass conservation in  $x, z$  and  $t$  (time) completed with appropriate surface and bottom boundary conditions. Before perturbations the fluid lies between the impermeable bottom at  $z = 0$  and the still water free surface at  $z = h$  with  $h$  constant. The undisturbed initial state is incompressible, inviscid, with zero surface tension and with a superimposed current profile in the  $x$  direction and depending only linearly on  $z$  i.e  $\vec{U} = \Omega z \vec{e}_x$  with  $\vec{e}_x$  the unitary vector in the  $x$  direction. The perturbations to the free surface, horizontal and vertical velocities and pressure (relative to the hydrostatic pressure in the undisturbed initial state) are  $\eta(x, t)$ ,  $u(x, z, t)$ ,  $w(x, z, t)$  and  $p(x, z, t)$  respectively. The governing equations are nondimensionalized as follow (primes mean dimensionless variables):  $z' = z/h, x' = kx, t' = kt\sqrt{gh}, \eta' = \eta/a, u' = 1/\sqrt{gh}u, w' = kh\sqrt{gh}w, \Omega' = \Omega/k\sqrt{gh}$ , with  $k$  an average or a typical wavenumber,  $a$  the amplitude and  $g$  the gravity. This step introduces the two parameters strictly smaller than 1;  $\alpha = a/h < 1$  and  $\delta = kh < 1$ , familiar and fundamental in the study of the classical surface water waves problem.

The final form of the governing equations require to scale  $u'$  and  $w'$  with  $\alpha$ . As a result we obtain for  $0 \leq z \leq 1 + \alpha\eta$  and dropping the primes

$$u_x + w_z = 0, \quad u_t + (\delta\Omega z + \alpha u) u_x + w(\delta\Omega + \alpha u_z) + p_x = 0, \quad w_t + (\alpha\Omega z + \alpha u) w_x + \alpha w w_z + \frac{p_z}{\delta^2} = 0 \quad (1)$$

$$p = \eta \quad \text{and} \quad \eta_t + [\delta\Omega(1 + \alpha\eta) + \alpha u] \eta_x - w = 0, \quad \text{on} \quad z = 1 + \alpha\eta; \quad \text{and} \quad w = 0, \quad \text{on} \quad z = 0. \quad (2)$$

Linearizing the system (1-2) we obtain the linear solutions for the horizontal and vertical velocities  $u_L$  and  $w_L$  as:  $u_L(x, z, t) = 2a i \delta \exp\{i\theta(x, t)\} \cosh \delta z$  and  $w_L(x, z, t) = 2a \exp\{i\theta(x, t)\} \sinh \delta z$  with  $\exp\{i\theta(x, t)\} = \exp\{i(x - \sigma t)\}$  and  $\sigma$  the linear frequency of expression

$$\sigma = \delta\Omega - \frac{\Omega}{2} \tanh \delta \pm \left\{ \frac{\Omega^2 \tanh^2 \delta}{4} + \frac{\tanh \delta}{2} \right\}^{\frac{1}{2}}. \quad (3)$$

Now we assume that the nonlinear horizontal velocity can be expanded as  $u(x, z, t) = \sum_{n=0}^{\infty} u_{2n} \delta^{2n} z^{2n}$ ,  $u_{2n} = u_{2n}(x, t)$ ,  $\forall n$ . The Johnson derivation of the *single-layer model* equations (in reference [23]) means, from a physical mathematical point of view, the lower order in  $\delta$  of the linear horizontal velocity solution  $u_L$  proportional to  $\exp i\theta(x, t)$  is replaced by an unknown function  $u(x, t)$  which is supposed to bring an *approximate* solution of the entire nonlinear problem. Our approach can be viewed as a generalisation of this columnar hypothesis where each one of the orders  $\delta^0 z^0 \exp\{i\theta(x, t)\}$ ,  $\delta^2 z^2 \exp\{i\theta(x, t)\}$ ,  $\delta^4 z^4 \exp\{i\theta(x, t)\}$ ,  $\dots$  are replaced by  $u_0(x, t)$ ,  $\delta^2 z^2 u_2(x, t)$ ,  $\delta^4 z^4 u_4(x, t)$ ,  $\dots$ . From the continuity equation and the boundary condition  $w = 0$  on  $z = 0$  it follow that  $w(x, z, t) = -\sum_{n=0}^{\infty} u_{2n,x} \frac{\delta^{2n} z^{2n+1}}{2n+1}$ . Substituting  $u(x, z, t)$  and  $w(x, z, t)$  in the equation for  $w_t$  integrating for  $z \in [z, 1 + \alpha\eta]$  and using  $p = \eta$  at  $z = 1 + \alpha\eta$  we obtain  $p(x, z, t)$ . Using the expression of  $p$  in  $u_t$  and integrating for  $z \in [0, 1 + \alpha\eta]$  lead to

$$u_{0,t} + \alpha u_0 u_{0,x} + \eta_x = \frac{\delta^2}{3(1 + \alpha\eta)} \left\{ \left[ (1 + \alpha\eta)^3 (u_{0,xt} + \alpha u_0 u_{0,xx} - \alpha u_{0,x}^2) \right]_x - (1 + \alpha\eta)^3 (u_{2,t} + \alpha (u_0 u_{2,x} - u_2 u_{0,x})) \right\} - \delta^3 \Omega \left\{ (1 + \alpha\eta)^3 \left( \frac{u_{2,x}}{6} - \frac{u_{0,xxx}}{4} - \alpha u_{0,xx} \eta_x (1 + \alpha\eta)^{-1} \right) \right\} + \mathcal{O}(\Omega \delta^5, \delta^4). \quad (4)$$

Finally equation for  $\eta(x, t)$  using  $u(x, z, t)$  and  $w(x, z, t)$  at  $z = 1 + \alpha\eta$  gives

$$\eta_t + (1 + \alpha\eta) \delta\Omega \eta_x + [u_0 (1 + \alpha\eta)]_x + \frac{\delta^2}{3} \left[ u_2 (1 + \alpha\eta)^3 \right]_x + \mathcal{O}(\delta^4) = 0. \quad (5)$$

Expressions (4) and (5) are a system of two coupled equations in three fields (and their derivatives):  $u_0$ ,  $u_2$  and  $\eta$ . In order to reduce this system to only two fields  $u_0$  and  $\eta$  we use the vorticity equation:  $\omega_t + (\delta\Omega z + \alpha u) \omega_x + \alpha w \omega_z = 0$ , with  $\omega = |\vec{\omega}|$ ,  $\vec{\omega} = \left( \frac{\delta}{\alpha} \Omega + u_z - \delta^2 w_x \right) \vec{e}_y$ , and  $\vec{e}_y$  the unitary vector in the  $y$  direction. Hence we obtain  $u_{2n+2} = -\frac{u_{2n,xx}}{(2n+1)(2n+2)}$ ,  $n = 0, 1, 2, \dots$ . This recurrence allows us to reduce (4) and (5), at any degree of approximation, in a system in  $u_0$ ,  $\eta$  and derivatives. The  $\alpha$  parameter in (4) and (5) is only a non-dimensionalization parameter. So theses equations can be viewed as strongly nonlinear evolution equations with weakly linear and non linear dispersion ( $\delta < 1$ ).

### III. THE VORTICITY NUMBER $\Omega$

Expressions (4) and (5) were derived assuming (implicitly) that  $\Omega$  is order zero in  $\delta$ . In fact, the amplitude of  $\Omega$  can dramatically change not only the orders  $\delta$  in (4) and (5) but the very nature of the physical origin of dispersion and nonlinear dispersion themselves. Table I shows, in its two columns, the successive degrees in dispersion (rows of the left column) and nonlinear dispersions (rows of the right column) of some equations which can be obtained from (4) retaining only terms of order one in  $\alpha$ . The successive degrees in linear dispersion are  $(\delta^2)$ ,  $(\Omega \delta^3; \delta^2)$ ,  $\dots$ , while the corresponding successive degrees in nonlinear dispersion are  $(\alpha \delta^2)$ ,  $(\alpha \Omega \delta^3; \alpha \delta^2)$ ,  $(\dots)$ . If we take  $\Omega$  large i.e.  $\Omega = \Omega_0 / \delta$  ( $\Omega_0$  order zero in delta) the associated equation with the first row exhibits not only pure dispersive and nonlinear dispersive terms i.e.; orders  $\delta^2$  and  $\alpha \delta^2$  but it also contains a new dispersive term  $\Omega_0 \delta^2$  and a new nonlinear dispersive term  $\Omega_0 \alpha \delta^2$  both involving vorticity. In equation (5) the  $\delta$  dependence of  $\Omega$  changes the coefficients of the terms of linear velocity and nonlinearity. Thus the choice of the  $\delta$  dependence of  $\Omega$  has consequences over every one of the equations. Therefore we can think about  $\Omega$  as a *dimensionless vorticity number* the value of which can add or remove

Linear dispersion	Nonlinear dispersion
$\delta^2$	$\alpha \delta^2$
$\Omega\delta^3, \delta^2$	$\alpha(\Omega\delta^3, \delta^2)$
$\delta^4, \Omega\delta^3, \delta^2$	$\alpha(\delta^4, \Omega\delta^3, \delta^2)$
$\Omega\delta^5, \delta^4, \Omega\delta^3, \delta^2$	$\alpha(\Omega\delta^5, \delta^4, \Omega\delta^3, \delta^2)$
...	...

TABLE I. Dispersion and nonlinear-dispersion

nonlinearity, dispersion or nonlinear dispersion associated with the superimposed shear. In our example we see that strong vorticity brings dispersion and nonlinear dispersion whose physically origin is the fluid vorticity. Consequently we proposed the following scaling for  $\Omega$ :  $\Omega = \Omega_0\delta^n$  with  $\Omega_0$  order zero in  $\delta$  and  $n = 0, 1, 2, \dots$ . Substituting  $u_2(x, t)$  in function of  $u_0(x, t)$  and rearranging the terms we have

$$u_{0,t} + \alpha u_0 u_{0,x} + \eta_x - \frac{\delta^2}{2} \left\{ (1 + \alpha\eta)^2 (u_{0,xt} + \alpha u_0 u_{0,xx} - \alpha u_{0,x}^2) \right\}_x - \delta^3 \Omega \left\{ \frac{1}{3} (1 + \alpha\eta)^3 u_{0,xxx} + \alpha (1 + \alpha\eta)^2 u_{0,xx} \eta_x \right\} + \mathcal{O}(\Omega\delta^5, \delta^4) = 0, \quad (6)$$

$$\eta_t + (1 + \alpha\eta)\delta\Omega\eta_x + [u_0(1 + \alpha\eta)]_x - \frac{\delta^2}{6} [u_{0,xx}(1 + \alpha\eta)^3]_x + \mathcal{O}(\delta^4) = 0. \quad (7)$$

In this work, we analyze four cases: the irrotational case with  $\Omega = 0$ , the normal case with  $\Omega = \Omega_0$ , the weak vorticity cases with  $\Omega = \Omega_0\delta$ , and the strong vorticity case with  $\Omega = \Omega_0\delta^{-1}$ . In all cases we make a further simplification and will only consider nonlinearity of order  $\alpha$ , dispersion of order  $\delta^2$  and nonlinear dispersion of order  $\alpha\delta^2$ . The balance between these factors are represented by soliton solutions for  $u_0(x, t)$  and  $\eta(x, t)$ . In this letter analytical expressions of  $u_0(x, t)$  and  $\eta(x, t)$  are only given for the strong vorticity case. The cases of nil, normal and weak vorticity can be easily obtained from the strong case through the transformations  $\Omega_0 \rightarrow 0$ ,  $\Omega_0 \rightarrow \Omega_0\delta^{-1}$  and  $\Omega_0 \rightarrow \Omega_0\delta^2$ .

### A. The strong vorticity case, $\Omega = \Omega_0\delta^{-1}$

The vorticity in this case is order  $\delta^{-1}$  and we have the system

$$u_{0,t} + \alpha u_0 u_{0,x} + \eta_x - \frac{\delta^2}{2} \left\{ u_{0,xt} + \alpha u_0 u_{0,xx} - \alpha u_{0,x}^2 + 2\alpha\eta u_{0,xt} \right\}_x - \Omega_0\delta^2 \left\{ u_{0,xxx} \frac{1 + 3\alpha\eta}{3} + \alpha\eta_x u_{0,xx} \right\} = 0 \quad (8)$$

$$\eta_t + (1 + \alpha\eta)\Omega_0\eta_x + u_{0,x} + \alpha(u_0\eta)_x - \frac{\delta^2}{6} u_{0,xxx} - \frac{\alpha\delta^2}{2} (u_{0,xx}\eta)_x = 0. \quad (9)$$

The effects of the strong vorticity on non linearity, dispersion and nonlinear dispersion can be viewed in both  $u_0$  and  $\eta$  dynamics. Now we suppose progressive solutions in  $r = x - ct$  going to zero for  $|x| \rightarrow \pm\infty$  and  $c$  the soliton velocity to be determined. Integrating (9) in  $r$  we obtain  $\eta(r)$  and substituting its expression in (8) we have

$$\eta = cu_0 - \frac{1}{2}\alpha u_0^2 + \alpha\delta^2 \left( \frac{1}{2} - c^2 + c\Omega_0 \right) u_0 u_{0,rr} - \frac{1}{2}\alpha\delta^2 u_{0,r}^2 - \delta^2 \left( -\frac{1}{3}\Omega_0 + \frac{1}{2}c \right) u_{0,rr}, \quad (10)$$

$$u_0(1 - c^2 + \Omega_0 c) + \alpha u_0^2 \left( \frac{3}{2}c + \frac{\Omega_0}{2}c^2 - \frac{\Omega_0}{2} \right) - \frac{1}{2}\alpha\delta^2 u_{0,r}^2 (\Omega_0 - c) - \delta^2 u_{0,rr} \left[ \frac{1}{6} + (\Omega_0 - c) \left( \frac{1}{3}\Omega_0 + \frac{1}{2}c \right) \right] + \alpha\delta^2 u_0 u_{0,rr} \left[ \frac{1}{2}(\Omega_0 - c) + (c - \Omega_0)^2 c - (\Omega_0 c + 1) \left( \frac{1}{3}\Omega_0 + \frac{1}{2}c \right) - \frac{c}{2} \right] = 0 \quad (11)$$

From the linear and dispersionless equation associated with (11) it follows that the critical positive velocity  $c_0$  is  $c_0 = \frac{\Omega_0}{2} + \sqrt{1 + \frac{\Omega_0^2}{4}}$ . Now we look for a solitary wave solution  $u_0(r)$  in the form:  $u_0(r) = A \operatorname{sech}^2(pr) + \alpha B \operatorname{sech}^4(pr)$ , with  $c$  a *super critical velocity* defined by  $c = c_0 + \alpha V$ . The constants  $B$ ,  $p$  and  $V$  must be determined in function of  $A$ . We have obtained

$$V = A c_0 \frac{3 + \Omega_0^2}{3(2c_0 - \Omega_0)}, p = \frac{1}{2\delta} \sqrt{\frac{\alpha A c_0 (3 + \Omega_0^2)}{\Omega_0^2 - c_0^2 + 2}} \left[ 1 + \frac{\alpha A c_0 (3 + \Omega_0^2)}{2(2c_0 - \Omega_0)(\Omega_0^2 - c_0^2 + 2)} \left( \frac{5\Omega_0}{6} - c_0 \right) \right], \quad (12)$$

$$B = \frac{3A^2}{20} \left[ \frac{\Omega_0^2 c_0 + 6\Omega_0 + c_0}{\Omega_0^2 - c_0^2 + 2} \right]. \quad (13)$$

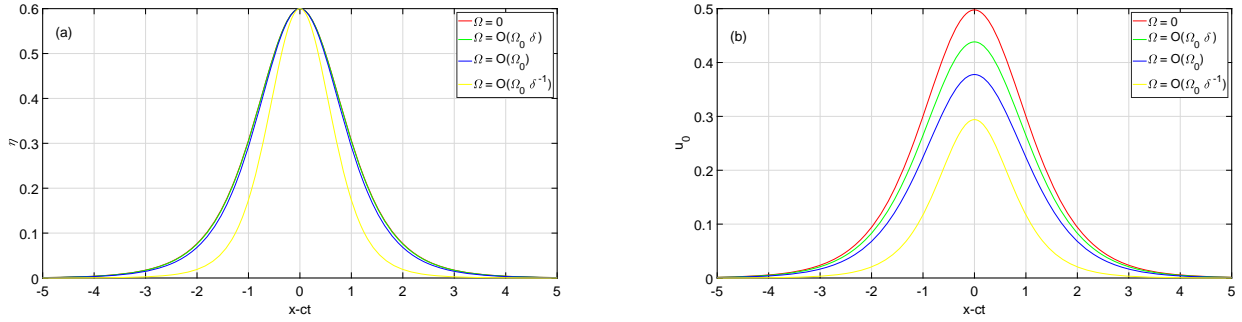


FIG. 1. Comparison of the free surface elevation  $\eta$  (a) and the mean velocity  $u_0$  (b) obtained with the four model equations. The red squares correspond to the case without any vorticity ( $\Omega = 0$ ), the green lines are the model with weak vorticity ( $\Omega = O(\Omega_0\delta)$ ), the blue lines describe the model with unit vorticity ( $\Omega = O(\Omega_0)$ ), and the yellow lines correspond to the strong vorticity case ( $\Omega = O(\Omega_0/\delta)$ ). The parameters used are  $\alpha = 0.6$ , and  $\delta = 0.5$ .

So the expressions of soliton solutions  $u_0(x, t)$  and  $\eta(x, t)$  read

$$u_0(x, t) = A \operatorname{sech}^2[p(x - ct)] + \alpha \frac{3A^2}{20} \left[ \frac{\Omega_0^2 c_0 + 6\Omega_0 + c_0}{\Omega_0^2 - c_0^2 + 2} \right] \operatorname{sech}^4[p(x - ct)], \quad (14)$$

$$\begin{aligned} \eta(x, t) = & c_0 A \operatorname{sech}^2[p(x - ct)] + c_0 (3 + \Omega_0^2) \left( \frac{1}{3(2c_0 - \Omega_0)} + \frac{2\Omega_0 - 3c_0}{6(\Omega_0^2 - c_0^2 + 2)} \right) \alpha A^2 \operatorname{sech}^2[p(x - ct)] \\ & + \frac{3c_0}{\Omega_0^2 - c_0^2 + 2} \left( \frac{\Omega_0^2 c_0 + 6\Omega_0 + c_0}{20} + \frac{(3 + \Omega_0^2)(3c_0 - 2\Omega_0)}{12} - \frac{1}{6} \frac{\Omega_0^2 - c_0^2 + 2}{c_0} \right) \alpha A^2 \operatorname{sech}^4[p(x - ct)] \end{aligned} \quad (15)$$

The parameter  $\alpha = a/h$  is by definition the dimensionless amplitude of  $\eta$ . The maximum of  $\eta$  is for  $x - ct = 0$  so taken  $\alpha = \eta(0)$ ,  $A$  is determined (in fonction of  $\alpha$ ) as the positive solution of the quadratic polynome

$$A^2 \left\{ \frac{c_0(3 + \Omega_0^2)}{3(2c_0 - \Omega_0)} - \frac{c_0(3 + \Omega_0^2)(2\Omega_0 - 3c_0)}{12(\Omega_0^2 - c_0^2 + 2)} + \frac{3c_0(\Omega_0^2 c_0 + 6\Omega_0 + c_0)}{20(\Omega_0^2 - c_0^2 + 2)} - \frac{1}{2} \right\} \alpha + c_0 A - \alpha = 0. \quad (16)$$

## B. Effect of the vorticity on the soliton properties

The purpose of this section is to describe the influence of vorticity magnitude on the properties of solitary waves. Fig. 1 presents the main characteristic of the solitary waves obtained for all cases of vorticity. Fig. 1 (a) corresponds to the water elevation  $\eta$ , while Fig. 1 (b) illustrates the distribution of the mean horizontal velocity  $u_0$  induced by the perturbation. In both subfigures, non dimensional parameters are chosen such as  $\alpha = 0.6$ ,  $\delta = 0.5$ . In every case considered the reference vorticity  $\Omega_0$  is chosen to be unity. In these figures red lines describe the results obtained when no vorticity is considered. Green lines correspond to the weak vorticity case, the blue lines are the result obtained with the normal vorticity, while the yellow lines describe the results obtained when the vorticity is strong. Since the value of  $\alpha$  is prescribed, the four curves describing water elevation (Fig.1 (a)) present the same maximum values. Differences appearing only concern the waves width, and thus the local slopes. The four curves presented in Fig. 1 (a) show a similar tendency. Solitary waves are narrower in the presence of vorticity. Indeed, if the curves obtained with no vorticity, and with weak vorticity are almost perfectly superimposed, one can notice that the model involving a unit vorticity describes a slightly narrower wave. The effect of vorticity when considering these two vortical conditions is almost insignificant, in terms of wave width. On the other, when considering the model assuming strong vorticity, the wave obtained is clearly narrower than the previous ones. The trend observed previously emphasized, and is more visible here. When considering the average velocity  $u_0$ , differences are much more significant induced by the solitary wave. These velocities are presented in Fig. 1 (b). From this figure, it clearly appears that the maximum velocity induced by the free surface disturbance is strongly affected by the magnitude of the vorticity. Namely, the stronger is the vorticity, the weaker is the fluid disturbance. It appears that the maximum of the velocity disturbance can be reduced up to 40% when the vorticity becomes strong. From equation (8) we observed that strong vorticity adds a negative dispersive term and a negative nonlinear dispersive term which read:  $-\Omega_0 \delta^2 \left\{ u_{0,xxx} \frac{1+3\alpha\eta}{3} + \alpha \eta_x u_{0,xx} \right\}$ . It is responsible for the trend leading to the weakest solitonic profil downstream propagating for strong vorticity. The effects on  $u_0(x, t)$  of normal and weak vorticities occur only through  $\eta(x, t)$ . For strong vorticity  $u_0(x, t)$  directly

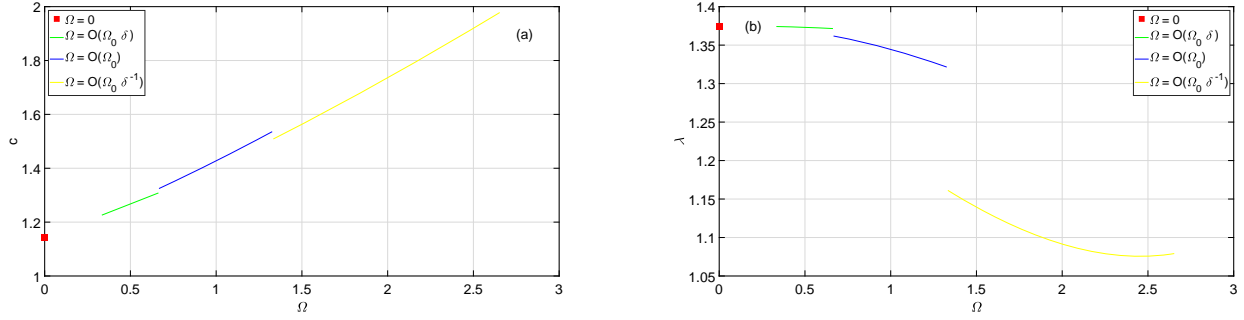


FIG. 2. Evolution of the wave velocity (a), and the wave width (b) as a function of the vorticity. The red squares correspond to the case without any vorticity ( $\Omega = 0$ ), the green lines are the model with weak vorticity ( $\Omega = O(\Omega_0 \delta)$ ), the blue lines describe the model with unit vorticity ( $\Omega = O(\Omega_0)$ ), and the yellow lines correspond to the strong vorticity case ( $\Omega = O(\Omega_0 \delta^{-1})$ ). The parameters used are  $\alpha = 0.6$ , and  $\delta = 0.5$ .

depends on  $\Omega$  through the new dispersive and nonlinear dispersive term. The evolution of the fluid velocity, observed in previous figures has to be related with the kinematic properties of the solitary wave. The evolution of its velocity and width are depicted in Fig. 2. On Fig. 2(a), the evolution of the wave celerity  $c$  of the solitary wave is plotted as a function of the vorticity  $\Omega$ , for the four models. Here again, the nonlinear and dispersive parameters ( $\alpha, \delta$ ) are set equal to  $(0.6, 0.5)$ , while  $\Omega$  varies. Since the models do not cover the same range of values for  $\Omega$ , four colors are used, in order to distinguish the area described by each model. Obviously, the model without vorticity corresponds to a single point in this diagram, while the green line correspond to the model with weak vorticity, the blue line is the model of vorticity of order unity, and the yellow line is the strong vorticity assumption. The first result observed is an opposite trend to the behaviour shown by the fluid disturbance velocity  $u_0$ . Namely, the celerity of the wave increases with vorticity, while the components of the fluid disturbance decreases with vorticity. For the parameters considered, the celerity is increased by about 85%. Another interesting remark, here, is to notice the relative continuity exhibited by the four models. Indeed, they have been derived independently, under various asymptotic assumptions, but present very compatible trends, and values. In figure 2 (b), the evolution of the width of the wave,  $\lambda$ , defined as  $\lambda = 1/p$ , is presented versus vorticity  $\Omega$  for the four models. This figure confirms the trend forebode on Fig. 1 (a). Here again, the four models are juxtaposed, where the model without vorticity corresponds to a single point, a red square, in this diagram, the green line correspond to the model with weak vorticity, the blue line is the model of vorticity of order unity, and the yellow line is the strong vorticity assumption. The three models involving vorticity present a similar trend, the width decreasing when vorticity increases. The behaviour of  $\lambda(\Omega)$  is quite difficult to analyse from the soliton expressions of  $u_0(x, t)$  and  $\eta(x, t)$ . However this becomes possible with soliton solutions of the Korteweg-de Vries equations associated to the system (6)-(7). Lets have a look at  $\lambda(\Omega)$  of the soliton solutions of KdV equations for  $u_0(x, t)$  in the two extreme cases  $\Omega = 0$  and  $\Omega = \Omega_0 \delta^{-1}$ . Using asymptotic methods (see [22]) in (8-9) we obtain  $u_{0,t}$  for these two extreme cases. They read

$$u_{0,t} + \frac{3}{2}\alpha u_0 u_{0,x} + \frac{\delta^2}{6} u_{0,xxx} = 0, \quad \Omega = 0, \quad (17)$$

$$(\Omega_0^2 + 4)^{\frac{1}{2}} u_{0,t} + c_0 (\Omega_0^2 + 3) \alpha u_0 u_{0,x} + \frac{\delta^2}{6} (c_0^2 - \Omega_0^2) u_{0,xxx} = 0, \quad \Omega = \Omega_0 / \delta, \quad c_0 = \frac{1}{2} [\Omega_0 + (\Omega_0^2 + 4)^{1/2}]. \quad (18)$$

The soliton solutions of (17) and (18) read

$$u_0 = A_s \operatorname{sech}^2 \left[ \frac{1}{2\delta} (3\alpha A_s)^{\frac{1}{2}} \left( x - \frac{A_s \alpha}{2} t \right) \right], \quad u_0 = A_s \operatorname{sech}^2 \left\{ \frac{1}{2\delta} (3\alpha A_s)^{\frac{1}{2}} \left[ \frac{c_0 (3 + \Omega_0^2)}{3(c_0^2 - \Omega_0^2)} \right]^{\frac{1}{2}} \left[ x - \left( \frac{\alpha A_s c_0 (3 + \Omega_0^2)}{3(4 + \Omega_0^2)} \right) t \right] \right\}, \quad (19)$$

with  $A_s$  an arbitrary constant amplitude. The associated soliton widths read

$$\lambda(\Omega = 0) = \frac{2\delta}{(3\alpha A_s)^{\frac{1}{2}}}, \quad \lambda(\Omega = \frac{\Omega_0}{\delta}) = \frac{2\delta}{(3\alpha A_s)^{\frac{1}{2}}} \left[ \frac{c_0^2 - \Omega_0^2}{c_0 (1 + \frac{\Omega_0^2}{3})} \right]^{\frac{1}{2}}. \quad (20)$$

From expressions (20) (taken  $\Omega_0 = 1$ ) we obtain  $\frac{\lambda(\Omega = \frac{1}{\delta})}{\lambda(\Omega = 0)} \sim 0, 5 < 1$ , and (19) in the case  $\Omega = \Omega_0 \delta^{-1}$  is a downstream wave narrower than in the case  $\Omega = 0$ . So for downstream propagation the stronger the vorticity the narrower the width. This is a very well known result not only for KdV but also for the Boussinesq equation and the Camassa-Holm equation (see [23]).

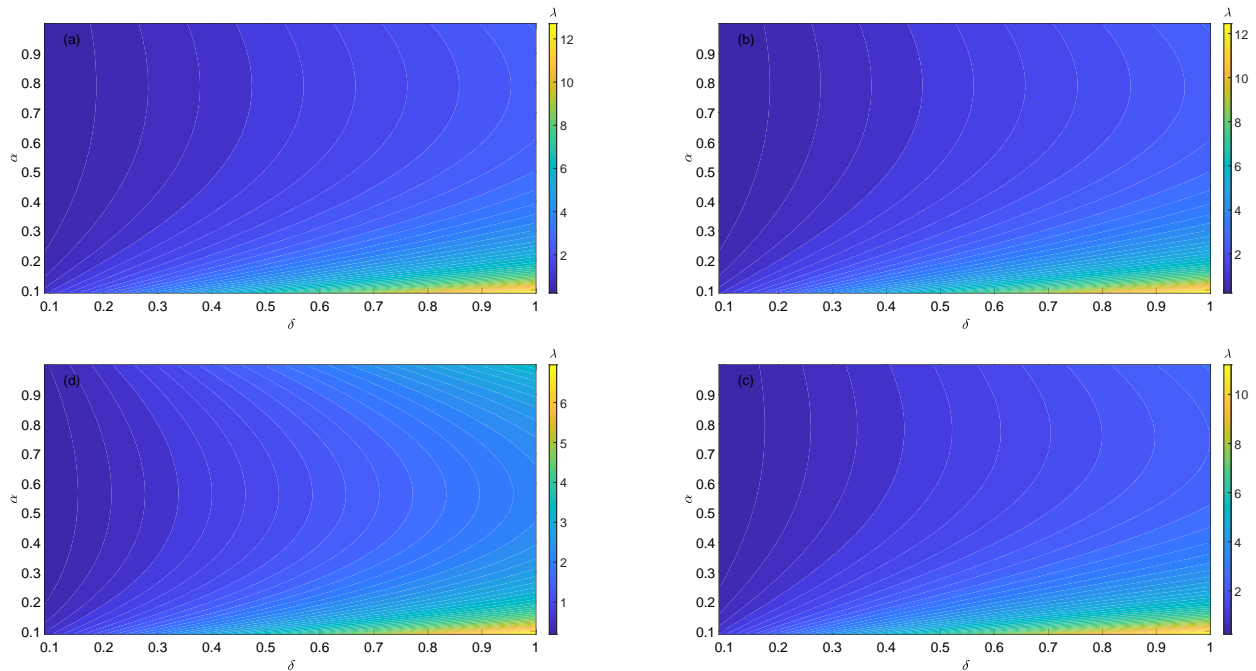


FIG. 3. Evolution of the width of the wave as a function of the dispersive parameter  $\delta$  and the nonlinear parameter  $\alpha$ . (a)  $\Omega = 0$ , (b)  $\Omega = O(\Omega_0\delta)$ , (c)  $\Omega = O(\Omega_0)$ , (d)  $\Omega = O(\Omega_0\delta^{-1})$ .

### C. Effect of $\alpha$ and $\delta$ on the soliton properties

Another interesting aspect of this approach is to allow comparisons of the influence of the parameters  $\alpha$  and  $\delta$  on the wave properties. Following that purpose, Fig. 3 describes the evolution of the wave width  $\lambda$  in the parameter map  $(\alpha, \delta)$ . Sub-figure (a) corresponds to the behaviour of a solitary wave propagating in the absence of vorticity, sub-figure (b) is the weak vorticity case, sub-figure (c) is the unit vorticity model, while sub-figure (d) is the strong vorticity configuration. First, it has to be emphasized that these figures present very different colour scales, confirming the trends observed in Fig. 2. In the meantime, it is striking to notice that the qualitative dependence on both the nonlinear and the dispersive parameters are very similar. Oppositely, Fig. 4 illustrates the evolution of the wave celerity in the parameter map  $(\alpha, \delta)$ . Sub-figure (a) corresponds to the behaviour of a solitary wave propagating in the absence of vorticity, sub-figure (b) is the weak vorticity case, sub-figure (c) is the unit vorticity model, while sub-figure (d) is the strong vorticity configuration. It is interesting to notice that, in the absence of vorticity, the travelling celerity does not depend on the dispersive parameter  $\delta$ , which confirms classical behaviour of travelling solitary waves in the absence of vorticity. Noteworthy, this result does not hold when vorticity is involved. Indeed, it is interesting to notice that, when weak or unit vorticity are considered, the wave celerity exhibits a strong dependence on  $\alpha$  and  $\delta$ . However, the respective influence of  $\alpha$  and  $\delta$  on this quantity is not completely similar. In the weak vorticity model, the dispersive parameter  $\delta$  seems to play a less significant role than in the unit vorticity approach. Finally, when considering the strong vorticity case, the celerity becomes constant again with respect to  $\delta$ . This asymptotic case exhibits characteristics very similar to those observed in the absence of vorticity.

## IV. CONCLUSIONS AND PERSPECTIVES

We have introduced a new hierarchy of weakly nonlinear equations for surface gravity in finite depth in the presence of constant vorticity  $\Omega$ . The members of the hierarchy are systems of equations for the  $z$ -integrate horizontal velocity  $u(x, z, t)$  and the wave amplitude  $\eta(x, t)$ . They involve weak nonlinearities, dispersion and nonlinear dispersion and are characterized by the powers of two perturbative parameters  $\alpha$  and  $\delta$  (nonlinearity and dispersion) and their products (nonlinear dispersion). The vorticity  $\Omega$  appears in the equations of motion in factors  $\Omega\delta^s$  ( $s = 1, 3, \dots$ ), so we have scaled  $\Omega$  as  $\Omega = \Omega_0\delta^q$  with  $\Omega_0 \sim \mathcal{O}(\delta^0)$  and  $q$  integer. The negatives values of  $q$  must be chosen such that the neglected term  $\mathcal{O}(\Omega\delta^5, \delta^4)$  in equation (4) remains negligible. A remarkable finding is that the physical origin of dispersion and nonlinear dispersion for  $u_0(x, t)$  can be associated or not with the vorticity  $\Omega$ . In the  $\eta(x, t)$  dynamics,



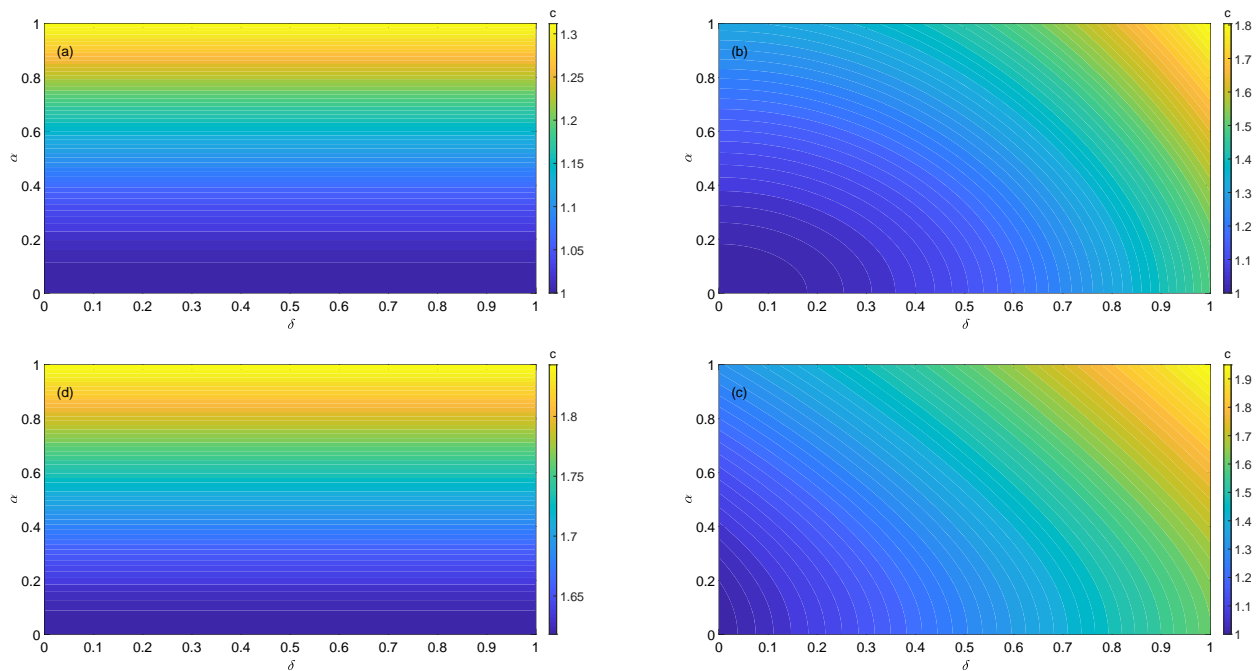


FIG. 4. Evolution of the wave celerity as a function of the dispersive parameter  $\delta$  and the nonlinear parameter  $\alpha$ . (a)  $\Omega = 0$ , (b)  $\Omega = O(\Omega_0\delta)$ , (c)  $\Omega = O(\Omega_0)$ , (d)  $\Omega = O(\Omega_0\delta^{-1})$ .

vorticity is never directly associated with nonlinear dispersion.

In order to analyse some physical-mathematical consequences while remaining relatively simple in formalism, we have only discussed equations at orders  $\alpha$ ,  $\delta^n$  and  $\alpha\delta^n$  with  $n = 1, 2$ . We have studied analytically and numerically the case of strong vorticity. The cases of null, normal and weak vorticity were only numerically analyzed. Non zero cases clearly shown that vorticity changes linearity, nonlinearity, dispersion and nonlinear-dispersion in relation to the zero vorticity case. Normal vorticity adds a nonlinear term and changes the linear associated equation for  $\eta(x, t)$  as the equation for  $u_0(x, t)$  does not change. Weak vorticity has nearly the same effects: changes in  $\eta(x, t)$  are the same but weaker and the equation for  $u_0(x, t)$  does not change. For strong vorticity the changes in  $\eta(x, t)$  are the same but order  $\delta^0$  and the changes for  $u_0(x, t)$  are very large because new dispersive and nonlinear terms appear. For each case we have numerically analyzed their solitonic solutions (celerity, amplitude and wave width).

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